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General optimal L^p-Nash inequalities on Riemannian manifolds

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Abstract. Let (M, g) be a smooth compact Riemannian manifold of dimension $2 \leq n$, let $1 < p$ and $1 \leq q < p$. In this paper, we establish the validity of the optimal Nash inequality

$$\left(\int_M |u|^p \, dv_g\right)^{\frac{\tau}{\theta p}} \leq \left(A_{\mathrm{opt}} \left(\int_M |\nabla_g u|^p \, dv_g\right)^{\frac{\tau}{p}} + B_{\mathrm{opt}} \left(\int_M |u|^p \, dv_g\right)^{\frac{\tau}{p}}\right) \left(\int_M |u|^q \, dv_g\right)^{\frac{\tau(1-\theta)}{\theta q}},$$

and the existence of extremal functions for this optimal inequality.

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1. Introduction and the main result

In 1958, Nash [14] showed the validity of the inequality

$$\left(\int_{\mathbb{R}^n} |u|^2 dx\right)^{\frac{n+2}{n}} \leq c \int_{\mathbb{R}^n} |\nabla u|^2 dx \left(\int_{\mathbb{R}^n} |u| dx\right)^{\frac{4}{n}}, \tag{1.1}$$

for every function $u \in C_0^\infty(\mathbb{R}^n)$ and some constant $c > 0$. This inequality was used to obtain *a priori* estimates for parabolic problems. The proof of the validity of this inequality essentially involves techniques of Fourier-transform. In this paper we are interested in a Nash inequality involving more general parameters. In order to study the inequality (1.1) using more general parameters, the Fourier transform is not appropriate. Thus we will use a combination of Jensen's inequality and the entropy inequality, as already observed by Beckner [3], to produce general Nash inequality.

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Let $\mathbf{Ent}_{dx}(|u|^p) = \int_{\mathbb{R}^n} |u|^p \ln(|u|^p)dx$ and $p > 1$. The entropy inequality states that there is a constant c such that

$$\mathbf{Ent}_{dx}(|u|^p) \leq \frac{n}{p} \ln \left(c \int_M |\nabla u|^p dx \right) ,$$

for all functions $u \in C_0^\infty(\mathbb{R}^n)$ with $\|u\|_{L^p(\mathbb{R}^n)} = 1$. We define the optimal constant of this inequality by

$$L(p,n)^{-1} = \inf_{u \in C_0^\infty(\mathbb{R}^n)} \left\{ \frac{\int_{\mathbb{R}^n} |\nabla u|^p dx}{\exp\left(\frac{p}{n} \mathbf{Ent}_{dx}(|u|^p)\right)} ; \|u\|_{L^p(\mathbb{R}^n)} = 1 \right\} .$$

We produce therefore the optimal entropy inequality

$$\mathbf{Ent}_{dx}(|u|^p) \leq \frac{n}{p} \ln \left(L(p,n) \int_{\mathbb{R}^n} |\nabla u|^p dx \right) . \tag{1.2}$$

The optimal constant $L(p,n)$ was calculated by del Pino and Dolbeault [7] for $1 < p < n$ and by Gentil [9] for $n \leq p$.

Consider $u \in C_0^\infty(\mathbb{R}^n)$ such that $\|u\|_{L^p(\mathbb{R}^n)} = 1$ and $1 \leq q < p$. Using Jensen's inequality, we find

$$\begin{aligned} -\ln\left(\int_{\mathbb{R}^n} |u|^q dx \right) &= -\ln \left(\int_{\mathbb{R}^n} |u|^{q-p} |u|^p dx \right) \\ &\leq -\int_{\mathbb{R}^n} \ln(|u|^{q-p}) |u|^p dx = \frac{p-q}{p} \int_{\mathbb{R}^n} \ln(|u|^p) |u|^p dx . \end{aligned}$$

When this inequality and (1.2) are coupled together, we obtain

$$\left(\int_{\mathbb{R}^n} |u|^q dx \right)^{-\frac{p^2}{n(p-q)}} \leq L(p,n) \int_{\mathbb{R}^n} |\nabla u|^p dx ,$$

for all functions $u \in C_0^\infty(\mathbb{R}^n)$ such that $\|u\|_{L^p(\mathbb{R}^n)} = 1$. By homogeneity and defining

$$\theta = \frac{n(p-q)}{qp - qn + np} ,$$

we obtain the *general Nash inequality*

$$\left(\int_{\mathbb{R}^n} |u|^p dx \right)^{\frac{1}{\theta}} \leq L(p,n) \int_{\mathbb{R}^n} |\nabla u|^p dx \left(\int_{\mathbb{R}^n} |u|^q dx \right)^{\frac{p(1-\theta)}{\theta q}} ,$$

for all functions $u \in C_0^\infty(\mathbb{R}^n)$ where $1 < p$ and $1 \leq q < p$. The *Euclidean optimal Nash inequality* states that, for any function $u \in C_0^\infty(\mathbb{R}^n)$, we have that

$$\left(\int_{\mathbb{R}^n} |u|^p \, dx \right)^{\frac{1}{\theta}} \leq N(p,q,n) \int_{\mathbb{R}^n} |\nabla u|^p \, dx \left(\int_{\mathbb{R}^n} |u|^q \, dx \right)^{\frac{p(1-\theta)}{\theta q}} , \tag{1.3}$$

occurs, where

$$N(p,q,n)^{-1} = \inf_{u \in C_0^\infty(\mathbb{R}^n)} \left\{ \|\nabla u\|^p_{L^p(\mathbb{R}^n)} \|u\|^{\frac{p(1-\theta)}{\theta}}_{L^q(\mathbb{R}^n)} ; \|u\|_{L^p(\mathbb{R}^n)} = 1 \right\}$$

is the best possible constant in the above inequality. In particular, we have

$$L(p,n) \geq N(p,q,n)$$

for all $p > 1$ and $n \geq 2$. Making use of an asymptotic argument contained in Bakry, Coulhon, Ledoux and Sallof-Coste [2] we can check that the optimal entropy inequality will be achieved through the optimal Nash inequality, particularly

$$\lim_{q \to p^-} N(p,q,n) = L(p,n) \ .$$

Thus, Nash's inequality (1.3) and the entropy inequality (1.2) are intimately related. These inequalities are important tools which are needed to establish *a priori* estimates to control the behavior of the solutions. In the famous paper of Nash [14], the inequality (1.1) and the optimal entropy inequality (1.2) were used as a key points in obtaining control over estimates of the solution of parabolic equations. An interesting discussion on the use of methods involving the entropy inequality and Nash's inequality for obtaining estimates of ultracontractive semigroups can be found in Coulhon [6]. Nash's inequality is also a relevant instrument for studying the smoothness properties of the Markov semigroup, (see Bakry, Bolley, Gentil and Maheux [1]).

Our main goal in this work is to study the inequality (1.3) in a Riemannian context. For this purpose, consider (M, g) a smooth compact Riemannian manifold. Combining inequality (1.3) with a local-to-global-type argument based on the partition of unity, such as the one made by Druet, Hebey and Vaugon [8], produces for each $\tilde{\varepsilon} > 0$ a constant $B_{\tilde{\varepsilon}}$, such that

$$\left(\int_M |u|^p \, dv_g \right)^{\frac{1}{\theta}} \leq \left((N(p,q,n) + \tilde{\varepsilon}) \int_M |\nabla_g u|^p \, dv_g + B_{\tilde{\varepsilon}} \int_M |u|^p \, dv_g \right) \left(\int_M |u|^q \, dv_g \right)^{\frac{p(1-\theta)}{\theta q}} ,$$

for all functions u in the Sobolev space $H^{1,p}(M)$, where dv_g and ∇_g denote, respectively, the Riemannian volume element and the gradient operator, for $p > 1$, $1 \leq q < p$ and $\theta = \frac{n(p-q)}{qp-qn+np}$. Considering $0 < \tau \leq p$, by direct computation, for all $u \in H^{1,p}(M)$ we have

$$\left(\int_M |u|^p dv_g \right)^{\frac{\tau}{\theta p}} \leq \left((N(p,q,n)^{\frac{\tau}{p}} + \varepsilon) \left(\int_M |\nabla_g u|^p \, dv_g \right)^{\frac{\tau}{p}} + B_\varepsilon \left(\int_M |u|^p dv_g \right)^{\frac{\tau}{p}} \right) \left(\int_M |u|^q dv_g \right)^{\frac{\tau(1-\theta)}{\theta q}} ,$$

denote this inequality by

$$\left(N(N(p,q,n)^{\frac{\tau}{p}} + \varepsilon, B_\varepsilon\right)$$

where $\varepsilon > 0$ can be taken as close to zero as we wish. This inequality shows that there exist constants $A, B \in \mathbb{R}$ such that $N(A, B)$ is valid for all functions in $H^{1,p}(M)$. We will use again an adaptation of the ideas in [8] and a local argument in normal coordinates, to showing that

$$N(p,q,n)^{\frac{\tau}{p}} \leq A\,. \tag{1.4}$$

Thus we can define the notion of best constant for the Riemannian Nash inequality. Due to the two constants of present in the Riemannian Nash inequality, the optimality can be defined in two ways. We follow the more interesting one from the viewpoint of partial differential equations (PDE's) (see Hebey [10]). The *first Riemannian L^p-Nash best constant* is defined by

$$A_{\mathrm{opt}} = \inf\{A \in \mathbb{R} : \text{ there exists } B \in \mathbb{R} \text{ such that } N(A,B) \text{ is valid}\}\,.$$

By (1.4), we have

$$N(p,q,n)^{\frac{\tau}{p}} \leq A_{\mathrm{opt}}\,.$$

Moreover, since $N(N(p,q,n)^{\frac{\tau}{p}} + \varepsilon, B_\varepsilon)$ is valid for all $\varepsilon > 0$, it follows that

$$A_{\mathrm{opt}} = N(p,q,n)^{\frac{\tau}{p}}\,.$$

We note that the first Riemannian L^p-Nash best constant is independent of the geometry of M. From this constant which we can define the *first optimal Riemannian L^p-Nash inequality* as follows: there exists a constant $B \in \mathbb{R}$ such that, for any $u \in H^{1,p}(M)$,

$$\left(\int_M |u|^p\, dv_g\right)^{\frac{\tau}{\theta p}} \leq \left(N(p,q,n)^{\frac{\tau}{p}} \left(\int_M |\nabla_g u|^p\, dv_g\right)^{\frac{\tau}{p}} + B\left(\int_M |u|^p\, dv_g\right)^{\frac{\tau}{p}}\right)\left(\int_M |u|^q\, dv_g\right)^{\frac{\tau(1-\theta)}{\theta q}}\,.$$

We should note that the validity of this inequality is not evident because when we make $A \to A_{\mathrm{opt}}$ the constant $B = B(A)$ could diverge to infinity.

Independently, the validity of optimal Riemannian L^p-Nash inequalities has been widely discussed in some special cases over the last few years. The optimal Nash inequality $N(N(p,q,n)^{\frac{\tau}{p}}, B)$ with $\tau = p = 2$ and $q = 1$ was obtained for some B by Humbert in [11]. Later, Brouttelande [4] proved Nash's inequality for $\tau = p = 2$ and $1 \leq q < p$. Recently Ceccon-Montenegro [5] proved Nash's inequality for $1 < p = \tau \leq 2$ and $1 \leq q < p$. In this paper, we extend the validity of optimal inequality $N(N(p,q,n)^{\frac{\tau}{p}}, B)$ for $p > 2$.

Theorem 1. *Let (M,g) be a smooth, compact Riemannian manifold without boundary of dimension $n \geq 2$ and let $1 \leq q < p$. If $0 < \tau \leq \min\{p, 2\}$, then $N(N(p,q,n)^{\frac{\tau}{p}}, B)$ is always valid for some B.*

In view of this theorem, we can define the *second Riemannian L^p-Nash best constant* by

$$B_{\text{opt}} = \inf\left\{B \in \mathbb{R};\, N\left(N(p,q,n)^{\frac{\tau}{p}}, B\right) \text{ is valid}\right\} .$$

Since the non-zero constant functions belong to the Sobolev space $H^{1,p}(M)$, the constant B_{opt} satisfies

$$B_{\text{opt}} \geq |M|^{-\frac{\tau}{n}} , \tag{1.5}$$

where $|M|$ denotes the volume of M.

A non-zero function in $H^{1,p}(M)$ satisfying the equality $N(N(p,q,n)^{\frac{\tau}{p}}, B_{\text{opt}})$ is called an *extremal function*. The extremal functions for the Riemannian Nash inequality in the case where $\tau = p = 2$ and $q = 1$ were studied by Humbert [12]. In this work, Humbert found that the existence of extremal functions depends on the geometry of the Riemannian manifold (M, g). However, when $\tau < 2$ we have the existence of extremal functions regardless of the geometry of the Riemannian manifold (M, g).

Theorem 2. *Let (M, g) be a smooth, compact Riemannian manifold without boundary of dimension $n \geq 2$ and let $1 \leq q < p$. If $0 < \tau \leq p$ and $\tau < 2$. Then $N(N(p,q,n)^{\frac{\tau}{p}}, B_{\text{opt}})$ admits an extremal function.*

2. Proof of Theorem 1

The proof of Theorem 1 proceeds by contradiction. So, for each α positive in $\mathbb{N}$, suppose one has

$$\nu_\alpha = \sup_{u \in E} J_\alpha(u) > N(p,q,n)^{\frac{\tau}{p}} , \tag{2.1}$$

where $E = \{u \in H^{1,p}(M) : \; \|\nabla_g u\|_{L^p(M)} = 1\}$ and

$$J_\alpha(u) = \left(\int_M |u|^p \, dv_g\right)^{\frac{\tau}{p\theta}} \left(\int_M |u|^q \, dv_g\right)^{-\frac{\tau(1-\theta)}{\theta q}} - \alpha \left(\int_M |u|^p \, dv_g\right)^{\frac{\tau}{p}} .$$

Our goal will be to use the method of Lagrange multipliers for the functional J_α; this method requires that the functional J_α is of class C^1. Thus we will consider two situations: assume first $q > 1$. By using standard arguments, we find a maximizer $\tilde{u}_\alpha \in E$ of J_α, *i.e.*

$$J_\alpha(\tilde{u}_\alpha) = \nu_\alpha = \sup_{u \in E} J_\alpha(u) . \tag{2.2}$$

When $q = 1$, note that the functional J_α is not of class C^1. In this case, we imitate an idea of Humbert [11].

Define, for each $\varepsilon > 0$, the functional

$$J_{\alpha,\varepsilon}(u) = \left(\int_M |u|^p \, dv_g\right)^{\frac{\tau}{p\theta}} \left(\int_M |u|^{1+\varepsilon} \, dv_g\right)^{-\frac{\tau(1-\theta)}{\theta(1+\varepsilon)}} - \alpha \left(\int_M |u|^p \, dv_g\right)^{\frac{\tau}{p}} .$$

It is clear that $J_{\alpha,\varepsilon}$ is of class C^1. Choose now a sequence (ε_α) such that $\varepsilon_\alpha \to 0$ as $\alpha \to \infty$, so that

$$\nu_{\varepsilon_\alpha,\alpha} = \sup_{u \in E} J_{\alpha,\varepsilon_\alpha}(u) > N(p,1,n)^{\frac{\tau}{p}} .$$

As usual, the preceding inequality leads to a maximizer $\tilde{u}_\alpha \in E$ of $J_{\alpha,\varepsilon_\alpha}$. From now on, the arguments are similar in the two cases $q > 1$ and $q = 1$ by working with the functionals J_α and $J_{\alpha,\varepsilon_\alpha}$ in each case. Thereby, we will focus our attention only on the case $q > 1$.

One may assume $\tilde{u}_\alpha \geq 0$, since $\nabla_g|\tilde{u}_\alpha| = \pm\nabla_g\tilde{u}_\alpha$ almost everywhere. By (2.2), $\tilde{u}_\alpha$ satisfies the Euler-Lagrange equation

$$\begin{aligned} &\frac{1}{\theta} \|\tilde{u}_\alpha\|_{L^p(M)}^{\frac{\tau}{\theta}-p} \|\tilde{u}_\alpha\|_{L^q(M)}^{-\frac{\tau(1-\theta)}{\theta}} \tilde{u}_\alpha^{p-1} - \frac{(1-\theta)}{\theta} \|\tilde{u}_\alpha\|_{L^p(M)}^{\frac{\tau}{\theta}} \|\tilde{u}_\alpha\|_{L^q(M)}^{-\frac{\tau(1-\theta)}{\theta}-q} \tilde{u}_\alpha^{q-1} \\ &- \alpha \|\tilde{u}_\alpha\|_{L^p(M)}^{\tau-p} \tilde{u}_\alpha^{p-1} = \nu_\alpha \Delta_{p,g} \tilde{u}_\alpha , \end{aligned} \tag{2.3}$$

where $\Delta_{p,g} = -\mathrm{div}_g(|\nabla_g|^{p-2}\nabla_g)$ is the p-Laplace operator of g. Taking $u_\alpha = \frac{\tilde{u}_\alpha}{\|\tilde{u}_\alpha\|_{L^p(M)}}$, we have

$$\lambda_\alpha^{-1} A_\alpha \Delta_{p,g} u_\alpha + \alpha A_\alpha^{\frac{\tau}{p}} u_\alpha^{p-1} + \frac{1-\theta}{\theta} \|u_\alpha\|_{L^q(M)}^{-q} u_\alpha^{q-1} = \frac{1}{\theta} u_\alpha^{p-1} \quad \text{on } M , \tag{2.4}$$

where $\|u_\alpha\|_{L^p(M)} = 1$,

$$A_\alpha = \left(\int_M u_\alpha^q \, dv_g\right)^{\frac{p(1-\theta)}{\theta q}}$$

and

$$\lambda_\alpha = \nu_\alpha^{-1} \|u_\alpha\|_{L^q(M)}^{\frac{(p-\tau)(1-\theta)}{\theta}} \|\tilde{u}_\alpha\|_{L^p(M)}^{\tau-p} .$$

We now highlight two important consequences of the Euler-Lagrange equation. Note that taking $\tilde{u}_\alpha$ as test function in (2.3), we have

$$\nu_\alpha \leq \|\tilde{u}_\alpha\|_{L^p(M)}^{\frac{\tau}{\theta}} \|\tilde{u}_\alpha\|_{L^q(M)}^{-\frac{\tau(1-\theta)}{\theta}} .$$

Putting together the previous inequality, (2.1) and the observation that $\tau \leq p$, we get

$$\begin{aligned} \lambda_\alpha &= \nu_\alpha^{-1} \|\tilde{u}_\alpha\|_{L^p(M)}^{\frac{(\tau-p)(1-\theta)}{\theta}} \|\tilde{u}_\alpha\|_{L^q(M)}^{-\frac{(\tau-p)(1-\theta)}{\theta}} \|\tilde{u}_\alpha\|_{L^p(M)}^{\tau-p} \\ &= \nu_\alpha^{-1} \left(\|\tilde{u}_\alpha\|_{L^p(M)}^{\frac{\tau}{\theta}} \|\tilde{u}_\alpha\|_{L^q(M)}^{-\frac{\tau(1-\theta)}{\theta}} \right)^{\frac{\tau-p}{\tau}} \leq \nu_\alpha^{-\frac{p}{\tau}} \leq N(p,q,n)^{-1} , \end{aligned}$$

so that

$$N(p,q,n) \le \lambda_\alpha^{-1}\,. \tag{2.5}$$

We also have, by (2.4), that $\alpha A_\alpha^{\frac{\tau}{p}} \le 1$. Therefore, taking a subsequence,

$$\lim_{\alpha\to\infty} A_\alpha = 0\,. \tag{2.6}$$

The proof of this theorem consists of three steps:

1) We will show that the sequence of normalized functions u_α has a point of explosion when $\alpha \to \infty$. Hence u_α will concentrate around its point of maximum, this will be made precise in Subsection 2.1.

2) The concentration established in Subsection 2.1 will be used to obtain a global and uniform estimate for the sequence u_α. Besides, the speed with which u_α tends to zero is of exponential type. This will be studied in Subsection 2.2.

3) In Subsection 2.3 we will use an appropriate test function to combine the optimal Euclidean inequality involving u_α with the Euler-Lagrange equation satisfied by u_α. This will lead to a comparison between $\|u_\alpha\|_{L^q(M)}$ and a piece of $\|u_\alpha\|_{L^p}$. This comparison, due to the speed with which the sequence u_α goes to zero (Subsection 2.2), will generate a contradiction. With this the proof of Theorem 1 is concluded.

2.1. L^p-concentration

Applying the Tolksdorf's regularity [16] in (2.4), it follows that u_α is of class $C^1(M)$. Then we can consider $x_\alpha \in M$ a maximum point of u_α, *i.e.*

$$u_\alpha(x_\alpha) = \|u_\alpha\|_{L^\infty(M)}\,.$$

Our aim here is to establish that

$$\lim_{\sigma\to\infty}\lim_{\alpha\to\infty}\int_{B(x_\alpha,\sigma A_\alpha^{\frac{1}{p}})} u_\alpha^p\, dv_g = 1\,. \tag{2.7}$$

From now on, c will denote several possibly different positive constants independent of α.

For each $x \in B(0,\sigma)$, define

$$\begin{aligned} h_\alpha(x) &= g\left(\exp_{x_\alpha}(A_\alpha^{\frac{1}{p}} x)\right), \\ \varphi_\alpha(x) &= A_\alpha^{\frac{n}{p^2}} u_\alpha\left(\exp_{x_\alpha}(A_\alpha^{\frac{1}{p}} x)\right). \end{aligned} \tag{2.8}$$

By (2.4) and the definition of θ, one easily deduces that

$$\lambda_\alpha^{-1}\Delta_{p,h_\alpha}\varphi_\alpha + \alpha A_\alpha^{\frac{\tau}{p}}\varphi_\alpha^{p-1} + \frac{1-\theta}{\theta}\varphi_\alpha^{q-1} = \frac{1}{\theta}\varphi_\alpha^{p-1} \;\text{ on }\; B(0,\sigma)\,. \tag{2.9}$$

Applying the Moser's iterative scheme, for $p \leq n$ see ([13] or [15]), or Morrey's inequality, for $p > n$, to this last equation, we see that for α large enough and $\sigma > 1$

$$\left(A_\alpha^{\frac{n}{p^2}} \|u_\alpha\|_{L^\infty(M)}\right)^p = \sup_{B(0,\frac{1}{2})} \varphi_\alpha^p \leq c \int_{B(0,1)} \varphi_\alpha^p \, dh_\alpha = c \int_{B(x_\alpha, A_\alpha^{\frac{1}{p}})} u_\alpha^p dv_g \leq c \, .$$

Using the definition of A_α and this estimate coupled together with

$$1 = \int_M u_\alpha^p \, dv_g \leq \|u_\alpha\|_{L^\infty(M)}^{p-q} \int_M u_\alpha^q dv_g = \left(\|u_\alpha\|_{L^\infty(M)} \, A_\alpha^{\frac{n}{p^2}}\right)^{p-q}$$

produces

$$1 \leq \|u_\alpha\|_{L^\infty(M)} \, A_\alpha^{\frac{n}{p^2}} \leq c \tag{2.10}$$

for α large enough. In particular, there exists a constant $c > 0$ such that

$$\int_{B(0,1)} \varphi_\alpha^p \, dh_\alpha \geq c \tag{2.11}$$

for all α large enough.

On the other hand, using Cartan's expansion in normal coordinates, we have for each $\sigma > 0$, that

$$\int_{B(0,\sigma)} \varphi_\alpha^p dx \leq c \int_{B(0,\sigma)} \varphi_\alpha^p dh_\alpha = c \int_{B(x_\alpha, \sigma A_\alpha^{\frac{1}{p}})} u_\alpha^p dv_g \leq c \, ,$$

and we also have, by (2.4),

$$\begin{aligned} \int_{B(0,\sigma)} |\nabla \varphi_\alpha|^p \, dx &\leq c \int_{B(0,\sigma)} |\nabla_{h_\alpha} \varphi_\alpha|^p \, dh_\alpha \\ &= c A_\alpha \int_{B(x_\alpha, \sigma A_\alpha^{\frac{1}{p}})} |\nabla_g u_\alpha|^p \, dv_g \leq c N(p,q,n)^{-1} \, , \end{aligned}$$

for α large enough. Therefore by Cantor's diagonalization process, there exists $\varphi \in W^{1,p}(\mathbb{R}^n)$ such that, for some suitable subsequence, $\varphi_\alpha \rightharpoonup \varphi$ in $W^{1,p}_{loc}(\mathbb{R}^n)$. For each $\sigma > 0$, we can conclude then

$$\int_{B(0,\sigma)} \varphi^q \, dx = \lim_{\alpha \to \infty} \int_{B(0,\sigma)} \varphi_\alpha^q \, dh_\alpha = \lim_{\alpha \to \infty} \frac{\int_{B(x_\alpha, \sigma A_\alpha^{\frac{1}{p}})} u_\alpha^q \, dv_g}{\int_M u_\alpha^q \, dv_g} \leq 1 \, ,$$

and

$$\int_{B(0,\sigma)} \varphi^p \, dx = \lim_{\alpha \to \infty} \int_{B(0,\sigma)} \varphi_\alpha^p \, dh_\alpha = \lim_{\alpha \to \infty} \int_{B(x_\alpha, \sigma A_\alpha^{\frac{1}{p}})} u_\alpha^p \, dv_g \leq 1 \, .$$

In particular,

$$\varphi \in L^q(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)\ . \tag{2.12}$$

Let $\eta \in C_0^1(\mathbb{R})$ be a cut-off function such that $\eta = 1$ on $[0, \frac{1}{2}]$, $\eta = 0$ on $[1, \infty)$ and $0 \leq \eta \leq 1$. Let $\eta_{\alpha,\sigma}(x) = \eta((\sigma A_\alpha^{\frac{1}{p}})^{-1} d_g(x, x_\alpha))$. Choosing $u_\alpha \eta_{\alpha,\sigma}^p$ as a test function in (2.4), one gets

$$\begin{aligned}\lambda_\alpha^{-1} A_\alpha \int_M |\nabla_g u_\alpha|^p \eta_{\alpha,\sigma}^p\, dv_g + A_\alpha \int_M |\nabla_g u_\alpha|^{p-2} \nabla_g u_\alpha \cdot \nabla_g(\eta_{\alpha,\sigma}^p) u_\alpha\, dv_g& \\ + \frac{1-\theta}{\theta} \frac{\int_M u_\alpha^q \eta_{\alpha,\sigma}^p\, dv_g}{\int_M u_\alpha^q dv_g}& \\ \leq \frac{1}{\theta} \int_M u_\alpha^p \eta_{\alpha,\sigma}^p\, dv_g\ .&\end{aligned} \tag{2.13}$$

We now show that

$$\lim_{\sigma\to\infty} \lim_{\alpha\to\infty} A_\alpha \int_M |\nabla_g u_\alpha|^{p-2} \nabla_g u_\alpha \cdot \nabla_g(\eta_{\alpha,\sigma}^p) u_\alpha\, dv_g = 0\,. \tag{2.14}$$

Taking u_α as a test function in (2.4), we have

$$A_\alpha \int_M |\nabla_g u_\alpha|^p dv_g \leq \lambda_\alpha \leq N(p, q, n)^{-1}\ .$$

Therefore, by Hölder's inequality, it suffices to establish that

$$A_\alpha \int_M u_\alpha^p |\nabla_g \eta_{\alpha,\sigma}|^p\, dv_g \leq \frac{c}{\sigma^p}\ . \tag{2.15}$$

By the definition of the function $\eta_{\alpha,\sigma}$ we derive

$$A_\alpha \int_M u_\alpha^p |\nabla_g \eta_{\alpha,\sigma}|^p\, dv_g \leq c \frac{1}{\sigma^p} \int_M u_\alpha^p\, dv_g = \frac{c}{\sigma^p}\ .$$

Therefore (2.15) holds and (2.14) is valid. Replacing (2.5) and (2.14) in (2.13), one arrives at

$$\begin{aligned}&\theta \lim_{\sigma\to\infty} \lim_{\alpha\to\infty} \left(N(p, q, n)\ A_\alpha \int_M |\nabla_g u_\alpha|^p \eta_{\alpha,\sigma}^p\, dv_g \right) \\ &\quad + (1-\theta) \lim_{\sigma\to\infty} \lim_{\alpha\to\infty} \frac{\int_M u_\alpha^q \eta_{\alpha,\sigma}^p\, dv_g}{\int_M u_\alpha^q\, dv_g} \\ &\leq \lim_{\sigma\to\infty} \lim_{\alpha\to\infty} \int_M u_\alpha^p \eta_{\alpha,\sigma}^p\, dv_g\ .\end{aligned}$$

In order to rewrite this inequality in a suitable format, we first remark that

$$\left| \frac{\int_M u_\alpha^q \eta_{\alpha,\sigma}^p \, dv_g}{\int_M u_\alpha^q \, dv_g} - \frac{\int_M u_\alpha^q \eta_{\alpha,\sigma}^q \, dv_g}{\int_M u_\alpha^q \, dv_g} \right| \le \frac{\int_{B(x_\alpha, \sigma A_\alpha^{\frac{1}{p}}) \setminus B(x_\alpha, \sigma A_\alpha^{\frac{1}{p}}/2)} u_\alpha^q \, dv_g}{\int_M u_\alpha^q \, dv_g}$$
$$= \int_{B(0,\sigma)\setminus B(0,\sigma/2)} \varphi_\alpha^q \, dh_\alpha$$

and by Cartan's expansion, the above right-hand side converges to 0 as $\alpha \to \infty$ and $\sigma \to \infty$, by (2.12). Thus,

$$\lim_{\sigma\to\infty} \lim_{\alpha\to\infty} \frac{\int_M u_\alpha^q \eta_{\alpha,\sigma}^p \, dv_g}{\int_M u_\alpha^q \, dv_g} = \lim_{\sigma\to\infty} \lim_{\alpha\to\infty} \frac{\int_M u_\alpha^q \eta_{\alpha,\sigma}^q \, dv_g}{\int_M u_\alpha^q \, dv_g} .$$

Consequently, we can write

$$\begin{aligned} &\theta \lim_{\sigma\to\infty} \lim_{\alpha\to\infty} \left(N(p,q,n)\, A_\alpha \int_M |\nabla_g u_\alpha|^p \eta_{\alpha,\sigma}^p \, dv_g \right) \\ &\quad + (1-\theta) \lim_{\sigma\to\infty} \lim_{\alpha\to\infty} \frac{\int_M u_\alpha^q \eta_{\alpha,\sigma}^q \, dv_g}{\int_M u_\alpha^q \, dv_g} \\ &\le \lim_{\sigma\to\infty} \lim_{\alpha\to\infty} \int_M u_\alpha^p \eta_{\alpha,\sigma}^p \, dv_g . \end{aligned} \tag{2.16}$$

On the other hand, we have the validity of $N(N(p,q,n)^{\frac{\tau}{p}} + \varepsilon, B_\varepsilon)$. So, for each $\varepsilon > 0$, let $B_\varepsilon > 0$, with B_ε independent of α, such that

$$\begin{aligned} &\left(\int_M u_\alpha^p \eta_{\alpha,\sigma}^p \, dv_g \right)^{\frac{\tau}{\theta p}} \\ &\le \left(N(p,q,n)^{\frac{\tau}{p}} + \varepsilon \right) \Bigg((1+\varepsilon) A_\alpha \int_M |\nabla_g u_\alpha|^p \eta_{\alpha,\sigma}^p \, dv_g \\ &\qquad + c(\varepsilon) A_\alpha \int_M u_\alpha^p |\nabla_g \eta_{\alpha,\sigma}|^p dv_g \Bigg)^{\frac{\tau}{p}} \left(\frac{\int_M u_\alpha^q \eta_{\alpha,\sigma}^q dv_g}{\int_M u_\alpha^q dv_g} \right)^{\frac{\tau(1-\theta)}{\theta q}} + c A_\alpha^{\frac{\tau}{p}} , \end{aligned}$$

where we have used the definition of A_α and Young's inequality. Then, using (2.6) and (2.15) and letting $\alpha \to \infty, \sigma \to \infty$ in this order and letting $\varepsilon \to 0$, one gets

$$\begin{aligned} &\lim_{\sigma\to\infty} \lim_{\alpha\to\infty} \left(\int_M u_\alpha^p \eta_{\alpha,\sigma}^p \, dv_g \right)^{\frac{1}{\theta}} \\ &\le \lim_{\sigma\to\infty} \lim_{\alpha\to\infty} \left(N(p,q,n) A_\alpha \int_M |\nabla_g u_\alpha|^p \eta_{\alpha,\sigma}^p \, dv_g \right) \\ &\quad \lim_{\sigma\to\infty} \lim_{\alpha\to\infty} \left(\frac{\int_M u_\alpha^q \eta_{\alpha,\sigma}^q \, dv_g}{\int_M u_\alpha^q \, dv_g} \right)^{\frac{p(1-\theta)}{\theta q}} . \end{aligned} \tag{2.17}$$

Let

$$X = \lim_{\sigma\to\infty}\lim_{\alpha\to\infty}\left(N(p,q,n)\ A_\alpha\ \int_M |\nabla_g u_\alpha|^p \eta^p_{\alpha,\sigma} dv_g\right),$$

$$Y = \lim_{\sigma\to\infty}\lim_{\alpha\to\infty}\frac{\int_M u^q_\alpha \eta^q_{\alpha,\sigma} dv_g}{\int_M u^q_\alpha dv_g}$$

and

$$Z = \lim_{\sigma\to\infty}\lim_{\alpha\to\infty}\int_M u^p_\alpha \eta^p_{\alpha,\sigma} dv_g\,.$$

The inequalities (2.16) and (2.17) may be rewritten as

$$\begin{cases}\theta X + (1-\theta)Y \le Z\\ Z \le X^\theta Y^{\frac{p(1-\theta)}{q}}\end{cases}. \tag{2.18}$$

By (2.11), one also has $Z > 0$, so that $X, Y > 0$. The assertion (2.7) follows readily once one has proved that $Z = 1$. By Young's inequality, (2.18) immediately yields

$$\begin{cases}X^\theta Y^{1-\theta} \le Z\\ Z \le X^\theta Y^{\frac{p(1-\theta)}{q}}\end{cases}$$

These two inequalities produce

$$X^\theta Y^{1-\theta} \le X^\theta Y^{\frac{p(1-\theta)}{q}}\,,$$

which implies that $Y = 1$ because $\frac{p(1-\theta)}{q} - 1 + \theta > 0$.

On the other hand, using (2.10), we have

$$\begin{aligned}\int_{M\setminus B(x_\alpha,\sigma A_\alpha^{\frac{1}{p}})} u^p_\alpha\ dv_g &\le \|u_\alpha\|^{p-q}_{L^\infty(M)} A_\alpha^{\frac{n(p-q)}{p^2}} \frac{\int_{M\setminus B(x_\alpha,\sigma A_\alpha^{\frac{1}{p}})} u^q_\alpha\ dv_g}{\int_M u^q_\alpha\ dv_g}\\ &\le c\frac{\int_{M\setminus B(x_\alpha,\sigma A_\alpha^{\frac{1}{p}})} u^q_\alpha\ dv_g}{\int_M u^q_\alpha\ dv_g}\,.\end{aligned}$$

This estimate coupled together with $Y = 1$, allows to conclude that

$$\lim_{\sigma\to\infty}\lim_{\alpha\to\infty}\int_{M\setminus B(x_\alpha,\sigma A_\alpha^{\frac{1}{p}})} u^p_\alpha\ dv_g = 0\,.$$

Therefore, because $Z \le 1$, it follows that $Z = 1$.

2.2. Uniform estimation

For any constant $\lambda > 0$ there exists a constant $c_\lambda > 0$, independent of α, such that

$$d_g(x, x_\alpha)^\lambda u_\alpha(x) \le c_\lambda A_\alpha^{\frac{\lambda}{p} - \frac{n}{p^2}} \tag{2.19}$$

for all $x \in M$ and $\alpha > 0$ large enough.

Suppose, by contradiction, that the above assertion is false. Then, there exist $\lambda_0 > 0$ and $y_\alpha \in M$ such that $f_\alpha(y_\alpha) \to \infty$ as $\alpha \to \infty$, where

$$f_\alpha(x) = d_g(x, x_\alpha)^{\lambda_0} u_\alpha(x)\ A_\alpha^{-\frac{\lambda_0}{p} + \frac{n}{p^2}}\,.$$

Assume, without loss of generality, that $f_\alpha(y_\alpha) = \|f_\alpha\|_{L^\infty(M)}$. From (2.10), we have

$$f_\alpha(y_\alpha) \le c \frac{u_\alpha(y_\alpha)}{\|u_\alpha\|_{L^\infty(M)}} d_g(x_\alpha, y_\alpha)^{\lambda_0} \|u_\alpha\|_{L^\infty(M)}^{\frac{\lambda_0 p}{n}} \le c d_g(x_\alpha, y_\alpha)^{\lambda_0} \|u_\alpha\|_{L^\infty(M)}^{\frac{\lambda_0 p}{n}}\,,$$

so that

$$d_g(x_\alpha, y_\alpha) \|u_\alpha\|_{L^\infty(M)}^{\frac{p}{n}} \to \infty\,. \tag{2.20}$$

Next, for any fixed $\sigma > 0$ and $\varepsilon \in (0, 1)$, we show that

$$B(y_\alpha, \varepsilon d_g(x_\alpha, y_\alpha)) \cap B\left(x_\alpha, \sigma \|u_\alpha\|_{L^\infty(M)}^{-\frac{p}{n}}\right) = \emptyset \tag{2.21}$$

for $\alpha > 0$ large enough. Clearly, this assertion follows from

$$d_g(x_\alpha, y_\alpha) \ge \sigma \|u_\alpha\|_{L^\infty(M)}^{-\frac{p}{n}} + \varepsilon d(x_\alpha, y_\alpha)\,.$$

The above inequality is equivalent to

$$(1 - \varepsilon) d_g(x_\alpha, y_\alpha) \|u_\alpha\|_{L^\infty(M)}^{\frac{p}{n}} \ge \sigma\,,$$

which is clearly satisfied, since $d_g(x_\alpha, y_\alpha) \|u_\alpha\|_{L^\infty(M)}^{\frac{p}{n}} \to \infty$ as $\alpha \to \infty$ and $1 - \varepsilon > 0$.

We claim that there exists a constant $c > 0$ such that

$$u_\alpha(x) \le c u_\alpha(y_\alpha) \tag{2.22}$$

for all $x \in B(y_\alpha, \varepsilon d_g(x_\alpha, y_\alpha))$ and $\alpha > 0$ large enough. In fact, for each $x \in B(y_\alpha, \varepsilon d_g(x_\alpha, y_\alpha))$, we have

$$d_g(x, x_\alpha) \ge d_g(x_\alpha, y_\alpha) - d_g(x, y_\alpha) \ge (1 - \varepsilon) d_g(x_\alpha, y_\alpha)\,.$$

Thus,

$$\begin{aligned} d_g(y_\alpha, x_\alpha)^{\lambda_0} u_\alpha(y_\alpha) A_\alpha^{-\frac{\lambda_0}{p}+\frac{n}{p^2}} &= f_\alpha(y_\alpha) \geq f_\alpha(x) \\ &= d_g(x, x_\alpha)^{\lambda_0} u_\alpha(x) A_\alpha^{-\frac{\lambda_0}{p}+\frac{n}{p^2}} \\ &\geq (1-\varepsilon)^{\lambda_0} d_g(y_\alpha, x_\alpha)^{\lambda_0} u_\alpha(x) A_\alpha^{-\frac{\lambda_0}{p}+\frac{n}{p^2}}, \end{aligned}$$

so that

$$u_\alpha(x) \leq \left(\frac{1}{1-\varepsilon}\right)^{\lambda_0} u_\alpha(y_\alpha)$$

for all $x \in B(y_\alpha, \varepsilon d_g(x_\alpha, y_\alpha))$ and $\alpha > 0$ large enough. This proves our claim.

Define

$$\begin{aligned} h_\alpha(x) &= g\left(\exp_{y_\alpha}\left(A_\alpha^{\frac{1}{p}} x\right)\right) \\ \psi_\alpha(x) &= A_\alpha^{\frac{n}{p^2}} u_\alpha\left(\exp_{y_\alpha}\left(A_\alpha^{\frac{1}{p}} x\right)\right) \end{aligned}$$

for each $x \in B(0,2)$ and $\alpha > 0$ large enough. From (2.4), it readily follows that

$$\lambda_\alpha^{-1} \Delta_{p,h_\alpha} \psi_\alpha + \alpha A_\alpha^{\frac{\tau}{p}} \psi_\alpha^{p-1} + \frac{1-\theta}{\theta} \psi_\alpha^{q-1} = \frac{1}{\theta} \psi_\alpha^{p-1} \text{ on } B(0,2). \tag{2.23}$$

In particular,

$$\int_{B(0,2)} |\nabla_{h_\alpha} \psi_\alpha|^{p-2} \nabla_{h_\alpha} \psi_\alpha \cdot \nabla_{h_\alpha} \phi \, dv_{h_\alpha} \leq c \int_{B(0,2)} \psi_\alpha^{p-1} \phi \, dv_{h_\alpha}$$

for any positive test functions $\phi \in C_0^1(B(0,2))$. Thus, by Moser's iterative scheme or Morrey's inequality, one deduces that

$$A_\alpha^{\frac{n}{p}} u_\alpha(y_\alpha)^p \leq \sup_{B(0,\frac{1}{4})} \psi_\alpha^p \leq c \int_{B(0,\frac{1}{2})} \psi_\alpha^p \, dv_{h_\alpha} = c \int_{B(y_\alpha, \frac{1}{2} A_\alpha^{\frac{1}{p}})} u_\alpha^p \, dv_g .$$

By (2.22) and (2.10), we rewrite this last inequality as

$$\left(\frac{\|u_\alpha\|_{L^\infty(D_\alpha)}}{\|u_\alpha\|_{L^\infty(M)}}\right)^p \leq c \int_{B(y_\alpha, \frac{1}{2} A_\alpha^{\frac{1}{p}})} u_\alpha^p \, dv_g , \tag{2.24}$$

where $D_\alpha = B(y_\alpha, \varepsilon d_g(x_\alpha, y_\alpha))$.

Let $k > 0$ be a constant. By (2.10) and (2.20) we obtain that

$$k A_\alpha^{\frac{1}{p}} \leq \varepsilon d(x_\alpha, y_\alpha) , \tag{2.25}$$

for α large enough. Then, using (2.25), the concentration property (2.7), combined with (2.21) and $k = \frac{1}{2}$, provides that

$$\int_{B(y_\alpha, \frac{1}{2} A_\alpha^{\frac{1}{p}})} u_\alpha^p \, dv_g \to 0 \, .$$

Then, by (2.24), we obtain

$$\lim_{\alpha \to \infty} \frac{\|u_\alpha\|_{L^\infty(D_\alpha)}}{\|u_\alpha\|_{L^\infty(M)}} = 0 \, . \tag{2.26}$$

Consider the function $\eta_\alpha(x) = \eta(A_\alpha^{-\frac{1}{p}} d_g(x, y_\alpha))$, where $\eta \in C_0^1(\mathbb{R})$ is a cut-off function such that $\eta = 1$ on $[0, \frac{1}{2}]$, $\eta = 0$ on $[1, \infty)$ and $0 \le \eta \le 1$. Taking $u_\alpha \eta_\alpha^p$ as a test function in (2.4), one has

$$\begin{aligned} &\lambda_\alpha^{-1} A_\alpha \int_M |\nabla_g u_\alpha|^p \eta_\alpha^p \, dv_g + \alpha A_\alpha^{\frac{\tau}{p}} \int_M u_\alpha^p \eta_\alpha^p \, dv_g + \frac{1-\theta}{\theta} \frac{\int_M u_\alpha^q \eta_\alpha^p \, dv_g}{\int_M u_\alpha^q dv_g} \\ &= \frac{1}{\theta} \int_M u_\alpha^p \eta_\alpha^p \, dv_g + p A_\alpha \int_M |\nabla_g u_\alpha|^{p-2} u_\alpha \eta_\alpha^{p-1} \nabla_g u_\alpha \cdot \nabla_g \eta_\alpha \, dv_g \, . \end{aligned}$$

Now, we will estimate each term on the right-hand side of this inequality. First note that from (2.10), (2.22) and (2.25), we have

$$\int_M u_\alpha^p \eta_\alpha^p \, dv_g \le c \|u_\alpha\|_{L^\infty(D_\alpha)}^p (A_\alpha^{\frac{1}{p}})^n \le c \left(\frac{\|u_\alpha\|_{L^\infty(D_\alpha)}}{\|u_\alpha\|_{L^\infty(M)}} \right)^p \, .$$

Of course, by Hölder's and Young's inequalities

$$\begin{aligned} &\left| \int_M |\nabla_g u_\alpha|^{p-2} u_\alpha \eta_\alpha^{p-1} \nabla_g u_\alpha \cdot \nabla_g \eta_\alpha \, dv_g \right| \\ &\le \varepsilon \int_M |\nabla_g u_\alpha|^p \eta_\alpha^p \, dv_g + c_\varepsilon \int_M |\nabla_g \eta_\alpha|^p u_\alpha^p \, dv_g \, . \end{aligned}$$

Also, by (2.10), (2.22) and (2.25), it follows that

$$\begin{aligned} A_\alpha \int_M |\nabla_g \eta_\alpha|^p u_\alpha^p \, dv_g &\le A_\alpha (A_\alpha^{-\frac{1}{p}})^p \int_{B(y_\alpha, A_\alpha^{\frac{1}{p}})} u_\alpha^p \, dv_g \\ &\le c \|u_\alpha\|_{L^\infty(D_\alpha)}^p (A_\alpha^{\frac{1}{p}})^n \le c \left(\frac{\|u_\alpha\|_{L^\infty(D_\alpha)}}{\|u_\alpha\|_{L^\infty(M)}} \right)^p \, . \end{aligned} \tag{2.27}$$

Consequently,

$$\begin{aligned} &A_\alpha \int_M |\nabla_g u_\alpha|^p \eta_\alpha^p \, dv_g + \alpha A_\alpha^{\frac{\tau}{p}} \int_M u_\alpha^p \eta_\alpha^p \, dv_g + \frac{\int_M u_\alpha^q \eta_\alpha^p \, dv_g}{\int_M u_\alpha^q \, dv_g} \\ &\le c \left(\frac{\|u_\alpha\|_{L^\infty(D_\alpha)}}{\|u_\alpha\|_{L^\infty(M)}} \right)^p \, . \end{aligned} \tag{2.28}$$

On the other hand, the non-sharp Riemannian Nash inequality produces

$$\begin{aligned}\left(\int_{B(y_\alpha,\frac{1}{2}A_\alpha^{\frac{1}{p}})} u_\alpha^p \, dv_g\right)^{\frac{1}{\theta}} &\le \left(\int_M (u_\alpha\eta_\alpha^p)^p \, dv_g\right)^{\frac{1}{\theta}} \\ &\le c\left(\int_M |\nabla_g u_\alpha|^p \eta_\alpha^{p^2} \, dv_g\right)\left(\int_M (u_\alpha\eta_\alpha^p)^q \, dv_g\right)^{\frac{p(1-\theta)}{\theta q}} \\ &\quad + c\left(\int_M |\nabla_g \eta_\alpha|^p u_\alpha^p \, dv_g\right)\left(\int_M (u_\alpha\eta_\alpha^p)^q \, dv_g\right)^{\frac{p(1-\theta)}{\theta q}} \\ &\quad + c\left(\int_M (u_\alpha\eta_\alpha^p)^p \, dv_g\right)\left(\int_M (u_\alpha\eta_\alpha^p)^q \, dv_g\right)^{\frac{p(1-\theta)}{\theta q}} .\end{aligned} \tag{2.29}$$

Due to (2.27) and (2.28), we can estimate each term a the right-hand side of (2.29). Indeed, by hypothesis we have $p > q \ge 1$, so

$$\begin{aligned}&\left(\int_M |\nabla_g u_\alpha|^p \eta_\alpha^{p^2} \, dv_g\right)\left(\int_M (u_\alpha\eta_\alpha^p)^q \, dv_g\right)^{\frac{p(1-\theta)}{\theta q}} \\ &\le A_\alpha \int_M |\nabla_g u_\alpha|^p \eta_\alpha^p \, dv_g \left(\frac{\int_M u_\alpha^q \eta_\alpha^p \, dv_g}{\int_M u_\alpha^q \, dv_g}\right)^{\frac{p(1-\theta)}{\theta q}} \\ &\le c\left(\frac{\|u_\alpha\|_{L^\infty(D_\alpha)}}{\|u_\alpha\|_{L^\infty(M)}}\right)^{p(1+\frac{p(1-\theta)}{\theta q})},\end{aligned}$$

$$\begin{aligned}&\left(\int_M |\nabla_g \eta_\alpha|^p u_\alpha^p \, dv_g\right)\left(\int_M (u_\alpha\eta_\alpha^p)^q \, dv_g\right)^{\frac{p(1-\theta)}{\theta q}} \\ &\le A_\alpha \int_M |\nabla_g \eta_\alpha|^p u_\alpha^p \, dv_g \left(\frac{\int_M u_\alpha^q \eta_\alpha^p \, dv_g}{\int_M u_\alpha^q \, dv_g}\right)^{\frac{p(1-\theta)}{\theta q}} \\ &\le c\left(\frac{\|u_\alpha\|_{L^\infty(D_\alpha)}}{\|u_\alpha\|_{L^\infty(M)}}\right)^{p(1+\frac{p(1-\theta)}{\theta q})}\end{aligned}$$

and because $p \ge \tau$, we have

$$\begin{aligned}&\left(\int_M (u_\alpha\eta_\alpha^p)^p \, dv_g\right)\left(\int_M (u_\alpha\eta_\alpha^p)^q \, dv_g\right)^{\frac{p(1-\theta)}{\theta q}} \\ &\le A_\alpha \int_M u_\alpha^p \eta_\alpha^p \, dv_g \left(\frac{\int_M u_\alpha^q \eta_\alpha^p \, dv_g}{\int_M u_\alpha^q \, dv_g}\right)^{\frac{p(1-\theta)}{\theta q}} \le c\left(\frac{\|u_\alpha\|_{L^\infty(D_\alpha)}}{\|u_\alpha\|_{L^\infty(M)}}\right)^{p(1+\frac{p(1-\theta)}{\theta q})} .\end{aligned}$$

Substituting these three estimates in (2.29), one gets

$$\left(\int_{B(y_\alpha,\frac{1}{2}A_\alpha^{\frac{1}{p}})} u_\alpha^p \, dv_g\right)^{\frac{1}{\theta}} \leq c\left(\frac{\|u_\alpha\|_{L^\infty(D_\alpha)}}{\|u_\alpha\|_{L^\infty(M)}}\right)^{p(1+\frac{p(1-\theta)}{\theta q})},$$

so that

$$\int_{B(y_\alpha,\frac{1}{2}A_\alpha^{\frac{1}{p}})} u_\alpha^p \, dv_g \leq c\left(\frac{\|u_\alpha\|_{L^\infty(D_\alpha)}}{\|u_\alpha\|_{L^\infty(M)}}\right)^{p(\theta+\frac{p(1-\theta)}{q})}.$$

Combining this inequality with (2.24), we obtain

$$1 \leq c\left(\frac{\|u_\alpha\|_{L^\infty(D_\alpha)}}{\|u_\alpha\|_{L^\infty(M)}}\right)^{p\left(-1+\theta+\frac{p(1-\theta)}{q}\right)}.$$

By definition $\theta < 1$ and by hypotesis $q < p$, so that

$$-1+\theta+\frac{p(1-\theta)}{q} = (1-\theta)\left(\frac{p}{q}-1\right) > 0,$$

but this inequality contradicts (2.26).

2.3. Conclusion of the proof of the Theorem 1

In the sequel, we will perform several estimates by using the uniform estimation.

Let us suppose that the radius of injectivity of M is grater than $r > 0$ and let $\eta \in C_0^1(\mathbb{R})$ is a cut-off function as in the previous section, define $\eta_\alpha(x) = \eta\left(\frac{d_g(x,x_\alpha)}{r}\right)$. From the Euclidean Nash inequality (1.3), we have

$$\left(\int_{B(0,r)} u_\alpha^p \eta_\alpha^p \, dx\right)^{\frac{1}{\theta}} \leq N(p,q,n)\left(\int_{B(0,r)} |\nabla(u_\alpha\eta_\alpha)|^p \, dx\right)\left(\int_{B(0,r)} u_\alpha^q \eta_\alpha^q \, dx\right)^{\frac{p(1-\theta)}{\theta q}}.$$

Expanding the metric g in normal coordinates around x_α, one locally gets

$$\left(1-cd_g(x,x_\alpha)^2\right) dv_g \leq dx \leq \left(1+cd_g(x,x_\alpha)^2\right) dv_g \tag{2.30}$$

and

$$|\nabla(u_\alpha\eta_\alpha)|^p \leq |\nabla_g(u_\alpha\eta_\alpha)|^p\left(1+cd_g(x,x_\alpha)^2\right).$$

Thus,

$$\left(\int_{B(0,r)} u_\alpha^p \eta_\alpha^p dx\right)^{\frac{1}{\theta}} \leq \left(A_\alpha N(p,q,n)\int_M |\nabla_g(u_\alpha\eta_\alpha)|^p \, dv_g \right. \\ \left. + cA_\alpha \int_M |\nabla_g(u_\alpha\eta_\alpha)|^p d_g(x,x_\alpha)^2 dv_g\right)\left(\frac{\int_{B(0,r)} u_\alpha^q \eta_\alpha^q \, dx}{\int_M u_\alpha^q \, dv_g}\right)^{\frac{p(1-\theta)}{\theta q}}. \tag{2.31}$$

On the other hand taking u_α as a test function in (2.4), and enclosing (2.5), we have

$$N(p,q,n)\ A_\alpha \int_M |\nabla_g u_\alpha|^p\ dv_g \le 1 - \alpha A_\alpha^{\frac{\tau}{p}}\ .$$

Coupling the above inequality together with inequality

$$|\nabla_g(u_\alpha \eta_\alpha)|^p \le |\nabla_g u_\alpha|^p \eta_\alpha^p + c|\eta_\alpha \nabla_g u_\alpha|^{p-1}|u_\alpha \nabla_g \eta_\alpha| + c|u_\alpha \nabla_g \eta_\alpha|^p\ ,$$

in (2.31), we obtain

$$\left(\int_{B(0,r)} u_\alpha^p \eta_\alpha^p\ dx\right)^{\frac{1}{\theta}} \le \Bigg(1-\alpha A_\alpha^{\frac{\tau}{p}} + cF_\alpha + cG_\alpha + cA_\alpha \int_{B(x_\alpha,r)\setminus B(x_\alpha,\frac{r}{2})} u_\alpha^p dv_g\Bigg)\left(\frac{\int_{B(0,r)} u_\alpha^q \eta_\alpha^q dx}{\int_M u_\alpha^q dv_g}\right)^{\frac{p(1-\theta)}{\theta q}}, \tag{2.32}$$

where

$$F_\alpha = A_\alpha \int_M |\nabla_g u_\alpha|^p \eta_\alpha^p d_g(x,x_\alpha)^2\ dv_g$$

and

$$G_\alpha = A_\alpha \int_M |\nabla_g u_\alpha|^{p-1} \eta_\alpha^{p-1} u_\alpha |\nabla_g \eta_\alpha|\ dv_g\ .$$

We now estimate F_α and G_α. Note that by (2.4), taking u_α as a test function, we have

$$A_\alpha \int_M |\nabla_g u_\alpha|^p dv_g \le \lambda_\alpha \le N(p,q,n)^{-1}\ .$$

Using this uniform limitation, Hölder's inequality, the definition of φ_α and uniform estimation (2.19), we have

$$\begin{aligned} G_\alpha &\le \left(A_\alpha \int_M |\nabla_g u_\alpha|^p dv_g\right)^{\frac{p-1}{p}} \left(A_\alpha \int_{B(x_\alpha,r)\setminus B(x_\alpha,\frac{r}{2})} u_\alpha^p dv_g\right)^{\frac{1}{p}} \\ &\le c\left(A_\alpha \int_{B(x_\alpha,r)\setminus B(x_\alpha,\frac{r}{2})} u_\alpha^p d_g(x,x_\alpha)^p dv_g\right)^{\frac{1}{p}} \\ &\le c\left(A_\alpha^2 \int_{B(0,rA_\alpha^{-\frac{1}{p}})\setminus B(0,\frac{rA_\alpha^{-\frac{1}{p}}}{2})} \varphi_\alpha^p |x|^p dh_\alpha\right)^{\frac{1}{p}} \\ &\le c_\lambda A_\alpha^{\frac{2}{p}} \left(\int_{\mathbb{R}^n\setminus B(0,1)} |x|^{p(1-\lambda)} dx\right)^{\frac{1}{p}} \le c\ A_\alpha^{\frac{2}{p}}\ , \end{aligned} \tag{2.33}$$

for λ sufficiently large and normal coordinates around x_α. Similarly, we have

$$\begin{aligned} A_\alpha &\int_M |\nabla_g u_\alpha|^{p-1}\eta_\alpha^p u_\alpha d_g(x,x_\alpha)\,dv_g \\ &\le \left(A_\alpha \int_M |\nabla_g u_\alpha|^p dv_g\right)^{\frac{p-1}{p}} \left(A_\alpha \int_{B(x_\alpha,r)} u_\alpha^p d_g(x,x_\alpha)^p dv_g\right)^{\frac{1}{p}} \\ &\le c A_\alpha^{\frac{2}{p}} \left(\int_{B(0,rA_\alpha^{-\frac{1}{p}})} \varphi_\alpha^p |x|^p dh_\alpha\right)^{\frac{1}{p}} \\ &\le c\ A_\alpha^{\frac{2}{p}} \left(1+\int_{\mathbb{R}^n\setminus B(0,1)} |x|^{p(1-\lambda)} dx\right)^{\frac{1}{p}} \le c\ A_\alpha^{\frac{2}{p}}\ . \end{aligned} \tag{2.34}$$

Taking $u_\alpha d_g^2 \eta_\alpha^p$ as a test function in (2.4), one easily checks that

$$\begin{aligned} F_\alpha = A_\alpha &\int_M |\nabla_g u_\alpha|^p \eta_\alpha^p d_g(x,x_\alpha)^2\,dv_g \\ &\le c\int_{B(x_\alpha,r)} u_\alpha^p d_g(x,x_\alpha)^2 dv_g + cA_\alpha \int_M |\nabla_g u_\alpha|^{p-1}\eta_\alpha^p u_\alpha d_g(x,x_\alpha)\,dv_g + cG_\alpha\,. \end{aligned}$$

Therefore, by (2.33) and (2.34),

$$F_\alpha \le c\int_{B(x_\alpha,r)} u_\alpha^p d_g(x,x_\alpha)^2\,dv_g + cA_\alpha^{\frac{2}{p}}\,.$$

Applying now the uniform estimation (2.19), one gets

$$\begin{aligned} \int_{B(x_\alpha,r)} u_\alpha^p d_g(x,x_\alpha)^2\,dv_g &\le cA_\alpha^{\frac{2}{p}}\left(1+\int_{B(0,rA_\alpha^{-\frac{1}{p}})\setminus B(0,1)} \varphi_\alpha^p |x|^2 dx\right) \\ &\le cA_\alpha^{\frac{2}{p}}\left(1+\int_{\mathbb{R}^n\setminus B(0,1)} |x|^{2-\lambda p} dx\right) \le cA_\alpha^{\frac{2}{p}} \end{aligned} \tag{2.35}$$

for λ large enough. Consequently,

$$F_\alpha \le cA_\alpha^{\frac{2}{p}} \quad \text{and} \quad G_\alpha \le cA_\alpha^{\frac{2}{p}}\,. \tag{2.36}$$

Using ideas similar to (2.33) for $p > 1$, we obtain

$$A_\alpha \int_{B(x_\alpha,r)\setminus B(x_\alpha,\frac{r}{2})} u_\alpha^p dv_g \le cA_\alpha \int_{B(x_\alpha,r)\setminus B(x_\alpha,\frac{r}{2})} u_\alpha^p d_g(x,x_\alpha)^p dv_g \le cA_\alpha^2 \le cA_\alpha^{\frac{2}{p}}.$$

Putting this estimate and (2.36) in (2.32), one arrives at

$$\left(\int_{B(x_\alpha,r)} u_\alpha^p \eta_\alpha^p\,dx\right)^{\frac{1}{\theta}} \le \left(1-\alpha A_\alpha^{\frac{\tau}{p}} + cA_\alpha^{\frac{2}{p}}\right)\left(\frac{\int_{B(x_\alpha,r)} u_\alpha^q \eta_\alpha^q\,dx}{\int_M u_\alpha^q\,dv_g}\right)^{\frac{p(1-\theta)}{\theta q}}\ . \tag{2.37}$$

On the other hand, by (2.30) and the mean value theorem, we obtain

$$\left(\int_M u_\alpha^p \eta_\alpha^p \, dx\right)^{\frac{1}{\theta}} \ge \left(\int_M u_\alpha^p \eta_\alpha^p dv_g - c\int_M u_\alpha^p \eta_\alpha^p d_g(x,x_\alpha)^2 dv_g\right)^{\frac{1}{\theta}}$$
$$\ge 1 - c\int_{M\setminus B(x_\alpha,r)} u_\alpha^p \, dv_g - c\int_M u_\alpha^p \eta_\alpha^p d_g(x,x_\alpha)^2 \, dv_g$$

and

$$\left(\frac{\int_{B(x_\alpha,r)} u_\alpha^q \eta_\alpha^q \, dx}{\int_M u_\alpha^q \, dv_g}\right)^{\frac{p(1-\theta)}{\theta q}} \le \left(\frac{\int_M u_\alpha^q \eta_\alpha^q \, dv_g + c\int_M u_\alpha^q \eta_\alpha^q d_g(x,x_\alpha)^2 \, dv_g}{\int_M u_\alpha^q \, dv_g}\right)^{\frac{p(1-\theta)}{\theta q}}$$
$$\le \left(\frac{\int_M u_\alpha^q \eta_\alpha^q \, dv_g}{\int_M u_\alpha^q \, dv_g}\right)^{\frac{p(1-\theta)}{\theta q}} + c\,\frac{\int_M u_\alpha^q \eta_\alpha^q d_g(x,x_\alpha)^2 \, dv_g}{\int_M u_\alpha^q \, dv_g}$$
$$\le 1 + c\,\frac{\int_M u_\alpha^q \eta_\alpha^q d_g(x,x_\alpha)^2 \, dv_g}{\int_M u_\alpha^q \, dv_g}\,.$$

Replacing these two estimates in (2.37), one gets

$$\alpha A_\alpha^{\frac{\tau}{p}} \le cA_\alpha^{\frac{2}{p}} + c\frac{\int_M u_\alpha^q \eta_\alpha^q d_g(x,x_\alpha)^2 \, dv_g}{\int_M u_\alpha^q \, dv_g}$$
$$+ c\int_M u_\alpha^p \eta_\alpha^p d_g(x,x_\alpha)^2 \, dv_g + c\int_{M\setminus B(x_\alpha,r)} u_\alpha^p \, dv_g\,.$$

By uniform estimate (2.19), compactness of M and taking $\lambda p - n = 2$, we have

$$\int_{M\setminus B(x_\alpha,r)} u_\alpha^p \, dv_g \le c\int_{M\setminus B(x_\alpha,r)} u_\alpha^p d(x,x_\alpha)^{\lambda p} \, dv_g \le c_\lambda A_\alpha^{\frac{\lambda p - n}{p}} = c\, A_\alpha^{\frac{2}{p}}\,.$$

So, by this inequality and (2.35), one concludes that

$$\alpha A_\alpha^{\frac{\tau}{p}} \le cA_\alpha^{\frac{2}{p}} + c\frac{\int_M u_\alpha^q \eta_\alpha^q d_g(x,x_\alpha)^2 \, dv_g}{\int_M u_\alpha^q \, dv_g}\,. \tag{2.38}$$

By uniform estimation (2.19) and λ greater enough, we obtain

$$\frac{\int_M u_\alpha^q \eta_\alpha^q d_g(x,x_\alpha)^2 \, dv_g}{\int_M u_\alpha^q \, dv_g} \le \frac{\int_{B(x_\alpha,r)} u_\alpha^q d_g(x,x_\alpha)^2 \, dv_g}{\int_M u_\alpha^q \, dv_g}$$
$$= A_\alpha^{\frac{2}{p}} \int_{B(0,rA_\alpha^{-\frac{1}{p}})} \varphi_\alpha^q |x|^2 dh_\alpha$$
$$\le cA_\alpha^{\frac{2}{p}}\left(1 + c\int_{\mathbb{R}^n\setminus B(0,1)} |x|^{2-\lambda q} dx\right) \le c\, A_\alpha^{\frac{2}{p}}\,.$$

Introducing this inequality in (2.38), we readily deduce that

$$\alpha A_\alpha^{\frac{\tau}{p}} \leq c\ A_\alpha^{\frac{2}{p}} .$$

Finally, because $0 < \tau \leq \min\{p, 2\}$, we get at the contradiction

$$\alpha \leq c\ A_\alpha^{\frac{2-\tau}{p}} .$$

3. Proof of Theorem 2

By Theorem 1 we have that

$$\begin{aligned}
&\left(\int_M |u|^p\, dv_g\right)^{\frac{\tau}{p\theta}} \\
&\leq \left(N(p,q,n)^{\frac{\tau}{p}} \left(\int_M |\nabla_g u|^p dv_g\right)^{\frac{\tau}{p}} + B_{\rm opt} \left(\int_M |u|^p\, dv_g\right)^{\frac{\tau}{p}}\right) \left(\int_M |u|^q\, dv_g\right)^{\frac{\tau(1-\theta)}{\theta q}},
\end{aligned}$$

is valid for all $u \in H^{1,p}(M)$.

Let $\alpha > 0$ and $c_\alpha = B_{\rm opt} - \alpha^{-1}$. Define

$$J_\alpha(u) = \left(\int_M |u|^p\, dv_g\right)^{\frac{\tau}{p\theta}} \left(\int_M |u|^q dv_g\right)^{-\frac{\tau(1-\theta)}{\theta q}} - c_\alpha \left(\int_M |u|^p\, dv_g\right)^{\frac{\tau}{p}} .$$

By definition of $B_{\rm opt}$, we have

$$\nu_\alpha = \sup_{u\in E} J_\alpha(u) > N(p,q,n)^{\frac{\tau}{p}} ,$$

where $E = \{u \in H^{1,p}(M) : \ \|\nabla_g u\|_{L^p(M)} = 1\}$. This supreme is well defined.

By the observation made in the proof of the Theorem 1, we will present the proof only in the case $q > 1$. In the case $q = 1$ we follow the ideas contained in [11]. Using standard arguments, we find a maximizer $\tilde{u}_\alpha \in E$ of J_α, therefore

$$J_\alpha(\tilde{u}_\alpha) = \nu_\alpha = \sup_{u\in E} J_\alpha(u) .$$

Since J_α is of class C^1, the function $\tilde{u}_\alpha$ satisfies the Euler-Lagrange equation

$$\begin{aligned}
&\frac{1}{\theta} \|\tilde{u}_\alpha\|_{L^p(M)}^{\frac{\tau-p\theta}{\theta}} \|\tilde{u}_\alpha\|_{L^q(M)}^{-\frac{\tau(1-\theta)}{\theta}} \tilde{u}_\alpha^{p-1} \\
&\quad - \frac{(1-\theta)}{\theta} \|\tilde{u}_\alpha\|_{L^p(M)}^{\frac{\tau}{\theta}} \|\tilde{u}_\alpha\|_{L^q(M)}^{-\frac{\tau(1-\theta)+\theta q}{\theta}} \tilde{u}_\alpha^{q-1} - c_\alpha \|\tilde{u}_\alpha\|_{L^p(M)}^{\tau-p} \tilde{u}_\alpha^{p-1} \\
&= \nu_\alpha \Delta_{p,g} \tilde{u}_\alpha,
\end{aligned}$$

where $\Delta_{p,g} = -\mathrm{div}_g(|\nabla_g|^{p-2}\nabla_g)$ is the p-Laplace operator of g. Provided that $\nabla_g|\tilde{u}_\alpha| = \pm\nabla_g\tilde{u}_\alpha$ almost everywhere, we can assume $\tilde{u}_\alpha \geq 0$ and following Tolksdorf (see [16]) we have that $\tilde{u}_\alpha \in C^1(M)$. Taking $u_\alpha = \frac{\tilde{u}_\alpha}{\|\tilde{u}_\alpha\|_{L^r(M)}}$, we find

$$\lambda_\alpha^{-1} A_\alpha \Delta_{p,g} u_\alpha + c_\alpha A_\alpha^{\frac{\tau}{p}} u_\alpha^{p-1} + \frac{1-\theta}{\theta}\|u_\alpha\|_{L^q(M)}^{-q} u_\alpha^{q-1} = \frac{1}{\theta} u_\alpha^{p-1} \text{ on } M\,, \tag{3.1}$$

where $\|u_\alpha\|_{L^p(M)} = 1$,

$$A_\alpha = \left(\int_M u_\alpha^q \, dv_g\right)^{\frac{p(1-\theta)}{\theta q}}$$

and

$$\lambda_\alpha = \nu_\alpha^{-1} \|u_\alpha\|_{L^q(M)}^{\frac{(p-\tau)(1-\theta)}{\theta}} \|\tilde{u}_\alpha\|_{L^p(M)}^{\tau-p}\,.$$

Taking a subsequence, we can assume that there exists $A \in \mathbb{R}$ such that

$$\lim_{\alpha\to\infty} A_\alpha = A\,.$$

Then, we have two possibilities:

(i) $A = 0$ or
(ii) $A > 0$.

We will show that (i) cannot occur. Otherwise, A = 0 would lead to (2.6) of Theorem 1. Therefore we can follow step by step the proof of Theorem 1 and will regain the same result as in Section 2.3, that is, we obtain

$$c_\alpha \leq c\, A_\alpha^{\frac{2-\tau}{p}}\,.$$

This results in a contradiction, because $0 < \tau < 2$, $c_\alpha = B_{\mathrm{opt}} - \alpha^{-1}$, $A = 0$ and because of (1.5). Therefore (ii) occurs. Using u_α as a test function and using, (ii) and (2.5) in (3.1), we see that there is $c > 0$ such that

$$\int_M |\nabla_g u_\alpha|^p dv_g \leq c$$

for all α. Since $\|u_\alpha\|_{L^p(M)} = 1$, up to subsequences $u_\alpha \rightharpoonup u_0$ in $H^{1,p}(M)$. Using again that $\|u_\alpha\|_{L^p(M)} = 1$ for all α, we have $\|u_0\|_{L^p(M)} = 1$. Furthermore, by equation (3.1), we have that

$$\int_M |\nabla_g u_\alpha|^{p-2}\nabla_g u_\alpha \nabla_g h dv_g \leq c \int_M u_\alpha^{p-1} h dv_g\,,$$

for any test function $h \geq 0$. Following the iterative scheme due to Moser, we find that

$$\sup_{x\in M} u_\alpha \leq c\,,$$

for all α. Applying the result of Tolksdorf in (3.1), we have $u_\alpha \to u_0$ in $C^1(M)$.

On the other hand, we know that $\tilde{u}_\alpha$ satisfies

$$\left(\int_M \tilde{u}_\alpha^p dv_g\right)^{\frac{\tau}{p\theta}} \geq \left(N(p,q,n)^{\frac{\tau}{p}} \left(\int_M |\nabla_g \tilde{u}_\alpha|^p dv_g\right)^{\frac{\tau}{p}} + \left(B_{\text{opt}} - \frac{1}{\alpha}\right)\left(\int_M \tilde{u}_\alpha^p dv_g\right)^{\frac{\tau}{p}}\right)\left(\int_M \tilde{u}_\alpha^q dv_g\right)^{\frac{\tau(1-\theta)}{\theta q}},$$

and for $u_\alpha = \frac{\tilde{u}_\alpha}{\|\tilde{u}_\alpha\|_{L^r(M)}}$, we have

$$1 \geq \left(N(p,q,n)^{\frac{\tau}{p}} \left(\int_M |\nabla_g u_\alpha|^p dv_g\right)^{\frac{\tau}{p}} + B_{\text{opt}} - \frac{1}{\alpha}\right)\left(\int_M u_\alpha^q dv_g\right)^{\frac{\tau(1-\theta)}{\theta q}}.$$

Taking the limit in α in this inequality, we obtain

$$1 \geq \left(N(p,q,n)^{\frac{\tau}{p}} \left(\int_M |\nabla_g u_0|^p dv_g\right)^{\frac{\tau}{p}} + B_{\text{opt}}\right)\left(\int_M u_0^q dv_g\right)^{\frac{\tau(1-\theta)}{\theta q}}.$$

Therefore, u_0 is an extremal function for $N(N(p,q,n)^{\frac{\tau}{p}}, B_{\text{opt}})$.

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