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Boundary asymptotic expansions of analytic self-mappings of the unit disk

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Abstract. We present necessary and sufficient conditions for the existence and for the uniqueness of an analytic self-mapping of the open unit disk having prescribed non-tangential boundary asymptotics at finitely many preassigned boundary points.

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1. Introduction

Let $\mathbb{D} = \{z : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$. Given a set $\Omega \subset \mathbb{C}$, we denote by $\mathrm{Hol}(\mathbb{D}, \Omega)$ the set of holomorphic functions $F : \mathbb{D} \to \Omega$. If $\Omega = \mathbb{C}$, we will simply write $\mathrm{Hol}(\mathbb{D})$ for the set of functions holomorphic in $\mathbb{D}$.

The class $\mathrm{Hol}(\mathbb{D}, \mathbb{D})$ consisting of holomorphic self-mappings of $\mathbb{D}$ is of particular interest. Functions of this class have been the subject of intensive study for over a century and have strong analytic, geometric and dynamic properties. In the interpolation context, it is more convenient to extend the class $\mathrm{Hol}(\mathbb{D}, \mathbb{D})$ by the constant unimodular functions. This extended class $\mathcal{S}$ (sometimes called the *Schur class*) coincides with $\mathrm{Hol}(\mathbb{D}, \overline{\mathbb{D}})$ by the maximum modulus principle (alternatively, $\mathcal{S}$ is the closed unit ball of the Hardy space $H^\infty(\mathbb{D})$ of bounded holomorphic functions on $\mathbb{D}$).

Two celebrated problems in complex analysis from early in the last century are the Nevanlinna-Pick and the Carathéodory-Fejér interpolation problems which naturally merge into the following combined Nevanlinna-Pick-Carathéodory-Fejér problem:

Given N distinct points $z_1, \dots, z_N \in \mathbb{D}$ along with non-negative integers $k_1, \dots, k_N$ and the jets $\{w_{i,j}\}_{j=0}^{k_i}$ of complex numbers assigned to each z_i,

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find an $F \in \mathcal{S}$ such that

$$F^{(j)}(z_i) = j!\, w_{i,j} \quad \textit{for} \quad i = 1, \ldots, N \quad \textit{and} \quad j = 0, \ldots, k_i,$$

or, equivalently, such that

$$F(z) = \sum_{j=0}^{k_i} w_{i,j}(z - z_i)^j + O\left(|z - z_i|^{k_i+1}\right) \quad \text{for} \quad i = 1, \ldots, N. \tag{1.1}$$

The Carathéodory-Fejér problem (where $N = 1$) was considered by Schur [29], Carathéodory [18] and Fejér [20], while the Nevanlinna-Pick problem ($k_i = 0$ for $i = 1, \ldots, N$) was studied by Pick [25] and Nevanlinna [24]. Numerous generalizations of these two classical problems (of which the combined problem (1.1) is the most immediate and straightforward generalization) have been the subject of much study. Of many different approaches to these and related problems we may mention [1,5,6,22,26,27]. Well-known results on the problem (1.1) include the following. The problem has a solution if and only if the *Pick matrix P* (constructed explicitly in terms of interpolation data) is positive semidefinite; some details are recalled in Remark 2.5 below. In case P is positive definite, the problem has infinitely many solutions which can be described in terms of a linear fractional formula. If $P \geq 0$ is singular, the problem has a unique solution which is a Blaschke product of degree equal to the rank of P. Finally, a solvable problem has a unique solution with the minimally possible H^∞-norm and this solution is necessarily a scalar multiple of a finite Blaschke product.

The objective of this paper is to study the boundary analog of the problem (1.1) where the interpolation nodes z_i are taken on the boundary $\mathbb{T}$ of the unit disk and where the prescribed Taylor expansions (1.1) of an unknown interpolant are replaced by non-tangential boundary asymptotics up to a certain order. Thus we are given an N-tuple $\boldsymbol{\zeta}$ of distinct points on $\mathbb{T}$ along with the tuple $\mathbf{k}$ of respective multiplicities, and a doubly-indexed collection $\mathbf{s}$ of complex numbers:

$$\boldsymbol{\zeta} = \{\zeta_1, \ldots, \zeta_N\}, \quad \mathbf{k} = \{k_1, \ldots, k_N\}, \quad \mathbf{s} = \{s_{i,j}\}_{i=1,\ldots,N}^{j=0,\ldots,k_i}, \tag{1.2}$$

and we are wondering *if there exists a function $F \in \mathcal{S}$ which admits the asymptotic expansions*

$$F(z) = s_{i,0} + s_{i,1}(z-\zeta_i) + \ldots + s_{i,k_i}(z-\zeta_i)^{k_i} + o\left(|z - \zeta_i|^{k_i}\right) \quad (i = 1, \ldots, N) \tag{1.3}$$

as z tends to ζ_i non-tangentially?

Observe that the asymptotic equalities (1.3) are equivalent to the existence of the following non-tangential boundary limits $F_j(\zeta_i)$ and the equalities

$$F_j(\zeta_i) := \angle \lim_{z \to \zeta_i} \frac{F^{(j)}(z)}{j!} = s_{i,j} \quad \text{for} \quad j = 0, \ldots, k_i; \; i = 1, \ldots, N. \tag{1.4}$$

Here and in what follows, the symbol $\angle \lim_{z\to\zeta}$ means that $z \in \mathbb{D}$ approaches the boundary point $\zeta \in \mathbb{T}$ non-tangentially; the regular notation $\lim_{z\to\zeta}$ will be used if z tends to ζ unrestrictedly in $\mathbb{D}$. We denote by $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$ the following boundary interpolation problem:

Problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$. Given data (1.2), find a function $F \in \mathcal{S}$ satisfying conditions (1.3) or equivalently, conditions (1.4).

We will call the problem *determinate* if it has only one solution. If the problem admits more than one solution, it has infinitely many solutions by the convexity of the solution set; in this case we will call the problem *indeterminate*. The single-point Carathéodory-Fejér boundary problem ($N = 1$) was studied in [10] and in [2] (in a related class of functions). The infinite case $N = 1, k_1 = \infty$ has been recently settled in [16].

The main objective of this paper is to extend the results from [10] concerning the solvability and determinacy of the problem to the multi-point setting of the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$. In Section 3, we introduce the *Pick matrix* and *companion numbers* associated with the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$, and formulate the main results of the paper (the solvability and the determinacy criteria) in terms of these objects. More specifically, Theorem 3.5 presents necessary and sufficient conditions for the problem to have infinitely many solutions, in which case (1) the problem has infinitely many *rational* solutions and (2) any solution satisfies certain Carathéodory-Julia type conditions (these conditions are recalled in Section 2 along with some other necessary background on the local boundary behavior of Schur-class functions). Theorem 3.6 presents necessary and sufficient conditions for the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$ to have a unique solution which is necessarily a finite Blaschke product of degree equal to the rank of the Pick matrix associated with the problem. Being combined, Theorems 3.5 and 3.6 cover all the cases in which the problem has a solution. In Section 4 we recall several special particular cases of the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$ already known from literature, and then we show in Section 5 that a general case of the problem can always be reduced to a special one. The formal proofs of the main results are given in Section 6. In Section 7, we make use of the Cayley transform to derive similar results on boundary interpolation for Carathéodory-class functions (analytic and with non-negative real part on $\mathbb{D}$).

2. Julia-Wolff-Carathéodory type conditions

Although every function $F \in \mathcal{S}$ admits non-tangential boundary limits at almost all boundary points, one cannot guarantee the existence of such a limit at a preassigned point $\zeta \in \mathbb{T}$. The situation with the boundary limits of F' is even worse: there are functions $F \in \mathcal{S}$ whose derivatives F' have a finite non-tangential boundary limit at no point on $\mathbb{T}$; see *e.g.*, [30, page 184]. However, there are conditions which guarantee the existence of finite boundary limits for F and for its derivatives.

Definition 2.1. Given a point $\zeta \in \mathbb{T}$, let us say that a function $F \in \mathcal{S}$ is in the class $\mathcal{S}^{(n)}(\zeta)$ if

$$\liminf_{z\to\zeta} \frac{\partial^{2n-2}}{\partial z^{n-1}\partial \bar{z}^{n-1}} \frac{1-|F(z)|^2}{1-|z|^2} < \infty. \tag{2.1}$$

Condition (2.1) was introduced in [12] and studied later in [15] and [14]; in case $n = 1$, it amounts to the classical Julia-Wolff-Carathéodory condition (see, for example, [30] and [31])

$$\liminf_{z\to\zeta} \frac{1-|F(z)|}{1-|z|} < \infty.$$

Condition (2.1) can be equivalently reformulated in terms of the de Branges-Rovnyak space $\mathcal{H}(F)$ (we refer to [17] for the definition) associated with the function $F \in \mathcal{S}$ as follows: *a function $F \in \mathcal{S}$ belongs to $\mathcal{S}^{(n)}(\zeta)$ if and only if for every function h from $\mathcal{H}(F)$, the boundary limits $h_j(\zeta)$ exist for $j = 0, \dots, n-1$.*

As was shown in [21] (and earlier in [3] for inner functions), the latter de Branges-Rovnyak space property (and therefore, the membership of F in $\mathcal{S}^{(n)}(\zeta)$) is equivalent to the condition

$$\sum_k \frac{1-|a_k|^2}{\left|\zeta - a_k\right|^{2n}} + \int_0^{2\pi} \frac{d\mu(\theta)}{\left|\zeta - e^{i\theta}\right|^{2n}} < \infty,$$

where the points $a_k \in \mathbb{D}$ and the measure μ come from the inner-outer factorization of F:

$$F(z) = \left(\prod_k \frac{\bar{a}_k}{a_k} \cdot \frac{z-a_k}{1-z\bar{a}_k}\right) \cdot \exp\left\{-\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} d\mu(\theta)\right\}.$$

Several other equivalent characterizations of the class $\mathcal{S}^{(n)}(\zeta)$ are recalled in Theorem 2.2 below. For every point $\zeta \in \mathbb{T}$ and every integer $n \geq 1$, we let

$$\Psi_n(\zeta) = \begin{bmatrix} \zeta & -\zeta^2 & \cdots & (-1)^{n-1}\binom{n-1}{0}\zeta^n \\ 0 & -\zeta^3 & \cdots & (-1)^{n-1}\binom{n-1}{1}\zeta^{n+1} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & (-1)^{n-1}\binom{n-1}{n-1}\zeta^{2n-1} \end{bmatrix} \tag{2.2}$$

be the $n \times n$ upper triangular matrix with the entries

$$\Psi_{r\ell}(\zeta) = \begin{cases} 0 & \text{if } r > \ell \\ (-1)^{\ell-1}\binom{\ell-1}{r-1}\zeta^{\ell+r-1} & \text{if } r \leq \ell \end{cases} \quad (r, \ell = 1, \dots, n). \tag{2.3}$$

To a function F admitting the boundary limits $F_j(\zeta) := \angle\lim_{z\to\zeta} \frac{F^{(j)}(z)}{j!}$, for $j = 0, \dots, 2n-1$, we associate the structured matrix

$$\mathbb{P}_n^F(\zeta) = \begin{bmatrix} F_1(\zeta) & \dots & F_n(\zeta) \\ \vdots & & \vdots \\ F_n(\zeta) & \dots & F_{2n-1}(\zeta) \end{bmatrix} \Psi_n(\zeta) \begin{bmatrix} \overline{F_0(\zeta)} & \dots & \overline{F_{n-1}(\zeta)} \\ & \ddots & \vdots \\ 0 & & \overline{F_0(\zeta)} \end{bmatrix}, \tag{2.4}$$

where the leftmost factor is a Hankel matrix and the rightmost factor is a Toeplitz upper triangular matrix. Furthermore, it is well-known that for any $F \in \mathcal{S}$ the associated kernel

$$K_F(z,\lambda) = \frac{1 - F(z)\overline{F(\lambda)}}{1 - z\overline{\lambda}} \tag{2.5}$$

is positive on $\mathbb{D} \times \mathbb{D}$ and therefore, the Schwarz-Pick matrix

$$\mathbf{P}_n^F(z) := \left[\frac{1}{i!j!}\frac{\partial^{i+j}}{\partial z^i \partial \bar{z}^j}\frac{1-|F(z)|^2}{1-|z|^2}\right]_{i,j=0}^{n-1} \tag{2.6}$$

is positive semidefinite for every $n \geq 1$ and $z \in \mathbb{D}$. Given a point $\zeta \in \mathbb{T}$, the *boundary Schwarz-Pick matrix* is defined by

$$\mathbf{P}_n^F(\zeta) := \angle\lim_{z\to\zeta} \mathbf{P}_n^F(z), \tag{2.7}$$

provided the non-tangential (finite) limit in (2.7) exists. Thus, once the boundary Schwarz-Pick matrix $\mathbf{P}_n^F(\zeta)$ exists, it is positive semidefinite as the limit of positive semidefinite matrices. It is readily seen from (2.1) that the membership $F \in \mathcal{S}^{(n)}(\zeta)$ is necessary for the limit (2.7) to exist, since it is necessary for the non-tangential convergence of the rightmost diagonal entry in $\mathbf{P}_n^F(z)$. In fact, it is also sufficient due the following theorem established in [12].

Theorem 2.2. *Let $F \in \mathcal{S}$, $\zeta \in \mathbb{T}$ and $n \in \mathbb{N}$. The following are equivalent*:

(1) $F \in \mathcal{S}^{(n)}(\zeta)$;
(2) *The boundary Schwarz-Pick matrix $\mathbf{P}_n^F(\zeta)$ exists*;
(3) *The non-tangential boundary limits $F_j(\zeta)$ exist for $j = 0, \dots, 2n-1$ and satisfy*

$$|F_0(\zeta)| = 1 \quad \textit{and} \quad \mathbb{P}_n^F(\zeta) \geq 0,$$

where $\mathbb{P}_n^F(\zeta)$ is the matrix defined in (2.4).

Moreover, if this is the case, then $\mathbf{P}_n^F(\zeta) = \mathbb{P}_n^F(\zeta)$.

We remark that in contrast to the boundary Schwarz-Pick matrix $\mathbf{P}_n^F(\zeta)$, which is positive semidefinite whenever it exists, the structured matrix $\mathbb{P}_n^F(\zeta)$ defined in terms of the angular limits $F_j(\zeta)$ by formula (2.4) does not have to be positive semidefinite and even Hermitian. Theorem 2.2 states in particular, that the positivity of this structured matrix is an exclusive property of the class $\mathcal{S}^{(n)}(\zeta)$. The following stronger version of the implication (3) $\Rightarrow$ (1) in Theorem 2.2 appears in [15, Theorem 1.7]. It is indeed stronger, since F is not assumed to be in $\mathcal{S}$ and the structured matrix $\mathbb{P}_n^F(\zeta)$ is not assumed to be positive semidefinite.

Theorem 2.3. *Given $F \in \mathrm{Hol}(\mathbb{D})$ and $\zeta \in \mathbb{T}$, let us assume that the non-tangential boundary limits $F_j(\zeta)$ exist for $j = 0, \dots, 2n-1$ and are such that $|F_0(\zeta)| = 1$ and $\mathbb{P}_n^F(\zeta) = \mathbb{P}_n^F(\zeta)^*$. Then F satisfies condition* (2.1).

Let us now turn to the multi-point situation. Given a tuple $\boldsymbol{\zeta} = \{\zeta_1, \dots, \zeta_N\}$ of distinct points in $\mathbb{T}$ and a tuple $\mathbf{d} = \{d_1, \dots, d_N\}$ of respective multiplicities, we may characterize functions from the class $\bigcap_{i=1}^N \mathcal{S}^{(d_i)}(\zeta_i)$, that is, the Schur-class functions F satisfying the Julia-Wolff-Carathéodory type conditions

$$\liminf_{z\to\zeta_i} \frac{\partial^{2d_i-2}}{\partial z^{d_i-1}\partial\bar{z}^{d_i-1}} \frac{1-|F(z)|^2}{1-|z|^2} < \infty \quad \text{for} \quad i = 1, \dots, N. \tag{2.8}$$

To this end, we apply Theorem 2.2 separately to each condition in (2.8) to conclude that the boundary limits $F_j(z_i)$ exist for all $j = 0, \dots, 2d_i - 1$ and $i = 1, \dots, N$ and are such that $|F_0(z_i)| = 1$ and $\mathbb{P}_{d_i}^F(\zeta_i) \ge 0$ for all $i = 1, \dots, N$. In fact we have some more as we will now explain.

Given tuples $\boldsymbol{\zeta} \in \mathbb{T}^N$ and $\mathbf{d} \in \mathbb{Z}_+^N$ as above, we introduce the multi-point analog of the structured matrix (2.4) as the block matrix

$$\mathbb{P}_{\mathbf{d}}^F(\boldsymbol{\zeta}) = \left[P_{ij}^F\right]_{i,j=1}^N \tag{2.9}$$

with the diagonal blocks $P_{ii}^F = \mathbb{P}_{d_i}^F(\zeta_i)$ defined via formula (2.9) and the $d_i \times d_j$ non-diagonal blocks P_{ij}^F defined entry-wise as follows

$$\begin{aligned}\left[P_{ij}^F\right]_{r+1,\ell+1} = &\sum_{t=0}^{\min\{\ell,r\}} \frac{(\ell+r-t)!}{(\ell-t)!t!(r-t)!} \frac{\zeta_i^{r-t}\bar{\zeta}_j^{\ell-t}}{\left(1-\zeta_i\bar{\zeta}_j\right)^{\ell+r-t+1}} \\ &-\sum_{\alpha=0}^{\ell}\sum_{\beta=0}^{r}\sum_{t=0}^{\min\{\alpha,\beta\}} \frac{(\alpha+\beta-t)!}{(\alpha-t)!t!(\beta-t)!} \frac{\zeta_i^{\beta-t}\bar{\zeta}_j^{\alpha-t}F_{\ell-\alpha}(\zeta_i)\overline{F_{r-\beta}(\zeta_j)}}{\left(1-\zeta_i\bar{\zeta}_j\right)^{\alpha+\beta-t+1}}.\end{aligned} \tag{2.10}$$

Observe that the entries of the block P_{ij}^F are completely determined by the boundary limits $F_0(z_i), \dots, F_{d_i-1}(z_i)$ and $F_0(z_j), \dots, F_{d_j-1}(z_j)$. The formula (2.10) is well justified by the the next result.

Theorem 2.4. *Let $F \in \bigcap_{i=1}^N \mathcal{S}^{(d_i)}(\zeta_i)$, where $\boldsymbol{\zeta} = \{\zeta_1, \dots, \zeta_N\} \subset \mathbb{T}$. Then*:

(1) *The non-tangential boundary limits $F_j(\zeta_i)$ exist and $|F_0(\zeta_i)| = 1$ for all $j = 0, \dots, 2d_i - 1$ and $i = 1, \dots, N$;*
(2) *The matrix $\mathbb{P}_{\mathbf{d}}^F(\boldsymbol{\zeta})$ given by* (2.9), (2.10) *is positive semidefinite.*

Proof. The first statement follows from Theorem 2.2 applied individually to every $\zeta_i \in \mathbb{T}$. Given a tuple $\mathbf{z} = \{z_1, \dots, z_N\}$ of distinct points in $\mathbb{D}$ and the tuple $\mathbf{d} = \{d_1, \dots, d_N\}$ as above, the multi-point counterpart of the Schwarz-Pick matrix (2.6) is the block matrix

$$\mathbf{P}^F_{\mathbf{d}}(\mathbf{z}) = \left[\mathbf{P}^F_{d_i,d_j}(z_i, z_j)\right]_{i,j=1}^N \tag{2.11}$$

with the $d_i \times d_j$ block entries

$$\mathbf{P}^F_{d_i,d_j}(z_i, z_j) = \left[\frac{1}{\ell! r!} \frac{\partial^{\ell+r}}{\partial z^\ell \partial \bar{\lambda}^r} \frac{1 - F(z)\overline{F(\lambda)}}{1 - z\bar{\lambda}} \Bigg|_{\substack{z = z_i \\ \lambda = z_j}} \right]_{\ell=0,\dots,d_i-1}^{r=0,\dots,d_j-1}. \tag{2.12}$$

Since the kernel (2.5) is positive on $\mathbb{D} \times \mathbb{D}$ for $F \in \mathcal{S}$, the matrix $\mathbf{P}^F_{\mathbf{d}}(\mathbf{z})$ is positive semidefinite for every choice of $\mathbf{z}$ and $\mathbf{d}$. The straightforward differentiation in (2.12) gives explicit formulas for the entries in $\mathbf{P}^F_{d_i,d_j}(z_i, z_j)$ which have the same form as in (2.10) but with z_i instead of ζ_i and with $F^{(j)}(z_i)/j!$ rather than $F_j(\zeta_i)$. Taking then the limit as $z_i \to \zeta_i$ and $z_j \to \zeta_j$ non-tangentially for $i \neq j$ we conclude that the non-diagonal blocks in $\mathbf{P}^F_{d_i,d_j}(z_i, z_j)$ converge to the corresponding blocks in $\mathbb{P}^F_{\mathbf{d}}(\boldsymbol{\zeta})$ (which now explains the definition (2.10)). The similar convergence for the diagonal blocks is guaranteed by Theorem 2.2. Therefore, the boundary Schwarz-Pick matrix

$$\mathbf{P}^F_{\mathbf{d}}(\boldsymbol{\zeta}) = \angle \lim_{\mathbf{z} \to \boldsymbol{\zeta}} \mathbf{P}^F_{\mathbf{d}}(\mathbf{z}) \tag{2.13}$$

exists and equals $\mathbb{P}^F_{\mathbf{d}}(\boldsymbol{\zeta})$. As the limit of positive semidefinite matrices, this matrix is positive semidefinite as well. □

Remark 2.5. The matrix $\mathbf{P}^F_{\mathbf{k}}(\mathbf{z})$ constructed as in (2.12) (with $\mathbf{k} = (k_1+1, \dots, k_N+1)$ rather than $\mathbf{d}$) plays the central role in solving the interior problem (1.1). The explicit differentiation in (2.8) shows that this matrix is completely determined by the Taylor coefficients $F^{(j)}(z_i)/j!$ of F for $j = 0, \dots, k_i$ and $i = 1, \dots, N$. For every solution F to the problem (1.1), these Taylor coefficients can be replaced by the corresponding target values $w_{i,j}$ producing the matrix $P = \left[P_{ij}\right]_{i,j=1}^N$ with $(k_i + 1) \times (k_j + 1)$ blocks P_{ij} defined entry-wise by

$$\begin{aligned}\left[P_{ij}\right]_{r+1,\ell+1} = &\sum_{s=0}^{\min\{\ell,r\}} \frac{(\ell + r - s)!}{(\ell - s)! s! (r - s)!} \frac{z_i^{r-s} \bar{z}_j^{\ell-s}}{\left(1 - z_i \bar{z}_j\right)^{\ell+r-s+1}} \\ &- \sum_{\alpha=0}^{\ell} \sum_{\beta=0}^{r} \sum_{s=0}^{\min\{\alpha,\beta\}} \frac{(\alpha + \beta - s)!}{(\alpha - s)! s! (\beta - s)!} \frac{z_i^{\beta-s} \bar{z}_j^{\alpha-s} w_{i,\ell-\alpha} \overline{w}_{j,r-\beta}}{\left(1 - z_i \bar{z}_j\right)^{\alpha+\beta-s+1}}.\end{aligned}$$

This matrix P is called the *Pick matrix* of the problem (1.1). It is now clear that for every solution $F \in \mathcal{S}$ to the problem (1.1), the positive semidefinite matrix $\mathbf{P}_{\mathbf{k}}^F(\mathbf{z})$ equals P. Another conclusion is that the condition $P \geq 0$ is necessary for the problem (1.1) to have a solution. A remarkable fact, though, is that this condition is also sufficient.

Let us remark that if F is a finite Blaschke product, it is analytic at every point $\zeta \in \mathbb{T}$ and thus, the boundary limits $F_j(\zeta)$ amount to the Taylor coefficients of F at ζ. Thus the structured matrix $\mathbb{P}_{\mathbf{d}}^F(\boldsymbol{\zeta})$ is defined for every choice of tuples $\boldsymbol{\zeta}$ and $\mathbf{d}$. The following result is known (see *e.g.*, [8, Lemma 2.1]).

Theorem 2.6. *Let F be a finite Blaschke product, let $\boldsymbol{\zeta} = \{\zeta_1, \dots, \zeta_N\} \subset \mathbb{T}$ and let $\mathbf{d} = \{d_1, \dots, d_N\} \subset \mathbb{N}$. Then the matrix $\mathbb{P}_{\mathbf{d}}^F(\boldsymbol{\zeta})$ is positive semidefinite and* $\operatorname{rank} \mathbb{P}_{\mathbf{d}}^F(\boldsymbol{\zeta}) = \min\{d_1 + \ldots + d_N, \ \deg F\}$.

3. Problem BIP($\boldsymbol{\zeta}$, k, s): main results

We start this section with simple sufficient conditions for the determinacy of the problem; see [9, Theorem 1.4] for the proof.

Lemma 3.1. *If $|s_{i,0}| < 1$ for $i = 1, \dots, N$, then the problem* **BIP**($\boldsymbol{\zeta}$, **k**, **s**) *has infinitely many rational solutions.*

Lemma 3.1 tells us that if $|s_{i,0}| < 1$ for $i = 1, \dots, N$, then the problem is indeterminate no matter what the rest of the data set is. On the other hand, by the very definition of the class $\mathcal{S}$, conditions $|s_{i,0}| \leq 1$ for $i = 1, \dots, N$ are necessary for the problem to have a solution. The Julia-Wolff-Carathéodory theorem states that if a function $F \in \mathcal{S}$ admits finite boundary limits $F_0(\zeta_i) \in \mathbb{T}$ and $F_1(\zeta_i)$, then necessarily

$$\zeta_i F_1(\zeta_i)\overline{F_0(\zeta_i)} = \angle \lim_{z \to \zeta_i} \frac{1 - |F(z)|}{1 - |z|} \geq 0.$$

We thus have another necessary condition for the problem **BIP**($\boldsymbol{\zeta}$, **k**, **s**) to have a solution: $\zeta_i s_{i,1} \overline{s}_{i,0} \geq 0$ whenever $|s_{i,0}| = 1$ and $k_i \geq 1$ (*i.e.*, $s_{i,1}$ is given).

3.1. The Pick matrix of the problem and the companion numbers

In the previous section we introduced the structured matrix (2.9) in terms of the boundary limits $F_j(\zeta_i)$. Since the problem **BIP**($\boldsymbol{\zeta}$, **k**, **s**) prescribes the values of these limits, we may try to plug them into (2.9) to construct the Pick matrix of the problem **BIP**($\boldsymbol{\zeta}$, **k**, **s**) and then to establish (as in Remark 2.5) the solvability criterion in terms of this matrix.

Before to proceed we will mention several distinctions between the interior problem (1.1) and its boundary counter-part (1.3).

(1) The Pick matrix of the interior problem incorporates all given data (equivalently, the matrix $\mathbf{P}_{\mathbf{k}}^{F}(\mathbf{z})$ contains all Taylor coefficients of F we want to match), while the matrix (2.9) incorporates an *even* number of boundary derivatives assigned to each point ζ_i and therefore, it may miss some data (for example, if one of k_i's in (1.2) is even).
(2) The Pick matrix of the interior problem is Hermitian by construction, while the matrix (2.9) does not have to be Hermitian (unless F satisfies quite restrictive Julia-Wolff-Carathéodory conditions (2.8)).
(3) Lemma 3.1 shows that the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$ may have solutions regardless of most of data.

The formal definition of the Pick matrix (given by formulas (3.9)-(3.11) below) needs some preliminary work. First, we introduce the index set

$$I = \{i :\ |s_{i,0}| = 1 \ \text{ and } \ k_i \geq 1\} \subset \{1, \dots, N\} \tag{3.1}$$

and recall that the conditions

$$|s_{i,0}| \leq 1 \quad \text{for } \ i = 1, \dots, N \ \text{ and } \quad \zeta_i s_{i,1} \overline{s}_{i,0} \geq 0 \quad \text{for all} \quad i \in I \tag{3.2}$$

are necessary for the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$ to have a solution. In what follows, we assume that these conditions are met. The whole index set $\{1, \dots, N\}$ can be split into three disjoint sets I, J, K where I is given in (3.1),

$$J = \{i : |s_{i,0}| = 1 \ \text{ and } \ k_i = 0\} \quad \text{and} \quad K = \{i : |s_{i,0}| < 1\}. \tag{3.3}$$

For every $i \in I$, we use the given string $\{s_{i,0}, \dots, s_{i,k_i}\}$ to define the lower triangular Toeplitz matrix

$$\mathbb{U}_{i,n}^{\mathbf{s}} = \begin{bmatrix} s_{i,0} & 0 & \dots & 0 \\ s_{i,1} & s_{i,0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ s_{i,n-1} & \dots & s_{i,1} & s_{i,0} \end{bmatrix} \tag{3.4}$$

and the Hankel matrix

$$\mathbb{H}_{i,n}^{\mathbf{s}} = \begin{bmatrix} s_{i,1} & s_{i,2} & \dots & s_{i,n} \\ s_{i,2} & s_{i,3} & \dots & s_{i,n+1} \\ \vdots & \vdots & & \vdots \\ s_{i,n} & s_{i,n+1} & \dots & s_{i,2n-1} \end{bmatrix} \tag{3.5}$$

for every appropriate integer $n \geq 1$ (*i.e.*, for every $n \leq k_i + 1$ in (3.4) and for every $n \leq (k_i + 1)/2$ in (3.5)). For every $n \leq (k_i + 1)/2$ and $i \in I$, we then define the matrix (*cf.* (2.4))

$$\mathbb{P}_n^{\mathbf{s}}(\zeta_i) = \left[p_{r\ell}^{\mathbf{s}}(\zeta_i)\right]_{r,\ell=1}^{n} = \mathbb{H}_{i,n}^{\mathbf{s}} \Psi_n(\zeta_i) \mathbb{U}_{i,n}^{\mathbf{s}*} \tag{3.6}$$

with the entries (as it follows from (2.2), (3.4) and (3.5))

$$p^{\mathbf{s}}_{r\ell}(\zeta_i) = \sum_{\alpha=1}^{\ell}\left(\sum_{\beta=1}^{\alpha} s_{i,r+\beta-1}\Psi_{\beta\alpha}(\zeta_i)\right)\overline{s}_{i,\ell-\alpha}. \tag{3.7}$$

Remark 3.2. Due to the upper triangular structure of the factors $\Psi_n(\zeta_i)$ and $\mathbb{U}^{\mathbf{s}*}_{i,n}$ in (3.6), it follows that $\mathbb{P}^{\mathbf{s}}_k(\zeta_i)$ is the leading submatrix of $\mathbb{P}^{\mathbf{s}}_n(\zeta_i)$ for every $k < n$. In particular, if $\mathbb{P}^{\mathbf{s}}_n(\zeta_i)$ is Hermitian, then $\mathbb{P}^{\mathbf{s}}_k(\zeta_i)$ is Hermitian for every $k < n$.

Remark 3.3. We observe that formula (3.7) defines the numbers $p^{\mathbf{s}}_{r\ell}(\zeta_i)$ in terms of $\mathbf{s}$ for every pair of indices (r, ℓ) subject to $r + \ell \le k_i + 1$.

The structured matrix $\mathbb{P}^{\mathbf{s}}_n(\zeta_i)$ does not have to be Hermitian. However, the results presented in the previous section indicate that the hermitian case is of particular interest. This suggests the following construction. Let us define the tuple $\mathbf{d} = \{d_1, \dots, d_N\}$ where

$$d_i = \begin{cases} \max\left\{n :\ \mathbb{P}^{\mathbf{s}}_n(\zeta_i) = \mathbb{P}^{\mathbf{s}}_n(\zeta_i)^*\right\} & \text{if } i \in I \\ 0 & \text{if } i \in J \cup K. \end{cases} \tag{3.8}$$

Observe that for $n = 1$, formula (3.6) takes the form $\mathbb{P}^{\mathbf{s}}_1(\zeta_i) = \zeta_i s_{i,1}\overline{s}_{i,0}$. Since we assume that conditions (3.2) are in force, it follows that $d_i > 0$ for every $i \in I$. Furthermore, for every $i \in I$, the structured matrix $\mathbb{P}^{\mathbf{s}}_{d_i}(\zeta_i)$ is Hermitian while the extended matrix $\mathbb{P}^{\mathbf{s}}_{d_i+1}(\zeta_i)$ is not (in case $d_i < k_i/2$).

We now define the *Pick matrix* of the problem $\mathbf{BIP}(\boldsymbol{\zeta}, \mathbf{k}, \mathbf{s})$ by

$$\mathbb{P}^{\mathbf{s}}_{\mathbf{d}} = \left[P^{\mathbf{s}}_{ij}\right]_{i,j\in I} \tag{3.9}$$

where the $d_i \times d_i$ diagonal blocks are given by

$$P^{\mathbf{s}}_{ii} = \mathbb{P}^{\mathbf{s}}_{d_i}(\zeta_i) = \mathbb{H}^{\mathbf{s}}_{i,d_i}\Psi_{d_i}(\zeta_i)\mathbb{U}^{\mathbf{s}*}_{i,d_i} \tag{3.10}$$

and where the $d_i \times d_j$ non-diagonal blocks $P^{\mathbf{s}}_{ij} = \left[p^{\mathbf{s}}_{r\ell}(\zeta_i, \zeta_j)\right]_{r=1,\dots,d_i}^{\ell=1,\dots,d_j}$ are defined entry-wise by

$$\begin{aligned} p^{\mathbf{s}}_{r+1,\ell+1}(\zeta_i, \zeta_j) = & \sum_{s=0}^{\min\{\ell,r\}} \frac{(\ell+r-s)!}{(\ell-s)!s!(r-s)!}\frac{\zeta_i^{r-s}\bar{\zeta}_j^{\ell-s}}{\left(1-\zeta_i\bar{\zeta}_j\right)^{\ell+r-s+1}} \\ & -\sum_{\alpha=0}^{\ell}\sum_{\beta=0}^{r}\sum_{s=0}^{\min\{\alpha,\beta\}} \frac{(\alpha+\beta-s)!}{(\alpha-s)!s!(\beta-s)!}\frac{\zeta_i^{\beta-s}\bar{\zeta}_j^{\alpha-s}s_{i,\ell-\alpha}\overline{s}_{j,r-\beta}}{\left(1-\zeta_i\bar{\zeta}_j\right)^{\alpha+\beta-s+1}}. \end{aligned} \tag{3.11}$$

Remark 3.4. Observe that the matrix $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ is Hermitian by construction, since the diagonal blocks $P^{\mathbf{s}}_{ii} = \mathbb{P}^{\mathbf{s}}_{d_i}(\zeta_i)$ are Hermitian due to the choice (3.8) of d_i while the non-diagonal blocks satisfy $P^{\mathbf{s}}_{ij} = P^{\mathbf{s}*}_{ji}$ according to (3.11).

We also remark that the Pick matrix $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ is non-trivial only if the index set I in (3.1) is non-empty. In this case we use the numbers (3.8) to break I into three disjoint parts:

$$I' = \{i \in I : 2d_i < k_i\}, \quad I'' = \{i \in I : 2d_i = k_i\}, \\ I''' = \{i \in I : 2d_i - 1 = k_i\}, \tag{3.12}$$

so that, for every $i \in I'''$, all data assigned to ζ_i is incorporated in the Hermitian matrix $\mathbb{P}^{\mathbf{s}}_{d_i}(\zeta_i)$. In case $i \in I''$, the matrix $\mathbb{P}^{\mathbf{s}}_{d_i}(\zeta_i)$ has the maximally possible size and incorporates all data assigned to ζ_i but s_{i,k_i}. Finally, if $i \in I'$, then we have enough data to construct the extended structured matrix $\mathbb{P}^{\mathbf{s}}_{d_i+1}(\zeta_i)$, but this matrix turns out to be non-Hermitian.

By Remark 3.3, for every $i \in I' \cup I''$ (that is, for every i such that $2d_i \leq k_i$), we can use formula (3.7) to define the numbers $p^{\mathbf{s}}_{d_i,d_i+1}(\zeta_i)$ and $p^{\mathbf{s}}_{d_i+1,d_i}(\zeta_i)$ which are not the entries of $\mathbb{P}^{\mathbf{s}}_{d_i}(\zeta_i)$. We then define the *companion numbers* γ_i of the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$ by

$$\gamma_i = \zeta_i \cdot \left(p^{\mathbf{s}}_{d_i+1,d_i}(\zeta_i) - \overline{p^{\mathbf{s}}_{d_i,d_i+1}(\zeta_i)} \right) \quad \text{for} \quad i \in I' \cup I''. \tag{3.13}$$

3.2. Main results

The Pick matrix of the boundary interpolation problem and the companion numbers is all we need to present the main results of this paper. The first theorem gives the indeterminacy criterion for the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$.

Theorem 3.5. *Given the data set* (1.2) *with* $|s_{i,0}| \leq 1$ *for* $i = 1, \ldots, N$, *let* I *be defined as in* (3.1), *let* $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ *be the Pick matrix defined in* (3.9)-(3.11) *and let* γ_i *be the companion numbers defined via formulas* (3.13) *and* (3.7). *Then the problem* $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$ *is indeterminate if and only if either* I *is empty or the following three conditions hold*:

(1) $I \neq \emptyset$ *and* $\zeta_i s_{i,1} \bar{s}_{i,0} > 0$ *for all* $i \in I$;
(2) *the matrix* $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ *is positive definite*;
(3) $\gamma_i > 0$ *for every* $i \in I'$ *and* $\gamma_i \geq 0$ *for every* $i \in I''$.

Furthermore, the indeterminate problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$ *admits infinitely many rational solutions. Finally, every solution* F *of the problem belongs to the class* $\bigcap_{i\in I} \mathcal{S}^{(d_i)}(\zeta_i)$ *and does not belong to* $\bigcup_{i\in I'} \mathcal{S}^{(d_i+1)}(\zeta_i)$.

It is seen from Theorem 3.5 that the indeterminacy of the problem does not depend on the part of data associated with the indices $i \in J \cup K$. To present the determinacy criterion, we introduce the notion of a *structured extension*. We let $\mathbf{e}_i$ to be a "coordinate" tuple so that all the integers in $\mathbf{d}+\mathbf{e}_i$ are the same as in $\mathbf{d}$ except for the i-th integer which is equal to $d_i + 1$. To construct the extended Pick matrix $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}+\mathbf{e}_i}$, we need some more data beyond (1.2). Namely, we need

$$(1)\ s_{i,1},\ \text{if}\ i \in J; \quad (2)\ s_{i,2d_i},\ \text{if}\ i \in I''; \quad (3)\ s_{i,2d_i},\ s_{i,2d_i+1}\ \text{if}\ i \in I'''. \tag{3.14}$$

We will refer to the matrices $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}+\mathbf{e}_i}$ as to the *structured extensions* of $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$. Finally, let us define the column-vector

$$C_i = \mathrm{Col}_{k\in I}\begin{bmatrix} p^{\mathbf{s}}_{1,d_i+1}(\zeta_k,\zeta_i) \\ p^{\mathbf{s}}_{2,d_i+1}(\zeta_k,\zeta_i) \\ \vdots \\ p^{\mathbf{s}}_{d_i,d_i+1}(\zeta_k,\zeta_i) \end{bmatrix} \quad \text{for all} \quad i \in I' \cup I'' \cup J, \tag{3.15}$$

where $p^{\mathbf{s}}_{r,d_i+1}(\zeta_k,\zeta_i)$ is defined for $r = 1,\dots,d_i$ by formula (3.11) in case $k \neq i$ or by

$$p^{\mathbf{s}}_{r,d_i+1}(\zeta_i,\zeta_i) := p^{\mathbf{s}}_{r,d_i+1}(\zeta_i) = \sum_{\alpha=1}^{d_i+1}\left(\sum_{\beta=1}^{\alpha} s_{i,r+\beta-1}\Psi_{\beta\alpha}(\zeta_i)\right)\overline{s}_{i,d_i+1-\alpha} \tag{3.16}$$

if $k = i$ (*cf.* (3.7)). Note that the entries of C_i do not appear in $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ but they do appear in the extended matrix $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}+\mathbf{e}_i}$. The next theorem gives the determinacy criterion for the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$.

Theorem 3.6. *The problem* $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$ *is determinate if and only if*:

(1) $|s_{i,0}| = 1$ *for every* $i = 1,\dots,N$ (*i.e., the set* K *is empty*);
(2) $2d_i \in \{k_i, k_i+1\}$ *for every* $i = 1,\dots,N$ (*i.e., the set* I' *is empty*);
(3) *The matrix* $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ *is positive semidefinite* (*singular*) *and admits positive semidefinite extensions* $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}+\mathbf{e}_i}$ *for* $i = 1,\dots,N$.

The third condition can be equivalently replaced by the following three conditions:

(3a) $\operatorname{rank}\mathbb{P}^{\mathbf{s}}_{\mathbf{d}} = \operatorname{rank}\left[\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}\ \ C_i\right]$ *for all* $i \in J$, *where* C_i *is given by* (3.15);
(3b) $p^{\mathbf{s}}_{d_i+1,d_i}(\zeta_i) = \overline{p^{\mathbf{s}}_{d_i,d_i+1}(\zeta_i)}$ *and* $\operatorname{rank}\mathbb{P}^{\mathbf{s}}_{\mathbf{d}} = \operatorname{rank}\left[\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}\ \ C_i\right]$ *for all* $i \in I''$;
(3c) $\operatorname{rank}\mathbb{P}^{\mathbf{s}}_{\mathbf{d}} = \operatorname{rank}\mathbb{P}^{\mathbf{s}}_{\mathbf{d}-\mathbf{e}_i}$ *for all* $i \in I'''$.

If this is the case, then the unique solution is a Blaschke product of degree equal to the rank of $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$.

Our further strategy is the following. In the next section we consider two special cases of the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$. In Section 5 we present the reduction step which will be used in Section 6 to prove Theorems 3.5 and 3.6.

4. Special cases

The special case of the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$ where the sets J, K, I', I'' defined in (3.12) and (3.3) are all empty has appeared in literature; see *e.g.*, [4,5,7,11,13, 23,28]. The main feature of this case is that its Pick matrix incorporates all interpolation data and that the solution set admits a nice linear fractional parametrization.

Although the generic problem **BIP**($\boldsymbol{\zeta}$, **k**, **s**) is not of this type, it contains a subproblem of this type whenever $I \neq \emptyset$. Indeed, it is seen from formulas (3.9)–(3.11) that the matrix $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ is constructed from ζ_i and $s_{i,0}, \dots, s_{i,2d_i-1}$ for all $i \in I$. The boundary interpolation problem associated to this partial data, *i.e.*, with

$$\widetilde{\boldsymbol{\zeta}} = \left\{\zeta_i\right\}_{i\in I}, \quad \widetilde{\mathbf{d}} = \left\{d_i\right\}_{i\in I}, \quad \widetilde{\mathbf{s}} = \left\{s_{i,j}\right\}_{i\in I}^{j=0,\dots,2d_i-1} \tag{4.1}$$

will be called the *Symmetric Boundary Interpolation Problem* and will be denoted by **SBIP**($\widetilde{\boldsymbol{\zeta}}$, $\widetilde{\mathbf{d}}$, $\widetilde{\mathbf{s}}$). The term "symmetric" is chosen due to the symmetry relations (4.5) (see below) satisfied by its data.

SBIP($\widetilde{\boldsymbol{\zeta}}$, $\widetilde{\mathbf{d}}$, $\widetilde{\mathbf{s}}$): *Given the data set* (4.1)*, find all functions $F \in \mathcal{S}$ such that*

$$F_j(\zeta_i) := \angle \lim_{z\to\zeta_i} \frac{F^{(j)}(z)}{j!} = s_{i,j} \quad \textit{for} \;\; j = 0, \dots, 2d_i - 1; \; i \in I, \tag{4.2}$$

or equivalently, such that

$$F(z) = s_{i,0} + s_{i,1}(z-\zeta_i) + \dots + s_{i,2d_i-1}(z-\zeta_i)^{2d_i-1} + o\left(|z-\zeta_i|^{2d_i-1}\right) \tag{4.3}$$

for all $i \in I$ as z tends to ζ_i non-tangentially.

By the definition (3.8), the data set (4.1) satisfies conditions

$$|s_{i,0}| = 1 \quad \text{and} \quad \mathbb{P}^{\mathbf{s}}_{d_i}(\zeta_i) = \mathbb{P}^{\mathbf{s}}_{d_i}(\zeta_i)^* \quad \text{for all} \quad i \in I \tag{4.4}$$

which turn out to be equivalent (see [15, Theorem 1.5] for the proof) to equalities

$$\begin{bmatrix} s_{i,0} & \dots & s_{i,2d_i-1} \\ & \ddots & \vdots \\ 0 & & s_{i,0} \end{bmatrix} \Psi_{2d_i}(\zeta_i) \begin{bmatrix} \overline{s}_{i,0} & \dots & \overline{s}_{i,2d_i-1} \\ & \ddots & \vdots \\ 0 & & \overline{s}_{i,0} \end{bmatrix} = \Psi_{2d_i}(\zeta_i) \tag{4.5}$$

for all $i \in I$, where the leftmost and the rightmost factors are upper triangular Toeplitz matrices and $\Psi_{2d_i}(\zeta_i)$ is defined via formula (2.2). By a result from [11], the matrix equality (4.5) is equivalent to the system of d_i equalities

$$\sum_{r=0}^{j}\sum_{\ell=0}^{j-r}(-1)^{\ell}\binom{j-\ell}{r}\zeta_i^{r-\ell}s_{i,r}\overline{s}_{i,\ell} = 1 \quad \text{for} \;\; j = 0, \dots, 2d_i - 1.$$

The following "symmetry" result (see [10, Lemma 3.8]) will be useful for our subsequent analysis.

Lemma 4.1. *Let us assume that conditions* (4.4) *are satisfied and let $p^{\mathbf{s}}_{r,\ell}(\zeta_i)$ be the numbers defined via formula* (3.7) *for*

$$r, \ell \in \{1, \dots, 2d_i - 2\}, \quad \textit{subject to} \quad 2 \le r + \ell + j \le 2d_i - 2. \tag{4.6}$$

Then $p^{\mathbf{s}}_{r,\ell}(\zeta_i) = \overline{p^{\mathbf{s}}_{\ell,r}(\zeta_i)}$ for all r, ℓ as in (4.6).

In other words, relations (4.4) force certain symmetries for the numbers $p^{\mathbf{s}}_{ij}(\zeta_i)$ which are not the entries of $\mathbb{P}^{\mathbf{s}}_{d_i}(\zeta_i)$. The next theorem can be found in [13].

Theorem 4.2. *If the problem* $\mathbf{SBIP}(\widetilde{\boldsymbol{\zeta}},\ \widetilde{\mathbf{d}},\ \widetilde{\mathbf{s}})$ *has a solution, then the Pick matrix* $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ *is positive semidefinite. Furthermore*:

(1) *If* $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ *is positive definite, then the problem* $\mathbf{SBIP}(\widetilde{\boldsymbol{\zeta}},\ \widetilde{\mathbf{d}},\ \widetilde{\mathbf{s}})$ *is indeterminate*;
(2) *If* $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ *is positive semidefinite and singular, then there exists a unique function* $F\in\mathcal{S}$ *subject to boundary interpolation conditions*

$$F_j(\zeta_i)=s_{i,j}\quad for\quad j=0,\dots,2d_i-2;\quad i\in I. \tag{4.7}$$

and

$$(-1)^{d_i}\zeta_i^{2d_i-1}\overline{s}_{i,0}\left(F_{2d_i-1}(\zeta_i)-s_{i,2d_i-1}\right)\geq 0\quad for\quad i\in I. \tag{4.8}$$

This unique F *is a Blaschke product of degree equal to the rank of* $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$.

A curious point here is that in case the Pick matrix $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}\geq 0$ is singular, the problem $\mathbf{SBIP}(\widetilde{\boldsymbol{\zeta}},\ \widetilde{\mathbf{d}},\ \widetilde{\mathbf{s}})$ may have no solutions. However, if the last condition at each point is relaxed as in (4.8), then this relaxed problem will have a unique solution. Since every solution of the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$ also solves the associated symmetric problem $\mathbf{SBIP}(\widetilde{\boldsymbol{\zeta}},\ \widetilde{\mathbf{d}},\ \widetilde{\mathbf{s}})$ and since every solution of the problem $\mathbf{SBIP}(\widetilde{\boldsymbol{\zeta}},\ \widetilde{\mathbf{d}},\ \widetilde{\mathbf{s}})$ satisfies the relaxed interpolation conditions (4.7), (4.8), we can make some conclusions concerning the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$.

Corollary 4.3. *Assume that the set* I *is not empty so that the Pick matrix* $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ *is non-trivial. Then*:

(1) *If the problem* $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$ *has a solution, then* $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ *is positive semidefinite*;
(2) *If* $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}\geq 0$ *is singular, then the problem* $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$ *has at most one solution, which is necessarily a Blaschke product of degree equal to* rank $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$.

Recall that the Pick matrix $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ was modeled from its prototype $\mathbb{P}^{F}_{\mathbf{d}}(\boldsymbol{\zeta})$ by simply replacing the boundary limits $F_j(\zeta_i)$ by the target values $s_{i,j}$. Hence, if F satisfies interpolation conditions (4.2), then clearly $\mathbb{P}^{F}_{\mathbf{d}}(\boldsymbol{\zeta})=\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$.

Remark 4.4. It can be shown (we refer to [13] for details) that conditions (4.7) guarantee that all the corresponding entries in $\mathbb{P}^{F}_{\mathbf{d}}(\boldsymbol{\zeta})$ and $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ are equal except for the rightmost diagonal entries in the diagonal blocks. For these entries, we have by (2.4) and (3.7),

$$p^{F}_{d_i,d_i}(\zeta_i)=\sum_{\ell=1}^{d_i}\left(\sum_{r=1d}^{\ell}F_{d_i+r-1}(\zeta_i)\Psi_{r\ell}(\zeta_i)\right)\overline{F_{i,d_i-\ell}(\zeta_i)},$$

$$p^{\mathbf{s}}_{d_i,d_i}(\zeta_i)=\sum_{\ell=1}^{d_i}\left(\sum_{r=1d}^{\ell}s_{i,d_i+r-1}\Psi_{r\ell}(\zeta_i)\right)\overline{s}_{i,d_i-\ell},$$

and, due to conditions (4.7),

$$\begin{aligned}p^{\mathbf{s}}_{d_i,d_i}(\zeta_i)-p^{F}_{d_i,d_i}(\zeta_i)&=\left(s_{i,2d_i-1}-F_{2d_i-1}(\zeta_i)\right)\Psi_{d_i,d_i}(\zeta_i)\overline{s}_{i,0}\\&=(-1)^{d_i}\zeta_i^{2d_i-1}\overline{s}_{i,0}\left(F_{2d_i-1}(\zeta_i)-s_{i,2d_i-1}\right).\end{aligned}$$

We thus conclude that if F satisfies relaxed interpolation conditions (4.7), (4.8), then $\mathbb{P}^F_{\mathbf{d}}(\boldsymbol{\zeta})\le\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$.

We next recall the linear fractional parametrization of all solutions of the problem $\mathbf{SBIP}(\widetilde{\boldsymbol{\zeta}},\,\widetilde{\mathbf{d}},\,\widetilde{\mathbf{s}})$ in the indeterminate case. With the data set (4.1) we associate the block diagonal matrix T and two column-vectors E and M with the block-matrix representations

$$T=\operatorname{diag}_{i\in I}J_{d_i}(\zeta_i),\quad E=\operatorname{Col}_{i\in I}E_{d_i},\quad M=\operatorname{Col}_{i\in I}M_i$$

conformal with (3.9), where the blocks $J_{d_i}(\zeta_i)\in\mathbb{C}^{d_i\times d_i}$, and E_{d_i}, $M_i\in\mathbb{C}^{d_i}$ are given by

$$J_{d_i}(\zeta_i)=\begin{bmatrix}\zeta_i & 0 & \dots & 0\\ 1 & \zeta_i & \ddots & \vdots\\ & \ddots & \ddots & 0\\ 0 & & 1 & \zeta_i\end{bmatrix},\quad E_{d_i}=\begin{bmatrix}1\\0\\ \vdots\\0\end{bmatrix},\quad M_i=\begin{bmatrix}s_{i,0}\\ s_{i,1}\\ \vdots\\ s_{i,d_i-1}\end{bmatrix}.$$

It turns out that the Pick matrix $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ satisfies the Stein equality

$$\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}-T\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}T^*=EE^*-MM^*.\tag{4.9}$$

We observe that equality (4.9) is a consequence of a particular structure of $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ as well as of conditions (4.4) and refer to [13, Section 3] for details. We now let

$$\widetilde{\mathbb{P}}:=\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}+MM^*=T\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}T^*+EE^*\tag{4.10}$$

where the second equality follows from (4.9). Since $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ is assumed to be positive definite, the matrix $\widetilde{\mathbb{P}}$ is positive definite as well. It follows from (4.10) that

$$1-M^*\widetilde{\mathbb{P}}^{-1}M=\det\left(I-MM^*\widetilde{\mathbb{P}}^{-1}\right)=\det\left(\widetilde{\mathbb{P}}-MM^*\right)\cdot\det\widetilde{\mathbb{P}}^{-1}=\frac{\det\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}}{\det\widetilde{\mathbb{P}}}>0$$

and similarly, $1-E^*\widetilde{\mathbb{P}}^{-1}E>0$. We then let

$$\alpha=\sqrt{1-M^*\widetilde{\mathbb{P}}^{-1}M},\quad\text{and}\quad\beta=\sqrt{1-E^*\widetilde{\mathbb{P}}^{-1}E}$$

and introduce the 2×2 matrix-function

$$\mathbf{S}=\begin{bmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{c} & \mathbf{g}\end{bmatrix}\tag{4.11}$$

with the entries

$$\mathbf{a}(z) = E^*\big(\widetilde{\mathbb{P}} - z\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}T^*\big)^{-1}M, \tag{4.12}$$

$$\mathbf{b}(z) = \beta\left(1 - zE^*\big(\widetilde{\mathbb{P}} - z\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}T^*\big)^{-1}T^{-1}E\right), \tag{4.13}$$

$$\mathbf{c}(z) = \alpha\left(1 - zM^*T^*\big(\widetilde{\mathbb{P}} - z\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}T^*\big)^{-1}M\right), \tag{4.14}$$

$$\mathbf{g}(z) = z\alpha\beta M^*\big(\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}\big)^{-1}\widetilde{\mathbb{P}}\big(\widetilde{\mathbb{P}} - \mathbb{P}^{\mathbf{s}}_{\mathbf{d}}T^*\big)^{-1}T^{-1}E. \tag{4.15}$$

It was shown in [13, Theorem 6.4] that $\mathbf{S}$ is a rational function of McMillan degree $|\mathbf{d}| = \sum_{i\in I} d_i$ which is inner in $\mathbb{D}$; these properties of $\mathbf{S}$ follow solely from the Stein equality (4.9) and the positivity of $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$. Therefore, the entries (4.11)-(4.15) of $\mathbf{S}$ are rational Schur-class functions analytic on $\mathbb{T}$. Some properties of their Taylor coefficients at the interpolation nodes ζ_i are listed in Theorem 4.5 below (see Lemma 6.5 in [13] for the proof). In its formulation we use notation $f_j(z) = f^{(j)}(z)/j!$.

Theorem 4.5. *Let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{g}$ *be defined as in* (4.12)-(4.15). *Then*:

(1) $\mathbf{a}_j(\zeta_i) = s_{i,j}$ *for* $j = 0, \dots, 2d_i - 1$ *and* $i \in I$;
(2) $|\mathbf{g}(\zeta_i)| = 1$ *for* $i \in I$;
(3) $\mathbf{b}_j(\zeta_i) = \mathbf{c}_j(\zeta_i) = 0$ *for* $j = 0, \dots, d_i - 1$ *and* $i \in I$;
(4) $\mathbf{b}_{d_i}(\zeta_i) \neq 0$, $\mathbf{c}_{d_i}(\zeta_i) \neq 0$ *and moreover,*

$$\zeta_i^{2d_i}\mathbf{b}_{d_i}(\zeta_i) = (-1)^{d_i-1}\overline{\mathbf{c}_{d_i}(\zeta_i)}\mathbf{g}(\zeta_i)s_{i,0}; \tag{4.16}$$

(5) *For every* $z \in \mathbb{C}\backslash\{\zeta_i : i \in I\}$, *it holds that* $\mathbf{b}(z) \neq 0$ *and* $\mathbf{c}(z) \neq 0$.

The next theorem (see [13, Theorem 1.6] for the proof) describes the solution set of the problem $\mathbf{SBIP}(\widetilde{\boldsymbol{\zeta}}, \widetilde{\mathbf{d}}, \widetilde{\mathbf{s}})$. In more detail, we first parametrize the solution set of the relaxed problem (4.7), (4.8) where the last interpolation conditions prescribing the values of $F_{2d_i-1}(\zeta_i)$ are relaxed to inequalities (4.8), and then we identify the functions from this solution set for which the equality prevails in (4.8) for every fixed $i \in I$. We will also identify the functions satisfying the non-tangential asymptotics

$$F(z) = s_{i,0} + s_{i,1}(z-\zeta_i) + \ldots + s_{i,2d_i-1}(z-\zeta_i)^{2d_i-1} + O\left(|z-\zeta_i|^{2d_i}\right) \tag{4.17}$$

which is slightly stronger than that in (4.3).

Theorem 4.6. *Let us assume that* $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ *is positive definite.*

(1) *A function* F *belongs to* $\mathcal{S}$ *and satisfies conditions* (4.7), (4.8) *if and only if it is of the form*

$$F(z) = \mathbf{T}_{\mathbf{S}}[\mathcal{E}](z) := \mathbf{a}(z) + \frac{\mathbf{b}(z)\mathbf{c}(z)\mathcal{E}(z)}{1 - \mathbf{g}(z)\mathcal{E}(z)} \tag{4.18}$$

where the coefficient matrix $\mathbf{S}$ *is given in* (4.11)–(4.15) *and where the parameter* $\mathcal{E}$ *is a Schur-class function.*

(2) *A function F of the form* (4.18) *satisfies condition* (4.17) *for a fixed $i \in I$ if and only if the corresponding parameter $\mathcal{E} \in \mathcal{S}$ is such that either*

$$\mathcal{E}(\zeta_i) := \angle \lim_{z \to \zeta_i} \mathcal{E}(z) \neq \overline{\mathbf{g}(\zeta_i)} \tag{4.19}$$

or the non-tangential boundary limit $\mathcal{E}(\zeta_i)$ does not exist.

(3) *A function F of the form* (4.18) *satisfies condition* (4.3) *for a fixed $i \in I$ if and only if the corresponding parameter $\mathcal{E} \in \mathcal{S}$ is either as in part* (2) *or is subject to equalities*

$$\mathcal{E}(\zeta_i) = \overline{\mathbf{g}(\zeta_i)} \quad \textit{and} \quad \liminf_{z \to \zeta_i} \frac{1 - |\mathcal{E}(z)|}{1 - |z|} = \infty. \tag{4.20}$$

Remark 4.7. The correspondence $\mathcal{E} \to F$ established by formula (4.18) is one-to-one and the inverse transformation is given by

$$\mathcal{E}(z) = \mathbf{T}_{\mathbf{S}}^{-1}[f](z) = \frac{F(z) - \mathbf{a}(z)}{\mathbf{b}(z)\mathbf{c}(z) + \mathbf{g}(z)(F(z) - \mathbf{a}(z))}. \tag{4.21}$$

Therefore, condition (4.20) explicitly describes the dichotomy between condition (4.3) and a weaker condition (4.17). Although condition (4.17) does not have a clear interpolation interpretation in general, it gets one while being restricted to *rational* Schur functions. In this case, (4.17) is equivalent to (4.3) and therefore, to conditions (4.2). The formula (4.18) parametrizes all *rational solutions* of the problem $\mathbf{SBIP}(\widetilde{\boldsymbol{\zeta}}, \widetilde{\mathbf{d}}, \widetilde{\mathbf{s}})$ if the parameter $\mathcal{E}$ runs through the set of all rational Schur functions satisfying condition (4.19) for every $i \in I$; we refer to [5] for the detailed treatment of the rational Schur-class boundary interpolation.

Remark 4.8. The coefficient matrix $\mathbf{S}(\zeta)$ is unitary for every $\zeta \in \mathbb{T}$ (since $\mathbf{S}$ is inner in $\mathbb{D}$). If in addition, $\zeta \notin \{\zeta_i : i \in I\}$, then we also have $\mathbf{b}(\zeta)\mathbf{c}(\zeta) \neq 0$ (by part (5) in Theorem 4.5) and therefore $|\mathbf{g}(\zeta))| < 1$. If F and $\mathcal{E}$ are rational Schur-class functions related as in (4.18), then the n first Taylor coefficients of F at ζ are completely determined by the n first Taylor coefficients of $\mathcal{E}$ at ζ and vice versa. Moreover, $|F(\zeta| = 1$ (respectively $|F(\zeta| < 1$) if and only if $|\mathcal{E}(\zeta)| = 1$ (respectively $|\mathcal{E}(\zeta)| < 1$). The latter follows from the identity

$$\begin{aligned} 1 - |F(\zeta)|^2 = {} & \frac{|\mathbf{b}(\zeta)|^2 \big(1 - |\mathcal{E}(\zeta)|^2\big)}{|1 - \mathbf{g}(\zeta)\mathcal{E}(\zeta)|^2} \\ & + \begin{bmatrix} 1 & \dfrac{\mathbf{b}(\zeta)\mathcal{E}(\zeta)}{1 - \mathbf{g}(\zeta)\mathcal{E}(\zeta)} \end{bmatrix} \big(I - \mathbf{S}(\zeta)\mathbf{S}(\zeta)^*\big) \begin{bmatrix} 1 \\ \\ \dfrac{\overline{\mathbf{b}(\zeta)\mathcal{E}(\zeta)}}{1 - \overline{\mathbf{g}(\zeta)\mathcal{E}(\zeta)}} \end{bmatrix} \end{aligned}$$

(which in turn is a straightforward consequence of equality (4.18)), since $\mathbf{b}(\zeta) \neq 0$, $1 - \mathbf{g}(\zeta)\mathcal{E}(\zeta) \neq 0$ and since the matrix $\mathbf{S}(\zeta)$ is unitary.

Remark 4.9. The relaxed interpolation problem (4.7), (4.8) was studied in [11,13, 23] and the linear fractional parametrization of its solution set with the free Schur-class parameter has been known for a while. Theorem 4.6 also describes the gap between the problem $\mathbf{SBIP}(\widetilde{\boldsymbol{\zeta}}, \widetilde{\mathbf{d}}, \widetilde{\mathbf{s}})$ and its relaxed version: the strict inequality holds in (4.8) for a function F of the form (4.18) if and only if the corresponding parameter $\mathcal{E}$ is subject to condition (4.20).

A particular case of the symmetric boundary interpolation problem is the boundary Nevanlinna-Pick problem studied in [28].

The problem BNPP. Given points $\zeta_i \in \partial\mathbb{D}$ and given complex numbers $s_{i,0}$ and $s_{i,1}$ subject to conditions

$$|s_{i,0}| = 1 \quad \text{and} \quad \zeta_i s_{i,1}\overline{s}_{i.0} \geq 0 \quad \text{for} \quad i = 1, \dots, M,$$

find all functions $F \in \mathcal{S}$ such that

$$F_0(\zeta_i) = s_{i,0} \quad \text{and} \quad F_1(\zeta_i) = s_{i,1} \quad \text{for} \quad i = 1, \dots, M. \tag{4.22}$$

Formulas (3.9)-(3.11) adapted to the present simple setting tell us that the Pick matrix of the problem **BNPP** is equal to

$$\mathbb{P} = \left[P_{ij}\right]_{i,j=1}^{M}, \quad \text{where} \quad P_{ij} = \begin{cases} \zeta_i s_{i,1}\overline{s}_{i.0} & \text{if } i = j, \\ \dfrac{1 - s_{i,0}\overline{s}_{j,0}}{1 - \zeta_i\overline{\zeta}_j} & \text{if } i \neq j. \end{cases} \tag{4.23}$$

The problem **BNPP** is indeterminate if and only if the Pick matrix (4.23) is positive definite. In this case one can construct the functions (4.12)-(4.15) and then use part (3) in Theorem 4.6 to get the description of all solutions to the problem **BNPP**. In particular, all rational solutions of the problem are of the form (4.18) where $\mathcal{E}$ is a rational Schur-class function satisfying condition (4.19) for every $i \in \{1, \dots, M\}$.

The symmetric interpolation problem is a special case of problem $\mathbf{BIP}(\boldsymbol{\zeta}, \mathbf{k}, \mathbf{s})$ where the Pick matrix contains all data and where the positivity of this matrix is necessary and sufficient for the existence of a solution. We now turn to another extremal case where the Pick matrix is null-dimensional.

Theorem 4.10. *Let us assume that the set I* (3.1) *is empty. Then the problem* $\mathbf{BIP}(\boldsymbol{\zeta}, \mathbf{k}, \mathbf{s})$ *has infinitely many rational solutions.*

Proof. By the assumption, $J \cup K = \{1, \dots, N\}$. We may assume without loss of generality that $J = \{1, \dots, M\}$ and $K = \{M + 1, \dots, N\}$. Observe that all non-diagonal entries in the matrix (4.23) are specified. We now let $s_{i,1} := \overline{\zeta}_i s_{i,0} r$ for every $i \in J$ where $r > 0$ is large enough to make the matrix (4.23) positive definite. Then the Nevanlinna-Pick problem (4.22) has infinitely many rational solutions which are parametrized by formula (4.18) with the parameter $\mathcal{E}$ an arbitrary rational Schur-class function such that $\mathcal{E}(\zeta_i)\mathbf{g}(\zeta_i) \neq 1$ for every $i \in J$. In particular, every parameter $\mathcal{E}$ satisfying conditions

$$\mathcal{E}(\zeta_i) = 0 \quad \text{for} \quad i \in J, \tag{4.24}$$

leads via formula (4.18) to a rational solution F to the Nevanlinna-Pick problem (4.22). In particular, $F(\zeta_i) = s_{i,0}$ for every $i \in J$.

By Remark 4.8, the Taylor coefficients $F_j(\zeta_i) = s_{i,j}$ $(j = 0, \dots, k_i,\ i \in K)$ determine those of $\mathcal{E}$:

$$\mathcal{E}_j(\zeta_i) = c_{i,j} \quad \text{for} \quad j = 0, \dots, k_i \quad \text{and} \quad i \in K. \tag{4.25}$$

Thus, the numbers $c_{i,j}$ are uniquely determined by the corresponding $s_{i,j}$'s. The explicit formula for $c_{i,j}$ is not that important. However, since $|s_{i,0}| < 1$, we also have $|c_{i,0}| < 1$ (by Remark 4.8). By Lemma 3.1, there are infinitely many rational Schur-class functions satisfying conditions (4.24) and (4.25). Every such $\mathcal{E}$ leads via formula (4.18) to a rational solution F of the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$. □

5. Reduction

In Section 4 we discussed two extremal cases of the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$. The generic case is such that the Pick matrix exists (*i.e.*, is non-trivial) but does not incorporate all of interpolation data. We thus assume that the set (3.1) is not empty and we still assume that the Pick matrix $\mathbb{P}_{\mathbf{d}}^{\mathbf{s}}$ defined in (3.9)-(3.11) is positive definite. We then show that the original problem can be reduced to a problem with a fewer number of interpolation conditions and we will see that the new problem is either of the same type as the one discussed in Theorem 4.10 or it does not have a solution.

Every solution F to the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$, if it exists, also solves the maximal symmetric subproblem $\mathbf{SBIP}(\widetilde{\boldsymbol{\zeta}},\ \widetilde{\mathbf{d}},\ \widetilde{\mathbf{s}})$ and therefore it is of the form (4.18) for some $\mathcal{E} \in \mathcal{S}$ (by Theorem 4.6). Making use of the numbers d_i's defined in (3.8) and of the rational functions (4.12)-(4.15), we construct the polynomials A, B, C and G such that (we recall notation $f_j(\zeta_i) := f^{(j)}(\zeta_i)/j!$)

$$\begin{aligned} A_j(\zeta_i) &= s_{i,2d_i+j} - \mathbf{a}_{2d_i+j}(\zeta_i), \\ B_j(\zeta_i) &= \mathbf{b}_{d_i+j}(\zeta_i), \\ C_j(\zeta_i) &= \mathbf{c}_{d_i+j}(\zeta_i), \\ G_j(\zeta_i) &= \mathbf{g}_j(\zeta_i) \qquad \text{for} \quad j = 0, \dots, k_i - 2d_i; \quad i \in I' \cup I'' \cup J \cup K. \end{aligned} \tag{5.1}$$

We may think of the polynomials A, B, C, D as of the unique polynomials of degree at most $N - 1 + \sum_{i=1}^{N}(k_i - 2d_i)$ solving the Hermite-Lagrange interpolation problems (5.1). Observe that in view of statements (1) and (3) in Theorem 4.5, conditions (5.1) are equivalent to the following relations

$$\mathbf{a}(z) + (z-\zeta_i)^{2d_i} A(z) = \sum_{j=0}^{k_i} s_{i,j}(z-\zeta_i)^j + O\left(|z-\zeta_i|^{k_i+1}\right), \tag{5.2}$$

$$\mathbf{b}(z) = (z-\zeta_i)^{d_i} B(z) + O\left(|z-\zeta_i|^{k_i+1}\right), \tag{5.3}$$

$$\mathbf{c}(z) = (z-\zeta_i)^{d_i} C(z) + O\left(|z-\zeta_i|^{k_i+1}\right), \tag{5.4}$$

$$\mathbf{g}(z) = G(z) + O(|z-\zeta_i|)^{k_i-2d_i+1}, \tag{5.5}$$

as $z \to \zeta_i$ for $i \in I' \cup I'' \cup J \cup K$. We next define the rational function

$$R(z) = \frac{A(z)}{B(z)C(z) + G(z)A(z)}. \tag{5.6}$$

Evaluating (5.6) at $z = \zeta_i$, making use of (5.1) and taking into account Remark 4.8 we arrive at the following result.

Lemma 5.1. *It holds that*

$$R_{i,0} := R(\zeta_i) = \frac{s_{i,0} - \mathbf{a}(\zeta_i)}{\mathbf{b}(\zeta_i)\mathbf{c}(\zeta_i) + \mathbf{g}(\zeta_i)(s_{i,0} - \mathbf{a}(\zeta_i))} \quad \textit{for every} \quad i \in J \cup K.$$

Moreover $|R(\zeta_i)| = 1$ *for* $i \in J$ (*i.e., if* $|s_{i,0}| = 1$) *and* $|R(\zeta_i)| < 1$ *for* $i \in K$ (*i.e., if* $|s_{i,0}| < 1$).

We now pass to the interpolation nodes ζ_i where $i \in I' \cup I''$. Since

$$B(\zeta_i)C(\zeta_i) = \mathbf{b}_{d_i}(\zeta_i)\mathbf{c}_{d_i}(\zeta_i) \neq 0$$

by statement (4) in Theorem 4.5, the numerator and the denominator in (5.6) cannot have a common zero at ζ_i for $i \in I' \cup I''$. Thus, $R(z)$ is analytic at ζ_i if and only if

$$B(\zeta_i)C(\zeta_i) + G(\zeta_i)A(\zeta_i) = \mathbf{b}_{d_i}(\zeta_i)\mathbf{c}_{d_i}(\zeta_i) + \mathbf{g}(\zeta_i)\big(s_{i,2d_i} - \mathbf{a}_{2d_i}(\zeta_i)\big) \neq 0. \tag{5.7}$$

Remark 5.2. If condition (5.7) is satisfied for some $i \in I' \cup I''$, then

$$R_{i,0} := R(\zeta_i) = \frac{\overline{\mathbf{g}(\zeta_i)}\big(s_{i,2d_i} - \mathbf{a}_{2d_i}(\zeta_i)\big)}{(-1)^{d_i-1}\overline{\zeta}_i^{2n}|\mathbf{c}_{d_i}(\zeta_i)|^2 s_{i,0} + s_{i,2d_i} - \mathbf{a}_{2d_i}(\zeta_i)} \neq \overline{\mathbf{g}(\zeta_i)}. \tag{5.8}$$

Proof. Evaluating (5.6) at $z = \zeta_i$ gives, on account of (5.1),

$$R(\zeta_i) = \frac{s_{i,2d_i} - \mathbf{a}_{2d_i}(\zeta_i)}{\mathbf{b}_{d_i}(\zeta_i)\mathbf{c}_{d_i}(\zeta_i) + \mathbf{g}(\zeta_i)\big(s_{i,2d_i} - \mathbf{a}_{2d_i}(\zeta_i)\big)},$$

and substituting (4.16) into the right-hand side part of the latter equality gives

$$R(\zeta_i) = \frac{s_{i,2d_i} - \mathbf{a}_{2d_i}(\zeta_i)}{(-1)^{n-1}\bar{\zeta}_i^{2d_i}\mathbf{g}(\zeta_i)s_{i,0}|\mathbf{c}_{d_i}(\zeta_i)|^2 + \mathbf{g}(\zeta_i)(s_{i,2d_i} - \mathbf{a}_{2d_i}(\zeta_i))}$$

which is equivalent to the second equality in (5.8) since $|\mathbf{g}(\zeta_i)| = 1$ (by part (1) in Theorem 4.5). Since $s_{i,0} \neq 0$ (as $i \in I$) and $\mathbf{c}_{d_i}(\zeta_i) \neq 0$ (by part (3) in Theorem 4.5), the inequality in (5.8) follows. □

Theorem 5.3. *Let us assume that the set I* (3.1) *is not empty and that the Pick matrix $\mathbb{P}_{\mathbf{d}}^{\mathbf{s}}$ defined in* (3.9)-(3.11) *is positive definite. Let A, B, C, G be the polynomials solving Hermite interpolation problems* (5.1) *and let R be given by* (5.6).

(1) *If the problem* $\mathbf{BIP}(\boldsymbol{\zeta}, \mathbf{k}, \mathbf{s})$ *admits a solution, then* (5.7) *holds for every $i \in I' \cup I''$.*
(2) *If condition* (5.7) *is satisfied for every $i \in\in I' \cup I''$, then a function F is a solution of the problem* $\mathbf{BIP}(\boldsymbol{\zeta}, \mathbf{k}, \mathbf{s})$ *if and only if it is of the form* (4.18) *for some $\mathcal{E} \in \mathcal{S}$ such that*

 (a) *For every $i \in I' \cup I'' \cup J \cup K$,*

$$\mathcal{E}(z) = R(z) + o\big(|z - \zeta_i|^{k_i - 2d_i}\big). \tag{5.9}$$

 (b) *For every $i \in I'''$, the boundary limit $\mathcal{E}(\zeta_i)$ either satisfies condition* (4.19) *or conditions* (4.20) *or does not exist.*

Proof. Let F be a solution to the problem $\mathbf{BIP}(\boldsymbol{\zeta}, \mathbf{k}, \mathbf{s})$. Then it also solves the maximal symmetric subproblem $\mathbf{SBIP}(\widetilde{\boldsymbol{\zeta}}, \widetilde{\mathbf{d}}, \widetilde{\mathbf{s}})$ and therefore it is of the form (4.18) for some $\mathcal{E} \in \mathcal{S}$ with the properties listed in part (2*b*) (by Theorem 4.6). By (5.2), the asymptotic equalities (1.3) for $i \in I' \cup I'' \cup J \cup K$ can be equivalently written as

$$F(z) = \mathbf{a}(z) + (z - \zeta_i)^{2d_i} A(z) + o\big(|z - \zeta_i|^{k_i}\big) \quad (i \in I' \cup I'' \cup J \cup K). \tag{5.10}$$

Substituting (5.10) and (5.3)-(5.5) into (4.21) (which is equivalent to (4.18)) gives

$$\mathcal{E}(z) = \frac{A(z) + o\big(|z - \zeta_i|^{k_i - 2d_i}\big)}{B(z)C(z) + G(z)A(z) + o\big(|z - \zeta_i|^{k_i - 2d_i}\big)} \tag{5.11}$$

as $z \to \zeta_i$ non-tangentially for $i \in I' \cup I'' \cup J \cup K$. Since A, B, C, G are polynomials, the limit (as z tends to ζ_i) of the expression on the right hand side of (5.11) exists (finite of infinite) and therefore the limit $\mathcal{E}(\zeta_i)$ exists as well. Since $\mathcal{E}$ is a Schur-class function, this limit is finite and therefore, (5.7) holds for every $i \in I' \cup I'' \cup J \cup K$. Asymptotic equalities (5.9) follow from (5.6) and (5.11) due to (5.7).

It remains to prove the "if" part in statement (2) of the theorem. To this end, let us assume that conditions (5.7) are met for every $i \in I' \cup I''$, so that R is analytic at ζ_i for every $i \in I' \cup I''$. It is also analytic at ζ_i for every $i \in J \cup K$, by Lemma 5.1. Note also that

$$G(\zeta_i)R(\zeta_i) = \mathbf{g}(\zeta_i)R(\zeta_i) \neq 1 \quad \text{for every} \quad i \in I' \cup I'' \cup J \cup K. \tag{5.12}$$

Indeed, if $i \in I' \cup I''$, then (5.12) follows from Remark 5.2 since $|\mathbf{g}(\zeta_i)| = 1$. If $i \in J \cup K$ then $|\mathbf{g}(\zeta_i)| < 1$ (see Remark 4.8), $|R(\zeta_i)| \leq 1$ (see Remark 5.1), and (5.12) follows as well.

Let us assume that $\mathcal{E}$ is a Schur-class function subject to asymptotic equalities (5.9) and let F be defined by formula (4.18). Then $F \in \mathcal{S}$ since $\mathcal{E} \in \mathcal{S}$ and the coefficient matrix (4.11) is inner. Substituting (5.9), (5.3)-(5.5) into (4.18) we get

$$\begin{aligned} F(z) &= \mathbf{a}(z) + \frac{\left[(z-\zeta_i)^{2d_i} B(z)C(z) + o\left(|z-\zeta_i|^{k}_{i}\right)\right]\cdot\left[R(z)+o\left(|z-\zeta_i|^{k_i-2d_i}\right)\right]}{1-\left[G(z)+o\left(|z-\zeta_i|^{k_i-2d_i}\right)\right]\cdot\left[R(z)+o(|z-\zeta_i|^{k_i-2d_i})\right]} \\ &= \mathbf{a}(z) + \frac{(z-\zeta_i)^{2d_i} B(z)C(z)R(z) + o\left(|z-\zeta_i|^{k_i}\right)}{1-G(z)R(z)+o\left(|z-\zeta_i|^{k_i-2d_i}\right)} \\ &= \mathbf{a}(z) + \frac{(z-\zeta_i)^{2d_i} B(z)C(z)R(z)}{1-G(z)R(z)} + o\left(|z-\zeta_i|^{k_i}\right), \end{aligned} \tag{5.13}$$

where the last equality follows by (5.12). Now we substitute formula (5.6) for R into (5.13) and arrive at (5.10) which is equivalent to equalities (1.3) for $i \in I' \cup I'' \cup J \cup K$. On the other hand, if $i \in I'''$, then equalities (1.3) hold due to assumptions from part ($2b$) of the theorem according to Theorem 4.6. Thus, F solves the problem $\mathbf{BIP}(\boldsymbol{\zeta}, \mathbf{k}, \mathbf{s})$, which completes the proof. □

Corollary 5.4. *Let F be of the form* (4.18) *for some function $\mathcal{E} \in \mathcal{S}$. Then for a fixed $i \in I' \cup I''$, the boundary limit $F_{2d_i}(\zeta_i)$ exists if and only if the boundary limit $\mathcal{E}(\zeta_i)$ exists and is not equal to $\overline{\mathbf{g}(\zeta_i)}$. In this case,*

$$F_{2d_i}(\zeta_i) = \mathbf{a}_{2d_i}(\zeta_i) + \frac{(-1)^{d_i-1}\overline{\zeta}_i^{2d_i}|\mathbf{c}_{d_i}(\zeta_i)|^2 s_{i,0}\mathcal{E}(\zeta_i)}{\overline{\mathbf{g}(\zeta_i)} - \mathcal{E}(\zeta_i)}. \tag{5.14}$$

Proof. Simultaneous existence of the limits follows from Theorem 5.3. Since $\mathcal{E}(\zeta_i) = R(\zeta_i)$ by (5.9), we have from (5.8),

$$\mathcal{E}(\zeta_i) = \frac{\overline{\mathbf{g}(\zeta_i)}\left(F_{2d_i}(\zeta_i) - \mathbf{a}_{2d_i}(\zeta_i)\right)}{(-1)^{d_i-1}\overline{\zeta}_i^{2d_i}|\mathbf{c}_{d_i}(\zeta_i)|^2 s_{i,0} + F_{2d_i}(\zeta_i) - \mathbf{a}_{2d_i}(\zeta_i)}.$$

Solving the latter equality for $F_{2d_i}(\zeta_i)$ gives (5.14). □

Now we take another look at formula (5.8). If we will think of

$$\{s_{i,j}: \quad j=0,\dots,2d_i-1,\ i\in I'''\} \tag{5.15}$$

as of given numbers satisfying conditions $|s_{i,0}|=1$ and $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}>0$, then for each $i\in I'''$, the formula (5.8) establishes a linear fractional map $Y_i : s_{i,2d_i}\mapsto R_{i,0}$ on the Riemann sphere (recall that the entries $\mathbf{g}(\zeta_i)$, $\mathbf{c}_{d_i}(\zeta_i)$ and $\mathbf{a}_{2d_i}(\zeta_i)$ in (5.8) are uniquely determined by ζ_i and the fixed numbers (5.15)). The only value of the argument $s_{i,2d_i}$ in (5.8) which does not satisfy condition (5.7) is

$$s^0_{i,2d_i}=\mathbf{a}_{2d_i}(\zeta_i)-\mathbf{b}_{d_i}(\zeta_i)\mathbf{c}_{d_i}(\zeta_i)\overline{\mathbf{g}(\zeta_i)}.$$

One can see from (5.7) that

$$Y_i\left(s^0_{i,2d_i}\right)=\infty \quad \text{and} \quad Y_i(\infty)=\overline{\mathbf{g}(\zeta_i)}.$$

Thus, if we consider Y_i as a map from $\mathbb{C}\setminus\{s^0_{i,2d_i}\}$ into $\mathbb{C}$, then condition (5.7) and inequality in (4.10) will be satisfied automatically. Still assuming that ζ_i and $s_{i,j}$ in (5.15) are fixed, we can define two linear functions

$$s_{i,2d_i}\mapsto p^{\mathbf{s}}_{d_i+1,d_i}(\zeta_i) \quad \text{and} \quad s_{i,2d_i}\mapsto p^{\mathbf{s}}_{d_i,d_i+1}(\zeta_i)$$

by the formula (3.7). Indeed, letting $(r,\ell)=(d_i+1,d_i)$ and $(r,\ell)=(d_i,d_i+1)$ in (3.7) and taking into account that

$$\Psi_{nn}(\zeta)=(-1)^{n-1}\zeta^{2n-1} \quad \text{and} \quad \Psi_{n+1,n+1}(\zeta)=(-1)^n\zeta^{2n+1}$$

(according to formula (2.3)), we get

$$\begin{aligned} p^{\mathbf{s}}_{d_i+1,d_i}(\zeta_i) &= (-1)^{d_i-1}\zeta_i^{2d_i-1}s_{i,2d_i}\overline{s}_{i,0}+\Phi_i,\\ p^{\mathbf{s}}_{d_i,d_i+1}(\zeta_i) &= (-1)^{d_i}\zeta_i^{2d_i+1}s_{i,2d_i}\overline{s}_{i,0}+\Upsilon_i, \end{aligned} \tag{5.16}$$

where the terms

$$\begin{aligned} \Phi_i &= \sum_{r=1}^{d_i-1}\sum_{\ell=1}^{r} s_{i,d_i+\ell}\Psi_{\ell r}(\zeta_i)\overline{s}_{i,d_i-r}+\sum_{\ell=1}^{d_i-1}s_{i,d_i+\ell}\Psi_{\ell d_i}(\zeta_i)\overline{s}_{i,0},\\ \Upsilon_i &= \sum_{r=1}^{d_i}\sum_{\ell=1}^{r} s_{i,d_i+\ell-1}\Psi_{\ell r}(\zeta_i)\overline{s}_{i,d_i+1-r}+\sum_{\ell=1}^{d_i}s_{i,d_i+\ell-1}\Psi_{\ell,d_i+1}(\zeta_i)\overline{s}_{i,0} \end{aligned} \tag{5.17}$$

are completely determined from ζ_i and $s_{i,0},\dots,s_{i,2d_i-1}$. We next establish the formula relating the companion numbers (3.13) and the numbers $R_{i,0}$ given by (5.8). This formula shows in particular, that the companion numbers are necessarily real.

Lemma 5.5. *For a fixed $i \in I' \cup I''$, let $R_{i,0}$, $p^{\mathbf{s}}_{d_i+1,d_i}(\zeta_i)$ and $p^{\mathbf{s}}_{d_i,d_i+1}(\zeta_i)$ be defined by formulas* (5.8), (5.17) *for some fixed $s_{i,2d_i}$. Then*

$$\gamma_i := \zeta_i \cdot \left(p^{\mathbf{s}}_{d_i+1,d_i}(\zeta_i) - \overline{p^{\mathbf{s}}_{d_i,d_i+1}(\zeta_i)}\right) = \frac{|\mathbf{c}_{d_i}(\zeta_i)|^2\left(1 - |R_{i,0}|^2\right)}{|\overline{\mathbf{g}(\zeta_i)} - R_{i,0}|^2}. \tag{5.18}$$

Proof. Let us substitute the constant unimodular function $\mathcal{E}(z) \equiv -\overline{\mathbf{g}(\zeta_i)}$ into (4.18):

$$h(z) := \mathbf{T_S}[-\overline{\mathbf{g}(\zeta_i)}](z) = \mathbf{a}(z) - \frac{\mathbf{b}(z)\mathbf{c}(z)\overline{\mathbf{g}(\zeta_i)}}{1 + \mathbf{g}(z)\overline{\mathbf{g}(\zeta_i)}}.$$

Since $\mathcal{E}$ is a unimodular constant function and since the matrix $\mathbf{S}$ of coefficients in (4.18) is inner, it follows that h is a rational inner function, *i.e.*, a finite Blaschke product. Since $\mathcal{E}(z) \equiv -\overline{\mathbf{g}}(\zeta_i)$ satisfies condition (4.19), the function h solves the problem $\mathbf{SBIP}(\widetilde{\boldsymbol{\zeta}}, \widetilde{\mathbf{d}}, \widetilde{\mathbf{s}})$ by Theorem 4.6. Thus, in particular,

$$h_j(\zeta_i) = s_{i,j} \quad \text{for} \quad j = 0, \ldots, 2d_i - 1 \tag{5.19}$$

and therefore $\mathbb{P}^h_{d_i}(\zeta_i) = \mathbb{P}^{\mathbf{s}}_{d_i}(\zeta_i)$ where the matrices $\mathbb{P}^h_n(\zeta_i)$ and $\mathbb{P}^{\mathbf{s}}_{d_i}(\zeta_i)$ are defined via formulas (2.4) and (3.6), respectively. The extended matrix $\mathbb{P}^h_{d_i+1}(\zeta_i)$ is positive semidefinite, since h is a finite Blaschke product. In particular, the $(d_i + 1, d_i)$ and $(d_i, d_i + 1)$ entries in this matrix are complex conjugates of each other:

$$p^h_{d_i+1,d_i}(\zeta_i) = \overline{p^h_{d_i,d_i+1}(\zeta_i)}. \tag{5.20}$$

These entries are defined via formula (3.7) with $s_{i,j}$ replaced by $h_j(\zeta_i)$. By (5.19),

$$\begin{aligned} p^h_{d_i+1,d_i}(\zeta_i) &= (-1)^{d_i-1}\zeta_i^{2d_i-1}h_{2d_i}(\zeta_i)\overline{s}_{i,0} + \Phi_i, \\ p^h_{d_i,d_i+1}(\zeta_i) &= (-1)^{d_i}\zeta_i^{2d_i+1}h_{2d_i}(\zeta_i)\overline{s}_{i,0} + \Upsilon_i \end{aligned}$$

where Φ_i and Υ_i are the same as in (5.17). Substituting the two latter equalities into (5.20) we have, after simple rearrangements,

$$\Phi_i - \overline{\Upsilon}_i = (-1)^{d_i}\zeta_i^{2d_i-1}h_{2d_i}(\zeta_i)\overline{s}_{i,0} + (-1)^{d_i}\overline{\zeta}_i^{2d_i+1}\overline{h_{2d_i}(\zeta_i)}s_{i,0}. \tag{5.21}$$

The formula for $h_{2d_i}(\zeta_i)$ can be obtained from Corollary 5.4 by plugging $\mathcal{E}(\zeta_i) = -\overline{\mathbf{g}(\zeta_i)}$ into (5.14):

$$h_{2d_i}(\zeta_i) = \mathbf{a}_{2d_i}(\zeta_i) + \frac{(-1)^{d_i}}{2}\overline{\zeta}_i^{2d_i}|\mathbf{c}_{d_i}(\zeta_i)|^2 s_{i,0}.$$

On the other hand, we have from (5.8),

$$s_{i,2d_i} = \mathbf{a}_{2d_i}(\zeta_i) + \frac{(-1)^{d_i-1}\overline{\zeta}_i^{2d_i}|\mathbf{c}_{d_i}(\zeta_i)|^2 s_{i,0}R_{i,0}}{\overline{\mathbf{g}(t_0)} - R_{i,0}},$$

and we conclude from the two last equalities that

$$\begin{aligned}\zeta_i^{2d_i}\overline{s}_{i,0}(s_{i,2d_i}-h_{2d_i}(\zeta_i)) &= (-1)^{d_i-1}|\mathbf{c}_{d_i}(\zeta_i)|^2\left[\frac{R_{i,0}}{\overline{\mathbf{g}(\zeta_i)}-R_{i,0}}+\frac{1}{2}\right]\\ &= \frac{(-1)^{d_i-1}|\mathbf{c}_{d_i}(\zeta_i)|^2}{2}\cdot\frac{\overline{\mathbf{g}(\zeta_i)}+R_{i,0}}{\overline{\mathbf{g}(\zeta_i)}-R_{i,0}}.\end{aligned} \tag{5.22}$$

Now we make subsequent use of (3.13), (5.16), (5.21) and (5.22) to get

$$\begin{aligned}\gamma_i &:= \zeta_i\left(p^{\mathbf{s}}_{d_i+1,d_i}(\zeta_i)-\overline{p^{\mathbf{s}}_{d_i,d_i+1}(\zeta_i)}\right)\\ &= (-1)^{d_i-1}\left[\zeta_i^{2d_i}s_{i,2d_i}\overline{s}_{i,0}+\overline{\zeta}_i^{2d_i}\overline{s}_{i,2d_i}s_{i,0}\right]+\zeta_i\left[\Phi_i-\overline{\Upsilon}_i\right]\\ &= (-1)^{d_i-1}\left[\zeta_i^{2d_i}\overline{s}_{i,0}(s_{i,2d_i}-h_{2d_i}(\zeta_i))+\overline{\zeta}_i^{2d_i}s_{i,0}\left(\overline{s}_{i,2d_i}-\overline{h_{2d_i}(\zeta_i)}\right)\right]\\ &= |\mathbf{c}_{d_i}(\zeta_i)|^2\cdot\operatorname{Re}\left(\frac{\overline{\mathbf{g}(\zeta_i)}+R_{i,0}}{\overline{\mathbf{g}(\zeta_i)}-R_{i,0}}\right)\\ &= \frac{\left|\mathbf{c}_{d_i}(\zeta_i)\right|^2\left(1-\left|R_{i,0}\right|^2\right)}{\left|\overline{\mathbf{g}(\zeta_i)}-R_{i,0}\right|^2}\end{aligned}$$

which completes the proof. □

6. Proof of the main results

We are now able to complete the proofs of Theorems 3.5 and 3.6.

Proof of Theorem 3.5. We will check all possible cases for the given data set (1.2). We assume that $|s_{i,0}| \le 1$ since otherwise the problem **BIP**($\boldsymbol{\zeta}$, **k**, **s**) has no solutions.

Case 1: Assume that the set I defined in (3.1) is empty. Then the problem is indeterminate and has infinitely many rational solutions by Lemma 3.1.

If $I \neq \emptyset$, then conditions (3.2) and $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}} \ge 0$ are necessary for the problem **BIP**($\boldsymbol{\zeta}$, **k**, **s**) to have a solution.

Case 2: If $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}} \ge 0$ is singular, the problem **BIP**($\boldsymbol{\zeta}$, **k**, **s**) has at most one solution (by Corollary 4.3) and, therefore, cannot be indeterminate.

Case 3: Let us assume that $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}} > 0$ and that

$$\gamma_i > 0 \quad \text{for every} \quad i \in I' \quad \text{and} \quad \gamma_i \ge 0 \quad \text{for every} \quad i \in I''. \tag{6.1}$$

We will show that in this case the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$ is indeterminate. By Theorem 5.3, it suffices to verify that there are infinitely many rational functions $\mathcal{E}\in\mathcal{S}$ satisfying conditions

$$\mathcal{E}_j(\zeta_i)=R_j(\zeta_i)\quad\text{for}\quad i\in I'\cup I''\cup J\cup K;\ j=0,\dots,k_i-2d_i, \tag{6.2}$$

$$\mathcal{E}(\zeta_i)=0\quad\text{for}\quad i\in I''', \tag{6.3}$$

where R is the rational function constructed as in (5.6). Indeed, conditions (6.2) are equivalent to non-tangential asymptotic equalities (5.9), while equalities (6.3) guarantee that conditions (4.19) are satisfied.

By Lemma 5.1, $|R(\zeta_i)|=1$ for every $i\in J$ and $|R(\zeta_i)|<1$ for every $i\in K$. By relation (5.18) in Lemma 5.5 and due to assumptions (6.1), $|R(\zeta_i)|=1$ for every $i\in I''$ and $|R(\zeta_i)|<1$ for every $i\in I'$. Hence we see from (6.2) that for every $i\in I''\cup J$, there is only one interpolation condition at ζ_i assigning a unimodular boundary value $R_0(\zeta_i)$ to the unknown interpolant $\mathcal{E}$. At all other points (that is, for $i\in I'\cup I'''\cup K$), there is one or more interpolation conditions but in any event, the prescribed boundary value for $\mathcal{E}$ at ζ_i is less than one in modulus. Therefore, the interpolation problem with conditions (6.2), (6.3) is of the type considered in Theorem 4.10. It has infinitely many rational solutions $\mathcal{E}$, and every such $\mathcal{E}$ leads via formula (4.18) to a rational solution F of the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$.

Case 4: Let us assume that $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}>0$ and that at least one of the conditions in (6.1) fails to be in force.

If $\gamma_i<0$ for some $i\in I'\cup I''$, then we conclude from formula (5.18) that $|R_{i,0}|=|R(\zeta_i)|>1$ and therefore conditions (6.2) cannot be matched by a Schur-class function $\mathcal{E}$. Then the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$ has no solutions by Theorem 5.3.

If $\gamma_i=0$ for some $i\in I'$, then the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$ also has no solutions. To see this, we recall a result from [15] (see Theorem 1.8 there):

Let $F\in\mathcal{S}$ admit the non-tangential boundary limits $F_j(\zeta_i)$ for $j=0,\dots,2d_i$ which are such that

$$|F_0(\zeta_i)|=1,\quad \mathbb{P}^F_{d_i}(\zeta_i)\geq 0\quad\textit{and}\quad p^F_{d_i+1,d_i}(\zeta_i)=\overline{p^F_{d_i,d_i+1}(\zeta_i)}. \tag{6.4}$$

If the boundary limit $F_{2d_i+1}(\zeta_i)$ exists then necessarily $\mathbb{P}^F_{d_i+1}(\zeta_i)\geq 0$.

Since $i\in I'$, we have $k_i>2d_i$ (by definition (3.12) of I'). Let us assume that F is a solution to the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$. Since $k_i>2d_i$, we have enough data to construct $\mathbb{P}^{\mathbf{s}}_{d_i+1}(\zeta_i)$ which must be equal to $\mathbb{P}^F_{d_i+1}(\zeta_i)$. By the assumptions of the current case, conditions (6.4) are met and the limit $F_{i,2d_i+1}(\zeta_i)$ exists. Therefore, the matrix $\mathbb{P}^F_{d_i+1}=\mathbb{P}^{\mathbf{s}}_{d_i+1}$ is positive semidefinite which contradicts to the choice (3.8) of d_i.

All possible cases have been verified. We see that the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$ is indeterminate only in cases (**1**) and (**3**). In case (**3**), any solution F of the problem belongs to $\bigcap_{i\in I}\mathcal{S}^{(d_i)}(\zeta_i)$, by Theorem 2.3. Finally, if for some $i\in I'$, there existed an $F \in \mathcal{S}^{(d_i+1)}(\zeta_i)$ solving the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$, then the structured matrix $\mathbb{P}^{\mathbf{s}}_{d_i+1}$ would be positive semidefinite which would contradict the choice of d_i. Thus, every solution of the problem does not belong to $\mathcal{S}^{(d_i+1)}(\zeta_i)$ for every $i \in I'$. This completes the proof of Theorem 3.5. □

Proof of Theorem 3.6. By the proof of the previous theorem, the determinacy may occur only if the Pick matrix $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ is positive semidefinite (singular). The unique solution (if exists) is a Blaschke product F of $\deg F = \operatorname{rank} \mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ by Corollary 4.3.

Necessity. Let us assume that the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$ is determinate and that F ($\deg F = \operatorname{rank} \mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$) is its only solution. Since a finite Blaschke product is unimodular on $\mathbb{T}$, the necessity of condition (1) in Theorem 3.6 follows.

We next observe that for every $i \in I'$, the matrix $\mathbb{P}^{\mathbf{s}}_{d_i+1}(\zeta_i)$ exists and is not Hermitian. On the other hand, for a finite Blaschke product F, the matrix $\mathbb{P}^{F}_{d_i+1}(\zeta_i)$ is positive semidefinite by Theorem 2.6. As a solution of the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$, the function F must satisfy the equality $\mathbb{P}^{F}_{d_i+1}(\zeta_i) = \mathbb{P}^{\mathbf{s}}_{d_i+1}(\zeta_i)$ which is not the case. Therefore, no finite Blaschke product solves the problem in case $I' \neq \emptyset$. This proves the necessity of condition (2).

Making use of the Taylor coefficients of F we define the requested numbers in (3.14) by letting $s_{i,j} = F_j(\zeta_i)$. Then the extended matrix $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}+\mathbf{e}_i}$ will be equal to $\mathbb{P}^{F}_{\mathbf{d}+\mathbf{e}_i}(\boldsymbol{\zeta})$ which in turn is positive semidefinite for every $i \in \{1, \dots, N\}$ by Theorem 2.6. Thus $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ admits a positive semidefinite extension $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}+\mathbf{e}_i}$ for $i = 1, \dots, N$ and condition (3) is also necessary.

Since F is a solution of the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$, we have $\mathbb{P}^{F}_{\mathbf{d}}(\boldsymbol{\zeta}) = \mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$. Since the latter matrices are singular, we have $\deg\ F < d_1 + \ldots + d_N$, and we conclude from Theorem 2.6 that for every $i \in I'''$,

$$\begin{aligned}\operatorname{rank} \mathbb{P}^{\mathbf{s}}_{\mathbf{d}-\mathbf{e}_i} &= \operatorname{rank}\left(\mathbb{P}^{F}_{\mathbf{d}-\mathbf{e}_i}(\boldsymbol{\zeta})\right)\\ &= \min\{d_1 + \ldots + d_N - 1, \deg\ F\}\\ &= \deg\ F = \operatorname{rank} \mathbb{P}^{\mathbf{F}}_{\mathbf{d}}(\boldsymbol{\zeta}) = \operatorname{rank} \mathbb{P}^{\mathbf{s}}_{\mathbf{d}}\end{aligned}$$

which proves the necessity of condition (3c).

On the other hand, for every $i \in I'' \cup J$ (and up to an appropriate re-indexing the blocks), the extended matrix $\mathbb{P}^{\mathbf{F}}_{\mathbf{d}+\mathbf{e}_i}(\boldsymbol{\zeta})$ is necessarily of the form

$$\mathbb{P}^{F}_{\mathbf{d}+\mathbf{e}_i}(\boldsymbol{\zeta}) = \begin{bmatrix} \mathbb{P}^{\mathbf{s}}_{\mathbf{d}} & C_i \\ C_i^* & p^{F}_{d_i+1,d_i+1}(\zeta_i) \end{bmatrix}.$$

Since the latter matrix is positive semidefinite, conditions (3a) and (3b) follow.

Sufficiency of conditions (1), (2), (3): Since $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ is singular, there is only one function $F \in \mathcal{S}$ (a finite Blaschke product) satisfying the relaxed conditions (4.7), (4.8). It remains to show that conditions (1), (2), (3) in Theorem 3.6 are sufficient for this only candidate to satisfy conditions (1.4). Since $K \cup I' = \emptyset$, it remains to show that conditions (1.4) are met for every $i \in I''' \cup I'' \cup J$.

Case 1: Let $r \in I'' \cup I'''$. By condition (3), there exist a number $s_{r,2d_r+1}$ (if $r \in I''$ in which case $s_{r,2d_r}$ is already specified in (1.2)) or two numbers $s_{r,2d_r}$ and $s_{r,2d_r+1}$ (if $r \in I'''$) so that the extended Pick matrix $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}+\mathbf{e}_r}$ is positive semidefinite (and singular). Then by Theorem 4.2, there exists a unique function $B \in \mathcal{S}$ satisfying conditions (4.7) for every $i \in I$, conditions (4.8) for $i \in I \setminus \{r\}$, and three additional conditions

$$\begin{gathered} B_{2d_r-1}(\zeta_r) = s_{r,2d_r-1}, \quad B_{2d_r}(\zeta_r) = s_{r,2d_r}, \\ (-1)^{d_r+1}\zeta_r^{2d_r+1}\overline{s}_{r,0}(B_{2d_r+1}(\zeta_r) - s_{r,2d_r+1}) \geq 0. \end{gathered} \tag{6.5}$$

Clearly, B satisfies all the conditions in (4.7), (4.8) (as well as F does) and therefore, by the uniqueness, $B \equiv F$. Therefore F satisfies additional conditions (6.5). If $r \in I''$, this means that all conditions in (1.4) are satisfied for $i = r$. If $r \in I'''$, then the number $s_{r,2d_r}$ is not specified, but in this case the first equality in (6.5) along with (4.7) means that F satisfies all conditions in (1.4) for $i = r$.

Case 2: Let $r \in J$. By condition (3), there exists a number $s_{r,1}$ so that the extended Pick matrix $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}+\mathbf{e}_r}$ is positive semidefinite (and singular). Again by Theorem 4.2, there exists a unique function $B \in \mathcal{S}$ satisfying conditions (4.7), (4.8) for $i \in I$ and two additional conditions

$$B_0(\zeta_r) = s_{r,0}, \quad -\zeta_r\overline{s}_{r,0}(B_1(\zeta_r) - s_{r,1}) \geq 0.$$

By the same argument as in Case 1, $B \equiv F$ and therefore F satisfies all interpolation conditions in (3.4) for $i = r$. This completes the proof of the asserted sufficiency.

Sufficiency of conditions (1), (2), (3a), (3b), (3c): As in the preceding proof we will show that the only function $F \in \mathcal{S}$ (a finite Blaschke product) satisfying the relaxed conditions (4.7), (4.8) satisfies conditions (1.4) for every $i \in J \cup I'' \cup I'''$.

We first show that conditions (3a) and (3b) guarantee that $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ admits a positive semidefinite extension $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}+\mathbf{e}_i}$ for every $i \in J \cup I''$. Indeed, up to an appropriate re-indexing of the blocks, the extended matrix $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}+\mathbf{e}_i}$ is necessarily of the form

$$\mathbb{P}^{\mathbf{s}}_{\mathbf{d}+\mathbf{e}_i} = \begin{bmatrix} \mathbb{P}^{\mathbf{s}}_{\mathbf{d}} & C_i \\ D_i & p^{\mathbf{s}}_{d_i+1,d_i+1}(\zeta_i) \end{bmatrix} \tag{6.6}$$

where C_i is defined as in (3.15), where D_i is the row-vector given by

$$D_i = \mathrm{Row}_{k \in I}\left[\, p^{\mathbf{s}}_{d_i+1,1}(\zeta_i, \zeta_k) \ \cdots \ p^{\mathbf{s}}_{d_i+1,d_i}(\zeta_i, \zeta_k) \,\right].$$

It follows from formula (3.11) that

$$p^{\mathbf{s}}_{r,d_i+1}(\zeta_k, \zeta_i) = \overline{p^{\mathbf{s}}_{d_i+1,r}(\zeta_i, \zeta_k)} \quad \text{for every} \quad k \neq i. \tag{6.7}$$

If $i \in J$, then equalities (6.7) hold for every $k \in I$ and therefore $D_i = C_i^*$. On the other hand, if $i \in I''$, then it follows from (6.7) and Lemma 4.1 that $D_i = C_i^*$ if and only if the rightmost entry in D_i and the bottom entry in C_i are complex-conjugates of each other. But this is guaranteed by condition (3b) in Theorem 3.6. Thus, formula (6.6) takes the form

$$\mathbb{P}^{\mathbf{s}}_{\mathbf{d}+\mathbf{e}_i} = \begin{bmatrix} \mathbb{P}^{\mathbf{s}}_{\mathbf{d}} & C_i \\ C_i^* & p^{\mathbf{s}}_{d_i+1,d_i+1}(\zeta_i) \end{bmatrix} \quad \text{for} \quad i \in J \cup I''. \tag{6.8}$$

According to (3.7),

$$\begin{aligned} p^{\mathbf{s}}_{d_i+1,d_i+1}(\zeta_i) &= \sum_{r=1}^{d_i-1} \sum_{\ell=1}^{r} s_{i,d_i+\ell} \Psi_{\ell,r}(\zeta_i) \overline{s}_{i,d_i+1-r} \\ &\quad + \sum_{\ell=1}^{d_i} s_{i,d_i+\ell} \Psi_{\ell,d_i+1}(\zeta_i) \overline{s}_{i,d_i+1-r} + (-1)^{d_i} \zeta_i^{2d_i+1} s_{i,2d_i+1} \overline{s}_{i,0}. \end{aligned}$$

Since the coefficient of $s_{i,2d_i+1}$ in (6.7) is nonzero, formula (6.7) shows that, by an appropriate choice of $s_{i,2d_i+1}$, we can make $p^{\mathbf{s}}_{d_i+1,d_i+1}(\zeta_i)$ be equal to any positive number. If this number is large enough, then the matrix (6.8) is positive semidefinite if and only if $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}} \geq 0$ and $\operatorname{rank} \mathbb{P}^{\mathbf{s}}_{\mathbf{d}} = \operatorname{rank} \begin{bmatrix} \mathbb{P}^{\mathbf{s}}_{\mathbf{d}} & C_i \end{bmatrix}$. The positivity of $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}} \geq 0$ is assumed while the rank equality is guaranteed by conditions (3a) and (3b) in Theorem 3.6. Thus, $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ admits a positive semidefinite extension $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}+\mathbf{e}_i}$ for every $i \in J \cup I''$ which implies (as we have seen in the preceding proof) that F satisfies interpolation conditions (3.4) for all $i \in J \cup I''$.

It remains to show that F also satisfies conditions $F_{2d_i-1}(\zeta_i) = s_{i,2d_i-1}$ (rather than inequalities (4.8)) for every $i \in I'''$. We will argue via contradiction: let us assume that

$$(-1)^{d_i} \zeta_r^{2d_i-1} \overline{s}_{i,0} \left(F_{2d_i-1}(\zeta_i) - s_{i,2d_i-1} \right) > 0.$$

Write the matrices $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ and $\mathbb{P}^{F}_{\mathbf{d}}(\boldsymbol{\zeta})$ in the block form as

$$\mathbb{P}^{\mathbf{s}}_{\mathbf{d}} = \begin{bmatrix} \mathbb{P}^{\mathbf{s}}_{\mathbf{d}-\mathbf{e}_i} & B \\ B^* & p^{\mathbf{s}}_{d_i,d_i}(\zeta_i) \end{bmatrix}, \quad \mathbb{P}^{F}_{\mathbf{d}}(\boldsymbol{\zeta}) = \begin{bmatrix} \mathbb{P}^{F}_{\mathbf{d}-\mathbf{e}_i}(\boldsymbol{\zeta}) & B \\ B^* & p^{F}_{d_i,d_i}(\zeta_i) \end{bmatrix}.$$

The equality of non-diagonal entries in the two latter matrices as well as inequalities

$$\mathbb{P}^{F}_{\mathbf{d}-\mathbf{e}_i}(\boldsymbol{\zeta}) \leq \mathbb{P}^{\mathbf{s}}_{\mathbf{d}-\mathbf{e}_i} \quad \text{and} \quad p^{F}_{d_i,d_i}(\zeta_i) < p^{\mathbf{s}}_{d_i,d_i}(\zeta_i) \tag{6.9}$$

were discussed in Remark 4.4 above. Since the matrix $\mathbb{P}^{F}_{\mathbf{d}}(\boldsymbol{\zeta})$ is positive semidefinite, the matrix

$$\widetilde{\mathbb{P}} := \begin{bmatrix} \mathbb{P}^{\mathbf{s}}_{\mathbf{d}-\mathbf{e}_i} & B \\ B^* & p^{F}_{d_i,d_i}(\zeta_i) \end{bmatrix} \geq \mathbb{P}^{F}_{\mathbf{d}}(\zeta)$$

is also positive semidefinite and by the standard Schur complement argument,

$$p^F_{d_i,d_i}(\zeta_i) \geq X^* \mathbb{P}^{\mathbf{s}}_{\mathbf{d}-\mathbf{e}_i} X \tag{6.10}$$

where X is any column-vector solving the equation $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}-\mathbf{e}_i} X = B$. The same Schur complement argument applied to the positive semidefinite matrix $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ decomposed as above, gives on account of (6.10) and (6.9),

$$\begin{aligned} \operatorname{rank} \mathbb{P}^{\mathbf{s}}_{\mathbf{d}} &= \operatorname{rank} \mathbb{P}^{\mathbf{s}}_{\mathbf{d}-\mathbf{e}_i} + p^{\mathbf{s}}_{d_i,d_i}(\zeta_i) - X^* \mathbb{P}^{\mathbf{s}}_{\mathbf{d}-\mathbf{e}_i} X \\ &> \operatorname{rank} \mathbb{P}^{\mathbf{s}}_{\mathbf{d}-\mathbf{e}_i} + p^F_{d_i,d_i}(\zeta_i) - X^* \mathbb{P}^{\mathbf{s}}_{\mathbf{d}-\mathbf{e}_i} X \geq \operatorname{rank} \mathbb{P}^{\mathbf{s}}_{\mathbf{d}-\mathbf{e}_i} \end{aligned}$$

which contradicts the assumption in condition (3c) in Theorem 3.6. This completes the proof. □

7. Boundary interpolation for Carathéodory-class functions

Let us say that a function H is of the *Carathéodory class* $\mathcal{C}$ if H is analytic and $\Re H(z) \geq 0$ on $\mathbb{D}$. By the Herglotz representation theorem, for every $H \in \mathcal{C}$, there exists a unique positive measure μ on $\mathbb{T}$ such that

$$H(z) = \int_{\mathbb{T}} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) + ic, \quad c = \Im h(0). \tag{7.1}$$

The Carathéodory-class analog of the boundary interpolation problem $\mathbf{BIP}(\boldsymbol{\zeta}, \mathbf{k}, \mathbf{s})$ is the following.

Problem $\mathbf{BIP}_{\mathcal{C}}(\boldsymbol{\zeta}, \mathbf{k}, \mathbf{h})$. *Given the data set*

$$\boldsymbol{\zeta} = \{\zeta_1, \ldots, \zeta_N\}, \quad \mathbf{k} = \{k_1, \ldots, k_N\}, \quad \mathbf{h} = \{h_{i,j}\}_{i=1,\ldots,N}^{j=0,\ldots,k_i}, \tag{7.2}$$

find a function $H \in \mathcal{C}$ *such that*

$$H_j(\zeta_i) := \angle \lim_{z \to \zeta_i} \frac{H^{(j)}(z)}{j!} = h_{i,j} \quad \textit{for} \;\; j = 0, \ldots, k_i; \;\; i = 1, \ldots, N. \tag{7.3}$$

This problem is equivalent to the problem $\mathbf{BIP}(\boldsymbol{\zeta}, \mathbf{k}, \mathbf{s})$ due to the Cayley transform

$$\mathbf{C}: \; H(z) \;\to\; F(z) = \frac{H(z) - 1}{H(z) + 1} \tag{7.4}$$

establishing a one-to-one correspondence between the sets $\mathcal{C}$ and $\mathcal{S} \backslash \{1\}$. We will use this correspondence to translate Theorems 3.5 and 3.6 to the Carathéodory-class setting.

We first remark that the inverse Cayley transform of a finite Blaschke product of degree k is a Carathéodory-class function whose associated measure $\mu(\theta)$ is discrete with k atoms. On the other hand, the preimage of the class $\mathcal{S}^{(n)}(\zeta)$ defined in (2.1) is the class $\mathcal{C}^{(n)}(\zeta)$ which is defined as the set of all functions $H \in \mathcal{C}$ such that

$$\liminf_{z\to\zeta} \frac{\partial^{2n-2}}{\partial z^{n-1}\partial \bar{z}^{n-1}} \frac{H(z)+\overline{H(z)}}{1-|z|^2} < \infty$$

or equivalently, the set of all Carathéodory-class functions with the associated measure μ satisfying the condition $\int_{\partial\mathcal{D}} \frac{d\mu(\theta)}{|e^{i\theta}-\zeta|^{2n}} < \infty$.

It follows from (7.4) that a function H solves the problem $\mathbf{BIP}_{\mathcal{C}}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{h})$ if and only if its Cayley transform F solves the problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$, where the numbers $s_{i,j}$ are given by

$$s_{i,0} = \frac{h_{i,0}-1}{h_{i,0}+1}, \quad s_{i,j} = \frac{1}{h_{i,0}+1}\cdot\left(h_{i,j} - \sum_{k=0}^{j-1} s_{i,k}h_{i,j-k}\right) \quad (j \geq 1). \tag{7.5}$$

Thus, the problem $\mathbf{BIP}_{\mathcal{C}}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{h})$ has a solution if and only if the Schur-class problem $\mathbf{BIP}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$ with the data set $\mathbf{s} = \{s_{i,j}\}$ defined as in (7.5) has a solution, and hence, Theorem 3.5 provides the indeterminacy criteria for the problem $\mathbf{BIP}_{\mathcal{C}}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{h})$ in terms of the numbers (7.5). However, it is desirable to get the answer directly in terms of $\mathbf{h} = \{h_{i,j}\}$. To this end, we first observe the equalities

$$1-|s_{i,0}|^2 = \frac{4\Re h_{i,0}}{|h_{i,0}+1|^2} \quad \text{and} \quad \zeta_i s_{i,1}\bar{s}_{i,0} = -\frac{\zeta_i h_{i,1}}{|h_{i,0}+1|^2}$$

which imply the equivalences

$$\begin{gathered} |s_{i,0}| = 1 \ \Leftrightarrow\ \Re\, h_{i,0} = 0, \qquad |s_{i,0}| < 1 \ \Leftrightarrow\ \Re\, h_{i,0} > 0. \\ \zeta_i s_{i,1}\bar{s}_{i,0} \geq 0 \ \Leftrightarrow\ \zeta_i h_{i,1} \leq 0. \end{gathered} \tag{7.6}$$

We next define the matrices $\mathbb{H}^{\mathbf{h}}_{i,n} = \left[h_{i,\ell+r-1}\right]_{\ell,r=1}^{n}$ and

$$\mathbb{Q}^{\mathbf{h}}_{n}(\zeta_i) = \left[q^{\mathbf{h}}_{r,\ell}(\zeta_i)\right] = -\mathbb{H}^{\mathbf{h}}_{i,n}\Psi_n(\zeta_i) \tag{7.7}$$

with the entries

$$q^{\mathbf{h}}_{r,\ell}(\zeta_i) = -\sum_{\alpha=1}^{\ell} h_{i,r+\alpha-1}\Psi_{\alpha\ell}(\zeta_i) \tag{7.8}$$

for all appropriate integers $n \geq 0$ (see formulas (2.2) and (2.3)).

Proposition 7.1. *Let the tuples* $\mathbf{h} = \{h_{i,j}\}$ *and* $\mathbf{s} = \{s_{i,j}\}$ *be related as in* (7.5) *and let* $\mathbb{U}^{\mathbf{h}}_{i,n}$, $\mathbb{P}^{\mathbf{s}}_n(\zeta_i)$ *and* $\mathbb{Q}^{\mathbf{h}}_n(\zeta_i)$ *be defined via formulas* (3.5), (3.6), *and* (7.5), *respectively. Let us assume that* $\Re h_{i,0} = 0$ *so that* $|s_{i,0}| = 1$*. Then*:

(1) *The matrix* $\mathbb{Q}^{\mathbf{h}}_n(\zeta_i)$ *is Hermitian if and only if* $\mathbb{P}^{\mathbf{s}}_n(\zeta_i)$ *is Hermitian in which case they are related by*

$$\mathbb{Q}^{\mathbf{h}}_n(\zeta_i) = \frac{1}{2} \cdot \left(I + \mathbb{U}^{\mathbf{h}\top}_{i,n}\right) \mathbb{P}^{\mathbf{s}}_n(\zeta_i) \left(I + \overline{\mathbb{U}}^{\mathbf{h}}_{i,n}\right). \tag{7.9}$$

(2) *If* $\mathbb{Q}^{\mathbf{h}}_n(\zeta_i)$ *and* $\mathbb{P}^{\mathbf{s}}_n(\zeta_i)$ *are Hermitian, then the numbers* $q^{\mathbf{h}}_{n+1,n}(\zeta_i)$, $q^{\mathbf{h}}_{n,n+1}(\zeta_i)$ *defined via formula* (7.8) *and* $p^{\mathbf{s}}_{n+1,n}(\zeta_i)$, $p^{\mathbf{s}}_{n,n+1}(\zeta_i)$ *defined via formula* (3.7) *are related by*

$$q^{\mathbf{h}}_{n+1,n}(\zeta_i) - \overline{q^{\mathbf{h}}_{n,n+1}(\zeta_i)} = |h_{i,0} + 1|^2 \cdot \left(p^{\mathbf{s}}_{n+1,n}(\zeta_i) - \overline{p^{\mathbf{s}}_{n,n+1}(\zeta_i)}\right). \tag{7.10}$$

By $\mathbb{U}^{\mathbf{h}\top}_{i,n}$ and $\overline{\mathbb{U}}^{\mathbf{h}}_{i,n}$ in formula (7.9) we mean the transpose and the complex conjugate matrices of $\mathbb{U}^{\mathbf{h}}_{i,n}$. The proof of the above statements is straightforward and will be omitted.

We now repeat the construction from Section 3 adapted to the present Carathéodory-class setting. Conditions $\Re\, h_{i,0} \geq 0$ and $\zeta_i h_{i,1} \leq 0$ (whenever $\Re\, h_{i,0} = 0$) are necessary for the problem $\mathbf{BIP}_{\mathcal{C}}(\boldsymbol{\zeta},\, \mathbf{k},\, \mathbf{h})$ to have a solution and we assume that these conditions are met. We then break the index set $\{1, \dots, N\}$ into three disjoint sets

$$\begin{gathered} I = \{i \, : \, \Re\, h_{i,0} = 0 \ \text{ and } \ k_i \geq 1\}, \quad J = \{i \, : \, \Re\, h_{i,0} = 0 \ \text{ and } \ k_i = 0\}, \\ K = \{i \, : \, \Re\, h_{i,0} > 0\}. \end{gathered} \tag{7.11}$$

We then use the structured matrices $\mathbb{Q}^{\mathbf{h}}_n(\zeta_i)$ to define the tuple $\mathbf{d} = \{d_1, \dots, d_N\}$ where

$$d_i = \begin{cases} \max\left\{n : \ \mathbb{Q}^{\mathbf{h}}_n(\zeta_i) = \mathbb{Q}^{\mathbf{h}}_n(\zeta_i)^*\right\} & \text{if } \ i \in I \\ 0 & \text{if } \ i \in J \cup K, \end{cases} \tag{7.12}$$

and we use this tuple to further split I into the three disjoint sets I', I'', I''' as in (3.12). We next define the Pick matrix of the problem $\mathbf{BIP}_{\mathcal{C}}(\boldsymbol{\zeta},\, \mathbf{k},\, \mathbf{h})$ by

$$\mathbb{Q}^{\mathbf{h}}_{\mathbf{d}} = \left[Q^{\mathbf{h}}_{ij}\right]_{i,j\in I} \tag{7.13}$$

where the $d_i \times d_i$ diagonal blocks are given by

$$Q^{\mathbf{s}}_{ii} = \mathbb{Q}^{\mathbf{h}}_{d_i}(\zeta_i) = -\mathbb{H}^{\mathbf{h}}_{i,d_i} \Psi_{d_i}(\zeta_i) \tag{7.14}$$

and the $d_i \times d_j$ non-diagonal blocks $Q^{\mathbf{h}}_{ij} = \left[q^{\mathbf{h}}_{r\ell}(\zeta_i, \zeta_j)\right]_{r=1,\dots,d_i}^{\ell=1,\dots,d_j}$ are defined entry-wise by

$$q^{\mathbf{h}}_{r+1,\ell+1}(\zeta_i, \zeta_j) = \sum_{\alpha=0}^{r} \sum_{\beta=0}^{\min\{\ell,\alpha\}} \frac{(\alpha+\ell-\beta)!}{(\ell-\beta)!\beta!(\alpha-\beta)!} \frac{\zeta_i^{\ell-\beta}\bar{\zeta}_j^{\alpha-\beta} h_{i,r-\alpha}}{(1-\zeta_i\bar{\zeta}_j)^{\alpha+\ell-\beta+1}} + \sum_{\alpha=0}^{\ell} \sum_{\beta=0}^{\min\{r,\alpha\}} \frac{(\alpha+r-\beta)!}{(r-\beta)!\beta!(\alpha-\beta)!} \frac{\zeta_i^{\alpha-\beta}\bar{\zeta}_j^{r-\beta}\overline{h_{j,\ell-\alpha}}}{(1-\zeta_i\bar{\zeta}_j)^{\alpha+r-\beta+1}}.$$

It follows from part (1) in Proposition 7.1 and from the equivalences (7.6), that if the tuples **h** and **s** are related as in (7.5) then the integers defined in (7.12) and (3.8) are equal and, therefore, the block decompositions (7.13) and (3.9) of the corresponding Pick matrices are conformal.

Proposition 7.2. *Let the tuples* $\mathbf{h} = \{h_{i,j}\}$ *and* $\mathbf{s} = \{s_{i,j}\}$ *be related as in* (7.5)*, let* $\mathbb{Q}^{\mathbf{h}}_{\mathbf{d}}$ *and* $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ *be the Pick matrices of the corresponding problems* $\mathbf{BIP}_{\mathcal{C}}(\boldsymbol{\zeta}, \mathbf{k}, \mathbf{h})$ *and* $\mathbf{BIP}(\boldsymbol{\zeta}, \mathbf{k}, \mathbf{s})$*, and let*

$$\mathbb{U}^{\mathbf{h}}_{\mathbf{d}} = \operatorname{diag}_{i\in I}\mathbb{U}^{\mathbf{h}}_{i,d_i}$$

be the diagonal block-matrix decomposed conformally with (7.13) *and* (3.9) *and whose diagonal blocks* $\mathbb{U}^{\mathbf{h}}_{i,d_i}$ *are difined via formula* (3.4)*. Then*

$$\mathbb{Q}^{\mathbf{h}}_{\mathbf{d}} = \frac{1}{2}\cdot\left(I + \mathbb{U}^{\mathbf{h}\top}_{\mathbf{d}}\right)\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}\left(I + \overline{\mathbb{U}}^{\mathbf{h}}_{\mathbf{d}}\right). \tag{7.15}$$

The equality of the diagonal blocks in (7.15) follows from (7.9). The equality of non-diagonal blocks follows by a long but straightforward verification. Observe that by the equality (7.9), the matrix $\mathbb{Q}^{\mathbf{h}}_{\mathbf{d}}$ is positive semidefinite if and only if $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ is positive semidefinite, and since the matrix $I + \mathbb{U}^{\mathbf{h}}_{\mathbf{d}}$ is invertible, it follows that $\mathbb{Q}^{\mathbf{h}}_{\mathbf{d}}$ and $\mathbb{P}^{\mathbf{s}}_{\mathbf{d}}$ are of the same rank. Finally, we define the companion numbers δ_i of the problem $\mathbf{BIP}_{\mathcal{C}}(\boldsymbol{\zeta}, \mathbf{k}, \mathbf{h})$ by

$$\delta_i = \zeta_i\cdot\left(q^{\mathbf{h}}_{d_i+1,d_i}(\zeta_i) - \overline{q^{\mathbf{h}}_{d_i,d_i+1}(\zeta_i)}\right) \quad \text{for} \quad i \in I'\cup I'' \tag{7.16}$$

and observe that in view of (7.10), they are related to the numbers (3.13) as follows: $\delta_i = |h_{i,0}+1|^2\cdot\gamma_i$. On account of the above observations we arrive at the following Carathéodory-class counterpart of Theorem 3.5.

Theorem 7.3. *Given the data set* (7.2) *with* $\Re\, h_{i,0} \geq 0$ *for* $i = 1, \dots, N$*, let* I *be defined as in* (7.11)*, let* $\mathbb{Q}^{\mathbf{h}}_{\mathbf{d}}$ *be the Pick matrix defined in* (7.13) *and let* δ_i *be the companion numbers given defined via formulas* (7.16) *and* (7.8)*. Then the problem* $\mathbf{BIP}_{\mathcal{C}}(\boldsymbol{\zeta}, \mathbf{k}, \mathbf{s})$ *is indeterminate if and only if either* I *is empty or the following conditions hold*:

(1) $I \neq \emptyset$ *and* $\zeta_i h_{i,1} < 0$ *for all* $i \in I$;
(2) *the matrix* $\mathbb{Q}^{\mathbf{h}}_{\mathbf{d}}$ *is positive definite*;
(3) $\delta_i > 0$ *for every* $i \in I'$ *and* $\delta_i \geq 0$ *for every* $i \in I''$.

Furthermore, the indeterminate problem $\mathbf{BIP}_{\mathcal{C}}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{s})$ *admits infinitely many rational solutions. Finally, every solution* H *of the problem belongs to the class* $\bigcap_{i\in I}\mathcal{C}^{(d_i)}(\zeta_i)$ *and does not belong to* $\bigcup_{i\in I'}\mathcal{C}^{(d_i+1)}(\zeta_i)$.

Here is the uniqueness criterion; we skip the rank conditions in its formulation.

Theorem 7.4. *The problem* $\mathbf{BIP}_{\mathcal{C}}(\boldsymbol{\zeta},\ \mathbf{k},\ \mathbf{h})$ *is determinate if and only if*:

(1) $\Re\, h_{i,0}=0$ *for every* $i=1,\dots,N$ (*i.e., the set* K *is empty*);
(2) $2d_i\in\{k_i,k_i+1\}$ *for every* $i=1,\dots,N$ (*i.e., the set* I' *is empty*);
(3) *The matrix* $\mathbb{Q}^{\mathbf{h}}_{\mathbf{d}}$ *is positive semidefinite* (*singular*) *and admits positive semidefinite extensions* $\mathbb{Q}^{\mathbf{h}}_{\mathbf{d}+\mathbf{e}_i}$ *for* $i=1,\dots,N$.

The associated measure for the unique solution is discrete with the number of atoms equal to rank $\mathbb{Q}^{\mathbf{h}}_{\mathbf{d}}$.

References

[1] V. M. ADAMJAN, D. Z. AROV and M. G. KREIN, *Infinite Hankel matrices and generalized Carathéodory-Fejér and I. Schur problems*, Funkcional. Anal. i Priložen. **2** (1968), 1–17.

[2] J. AGLER, Z. A. LYKOVA and N. J. YOUNG, *Pseudo-Taylor expansions and the Carathéodory-Fejér problem*, J. Math. Anal. Appl. **386** (2012), 308–318.

[3] P. R. AHERN and D. N. CLARK, *Radial limits and invariant subspaces*, Amer. J. Math. **92** (1970), 332–342.

[4] J. A. BALL, *Interpolation problems of Pick-Nevanlinna and Loewner type for meromorphic matrix functions*, Integral Equations Operator Theory **6** (1983), 804–840.

[5] J. A. BALL, I. GOHBERG and L. RODMAN, "Interpolation of Rational Matrix Functions", OT45, Birkhäuser-Verlag, Basel-Boston, 1990.

[6] J. A. BALL and J. W. HELTON, *A Beurling-Lax theorem for the Lie group which contains most classical interpolation theory*, J. Operator Theory **9** (1983), 107–142.

[7] J. A. BALL and J. W. HELTON, *Interpolation problems of Pick-Nevanlinna and Loewner types for meromorphic matrix-functions: parametrization of the set of all solutions*, Integral Equations Operator Theory **9** (1986), 155–203.

[8] V. BOLOTNIKOV, *A uniqueness result on boundary interpolation*, Proc. Amer. Math. Soc. **136** (2008), 1705–1715.

[9] V. BOLOTNIKOV, *On a boundary analogue of the Carathéodory-Schur interpolation problem*, Proc. Amer. Math. Soc. **136** (2008), 3121–3131.

[10] V. BOLOTNIKOV, *On a boundary analog of the Carathéodory-Schur interpolation problem*, J. Approx. Theory **163** (2011), 568–589.

[11] V. BOLOTNIKOV and H. DYM, "On boundary interpolation for matrix valued Schur functions", Mem. Amer. Math. Soc. **181** (2006), vi + 107 pp.

[12] V. BOLOTNIKOV and A. KHEIFETS, *A higher multiplicity analogue of the Carathéodory–Julia theorem*, J. Funct. Anal. **237** (2006), 350–371.

[13] V. BOLOTNIKOV and A. KHEIFETS, *The higher order Carathéodory–Julia theorem and related boundary interpolation problems*, In: "Recent Advances in Matrix and Operator Theory", Oper. Theory Adv. Appl., Vol. 179, Biskhäuser, Basel, 2008, 63–102.

[14] V. BOLOTNIKOV and A. KHEIFETS, *Carathéodory-Julia type theorems for operator valued Schur functions*, J. Analyse Math. **106** (2008), 237–270.

[15] V. BOLOTNIKOV and A. KHEIFETS, *Carathéodory-Julia type conditions and symmetries of boundary asymptotics for analytic functions on the unit disk*, Math. Nachr. **282** (2009), 1513–1536.
[16] V. BOLOTNIKOV and N. ZOBIN, *Boundary asymptotic expansions of analytic self-maps of the unit disk*, Rev. Mat. Iberoam. **29** (2013), 357–371.
[17] L. DE BRANGES and J. ROVNYAK, "Square Summable Power Series", Holt, Rinehart and Winston, New York, 1966.
[18] C. CARATHÉODORY, *Über den variabilitasbereich der koeffizienten von potenzreihen die egebene werte nicht annehmen*, Math. Ann. **64** (1907), 95–115.
[19] C. CARATHÉODORY, *Über die Winkelderivierten von beschränkten Funktionen*, Sitzungber. Preuss. Akad. Wiss. (1929), 39–52.
[20] C. CARATHÉODORY and L. FEJÉR, *Über den zusammenhang der extremen von harmonischen funktionen mit ihren Keoffizienten und über den Picard-Landauschen*, Satz, Rend. Circ. Mat. Palermo **32** (1911), 218–239.
[21] E. FRICAIN and J. MASHREGHI, *Boundary behavior of functions in the de Branges-Rovnyak spaces*, Complex Anal. Oper. Theory **2** (2008), 87–97.
[22] A. KHEIFETS, *The abstract interpolation problem and applications*, In: "Holomorphic Spaces", Math. Sci. Res. Inst. Publ., 33, Cambridge Univ. Press, Cambridge, 1998, 351–379.
[23] I. V. KOVALISHINA, *Multiple boundary interpolation problem for contractive matrix–valued functions in the unit circle*, J. Soviet Math. **52** (1990), 3467–3481.
[24] R. NEVANLINNA, *Über beschränkte Funktionen die in gegebenen Punkten vorgeschriebene Werte annehmen* Ann. Acad. Sci. Fenn. **13** (1919), 1–71.
[25] G. PICK, *Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionswerte bewirkt werden*, Math. Ann. **77** (1916), 7–23.
[26] V. P. POTAPOV, "Collected Papers of V. P. Potapov", Hokkaido University, Sapporo, 1982.
[27] D. SARASON, *Generalized interpolation in* H^∞, Trans. Amer. Math. Soc. **127** (1967), 179–203.
[28] D. SARASON, *Nevanlinna–Pick interpolation with boundary data*, Integral Equations Operator Theory **30** (1998), 231–250.
[29] I. SCHUR, *Über Potenzreihen die im Innern des Einheitskreises beschränt sind*, J. Reine Angew. Math. **47** (1917), 205–232.
[30] J. H. SHAPIRO, "Composition Operators and Classical Function Theory", Tracts in Mathematics, Springer-Verlag, New York, 1993.
[31] D. SHOIKHET, "Semigroups in Geometrical Function Theory", Kluwer, Dordrecht, NL, 2001.

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