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Schanuel's theorem for heights defined via extension fields

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Abstract. Let k be a number field, let θ be a nonzero algebraic number, and let $H(\cdot)$ be the Weil height on the algebraic numbers. In response to a question by T. Loher and D. W. Masser, we prove an asymptotic formula for the number of $\alpha \in k$ with $H(\alpha\theta) \le X$, and we analyze the leading constant in our asymptotic formula. In particular, we prove a sharp upper bound in terms of the classical Schanuel constant.

We also prove an asymptotic counting result for a new class of height functions defined via extension fields of k with a fairly explicit error term. This provides a conceptual framework for Loher and Masser's problem and generalizations thereof.

Finally, we establish asymptotic counting results for varying θ, namely, for the number of $\sqrt{p}\alpha$ of bounded height, where $\alpha \in k$ and p is any rational prime inert in k.

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1. Introduction

Let k be a number field. A well-known result due to Schanuel [12] shows that the subset of k^n of points with absolute multiplicative Weil height no larger than X has cardinality

$$S_k(n)X^{d(n+1)} + O\left(X^{d(n+1)-1}\log X\right),$$

as X tends to infinity. Here d is the degree of k and the positive constant $S_k(n)$ involves all the classical number field invariants; for the definition see (1.2).

In the present article we generalize this result in various respects motivated by a question of Loher and Masser. Let θ be a nonzero algebraic number, let $H(\cdot)$

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denote the absolute multiplicative Weil height on the algebraic numbers $\overline{\mathbb{Q}}$, and write $N(\theta k, X)$ for the number of $\alpha \in k$ with $H(\theta\alpha) \leq X$.

Evertse was the first one to consider the quantity $N(\theta k, X)$. The proof of his celebrated uniform upper bounds [4] for the number of solutions of S-unit equations over k involves the following uniform upper bound

$$N(\theta k, X) \leq 5 \cdot 2^d X^{3d} + 1.$$

Later Schmidt [13, Lemma 8B, page 29] refined Evertse's argument to get the correct exponent on X. Schmidt used a different height but elementary inequalities between them imply

$$N(\theta k, X) \leq 36 \cdot 2^{3d} X^{2d}.$$

But the constant is fairly large. Indeed, the constant's exponential dependence on d can be removed, as shown by Loher and Masser. More precisely, they proved

$$N(\theta k, X) \leq 68(d \log d) X^{2d}, \tag{1.1}$$

provided $d > 1$, and $N(\theta\mathbb{Q}, X) \leq 17X^2$. (In the special case $\theta \in k$ a similar result was obtained earlier by Loher in his Ph.D. thesis [8].) By counting roots of unity they also showed that an upper bound with a constant of the form $o(d \log\log d)$ cannot hold, and hence regarding the degree their result is nearly optimal. Loher and Masser's result (1.1) played also an important role in the recent proof of a longstanding conjecture of Erdős on the largest prime divisor of $2^n - 1$ by Stewart [16]. Stewart's strategy builds up on work of Yu [21,22] on p-adic logarithm forms in which Yu applies a consequence of (1.1) to obtain a significant improvement. It is this improvement that makes Stewart's approach work (*cf.* [22, page 378]).

All the proofs of these upper bounds for $N(\theta k, X)$ rely in an essential way on the box-principle, which works well for upper bounds but seems inappropriate to produce asymptotic results. This may have motivated Loher and Masser's following statement [6, page 279] regarding their bound on $N(\theta k, X)$: *It would be interesting to know if there are asymptotic formulae like Schanuel's for the cardinalities here, at least for fixed θ not in k.*

Our Theorem 1.1 responds to this problem for fixed θ not in k, and our Theorem 1.4 generalizes Theorem 1.1 to arbitrary dimensions. Theorem 1.2 gives a sharp upper bound for the leading constant in these asymptotics in terms of Schanuel's constant $S_k(n)$. In Theorem 1.3, we shall see asymptotic results for varying θ not in k.

To provide a more general framework for Loher and Masser's, and similar questions, we introduce a new class of heights on $\mathbb{P}^n(k)$, using finite extensions of the base field k. As usual, these heights decompose into local factors, one for each place v of k. However, at a finite number of non-Archimedean places, the local factors of these heights do not necessarily arise from norms, and moreover, their values do not necessarily lie in the value groups of the corresponding places

v. Theorem 6.1 (in Section 6), from which we will deduce Theorem 1.4 (and thus also Theorem 1.1), is a counting result, in the style of Schanuel's, for these heights.

Our heights are special cases of the heights used by Peyre [11, Définition 1.2]. Peyre gives asymptotic counting results [11, Corollaire 6.2.18] but no error estimates for his general heights. Therefore the main terms in our Theorem 1.4 and Theorem 6.1 could likely be derived from Peyre's result, although with a different representation of the constant. Indeed, a significant part of this work consists of finding the right representation which enables us to prove the sharp upper bound in Theorem 1.2, as well as some invariance properties. Furthermore, Peyre's approach does not seem to provide comparable error terms, and the latter are essential for the proof of our Theorem 1.3. A very recent result due to Ange [1, Théorème 1.1] provides a Schanuel type counting result for another special case of Peyre's heights. Ange also gives a completely explicit and fairly sharp error term. However, his heights require Euclidean/Hermitian norms at the Archimedean places and thus do not include the usual Weil height.

Next we introduce some notation. We start with Schanuel's constant $S_k(n)$, which is defined as follows

$$S_k(n) = \frac{h_k R_k}{w_k \zeta_k(n+1)} \left(\frac{2^r (2\pi)^s}{\sqrt{|\Delta_k|}} \right)^{n+1} (n+1)^{r+s-1}. \tag{1.2}$$

Here h_k is the class number, R_k the regulator, w_k the number of roots of unity in k, ζ_k the Dedekind zeta-function of k, Δ_k the discriminant, $r = r_k$ is the number of real embeddings of k and $s = s_k$ is the number of pairs of complex conjugate embeddings of k.

For each place v of k (or w of $K := k(\theta)$) we choose the unique absolute value $|\cdot|_v$ on k (or $|\cdot|_w$ on K) that extends either the usual Euclidean absolute value on $\mathbb{Q}$ or a usual p-adic absolute value. We also fix a completion k_v of k at v and for each Archimedean place v of k we define a set of points $(z_0, \dots, z_n) \in k_v^{n+1}$ by

$$\prod_{w|v} \max\{|z_0|_v, |\theta|_w |z_1|_v, \dots, |\theta|_w |z_n|_v\}^{\frac{[K_w:k_v]}{[K:k]}} < 1,$$

where the product runs over all places w of $K = k(\theta)$ extending v. As these sets are open, bounded, and not empty, they are measurable and have a finite, positive volume, which we denote by V_v. Here we identify k_v with $\mathbb{R}$ or with $\mathbb{C}$, and we identify the latter with $\mathbb{R}^2$. We define

$$V = V(\theta, k, n) := \left(2^r \pi^s\right)^{-(n+1)} \prod_{v|\infty} V_v. \tag{1.3}$$

Write $\mathcal{O}_k$ for the ring of integers of k and let μ_k be the Möbius function for nonzero ideals of $\mathcal{O}_k$. For ideals A, B of $\mathcal{O}_k$, we write $(A, B) := A + B$. Moreover, $\mathfrak{N}_k A$ denotes the absolute norm of the fractional ideal A of k. For $\alpha \in k$, we also write $\mathfrak{N}_k(\alpha) := \mathfrak{N}_k(\alpha \mathcal{O}_k)$. Analogous notation is used for K instead of k.

For an ideal B of $\mathcal{O}_k$, we write ${}^{\mathfrak{u}}B := B\mathcal{O}_K$ for the extension of B to $\mathcal{O}_K$ ("up"). Similarly, for an ideal $\mathfrak{D}$ of $\mathcal{O}_K$, we write ${}^{\mathfrak{d}}\mathfrak{D} := \mathfrak{D} \cap \mathcal{O}_k$ for the contraction of $\mathfrak{D}$ to $\mathcal{O}_k$ ("down").

The dependence on θ comes in two flavors; while V amounts only to the Archimedean part the following constant captures both parts.

Let α be nonzero and in $\mathcal{O}_k$ such that $\alpha\theta \in \mathcal{O}_K$, let $\mathfrak{D} := \alpha\theta\mathcal{O}_K$, and $D := {}^{\mathfrak{d}}\mathfrak{D}$. We define

$$g_k(\theta, n) := \frac{V}{\mathfrak{N}_k(\alpha)^n} \sum_{B \mid D} \frac{\mathfrak{N}_K\left(\mathfrak{D}, {}^{\mathfrak{u}}B\right)^{\frac{n+1}{[K:k]}}}{\mathfrak{N}_k B} \sum_{A \mid B^{-1}D} \frac{\mu_k(A)}{\mathfrak{N}_k A} \prod_{P \mid AB} \frac{\mathfrak{N}_k P^{n+1} - \mathfrak{N}_k P}{\mathfrak{N}_k P^{n+1} - 1}. \tag{1.4}$$

In the product, P runs over all prime ideals of $\mathcal{O}_k$ dividing AB. It will follow from Lemma 2.3 that this definition does not depend on the choice of α, and from Proposition 2.2 that $g_k(\theta, n) > 0$.

Theorem 1.1. *Let θ be a nonzero algebraic number, let k be a number field and denote its degree by d. Then, as $X \geq 1$ tends to infinity, we have*

$$N(\theta k, X) = g_k(\theta, 1) S_k(1) X^{2d} + O\left(X^{2d-1}\mathfrak{L}\right),$$

where $\mathfrak{L} := \log(X+1)$ if $d = 1$ and $\mathfrak{L} := 1$ otherwise. The implicit constant in the O-term depends on θ and on k.

Let us briefly discuss some properties of the constant $g_k(\theta, 1)$ and then illustrate the theorem by some examples.

For any nonzero α in k we have $\theta k = \alpha\theta k$. Also, the height is invariant under multiplication by a root of unity. Therefore $N(\theta k, X) = N(\zeta\alpha\theta k, X)$ for any $\alpha \in k^*$ and any root of unity ζ, in particular we have

$$g_k(\theta, 1) = g_k(\zeta\alpha\theta, 1). \tag{1.5}$$

This can also be proved directly from the definition as we shall see in Section 2. By Schanuel's theorem we conclude that $g_k(\zeta\alpha, 1) = 1$. But, as is straightforward to check, the theorem implies even $g_k(\zeta\alpha, 1) = 1$ for ζ a root of any unit in $\mathcal{O}_k$ and $\alpha \in k^*$.

The fact that $H(\alpha\theta) = H(\alpha^{-1}\theta^{-1})$ implies

$$g_k(\theta, 1) = g_k\left(\theta^{-1}, 1\right).$$

Next we consider the problem of uniformly bounding $g_k(\theta, 1)$. From Schanuel's theorem and the standard inequalities $H(\alpha)/H(\theta) \leq H(\theta\alpha) \leq H(\theta)H(\alpha)$ we conclude

$$H(\theta)^{-2d} \leq g_k(\theta, 1) \leq H(\theta)^{2d}.$$

This raises the question of the existence of bounds that are uniform in θ or in d, or even uniform in both quantities θ and d. From (1.1) we obtain an upper bound that is uniform in θ, *i.e.*, for $d > 1$

$$g_k(\theta, 1) \leq \frac{68d \log d}{S_k(1)}.$$

Now if we fix $d > 1$ and vary the fields k then by the Siegel-Brauer theorem the right hand-side tends to infinity, so this bound really depends on Δ_k and not only on d. However, intuitively one might guess that for most $\alpha \in k$ one has $H(\theta\alpha) \geq H(\alpha)$, so one might even expect that $g_k(\theta, 1) \leq 1$ holds true, which, of course, would be best-possible. We shall answer here all of these questions. We start with the upper bound and confirm the intuitive guess.

Theorem 1.2. *Let θ be a nonzero algebraic number. Then $g_k(\theta, n) \leq 1$. Moreover, equality holds if and only if for every place v of k there is an $\alpha_v \in k_v$ such that $|\theta|_w = |\alpha_v|_v$ holds for all places w of K above v.*

Let us now illustrate Theorem 1.1 with an example, and thereby explain also the situation regarding lower bounds for $g_k(\theta, 1)$. Let us first take $k = \mathbb{Q}$, and $\theta = \sqrt{p}$ for a prime number p. Then we get the asymptotics

$$\frac{2\sqrt{p}}{p+1} S_{\mathbb{Q}}(1) X^2 = \frac{24\sqrt{p}}{\pi^2(p+1)} X^2.$$

More generally, if p is inert in k and $\theta = \sqrt{p}$ then we get the asymptotics

$$\frac{2p^{d/2}}{p^d + 1} S_k(1) X^{2d}. \tag{1.6}$$

Letting p tend to infinity shows that there is no lower bound for $g_k(\theta, 1)$ that is uniform in θ. Likewise, fixing a p and taking a sequence $\mathbb{Q}, k_1, k_2, \ldots$ of number fields with p inert in k_i and $[k_i : \mathbb{Q}] \to \infty$ shows that there is no lower bound for $g_k(\theta, 1)$ that is uniform in d.

The fast decay of $g_k(\sqrt{p}, 1)$ as p runs over the set $\mathbf{P}_k$ (which we define as the set of positive rational primes inert in k) suggests another problem. Let

$$\sqrt{\mathbf{P}_k}k := \left\{\sqrt{p}\alpha : p \in \mathbf{P}_k, \alpha \in k\right\} = \bigcup_{p \in \mathbf{P}_k} \sqrt{p}k.$$

The above set has uniformly bounded degree, and thus, by Northcott's theorem, we may consider its counting function $N(\sqrt{\mathbf{P}_k}k, X) := |\{\beta \in \sqrt{\mathbf{P}_k}k : H(\beta) \leq X\}|$. Now if $d > 2$ then the sum over the terms in (1.6) converges, so it is natural to ask whether the asymptotics of $N(\sqrt{\mathbf{P}_k}k, X)$ are given simply by summing the asymptotics of $N(\sqrt{p}k, X)$ over $\mathbf{P}_k$. The following result positively answers this question.

Theorem 1.3. *Let k be a number field of degree d. Then, as $X \geq 3$ tends to infinity, we have*

$$N\left(\sqrt{\mathbf{P}_k}k, X\right) = \begin{cases} S_k(1)X^4 \log\log X + O\left(X^4\right) & \text{if } d = 2, \\ \left(\sum_{\mathbf{P}_k} \frac{2p^{d/2}}{p^d+1}\right) S_k(1)X^{2d} + O\left(X^{2d-1}\mathcal{L}\right) & \text{if } d > 2, \end{cases}$$

where $\mathcal{L} = \log\log X$ if $d = 3$ and $\mathcal{L} = 1$ if $d > 3$. The implicit constant in the O-term depends on k.

The case $d = 2$ is just slightly more difficult than $d > 2$ and requires additionally Chebotarev's density theorem and partial summation. However, it is not clear to us how to handle the case $d = 1$.

Finally, let us mention that Theorem 1.1 can also be used to count the elements in the nonzero, *e.g.*, square classes $k^*/(k^*)^2$. Each class has the form $\gamma \cdot (k^*)^2$ with some $\gamma \in k^*$. To count the number $N(\gamma \cdot (k^*)^2, X)$ of elements in this square class with height no larger than X we note that $H(\gamma\alpha^2) = H(\sqrt{\gamma}\alpha)^2$, and thus $N(\gamma \cdot (k^*)^2, X) = (1/2)(N(\sqrt{\gamma}k, \sqrt{X}) - 1)$. *E.g.*, the square class $(\mathbb{Q}^*)^2$ has asymptotically $(6/\pi^2)X$ elements whereas the square class $11 \cdot (\mathbb{Q}^*)^2$ has asymptotically only $(\sqrt{11}/\pi^2)X$ elements of height bounded by X.

Next we generalize Theorem 1.1 to higher dimensions. Let $N(\theta k^n, X)$ be the number of points $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in k^n$ with $H((\theta\alpha_1, \ldots, \theta\alpha_n)) \leq X$. Of course, here $H : \overline{\mathbb{Q}}^n \to [1, \infty)$ is the (affine) absolute multiplicative Weil height, defined by

$$H(\omega_1, \ldots, \omega_n)^{[K:\mathbb{Q}]} := \prod_{w \in M_K} \max\{1, |\omega_1|_w, \ldots, |\omega_n|_w\}^{d_w},$$

where K is any number field containing $\omega_1, \ldots, \omega_n$, the index w runs over the set M_K of all places of K, and $d_w := [K_w : \mathbb{Q}_w]$ denotes the local degree, where $\mathbb{Q}_w$ is the completion of $\mathbb{Q}$ with respect to the place below w.

Theorem 1.4. *Let θ be a nonzero algebraic number, let k be a number field, denote its degree by d, and let n be a positive rational integer. Then, as $X \geq 1$ tends to infinity, we have*

$$N\left(\theta k^n, X\right) = g_k(\theta, n)S_k(n)X^{d(n+1)} + O\left(X^{d(n+1)-1}\mathfrak{L}\right),$$

where $\mathfrak{L} := \log(X+1)$ if $(n, d) = (1, 1)$, and $\mathfrak{L} := 1$ otherwise. The implicit constant in the O-term depends on θ, on k, and on n.

Of course the invariance property (1.5) remains valid for arbitrary n instead of 1. Ange [1, Corollaire 1.6] has shown a related result (although with different choice of the height); instead of fixing one θ he allows a different θ for each coordinate and his error term is completely explicit and quite sharp. On the other hand he requires that a (positive) power of each θ lies in the ground field k.

So far we have counted elements $\theta\boldsymbol{\alpha}$ in θk^n of bounded height. What if we replace the set θk by $\theta + k$? Or θk^2 by $\theta_1 k \times \theta_2 k$? More generally, we suppose $L_1, \ldots, L_n$ are linearly independent linear forms in n variables with coefficients in $\overline{\mathbb{Q}}$ and $\theta_1, \ldots, \theta_n$ are in $\overline{\mathbb{Q}}$. Suppose we want to count elements of bounded height in the set

$$\left\{(L_1(\boldsymbol{\alpha}) + \theta_1, \ldots, L_n(\boldsymbol{\alpha}) + \theta_n) : \boldsymbol{\alpha} \in k^n\right\}.$$

Now let $\boldsymbol{\alpha} := (\omega_1/\omega_0, \ldots, \omega_n/\omega_0) \in k^n$ and define $\boldsymbol{\omega} := (\omega_0, \ldots, \omega_n)$. Then

$$H\Big(\big(L_1(\boldsymbol{\alpha}) + \theta_1, \ldots, L_n(\boldsymbol{\alpha}) + \theta_n\big)\Big) = \prod_w \max\left\{|\mathcal{L}_0(\boldsymbol{\omega})|_w, \ldots, |\mathcal{L}_n(\boldsymbol{\omega})|_w\right\}^{\frac{[K_w:\mathbb{Q}_w]}{[K:\mathbb{Q}]}},$$

where $\mathcal{L}_0(\boldsymbol{\omega}) = \omega_0$ and $\mathcal{L}_i(\boldsymbol{\omega}) = L_i(\omega_1, \ldots, \omega_n) + \theta_i\omega_0$ (for $1 \le i \le n$), which give us $n+1$ linearly independent linear forms. Here the right hand-side defines a special case of a so-called adelic Lipschitz height $H_{\mathcal{N}}$ (introduced in [19]) on $\mathbb{P}^n(K)$, where K is any number field containing k, and the coefficients of $\mathcal{L}_0, \ldots, \mathcal{L}_n$, and the product runs over all places w of K. Thus, we need to count the points $P = (\omega_0 : \cdots : \omega_n) \in \mathbb{P}^n(k)$ with $\omega_0 \neq 0$ and $H_{\mathcal{N}}(P) \le X$.

These generalizations of Loher and Masser's problem naturally motivate our general theorem (Theorem 6.1), which is as follows. Given two number fields $k \subseteq K$ and an adelic Lipschitz height $H_{\mathcal{N}}$ on K, we give an asymptotic formula for the number of points $P \in \mathbb{P}^n(k)$ with $H_{\mathcal{N}}(P) \le X$, as the parameter X tends to infinity. To be more accurate, we also impose a minor additional assumption on the adelic Lipschitz height $H_{\mathcal{N}}$, which seems fulfilled in all natural applications, in particular, it holds in the aforementioned examples.

The special case $K = k$ of our general theorem follows from a result in [19]. There, a complementary result was proved, in the sense that points of $\mathbb{P}^n(K)$ defined over a proper subextension of K/k were excluded from the counting (which is insignificant for the main term but was needed to obtain good error terms).

Now already with general linear forms as above it seems unlikely that the main term can be brought into an as civilized form as for Theorem 1.4 (see also the remark in [18, page 1766 third paragraph]). Indeed, a considerable part of our work consists of finding the simple representation of the constant in the special case of Theorem 1.4. However, it turns out that the given representation is not so convenient for theoretical considerations. Indeed, even the most obvious properties, such as the invariance property (1.5), are not immediately clear from the present definition. In Section 2 we establish a representation of $g_k(\theta, n)$ as a product of local factors (Proposition 2.2), which is a first step in the proof of Theorem 1.2 and also reveals the invariance property (1.5).

At any rate, a situation involving linear forms similar to the above turns up if we want to count solutions of a system of linear equations with certain restrictions to the coordinates of the solutions. Here is an example. Consider the equation

$$\sqrt{2}x + \sqrt{3}y + \sqrt{5}z = 0, \tag{1.7}$$

defined over $K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$. Using arguments from [18] one can easily compute that the number of solutions $(x, y, z) \in K^3$ with $H((x, y, z)) \leq X$ is asymptotically given by

$$\left(\frac{\sqrt{96} - (\sqrt{2} + \sqrt{3} - \sqrt{5})^2}{\sqrt{480}} \right)^8 S_K(2) X^{24} + O(X^{23}).$$

But what about the number of such solutions whose first two coordinates are rational? This question reduces to counting the elements $(\omega_0 : \omega_1 : \omega_2) \in \mathbb{P}^2(\mathbb{Q})$ with bounded adelic Lipschitz height

$$H_{\mathcal{N}}((\omega_0 : \omega_1 : \omega_2)) = \prod_w \max\left\{ |\omega_0|_w, |\omega_1|_w, |\omega_2|_w, \left| \frac{\sqrt{2}\omega_1 + \sqrt{3}\omega_2}{\sqrt{5}} \right|_w \right\}^{\frac{[K_w:\mathbb{Q}_w]}{[K:\mathbb{Q}]}}.$$

Applying our general theorem gives the following asymptotic formula

$$N_L(X) = V_{\mathcal{N}'} \cdot \frac{1}{62\zeta(3)} \cdot \left(1 + 2 \cdot 5^{1/4} + 4 \cdot 5^{-1/2}\right) X^3 + O\left(X^2\right) \tag{1.8}$$

for the number $N_L(X)$ of solutions (x, y, z) of (1.7) of height bounded by X and with $x, y \in \mathbb{Q}$. Here $V_{\mathcal{N}'}$ denotes the volume of the set of points (z_0, z_1, z_2) in $\mathbb{R}^3$ that satisfy the inequality

$$\max\{|z_0|, |z_1|, |z_2|, |\sqrt{2}z_1 + \sqrt{3}z_2|/\sqrt{5}\} \max\{|z_0|, |z_1|, |z_2|, |\sqrt{2}z_1 - \sqrt{3}z_2|/\sqrt{5}\} < 1.$$

For the computations we refer the reader to the appendix.

Finally, by Northcott's theorem there is no need to restrict to a fixed number field, and one could also consider all number fields of a given fixed degree simultaneously. Let us define the set

$$\theta k(n; e) = \{(\theta\alpha_1, \ldots, \theta\alpha_n) : [k(\alpha_1, \ldots, \alpha_n) : k] = e\}.$$

So Theorem 1.4 gives the asymptotics for the counting function $N(\theta k(n; 1), X) = N(\theta k^n, X)$, and more generally, one could ask for the asymptotics of $N(\theta k(n; e), X)$. The special case $\theta \in k$ was considered in [5, 9, 10, 14, 15], and [17]. Indeed, it is likely that the methods from [19] and [17], combined with those of the present article, are sufficient to solve this problem, provided n is large enough. On the other hand, it would be interesting to know whether Masser and Vaaler's approach from [9] can be combined with ours to handle the case $n = 1$.

The plan of the paper is as follows. In Section 2 we establish a product representation of $g_k(\theta, n)$, and we use this to deduce some of its properties. This product form is also the starting point in the proof of Theorem 1.2, which we give in Section 3. Then in Section 4 we state and prove some basic facts about lattice points, which are required for the proofs of Theorem 6.1 and Theorem 1.3. Section 5 provides the necessary notions such as adelic Lipschitz systems to state our general theorem. Then in Section 6 we state the general theorem (Theorem 6.1), and in Section 7 we give its proof. From Theorem 6.1 we deduce Theorem 1.4, which is done in Section 8. The proof of Theorem 1.3 is carried out in Section 9. Finally, in the appendix we calculate formula (1.8) using Theorem 6.1.

By a prime ideal we always mean a nonzero prime ideal. By $E \trianglelefteq \mathcal{O}_k$, we mean that E is a nonzero ideal of $\mathcal{O}_k$. An empty product is always interpreted as 1, and an empty sum is interpreted as 0.

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2. Product representation and invariance properties of the constant

In this section, we use a product representation for the constant $g_k(\theta, n)$ to derive some of its properties. Let $\mathfrak{D}$, B be nonzero ideals of $\mathcal{O}_K$ or $\mathcal{O}_k$, respectively. For convenience, we define

$$q(\mathfrak{D}, B) := q(\mathfrak{D}, B, n) := \frac{\mathfrak{N}_K(\mathfrak{D}, {}^{\mathfrak{u}}B)^{(n+1)/[K:k]}}{\mathfrak{N}_k B}.$$

Clearly, $q(\mathfrak{D}, B)$ is multiplicative in B, by which we mean that $q(\mathfrak{D}, B_1B_2) = q(\mathfrak{D}, B_1)q(\mathfrak{D}, B_2)$ whenever $(B_1, B_2) = 1$. Moreover, $q(\mathfrak{D}, B) = q((\mathfrak{D}, {}^{\mathfrak{u}}B), B)$, and if $B_1 \mid B_2$, then $q({}^{\mathfrak{u}}B_2\mathfrak{D}, B_1) = \mathfrak{N}_k B_1^n$ and $q({}^{\mathfrak{u}}B_1\mathfrak{D}, B_2) = \mathfrak{N}_k B_1^n q(\mathfrak{D}, B_1^{-1}B_2)$.

We now define local factors at prime ideals P of $\mathcal{O}_k$, by

$$g_P(\mathfrak{D}, n) := \frac{\mathfrak{N}_k P - 1}{\mathfrak{N}_k P^{n+1} - 1}\left(1 + (\mathfrak{N}_k P^n - 1)\sum_{j=0}^{\infty} q(\mathfrak{D}, P^j)\right).$$

Let v_P denote the P-adic valuation on k, normalized by $v_P(k^*) = \mathbb{Z}$. The infinite sum converges, since

$$q(\mathfrak{D}, P^j) = \mathfrak{N}_k P^{v_P({}^{\mathfrak{d}}\mathfrak{D})-j} q\left(\mathfrak{D}, P^{v_P({}^{\mathfrak{d}}\mathfrak{D})}\right) \tag{2.1}$$

holds for all $j \geq v_P({}^{\mathfrak{d}}\mathfrak{D})$. Clearly, $g_P(\mathfrak{D}, n) = g_P(\mathfrak{D}_P, n)$, where $\mathfrak{D}_P := \prod_{\mathfrak{P}|P} \mathfrak{P}^{v_{\mathfrak{P}}(\mathfrak{D})}$ is the part of $\mathfrak{D}$ lying over P.

Lemma 2.1. *Let $\mathfrak{D}$ be a nonzero ideal of $\mathcal{O}_K$ and $D := {}^{\flat}\mathfrak{D}$. Then*

$$\sum_{B \mid D} q(\mathfrak{D}, B) \sum_{A \mid B^{-1}D} \frac{\mu_k(A)}{\mathfrak{N}_k A} \prod_{P \mid AB} \frac{\mathfrak{N}_k P^{n+1} - \mathfrak{N}_k P}{\mathfrak{N}_k P^{n+1} - 1} = \prod_P g_P(\mathfrak{D}, n).$$

Proof. We start by investigating the expression

$$S(D, B) := \sum_{A \mid B^{-1}D} \frac{\mu_k(A)}{\mathfrak{N}_k A} \prod_{P \mid AB} \frac{\mathfrak{N}_k P^{n+1} - \mathfrak{N}_k P}{\mathfrak{N}_k P^{n+1} - 1}$$

for a given ideal B of $\mathcal{O}_k$ dividing D. Clearly,

$$S(D, B) = \prod_{P \mid B} \frac{\mathfrak{N}_k P^{n+1} - \mathfrak{N}_k P}{\mathfrak{N}_k P^{n+1} - 1} \sum_{A \mid B^{-1}D} f(A),$$

where

$$f(A) := \frac{\mu_k(A)}{\mathfrak{N}_k A} \prod_{\substack{P \mid A \\ P \nmid B}} \frac{\mathfrak{N}_k P^{n+1} - \mathfrak{N}_k P}{\mathfrak{N}_k P^{n+1} - 1}.$$

The function f is multiplicative and $f(\mathcal{O}_k) = 1$. For any prime ideal P dividing $B^{-1}D$, we have

$$f(P) = \begin{cases} -\mathfrak{N}_k P^{-1} & \text{if } P \mid B, \\ -\big(\mathfrak{N}_k P^n - 1\big)/\big(\mathfrak{N}_k P^{n+1} - 1\big) & \text{if } P \nmid B. \end{cases}$$

Moreover, $f(P^e) = 0$ if $e > 1$. We use

$$\sum_{A \mid B^{-1}D} f(A) = \prod_{P \mid B^{-1}D} (1 + f(P))$$

to obtain

$$\begin{aligned} S(D, B) &= \prod_{P \mid B} \frac{\mathfrak{N}_k P^{n+1} - \mathfrak{N}_k P}{\mathfrak{N}_k P^{n+1} - 1} \prod_{\substack{P \mid B^{-1}D \\ P \nmid B}} \frac{\mathfrak{N} P^{n+1} - \mathfrak{N}_k P^n}{\mathfrak{N}_k P^{n+1} - 1} \prod_{P \mid (B^{-1}D, B)} \frac{\mathfrak{N}_k P - 1}{\mathfrak{N}_k P} \\ &= \prod_{P \mid D} \frac{\mathfrak{N}_k P^{n+1} - \mathfrak{N}_k P^n}{\mathfrak{N}_k P^{n+1} - 1} \prod_{P \mid B} \frac{\mathfrak{N}_k P^{n+1} - \mathfrak{N}_k P}{\mathfrak{N}_k P^{n+1} - \mathfrak{N}_k P^n} \prod_{P \mid (B^{-1}D, B)} \frac{\mathfrak{N}_k P - 1}{\mathfrak{N}_k P}. \end{aligned}$$

Let $T(D, B) := S(D, B) / \prod_{P \mid D} \frac{\mathfrak{N}_k P^{n+1} - \mathfrak{N}_k P^n}{\mathfrak{N}_k P^{n+1} - 1}$. Then the expression on the left-hand side of the Lemma is given by

$$\left(\prod_{P \mid D} \frac{\mathfrak{N}_k P^{n+1} - \mathfrak{N}_k P^n}{\mathfrak{N}_k P^{n+1} - 1} \right) \sum_{B \mid D} q(\mathfrak{D}, B) T(D, B).$$

Since both $T(D, B)$ and $q(\mathfrak{D}, B)$ are multiplicative in B, this is equal to

$$\prod_{P|D}\left(\frac{\mathfrak{N}_k P^{n+1}-\mathfrak{N}_k P^n}{\mathfrak{N}_k P^{n+1}-1}\sum_{j=0}^{v_P(D)} q(\mathfrak{D}, P^j)T(D, P^j)\right). \tag{2.2}$$

Elementary manipulations show that

$$T\big(D, P^j\big)=\frac{(\mathfrak{N}_k P^n-1)(\mathfrak{N}_k P-1)}{\mathfrak{N}_k P^{n+1}-\mathfrak{N}_k P^n}\cdot\begin{cases}\dfrac{\mathfrak{N}_k P^{n+1}-\mathfrak{N}_k P^n}{(\mathfrak{N}_k P^n-1)(\mathfrak{N}_k P-1)} & \text{if } j=0,\\ 1 & \text{if } 1\le j<v_P(D),\\ \displaystyle\sum_{j=v_P(D)}^{\infty}\mathfrak{N}_k P^{v_P(D)-j} & \text{if } j=v_P(D).\end{cases}$$

Using (2.1), this shows that each of the factors in (2.2) has the form

$$\frac{(\mathfrak{N}_k P-1)(\mathfrak{N}_k P^n-1)}{\mathfrak{N}_k P^{n+1}-1}\left(\frac{\mathfrak{N}_k P^{n+1}-\mathfrak{N}_k P^n}{(\mathfrak{N}_k P^n-1)(\mathfrak{N}_k P-1)}+\sum_{j=1}^{\infty}q(\mathfrak{D}, P^j)\right)=g_P(\mathfrak{D},n). \qquad \square$$

Lemma 2.1 with $\mathfrak{D} := \alpha\theta\mathcal{O}_K$ yields the following formula for $g_k(\theta, n)$.

Proposition 2.2. *If α is nonzero and in $\mathcal{O}_k$ with $\alpha\theta \in \mathcal{O}_K$ then*

$$g_k(\theta, n) = \frac{V}{\mathfrak{N}_k(\alpha)^n}\prod_P g_P(\alpha\theta\mathcal{O}_K, n). \tag{2.3}$$

The next lemma shows that $g_k(\theta, n)$ does not depend on the choice of α.

Lemma 2.3. *Let A be a nonzero ideal of $\mathcal{O}_k$ and $\mathfrak{D}$ a nonzero ideal of $\mathcal{O}_K$. Then*

$$g_P\big({}^{\mathfrak{u}}A\mathfrak{D}, n\big) = \mathfrak{N}_k P^{nv_P(A)} g_P(\mathfrak{D}, n).$$

Proof. We have

$$q\big({}^{\mathfrak{u}}A\mathfrak{D}, P^j\big) = \begin{cases}\mathfrak{N}_k P^{nj} & \text{if } 0\le j<v_P(A),\\ \mathfrak{N}_k P^{nv_P(A)}q\big(\mathfrak{D}, P^{j-v_P(A)}\big) & \text{if } j\ge v_P(A).\end{cases}$$

The lemma follows by inserting these expressions for $q({}^{\mathfrak{u}}A\mathfrak{D}, P^j)$ in the definition of $g_P({}^{\mathfrak{u}}A\mathfrak{D}, n)$. $\square$

Given nonzero $\alpha, \beta \in \mathcal{O}_k$ such that $\alpha\theta, \beta\theta \in \mathcal{O}_K$, then we have

$$\mathfrak{N}_k(\alpha)^n \prod_P g_P(\beta\theta\mathcal{O}_K, n) = \prod_P g_P(\alpha\beta\theta\mathcal{O}_K, n) = \mathfrak{N}_k(\beta)^n \prod_P g_P(\alpha\theta\mathcal{O}_K, n),$$

which shows the independence of $g_k(\theta, n)$ from the choice of α.

To see invariance property (1.5) directly from (2.3), we need the following lemma.

Lemma 2.4. *Let $\alpha \in k^*$. Then*

$$V(\alpha\theta, k, n) = \frac{V(\theta, k, n)}{\mathfrak{N}_k(\alpha)^n}.$$

Proof. For any Archimedean place v of k, the map $\phi_v : k_v^{n+1} \to k_v^{n+1}$ defined by $\phi_v(z_0, z_1 \ldots, z_n) = (z_0, |\alpha|_v z_1, \ldots, |\alpha|_v z_n)$ is a linear automorphism of k_v^{n+1} (considered as $\mathbb{R}^{[k_v:\mathbb{R}](n+1)}$) of determinant $|\alpha|_v^{[k_v:\mathbb{R}]n}$.Therefore, $|\alpha|_v^{[k_v:\mathbb{R}]n} V_v(\alpha\theta, k, n) = V_v(\theta, k, n)$. □

Lemmas 2.3 and 2.4 imply that

$$g_k(\alpha\theta, n) = g_k(\theta, n) \tag{2.4}$$

for every nonzero $\alpha \in \mathcal{O}_k$, and hence for every $\alpha \in k^*$. In particular, it suffices to prove Theorem 1.2 and Theorem 1.4 for integral θ.

3. Proof of Theorem 1.2

We start off by estimating the volume $V(\theta, k, n)$.

Lemma 3.1. *We have*

$$V(\theta, k, n) \leq \mathfrak{N}_K(\theta)^{-n/[K:k]}.$$

Moreover, equality holds if and only if for every Archimedean place v of k the absolute values $|\theta|_w$ are equal for all $w \mid v$.

Proof. Let v be an Archimedean place of K, and let $p_v = p_v(\theta) := \prod_{w|v} |\theta|_w^{\frac{[K_w:k_v]}{[K:k]}}$. Consider the functions $f_v^{(1)}, f_v^{(2)} : k_v^{n+1} \to \mathbb{R}$ given by

$$f_v^{(1)}(z_0, \ldots, z_n) := \prod_{w|v} \max\{|z_0|_v, |\theta|_w|z_1|_v, \ldots, |\theta|_w|z_n|_v\}^{\frac{[K_w:k_v]}{[K:k]}},$$

$$f_v^{(2)}(z_0, \ldots, z_n) := \max\{|z_0|_v, p_v|z_1|_v, \ldots, p_v|z_n|_v\}.$$

Then $f_v^{(i)}(t\mathbf{z}) = |t|_v f_v^{(i)}(\mathbf{z})$ holds for all $t \in k_v, \mathbf{z} \in k_v^{n+1}$, and $i \in \{1, 2\}$. Moreover, $f_v^{(1)} \geq f_v^{(2)}$ as functions on k_v^{n+1}, with equality if and only if $|\theta|_w$ is constant on $w \mid v$.

Now $\mathrm{Vol}\{\mathbf{z} \in k_v^{n+1} : f_v^{(1)}(\mathbf{z}) < 1\} \leq \mathrm{Vol}\{\mathbf{z} \in k_v^{n+1} : f_v^{(2)}(\mathbf{z}) < 1\}$, with equality if and only if $f_v^{(1)} = f_v^{(2)}$. Evaluating both volumes gives

$$V_v \leq p_v^{-n[k_v:\mathbb{R}]} \cdot \begin{cases} 2^{n+1} & \text{if } v \text{ is real,} \\ \pi^{n+1} & \text{if } v \text{ is complex,} \end{cases} \tag{3.1}$$

with equality if and only if $|\theta|_w$ is constant on $w \mid v$. Thus,

$$V(\theta, k, n) \leq \prod_{w\mid\infty} |\theta|_w^{-\frac{n[K_w:\mathbb{R}]}{[K:k]}} = \mathfrak{N}_K(\theta)^{-n/[K:k]},$$

with equality if and only if the condition in the lemma is satisfied. □

Let us recall some simple facts, which will be used in the sequel without further notice. Let A, B be ideals of $\mathcal{O}_k$, and let $\mathfrak{A}, \mathfrak{B}$ be ideals of $\mathcal{O}_K$. Moreover, suppose that P is a prime ideal of $\mathcal{O}_k$ and that $\mathfrak{P}$ runs over all prime ideals of $\mathcal{O}_K$ above P. Then:

- $v_P({}^{\mathfrak{d}}\mathfrak{A}) = \max_{\mathfrak{P}\mid P}\{\lceil v_{\mathfrak{P}}(\mathfrak{A})/e_{\mathfrak{P}}\rceil\}$,
- ${}^{\mathfrak{d}\mathfrak{u}}A = A$,
- $\mathfrak{A} \mid {}^{\mathfrak{u}\mathfrak{d}}\mathfrak{A}$,
- ${}^{\mathfrak{u}}(AB) = {}^{\mathfrak{u}}A{}^{\mathfrak{u}}B$,
- $\mathfrak{A} \mid {}^{\mathfrak{u}}A$ if and only if ${}^{\mathfrak{d}}\mathfrak{A} \mid A$.

Lemma 3.2. *Let $\mathfrak{D}$ be a nonzero ideal of $\mathcal{O}_K$ and P a prime ideal of $\mathcal{O}_k$. Then*

$$g_P(\mathfrak{D}, n) \leq \mathfrak{N}_K(\mathfrak{D}_P)^{n/[K:k]},$$

with equality if and only if $\mathfrak{D}_P = {}^{\mathfrak{u}\mathfrak{d}}\mathfrak{D}_P$.

Proof. Lemma 2.3 and the fact that $g_P(\mathfrak{D}, n) = g_P(\mathfrak{D}_P, n)$ imply equality if $\mathfrak{D}_P = {}^{\mathfrak{u}\mathfrak{d}}\mathfrak{D}_P$. Therefore, let us assume that $v_P({}^{\mathfrak{d}}\mathfrak{D}) =: l \geq 1$ and that $\mathfrak{D}_P$ is a proper divisor of ${}^{\mathfrak{u}\mathfrak{d}}\mathfrak{D}_P = {}^{\mathfrak{u}}P^l$. Again by Lemma 2.3, we may assume that ${}^{\mathfrak{u}}P \nmid \mathfrak{D}$.

Let

$$u := \frac{1}{[K:k]} \sum_{\mathfrak{P}\mid P} f_{\mathfrak{P}} v_{\mathfrak{P}}(\mathfrak{D}),$$

where the sum runs over all prime ideals $\mathfrak{P}$ of $\mathcal{O}_K$ lying over P, and $f_{\mathfrak{P}} = f_{\mathfrak{P}\mid P} = [\mathcal{O}_K/\mathfrak{P} : \mathcal{O}_k/P]$ is the relative degree of $\mathfrak{P}$ over P. Then the right-hand side in the lemma is just $\mathfrak{N}_k(P)^{nu}$. Since $v_P({}^{\mathfrak{d}}\mathfrak{D}) = l \geq 1$, we get $u > 0$. Let $e_{\mathfrak{P}} = e_{\mathfrak{P}\mid P}$ be the ramification index of $\mathfrak{P}$ over P. As $\mathfrak{D}_P$ is a proper divisor of ${}^{\mathfrak{u}\mathfrak{d}}\mathfrak{D}_P$, we conclude that $v_{\mathfrak{P}}(\mathfrak{D}) < e_{\mathfrak{P}} v_P({}^{\mathfrak{d}}\mathfrak{D})$ for at least one $\mathfrak{P} \mid P$. Thus,

$$\sum_{\mathfrak{P}\mid P} f_{\mathfrak{P}} v_{\mathfrak{P}}(\mathfrak{D}) < \sum_{\mathfrak{P}\mid P} f_{\mathfrak{P}} e_{\mathfrak{P}} v_P\left({}^{\mathfrak{d}}\mathfrak{D}\right) = [K:k] \cdot l,$$

and therefore $u < l$. Similarly, we have

$$q(\mathfrak{D}, P^j) = \mathfrak{N}_k(P)^{\frac{n+1}{[K:k]}\left(\sum_{\mathfrak{P}|P} f_{\mathfrak{P}} \min\{v_{\mathfrak{P}}(\mathfrak{D}), je_{\mathfrak{P}}\}\right)-j},$$

for any $j \geq 0$. By our assumption that ${}^uP \nmid \mathfrak{D}$, we have $v_{\mathfrak{P}}(\mathfrak{D}) < je_{\mathfrak{P}}$ for some $\mathfrak{P} \mid P$ and all $j \geq 1$. Replacing all the minima in the above formula by their second arguments yields

$$q(\mathfrak{D}, P^j) < \mathfrak{N}_k P^{jn}. \tag{3.2}$$

Similarly, replacing the minima by their first arguments yields

$$q(\mathfrak{D}, P^j) \leq \mathfrak{N}_k(P)^{(n+1)u-j}, \tag{3.3}$$

and the inequality is strict if and only if $j < l$. Let $1 \leq L \leq l$ be the integer with $L-1 < u \leq L$. We use (3.2) for $j < L$ and (3.3) for $j \geq L$ to estimate $q(\mathfrak{D}, P^j)$ in the definition of $g_P(\mathfrak{D}, n)$. This shows that $g_P(\mathfrak{D}, n)$ is bounded from above by

$$\frac{1}{\mathfrak{N}_k P^{n+1}-1}\left(\mathfrak{N}_k P^{Ln+1} - \mathfrak{N}_k P^{Ln} + \mathfrak{N}_k P^{(n+1)u-L+n+1} - \mathfrak{N}_k P^{(n+1)u-L+1}\right),$$

with a strict inequality whenever $l > 1$. To prove the lemma, it is enough to show that this expression is bounded by $\mathfrak{N}_k P^{nu}$ (with strict inequality if $l = 1$). To this end, let h be the function given by

$$h(x) := x^{nu+n+1} - x^{nu} + x^{(n+1)u-L+1} - x^{(n+1)u-L+n+1} + x^{Ln} - x^{Ln+1}.$$

Hence, we need to show that $h(\mathfrak{N}_k P) \geq 0$, with a strict inequality if $l = 1$. With $\tilde{u} := u - L + 1 \in (0, 1]$ and

$$h_1(x) := x^{n\tilde{u}+n+1} - x^{n\tilde{u}} + x^{(n+1)\tilde{u}} - x^{(n+1)\tilde{u}+n} + x^n - x^{n+1},$$

we have $h(x) = x^{n(L-1)}h_1(x)$. If $\tilde{u} = 1$ then $h_1(x) \equiv 0$. We observe that $\tilde{u} = 1$ is impossible if $l = 1$, since $u < l$. Let us assume that $0 < \tilde{u} < 1$ and prove that, in this case, $h_1(x) > 0$ for all $x > 1$.

The function $h_1(x)$ is in fact a polynomial in $x^{1/[K:k]}$. We have

$$n\tilde{u} + n + 1 > \begin{cases} (n+1)\tilde{u} + n \\ n+1 \end{cases} > \begin{cases} (n+1)\tilde{u} \\ n \end{cases} > n\tilde{u}.$$

By Descartes' rule of signs, $h_1(x)$ has at most three positive zeros (with multiplicities). Since $h_1(1) = h_1'(1) = h_1''(1) = 0$ and $\lim_{x\to\infty} h_1(x) = \infty$, we have $h_1(x) > 0$ for $x > 1$. □

We can now easily finish the proof of Theorem 1.2. After multiplying with a suitable element from k^* we can assume that θ is an algebraic integer and choose $\alpha := 1$. From Proposition 2.2, Lemmata 3.1 and 3.2, and the observation that

$$\mathfrak{N}_K(\theta)^{\frac{n}{[K:k]}} = \prod_P \mathfrak{N}_K\left((\theta\mathcal{O}_K)_P\right)^{\frac{n}{[K:k]}},$$

we immediately get that $g_k(\theta, n) \leq 1$. Now $g_k(\theta, n) = 1$ holds if and only if we have equality in Lemmata 3.1 and 3.2. This is the case if and only if $\theta\mathcal{O}_K = {}^{\mathrm{u}\partial}\theta\mathcal{O}_K$ and for each Archimedean place v of k the $|\theta|_w$ for $w \mid v$ are all equal. The condition for equality in Theorem 1.2 is just a uniform reformulation of these two statements.

4. Preliminaries on lattices

In this section we establish a basic counting result for lattice points, which will be used in the proofs of Theorem 6.1 and Theorem 1.3.

For a vector $\mathbf{x}$ in $\mathbb{R}^m$ we write $|\mathbf{x}|$ for the Euclidean length of $\mathbf{x}$. For a lattice Λ in $\mathbb{R}^m$ we write $\lambda_i = \lambda_i(\Lambda)$ $(1 \leq i \leq m)$ for the successive minima of Λ with respect to the Euclidean distance.

Definition 4.1. Let M and $m > 1$ be positive integers and let L be a non-negative real. We say that a set S is in Lip(m, M, L) if S is a subset of $\mathbb{R}^m$, and if there are M maps $\phi_1, \ldots, \phi_M : [0, 1]^{m-1} \longrightarrow \mathbb{R}^m$ satisfying a Lipschitz condition

$$|\phi_i(\mathbf{x}) - \phi_i(\mathbf{y})| \leq L|\mathbf{x} - \mathbf{y}| \text{ for } \mathbf{x}, \mathbf{y} \in [0, 1]^{m-1}, i = 1, \ldots, M, \tag{4.1}$$

such that S is covered by the images of the maps ϕ_i.

We can now state and prove our counting result.

Lemma 4.2. *Let $m > 1$ be an integer, let Λ be a lattice in $\mathbb{R}^m$ with successive minima $\lambda_1, \ldots, \lambda_m$, and let $a \in \{1, \ldots, m\}$. Let S be a bounded set in $\mathbb{R}^m$ such that the boundary ∂S of S is in Lip(m, M, L), S is contained in the zero-centered ball of radius L, and $\mathbf{0} \notin S$. Then S is measurable and we have*

$$\left| |S \cap \Lambda| - \frac{\operatorname{Vol} S}{\det \Lambda} \right| \leq c_1(m) M \max\left\{ \frac{L^{a-1}}{\lambda_1{}^{a-1}}, \frac{L^{m-1}}{\lambda_1{}^{a-1}\lambda_a{}^{m-a}} \right\}.$$

The constant $c_1(m)$ depends only on m.

Proof. Applying [19, Theorem 5.4] proves measurability and gives

$$\left| |S \cap \Lambda| - \frac{\operatorname{Vol} S}{\det \Lambda} \right| \leq c_1(m) M \max_{0 \leq i \leq m-1} \frac{L^i}{\lambda_1 \cdots \lambda_i}.$$

First we assume $L/\lambda_1 \geq 1$.

Then we conclude

$$\begin{aligned} \max_{0 \leq i \leq m-1} \frac{L^i}{\lambda_1 \cdots \lambda_i} &\leq \max_{0 \leq i \leq m-a} \frac{L^{a-1}}{\lambda_1^{a-1}} \left(\frac{L}{\lambda_a} \right)^i \\ &= \frac{L^{a-1}}{\lambda_1^{a-1}} \max\left\{ 1, \frac{L^{m-a}}{\lambda_a^{m-a}} \right\} \\ &= \max\left\{ \frac{L^{a-1}}{\lambda_1{}^{a-1}}, \frac{L^{m-1}}{\lambda_1{}^{a-1}\lambda_a{}^{m-a}} \right\}. \end{aligned}$$

Next we assume $L/\lambda_1 < 1$. Then we have $|S \cap \Lambda| = 0$. Moreover, by Minkowski's second theorem,

$$\frac{\operatorname{Vol} S}{\det \Lambda} \leq c_1(m) \frac{L^m}{\lambda_1 \cdots \lambda_m}.$$

Furthermore,

$$\frac{L^m}{\lambda_1 \cdots \lambda_m} \leq \frac{L^m}{\lambda_1 \cdots \lambda_m} \frac{\lambda_1}{L} = \frac{L^{m-1}}{\lambda_2 \cdots \lambda_m} \leq \max \left\{ \frac{L^{a-1}}{{\lambda_1}^{a-1}}, \frac{L^{m-1}}{{\lambda_1}^{a-1}{\lambda_a}^{m-a}} \right\}. \qquad \square$$

We recall the following lemma, which is a special case of [3, Lemma VIII.1].

Lemma 4.3. *Let Λ be a lattice in $\mathbb{R}^m$. Then there exist linearly independent vectors $v_1, \ldots, v_m$ in Λ such that $|v_i| = \lambda_i$ for $1 \leq i \leq m$.*

Lemma 4.4. *Let Λ be a lattice in $\mathbb{R}^m$. Then there exists a basis $u_1, \ldots, u_m$ of Λ such that*

$$|u_i| \leq C_0(m) \lambda_1^{-m+1} \det \Lambda \quad \text{for } 1 \leq i \leq m,$$

where $C_0(m)$ is an explicit constant depending only on m.

Proof. Let $v_1, \ldots, v_m$ be linearly independent vectors as in Lemma 4.3. By a lemma of Mahler and Weyl (see [3, Lemma 8, page 135]) we obtain a basis $u_1, \ldots, u_m$ of Λ such that $|u_i| \leq \max\{1, m/2\}\lambda_i$. Observing that $|u_i| \leq |u_1| \cdots |u_m| / \lambda_1^{m-1}$, the lemma follows from Minkowski's second theorem. $\square$

The following result will be used only for the proof of Theorem 1.3 in Section 9.

Lemma 4.5. *Let Λ_1 and Λ_2 be lattices in $\mathbb{R}^d$, and consider the lattice $\Lambda := \Lambda_1 \times \Lambda_2$ in $\mathbb{R}^{2d}$. Then we have*

$$\lambda_1(\Lambda) = \min \{ \lambda_1(\Lambda_1), \lambda_1(\Lambda_2) \},$$
$$\lambda_{d+1}(\Lambda) \geq \max \{ \lambda_1(\Lambda_1), \lambda_1(\Lambda_2) \}.$$

Proof. The first assertion is obvious. For the second assertion we choose, by Lemma 4.3, $d+1$ linearly independent elements $v_j = (w_j^{(1)}, w_j^{(2)}) \in \Lambda$ $(1 \leq j \leq d+1)$ with $|v_j| = \lambda_j$. Clearly, not all of them can lie in $\mathbb{R}^d \times \{\mathbf{0}\}$, and similarly not all of them can lie in $\{\mathbf{0}\} \times \mathbb{R}^d$. Suppose $v_{j_1} \notin \mathbb{R}^d \times \{\mathbf{0}\}$ and $v_{j_2} \notin \{\mathbf{0}\} \times \mathbb{R}^d$. Hence $|v_{j_1}| \geq |w_{j_1}^{(2)}| \geq \lambda_1(\Lambda_2)$ and $|v_{j_2}| \geq |w_{j_2}^{(1)}| \geq \lambda_1(\Lambda_1)$. This proves the lemma. $\square$

5. Adelic Lipschitz heights

In [9] Masser and Vaaler have introduced what one may call Lipschitz heights on $\mathbb{P}^n(K)$. This notion generalizes the absolute Weil height and allows so-called Lipschitz distance functions instead of just the maximum norm at the Archimedean places. Nonetheless, this notion is sometimes too rigid, as one often also needs modification at a finite number of non-Archimedean places. This leads naturally to the concept of adelic Lipschitz heights, introduced in [19].

5.1. Adelic Lipschitz systems on a number field

Let K be a number field and recall that M_K denotes the set of places of K, and that for every place w we have fixed a completion K_w of K at w. We write $d_w = [K_w : \mathbb{Q}_w]$ for the local degree, where $\mathbb{Q}_w$ denotes the completion of $\mathbb{Q}$ with respect to the unique place of $\mathbb{Q}$ that extends to w. The value set of w, $\Gamma_w := \{|\alpha|_w : \alpha \in K_w\}$ is equal to $[0, \infty)$ if w is Archimedean, and to

$$\left\{0, (\mathfrak{N}_K\mathfrak{P}_w)^0, (\mathfrak{N}_K\mathfrak{P}_w)^{\pm 1/d_w}, (\mathfrak{N}_K\mathfrak{P}_w)^{\pm 2/d_w}, \ldots\right\}$$

(topologized by the trivial topology) if w is a non-Archimedean place corresponding to the prime ideal $\mathfrak{P}_w$ of $\mathcal{O}_K$. For $w \mid \infty$ we identify K_w with $\mathbb{R}$ or $\mathbb{C}$, respectively, and we identify $\mathbb{C}$ with $\mathbb{R}^2$.

Definition 5.1. An adelic Lipschitz system $\mathcal{N}$ on K (of dimension n) is a set of continuous maps

$$N_w : K_w^{n+1} \to \Gamma_w \quad w \in M_K \tag{5.1}$$

such that for $w \in M_K$ we have:

(i) $N_w(\mathbf{z}) = 0$ if and only if $\mathbf{z} = \mathbf{0}$,
(ii) $N_w(a\mathbf{z}) = |a|_w N_w(\mathbf{z})$ for all $a \in K_w$ and all $\mathbf{z} \in K_w^{n+1}$,
(iii) if $w \mid \infty$: $\{\mathbf{z} : N_w(\mathbf{z}) = 1\}$ is in $Lip(d_w(n+1), M_w, L_w)$ for some M_w, L_w,
(iv) if $w \nmid \infty$: $N_w(\mathbf{z}_1 + \mathbf{z}_2) \leq \max\{N_w(\mathbf{z}_1), N_w(\mathbf{z}_2)\}$ for all $\mathbf{z}_1, \mathbf{z}_2 \in K_w^{n+1}$.

Moreover, we assume that the equality of functions

$$N_w(\mathbf{z}) = \max\{|z_0|_w, \ldots, |z_n|_w\} \tag{5.2}$$

holds for all but a finite number of $w \in M_K$.

If we consider only the functions N_w for $w \mid \infty$ then we get a Lipschitz system (of dimension n) in the sense of Masser and Vaaler [9].

For all $w \in M_K$ there are $c_w \leq 1$ in the value group $\Gamma_w^* = \Gamma_w \backslash \{0\}$ with

$$c_w \max\{|z_0|_w, \ldots, |z_n|_w\} \leq N_w(\mathbf{z}) \leq c_w^{-1} \max\{|z_0|_w, \ldots, |z_n|_w\} \tag{5.3}$$

for all $\mathbf{z} = (z_0, \ldots, z_n)$ in K_w^{n+1}. Due to (5.2) we can and will assume that

$$c_w = 1 \tag{5.4}$$

for all but a finite number of places w. We define

$$C_{\mathcal{N}}^{\text{fin}} := \prod_{w \nmid \infty} c_w^{-\frac{d_w}{[K:\mathbb{Q}]}} \geq 1, \tag{5.5}$$

and

$$C_{\mathcal{N}}^{\text{inf}} := \max_{w \mid \infty} \left\{ c_w^{-1} \right\} \geq 1. \tag{5.6}$$

For a prime ideal $\mathfrak{P}$ of $\mathcal{O}_K$ we write $v_{\mathfrak{P}}$ for the corresponding valuation on K, normalized by $v_{\mathfrak{P}}(K^*) = \mathbb{Z}$. For a nonzero fractional ideal $\mathfrak{A}$ of K and a non-Archimedean place w of K, associated to the prime $\mathfrak{P}$, we define

$$|\mathfrak{A}|_w := \mathfrak{N}_K(\mathfrak{P})^{-v_{\mathfrak{P}}(\mathfrak{A})/d_w},$$

so that $|\alpha|_w = |\alpha \mathcal{O}_K|_w$ for $\alpha \in K^*$. For $w \in M_K$ let σ_w be the canonical embedding of K in K_w, extended component-wise to K^{n+1}. For any nonzero $\boldsymbol{\omega} \in K^{n+1}$, let $i_{\mathcal{N}}(\boldsymbol{\omega})$ be the unique fractional ideal of K defined by

$$N_w(\sigma_w \boldsymbol{\omega}) = |i_{\mathcal{N}}(\boldsymbol{\omega})|_w$$

for all non-Archimedean $w \in M_K$, and we set by convention $i_{\mathcal{N}}(\mathbf{0}) := \{0\}$.

Moreover, set

$$\mathcal{O}_K(\boldsymbol{\omega}) := \omega_0 \mathcal{O}_K + \cdots + \omega_n \mathcal{O}_K,$$

so that $\mathcal{O}_K(\boldsymbol{\omega})$ is simply $i_{\mathcal{N}}(\boldsymbol{\omega})$ for any adelic Lipschitz system with (5.2) for all finite places. Now by (5.3) we get

$$c_w \max\{|\omega_0|_w, \ldots, |\omega_n|_w\} \leq |i_{\mathcal{N}}(\boldsymbol{\omega})|_w \leq c_w^{-1} \max\{|\omega_0|_w, \ldots, |\omega_n|_w\}. \tag{5.7}$$

Recall that $c_w = 1$ up to finitely many exceptions and let

$$F_{\mathcal{N}} := \{\mathfrak{A} : \mathfrak{A} \text{ nonzero fractional ideal of } K \text{ and } c_w \leq |\mathfrak{A}|_w \leq c_w^{-1} \text{ for all } w \nmid \infty\}.$$

By unique factorization of fractional ideals, $F_{\mathcal{N}}$ is finite. Moreover, for any $\boldsymbol{\omega} \in K^{n+1}$, we have

$$i_{\mathcal{N}}(\boldsymbol{\omega}) = \mathcal{O}_K(\boldsymbol{\omega}) \mathfrak{F}(\boldsymbol{\omega}) \tag{5.8}$$

for some $\mathfrak{F}(\boldsymbol{\omega}) \in F_{\mathcal{N}}$. Taking the product in (5.7) over all finite places with multiplicities d_w shows that

$$C_{\mathcal{N}}^{\text{fin}\,-[K:\mathbb{Q}]} \mathfrak{N}_K \mathcal{O}_K(\boldsymbol{\omega}) \leq \mathfrak{N}_K i_{\mathcal{N}}(\boldsymbol{\omega}) \leq C_{\mathcal{N}}^{\text{fin}\,[K:\mathbb{Q}]} \mathfrak{N}_K \mathcal{O}_K(\boldsymbol{\omega}). \tag{5.9}$$

5.2. Adelic Lipschitz heights on $\mathbb{P}^n(K)$

Let $\mathcal{N}$ be an adelic Lipschitz system on K of dimension n. Then the height $H_{\mathcal{N}}$ on K^{n+1} is defined by

$$H_{\mathcal{N}}(\boldsymbol{\omega}) := \prod_{w \in M_K} N_w(\sigma_w(\boldsymbol{\omega}))^{\frac{d_w}{[K:\mathbb{Q}]}}.$$

Thanks to the product formula and (ii) from Definition 5.1 $H_{\mathcal{N}}(\boldsymbol{\omega})$ is invariant under scalar multiplication by elements of K^*. Therefore $H_{\mathcal{N}}$ is well-defined on $\mathbb{P}^n(K)$ by setting

$$H_{\mathcal{N}}(P) := H_{\mathcal{N}}(\boldsymbol{\omega}),$$

where $P = (\omega_0 : \cdots : \omega_n) \in \mathbb{P}^n(K)$ and $\boldsymbol{\omega} = (\omega_0, \ldots, \omega_n) \in K^{n+1}$. We note that by (5.3), (5.5) and (5.6) we have

$$\left(C_{\mathcal{N}}^{\text{fin}} C_{\mathcal{N}}^{\text{inf}}\right)^{-1} H(P) \leq H_{\mathcal{N}}(P) \leq C_{\mathcal{N}}^{\text{fin}} C_{\mathcal{N}}^{\text{inf}} H(P), \tag{5.10}$$

where $H(P)$ denotes the projective absolute multiplicative Weil height of P. Hence, by Northcott's theorem, $\{P \in \mathbb{P}^n(K) : H_{\mathcal{N}}(P) \leq X\}$ is a finite set for each X in $[0, \infty)$.

6. The general theorem

Let $k \subseteq K$ be number fields and let $\mathcal{N}$ be an adelic Lipschitz system of dimension n on K. Recall that the functions N_w, n, and K are all part of the data of $\mathcal{N}$. From $\mathcal{N}$ we obtain an adelic Lipschitz height $H_{\mathcal{N}}$ on $\mathbb{P}^n(K)$. Our goal in this section is to derive an asymptotic formula for the counting function

$$N_{\mathcal{N}}(\mathbb{P}^n(k), X) := |\{P \in \mathbb{P}^n(k) : H_{\mathcal{N}}(P) \leq X\}|.$$

Let us set some necessary notation first. For each Archimedean place v of k we define a function N_v on k_v^{n+1} by

$$N_v(\mathbf{z}) := \prod_{w|v} N_w(\mathbf{z})^{\frac{d_w}{d_v[K:k]}}. \tag{6.1}$$

Let $\mathcal{N}' = \mathcal{N}'(\mathcal{N}, k)$ be the collection of functions N_v, where N_v is as in (6.1) if v is an Archimedean place of k and

$$N_v(\mathbf{z}) := \max\{|z_0|_v, \ldots, |z_n|_v\}$$

if v is a non-Archimedean place of k. From now on we assume that $\mathcal{N}'$ is an adelic Lipschitz system (of dimension n) on k (the conditions (i), (ii) and (iv) are automatically satisfied but (iii) may possibly fail). Hence there exists a positive integer $M_{\mathcal{N}'}$ and a positive real number $L_{\mathcal{N}'}$ such that the sets defined by $N_v(\mathbf{z}) = 1$ lie in $\mathrm{Lip}(d_v(n+1), M_{\mathcal{N}'}, L_{\mathcal{N}'})$ for all Archimedean places v of k. The sets defined by $N_v(\mathbf{z}) < 1$ are measurable and have a finite, positive volume, which we denote by V_v, and set

$$V_{\mathcal{N}'} := \prod_{v|\infty} V_v. \tag{6.2}$$

We denote by $\sigma_1, \dots, \sigma_d$ the embeddings from k to $\mathbb{R}$ or $\mathbb{C}$ respectively, ordered such that $\sigma_{r+s+i} = \overline{\sigma}_{r+i}$ for $1 \le i \le s$. We define

$$\begin{aligned} \sigma : k &\longrightarrow \mathbb{R}^r \times \mathbb{C}^s = \mathbb{R}^d \\ \sigma(\omega) &= \big(\sigma_1(\omega), \dots, \sigma_{r+s}(\omega)\big) \end{aligned} \tag{6.3}$$

and extend σ component-wise to get a map

$$\sigma : k^{n+1} \longrightarrow \mathbb{R}^m, \tag{6.4}$$

where $m = d(n+1)$.

For nonzero fractional ideals C of k, and $\mathfrak{D}$ of K, we define the following subsets of $\mathbb{R}^m$:

$$\begin{aligned} \Lambda_C^*(\mathfrak{D}) : &= \big\{\sigma(\boldsymbol{\omega}) : \boldsymbol{\omega} \in k^{n+1}, \mathcal{O}_k(\boldsymbol{\omega}) = C, i_{\mathcal{N}}(\boldsymbol{\omega}) = \mathfrak{D}\big\}, \\ \Lambda_C(\mathfrak{D}) : &= \big\{\sigma(\boldsymbol{\omega}) : \boldsymbol{\omega} \in k^{n+1}, \mathcal{O}_k(\boldsymbol{\omega}) = C, i_{\mathcal{N}}(\boldsymbol{\omega}) \subseteq \mathfrak{D}\big\}, \\ \Lambda(\mathfrak{D}) : &= \big\{\sigma(\boldsymbol{\omega}) : \boldsymbol{\omega} \in k^{n+1}, i_{\mathcal{N}}(\boldsymbol{\omega}) \subseteq \mathfrak{D}\big\}. \end{aligned}$$

Note that by (5.8) we have

$$\mathfrak{D} \in {}^{\mathrm{u}}C F_{\mathcal{N}} \tag{6.5}$$

whenever $\Lambda_C^*(\mathfrak{D})$ is non-empty, where ${}^{\mathrm{u}}C F_{\mathcal{N}}$ denotes the finite set of fractional ideals of the form ${}^{\mathrm{u}}C\mathfrak{F}$ with $\mathfrak{F} \in F_{\mathcal{N}}$.

Let $\mathcal{R}$ be a set of integral representatives for the class group Cl_k. For any $C \in \mathcal{R}$, we choose a finite set S_C of nonzero fractional ideals of K such that

$$S_C \text{ contains all } \mathfrak{D} \text{ with } \Lambda_C^*(\mathfrak{D}) \neq \emptyset.$$

Moreover, we choose a finite set T in the following way. For any $\mathfrak{D} \in S_C$, let $T_{C,\mathfrak{D}}$ be the set of all nonzero ideals $\mathfrak{A}$ of $\mathcal{O}_K$ such that $\Lambda_C(\mathfrak{A}\mathfrak{D}) \neq \emptyset$. This set is finite, since, similar as above, we have $\mathfrak{A}\mathfrak{D}\mathfrak{C} \in {}^{\mathrm{u}}C F_{\mathcal{N}}$ for some ideal $\mathfrak{C}$ of $\mathcal{O}_K$ whenever $\Lambda_C(\mathfrak{A}\mathfrak{D}) \neq \emptyset$. Then we choose T to be any finite set of nonzero ideals of $\mathcal{O}_K$ such that

$$T \text{ contains all the sets } T_{C,\mathfrak{D}} \text{ for } C \in \mathcal{R} \text{ and } \mathfrak{D} \in S_C.$$

We define

$$g_k^{\mathcal{N}} := \sum_{C\in\mathcal{R}} \sum_{\mathfrak{D}\in S_C} \sum_{\mathfrak{A}\in T} \mu_K(\mathfrak{A}) \sum_{E\trianglelefteq\mathcal{O}_k} \mu_k(E) \frac{\mathfrak{N}_K \mathfrak{D}^{\frac{n+1}{[K:k]}}}{\det \Lambda(\mathfrak{A}\mathfrak{D}, CE)}, \tag{6.6}$$

where

$$\Lambda(\mathfrak{A}\mathfrak{D}, CE) = \Lambda(\mathfrak{A}\mathfrak{D}) \cap \sigma\left((CE)^{n+1}\right).$$

Note that the infinite sum in (6.6) taken over all nonzero ideals E converges absolutely, as $\det \Lambda(\mathfrak{A}\mathfrak{D}, CE) \geq (2^{-s}\mathfrak{N}_k CE)^{n+1}$. Although $g_k^{\mathcal{N}}$ seems to depend on the choice of $\mathcal{R}$, S_C and T, we will see that this is actually not the case. Of course, one could impose a minimality condition to render the choice of the sets S_C and T unique, but for the calculation of $g_k^{\mathcal{N}}$ it is convenient to have more flexibility for the choices of these sets. From Theorem 6.1, (5.10), and Schanuel's theorem it will follow that $g_k^{\mathcal{N}} > 0$.

Finally, we define

$$A_{\mathcal{N}} := A_{\mathcal{N}}(k) := |F_{\mathcal{N}}| M_{\mathcal{N}'}^{r+s} \left(\left(L_{\mathcal{N}'} + C_{\mathcal{N}'}^{\inf}\right) C_{\mathcal{N}}^{\mathrm{fin}}\right)^{d(n+1)-1}. \tag{6.7}$$

We can now state the theorem.

Theorem 6.1. *Let $k \subseteq K$ be number fields, $d := [k : \mathbb{Q}]$, let $\mathcal{N}$ be an adelic Lipschitz system (of dimension n) on K, and suppose that $\mathcal{N}' = \mathcal{N}'(\mathcal{N}, k)$ is an adelic Lipschitz system (of dimension n) on k. Then, as $X \geq 1$ tends to infinity, we have*

$$N_{\mathcal{N}}\big(\mathbb{P}^n(k), X\big) = \omega_k^{-1}(n+1)^{r+s-1} R_k V_{\mathcal{N}'} g_k^{\mathcal{N}} X^{d(n+1)} + O\left(|T| A_{\mathcal{N}} X^{d(n+1)-1} \mathfrak{L}\right),$$

where $\mathfrak{L} = 1 + \log(C_{\mathcal{N}'}^{\inf} C_{\mathcal{N}}^{\mathrm{fin}} X)$ if $(n, d) = (1, 1)$ and $\mathfrak{L} = 1$ otherwise. The implicit constant in the O-term depends only on k and on n.

The hypothesis of $\mathcal{N}'$ being an adelic Lipschitz system is a minor one. For instance, this hypothesis is certainly fulfilled when the functions N_w of $\mathcal{N}$ are norms, as we shall prove in the appendix (see Lemma A.1).

7. Proof of Theorem 6.1

The proof of Theorem 6.1 makes frequent use of arguments from [9] and [19] (some of which can be traced back to [12], or even to Dedekind and Weber).

Let $q := r + s - 1$, Σ the hyperplane in $\mathbb{R}^{q+1}$ defined by $x_1 + \cdots + x_{q+1} = 0$ and $\boldsymbol{\delta} = (d_1, \ldots, d_{q+1})$ with $d_i = 1$ for $1 \leq i \leq r$ and $d_i = 2$ for $r + 1 \leq i \leq r + s = q + 1$. The map $l(\eta) := (d_1 \log|\sigma_1(\eta)|, \ldots, d_{q+1} \log|\sigma_{q+1}(\eta)|)$ sends k^* to $\mathbb{R}^{q+1}$. For $q > 0$ the image of the unit group $\mathcal{O}_k^*$ under l is a lattice in Σ with determinant $\sqrt{q+1} R_k$.

We now define a set $S_F(t)$ using our adelic Lipschitz system $\mathcal{N}'$ on k. Let F be a bounded set in Σ and for $t > 0$ let $F(t)$ be the vector sum

$$F(t) := F + \boldsymbol{\delta}(-\infty, \log t]. \tag{7.1}$$

We denote by exp the diagonal exponential map from $\mathbb{R}^{q+1}$ to $(0,\infty)^{q+1}$. Any embedding σ_i $(1 \leq i \leq q+1)$ corresponds to an Archimedean place v, and thus gives rise to one of our Lipschitz distance functions $N_i := N_v$ from $\mathcal{N}'$. We use variables $\mathbf{z}_1, \ldots, \mathbf{z}_{q+1}$ with $\mathbf{z}_i$ in $\mathbb{R}^{d_i(n+1)}$. Exactly as in [9] we define $S_F(t)$ in $\mathbb{R}^m$ for $m = \sum_{i=1}^{q+1} d_i(n+1) = d(n+1)$ as the set of all $\mathbf{z}_1, \ldots, \mathbf{z}_{q+1}$ such that

$$\left(N_1(\mathbf{z}_1)^{d_1}, \ldots, N_{q+1}(\mathbf{z}_{q+1})^{d_{q+1}}\right) \in \exp(F(t)). \tag{7.2}$$

We note that

$$\mathbf{0} \notin S_F(t). \tag{7.3}$$

Using (ii) from Definition 5.1 it is easily seen that $S_F(t)$ is homogeneously expanding, *i.e.*,

$$S_F(t) = tS_F(1). \tag{7.4}$$

Moreover, if F lies in a zero-centered ball of radius r_F then

$$S_F(t) \subseteq \left\{\left(\mathbf{z}_1, \ldots, \mathbf{z}_{q+1}\right) : N_i(\mathbf{z}_i) \leq \exp(r_F)t \text{ for } 1 \leq i \leq q+1\right\}.$$

The latter set lies in the the zero-centered ball of radius $\sqrt{m}C_{\mathcal{N}'}^{\inf}\exp(r_F)t$, and thus

$$S_F(t) \subseteq B_0\left(\sqrt{m}C_{\mathcal{N}'}^{\inf}\exp(r_F)t\right). \tag{7.5}$$

Note that for $q = 0$ we automatically have $F = \{0\}$, and our set $S_F(t)$ is precisely the set defined by $N_1(\mathbf{z}) \leq t$.

We now specify our set F when $q > 0$. We choose a basis $u_1, \ldots, u_q$ of the lattice $l(O_k^*)$ as in Lemma 4.4. Set $F := [0,1)u_1 + \cdots + [0,1)u_q$. So F is measurable of (q-dimensional) volume

$$\mathrm{Vol}(F) = \sqrt{q+1}R_k \tag{7.6}$$

(and this remains true for $q = 0$). From the argument in [19] following (8.2), we see that $\lambda_1(l(\mathcal{O}_k^*)) \geq c_d$ for some positive constant c_d depending only on d. With the estimate from Lemma 4.4, we get

$$|u_i| \leq C_0(q)c_d^{-q+1}\,\mathrm{Vol}(F) \leq C_d R_k, \quad (1 \leq i \leq q) \tag{7.7}$$

for some positive constant C_d depending only on d. Note that F lies in the zero centered ball of radius qC_dR_k, and this remains trivially true for $q = 0$. Therefore by (7.5)

$$S_F(t) \subseteq B_0(\kappa t), \tag{7.8}$$

where

$$\kappa := \sqrt{m} C_{\mathcal{N}'}^{\inf} \exp(qC_dR_k). \tag{7.9}$$

Lemma 7.1. *There exists a constant $c_k(n)$ depending only on k and n, a positive integer $\widetilde{M}$, and a positive real $\widetilde{L}$ with $\widetilde{M} \leq c_k(n) M_{\mathcal{N}'}^{q+1}$, $\widetilde{L} \leq c_k(n)(L_{\mathcal{N}'} + C_{\mathcal{N}'}^{\inf})$, such that*

$$\partial S_F(t) \in \operatorname{Lip}(m, \widetilde{M}, \widetilde{L}t) \text{ and } S_F(t) \subseteq B_0(\widetilde{L}t). \tag{7.10}$$

Proof. The second part follows immediately from (7.8) and (7.9).

Let us now prove the first part. For $q = 0$ our set $S_F(t)$ is precisely the set defined by $N_v(\mathbf{z}) \leq t$, where v is the single Archimedean place of k. So the boundary of $S_F(t)$ is the set $\{\mathbf{z} : N_v(\mathbf{z}) = t\} = t\{\mathbf{z} : N_v(\mathbf{z}) = 1\}$. By assumption $\mathcal{N}'$ is an adelic Lipschitz system, and thus the latter set lies in $\operatorname{Lip}(m, M_{\mathcal{N}'}, L_{\mathcal{N}'}t)$. This proves the lemma for $q = 0$.

Suppose now that $q \geq 1$. Then we can find $2q$ linear maps $\psi_i : [0, 1]^{q-1} \to \Sigma$ parameterizing ∂F that, because of (7.7), will satisfy a Lipschitz condition with constant $(q-1)C_dR_k$ (for $q = 1$ this is simply interpreted as $|\partial F| \leq 2$). The claim now follows from [19, Lemma 7.1] by a simple computation. □

We conclude from [9, Lemma 4], (7.6), and (7.4) that $S_F(t)$ is measurable and has volume

$$\operatorname{Vol} S_F(t) = (n+1)^q R_k V_{\mathcal{N}'} t^m. \tag{7.11}$$

Lemma 7.2. *We have*

$$N_{\mathcal{N}}\left(\mathbb{P}^n(k), X\right) = \omega_k^{-1} \sum_{C \in \mathcal{R}} \sum_{\mathfrak{D} \in S_C} \left| \Lambda_C^*(\mathfrak{D}) \cap S_F\left(X \mathfrak{N}_K \mathfrak{D}^{1/[K:\mathbb{Q}]}\right) \right|.$$

Proof. Let $P \in \mathbb{P}^n(k)$ with homogeneous coordinates $(\omega_0, \ldots, \omega_n) = \boldsymbol{\omega} \in k^{n+1} \backslash \{\mathbf{0}\}$. Recall the definition of the adelic Lipschitz system $\mathcal{N}'$. The functions N_v (or N_i) will denote those associated with $\mathcal{N}'$, whereas N_w will denote a function associated with the adelic Lipschitz system $\mathcal{N}$ on K.

Now

$$i_{\mathcal{N}'}(\boldsymbol{\omega}) = \mathcal{O}_k(\boldsymbol{\omega}) \tag{7.12}$$

Suppose $\varepsilon \in k^*$. Then we have

$$i_{\mathcal{N}'}(\varepsilon \boldsymbol{\omega}) = \varepsilon i_{\mathcal{N}'}(\boldsymbol{\omega}).$$

Hence the ideal class of $i_{\mathcal{N}'}(\boldsymbol{\omega})$ is independent of the coordinates $\boldsymbol{\omega}$ we have chosen. In particular, we can choose $\boldsymbol{\omega}$ such that $i_{\mathcal{N}'}(\boldsymbol{\omega}) = C$ for some unique C in $\mathcal{R}$. Thus, $\boldsymbol{\omega}$ is unique up to scalar multiplication by units η, and moreover, $i_{\mathcal{N}}(\boldsymbol{\omega}) := \mathfrak{D} \in S_C$. The set $F(\infty) = F + \mathbb{R}\boldsymbol{\delta}$ is a fundamental set of $\mathbb{R}^{q+1}$ under the action of the additive subgroup $l(\mathcal{O}_k^*)$. Because of Definition 5.1, (ii) we have

$$\log N_i\big(\sigma_i(\eta\boldsymbol{\omega})\big)^{d_i} = \log N_i(\sigma_i\boldsymbol{\omega})^{d_i} + d_i \log|\sigma_i\eta|$$

for $1 \le i \le q+1$. Hence, there exist exactly ω_k representatives $\boldsymbol{\omega}$ of P with

$$\big(d_1 \log N_1(\sigma_1\boldsymbol{\omega}), \ldots, d_{q+1} \log N_{q+1}(\sigma_{q+1}\boldsymbol{\omega})\big) \in F(\infty).$$

But the above is equivalent with

$$\big(N_1(\sigma_1\boldsymbol{\omega})^{d_1}, \ldots, N_{q+1}(\sigma_{q+1}\boldsymbol{\omega})^{d_{q+1}}\big) \in \exp(F(\infty)).$$

Furthermore

$$\exp\big(F(t_0)\big) = \big\{(X_1, \ldots, X_{q+1}) \in \exp(F(\infty)) : X_1 \cdots X_{q+1} \le t_0^d\big\}.$$

Hence, for all ω_k representatives $\boldsymbol{\omega}$ of P as above, the inequality

$$\prod_{v|\infty} N_v(\sigma_v(\boldsymbol{\omega}))^{d_v/d} = \prod_{v|\infty}\prod_{w|v} N_w(\sigma_w(\boldsymbol{\omega}))^{d_w/[K:\mathbb{Q}]} \le t_0$$

is equivalent to

$$\sigma\boldsymbol{\omega} \in S_F(t_0).$$

On the other hand,

$$\prod_{w\nmid\infty} N_w(\sigma_w(\boldsymbol{\omega}))^{d_w/[K:\mathbb{Q}]} = \mathfrak{N}_K i_{\mathcal{N}}(\boldsymbol{\omega})^{-1/[K:\mathbb{Q}]} = \mathfrak{N}_K \mathfrak{D}^{-1/[K:\mathbb{Q}]}.$$

As

$$H_{\mathcal{N}}(P) = \prod_{v|\infty}\prod_{w|v} N_w(\sigma_w(\boldsymbol{\omega}))^{d_w/[K:\mathbb{Q}]} \prod_{w\nmid\infty} N_w(\sigma_w(\boldsymbol{\omega}))^{d_w/[K:\mathbb{Q}]},$$

the claim follows. □

Lemma 7.3. *We have*

$$N_{\mathcal{N}}(\mathbb{P}^n(k), X) = \omega_k^{-1} \sum_{C\in\mathcal{R}} \sum_{\mathfrak{D}\in S_C} \sum_{\mathfrak{A}\in T} \mu_K(\mathfrak{A}) \sum_{E\trianglelefteq\mathcal{O}_k} \mu_k(E) \left|\Lambda(\mathfrak{A}\mathfrak{D}, CE) \cap S_F\left(X\mathfrak{N}_K\mathfrak{D}^{1/[K:\mathbb{Q}]}\right)\right|,$$

where E runs over all nonzero ideals of $\mathcal{O}_k$.

Proof. We start off from Lemma 7.2 and we apply Möbius inversion twice to get rid of the two coprimality conditions $_C$ and *.

Directly from the definition we get

$$\Lambda_C(\mathfrak{A}\mathfrak{D}) = \bigcup_{\mathfrak{B}} \Lambda_C^*(\mathfrak{A}\mathfrak{B}\mathfrak{D}),$$

where $\mathfrak{B}$ runs over all nonzero ideals of $\mathcal{O}_K$. This is clearly a disjoint union. Note that $\Lambda_C^*(\mathfrak{A}\mathfrak{B}\mathfrak{D}) \neq \emptyset$ only when $\mathfrak{A}\mathfrak{B}\mathfrak{D}$ lies in the finite set S_C. Möbius inversion leads then to

$$\begin{aligned}\left|\Lambda_C^*(\mathfrak{D}) \cap S_F\left(X\mathfrak{N}_K\mathfrak{D}^{1/[K:\mathbb{Q}]}\right)\right| &= \sum_{\mathfrak{A}} \mu_K(\mathfrak{A}) \sum_{\mathfrak{B}} \left|\Lambda_C^*(\mathfrak{A}\mathfrak{B}\mathfrak{D}) \cap S_F\left(X\mathfrak{N}_K\mathfrak{D}^{1/[K:\mathbb{Q}]}\right)\right| \\ &= \sum_{\mathfrak{A}} \mu_K(\mathfrak{A}) \left|\Lambda_C(\mathfrak{A}\mathfrak{D}) \cap S_F\left(X\mathfrak{N}_K\mathfrak{D}^{1/[K:\mathbb{Q}]}\right)\right|,\end{aligned}$$

where the sums run over all nonzero ideals in $\mathcal{O}_K$. Next note that by definition of $T_{C,\mathfrak{D}}$ we have $\Lambda_C(\mathfrak{A}\mathfrak{D}) = \emptyset$ whenever $\mathfrak{A} \notin T_{C,\mathfrak{D}}$. As $T_{C,\mathfrak{D}} \subseteq T$ we can restrict the last sum to $\mathfrak{A} \in T$ and we get

$$\left|\Lambda_C^*(\mathfrak{D}) \cap S_F\left(X\mathfrak{N}_K\mathfrak{D}^{1/[K:\mathbb{Q}]}\right)\right| = \sum_{\mathfrak{A}\in T} \mu_K(\mathfrak{A}) \left|\Lambda_C(\mathfrak{A}\mathfrak{D}) \cap S_F\left(X\mathfrak{N}_K\mathfrak{D}^{1/[K:\mathbb{Q}]}\right)\right|.$$

We now deal with the second coprimality condition $_C$. Also directly from the definition we get

$$\Lambda(\mathfrak{A}\mathfrak{D}, EC) = \Lambda(\mathfrak{A}\mathfrak{D}) \cap \sigma\left((EC)^{n+1}\right) = \bigcup_{B \trianglelefteq \mathcal{O}_k} \Lambda_{ECB}(\mathfrak{A}\mathfrak{D}) \cup \{0\}.$$

Again, B runs over all nonzero ideals of $\mathcal{O}_k$ and the union is disjoint. As $\sigma((EC)^{n+1})$ is a lattice and $S_F(X\mathfrak{N}_K\mathfrak{D}^{1/[K:\mathbb{Q}]})$ is bounded we conclude from the latter equality that $\Lambda_{ECB}(\mathfrak{A}\mathfrak{D}) \cap S_F(X\mathfrak{N}_K\mathfrak{D}^{1/[K:\mathbb{Q}]})$ is empty for all but finitely many B. Möbius inversion and (7.3) lead therefore to

$$\begin{aligned}&\left|\Lambda_C(\mathfrak{A}\mathfrak{D}) \cap S_F\left(X\mathfrak{N}_K\mathfrak{D}^{1/[K:\mathbb{Q}]}\right)\right| \\ &= \sum_{E \trianglelefteq \mathcal{O}_k} \mu_k(E) \sum_{B \trianglelefteq \mathcal{O}_k} \left|\Lambda_{ECB}(\mathfrak{A}\mathfrak{D}) \cap S_F\left(X\mathfrak{N}_K\mathfrak{D}^{1/[K:\mathbb{Q}]}\right)\right| \\ &= \sum_{E \trianglelefteq \mathcal{O}_k} \mu_k(E) \left|\Lambda(\mathfrak{A}\mathfrak{D}, CE) \cap S_F\left(X\mathfrak{N}_K\mathfrak{D}^{1/[K:\mathbb{Q}]}\right)\right|.\end{aligned}$$

In view of Lemma 7.2 this proves the claim. □

We choose a positive real Γ such that for any $C \in \mathcal{R}$ and any $\mathfrak{D} \in S_C$

$$\Gamma \leq \frac{\mathfrak{N}_k C}{\mathfrak{N}_K(\mathfrak{D})^{1/[K:k]}}. \tag{7.13}$$

Before we proceed note that if S_C is chosen minimal for all $C \in \mathcal{R}$ (*i.e.* $S_C = \{i_{\mathcal{N}}(\boldsymbol{\omega}) : \boldsymbol{\omega} \in k^{n+1}, \mathcal{O}_k(\boldsymbol{\omega}) = C\}$) then it follows from (5.9) that we can choose $\Gamma = C_{\mathcal{N}}^{\mathrm{fin}\,-d}$, and moreover, $|S_C| \leq |F_{\mathcal{N}}|$.

Lemma 7.4. *Let $\lambda_1 = \lambda_1(\Lambda(\mathfrak{A}\mathfrak{D}, CE))$ be the first successive minimum of the lattice $\Lambda(\mathfrak{A}\mathfrak{D}, CE)$, and let $\widetilde{M}$ and $\widetilde{L}$ be as in Lemma* 7.1. *Then we have*

$$\left|\Lambda(\mathfrak{A}\mathfrak{D}, CE) \cap S_F\left(X\mathfrak{N}_K\mathfrak{D}^{1/[K:\mathbb{Q}]}\right)\right| = \frac{\operatorname{Vol} S_F(1)\mathfrak{N}_K\mathfrak{D}^{\frac{n+1}{[K:k]}}X^m}{\det \Lambda(\mathfrak{A}\mathfrak{D}, CE)} + O\left(\widetilde{M}\frac{\mathfrak{N}_K\mathfrak{D}^{\frac{m-1}{[K:\mathbb{Q}]}}(\widetilde{L}X)^{m-1}}{\lambda_1^{m-1}}\right),$$

where the constant in the O-term depends only on m. Moreover, with Γ as in (7.13) *we have*

$$\lambda_1 \geq \mathfrak{N}_K(\mathfrak{D})^{1/[K:\mathbb{Q}]}\Big(\Gamma\mathfrak{N}_k(E)\Big)^{1/d}.$$

And finally, with κ as in (7.9), *if $\mathfrak{N}_k E > (\kappa X)^d/\Gamma$ then*

$$\Lambda(\mathfrak{A}\mathfrak{D}, CE) \cap S_F\left(X\mathfrak{N}_K\mathfrak{D}^{1/[K:\mathbb{Q}]}\right) = \emptyset.$$

Proof. For the first assertion we use (7.3) and apply Lemma 4.2 with $a = m$. Thanks to (7.8) and Lemma 7.1 the required conditions are satisfied, and using (7.4) the first result drops out.

Now for the second statement we first observe that λ_1 is at least as large as the first successive minimum of the lattice $\sigma(CE)$. But it is well-known that the latter is at least $\mathfrak{N}_k(CE)^{1/d}$, see, *e.g.*, [9, Lemma 5]. Now as $\mathfrak{D} \in S_C$ and by the definition of Γ we get $\mathfrak{N}_k C \geq \Gamma\mathfrak{N}_K(\mathfrak{D})^{1/[K:k]}$ and this yields the second assertion.

The last claim follows upon combining the above estimate for λ_1 with (7.3), (7.8). ☐

We can now conclude the proof of Theorem 6.1. Let us first assume that $(n, d) \neq (1, 1)$. Combining Lemma 7.3, Lemma 7.4 and (7.11) gives the main term as in Theorem 6.1. The error term is bounded by

$$\begin{aligned}
&\sum_{C\in\mathcal{R}}\sum_{\mathfrak{D}\in S_C}\sum_{\mathfrak{A}\in T}\sum_{E\trianglelefteq\mathcal{O}_k} O\left(\widetilde{M}\frac{\mathfrak{N}_K\mathfrak{D}^{\frac{m-1}{[K:\mathbb{Q}]}}(\widetilde{L}X)^{m-1}}{\lambda_1^{m-1}}\right)\\
\leq &\sum_{C\in\mathcal{R}}\sum_{\mathfrak{D}\in S_C}\sum_{\mathfrak{A}\in T}\sum_{E\trianglelefteq\mathcal{O}_k} O\left(\frac{\widetilde{M}(\widetilde{L}X)^{m-1}}{\Gamma^{(m-1)/d}\mathfrak{N}_k E^{(n+1)-1/d}}\right)\\
\leq &\sum_{C\in\mathcal{R}}\sum_{\mathfrak{D}\in S_C}\sum_{\mathfrak{A}\in T} O\left(\frac{\widetilde{M}(\widetilde{L}X)^{m-1}}{\Gamma^{(m-1)/d}}\right)\\
= &O\left(\sum_{C\in\mathcal{R}}|S_C||T|\frac{\widetilde{M}(\widetilde{L}X)^{m-1}}{\Gamma^{(m-1)/d}}\right).
\end{aligned}$$

This proves the theorem in the case $(n, d) \neq (1, 1)$ except that the constant in the error term is different from the one in the statement of the theorem. In particular, it shows that the main term is independent of the particular choice of the sets S_C. However, if we choose all the sets S_C to be minimal then, by the remark just after (7.13), we can choose $\Gamma = C_{\mathcal{N}}^{fin^{-d}}$, and $|S_C| \leq |F_{\mathcal{N}}|$. This, and not forgetting the definition of $\widetilde{M}$ and $\widetilde{L}$ from Lemma 7.1, yields the desired error term.

We now assume $(n, d) = (1, 1)$ (which of course means $k = \mathbb{Q}$, $\mathcal{R} = \{C\}$, $\omega_k = 2$). Using also the last part of Lemma 7.4 we conclude

$$
\begin{aligned}
&N_{\mathcal{N}}(\mathbb{P}^1(\mathbb{Q}), X) = \\
&\frac{1}{2}\sum_{\mathfrak{D}\in S_C}\sum_{\mathfrak{A}\in T}\mu_K(\mathfrak{A}) \sum_{\substack{E\trianglelefteq\mathbb{Z}\\ \mathfrak{N}_{\mathbb{Q}}E\leq\kappa X/\Gamma}} \mu_{\mathbb{Q}}(E)\left|\Lambda(\mathfrak{A}\mathfrak{D}, CE)\cap S_F\left(X\mathfrak{N}_K\mathfrak{D}^{1/[K:\mathbb{Q}]}\right)\right| \\
&= \frac{1}{2}\sum_{\mathfrak{D}\in S_C}\sum_{\mathfrak{A}\in T}\mu_K(\mathfrak{A})\sum_{E\trianglelefteq\mathbb{Z}}\mu_{\mathbb{Q}}(E)\frac{\mathrm{Vol}\, S_F(1)\mathfrak{N}_K\mathfrak{D}^{\frac{2}{[K:\mathbb{Q}]}}X^2}{\det\Lambda(\mathfrak{A}\mathfrak{D}, CE)} \\
&+ O\left(\sum_{\mathfrak{D}\in S_C}\sum_{\mathfrak{A}\in T}\sum_{\substack{E\trianglelefteq\mathbb{Z}\\ \mathfrak{N}_{\mathbb{Q}}E>\kappa X/\Gamma}}\frac{\mathrm{Vol}\, S_F(1)\mathfrak{N}_K\mathfrak{D}^{\frac{2}{[K:\mathbb{Q}]}}X^2}{\det\Lambda(\mathfrak{A}\mathfrak{D}, CE)}\right) \\
&+ O\left(\sum_{\mathfrak{D}\in S_C}\sum_{\mathfrak{A}\in T}\sum_{\substack{E\trianglelefteq\mathbb{Z}\\ \mathfrak{N}_{\mathbb{Q}}E\leq\kappa X/\Gamma}}\frac{\widetilde{M}\mathfrak{N}_K\mathfrak{D}^{\frac{1}{[K:\mathbb{Q}]}}\widetilde{L}X}{\lambda_1}\right).
\end{aligned}
$$

Now the first term gives the main term as before. For the second term we use Minkowski's first theorem to estimate the determinant in terms of λ_1, and then a simple computation using Lemma 7.4 and (7.8) gives the error term $O(|S_C||T|(1 + \kappa X/\Gamma)$. For the last error term we use again Lemma 7.4, and again a simple computation yields the error term

$$O(|S_C||T|(\widetilde{M}\widetilde{L}/\Gamma)X(1 + \log(\kappa X/\Gamma)).$$

To get the right error term we choose again S_C to be minimal so that we can take $\Gamma = C_{\mathcal{N}}^{fin^{-1}}$, and $|S_C| \leq |F_{\mathcal{N}}|$. This proves Theorem 6.1.

8. Proof of Theorem 1.4

In this section, we deduce Theorem 1.4 from Theorem 6.1. Recall the simple facts mentioned just before Lemma 3.2.

As mentioned after Lemma 2.4, we can and will assume that θ is an algebraic integer. Let $K := k(\theta)$, and let $\mathcal{N}$ be the adelic Lipschitz system on K of dimension

n defined by

$$N_w(\mathbf{z}) := \max\{|z_0|_w, |\theta|_w|z_1|_w, \dots, |\theta|_w|z_n|_w\},$$

so

$$i_{\mathcal{N}}(\boldsymbol{\omega}) = \omega_0\mathcal{O}_K + \theta\omega_1\mathcal{O}_K + \cdots + \theta\omega_n\mathcal{O}_K. \tag{8.1}$$

Lemma 8.1. *We have*

$$N\big(\theta k^n, X\big) = N_{\mathcal{N}}\big(\mathbb{P}^n(k), X\big) + O\left(X^{nd}\right),$$

where the implicit constant in the error term depends only on k, θ, and n.

Proof. The points $\boldsymbol{\alpha} = (\omega_1/\omega_0, \dots, \omega_n/\omega_0) \in k^n$ with $H(\theta\boldsymbol{\alpha}) \le X$ are in one-to-one correspondence with the projective points $P = (\omega_0 : \cdots : \omega_n) \in \mathbb{P}^n(k)$ with $\omega_0 \ne 0$ and $H_{\mathcal{N}}(P) \le X$.

If $n > 1$ then we can apply Theorem 6.1 with $n-1$ and the adelic Lipschitz system given by the norm functions (see Lemma A.1 in the appendix)

$$N_w((z_1, \dots, z_n)) := \max\{|\theta|_w|z_1|_w, \dots, |\theta|_w|z_n|_w\} \tag{8.2}$$

(with $\mathcal{R}$, S_C and T chosen in such a way that $|T|$ is minimal) to see that the number of such points P with $\omega_0 = 0$ is $O(X^{nd})$. This trivially remains true for $n = 1$. □

Since the functions N_w are norms, the adelic Lipschitz system $\mathcal{N}$ satisfies the hypothesis of Theorem 6.1. As our choice of $\mathcal{R}$, S_C and T in Theorem 6.1 will depend only on k, n and θ, we obtain

$$N_{\mathcal{N}}(\mathbb{P}^n(k), X) = \omega_k^{-1}(n+1)^{r+s-1}R_k V_{\mathcal{N}'} g_k^{\mathcal{N}} X^{d(n+1)} + O\left(X^{d(n+1)-1}\mathfrak{L}\right), \tag{8.3}$$

where $\mathfrak{L} := \log(X+1)$ if $(n, d) = (1, 1)$ and $\mathfrak{L} := 1$ otherwise. The implicit constant in the error term depends only on k, θ, and n.

We notice that

$$V_{\mathcal{N}'} = (2^r\pi^s)^{n+1}V(\theta, k, n), \tag{8.4}$$

with $V(\theta, k, n)$ as in (1.3). To prove the theorem, we need to compute $g_k^{\mathcal{N}}$. First we choose the sets $\mathcal{R}$, S_C and T. Denote

$$D := {}^{\mathfrak{d}}(\theta\mathcal{O}_K).$$

For $\mathcal{R}$ we choose any system of integral representatives for the class group Cl_k with

$$(C, D) = \mathcal{O}_k \text{ for all } C \in \mathcal{R}. \tag{8.5}$$

We will see in Lemma 8.2, (i), that

$$S_C := \{{}^{\mathfrak{u}}C(\theta\mathcal{O}_K, {}^{\mathfrak{u}}B) : B \trianglelefteq \mathcal{O}_k, B \mid D\} \tag{8.6}$$

is a valid choice for S_C. For T, we take the finite set

$$T := \bigcup_{C\in\mathcal{R}}\bigcup_{\mathfrak{D}\in S_C} T_{C,\mathfrak{D}} \cup \{\mathfrak{A} \trianglelefteq \mathcal{O}_K : \mathfrak{A} \mid \theta\mathcal{O}_K\}. \tag{8.7}$$

Lemma 8.2.

(i) *Let* $\boldsymbol{\omega} \in k^{n+1}$ *with* $\mathcal{O}_k(\boldsymbol{\omega}) = C$. *Then* $i_{\mathcal{N}}(\boldsymbol{\omega}) \in S_C$.
(ii) *Let* $\mathfrak{A}$ *be an ideal of* $\mathcal{O}_K$ *and* B *an ideal of* $\mathcal{O}_k$. *Then* ${}^{\mathfrak{d}}(\mathfrak{A}, {}^{\mathfrak{u}}B) = ({}^{\mathfrak{d}}\mathfrak{A}, B)$.
(iii) *Let* B *be an ideal of* $\mathcal{O}_k$ *with* $B \mid D$. *Then* ${}^{\mathfrak{d}}(\theta\mathcal{O}_K, {}^{\mathfrak{u}}B) = B$.

Proof. (i): We have $\omega_0\mathcal{O}_K + \cdots + \omega_n\mathcal{O}_K = {}^{\mathfrak{u}}\mathcal{O}_k(\boldsymbol{\omega}) = {}^{\mathfrak{u}}C$, so

$$\begin{aligned} i_{\mathcal{N}}(\boldsymbol{\omega}) &= {}^{\mathfrak{u}}C\left(\omega_0({}^{\mathfrak{u}}C)^{-1} + \theta\left(\omega_1({}^{\mathfrak{u}}C)^{-1} + \cdots + \omega_n({}^{\mathfrak{u}}C)^{-1}\right)\right) \\ &= {}^{\mathfrak{u}}C\left(\omega_0({}^{\mathfrak{u}}C)^{-1}, \theta\mathcal{O}_K\right). \end{aligned}$$

Moreover, since $\theta\mathcal{O}_K \mid {}^{\mathfrak{u}}D$, we obtain

$$\left(\omega_0({}^{\mathfrak{u}}C)^{-1}, \theta\mathcal{O}_K\right) = \left(\omega_0({}^{\mathfrak{u}}C)^{-1}, {}^{\mathfrak{u}}D, \theta\mathcal{O}_K\right) = (\theta\mathcal{O}_K, {}^{\mathfrak{u}}B),$$

for $B := (\omega_0 C^{-1}, D) \mid D$.

(ii): Let P be a prime ideal of $\mathcal{O}_k$ and ${}^{\mathfrak{u}}P = \prod_{\mathfrak{P}} \mathfrak{P}^{e_{\mathfrak{P}}}$ its factorization in $\mathcal{O}_K$. Then

$$\begin{aligned} v_P({}^{\mathfrak{d}}(\mathfrak{A}, {}^{\mathfrak{u}}B)) &= \max_{\mathfrak{P}}\{\lceil \min\{v_{\mathfrak{P}}(\mathfrak{A}), v_{\mathfrak{P}}({}^{\mathfrak{u}}B)\}/e_{\mathfrak{P}}\rceil\} \\ &= \max_{\mathfrak{P}}\{\min\{\lceil v_{\mathfrak{P}}(\mathfrak{A})/e_{\mathfrak{P}}\rceil, v_P(B)\}\} \\ &= \min\left\{\max_{\mathfrak{P}}\{\lceil v_{\mathfrak{P}}(\mathfrak{A})/e_{\mathfrak{P}}\rceil\}, v_P(B)\right\} = v_P\left(({}^{\mathfrak{d}}\mathfrak{A}, B)\right). \end{aligned}$$

(iii): By (ii), we have ${}^{\mathfrak{d}}(\theta\mathcal{O}_K, {}^{\mathfrak{u}}B) = (D, B) = B$. □

The first step in our computation of $g_k^{\mathcal{N}}$ is to evaluate the determinant of the lattice $\Lambda(\mathfrak{A}\mathfrak{D}, CE) = \Lambda(\mathfrak{A}\mathfrak{D}) \cap \sigma((CE)^{n+1})$.

Lemma 8.3. *Let* $\mathfrak{A}$, B *be nonzero ideals of* $\mathcal{O}_K$ *and* $\mathcal{O}_k$, *respectively. Then*

$$\det \Lambda(\mathfrak{A}, B) = \left(2^{-s}\sqrt{|\Delta_k|}\right)^{n+1} \cdot \mathfrak{N}_k({}^{\mathfrak{d}}\mathfrak{A} \cap B) \cdot \mathfrak{N}_k\left({}^{\mathfrak{d}}\left(\mathfrak{A}(\theta\mathcal{O}_K, \mathfrak{A})^{-1}\right) \cap B\right)^n.$$

Proof. Let $\boldsymbol{\omega} = (\omega_0, \ldots, \omega_n) \in k^n$. Clearly, $\sigma\boldsymbol{\omega} \in \Lambda(\mathfrak{A}, B)$ if and only if $\omega_i \in B$ for all $0 \le i \le n$, $\omega_0 \in \mathfrak{A}$, and $\theta\omega_i \in \mathfrak{A}$ for all $1 \le i \le n$. For $\omega_i \in \mathcal{O}_k$, we have

$$\theta\omega_i \in \mathfrak{A} \quad \text{if and only if} \quad \mathfrak{A}(\theta\mathcal{O}_K, \mathfrak{A})^{-1} \mid \omega_i\mathcal{O}_K.$$

Therefore, we obtain

$$\Lambda(\mathfrak{A}, B) = \sigma\left(({}^{\mathfrak{d}}\mathfrak{A} \cap B) \times \left({}^{\mathfrak{d}}\left(\mathfrak{A}(\theta\mathcal{O}_K, \mathfrak{A})^{-1}\right) \cap B\right)^n\right).$$ □

Let $\mathfrak{A} \in T$ and let B be an ideal of $\mathcal{O}_k$ with $B \mid D$. To facilitate further notation, we define ideals A and A_1 of $\mathcal{O}_k$ by

$$A = A(\mathfrak{A}, B) := {}^{\mathfrak{d}}\left(\mathfrak{A}(\theta\mathcal{O}_K, {}^{\mathrm{u}}B)\right) \quad \text{and} \tag{8.8}$$

$$A_1 = A_1(\mathfrak{A}, B) := {}^{\mathfrak{d}}\left(\mathfrak{A}(\theta\mathcal{O}_K, {}^{\mathrm{u}}B)(\theta\mathcal{O}_K, \mathfrak{A}{}^{\mathrm{u}}B)^{-1}\right) \mid A. \tag{8.9}$$

For any $\mathfrak{D} = {}^{\mathrm{u}}C(\theta\mathcal{O}_K, {}^{\mathrm{u}}B) \in S_C$ and for any nonzero ideal E of $\mathcal{O}_k$ we have

$$\mathfrak{N}_k({}^{\mathfrak{d}}(\mathfrak{A}\mathfrak{D}) \cap CE) = \mathfrak{N}_k C \cdot \mathfrak{N}_k(A \cap E). \tag{8.10}$$

Clearly, we have $(\theta\mathcal{O}_K, \mathfrak{A}(\theta\mathcal{O}_K, {}^{\mathrm{u}}B)) = (\theta\mathcal{O}_K, \mathfrak{A}{}^{\mathrm{u}}B)$. Furthermore, by our choice of $\mathcal{R}$ with (8.5), we have $({}^{\mathrm{u}}C, \theta\mathcal{O}_K) = \mathcal{O}_K$. Therefore, we obtain

$$\mathfrak{N}_k\left({}^{\mathfrak{d}}\left((\mathfrak{A}\mathfrak{D})(\theta\mathcal{O}_K, \mathfrak{A}\mathfrak{D})^{-1}\right) \cap CE\right) = \mathfrak{N}_k C \cdot \mathfrak{N}_k\left(A_1 \cap E\right). \tag{8.11}$$

Moreover, we have

$$\mathfrak{N}_K \mathfrak{D}^{(n+1)/[K:k]} = \mathfrak{N}_k C^{n+1} \cdot \mathfrak{N}_K(\theta\mathcal{O}_K, {}^{\mathrm{u}}B)^{(n+1)/[K:k]}. \tag{8.12}$$

Lemma 8.4. *Let B be an ideal of $\mathcal{O}_k$ with $B \mid D$, let $\mathfrak{D} = {}^{\mathrm{u}}C(\theta\mathcal{O}_K, {}^{\mathrm{u}}B) \in S_C$, let $\mathfrak{A} \in T$, and let E be a nonzero ideal of $\mathcal{O}_k$. Then*

$$\frac{\mathfrak{N}_K \mathfrak{D}^{\frac{n+1}{[K:k]}}}{\det \Lambda(\mathfrak{A}\mathfrak{D}, CE)} = \left(2^{-s}\sqrt{|\Delta_k|}\right)^{-(n+1)} \cdot \frac{\mathfrak{N}_K(\theta\mathcal{O}_K, {}^{\mathrm{u}}B)^{\frac{n+1}{[K:k]}}}{\mathfrak{N}_k(A \cap E) \cdot \mathfrak{N}_k\left(A_1 \cap E\right)^n}.$$

Proof. We apply Lemma 8.3 and use (8.10), (8.11), and (8.12). □

Lemma 8.5. *We have*

$$g_k^{\mathcal{N}} = c_0 \sum_{B \mid D} \mathfrak{N}_K(\theta\mathcal{O}_K, {}^{\mathrm{u}}B)^{\frac{n+1}{[K:k]}} \sum_{\mathfrak{A} \in T} \mu_K(\mathfrak{A}) \sum_{E \trianglelefteq \mathcal{O}_k} \frac{\mu_k(E)}{\mathfrak{N}_k(A \cap E) \cdot \mathfrak{N}_k(A_1 \cap E)^n},$$

where $A = A(\mathfrak{A}, B)$, $A_1 = A_1(\mathfrak{A}, B)$, and $c_0 := h_k 2^{s(n+1)}(\sqrt{|\Delta_k|})^{-(n+1)}$ and E runs over all nonzero ideals of $\mathcal{O}_k$.

Proof. Recall the definition of $g_k^{\mathcal{N}}$ in (6.6). The expression on the right-hand side in Lemma 8.4 does not depend on C. With (8.6), a simple computation proves the lemma. □

The inner sum over E in Lemma 8.5 can be handled by the following lemma.

Lemma 8.6. *Let $J_1 \mid J$ be nonzero ideals of $\mathcal{O}_k$ and let*

$$\xi := \sum_{E \trianglelefteq \mathcal{O}_k} \frac{\mu_k(E)}{\mathfrak{N}_k(J \cap E) \cdot \mathfrak{N}_k(J_1 \cap E)^n}.$$

If $J_1 \neq \mathcal{O}_k$ then $\xi = 0$. If $J_1 = \mathcal{O}_k$ then

$$\xi = \frac{1}{\zeta_k(n+1)\mathfrak{N}_k(J)} \prod_{P \mid J} \frac{\mathfrak{N}_k P^{n+1} - \mathfrak{N}_k P}{\mathfrak{N}_k P^{n+1} - 1}.$$

Proof. Let $f(E) := \mu_k(E) \cdot \mathfrak{N}_k(J, E) \cdot \mathfrak{N}_k(J_1, E)^n$. Then f is multiplicative and

$$\xi = \frac{1}{\mathfrak{N}_k\left(J J_1^n\right)} \sum_{E \trianglelefteq \mathcal{O}_k} \frac{f(E)}{\mathfrak{N}_k E^{n+1}}.$$

Clearly, this Dirichlet series converges absolutely for all $n > 0$. Let us compute its Euler product expansion. For any prime ideal P of $\mathcal{O}_k$, we have $f(P^e) = 0$ if $e \geq 2$. Moreover, $f(\mathcal{O}_k) = 1$ and

$$f(P) = \begin{cases} -\mathfrak{N}_k P^{n+1} & \text{if } P \mid J_1, \\ -\mathfrak{N}_k P & \text{if } P \mid J \text{ and } P \nmid J_1, \\ -1 & \text{if } P \nmid J. \end{cases}$$

We obtain the formal expansion

$$\sum_{E \trianglelefteq \mathcal{O}_k} \frac{f(E)}{\mathfrak{N}_k E^s} = \prod_{P \mid J_1} \left(1 - \frac{\mathfrak{N}_k P^{n+1}}{\mathfrak{N}_k P^s}\right) \prod_{\substack{P \mid J \\ P \nmid J_1}} \left(1 - \frac{\mathfrak{N}_k P}{\mathfrak{N}_k P^s}\right) \prod_{P \nmid J} \left(1 - \frac{1}{\mathfrak{N}_k P^s}\right).$$

Since the infinite product $\prod_{P \nmid J} \left(1 - \mathfrak{N}_k P^{-s}\right)$ converges absolutely for $s > 1$, we obtain $\xi = 0$ whenever $J_1 \neq \mathcal{O}_k$. If $J_1 = \mathcal{O}_k$ and $s = n + 1$, the expression simplifies to

$$\sum_{E \trianglelefteq \mathcal{O}_k} \frac{f(E)}{\mathfrak{N}_k E^{n+1}} = \frac{1}{\zeta_k(n+1)} \prod_{P \mid J} \frac{\mathfrak{N}_k P^{n+1} - \mathfrak{N}_k P}{\mathfrak{N}_k P^{n+1} - 1}. \qquad \square$$

Recall the definition of A and A_1 from (8.8) and (8.9). We have $A_1 = \mathcal{O}_k$ if and only if $\mathfrak{A}(\theta\mathcal{O}_K, {}^{\mathrm{u}}B) = (\theta\mathcal{O}_K, \mathfrak{A}{}^{\mathrm{u}}B)$, which is equivalent to $\mathfrak{A}(\theta\mathcal{O}_K, {}^{\mathrm{u}}B) \mid \theta\mathcal{O}_K$, or

$$\mathfrak{A} \mid \theta\mathcal{O}_K(\theta\mathcal{O}_K, {}^{\mathrm{u}}B)^{-1}. \tag{8.13}$$

Recall that, by (8.7), the set T contains all ideals $\mathfrak{A}$ of $\mathcal{O}_K$ with $\mathfrak{A} \mid \theta\mathcal{O}_K$. Also, for every $\mathfrak{A}$ with (8.13), we have $A = {}^{\mathrm{d}}(\mathfrak{A}(\theta\mathcal{O}_K, {}^{\mathrm{u}}B)) \mid D$. We obtain

$$g_k^{\mathcal{N}} = c_1 \sum_{B \mid D} \mathfrak{N}_K(\theta\mathcal{O}_K, {}^{\mathrm{u}}B)^{\frac{n+1}{[K:k]}} \sum_{A \mid D} \frac{1}{\mathfrak{N}_k A} \prod_{P \mid A} \frac{\mathfrak{N}_k P^{n+1} - \mathfrak{N}_k P}{\mathfrak{N}_k P^{n+1} - 1} s_0(A, B),$$

where $c_1 := \zeta_k(n+1)^{-1} c_0 = h_k 2^{s(n+1)} \zeta_k(n+1)^{-1} (\sqrt{|\Delta_k|})^{-(n+1)}$ and

$$s_0(A, B) := \sum_{\substack{\mathfrak{A} \text{ with } (8.13) \\ {}^{\mathrm{d}}(\mathfrak{A}(\theta\mathcal{O}_K, {}^{\mathrm{u}}B)) = A}} \mu_K(\mathfrak{A}).$$

If $s_0(A, B)$ is not zero then there is at least one $\mathfrak{A}$ with

$$A = {}^{\mathrm{d}}\left(\mathfrak{A}(\theta\mathcal{O}_K, {}^{\mathrm{u}}B)\right) \subseteq {}^{\mathrm{d}}(\theta\mathcal{O}_K, {}^{\mathrm{u}}B) = B.$$

For the last equality, we used Lemma 8.2, (iii). We replace A by $B^{-1}A$ to obtain

$$g_k^{\mathcal{N}} = c_1 \sum_{B \mid D} \frac{\mathfrak{N}_K(\theta\mathcal{O}_K, {}^{\mathfrak{u}}B)^{\frac{n+1}{[K:k]}}}{\mathfrak{N}_k B} \sum_{A \mid B^{-1}D} \frac{1}{\mathfrak{N}_k A} \prod_{P \mid AB} \frac{\mathfrak{N}_k P^{n+1} - \mathfrak{N}_k P}{\mathfrak{N}_k P^{n+1} - 1} s(A, B),$$

where

$$s(A, B) := \sum_{\substack{\mathfrak{A} \text{ with (8.13)} \\ {}^{\mathfrak{d}}(\mathfrak{A}(\theta\mathcal{O}_K, {}^{\mathfrak{u}}B)) = AB}} \mu_K(\mathfrak{A}).$$

Lemma 8.7. *Let $\mathfrak{J}$, $\mathfrak{K}$ be nonzero ideals of $\mathcal{O}_K$ and J a nonzero ideal of $\mathcal{O}_k$. Then ${}^{\mathfrak{d}}(\mathfrak{J}\mathfrak{K}) = J\,{}^{\mathfrak{d}}\mathfrak{K}$ if and only if*

$$\mathfrak{J} \mid {}^{\mathfrak{u}}J\left({}^{\mathfrak{u}\mathfrak{d}}\mathfrak{K}\right)\mathfrak{K}^{-1} \text{ and } \mathfrak{J} \nmid {}^{\mathfrak{u}}\left(P^{-1}J\right)\left({}^{\mathfrak{u}\mathfrak{d}}\mathfrak{K}\right)\mathfrak{K}^{-1} \text{ for all prime ideals } P \mid J. \tag{8.14}$$

Proof. Clearly,

$$\mathfrak{J} \mid {}^{\mathfrak{u}}J\left({}^{\mathfrak{u}\mathfrak{d}}\mathfrak{K}\right)\mathfrak{K}^{-1} \iff \mathfrak{J}\mathfrak{K} \mid {}^{\mathfrak{u}}\left(J\,{}^{\mathfrak{d}}\mathfrak{K}\right) \iff {}^{\mathfrak{d}}(\mathfrak{J}\mathfrak{K}) \mid J\,{}^{\mathfrak{d}}\mathfrak{K}$$

and

$$\mathfrak{J} \nmid {}^{\mathfrak{u}}\left(P^{-1}J\right)\left({}^{\mathfrak{u}\mathfrak{d}}\mathfrak{K}\right)\mathfrak{K}^{-1} \iff \mathfrak{J}\mathfrak{K} \nmid {}^{\mathfrak{u}}\left(P^{-1}J\,{}^{\mathfrak{d}}\mathfrak{K}\right) \iff {}^{\mathfrak{d}}(\mathfrak{J}\mathfrak{K}) \nmid \left(P^{-1}J\right){}^{\mathfrak{d}}\mathfrak{K}. \qquad \square$$

Lemma 8.8. *If $A \mid B^{-1}D$ then $s(A, B) = \mu_k(A)$.*

Proof. By Lemma 8.2, (iii), we have ${}^{\mathfrak{d}}(\theta\mathcal{O}_K, {}^{\mathfrak{u}}B) = B$. By the previous lemma, ${}^{\mathfrak{d}}(\mathfrak{A}(\theta\mathcal{O}_K, {}^{\mathfrak{u}}B)) = AB$ is equivalent to

$$\mathfrak{A} \mid {}^{\mathfrak{u}}A\,{}^{\mathfrak{u}}B\left(\theta\mathcal{O}_K, {}^{\mathfrak{u}}B\right)^{-1} \text{ and } \mathfrak{A} \nmid {}^{\mathfrak{u}}\left(P^{-1}A\right){}^{\mathfrak{u}}B(\theta\mathcal{O}_K, {}^{\mathfrak{u}}B)^{-1} \text{ for all } P \mid A. \tag{8.15}$$

Clearly, conditions (8.13) and (8.15) imply

$$\mathfrak{A} \mid \left(\theta\mathcal{O}_K\left(\theta\mathcal{O}_K, {}^{\mathfrak{u}}B\right)^{-1}, {}^{\mathfrak{u}}A\,{}^{\mathfrak{u}}B(\theta\mathcal{O}_K, {}^{\mathfrak{u}}B)^{-1}\right) = \left(\theta\mathcal{O}_K(\theta\mathcal{O}_K, {}^{\mathfrak{u}}B)^{-1}, {}^{\mathfrak{u}}A\right) \tag{8.16}$$

and

$$\mathfrak{A} \nmid {}^{\mathfrak{u}}\left(P^{-1}A\right) \text{ for all prime ideals } P \mid A. \tag{8.17}$$

In fact, (8.13) and (8.15) are equivalent to (8.16) and (8.17). Indeed, (8.16) immediately implies (8.13) and the first part of (8.15). For the second part of (8.15), we use that every $\mathfrak{A} \mid \theta\mathcal{O}_K(\theta\mathcal{O}_K, {}^{\mathfrak{u}}B)^{-1}$ satisfies $(\mathfrak{A}, {}^{\mathfrak{u}}B(\theta\mathcal{O}_K, {}^{\mathfrak{u}}B)^{-1}) = \mathcal{O}_K$. Thus,

$$s(A, B) = \sum_{\substack{\mathfrak{A} \trianglelefteq \mathcal{O}_K \\ \text{(8.13) and (8.15)}}} \mu_K(\mathfrak{A}) = \sum_{\substack{\mathfrak{A} \trianglelefteq \mathcal{O}_K \\ \text{(8.16) and (8.17)}}} \mu_K(\mathfrak{A}).$$

By inclusion-exclusion for (8.17), we obtain

$$s(A,B)=\sum_{F\mid A}\mu_k(F)\sum_{\mathfrak{A}\mid(\theta\mathcal{O}_K(\theta\mathcal{O}_K,{}^{\mathrm{u}}B)^{-1},{}^{\mathrm{u}}(F^{-1}A))}\mu_K(\mathfrak{A}).$$

The last sum is 1 if $F=A$. Moreover,

$$F^{-1}A\mid B^{-1}D=\mathfrak{d}(\theta\mathcal{O}_K)\left(\mathfrak{d}(\theta\mathcal{O}_K,{}^{\mathrm{u}}B)\right)^{-1}\mid\mathfrak{d}\left(\theta\mathcal{O}_K(\theta\mathcal{O}_K,{}^{\mathrm{u}}B)^{-1}\right),$$

so $F\neq A$ implies that

$$\left(\theta\mathcal{O}_K(\theta\mathcal{O}_K,{}^{\mathrm{u}}B)^{-1},{}^{\mathrm{u}}(F^{-1}A)\right)\neq\mathcal{O}_K.$$

This shows that the last sum is 0 whenever $F\neq A$. □

We obtain

$$g_k^{\mathcal{N}}=c_1\sum_{B\mid D}\frac{\mathfrak{N}_K(\theta\mathcal{O}_K,{}^{\mathrm{u}}B)^{(n+1)/[K:k]}}{\mathfrak{N}_kB}\sum_{A\mid B^{-1}D}\frac{\mu_k(A)}{\mathfrak{N}_kA}\prod_{P\mid AB}\frac{\mathfrak{N}_kP^{n+1}-\mathfrak{N}_kP}{\mathfrak{N}_kP^{n+1}-1},$$

and Theorem 1.4 follows by substituting this and (8.4) in (8.3).

9. Proof of Theorem 1.3

In this section we will use not only Landau's O-notation but also Vinogradov's symbol $\ll$. All implied constants depend solely on k. As we will encounter expressions like $\log\log X$ we assume throughout the entire section that $X\geq 3$. Our main task will be to prove the following proposition.

Proposition 9.1. *Suppose $p\in\mathbf{P}_k$. Then, as $X\geq 3$ tends to infinity, we have*

$$N\left(\sqrt{p}k^*,X\right)=\frac{2p^{d/2}}{p^d+1}S_k(1)X^{2d}+O\left(\frac{X^{2d-1}}{p^{(d-1)/2}}+X^d\log X+X^d\log p\right).$$

We choose the adelic Lipschitz system $\mathcal{N}$ (of dimension 1) on $K:=k(\sqrt{p})$, defined by

$$N_w((z_0,z_1)):=\max\{|z_0|_w,|\sqrt{p}|_w|z_1|_w\}$$

for any place w of K. Recall the definition of $C_{\mathcal{N}}^{\mathrm{fin}}$ and $C_{\mathcal{N}}^{\mathrm{inf}}$ from (5.5) and (5.6), and note that we can take

$$C_{\mathcal{N}}^{\mathrm{fin}}=C_{\mathcal{N}}^{\mathrm{inf}}=\sqrt{p}. \tag{9.1}$$

The adelic Lipschitz system $\mathcal{N}$ on K leads to an adelic Lipschitz system $\mathcal{N}'$ on k as in Section 6. Note that for any Archimedean v from k and N_v from $\mathcal{N}'$ we have $N_v((z_0,z_1))=\max\{|z_0|_v,\sqrt{p}|z_1|_v\}$. Thus we can also take

$$C_{\mathcal{N}'}^{\mathrm{inf}}=\sqrt{p}. \tag{9.2}$$

Lemma 9.2. *We have*

$$N\left(\sqrt{p}k^*, X\right) = N_{\mathcal{N}}\left(\mathbb{P}^1(k); X\right) - 2.$$

Proof. The map $\alpha \mapsto (1 : \alpha)$ is a one-to-one correspondence between k^* and $\mathbb{P}^1(k)\backslash\{(0 : 1), (1 : 0)\}$ Moreover, $H(\sqrt{p}\alpha) = H_{\mathcal{N}}((1 : \alpha))$. Hence there is a one-to-one correspondence between $\{\alpha \in k^* : H(\sqrt{p}\alpha) \leq X\}$ and $\{P \in \mathbb{P}^1(k)\backslash\{(0 : 1), (1 : 0)\} : H_{\mathcal{N}}(P) \leq X\}$. As $H_{\mathcal{N}}((0 : 1)) = H_{\mathcal{N}}((1 : 0)) = 1$ the claim follows. □

We can now basically follow the proof of Theorem 6.1 using our specific adelic Lipschitz system. However, to get the good error terms regarding p an additional idea is required. We will use the same notation as in Sections 6 and 7. In particular, recall the definition of the set $S_F(t)$ introduced in (7.2). As in (8.5), we choose a system $\mathcal{R}$ of integral representatives for Cl_k such that $(C, p\mathcal{O}_k) = \mathcal{O}_k$ for all $C \in \mathcal{R}$.

Lemma 9.3. *We can choose* $S_C := \{{}^{\mathrm{u}}C, \sqrt{p}{}^{\mathrm{u}}C\}$.

Proof. As in (8.1) we have $i_{\mathcal{N}}(\boldsymbol{\omega}) = \omega_0\mathcal{O}_K + \sqrt{p}\omega_1\mathcal{O}_K$. So if $\mathcal{O}_k(\boldsymbol{\omega}) = C$ we get $\sqrt{p}{}^{\mathrm{u}}C \subseteq i_{\mathcal{N}}(\boldsymbol{\omega}) \subseteq {}^{\mathrm{u}}C$. As $\sqrt{p}\mathcal{O}_K$ is a prime ideal this proves the lemma. □

With this choice of the sets S_C we directly verify that Γ from (7.13) can be chosen to be

$$\Gamma := p^{-d/2}. \tag{9.3}$$

From now on C is always in $\mathcal{R}$, $\mathfrak{D}$ is always in S_C, and $\mathfrak{A}$ will always be in T.

Lemma 9.4. *We can choose T such that* $|T| \leq 2$.

Proof. Recall that we may choose $T = \cup_{C\in\mathcal{R}} \cup_{\mathfrak{D}\in S_C} T_{C,\mathfrak{D}}$. By definition we have

$$\begin{aligned} T_{C,\mathfrak{D}} &= \{\mathfrak{B} \trianglelefteq \mathcal{O}_K : \Lambda_C(\mathfrak{D}\mathfrak{B}) \neq \emptyset\} \\ &= \{\mathfrak{B} \trianglelefteq \mathcal{O}_K : \Lambda_C^*(\mathfrak{E}\mathfrak{D}\mathfrak{B}) \neq \emptyset \text{ for some } \mathfrak{E} \trianglelefteq \mathcal{O}_K\} \\ &\subseteq \{\mathfrak{B} \trianglelefteq \mathcal{O}_K : \mathfrak{E}\mathfrak{D}\mathfrak{B} \in S_C \text{ for some } \mathfrak{E} \trianglelefteq \mathcal{O}_K\}. \end{aligned}$$

Now using that $S_C = \{{}^{\mathrm{u}}C, \sqrt{p}{}^{\mathrm{u}}C\}$ and that $\sqrt{p}\mathcal{O}_K$ is a prime ideal we see that $T_{C,\mathfrak{D}} \subseteq \{\mathcal{O}_K, \sqrt{p}\mathcal{O}_K\}$ for any $\mathfrak{D} \in S_C$. Thus $|T| = |\cup_{C\in\mathcal{R}} \cup_{\mathfrak{D}\in S_C} T_{C,\mathfrak{D}}| \leq 2$. □

Lemma 9.5. *Let σ be as in* (6.3). *We have*

$$\Lambda(\mathfrak{A}\mathfrak{D}, CE) \subseteq \sigma(CE) \times \sigma(CE).$$

Moreover, if $\mathfrak{D} = \sqrt{p}{}^{\mathrm{u}}C$ then we have

$$\Lambda(\mathfrak{A}\mathfrak{D}, CE) \subseteq \sigma\left(CEp(CE, p\mathcal{O}_k)^{-1}\right) \times \sigma(CE).$$

Proof. The first assertion is clear from the definition. For the second assertion we could use the last equality in the proof of Lemma 8.3, but we prefer to give a direct argument here. Note that $\sigma\boldsymbol{\omega} \in \Lambda(\mathfrak{A}\mathfrak{D})$ implies $\mathfrak{D} \mid i_{\mathcal{N}}(\boldsymbol{\omega}) = (\omega_0\mathcal{O}_K, \sqrt{p}\omega_1\mathcal{O}_K)$. As $\mathfrak{D} = \sqrt{p}^{\mathrm{u}}C$ we conclude $\sqrt{p}\mathcal{O}_K \mid \omega_0\mathcal{O}_K$, and thus $p\mathcal{O}_k \mid \omega_0\mathcal{O}_k$. Therefore $\omega_0 \in CE \cap p\mathcal{O}_k$. This proves the second assertion. □

Next we use a trick, simpler but reminiscent of those used in [20, Section 6]. To this end we introduce a linear automorphism Φ of determinant 1 on $(\mathbb{R}^r \times \mathbb{C}^s)^2$ by

$$\Phi(\mathbf{z}_0, \mathbf{z}_1) := \left(p^{-1/4}\mathbf{z}_0, p^{1/4}\mathbf{z}_1\right). \tag{9.4}$$

Lemma 9.6. *Write $\Lambda := \Lambda(\mathfrak{A}\mathfrak{D}, CE)$. If $\mathfrak{D} = {}^{\mathrm{u}}C$ then we have*

$$\lambda_1(\Phi\Lambda) \geq p^{-1/4}\mathfrak{N}_k(CE)^{1/d},$$
$$\lambda_{d+1}(\Phi\Lambda) \geq p^{1/4}\mathfrak{N}_k(CE)^{1/d}.$$

If $\mathfrak{D} = \sqrt{p}^{\mathrm{u}}C$ then we have

$$\lambda_1(\Phi\Lambda) \geq \begin{cases} p^{-1/4}\mathfrak{N}_k(CE)^{1/d} & \text{if } p\mathcal{O}_k \mid E, \\ p^{1/4}\mathfrak{N}_k(CE)^{1/d} & \text{if } p\mathcal{O}_k \nmid E. \end{cases}$$

$$\lambda_{d+1}(\Phi\Lambda) \geq \begin{cases} p^{1/4}\mathfrak{N}_k(CE)^{1/d} & \text{if } p\mathcal{O}_k \mid E, \\ p^{3/4}\mathfrak{N}_k(CE)^{1/d} & \text{if } p\mathcal{O}_k \nmid E. \end{cases}$$

Proof. By Lemma 9.5 we have $\Phi\Lambda \subseteq \Lambda_1 \times \Lambda_2$, where $\Lambda_2 := p^{1/4}\sigma(CE)$ and Λ_1 is $p^{-1/4}\sigma(CE)$ if $\mathfrak{D} = {}^{\mathrm{u}}C$ and $p^{-1/4}\sigma\left(CEp(CE, p\mathcal{O}_k)^{-1}\right)$ if $\mathfrak{D} = \sqrt{p}^{\mathrm{u}}C$. Recall the fact (already used in Lemma 7.4) that $\lambda_1(\sigma A) \geq \mathfrak{N}_k A^{1/d}$ for any nonzero ideal A of k. Using this and applying Lemma 4.5 the result follows from an easy computation. □

Lemma 9.7. *There exist constants $c_1 = c_1(k)$ and $M = M(k)$ depending solely on k such that, with $L = c_1 p^{-1/4}t$, we have $\Phi S_F(t) \subseteq B_0(L)$ and the boundary $\partial\Phi S_F(t) \in \mathrm{Lip}(2d, M, L)$.*

Proof. The adelic Lipschitz system $\mathcal{N}$ on K leads to an adelic Lipschitz system $\mathcal{N}'$ on k as in Section 6. The latter is used to define $S_F(t)$.

Now notice that applying Φ to $S_F(t)$ gives the same as defining $S_F(t)$ using the standard adelic Lipschitz system defined by $N_v(z_0, z_1) = \max\{|z_0|_v, |z_1|_v\}$ for all v and then homogeneously shrinking this set by the factor $p^{-1/4}$. The claims then follow immediately from Lemma 7.1, (7.9), and (7.8) applied to the standard adelic Lipschitz system. □

Lemma 9.8. *Let* $\mathcal{E}_1 := X^d/\mathfrak{N}_k(E)$*, and let* $\mathcal{E}_2 := X^{2d-1}/(p^{(d-1)/2}\mathfrak{N}_k(E)^{2-1/d})$*. Then we have*

$$\left|\Lambda(\mathfrak{A}\mathfrak{D}, CE) \cap S_F\left(X\mathfrak{N}_K\mathfrak{D}^{1/(2d)}\right)\right| = \frac{\operatorname{Vol} S_F(1)\mathfrak{N}_K\mathfrak{D}X^{2d}}{\det \Lambda(\mathfrak{A}\mathfrak{D}, CE)} + O\left(\begin{cases} \mathcal{E}_1 + \mathcal{E}_2 & \text{if } p\mathcal{O}_k \nmid E \\ p^{d/2}\mathcal{E}_1 + p^{d-1/2}\mathcal{E}_2 & \text{if } p\mathcal{O}_k \mid E \end{cases}\right).$$

Moreover, there is a constant $\gamma = \gamma(k) \geq 1$ *depending only on k, such that* $|\Lambda(\mathfrak{A}\mathfrak{D}, CE) \cap S_F(X\mathfrak{N}_K\mathfrak{D}^{1/(2d)})| = 0$ *whenever* $\mathfrak{N}_k E > (\gamma p X)^d$*.*

Proof. First note that

$$\left|\Lambda\left(\mathfrak{A}\mathfrak{D}, CE) \cap S_F(X\mathfrak{N}_K\mathfrak{D}^{1/(2d)}\right)\right| = \left|\Phi\Lambda(\mathfrak{A}\mathfrak{D}, CE) \cap \Phi S_F\left(X\mathfrak{N}_K\mathfrak{D}^{1/(2d)}\right)\right|.$$

Now we apply Lemma 4.2 with $a = d + 1$ combined with Lemma 9.7 to conclude

$$\left|\Phi\Lambda(\mathfrak{A}\mathfrak{D}, CE) \cap \Phi S_F\left(X\mathfrak{N}_K\mathfrak{D}^{1/(2d)}\right)\right| = \frac{\operatorname{Vol} S_F(1)\mathfrak{N}_K\mathfrak{D}X^{2d}}{\det \Lambda(\mathfrak{A}\mathfrak{D}, CE)} + O\left(\max\left\{\frac{p^{-d/4}X^d\mathfrak{N}_K\mathfrak{D}^{1/2}}{\lambda_1(\Phi\Lambda)^d}, \frac{p^{-(2d-1)/4}X^{2d-1}\mathfrak{N}_K\mathfrak{D}^{1-1/(2d)}}{\lambda_1(\Phi\Lambda)^d\lambda_{d+1}(\Phi\Lambda)^{d-1}}\right\}\right).$$

Finally, we use Lemma 9.6 to estimate $\lambda_1(\Phi\Lambda)$ and $\lambda_{d+1}(\Phi\Lambda)$, and the first claim follows from a simple computation. The second claim follows from Lemma 7.4 combined with (9.2) and (9.3). □

We are now in the position to prove Proposition 9.1. In the introduction we already computed the main term, see (1.6). Proceeding exactly as in the proof of Theorem 6.1 in the case $(n, d) = (1, 1)$, we obtain

$$\begin{aligned} N_{\mathcal{N}}(\mathbb{P}^1(k); X) = {} & \frac{2p^{d/2}}{p^d+1}S_k(1)X^{2d} \\ & + O\left(\sum_{C\in\mathcal{R}}\sum_{\mathfrak{D}\in S_C}\sum_{\mathfrak{A}\in T}\sum_{\substack{E\trianglelefteq\mathcal{O}_k \\ \mathfrak{N}_k E>(\gamma pX)^d}} \frac{\operatorname{Vol} \Phi S_F(X\mathfrak{N}_K\mathfrak{D}^{1/(2d)})}{\det \Phi\Lambda(\mathfrak{A}\mathfrak{D}, CE)}\right) \\ & + O\left(\sum_{C\in\mathcal{R}}\sum_{\mathfrak{D}\in S_C}\sum_{\mathfrak{A}\in T}\sum_{\substack{E\trianglelefteq\mathcal{O}_k \\ \mathfrak{N}_k E\leq(\gamma pX)^d}} \mathcal{E}_1 + \mathcal{E}_2\right) \\ & + O\left(\sum_{C\in\mathcal{R}}\sum_{\mathfrak{D}\in S_C}\sum_{\mathfrak{A}\in T}\sum_{\substack{E\trianglelefteq\mathcal{O}_k \\ \mathfrak{N}_k E\leq(\gamma pX)^d \\ p\mathcal{O}_k\mid E}} p^{d/2}\mathcal{E}_1 + p^{d-1/2}\mathcal{E}_2\right). \end{aligned}$$

For the first error term we apply Minkowski's second theorem and Lemma 9.7 to get the upper bound

$$\frac{\operatorname{Vol} \Phi S_F\left(X \mathfrak{N}_K \mathfrak{D}^{1/(2d)}\right)}{\det \Phi\Lambda(\mathfrak{A}\mathfrak{D}, CE)} \ll \frac{L^{2d}}{\lambda_1(\Phi\Lambda)^d \lambda_{d+1}(\Phi\Lambda)^d},$$

where $L \ll p^{-1/4} X \mathfrak{N}_K \mathfrak{D}^{1/(2d)}$. Summing the above over the finite sums can be handled by Lemmata 9.3 and 9.4. Now for the infinite sum over the ideals E, we apply Lemma 9.6, and a straightforward computation (using the dichotomy $P \mid E$, $P \nmid E$) yields the upper bound

$$\ll \frac{X^d}{p^{3d/2}}.$$

For the second error term we note that

$$\sum_{\substack{E \trianglelefteq \mathcal{O}_k \\ \mathfrak{N}_k E \leq (\gamma p X)^d}} \mathcal{E}_1 = \sum_{\substack{E \trianglelefteq \mathcal{O}_k \\ \mathfrak{N}_k E \leq (\gamma p X)^d}} \frac{X^d}{\mathfrak{N}_k E} \ll X^d \log\left((\gamma p X)^d\right) \ll X^d \log X + X^d \log p,$$

and

$$\sum_{\substack{E \trianglelefteq \mathcal{O}_k \\ \mathfrak{N}_k E \leq (\gamma p X)^d}} \mathcal{E}_2 \leq \frac{X^{2d-1}}{p^{(d-1)/2}} \sum_{E \trianglelefteq \mathcal{O}_k} \mathfrak{N}_k E^{-2+1/d} \ll \frac{X^{2d-1}}{p^{(d-1)/2}}.$$

Then we apply Lemmata 9.3 and 9.4 to conclude

$$\sum_{C \in \mathcal{R}} \sum_{\mathfrak{D} \in S_C} \sum_{\mathfrak{A} \in T} \sum_{\substack{E \trianglelefteq \mathcal{O}_k \\ \mathfrak{N}_k E \leq (\gamma p X)^d}} \mathcal{E}_1 + \mathcal{E}_2 \ll X^d \log X + X^d \log p + \frac{X^{2d-1}}{p^{(d-1)/2}}.$$

Similar straightforward calculations yield

$$\sum_{\substack{E \trianglelefteq \mathcal{O}_k \\ \mathfrak{N}_k E \leq (\gamma p X)^d \\ p\mathcal{O}_k \mid E}} p^{d/2} \mathcal{E}_1 \ll \frac{X^d}{p^{d/2}} \log X,$$

and

$$\sum_{\substack{E \trianglelefteq \mathcal{O}_k \\ \mathfrak{N}_k E \leq (\gamma p X)^d \\ p\mathcal{O}_k \mid E}} p^{d-1/2} \mathcal{E}_2 \ll \frac{X^{2d-1}}{p^{3d/2-1}}.$$

Thus, applying again Lemmata 9.3 and 9.4, we see that

$$\sum_{C\in\mathcal{R}}\sum_{\mathfrak{D}\in S_C}\sum_{\mathfrak{A}\in T}\sum_{\substack{E\trianglelefteq\mathcal{O}_k\\ \mathfrak{N}_k E\leq(\gamma pX)^d\\ p\mathcal{O}_k\mid E}} p^{d/2}\mathcal{E}_1 + p^{d-1/2}\mathcal{E}_2 \ll X^d\log X + \frac{X^{2d-1}}{p^{(d-1)/2}}.$$

Combining these estimates and Lemma 9.2 completes the proof of Proposition 9.1.

We can now sum $N(\sqrt{p}k^*, X)$ over all $p \in \mathbf{P}_k$. The next lemma tells us that we can restrict the summation to $p \leq X^2$.

Lemma 9.9. *For any $\alpha \in k^*$ and any $p \in \mathbf{P}_k$ we have $H(\sqrt{p}\alpha) \geq \sqrt{p}$.*

Proof. Let $x \in K$ and let $\mathfrak{P}$ be the prime ideal $\sqrt{p}\mathcal{O}_K$. Then

$$H(x) \geq \max\{1, \mathfrak{N}_K\mathfrak{P}\}^{-v_{\mathfrak{P}}(x\mathcal{O}_K)/(2d)} = \max\left\{1, p^d\right\}^{-v_{\mathfrak{P}}(x\mathcal{O}_K)/(2d)}.$$

In particular, if $v_{\mathfrak{P}}(x\mathcal{O}_K) < 0$ we get $H(x) \geq \sqrt{p}$. As $H(x) = H(1/x)$ for any nonzero x whatsoever, it suffices to show that the order of $\sqrt{p}\alpha\mathcal{O}_K$ at $\mathfrak{P}$ is nonzero. As p is inert in k the order of $\alpha\mathcal{O}_K$ at $\mathfrak{P}$ is even. Hence the order of $\sqrt{p}\alpha\mathcal{O}_K$ at $\mathfrak{P}$ is odd. $\square$

We can now prove Theorem 1.3. Clearly, we have

$$\begin{aligned} N\left(\sqrt{\mathbf{P}_k}k, X\right) &= 1 + \sum_{\substack{p\in\mathbf{P}_k\\ p\leq X^2}} N(\sqrt{p}k^*, X) \\ &= \sum_{\substack{p\in\mathbf{P}_k\\ p\leq X^2}} \frac{2p^{d/2}}{p^d+1}S_k(1)X^{2d} + O\left(\frac{X^{2d-1}}{p^{(d-1)/2}} + X^d\log X + X^d\log p\right) \\ &= \sum_{\substack{p\in\mathbf{P}_k\\ p\leq X^2}} \frac{2p^{d/2}}{p^d+1}S_k(1)X^{2d} + O\left(\sum_{\substack{p\in\mathbf{P}_k\\ p\leq X^2}} \frac{X^{2d-1}}{p^{(d-1)/2}}\right) + O\left(\sum_{\substack{p\in\mathbf{P}_k\\ p\leq X^2}} X^d\log X\right). \end{aligned}$$

By the prime number theorem we have

$$\sum_{\substack{p\in\mathbf{P}_k\\ p\leq X^2}} X^d\log X \ll X^{d+2}.$$

A straightforward calculation yields

$$\sum_{\substack{p\in\mathbf{P}_k\\ p\leq X^2}} \frac{X^{2d-1}}{p^{(d-1)/2}} \ll \begin{cases} X^{2d-1} & \text{if } d\geq 4,\\ X^5\log\log X & \text{if } d=3,\\ X^4 & \text{if } d=2. \end{cases}$$

To handle the first term let us start with the simpler case $d \geq 3$. Then we have

$$\sum_{\substack{p\in\mathbf{P}_k\\ p\leq X^2}} \frac{2p^{d/2}}{p^d+1} S_k(1) X^{2d} = \sum_{p\in\mathbf{P}_k} \frac{2p^{d/2}}{p^d+1} S_k(1) X^{2d} + O\left(\sum_{\substack{p\in\mathbf{P}_k\\ p> X^2}} \frac{2p^{d/2}}{p^d+1} S_k(1) X^{2d} \right)$$
$$= \sum_{p\in\mathbf{P}_k} \frac{2p^{d/2}}{p^d+1} S_k(1) X^{2d} + O\left(X^{2d-1}\right).$$

This finishes the proof of Theorem 1.3 for $d \geq 3$.

Let us now assume $d = 2$. It remains to show that

$$\sum_{\substack{p\in\mathbf{P}_k\\ p\leq X^2}} \frac{2p}{p^2+1} = \log\log X + O(1).$$

Clearly, we have

$$\sum_{\substack{p\in\mathbf{P}_k\\ p\leq X^2}} \frac{2p}{p^2+1} = \sum_{\substack{p\in\mathbf{P}_k\\ p\leq X^2}} \frac{2}{p} + O(1).$$

By an explicit version of Chebotarev's density theorem (see, *e.g.*, [7]) we know that for $T \geq 3$ (using $\mathrm{Li}(T) = T/\log T + O(T/(\log T)^2)$)

$$\sum_{\substack{p\in\mathbf{P}_k\\ p\leq T}} 1 = \frac{T}{2\log T} + O\left(\frac{T}{(\log T)^2}\right).$$

Applying partial summation we get

$$\sum_{\substack{p\in\mathbf{P}_k\\ p\leq X^2}} \frac{2}{p} = \sum_{m=2}^{X^2} \frac{1}{(m+1)\log m} + O(1) = \log\log X + O(1).$$

This completes the proof of Theorem 1.3 for $d = 2$.

A. Appendix

We will now apply Theorem 6.1 to deduce the formula (1.8). We start by proving our claim that $\mathcal{N}'$ is an adelic Lipschitz system whenever all the functions N_w of $\mathcal{N}$ are norms. To this end we shall use the following simple observations.

Let $f_1, f_2, f : \mathbb{R}^q \to \mathbb{R}$ and $F : [0,1]^{q-1} \to \mathbb{R}^q$ be functions that satisfy a Lipschitz condition with Lipschitz constant L_{f_1}, L_{f_2}, L_f and L_F respectively. Then we have:

1. $|f(F(\mathbf{t})) - f(F(\mathbf{t}'))| \le L_f L_F |\mathbf{t} - \mathbf{t}'|$ for all $\mathbf{t}, \mathbf{t}' \in [0,1]^{q-1}$.
2. Suppose that $f(F(\mathbf{t})) \ge c > 0$ for all $\mathbf{t} \in [0,1]^{q-1}$ and let $\alpha \le 1$. Then $|f(F(\mathbf{t}))^\alpha - f(F(\mathbf{t}'))^\alpha| \le |\alpha| c^{\alpha-1} L_f L_F |\mathbf{t} - \mathbf{t}'|$ for all $\mathbf{t}, \mathbf{t}' \in [0,1]^{q-1}$. (We use the convention that $0^0 = 1$.)
3. Suppose that $|f_1(F(\mathbf{t}))|, |f_2(F(\mathbf{t}))|, |f(F(\mathbf{t}))|, |F(\mathbf{t})| \le C$ for all $\mathbf{t} \in [0,1]^{q-1}$. Then, for all $\mathbf{t}, \mathbf{t}' \in [0,1]^{q-1}$,
   (a) $|f_1(F(\mathbf{t}))f_2(F(\mathbf{t})) - f_1(F(\mathbf{t}'))f_2(F(\mathbf{t}'))| \le C(L_{f_1} + L_{f_2}) L_F |\mathbf{t} - \mathbf{t}'|$,
   (b) $|f(F(\mathbf{t}))F(\mathbf{t}) - f(F(\mathbf{t}'))F(\mathbf{t}')| \le C L_F (L_f + 1)|\mathbf{t} - \mathbf{t}'|$.

Here 1. is obvious, 2. follows from the mean value theorem and 1., and 3. (a) and (b) are consequences of the identity $fg - f'g' = (f - f')g + f'(g - g')$ and 1. (note that the assumption $|F(\mathbf{t})| \le C$ is needed only for (b)).

Lemma A.1. *Let $\mathcal{N}$ be an adelic Lipschitz system (of dimension n) on K and assume that for every Archimedean place w of K the function N_w satisfies a Lipschitz condition. Then $\mathcal{N}' = \mathcal{N}'(\mathcal{N}, k)$ is an adelic Lipschitz system (of dimension n) on k.*

Proof. The conditions (i), (ii) and (iv) in Definition 5.1 are obviously satisfied. It remains to prove (iii). Given an Archimedean place v of k, let $\rho : [0,1]^{d_v(n+1)-1} \to \mathbb{S}^{d_v(n+1)-1}$ be the (normalized) standard parameterization via polar coordinates of the $(d_v(n+1)-1)$-dimensional unit sphere in k_v^{n+1}. Then ρ is Lipschitz. The subset of k_v^{n+1} where $N_v(\mathbf{z}) = 1$ is parameterized by the function $\psi : [0,1]^{d_v(n+1)-1} \to k_v^{n+1}$, defined by $\psi(\mathbf{t}) := 1/N_v(\rho(\mathbf{t})) \cdot \rho(\mathbf{t})$. Let us show that ψ satisfies a Lipschitz condition.

For any Archimedean place w of K extending v, the function N_w is continuous and nonzero on the compact set $\mathbb{S}^{d_v(n+1)-1}$, whence $1 \ll_{\mathcal{N}} N_w(\rho(\mathbf{t})) \ll_{\mathcal{N}} 1$ on $[0,1]^{d_v(n+1)-1}$. Thus, $N_w(\rho(\mathbf{t}))^{-\frac{d_w}{d_v[K:k]}}$ is bounded, and by 2. satisfies a Lipschitz condition. Hence, by 3. (a) also $N_v(\rho(\mathbf{t}))^{-1}$ is Lipschitz. By 3. (b), we conclude that ψ satisfies a Lipschitz condition. □

Note that any norm $\|\cdot\|$ on $\mathbb{R}^q$ satisfies a Lipschitz condition. This follows from the reverse triangle inequality $|\|x\| - \|y\|| \le \|x - y\|$ and the equivalence of all norms on $\mathbb{R}^q$. Thus, if all the functions N_w are norms then Lemma A.1 applies and so $\mathcal{N}' = \mathcal{N}'(\mathcal{N}, k)$ is an adelic Lipschitz system (of dimension n) on k. More generally, let $B_w := \{\mathbf{z} \in K_w^{n+1} : N_w(\mathbf{z}) \le 1\}$ be the compact star-shaped body corresponding to N_w. Let $\ker(B_w)$ be the convex kernel of B_w, that is the set of all $\mathbf{z} \in B_w$ such that for all $\mathbf{z}' \in B_w$ the line segment $[\mathbf{z}, \mathbf{z}']$ is contained in B_w. Then $\mathbf{0} \in \ker(B_w)$ and B_w is convex if and only if $\ker(B_w) = B_w$. Moreover, [2, Lemma 1] tells us that N_w is Lipschitz whenever $\mathbf{0}$ is in the interior of $\ker(B_w)$.

Let us now show how the formula (1.8) follows from Theorem 6.1. We use the adelic Lipschitz system $\mathcal{N}$ (of dimension 2) on $K := \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ defined by

$$N_w(z_0, z_1, z_2) := \max\left\{|z_0|_w, |z_1|_w, |z_2|_w, \left|\frac{\sqrt{2}z_1 + \sqrt{3}z_2}{\sqrt{5}}\right|_w\right\},$$

for any place w of K. Hence all the N_w are norms so that, thanks to Lemma A.1, we can apply Theorem 6.1. With the notation from Section 6, we have $N_L(X) = N_{\mathcal{N}}(\mathbb{P}^2(\mathbb{Q}), X) + O(X^2)$, as already mentioned in the introduction. Here the error term accounts for the projective points of the form $(0 : \omega_1 : \omega_2)$. With Theorem 6.1, the only remaining task is to calculate $g_{\mathbb{Q}}^{\mathcal{N}}$.

Lemma A.2. *We have*

$$g_{\mathbb{Q}}^{\mathcal{N}} = \frac{1}{31\zeta(3)}\left(1 + 2 \cdot 5^{1/4} + 4 \cdot 5^{-1/2}\right).$$

Proof. For some tedious computations in K, we use the computer algebra system Sage[1]. We use the same notation as in Section 6. Clearly, we can choose $\mathcal{R} = \{\mathbb{Z}\}$. For any $\boldsymbol{\omega} = (\omega_0, \omega_1, \omega_2) \in \mathbb{Q}^3$, we have

$$i_{\mathcal{N}}(\boldsymbol{\omega}) = \omega_0\mathcal{O}_K + \omega_1\mathcal{O}_K + \omega_2\mathcal{O}_K + \frac{\sqrt{2}\omega_1 + \sqrt{3}\omega_2}{\sqrt{5}}\mathcal{O}_K.$$

If $\mathcal{O}_{\mathbb{Q}}(\boldsymbol{\omega}) = \mathbb{Z}$ then $\omega_0\mathcal{O}_K + \omega_1\mathcal{O}_K + \omega_2\mathcal{O}_K = \mathcal{O}_K$, so $i_{\mathcal{N}}(\boldsymbol{\omega}) \supseteq \mathcal{O}_K$. On the other hand, we clearly have $i_{\mathcal{N}}(\boldsymbol{\omega}) \subseteq (\sqrt{5})^{-1}O_K$. Thus, we can choose

$$S_{\mathbb{Z}} := \left\{\left(\sqrt{5}\right)^{-1}\mathfrak{D} : \mathfrak{D} \mid \sqrt{5}\mathcal{O}_K\right\}.$$

Moreover, if $\boldsymbol{\omega} \in \Lambda_{\mathbb{Z}}((\sqrt{5})^{-1}\mathfrak{D}\mathfrak{A})$, for some nonzero ideal $\mathfrak{A}$ of $\mathcal{O}_K$, then $i_{\mathcal{N}}(\boldsymbol{\omega}) = (\sqrt{5})^{-1}\mathfrak{D}_1$, for some nonzero ideal $\mathfrak{D}_1 \mid \sqrt{5}\mathcal{O}_K$. In particular, $\mathfrak{D}\mathfrak{A} \mid \mathfrak{D}_1$. This shows that $T_{\mathbb{Z},(\sqrt{5})^{-1}\mathfrak{D}}$ is contained in the finite set

$$T := \left\{\mathfrak{A} : \mathfrak{A} \mid \sqrt{5}\mathcal{O}_K\right\}.$$

With (6.6), we obtain

$$g_{\mathbb{Q}}^{\mathcal{N}} = \sum_{\mathfrak{D}\mid\sqrt{5}\mathcal{O}_K} \mathfrak{N}_K\left(\left(\sqrt{5}\right)^{-1}\mathfrak{D}\right)^{3/8} \sum_{\mathfrak{A}\mid\sqrt{5}\mathcal{O}_K} \mu_K(\mathfrak{A})\Sigma(\mathfrak{A}\mathfrak{D}), \tag{A.1}$$

[1] http://www.sagemath.org

where

$$\Sigma(\mathfrak{B}) := \sum_{n=1}^{\infty} \frac{\mu(n)}{\det \Lambda\left(\left(\sqrt{5}\right)^{-1}\mathfrak{B}, n\mathbb{Z}\right)}.$$

Let us evaluate this sum for any ideal $\mathfrak{B}$ of $\mathcal{O}_K$ dividing $5\mathcal{O}_K$. Elementary manipulations show that $\Lambda((\sqrt{5})^{-1}\mathfrak{B}, n\mathbb{Z})$ is the sublattice of $\mathbb{Z}^3$ consisting of all

$$\boldsymbol{\omega} = (\omega_0, \omega_1, \omega_2) \in \left(n\mathbb{Z} \cap \left(\sqrt{5}\right)^{-1}\mathfrak{B}\right)^3 \text{ such that } \sqrt{2}\omega_1 + \sqrt{3}\omega_2 \in \mathfrak{B}. \quad \text{(A.2)}$$

We have $5\mathcal{O}_K = \mathfrak{P}_1^2\mathfrak{P}_2^2$, where

$$\mathfrak{P}_1 := \left(5, \sqrt{15} - \sqrt{10} + \sqrt{6} - 1\right), \mathfrak{P}_2 := \left(5, \sqrt{15} - \sqrt{10} + \sqrt{6} + 1\right)$$

are distinct prime ideals of $\mathcal{O}_K$ with inertia degrees equal to 2.

For $\mathfrak{B} = \mathcal{O}_K$, the first condition in (A.2) amounts to $\boldsymbol{\omega} \in (n\mathbb{Z})^3$. Then the second condition is always satisfied, and $\det \Lambda((\sqrt{5})^{-1}\mathcal{O}_K, n\mathbb{Z}) = n^3$. Therefore,

$$\Sigma(\mathcal{O}_K) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^3} = \frac{1}{\zeta(3)}. \quad \text{(A.3)}$$

If $\mathfrak{B} = \mathfrak{P}_1$, then the first condition in (A.2) is equivalent to $\boldsymbol{\omega} \in (n\mathbb{Z})^3$. For the second condition, we find that $-(\sqrt{3})^{-1}\sqrt{2} \equiv 3 \mod \mathfrak{P}_1$, so this condition is equivalent to $\omega_2 = 3\omega_1 + a$, for an $a \in \mathfrak{P}_1 \cap n\mathbb{Z} = \operatorname{lcm}(5, n)\mathbb{Z}$. Therefore, $\Lambda((\sqrt{5})^{-1}\mathfrak{P}_1, n\mathbb{Z})$ has the basis

$$\{(n, 0, 0), (0, n, 3n), (0, 0, \operatorname{lcm}(5, n))\}$$

of determinant $n^2 \operatorname{lcm}(5, n)$. A similar computation shows that $-(\sqrt{3})^{-1}\sqrt{2} \equiv 2 \mod \mathfrak{P}_2$, so

$$\{(n, 0, 0), (0, n, 2n), (0, 0, \operatorname{lcm}(5, n))\}$$

is a basis of $\Lambda((\sqrt{5})^{-1}\mathfrak{P}_2, n\mathbb{Z})$ of the same determinant. Thus,

$$\Sigma(\mathfrak{P}_i) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^2 \operatorname{lcm}(5, n)} = \frac{1}{\zeta(3)} \frac{5^2 - 1}{5^3 - 1}. \quad \text{(A.4)}$$

For $\mathfrak{B} = \mathfrak{P}_1\mathfrak{P}_2 = \sqrt{5}\mathcal{O}_K$, the first condition in (A.2) is again equivalent to $\boldsymbol{\omega} \in (n\mathbb{Z})^3$. The second condition is equivalent to $\omega_2 \equiv -(\sqrt{3})^{-1}\sqrt{2}\omega_1 \mod \mathfrak{P}_1\mathfrak{P}_2$. By the Chinese remainder theorem and what we have seen before, this is equivalent to

$$\omega_2 \equiv 2\omega_1 \mod 5 \quad \text{and} \quad \omega_2 \equiv 3\omega_1 \mod 5,$$

so $\omega_1 \equiv \omega_2 \equiv 0 \mod 5$. Thus, $\Lambda((\sqrt{5})^{-1}\mathfrak{P}_1\mathfrak{P}_2, n\mathbb{Z}) = n\mathbb{Z} \times (\mathrm{lcm}(5,n)\mathbb{Z})^2$ has determinant $n\,\mathrm{lcm}(5,n)^2$. We obtain

$$\Sigma(\mathfrak{P}_1\mathfrak{P}_2) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n\,\mathrm{lcm}(5,n)^2} = \frac{1}{\zeta(3)}\frac{5-1}{5^3-1}. \tag{A.5}$$

In the other cases, that is $\mathfrak{P}_1^2 \mid \mathfrak{B}$ or $\mathfrak{P}_2^2 \mid \mathfrak{B}$, we have $\mathfrak{d}((\sqrt{5})^{-1}\mathfrak{B}) = 5\mathbb{Z}$, so the first condition in (A.2) is equivalent to $\boldsymbol{\omega} \in (\mathrm{lcm}(5,n)\mathbb{Z})^3$. In this case, the second condition is always satisfied, so we obtain $\det \Lambda((\sqrt{5})^{-1}\mathfrak{B}, n\mathbb{Z}) = \mathrm{lcm}(5,n)^3$ and

$$\Sigma(\mathfrak{B}) = \sum_{n=1}^{\infty} \frac{\mu(n)}{\mathrm{lcm}(5,n)^3} = 0. \tag{A.6}$$

A simple computation shows that

$$\mathfrak{N}_K\left(\left(\sqrt{5}\right)^{-1}\mathcal{O}_K\right)^{3/8} = 5^{-3/2}, \quad \mathfrak{N}_K\left(\left(\sqrt{5}\right)^{-1}\mathfrak{P}_i\right)^{3/8} = 5^{-3/4},$$
$$\mathfrak{N}_K(\mathcal{O}_K)^{3/8} = 1.$$

To prove the lemma, just substitute this and (A.3) – (A.6) in (A.1). □

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