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Obstructions to finite dimensional cohomology of abstract Cauchy-Riemann complexes

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Abstract. Let M be a compact abstract CR manifold of arbitrary CR codimension. Under certain conditions on the Levi form we prove the infinite dimensionality of some global cohomology groups of M.

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1. Introduction

Although there is a sizeable literature concerning various questions about CR embeddable CR manifolds, there appears to be very few results about the cohomology of abstract CR manifolds. We consider a $\mathcal{C}^\infty$ smooth compact orientable abstract CR manifold of type (n, k).

Here an abstract CR manifold of type (n, k) is a triple (M, HM, J), where M is a smooth real manifold of dimension $2n + k$, HM is a subbundle of rank $2n$ of the tangent bundle TM, and $J : HM \to HM$ is a smooth fiber-preserving bundle isomorphism with $J^2 = -\mathrm{Id}$. We also require that J be formally integrable; *i.e.* that we have

$$\left[T^{0,1}M, T^{0,1}M\right] \subset T^{0,1}M$$

where

$$T^{0,1}M = \{X + iJX \mid X \in \Gamma(M, HM)\} \subset \Gamma(M, \mathbb{C}TM),$$

with Γ denoting smooth sections.

The CR dimension of M is $n \geq 1$ and the CR codimension is $k \geq 1$.

We denote by $H^oM = \{\xi \in T^*M \mid \langle X, \xi\rangle = 0, \forall X \in H_{\pi(\xi)}M\}$ the *characteristic conormal bundle* of M. Here $\pi : TM \longrightarrow M$ is the natural projection. To each $\xi \in H^o_pM$, we associate the Levi form at ξ:

$$\mathcal{L}_p(\xi, X) = \xi\left(\left[J\tilde{X}, \tilde{X}\right]\right) = d\tilde{\xi}(X, JX) \text{ for } X \in H_pM$$

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which is Hermitian for the complex structure of H_pM defined by J. Here $\tilde{\xi}$ is a section of H^oM extending ξ and $\tilde{X}$ a section of HM extending X.

We denote by $\overline{\partial}_M$ the tangential Cauchy-Riemann operator on M. The associated cohomology groups of $\overline{\partial}_M$ acting on smooth forms will be denoted by $H^{p,q}(M), 0 \leq p \leq n+k,\ 0 \leq q \leq n$. For more details on the $\overline{\partial}_M$ complex, we refer the reader to [6] or [7].

Our results are as follows.

Theorem 1.1. *Let M be a compact orientable abstract CR manifold of type (n, k). Assume that there exists a point $p_0 \in M$ and a characteristic conormal direction $\xi \in H^o_{p_0}M$ such that the Levi form $\mathcal{L}_{p_0}(\xi, \cdot)$ has q negative and $n-q$ positive eigenvalues. Then for $0 \leq p \leq n+k$, the following holds: Either $H^{p,q}(M)$ or $H^{p,q+1}(M)$ is infinite dimensional, and either $H^{p,n-q}(M)$ or $H^{p,n-q+1}(M)$ is infinite dimensional.*

This result is proved in Section 2. Although here we are proving the infinite dimensionality of certain (global) cohomology groups of M, our argument follows the pattern of M. Nacinovich [11], where the emphasis was on demonstrating the absence of the (local) Poincaré lemma.

The following two theorems are consequences of Theorem 1.1. The simple arguments proving them are given at the end of Section 2.

Theorem 1.2. *Let M be a compact orientable abstract CR manifold of type $(n, 1)$. Assume that at each point $x \in M$, there exists a characteristic conormal direction $\xi \in H^o_x(M)$ such that $\mathcal{L}_x(\xi, \cdot)$ has q negative and $n-q$ positive eigenvalues. If moreover $2q \neq n-1$, then $H^{p,q}(M)$ is infinite dimensional; and if $2q \neq n+1$, then $H^{p,n-q}(M)$ is infinite dimensional for $0 \leq p \leq n+1$.*

Theorem 1.3. *Let M be a compact orientable abstract CR manifold of type (n, k) with n even. For $q = \frac{n}{2}$ we assume that at each point $x \in M$ and every characteristic conormal direction $\xi \in H^o_x(M) \setminus \{0\}$ the Levi form $\mathcal{L}_x(\xi, \cdot)$ has q negative and q positive eigenvalues. Then $H^{p,q}(M)$ is infinite dimensional for $0 \leq p \leq n+k$.*

In the case where M is CR embedded in some ambient complex manifold, related local and global results have been discussed in [1–3] and [7].

2. Proofs of the theorems

Our proof of Theorem 1.1 relies on a well-known construction for CR embedded CR manifolds at a point where there exists a characteristic conormal direction such that the associated Levi form has exactly q negative and $n-q$ positive eigenvalues. For the reader's convenience, we now sketch this construction in the case of a hypersurface in $\mathbb{C}^{n+1}$.

So let $S \ni 0$ be a piece of a smooth real hypersurface in $\mathbb{C}^{n+1}$ such that $\mathcal{L}_0(\xi, \cdot)$ has q negative and $n-q$ positive eigenvalues for some characteristic conormal direction ξ. Then we can choose a local real defining function ρ of S of the form

$$\rho(z) = \mathrm{Im}\big(z_{n+1}\big) - h(z) \quad \text{with } h(z) = O\left(|z|^2\right).$$

Here $O(|z|^{\ell})$ denotes a term vanishing to order ℓ at the point $z = 0$. Moreover, after a holomorphic change of coordinates, we may assume

$$h(z) = \sum_{\alpha=1}^{q} |z_\alpha|^2 - \sum_{\alpha=q+1}^{n} |z_\alpha|^2 + O\left(|z|^3\right) \quad \text{at } 0.$$

Set

$$\phi(z) = -i\mathrm{Re}\big(z_{n+1}\big) + h(z) - 2\sum_{\alpha=1}^{q} |z_\alpha|^2 - \big(\mathrm{Re}\big(z_{n+1}\big) + ih(z)\big)^2.$$

Then

$$\mathrm{Re}\phi(z) \leq -\frac{1}{2}\left(\sum_{\alpha=1}^{n} |z_\alpha|^2 + (\mathrm{Re}(z_{n+1}))^2\right) \quad \text{near } 0.$$

For $\lambda > 0$ we then define the following "peak forms"

$$f_\lambda = e^{\lambda\phi} dz_1 \wedge \ldots \wedge dz_p \wedge d\bar{z}_1 \wedge \ldots d\bar{z}_q,$$

which defines a smooth (p, q)-form on S satisfying $\bar{\partial}_S f_\lambda = 0$ (note that $\mathrm{Re}(z_{n+1}) + ih$ is the restriction to S of the holomorphic function z_{n+1}).

Similarly we set

$$\psi(z) = i\mathrm{Re}\big(z_{n+1}\big) - h(z) - 2\sum_{\alpha=q+1}^{n} |z_\alpha|^2 - \Big(\mathrm{Re}(z_{n+1}) + ih(z)\Big)^2.$$

Then we also have

$$\mathrm{Re}\psi(z) \leq -\frac{1}{2}\left(\sum_{\alpha=1}^{n} |z_\alpha|^2 + (\mathrm{Re}(z_{n+1}))^2\right) \quad \text{near } 0,$$

and we define the following "peak forms" of degree $(n + 1 - p, n - q)$ on S:

$$g_\lambda = e^{\lambda\psi} dz_{p+1} \wedge \ldots \wedge dz_{n+1} \wedge d\bar{z}_{q+1} \wedge \ldots \wedge d\bar{z}_n.$$

Again we have $\bar{\partial}_S g_\lambda = 0$.

In the proof of the nonvalidity of the Poincaré lemma for the $\bar{\partial}_S$-operator, the forms f_λ and g_λ play an essential role, because their properties contradict the existence of certain a priori estimates. Also our proof of Theorem 1.1 is based on the existence of forms with the analogous properties up to some terms vanishing to infinite order at the point under consideration.

For more details on the construction of the corresponding functions and forms in the higher codimensional situation, we refer the reader to the paper [1, page 388].

Proof of Theorem 1.1. Let us first consider the case $q > 0$, and assume by contradiction that both $H^{p,q}(M)$ and $H^{p,q+1}(M)$ are finite dimensional. In order to make the proof as clear as possible, we first assume that $k = 1$ (CR manifold of hypersurface type).

By the assumption that $H^{p,q}(M)$ is finite dimensional we get that

$$\overline{\partial}_M : \mathcal{C}^\infty_{p,q-1}(M) \longrightarrow \mathcal{C}^\infty_{p,q}(M)$$

has closed range. Then Banach's open mapping theorem implies that there exist a constant $C_1 > 0$ and an integer $m_1 > 0$ such that for all $f \in \overline{\partial}_M \mathcal{C}^\infty_{p,q-1}(M)$ there exists $u \in \mathcal{C}^\infty_{p,q-1}(M)$ satisfying $\overline{\partial}_M u = f$ and

$$\|u\|_0 \leq C_1 \|f\|_{m_1}. \tag{2.1}$$

Here $\|\cdot\|_m$ denotes the usual $\mathcal{C}^m$-norm on $\mathcal{C}^\infty_{\cdot,\cdot}(M)$.

Reasoning as before, the assumption that $H^{p,q+1}(M)$ is finite dimensional implies that there exist a constant $C_2 > 0$ and and integer $m_2 > 0$ such that for all $g \in \overline{\partial}_M \mathcal{C}^\infty_{p,q}(M)$ there exists $h \in \mathcal{C}^\infty_{p,q}(M)$ satisfying $\overline{\partial}_M h = g$ and

$$\|h\|_{m_1} \leq C_2 \|g\|_{m_2}. \tag{2.2}$$

Using Stokes' formula, we have for every $f = \overline{\partial}_M u \in \overline{\partial}_M \mathcal{C}^\infty_{p,q-1}(M)$ and every $g \in \mathcal{C}^\infty_{n+1-p,n-q}(M)$ that

$$\int_M f \wedge g = \int_M \overline{\partial}_M u \wedge g = (-1)^{p+q} \int_M u \wedge \overline{\partial}_M g.$$

Hence (2.1) implies

$$\left| \int_M f \wedge g \right| \lesssim C_1 \|f\|_{m_1} \cdot \|\overline{\partial}_M g\|_0 \tag{2.3}$$

for every $f \in \overline{\partial}_M \mathcal{C}^\infty_{p,q-1}(M)$ and every $g \in \mathcal{C}^\infty_{n+1-p,n-q}(M)$. Here $a \lesssim b$ means that there exists a constant $C > 0$ such that $a \leq C \cdot b$.

Now let $l := \dim H^{p,q}(M) < +\infty$, and let Ω be an open neighborhood of $p_0 \in M$ such that for every point $x \in \Omega$, there exists a characteristic conormal direction ξ_x such that $\mathcal{L}_x(\xi_x, \cdot)$ has q negative and $n - q$ positive eigenvalues.

We choose l different points $p_1, \ldots, p_l$ inside Ω, all different from p_0. Moreover, we choose cut-off functions χ_j, $j = 0, \ldots, l$, with $\chi_j \equiv 1$ near p_j, such that the χ_j's have disjoint supports. For each $0 \leq j \leq l$, we then make the following construction:

Choose local coordinates $z_1 = x_1 + ix_{n+1}, \ldots, z_n = x_n + ix_{2n}, x_{2n+1}$ for M so that p_j becomes the origin. By the formal Cauchy-Kowalewski procedure, we can find smooth complex valued functions $\varphi = (\varphi_1, \ldots, \varphi_{n+1})$ in an open neighborhood U of 0 with each $\varphi_i(0) = 0$, $d\varphi_1 \wedge \ldots \wedge \varphi_{n+1} \neq 0$ in U, and such that

$\overline{\partial}_M\varphi_i$ vanishes to infinite order at 0. Then $\varphi : U \longrightarrow \mathbb{C}^{n+1}$ gives a smooth local embedding $\tilde{M} = \varphi(U)$ of M into $\mathbb{C}^{n+1}$. On $\tilde{M}$ there is the CR structure induced from $\mathbb{C}^{n+1}$; it agrees to infinite order at 0 with the original CR structure on M. In particular $\tilde{M}$ is a smooth real hypersurface in $\mathbb{C}^{n+1}$ which is strictly q-convex and strictly $(n-q)$-concave with respect to the induced CR structure. As explained in the paragraphs preceeding this proof this means that after possibly shrinking U, there are smooth complex valued functions ϕ_j and ψ_j on U with $\overline{\partial}_M\phi_j$ and $\overline{\partial}_M\psi_j$ vanishing to infinite order at 0 satisfying

$$\mathrm{Re}\phi_j \leq -\frac{1}{2}|x|^2 \quad \text{on } U, \tag{2.4}$$

$$\mathrm{Re}\psi_j \leq -\frac{1}{2}|x|^2 \quad \text{on } U \tag{2.5}$$

and

$$\phi_j + \psi_j = -2|x|^2 + O\left(|x|^3\right) \tag{2.6}$$

(one constructs the corresponding functions ϕ and ψ on $\tilde{M}$ and considers the pull-back under φ.)

Moreover, T^*M is spanned by forms

$$\begin{aligned}\omega_1 &= dz_1 + O\big(|x|^\infty\big), \ldots, \omega_n = dz_n + O\big(|x|^\infty\big),\\ \overline{\omega}_1 &= d\overline{z}_1 + O\big(|x|^\infty\big), \ldots, \overline{\omega}_n = d\overline{z}_n = O\big(|x|^\infty\big),\\ \theta &= dx_{2n+1} + O\big(|x|^\infty\big)\end{aligned}$$

which are d-closed to infinite order at 0 (here, of course, $\Lambda T^{0,1}M$ is spanned by $\overline{\omega}_1, \ldots, \overline{\omega}_n$). Following again [1] or [7], by the geometric condition on the Levi-form of M we may also assume that $\overline{\partial}_M\phi_j\wedge\overline{\omega}_1\wedge\ldots\wedge\overline{\omega}_q$ and $\overline{\partial}_M\psi_j\wedge\overline{\omega}_{q+1}\wedge\ldots\wedge\overline{\omega}_n$ vanish to infinite order at 0.

For each real $\lambda > 0$ we now define

$$f_j^\lambda = \chi_j e^{\lambda\phi_j}\omega_1 \wedge \ldots \wedge \omega_p \wedge \overline{\omega}_1 \ldots \wedge \overline{\omega}_q.$$

This is a smooth (p,q)-form on M. Moreover the properties of ϕ_j imply that $\overline{\partial}_M(f_j^\lambda)$ is rapidly decreasing with respect to λ in the topology of $\mathcal{C}^\infty(M)$ as λ tends to infinity. Indeed, by (2.4) the function $\exp(\lambda\phi_j)$, and any derivative of it with respect to x, is rapidly decreasing as $\lambda \to +\infty$, while all other terms, and their derivatives with respect to x, have only polynomial growth in λ.

We also set

$$g_j^\lambda = \chi_j e^{\lambda\psi_j}\omega_{p+1} \wedge \ldots \wedge \omega_n \wedge \theta \wedge \overline{\omega}_{q+1} \wedge \ldots \wedge \overline{\omega}_n.$$

Then, arguing as before, $\overline{\partial}_M(g_j^\lambda)$ is rapidly decreasing with respect to λ in the topology of $\mathcal{C}^\infty(M)$ as λ tends to infinity.

Next, we solve $\overline{\partial}_M u_j^\lambda = \overline{\partial}_M f_j^\lambda$ with an estimate

$$\left\| u_j^\lambda \right\|_{m_1} \leq C_2 \left\| \overline{\partial}_M f_j^\lambda \right\|_{m_2}, \tag{2.7}$$

using (2.2). Hence $\|u_j^\lambda\|_{m_1}$ is rapidly decreasing with respect to λ. Defining $\tilde{f}_j^\lambda = f_j^\lambda - u_j^\lambda$, we obtain a smooth, $\overline{\partial}_M$-closed (p, q)-form on M.

Since $\dim H^{p,q}(M) = l$, there exist constants $c_0^\lambda, \ldots, c_l^\lambda$, not all equal to zero, such that

$$c_0^\lambda \tilde{f}_0^\lambda + \ldots + c_l^\lambda \tilde{f}_l^\lambda \in \mathrm{Im}\overline{\partial}_M.$$

To get a contradiction, we are going to use the estimate (2.3) with $f = \sum_{j=0}^l c_j^\lambda \tilde{f}_j^\lambda$ and $g = \sum_{j=0}^l \overline{c}_j^\lambda g_j^\lambda$. We have

$$\begin{aligned}
\int_M f \wedge g &= \int_M \left(\sum_{j=0}^l c_j^\lambda \tilde{f}_j^\lambda \right) \wedge \left(\sum_{j=0}^l \overline{c}_j^\lambda g_j^\lambda \right) \\
&= \int_M \left(\sum_{j=0}^l c_j^\lambda \left(f_j^\lambda - u_j^\lambda \right) \right) \wedge \left(\sum_{j=0}^l \overline{c}_j^\lambda g_j^\lambda \right) \\
&= \sum_{j=0}^l \left| c_j^\lambda \right|^2 \int_M f_j^\lambda \wedge g_j^\lambda - \int_M \sum_{i,j=0}^l c_i^\lambda \overline{c}_j^\lambda u_i^\lambda \wedge g_j^\lambda .
\end{aligned} \tag{2.8}$$

Note that for the third equality, we have used that the χ_j's have disjoint supports.

We are now going to estimate the term on the right of (2.8). We have

$$\begin{aligned}
\int_M f_j^\lambda \wedge g_j^\lambda &= \int_M \chi_j^2 e^{\lambda(\phi_j + \psi_j)} \omega_1 \wedge \ldots \wedge \omega_n \wedge \theta \wedge \overline{\omega}_1 \wedge \ldots \wedge \overline{\omega}_n \\
&= \int_M \Big\{ \chi_j^2 e^{\lambda(-2|x|^2 + O(|x|^3)} \\
&\qquad + O(|x|) \Big\} dz_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_n \wedge dx_{2n+1}.
\end{aligned}$$

Making the change of variables $y = \sqrt{\lambda} x$, and afterwards changing the name of y back to x, we get

$$\begin{aligned}
\int_M f_j^\lambda \wedge g_j^\lambda = \lambda^{-n-\frac{1}{2}} \Big\{ &\int_M \chi_j^2 \left(\frac{x}{\sqrt{\lambda}} \right) e^{-2|x|^2 + O(\lambda^{-\frac{1}{2}})} dz_1 \wedge \ldots \\
&\wedge dz_n \wedge d\overline{z}_1 \wedge \ldots \wedge d\overline{z}_n \wedge dx_{2n+1} + O(\lambda^{-\frac{1}{2}}) \Big\}.
\end{aligned}$$

Therefore we obtain

$$\left| \int_M f_j^\lambda \wedge g_j^\lambda \right| \geq c \lambda^{-n-\frac{1}{2}} \tag{2.9}$$

for some constant $c > 0$.

Also we can use (2.7) to get

$$\left|\int_M \sum_{i,j=0}^{l} c_i^\lambda \overline{c}_j^\lambda u_i^\lambda \wedge g_j^\lambda\right| \lesssim \sum_{j=0}^{l} \left|c_j^\lambda\right|^2 \sup_{i,j}\left(\|u_i^\lambda\|_0 \cdot \|g_j^\lambda\|_0\right)$$
$$\lesssim \sum_{j=0}^{l} \left|c_j^\lambda\right|^2 \sup_{i,j}\left(\left\|\overline{\partial}_M f_i^\lambda\right\|_{m_2} \cdot \|g_j^\lambda\|_0\right).$$

Now $\|\overline{\partial}_M f_i^\lambda\|_{m_2}$ is rapidly decreasing with respect to λ, whereas $\|g_j^\lambda\|_0$ is of polynomial growth with respect to λ, hence we get

$$\left|\int_M \sum_{i,j=0}^{l} c_i^\lambda \overline{c}_j^\lambda u_i^\lambda \wedge g_j^\lambda\right| \leq \sum_{j=0}^{l} \left|c_j^\lambda\right|^2 \lambda^{-n-1}$$

for sufficiently large λ. Combining this with (2.9), we get

$$\left|\int_M f \wedge g\right| \geq \frac{c}{2} \sum_{j=0}^{l} \left|c_j^\lambda\right|^2 \lambda^{-n-\frac{1}{2}} \tag{2.10}$$

for sufficiently large λ.

On the other hand, using (2.3), we can estimate $\int_M f \wedge g$ as follows:

$$\left|\int_M f \wedge g\right| \leq C_1 \|f\|_{m_1} \cdot \|\overline{\partial}_M g\|_0$$
$$\lesssim \sum_{j=0}^{l} \left|c_j^\lambda\right|^2 \sup_{i,j}\left(\left\|\tilde{f}_j^\lambda\right\|_{m_1} \cdot \left\|\overline{\partial}_M g_j^\lambda\right\|_0\right)$$
$$\lesssim \sum_{j=0}^{l} \left|c_j^\lambda\right|^2 \sup_{i,j}\left(\left\|f_j^\lambda\right\|_{m_2+1} \cdot \left\|\overline{\partial}_M g_j^\lambda\right\|_0\right).$$

Since $\|f_j^\lambda\|_{m_2+1}$ is of polynomial growth in λ whereas $\|\overline{\partial}_M g_j^\lambda\|_0$ is rapidly decreasing with respect to λ, we get that

$$\left|\int_M f \wedge g\right| \lesssim \sum_{j=0}^{l} \left|c_j^\lambda\right|^2 \lambda^{-n-1}.$$

This contradicts (2.10) and therefore proves that either $H^{p,q}(M)$ or $H^{p,q+1}(M)$ has to be infinite dimensional.

Now, replacing ξ by $-\xi$, it also follows that either $H^{p,n-q}(M)$ or $H^{p,n-q+1}(M)$ is infinite dimensional.

For $q = 0$, the statement is essentially Boutet de Monvel's result [4]: in this case, M is strictly pseudoconvex at p_0. If $H^{p,1}(M)$ was finite dimensional, then in particular the range of $\overline{\partial}_M$ was closed in $\mathcal{C}^\infty_{p,1}(M)$. But then one can construct infnitely many linearly independent CR functions on M as in [4].

Also, the Levi-form $\mathcal{L}_{p_o}(-\xi, \cdot)$ has $n > 0$ negative and 0 positive eigenvalues. By what already proved, we therefore know that $H^{p,n}(M)$ is infinite dimensional (note that $H^{p,n+1}(M)$ is always zero).

If $k > 1$, the proof is essentially as before, with $\mathbb{C}^{n+1}$ replaced by $\mathbb{C}^{n+k}$. The crucial point is to observe that the approximate CR embedding $\tilde{M}$ in $\mathbb{C}^{n+k}$, which now has real codimension k, is contained in a hypersurface which is strictly q-convex and strictly $(n-q)$-concave. Then ξ corresponds to + or – the conormal to the hypersurface at p_0. As before, this gives us the existence of smooth functions ϕ_j and ψ_j with the same properties that were essential in the proof for $k = 1$. □

Proof of Theorem 1.2. The theorem follows immediately from Theorem 1.1. Indeed, it suffices to note that the assumptions on M imply that the classical conditions $Y(q+1)$ and $Y(n-q+1)$ are satisfied. Hence the $\overline{\partial}_M$-complex is $\frac{1}{2}$-subelliptic in degree $(p, q+1)$ and $(p, n-q+1)$ (see [5, Theorem 5.4.9]), hence $H^{p,q+1}(M)$ and $H^{p,n-q+1}(M)$ are finite dimensional. □

Proof of Theorem 1.3. The assumptions on M imply that M is q-pseudoconcave (see [6] for the definition), hence the $\overline{\partial}_M$-complex is ϵ-subelliptic in degree $(p, q+1)$ for some $\epsilon > 0$ (see [6] for the proof), hence $H^{p,q+1}(M)$ is finite dimensional. Again the statement then follows from Theorem 1.1. □

3. Corollaries and remarks

We would like to emphasize that Theorems 1.1, 1.2, 1.3 are valid for an *abstract* CR manifold M, which might possibly be not even locally CR embeddable at any point. However, it is of some interest to consider the situation where M is globally CR embedded as a generic CR submanifold of some complex manifold X, and ask what these theorems imply about the pair (X, M). Then the complex dimension of X is $n+k$, and we have the usual Dolbeault cohomology groups $H^{p,q}(X)$, as well as the Dolbeault-like cohomology groups $H^{p,q}(X, \mathcal{I})$. The latter consists of smooth $\overline{\partial}$-closed forms on X modulo smooth $\overline{\partial}$-exact forms on X, in which all forms are required to have zero Cauchy data along the submanifold M. (Think of the real codimension k of M in X as corresponding to k "time variables".) More precisely, we consider the sheaf $\mathcal{I}_M$ of germs of $\mathcal{C}^\infty$ functions on X which vanish on M. Then we denote by $\mathcal{I}$ the sheaf of $\mathcal{C}^\infty_{\cdot,\cdot}(X)$-modules which is locally generated by $\mathcal{I}_M$ and $\overline{\partial}\mathcal{I}_M$.

The interpretation of $H^{p,q+1}(X, \mathcal{I})$ is that it is the obstruction to the solvability of the general inhomogeneous Cauchy problem

$$\begin{cases} \overline{\partial} u = f & \text{on } X \\ u = u_0 & \text{on } M. \end{cases} \tag{3.1}$$

Here f is a given smooth $\overline{\partial}$-closed $(p, q + 1)$-form on X, u_0 is a given smooth $\overline{\partial}_M$-closed tangential (p, q)-form on M, and it is assumed that the data $\{f, u_0\}$ are compatible (see [2, page 350–351]). The desired solution u to the problem (3.1) should be a smooth (p, q)-form on X. Then the solvability of (3.1) for all compatible data is equivalent to the vanishing of $H^{p,q+1}(X, \mathcal{I})$. If, for example, $H^{p,q+1}(X, \mathcal{I})$ is infinite dimensional, it means that there is an infinite dimensional set of equivalent classes of data for which (3.1) has no solution (see [2]). From Theorem 1.1 and standard exact sequences, such as the Mayer-Vietoris sequence, we obtain the following:

Corollary 3.1. *With M as in Theorem* 1.1, *assume X is either compact or Stein. Then*:

(a) *Either $H^{p,q+1}(X, \mathcal{I})$ or $H^{p,q+2}(X, \mathcal{I})$ is infinite dimensional*;
(b) *Either $H^{p,n-q+1}(X, \mathcal{I})$ or $H^{p,n-q+2}(X, \mathcal{I})$ is infinite dimensional*.

Proof. If X is compact, then we have $\dim H^{r,s}(X) < +\infty$ for $0 \le r, s \le n + k$, whereas $H^{r,s}(X) = 0$ for $0 \le r \le n + k$, $1 \le s \le n + k$ if X is Stein. Therefore we may use the following long exact sequence

$$\ldots \to H^{r,s}(X) \to H^{r,s}(M) \to H^{r,s+1}(X, \mathcal{I}) \to H^{r,s+1}(X) \to \ldots$$

and Theorem 1.1 to conclude. □

In the special case $k = 1$, we may assume M divides X into complex manifolds-with-boundary, which we call X^- and X^+ (the common boundary is, of course, M). Then, roughly speaking, the cohomology group $H^{p,j+1}(X, \mathcal{I})$ is isomorphic to the direct sum of $H^{p,j}(X^-)$ and $H^{p,j}(X^+)$ modulo some global Dolbeault cohomology groups of X. Therefore the hypersurface case of Corollary 3.1 then breaks down into:

(a) At least one of $H^{p,q}(X^+)$, $H^{p,q}(X^-)$, $H^{p,q+1}(X^+)$, $H^{p,q+1}(X^-)$ is infinite dimensional;
(b) At least one of $H^{p,n-q}(X^+)$, $H^{p,n-q}(X^-)$, $H^{p,n-q+1}(X^+)$, $H^{p,n-q+1}(X^-)$ is infinite dimensional.

We should emphasize here that the above corollary requires a hypothesis on M at only *one* single point p_0 on M and in only *one* single characteristic conormal direction ξ. This has the following consequence: Suppose M is generically CR embedded in X, as above, with X either compact or Stein, but that initially no other hypotheses are made about M. Then the situation is initially whatever it is. But if now we make arbitrarily small smooth modifications of M at a few points, we can produce a modified CR manifold $\tilde{M}$, such that for the new pair $(X, \tilde{M})$, there is a plethora of infinite dimensional cohomology groups $H^{p,*}(X, \tilde{\mathcal{I}})$.

Note that in Theorems 1.2 and 1.3 the situation is quite different. In those theorems some hypothesis is needed at each point of M, which puts a much greater constraint of the "shape" of M, and we are then in a territory that is less novel and has been much more discussed in the literature.

Indeed, with M as in Theorem 1.2, and X compact, we know from [3, page 805] that it is $H^{p,q}(X^-)$ and $H^{p,n-q}(X^+)$ that are infinite dimensional. Also $H^{p,j}(X^-)$ is finite for $j \neq q$ and $H^{p,j}(X^+)$ is finite for $j \neq n-q$. So in this context, given the finiteness theorems proved in [3], what Theorem 1.2 provides us in most cases is just a new proof of the infinite dimensionality of $H^{p,q}(X^-)$ and $H^{p,n-q}(X^+)$. We should also recall from [3] that when $2q \neq n$, we therefore have a good one sided global Cauchy problem in degree q from the X^- side, and another one in degree $n-q$ from the X^+ side. Both these Cauchy problems are almost always solvable. If $2q = n$, then we have an almost always solvable Riemann-Hilbert problem in degree $q = n-q$.

Now with M as in Theorem 1.3 and X compact, M is a maximally pseudoconcave generic CR submanifold of X, of codimension k. Theorem 1.3 gives us a new proof (in the maximally pseudoconcave case) of the infinite dimensionality of $H^{p,q}(M)$, which is related to [6, Theorem 4.2]. In that situation the global solvability of the inhomogeneous Cauchy problem (3.1) is obstructed by the infinite dimensional $H^{p,q+1}(X, \mathcal{I})$.

4. Examples

Standard examples of compact hypersurfaces satisfying the assumptions of Theorem 1.2 are the real projective hypersurfaces

$$M = \Big\{ \big(z_0 : z_1 : \ldots : z_{n+1}\big) \in \mathbb{CP}^{n+1} \mid \operatorname{Im}\big(z_0 z_{n+1}\big) \\ = \big|z_1\big|^1 + \ldots + \big|z_q\big|^2 - \big|z_{q+1}\big|^2 - \ldots - \big|z_n\big|^2 \Big\}.$$

Various other examples of CR manifolds satisfying the assumptions of Theorem 1.2 or Theorem 1.3 have been constructed by C. Medori and M. Nacinovich: They continued the investigations of Tanaka and developed the algebraic theory of Levi-Tanaka algebras in order to construct homogeneous CR manifolds of arbitrary codimension k. In [9, Theorem 4.5] they showed that if the Levi-Tanaka algebra $\mathfrak{g}$ is semisimple, then the associated homogeneous CR manifold $M_{\mathfrak{g}}$ is compact. Moreover, in [10] it was proved that the Levi-form of M is nondegenerate if and only if the Levi-Tanaka algebra is finite dimensional. A complete classification of semisimple Levi-Tanaka algebras was also given in [10]. So in those examples, we get from

Theorems 1.2 and 1.3 that for fixed p, one or two cohomology groups are infinite dimensional, while all others are finite dimensional.

Here is a method to construct compact CR manifolds with at least approximately half of its cohomology groups being infinite dimensional: we start with a compact CR manifold M of arbitrary type (n, k), which we assume to be generically CR embedded into some complex manifold X. Now we cut out a small piece of M near a given point and glue in a small modification, arranging that $q = 0$ at one point, that $q = 1$ at another point, ..., and that $q = n$ at still another point (all happening locally in $\mathbb{C}^{n+k}$). We denote the modified CR manifold by $\tilde{M}$. Then, using Theorem 1.1, we obtain that either $H^{p,j}(\tilde{M})$ or $H^{p,j+1}(\tilde{M})$ must be infinite dimensional for all $j = 0, \ldots, n$.

It is, however, far more difficult to produce examples of *abstract* CR structures having a characteristic conormal direction whose associated Levi-form is nondegenerate. Examples exist, but they are few. Here we would like to mention the following example from [8, Theorem 6.16]:

Let $Q \subset \mathbb{CP}^{n+1}$, $n \geq 1$, be the hyperquadric defined by

$$Q = \left\{ z_0\bar{z}_0 + z_1\bar{z}_1 = z_2\bar{z}_2 + \ldots + z_{n+1}\bar{z}_{n+1} \right\}.$$

Then one can find a new CR structure on Q, which is not locally CR-embeddable at all points of the divisor $D = \{z_0 = 0\}$. We denote Q with this new CR strucure by $\tilde{Q}$. The CR structure on $\tilde{Q}$ is such that at each point $x \in M$, there exists a characteristic conormal direction such that $\mathcal{L}_x(\xi, \cdot)$ has 1 negative and $n-1$ positive eigenvalues, *i.e.* $\tilde{Q}$ satisfies the assumptions of Theorem 1.2 with $q = 1$. In this situation, Theorem 1.2 yields that if $n \neq 3$, then $H^{p,1}(\tilde{Q})$ is infinite dimensional, and if $n \neq 1$, then $H^{p,n-1}(\tilde{Q})$ is infinite dimensional, $0 \leq p \leq n + 1$. This is a new result.

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