Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XV (2016), 209-247

Sobolev extension property for tree-shaped domains with self-contacting fractal boundary

THIBAUT DEHEUVELS

Abstract. In this paper, we investigate the existence of $W^{1,p}$-extension operators for a class of bidimensional ramified domains with a self-similar fractal boundary previously studied by Mandelbrot and Frame. When the fractal boundary has no self-contact, the domains have the (ϵ, δ)-property, and the extension results of Jones imply that there exist such extension operators for all $1 \leqslant p \leqslant \infty$. In the case where the fractal boundary self-intersects, this result does not hold. In this work we construct extension operators for $1 < p < p^\star$, where $p^\star$ depends only on the dimension of the self-intersection of the boundary. The construction of the extension operators is based on a Haar wavelet decomposition on the fractal part of the boundary. It relies mainly on the self-similar properties of the domain. The result is sharp in the sense that $W^{1,p}$-extension operators fail to exist when $p > p^\star$.

Mathematics Subject Classification (2010): 46E35(primary); 28A80 (secondary).

1. Introduction

This work deals with some $W^{1,p}$-extension results for a class of bidimensional ramified domains with a self-similar fractal boundary. The focus will be put on the special case where the boundary of the domain is self-contacting. Such a geometry (see Figure 2.2) can be seen as a bidimensional idealization of the bronchial tree. More precisely, we will investigate if the domains introduced below have the $W^{1,p}$-extension property ($1 < p < \infty$). A domain $D \subset \mathbb{R}^n$ is said to have the $W^{1,p}$-extension property for some $p \in [1, \infty]$ if there exists a bounded linear operator

$$\Lambda : W^{1,p}(D) \to W^{1,p}(\mathbb{R}^n),$$

such that $\Lambda u|_D = u$ for all $u \in W^{1,p}(D)$. Such domains are called $W^{1,p}$-extension domains. A domain that has the $W^{1,p}$-extension property for all $p \in [1, \infty]$ is sometimes referred to as a Sobolev extension domain.

Received July 12, 2013; accepted in revised form April 9, 2014.
Published online February 2016.

It is well known that every Lipschitz domain in $\mathbb{R}^n$, that is every domain whose boundary is locally the graph of a Lipschitz function, is a Sobolev extension domain. Calderón proved the $W^{1,p}$-extension property for $p \in (1, \infty)$, see [7], and Stein extended the result to the cases $p = 1$ and $p = \infty$, see [30]. In [14], Jones improved this result by introducing a class of domains in $\mathbb{R}^n$ satisfying a particular geometric condition called the (ε, δ)-property, every member of which is a Sobolev extension domain. These domains are referred to as (ε, δ)-domains, or sometimes Jones domains. Jones' proof uses as a main ingredient Whitney's extension theory. Such domains were also introduced by Martio and Sarvas who referred to them as locally uniform domains (see [25]). This result is almost optimal in the plane, in the sense that every plane finitely connected Sobolev extension domain is an (ε, δ)-domain, see [14,26]. The case where the domain $D \subset \mathbb{R}^2$ is unbounded has been studied in [31]. Extension properties for domains which do not have the (ε, δ)-property have been studied, *e.g.*, in [27], where the authors examine in particular the case of domains with cusps. In [20], Koskela proved that in the general case, if an open domain $D \subset \mathbb{R}^n$ has the $W^{1,n}$-extension property, then it has the $W^{1,p}$-extension property for all $p \geqslant n$. He also showed that if the embedding $W^{1,p}(D) \hookrightarrow C^{0,1-n/p}(\overline{D})$ holds for some $p > n$, then D is a $W^{1,q}$-extension domain for all $q > p$. The case $p < n$ is not as well understood. Hajłasz, Koskela and Tuominem proved that every $W^{1,p}$-extension domain in $\mathbb{R}^n$ for $1 \leqslant p < \infty$ is an n-set, as defined, *e.g.*, in [17], and provided several characterizations of $W^{1,p}$-extension domains for $1 < p < \infty$, see [12]. See also [21] for geometric properties of domains satisfying a $W^{1,p}$-extension property. The same questions were also investigated in the setting of metric measure spaces, see [11].

The domains we consider in the present work do not have the $W^{1,p}$-extension property for any $p > 2$, and we will study the case when $p < 2$. We focus on a class of tree-shaped domains Ω in $\mathbb{R}^2$ with a self-similar fractal boundary Γ^∞ which self-intersects, see for example Figure 2.2. The set Γ^∞ is defined as the unique compact set such that

$$\Gamma^\infty = f_1(\Gamma^\infty) \cup f_2(\Gamma^\infty),$$

where f_1 and f_2 are two contracting similarities with opposite rotation angles $\pm\theta$ $(0 \leqslant \theta < \frac{\pi}{2})$ and contraction ratio $a \in [0, 1)$. This type of fractal sets were first studied by Mandelbrot and Frame in [24]. We will see in Paragraph 2.1.1 that there exists a critical ratio $a^\star_\theta$ dependent on the rotation angle of the similitude such that for every $a < a^\star_\theta$, the set Γ^∞ is totally disconnected, and for $a = a^\star_\theta$, it is connected. In the first case, the domain Ω is an (ε, δ)-domain and Ω is a Sobolev extension domain. In this paper, emphasis will be put on the latter case, in which Ω is not an (ε, δ)-domain and Ω is not a Sobolev extension domain. In this case, the assumptions required in [14] for the construction of Whitney extension operators are not satisfied.

Particular care will be given to the notion of trace on the set Γ^∞ for functions in $W^{1,p}(\Omega)$. We will use two different definitions of trace on Γ^∞:

- The first one, referred to as the classical or strictly defined trace below, relies on the notion of the strict definition of a locally integrable function, see for instance [17, page 206]. For a function u in $W^{1,p}(\Omega)$ or $W^{1,p}(\mathbb{R}^d)$, this trace, noted $u|_{\Gamma^\infty}$ below, is defined as its strictly defined counterpart on the subset of Γ^∞ where u is strictly defined;

- The second one was first introduced in [3]. Its construction is recalled in Section 3. This trace operator, denoted ℓ^∞ below, is obtained by exploiting the self-similarity as the limit of a sequence of operators ℓ^n which map $W^{1,p}(\Omega)$ to piecewise constant functions on a partition of Γ^∞ into 2^n sets whose measure is 2^{-n}. Achdou and Tchou provided in [5] a characterization of functions in the trace space $\ell^\infty(W^{1,p}(\Omega))$ in terms of Lipschitz functions with jumps (see Section 4) that will be helpful in the proof of the main theorems. Embeddings of the trace space in some Sobolev spaces on the set Γ^∞ were studied especially in [1], see Subsection 4.3.

A consequence of the main result of this paper is that these two definitions of trace on Γ^∞ in fact coincide (almost everywhere) on the set Γ^∞; this is proved in [2].

Jonsson and Wallin have proved extension and trace results for Besov and Sobolev spaces on d-sets (see [17]). See Paragraph 2.2.3 for a definition of the Besov spaces on such sets in the special case of Γ^∞ which is a d-set where d is its Hausdorff dimension. In particular, see [17, page 183], $W^{1,p}(\mathbb{R}^2)|_{\Gamma^\infty} = W^{1-\frac{2-d}{p},p}(\Gamma^\infty)$ for $p \in (1,\infty)$, where the trace is meant in the classical sense. The extension part of the theorem mainly relies on Whitney's extension theorem.

On the other hand, it has been proved in [1] that there exists a real number $p^\star > 1$ such that $p^\star$ only depends on the Hausdorff dimension of the self-intersection of Γ^∞ and

- if $p < p^\star$, then $\ell^\infty(W^{1,p}(\Omega)) = W^{1-\frac{2-d}{p},p}(\Gamma^\infty)$,
- if $p > p^\star$, then the previous result does not hold.

The main extension result of this paper (Theorem B) states that when $p < p^\star$, the domains Ω in fact have the $W^{1,p}$-extension property. To prove this result, we start by proving in Theorem A that there exists a continuous lifting operator from $W^{1-\frac{2-d}{p},p}(\Gamma^\infty)$ to $W^{1,p}(\mathbb{R}^2)$ in the sense of the trace operator ℓ^∞. We prove this last result by constructing an extension operator based on the Haar wavelet decomposition of functions on Γ^∞. The strategy we propose can be seen as a self-similar adaptation of the Whitney decomposition (see [32]).

The proof of Theorem B relies on the construction of a sequence of operators converging to the desired extension operator. The construction uses the operator of Theorem A and self-similar properties of this operator that *a priori* are not guaranteed by the lifting operator of Jonsson and Wallin in the classical sense.

Theorem B is sharp in the sense that whenever $p > p^\star$, Ω is not a $W^{1,p}$-extension domain, see Remark 5.3 below. The case $p = p^\star$ is partially discussed in Remark 5.3.

An immediate consequence of this extension theorem is that, for $p \in (1, p^\star)$, the Sobolev embeddings hold in Ω.

Note that the question of extensions or traces naturally arises in boundary value or transmission problems in domains with fractal boundaries. Results in this direction have been given by Mosco and Vivaldi (see [29]), Lancia (see [22, 23]), and Capitanelli (see [8]) for the Koch snowflake. Boundary value problems posed in the domains Ω displayed in Figure 2.2 were studied, *e.g.*, in [3], and numerical methods were proposed to compute the solutions in subdomains of Ω. Such a geometry can be seen as a bidimensional idealization of the bronchial tree, for example. The problems studied in the latter papers aim at simulating the diffusion of medical sprays in human lungs.

The paper is organized as follows: the geometry of the studied domains is presented in Section 2. In Section 3, we briefly treat the less interesting sub-critical case when $a < a_\theta^\star$ and recall the construction of the trace operator introduced in [3]. The theory proposed in [16] is reviewed in Section 4, where we also recall the characterization of the trace space proved in [5] and the trace theorems proved in [1]. The main results of the paper are Theorems A and B which are stated in Section 5 and proved in Sections 6 and 7. For the ease of the reader, the geometrical lemmas, which are crucial but technical, are proved in the Appendix at the end of the paper.

2. The geometry

In this section we define the geometry of fractal self-similar sets Γ^∞, and ramified domains Ω whose boundary contains Γ^∞, see for example Figure 2.2.

2.1. The self-similar set Γ^∞

2.1.1. *Definitions and notation*

Consider four real numbers a, α, β, θ such that $0 < a < 1/\sqrt{2}, \alpha > 0, \beta > 0$ and $0 < \theta < \pi/2$. Let f_i, for $i = 1, 2$ be the two similarities in $\mathbb{R}^2$ given by

$$f_1\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}-\alpha\\\beta\end{pmatrix} + a\begin{pmatrix}x_1\cos\theta - x_2\sin\theta\\x_1\sin\theta + x_2\cos\theta\end{pmatrix},$$

$$f_2\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\end{pmatrix} + a\begin{pmatrix}x_1\cos\theta + x_2\sin\theta\\-x_1\sin\theta + x_2\cos\theta\end{pmatrix}.$$

The two similarities have the same dilation ratio a and opposite angles $\pm\theta$. One can obtain f_2 by composing f_1 with the symmetry with respect to the axis $\{x_1 = 0\}$. We denote by Γ^∞ the self-similar set associated to the similarities f_1 and f_2, *i.e.*, the unique compact subset of $\mathbb{R}^2$ such that

$$\Gamma^\infty = f_1(\Gamma^\infty) \cup f_2(\Gamma^\infty).$$

Notation We denote by:

- $\mathcal{A}_n$ the set containing all the 2^n mappings from $\{1,\dots,n\}$ to $\{1,2\}$ also called strings of length n for $n \geqslant 1$;
- $\mathcal{A}_0$ the set containing only one element called the empty string, that we agree to note ϵ;
- $\mathcal{A}$ the set defined by $\mathcal{A} = \cup_{n\geqslant 0}\mathcal{A}_n$ containing the empty string and all the finite strings;
- $\mathcal{A}_\infty = \{1,2\}^{\mathbb{N}\setminus\{0\}}$ the set of the sequences $\sigma = (\sigma(i)\ )_{i=1,\dots,\infty}$ with values $\sigma(i) \in \{1,2\}$, *i.e.*, the set of all infinite strings.

We will use the following notation:

- if n, m are non-negative integers and $\sigma \in \mathcal{A}_n$, $\sigma' \in \mathcal{A}_m$, define:
$$\sigma\sigma' = \big(\sigma(1),\dots,\sigma(n),\sigma'(1),\dots,\sigma'(m)\big) \in \mathcal{A}_{n+m}; \tag{2.1}$$
if $m = \infty$, we define similarly $\sigma\sigma' \in \mathcal{A}_\infty$;
- for $n > 0$, $\sigma \in \mathcal{A}_n$, and $k \geqslant 0$, we define
$$\sigma^k = \underbrace{\sigma\sigma\dots\sigma}_{k} \in \mathcal{A}_{nk}, \quad \sigma^\infty = \sigma\sigma\dots\sigma\dots \in \mathcal{A}_\infty; \tag{2.2}$$
- for $\sigma, \tau \in \mathcal{A}$, define
$$\sigma|\tau = \{\sigma,\tau\} \subset \mathcal{A}; \tag{2.3}$$
- for $\sigma \in \mathcal{A}$ and $\mathcal{X} \subset \mathcal{A} \cup \mathcal{A}_\infty$, define the set
$$\sigma\mathcal{X} = \{\sigma\tau,\ \tau \in \mathcal{X}\}; \tag{2.4}$$
similarly, if $\mathcal{X} \subset \mathcal{A}$, define the set $\mathcal{X}\sigma = \{\tau\sigma,\ \tau \in \mathcal{X}\}$;
- for $\mathcal{X} \subset \mathcal{A}$ and $k \in \mathbb{N}$, introduce the sets
$$\begin{aligned}\mathcal{X}^k &= \{\sigma_1\dots\sigma_k,\ \sigma_1,\dots,\sigma_k \in \mathcal{X}\},\\ \mathcal{X}^\infty &= \{\sigma_1\sigma_2\dots \in \mathcal{A}_\infty,\ \forall i,\ \sigma_i \in \mathcal{X}\},\\ \mathcal{X}^\star &= \bigcup_{k\in\mathbb{N}} \mathcal{X}^k.\end{aligned}$$

Example 2.1. The set $(12|21)^\infty$ is the set of infinite strings $\sigma \in \mathcal{A}_\infty$ such that $\sigma(2k) \neq \sigma(2k-1)$ for all positive integers k. For $n \geqslant 0$, the set $(12|21)^n(1|2|\epsilon)$ is the set of strings $\sigma \in \mathcal{A}_{2n} \cup \mathcal{A}_{2n+1}$ such that $\sigma(2k) \neq \sigma(2k-1)$ for all integers $k \in [1,n]$. The set $(12|21)^\star(1|2|\epsilon)$ is the set of strings $\sigma \in \mathcal{A}$ such that $\sigma \in (12|21)^n(1|2|\epsilon)$ for some $n \geqslant 0$.

We say that $\sigma \in \mathcal{A}$ is a prefix of $\tau \in \mathcal{A} \cup \mathcal{A}_\infty$ if $\tau = \sigma\sigma'$ for some $\sigma' \in \mathcal{A} \cup \mathcal{A}_\infty$. For $\sigma \in \mathcal{A}_n$ ($n \in \mathbb{N}$) and $k \leqslant n$, we denote by $\sigma_{\restriction k}$ the only prefix of σ in $\mathcal{A}_k$:
$$\sigma_{\restriction k} = (\sigma(1),\dots,\sigma(k)) \in \mathcal{A}_k. \tag{2.5}$$

Similarly, we say that $\sigma' \in \mathcal{A} \cup \mathcal{A}_\infty$ is a suffix of $\tau \in \mathcal{A} \cup \mathcal{A}_\infty$ if $\tau = \sigma\sigma'$ for some $\sigma \in \mathcal{A}$. For a positive integer n and $\sigma \in \mathcal{A}_n$, we define the similitude f_σ by

$$f_\sigma = f_{\sigma(1)} \circ \dots \circ f_{\sigma(n)}. \tag{2.6}$$

We also agree that $f_\epsilon = \mathrm{Id}$. Similarly, if $\sigma \in \mathcal{A}_\infty$, define

$$f_\sigma = \lim_{n\to\infty} f_{\sigma(1)} \circ \dots \circ f_{\sigma(n)}. \tag{2.7}$$

If $\sigma \in \mathcal{A}_\infty$ and $x \in \mathbb{R}^2$, we know that $f_\sigma(x) \in \Gamma^\infty$ does not depend on the point x. We call this point the *limit point* of the string σ and agree to write it $f_\sigma(O)$, where O is the origin $(0,0)$. We recall that the set of all limit points of strings in $\mathcal{A}_\infty$ is exactly Γ^∞ (see for example [19]). For $\sigma \in \mathcal{A}$, let the subset $\Gamma^{\infty,\sigma}$ of Γ^∞ be defined by

$$\Gamma^{\infty,\sigma} = f_\sigma(\Gamma^\infty). \tag{2.8}$$

The definition of Γ^∞ implies that for all $n > 0$, $\Gamma^\infty = \bigcup_{\sigma\in\mathcal{A}_n} \Gamma^{\infty,\sigma}$. We also define the set

$$\Xi = f_1(\Gamma^\infty) \cap f_2(\Gamma^\infty). \tag{2.9}$$

The critical contraction ratio $a^\star_\theta$ The following theorem was stated by Mandelbrot *et al.* in [24], and a complete proof is given in [9]:

Theorem 2.2. *For any θ, $0 < \theta < \pi/2$, there exists a unique positive number $a^\star_\theta < 1/\sqrt{2}$ which does not depend on (α, β) such that*

$$\begin{array}{lll} \text{if } 0 < a < a^\star_\theta & \text{then } \Xi = \emptyset \text{ and } \Gamma^\infty \text{ is totally disconnected,} \\ \text{if } a = a^\star_\theta & \text{then } \Xi \neq \emptyset \text{ and } \Gamma^\infty \text{ is connected.} \end{array} \tag{2.10}$$

The critical parameter $a^\star_\theta$ is the unique positive root of the polynomial equation

$$\sum_{i=0}^{\mathfrak{m}_\theta - 1} X^{i+2} \cos i\theta = \frac{1}{2}, \tag{2.11}$$

where

$$\mathfrak{m}_\theta \text{ is the smallest integer such that } \mathfrak{m}_\theta\theta \geqslant \pi/2. \tag{2.12}$$

Remark 2.3. From (2.11), it can be seen that $\theta \mapsto a^\star_\theta$ is a continuous and increasing function from $(0, \pi/2)$ onto $(1/2, 1/\sqrt{2})$ and that $\lim_{\theta\to 0} a^\star_\theta = 1/2$.

Hereafter, for a given θ, $0 < \theta < \pi/2$, we will write for brevity $\mathfrak{m}$ instead of $\mathfrak{m}_\theta$ and $a^\star$ instead of $a^\star_\theta$, and we will only consider a such that $0 < a \leqslant a^\star$.

2.1.2. *Characterization of* Ξ

We recall a geometric characterization of the set Ξ from [1] when $a = a^\star$. It is immediate that $\Xi = \Gamma^\infty \cap \Lambda$, where Λ is the axis of the ordinates $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0\}$. The following characterization was proved in [9]:

- ◇ if $\mathfrak{m}\theta > \pi/2$ and $a = a^\star$, then Ξ is reduced to a single point, which is the limit point of the string σ, where

$$\sigma = 12^{\mathfrak{m}+1}(12)^\infty \;\text{ or }\; \sigma = 21^{\mathfrak{m}+1}(21)^\infty, \tag{2.13}$$

see Paragraph 2.1.1 and Example 2.1 for the notation;

- ◇ if $\mathfrak{m}\theta = \pi/2$ and $a = a^\star$, then

$$\Xi = \left\{ f_\sigma(O) : \sigma \in 12^{\mathfrak{m}+1}(12|21)^\infty\} = \{ f_\sigma(O) : \sigma \in 21^{\mathfrak{m}+1}(12|21)^\infty \right\}. \tag{2.14}$$

2.2. Ramified domains

2.2.1. *The construction*

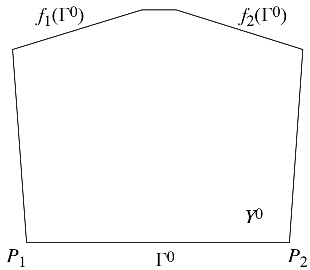

Figure 2.1. The first cell Y^0.

Call $P_1 = (-1, 0)$ and $P_2 = (1, 0)$ and Γ^0 the line segment $\Gamma^0 = [P_1 P_2]$. We impose that $f_2(P_1)$, and $f_2(P_2)$ have positive coordinates, *i.e.*, that

$$a \cos\theta < \alpha \;\text{ and }\; a \sin\theta < \beta. \tag{2.15}$$

We also impose that the open domain Y^0 inside the closed polygonal line joining the points P_1, P_2, $f_2(P_2)$, $f_2(P_1)$, $f_1(P_2)$, $f_1(P_1)$, P_1 in this order must be convex and hexagonal except if $\theta = 0$. With (2.15), this is true if and only if

$$(\alpha - 1)\sin\theta + \beta\cos\theta > 0. \tag{2.16}$$

Under assumptions (2.15) and (2.16), the domain Y^0 is contained in the half-plane $x_2 > 0$ and symmetric with respect to the vertical axis $x_1 = 0$.
We introduce $K^0 = \overline{Y^0}$. It is possible to glue together K^0, $f_1(K^0)$ and $f_2(K^0)$ and obtain a new polygonal domain, also symmetric with respect to the axis $\{x_1 = 0\}$.

The assumptions (2.15) and (2.16) imply that $Y^0 \cap f_1(Y^0) = \emptyset$ and $Y^0 \cap f_2(Y^0) = \emptyset$. We define the ramified open domain Ω (see Figure 2.2):

$$\Omega = \text{Interior}\left(\bigcup_{\sigma \in \mathcal{A}} f_\sigma(K^0) \right). \tag{2.17}$$

Note that Ω is symmetric with respect to the axis $x_1 = 0$.

For a given θ, with $a^\star$ defined as above, we shall make the following assumption on (α, β).

Assumption 2.4. For $0 < \theta < \pi/2$, the parameters α and β satisfy (2.15) and (2.16) for $a = a^\star$,
and are such that

$$\begin{cases} i.\ \text{for all } a, 0 < a \leqslant a^\star, \text{the sets } Y^0,\ f_\sigma(Y^0), \sigma \in \mathcal{A}_n, n > 0, \text{are disjoint}, \\ ii.\ \text{for all } a, 0 < a < a^\star,\ f_1(\overline{\Omega}) \cap f_2(\overline{\Omega}) = \emptyset, \\ iii.\ \text{for } a = a^\star,\ f_1(\overline{\Omega}) \cap f_2(\overline{\Omega}) \neq \emptyset. \end{cases}$$

Remark 2.5. Assumption 1 implies that if $a = a^\star$, then $f_1(\Omega) \cap f_2(\Omega) = \emptyset$.

The following theorem proved in [5] asserts that for all $\theta, 0 < \theta < \pi/2$, there exists (α, β) satisfying Assumption 1.

Theorem 2.6. *If $\theta \in (0, \pi/2)$, then for all $\alpha > a^\star \cos\theta$, there exists $\bar{\beta} > 0$ such that $\bar{\beta} > a^\star \sin\theta$ and $(\alpha - 1)\sin\theta + \bar{\beta}\cos\theta \geqslant 0$ and for all $\beta \geqslant \bar{\beta}$, (α, β) satisfies Assumption* 1.

Displayed in Figure 2.2 are two examples where Assumption 1 is satisfied. In the left part, we make the choice $\theta = \frac{\pi}{5}$, and in the right part, we chose $\theta = \frac{\pi}{4}$ (see also Example 2.1). Note the difference between the two cases: in the former case $\mathfrak{m}_\theta \theta > \pi/2$ and the set Ξ (defined in (2.9)) is a singleton, whereas in the latter case, $\mathfrak{m}_\theta \theta = \pi/2$ and the set Ξ is not countable.

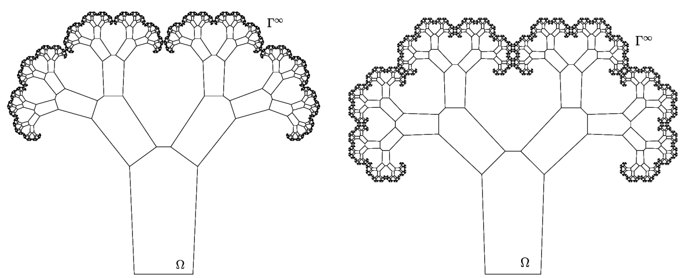

Figure 2.2. The ramified domain Ω for $\theta = \pi/5$ and $\theta = \pi/4$ when $a = a^\star, \alpha = 0.7$, $\beta = 1.5$.

2.2.2. *The Moran condition*

The Moran condition (or open set condition), see [19,28], is that there should exist a nonempty bounded open subset ω of $\mathbb{R}^2$ such that $f_1(\omega) \cap f_2(\omega) = \emptyset$ and $f_1(\omega) \cup f_2(\omega) \subset \omega$. For a given $\theta \in (0, \pi/2)$, let (α, β) satisfy Assumption 1; for $0 < a \leqslant a^\star$, the Moran condition is satisfied with $\omega = \Omega$, because:

- $f_1(\Omega) \cap f_2(\Omega) = \emptyset$, which stems from point *ii.* in Assumption 1 if $a < a^\star$, and from Remark 2.5 if $a = a^\star$;
- by construction of Ω, we also have $f_1(\Omega) \cup f_2(\Omega) \subset \Omega$.

The Moran condition implies that the Hausdorff dimension of Γ^∞ is

$$\dim_H(\Gamma^\infty) = d \equiv -\log 2/\log a,$$

see [19,28]. Note that if $a > 1/2$, then $d > 1$. For instance, if $\theta = \pi/4$ and $a = a^\star_{\pi/4}$, then $\dim_H(\Gamma^\infty) \simeq 1.3284371$. It can be shown that if $0 < \theta < \pi/2$, we have $0 < a \leqslant a^\star < 1/\sqrt{2}$ and thus $d < 2$. It can also be seen that if $\mathfrak{m}\theta = \pi/2$ and $a = a^\star$, then the Hausdorff dimension of Ξ is $d/2$.

2.2.3. *The self-similar measure μ and the spaces $W^{s,p}(\Gamma^\infty)$*

To define traces on Γ^∞, we recall the classical result on self-similar measures, see [10,13] and [19, page 26].

Theorem 2.7. *There exists a unique Borel regular probability measure μ on Γ^∞ such that for any Borel set $A \subset \Gamma^\infty$,*

$$\mu(A) = \frac{1}{2}\mu\left(f_1^{-1}(A)\right) + \frac{1}{2}\mu\left(f_2^{-1}(A)\right). \tag{2.18}$$

The measure μ is called the *self-similar measure defined in the self-similar triplet* $(\Gamma^\infty, f_1, f_2)$.

Proposition 2.8. *The measure μ is a d-measure on Γ^∞, with $d = -\log 2/\log a$, according to the definition in* [17, page 28]*: there exist two positive constants c_1 and c_2 such that*

$$c_1 r^d \leqslant \mu(B(x,r)) \leqslant c_2 r^d,$$

for any r $0 < r < 1$ and $x \in \Gamma^\infty$, where $B(x,r)$ is the Euclidean ball in Γ^∞ centered at x and with radius r. In other words, the closed set Γ^∞ is a d-set, see [17, page 28].

Proof. The proof stems from the Moran condition. It is due to Moran [28] and has been extended by Kigami, see [19, Section 1.5, especially Proposition 1.5.8 and Theorem 1.5.7]. □

We define $L^p(\Gamma^\infty)$, $p \in [1, +\infty)$ as the space of the measurable functions v on Γ^∞ such that $\int_{\Gamma^\infty} |v|^p \, \mathrm{d}\mu < \infty$, endowed with the norm $\|v\|_{L^p(\Gamma^\infty)} = \left(\int_{\Gamma^\infty} |v|^p \, \mathrm{d}\mu\right)^{1/p}$. We also introduce $L^\infty(\Gamma^\infty)$, the space of essentially bounded functions with respect to the measure μ. A Hilbertian basis of $L^2(\Gamma^\infty)$ can be constructed with *e.g.* Haar wavelets.

We also define the space $W^{s,p}(\Gamma^\infty)$ for $s \in (0, 1)$ and $p \in [1, \infty)$ as the space of functions $v \in L^p(\Gamma^\infty)$ such that $|v|_{W^{s,p}(\Gamma^\infty)} < \infty$, where

$$|v|_{W^{s,p}(\Gamma^\infty)} = \left(\int_{\Gamma^\infty} \int_{\Gamma^\infty} \frac{|v(x) - v(y)|^p}{|x - y|^{d+ps}} \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y)\right)^{\frac{1}{p}},$$

endowed with the norm $\|v\|_{W^{s,p}(\Gamma^\infty)} = \|v\|_{L^p(\Gamma^\infty)} + |v|_{W^{s,p}(\Gamma^\infty)}$.

2.2.4. *Additional notation*

For what follows, it is important to define the polygonal open domain Y^N obtained by stopping the above construction at step $N + 1$,

$$Y^N = \text{Interior}\left(K^0 \cup \left(\bigcup_{n=1}^{N} \bigcup_{\sigma \in \mathcal{A}_n} f_\sigma(K^0)\right)\right). \tag{2.19}$$

We introduce the open domains $Y^\sigma = f_\sigma(Y^0)$, $\Omega^\sigma = f_\sigma(\Omega)$ and $\Omega^N = \cup_{\sigma \in \mathcal{A}_N} \Omega^\sigma$, for $N > 0$. When needed, we will agree that $\Omega^0 = \Omega$. We define the sets $\Gamma^\sigma = f_\sigma(\Gamma^0)$ and $\Gamma^N = \cup_{\sigma \in \mathcal{A}_N} \Gamma^\sigma$. The one-dimensional Lebesgue measure of Γ^σ for $\sigma \in \mathcal{A}_N$ and of Γ^N are

$$|\Gamma^\sigma| = a^N |\Gamma^0| \text{ and } |\Gamma^N| = (2a)^N |\Gamma^0|.$$

We also introduce the sets $\Omega^\sigma = f_\sigma(\Omega)$ for all $\sigma \in \mathcal{A}$. We will sometimes use the notation $\lesssim$ or $\gtrsim$ to indicate that there may arise constants in the estimates, which are independent of the index n in Γ^n, or of the index σ in Γ^σ or $\Gamma^{\infty,\sigma}$. We may also write $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$.

3. The space $W^{1,p}(\Omega)$

Hereafter, we consider a domain Ω as defined in Section 2, with θ in $[0, \pi/2)$, $a \leqslant a^\star$ and we suppose that the parameters α, β are such that Assumption 1 is satisfied.

Basic facts For a real number $p \geqslant 1$, let $W^{1,p}(\Omega)$ be the space of functions in $L^p(\Omega)$ with first order partial derivatives in $L^p(\Omega)$. The space $W^{1,p}(\Omega)$ is a Banach space with the norm $\|u\|_{W^{1,p}(\Omega)} = \left(\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}$, where $\|\nabla u\|^p_{L^p(\Omega)} = \int_\Omega |\nabla u|^p \, \mathrm{d}x$ and $|\nabla u| = \sqrt{\left|\frac{\partial u}{\partial x_1}\right|^2 + \left|\frac{\partial u}{\partial x_2}\right|^2}$. The spaces $W^{1,p}(\Omega)$

as well as elliptic boundary value problems in Ω have been studied in [3], with, in particular Poincaré inequalities and a Rellich compactness theorem. The same results in a similar but different geometry were proved by Berger [6] with other methods.

Extension result in the case $a < a^\star$ We first briefly discuss the less interesting case when $a < a^\star$ and the self-similar set Γ^∞ is totally disconnected. In this case, the domain Ω is an (ε, δ)-domain as defined in [14] (see [4, Lemma 1, page 5]). Therefore, Theorem 1 in [14] yields a continuous extension operator from $W^{1,p}(\Omega)$ to $W^{1,p}(\mathbb{R}^2)$ for every $p \in (1, \infty)$.

The case $a = a^\star$ We will focus on the case $a = a^\star$ in the rest of the present paper. It can be seen that in this case, the domain Ω is not an (ε, δ)-domain, and the previous argument does not hold. However, it will be proved in Theorem B below that the same result is true when $1 < p < 2$ for angles $\theta \in [0, \frac{\pi}{2})$ such that $\mathfrak{m}\theta > \frac{\pi}{2}$, and when $1 < p < 2 - \frac{d}{2}$ for angles such that $\mathfrak{m}\theta = \frac{\pi}{2}$. It will also be seen (see Remark 5.3) that if $p > 2$ in the first case, and if $p > 2 - \frac{d}{2}$ in the second case, the extension result cannot hold.

A trace operator on Γ^∞ We construct a sequence $(\ell^n)_n$ of approximations of the trace operator: consider the sequence of linear operators $\ell^n : W^{1,p}(\Omega) \to L^p(\Gamma^\infty)$,

$$\ell^n(v) = \sum_{\sigma \in \mathcal{A}_n} \left(\frac{1}{|\Gamma^\sigma|} \int_{\Gamma^\sigma} v \, dx \right) \mathbb{1}_{f_\sigma(\Gamma^\infty)}, \tag{3.1}$$

where $|\Gamma^\sigma|$ is the one-dimensional Lebesgue measure of Γ^σ. The following result was proved in [3].

Proposition 3.1. *The sequence $(\ell^n)_n$ converges in $\mathcal{L}(W^{1,p}(\Omega), L^p(\Gamma^\infty))$ to an operator that we call ℓ^∞.*

4. The spaces $\mathrm{JLip}(t, p, q; 0; \Gamma^\infty)$ for $0 < t < 1$

In [16], A. Jonsson has introduced Haar wavelets of arbitrary order on self-similar fractal sets and has used these wavelets to construct a family of Lipschitz spaces. These function spaces are named $\mathrm{JLip}(t, p, q; m; \mathcal{S})$, where $\mathcal{S}$ is the fractal set, t is a non-negative real number, p, q are two real numbers not smaller than 1 and m is an integer (m is the order of the Haar wavelets used for the construction of the space). Here J stands for jumps, since the considered functions may jump at some points of $\mathcal{S}$. If the fractal set $\mathcal{S}$ is totally disconnected, then these spaces coincide with the Lipschitz spaces $\mathrm{Lip}(t, p, q; m; \mathcal{S})$ also introduced in [16]. The latter are a generalization of the more classical spaces $\mathrm{Lip}(t, p, q; \mathcal{S})$ introduced in [17] since $\mathrm{Lip}(t, p, q; [t]; \mathcal{S}) = \mathrm{Lip}(t, p, q; \mathcal{S})$. Note that $\mathrm{Lip}(t, p, p; [t]; \mathcal{S}) = W^{t,p}(\mathcal{S})$, see [18]. We will focus on the case when $\mathcal{S} = \Gamma^\infty$, $m = 0$ and $p = q$, since this is sufficient for what follows.

4.1. Definition of the spaces JLip$(t, p, p; 0; \Gamma^\infty)$

The definition of JLip$(t, p, p; 0; \Gamma^\infty)$ for $p \in [1, \infty)$ presented below is adapted to the class of fractal sets Γ^∞ considered in the present paper. It was proved in [5] that this definition coincides with the original and more general one that was proposed in [16].

Consider a real number $t, 0 < t < 1$. Following [16], it is possible to characterize JLip$(t, p, p; 0; \Gamma^\infty)$ by using expansions in the standard Haar wavelet basis on Γ^∞. Consider the Haar mother wavelet g_0 on Γ^∞,

$$g_0 = \mathbb{1}_{f_1(\Gamma^\infty)} - \mathbb{1}_{f_2(\Gamma^\infty)}, \tag{4.1}$$

and for $n \in \mathbb{N}, n > 0, \sigma \in \mathcal{A}_n$, let g_σ be given by

$$g_\sigma|_{\Gamma^{\infty,\sigma}} = 2^{n/2} g_0 \circ f_\sigma^{-1}, \text{ and } g_\sigma|_{\Gamma^\infty \setminus \Gamma^{\infty,\sigma}} = 0. \tag{4.2}$$

It is proved in [15] Section 5 that a function $f \in L^p(\Gamma^\infty)$ can be expanded in the Haar basis as follows:

$$f = P_0 f + \sum_{n \geqslant 0} \sum_{\sigma \in \mathcal{A}_n} \beta_\sigma g_\sigma, \tag{4.3}$$

where $P_0 f = \int_{\Gamma^\infty} f \, d\mu$. For any function $f \in L^p(\Gamma^\infty)$, we define $|f|_{\mathrm{JLip}(t,p,p;0;\Gamma^\infty)}$ by:

$$|f|_{\mathrm{JLip}(t,p,p;0;\Gamma^\infty)} = \left(\sum_{n \geqslant 0} 2^{n \frac{pt}{d}} 2^{n(\frac{p}{2}-1)} \sum_{\sigma \in \mathcal{A}_n} |\beta_\sigma|^p \right)^{\frac{1}{p}}, \tag{4.4}$$

where the numbers $\beta_\sigma, \sigma \in \mathcal{A}$ are the coefficients of f in the Haar wavelet basis expansion given in (4.3).

Definition 4.1. A function $f \in L^p(\Gamma^\infty)$ belongs to JLip$(t, p, p; 0; \Gamma^\infty)$ if and only if the norm

$$\|f\|_{\mathrm{JLip}(t,p,p;0;\Gamma^\infty)} = |P_0 f| + |f|_{\mathrm{JLip}(t,p,p;0;\Gamma^\infty)} \tag{4.5}$$

is finite.

Remark 4.2. An equivalent definition of JLip$(t, p, p; 0; \Gamma^\infty)$ can be given using the projection of f on the constants on $\Gamma^{\infty,\sigma}$, see [5,16].

Remark 4.3. Here, since we focus on the case $m = 0$, we do not need to suppose that Γ^∞ is not contained in a straight line, as was done in [16] in order to obtain that the fractal set satisfies a Markov property.

4.2. Characterization of the traces on Γ^∞ of functions in $W^{1,p}(\Omega)$

We recall the following theorem proved in [5] which provides a characterization of the trace space $\ell^\infty(W^{1,p}(\Omega))$ in terms of JLip spaces, and will prove very useful in the proof of the main Theorems A and B.

Theorem 4.4. *For a given θ, $0 \leqslant \theta < \pi/2$, if (α, β) satisfies Assumption 1 and Ω is constructed as in Paragraph 2.2.1, with $1/2 \leqslant a \leqslant a^\star$, then for all p, $1 < p < \infty$,*

$$\ell^\infty \left(W^{1,p}(\Omega) \right) = \mathrm{JLip} \left(1 - \frac{2-d}{p}, p, p; 0; \Gamma^\infty \right). \tag{4.6}$$

4.3. JLip versus Sobolev spaces on Γ^∞

We briefly recall the result obtained in [1], which compares the JLip spaces and the Sobolev spaces $W^{s,p}(\Gamma^\infty)$ on Γ^∞.

Theorem 4.5. *There exists a real number $p^\star_\theta > 0$ depending only on the dimension of the self-intersection of the fractal set Γ^∞, such that:*

◇ *if $a = a^\star$ and $1 < p < p^\star_\theta$, then*

$$\mathrm{JLip} \left(1 - \frac{2-d}{p}, p, p; 0; \Gamma^\infty \right) = W^{1-\frac{2-d}{p},p}(\Gamma^\infty); \tag{4.7}$$

◇ *if $a = a^\star$ and $p > p^\star_\theta$, then*

$$\mathrm{JLip} \left(1 - \frac{2-d}{p}, p, p; 0; \Gamma^\infty \right) \not\subset W^{1-\frac{2-d}{p},p}(\Gamma^\infty). \tag{4.8}$$

The number $p^\star_\theta$ is given by:

$$\begin{aligned} p^\star_\theta &= 2 && \text{if } \theta \notin \{\tfrac{\pi}{2k},\ k > 0\}, \\ p^\star_\theta &= 2 - d/2 && \text{if } \theta = \tfrac{\pi}{2k},\ k > 0, \end{aligned} \tag{4.9}$$

that is $p^\star_\theta = 2 - \dim_H \Xi$. Together with Theorem 4.4, Theorem 4.5 provides another characterization of the trace space $\ell^\infty(\Omega)$ when $p < p^\star_\theta$.

5. The main extension result

We will focus on the case $a = a^\star$ in the rest of the paper. As was seen in Section 3, the domain Ω is not an (ε, δ)-domain in this case, and the argument given for the case $a < a^\star$ does not hold. However, it will be proved in Theorem B that the extension result is true when $1 < p < p^\star_\theta$. It will also be seen (see Remark 5.3) that if $p > p^\star_\theta$, the extension result cannot hold.

Theorem A below states that when $p \in (1, p^\star_\theta)$, there exist liftings in $W^{1,p}(\mathbb{R}^2)$ for functions in the trace space $\mathrm{JLip}(1 - \frac{2-d}{p}, p, p; 0; \Gamma^\infty)$ of $W^{1,p}(\Omega)$ on Γ^∞ (see (4.6)), where the trace is meant in the sense of the operator ℓ^∞. This result is the main ingredient in the proof of Theorem B. The proof of Theorem A can be seen as a self-similar adaptation of the Whitney construction based on the wavelet decomposition of functions on Γ^∞.

Remark 5.1. Using Theorem 4.5 above and the trace theorem of Jonsson and Wallin stating that $W^{1-\frac{2-d}{p},p}(\Gamma^\infty) = W^{1,p}(\mathbb{R}^2)|_{\Gamma^\infty}$ for $p \in (1,\infty)$, it can be proved that

$$\mathrm{JLip}(1 - \tfrac{2-d}{p}, p, p; 0; \Gamma^\infty) = W^{1,p}(\mathbb{R}^2)|_{\Gamma^\infty}, \tag{5.1}$$

where we recall that the trace $u|_{\Gamma^\infty}$ of a function u is meant in the classical sense (see [17]). It should be noted that Theorem A below differs from the previous result in that the trace is meant in the sense of ℓ^∞, which will be of particular importance, especially in the proof of Theorem B below. The method proposed in the proof of Theorem B uses the lifting of Theorem A and the self-similar properties of the trace operator. Another choice could have been to work with the Whitney extension operator of (5.1), but we then could not have exploited the self-similar properties of the geometry as is done to prove Theorem B. In [2], Theorems A and B below are key ingredients in the proof that $u|_{\Gamma^\infty} = \ell^\infty(u)$ μ-almost everywhere for all $p > 1$ and $u \in W^{1,p}(\Omega)$.

The second result (Theorem B) states that there exists a continuous extension operator from $W^{1,p}(\Omega)$ to $W^{1,p}(\mathbb{R}^2)$ when $1 < p < p^\star_\theta$. The proof consists in the construction of a sequence of operators converging to the extension operator, with the help of the lifting introduced in Theorem A.

Theorem A.

1. *If $\theta \notin \frac{\pi}{2\mathbb{N}}$ and $p \in (1,2)$, then there exists a continuous linear lifting operator $\mathcal{E}$ from* JLip $\left(1 - \frac{2-d}{p}, p, p; 0; \Gamma^\infty\right)$ *to $W^{1,p}(\mathbb{R}^2)$ in the sense of ℓ^∞,* i.e.

$$\forall v \in \mathrm{JLip}\left(1 - \frac{2-d}{p}, p, p; 0; \Gamma^\infty\right), \quad \ell^\infty((\mathcal{E}v)|_\Omega) = v. \tag{5.2}$$

2. *If $\theta \in \frac{\pi}{2\mathbb{N}}$ and $p \in \left(1, 2 - \frac{d}{2}\right)$, then the conclusion remains true.*

We immediately deduce the following result for functions in $W^{1,p}(\Omega)$ with $1 < p < p^\star_\theta$.

Corollary 5.2. *If $p \in (1, p^\star_\theta)$ and $u \in W^{1,p}(\Omega)$, then the function $\bar{u} = \mathcal{E}\ell^\infty(u) \in W^{1,p}(\mathbb{R}^2)$ satisfies $\ell^\infty(\bar{u}) = \ell^\infty(u)$.*

We will deduce the main extension result of this paper:

Theorem B.

1. *If $\theta \notin \frac{\pi}{2\mathbb{N}}$ and $p \in (1,2)$, then there exists a continuous linear operator $\mathcal{F}$ from $W^{1,p}(\Omega)$ to $W^{1,p}(\mathbb{R}^2)$ such that, for all $u \in W^{1,p}(\Omega)$, $\mathcal{F}u|_\Omega = u$.*
2. *If $\theta \in \frac{\pi}{2\mathbb{N}}$ and $p \in (1, 2 - \frac{d}{2})$, then the extension result remains true.*

In other words, if $p \in (1, p^\star_\theta)$, then Ω has the $W^{1,p}$-extension property.

Remark 5.3. The extension result of Theorem B is sharp in the following sense. As was seen in Remark 5.1, it is proved *a posteriori* in [2] that the trace operator ℓ^∞ coincides with the trace operator introduced by Jonsson and Wallin in [17] μ-almost everywhere. Therefore, if Ω is a $W^{1,p}$-extension domain for some $p > p^\star_\theta$, then, by the trace theorem in [17] (p.182), $\ell^\infty(W^{1,p}(\Omega)) = W^{1-\frac{2-d}{p},p}(\Gamma^\infty)$, which contradicts (4.8). In the case when $\theta \notin \frac{\pi}{2\mathbb{N}}$, we have $p^\star_\theta = 2$, and we can conclude that Ω does not have the $W^{1,p^\star_\theta}$-extension property: if it did, then Koskela's theorem in [20] (Theorem B) would imply that Ω has the $W^{1,p}$-extension property for all $p > p^\star_\theta$. The case $\theta \in \frac{\pi}{2\mathbb{N}}$ is open.

6. Proof of the lifting theorem

In this section we prove Theorem A.

6.1. Proof of point 1

Recall that in this case, $\theta \notin \{\frac{\pi}{2k},\ k > 0\}$, and $\mathfrak{m}\theta > \frac{\pi}{2}$, where $\mathfrak{m}$ was introduced in (2.12). We start by lifting the Haar wavelets on Γ^∞ into functions in $W^{1,p}(\mathbb{R}^2)$, $1 < p < \infty$. This will yield a natural lifting for functions in $\mathrm{JLip}\left(1 - \frac{2-d}{p}, p, p; 0; \Gamma^\infty\right)$, defined as the lifting of their expansion in the Haar wavelets basis.

6.1.1. *Lifting of the Haar wavelets*

In this section, we define liftings $\bar{g}_\sigma$ of the Haar wavelets g_σ, $\sigma \in \mathcal{A}$, such that $\bar{g}_\sigma \in W^{1,p}(\mathbb{R}^2)$ for all $p \in (1,\infty)$. It will be of particular importance to control the pairwise intersections of the sets $\operatorname{supp} \nabla \bar{g}_\sigma$ in order to limit their contribution to the $W^{1,p}$-norm of the exended function.

To that end, we will make sure that these intersections are contained in some cones centered at the points $f_\eta(A)$ ($\eta \in \mathcal{A}$), where A is the single point contained in $f_1(\Gamma^\infty) \cap f_2(\Gamma^\infty)$. It is not hard to show that there exists an angle φ_0 small enough so that the vertical cone $\mathscr{C}$ with vertex A and angle φ_0 does not intersect $f_1(\Omega)$ or $f_2(\Omega)$ (in particular, $\varphi_0 < \min(\frac{\pi}{2} - (\mathfrak{m}-1)\theta, \mathfrak{m}\theta - \frac{\pi}{2})$). We impose that the liftings $\bar{g}_\sigma, \sigma \in \mathcal{A}$ should satisfy the following condition: if $\sigma \neq \tau$, then

$$\operatorname{supp} \nabla \bar{g}_\sigma \cap \operatorname{supp} \nabla \bar{g}_\tau \ \subset\ \bigcup_{\eta \in \mathcal{A}} f_\eta(\mathscr{C}). \tag{6.1}$$

Proposition 6.4 below will imply in particular that this condition is satisfied.

We proceed in three steps: we successively construct liftings for the constants, the Haar mother wavelet, and the other Haar wavelets.

Lifting of the constants We will introduce a lifting of the constant function 1 on Γ^∞, in order to define liftings for the Haar wavelets by self-similarity.

The proof of Lemma 3 in [1], see especially Figure 6.1 therein, can be easily modified to obtain the existence of a constant $c > 0$ such that for all $\sigma \in \mathcal{A}_n$ $(n > 0)$ with $\sigma \notin \mathcal{B}$,

$$d(f_\sigma(\Omega), \mathscr{C}) > ca^n, \tag{6.2}$$

where

$$\mathcal{B} = \{\sigma \in \mathcal{A},\ \sigma \text{ is a prefix of } 12^{\mathrm{m}+1}(12)^\infty \text{ or } 21^{\mathrm{m}+1}(21)^\infty\}, \tag{6.3}$$

see Paragraph 2.1.1 for the notation. Recall that the elements of $\mathcal{B}$ are those strings $\sigma \in \mathcal{A}$ such that $d(f_\sigma(\Omega), \Lambda) = 0$ where Λ is the axis given by $\{x_1 = 0\}$ (*cf.* Paragraph 2.1.2). We consider a smooth compactly supported function χ on $\mathbb{R}^2$ such that $\chi \equiv 1$ in a neighborhood of the closure of the ramified domain Ω, and such that χ is symmetric with respect to the axis Λ, see Figure 6.1.

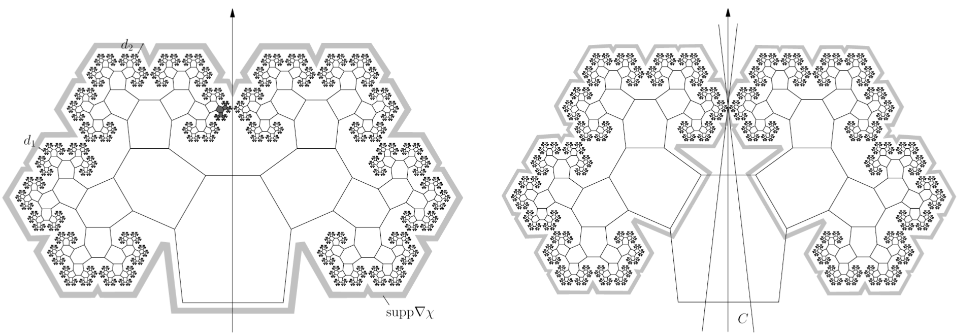

Figure 6.1. Left: the support of the function χ. Right: the support of the function $\bar{g}_0$.

Lifting of the Haar mother wavelet We introduce a function ψ in $\mathbb{R}^2$ such that

- $\psi \equiv 1$ in $\{x_1 \leqslant 0\} \setminus \mathscr{C}$ and $\psi \equiv 0$ in $\{x_1 > 0\} \setminus \mathscr{C}$;
- ψ is smooth in the interior of $\mathscr{C}$, $\frac{\partial \psi}{\partial \varphi}(r, \varphi)$ is constant and $\frac{\partial \psi}{\partial r}(r, \varphi) \equiv 0$ in $\operatorname{int}(\mathscr{C})$, where (r, φ) are the polar coordinates centered at the vertex A of the cone $\mathscr{C}$;
- ψ is continuous in $\mathbb{R}^2 \setminus \{A\}$.

Define the lifting $\bar{g}_0$ of the Haar mother wavelet by

$$\bar{g}_0 = \psi \cdot \left(\chi \circ f_1^{-1}\right) - (1 - \psi) \cdot \left(\chi \circ f_2^{-1}\right). \tag{6.4}$$

Note that $\bar{g}_0 \in W^{1,p}(\mathbb{R}^2)$ if and only if $p < 2$: it is enough to observe that

$$\begin{aligned} \|\nabla \psi\|^p_{L^p(\operatorname{supp} \bar{g}_0)} &= \|\nabla \psi\|^p_{L^p(\operatorname{supp} \bar{g}_0 \cap \mathscr{C})} \\ &\leqslant \int_{-R}^{R} \int_{-\varphi_0}^{\varphi_0} \frac{1}{r^p} r \,\mathrm{d}r \,\mathrm{d}\varphi \lesssim \int_{-R}^{R} \frac{\mathrm{d}r}{r^{p-1}}, \end{aligned} \tag{6.5}$$

where $R = \operatorname{diam} \operatorname{supp} \bar{g}_0$. Observe that the function χ can be chosen such that $\bar{g}_0$ satisfies the geometric condition:

$$\mathscr{C} \cap \operatorname{supp} \bar{g}_0 \subset \operatorname{conv}(\Omega), \tag{6.6}$$

where conv(Ω) refers to the convex hull of the ramified domain Ω. We assume this condition is fulfilled in the following.

Lifting of the Haar wavelets We use the function $\bar{g}_0$ and the self-similarity to define the liftings of the other Haar wavelets. We first define the natural lifting $\widetilde{g}_\sigma$ of g_σ, for $\sigma \in \mathcal{A}_n$ by

$$\widetilde{g}_\sigma \;=\; 2^{\frac{n}{2}} \bar{g}_0 \circ f_\sigma{}^{-1}.$$

Note that the functions $\widetilde{g}_\sigma$, $\sigma \in \mathcal{A}$ do not satisfy condition (6.1) (take for example $\sigma = 1$ and $\tau = 2$). Hence, we will define cut-off functions whose gradients are supported in the cones $f_\tau(\mathscr{C})$.

Take $\sigma \in \mathcal{A} \setminus \{\epsilon\}$. For any prefix $\tau \in \mathcal{A}_k$ of σ such that $\tau \neq \sigma$, we define

$$\gamma_\tau^\sigma = \mathbb{1}_{\sigma(k+1)=1} \psi \circ f_\tau{}^{-1} + \mathbb{1}_{\sigma(k+1)=2} (1-\psi) \circ f_\tau{}^{-1}. \tag{6.7}$$

Note that the function γ_τ^σ is 1 on one connected component of $\mathbb{R}^2 \setminus f_\tau(\mathscr{C})$, and 0 on the other. This definition is based on the observation that for all prefix $\tau \in \mathcal{A}_k$ of σ such that $\tau \neq \sigma$, $\Omega^\sigma \subset f_\tau(\{x_1 < 0\})$ if $\sigma(k+1) = 1$, and $\Omega^\sigma \subset f_\tau(\{x_1 > 0\})$ if $\sigma(k+1) = 2$. For $\sigma \in \mathcal{A}$, we introduce the set $\mathcal{M}(\sigma)$ of all those prefixes $\tau \in \mathcal{A}$ of σ such that $f_\tau(A) \in \Gamma^{\infty,\sigma}$. It is easily checked that

$$\mathcal{M}(\sigma) = \{\tau \in \mathcal{A},\ \sigma = \tau\sigma',\ \sigma' \in \mathcal{B}\}, \tag{6.8}$$

where $\mathcal{B}$ is defined in (6.3). If $\tau \in \mathcal{M}(\sigma) \setminus \{\sigma\}$, then the function $\widetilde{g}_\sigma$ needs to be multiplied by the cut-off function γ_τ^σ.

Remark 6.1. 1. If $\sigma \in \mathcal{B}$, then $\epsilon \in \mathcal{M}(\sigma)$.
2. For all $n > 0$ and $\sigma \in \mathcal{A}_n$, one has $\sigma, \sigma_{\restriction n-1} \in \mathcal{M}(\sigma)$, since the empty string and the string $(\sigma(n))$ belong to $\mathcal{B}$.

We may now define the cut-off lifting $\bar{g}_\sigma$ of the Haar wavelet g_σ, $\sigma \in \mathcal{A}_n$ by

$$\bar{g}_\sigma = \Bigg(\prod_{\substack{\tau \in \mathcal{M}(\sigma) \\ \tau \neq \sigma}} \gamma_\tau^\sigma \Bigg) \widetilde{g}_\sigma. \tag{6.9}$$

Example In Figure 6.2, we present an example where $\theta = \frac{\pi}{3}$ (hence $\mathfrak{m} = 2$), and $\sigma = 12^3 12$. Therefore, $\mathcal{M}(\sigma) = \{\epsilon, 12^3, 12^3 1, 12^3 12\}$. The gray area shows the support of $\nabla \bar{g}_\sigma$. In Figure 6.2, we have only represented the domain Ω^σ, which corresponds to the small area in dark gray in Figure 6.1.

In what follows, we will need a uniform bound on the cardinality of $\mathcal{M}(\sigma)$, $\sigma \in \mathcal{A}$:

Lemma 6.2. *There exists a constant C such that for all $\sigma \in \mathcal{A}$,*

$$\#\mathcal{M}(\sigma) \;\leqslant\; C. \tag{6.10}$$

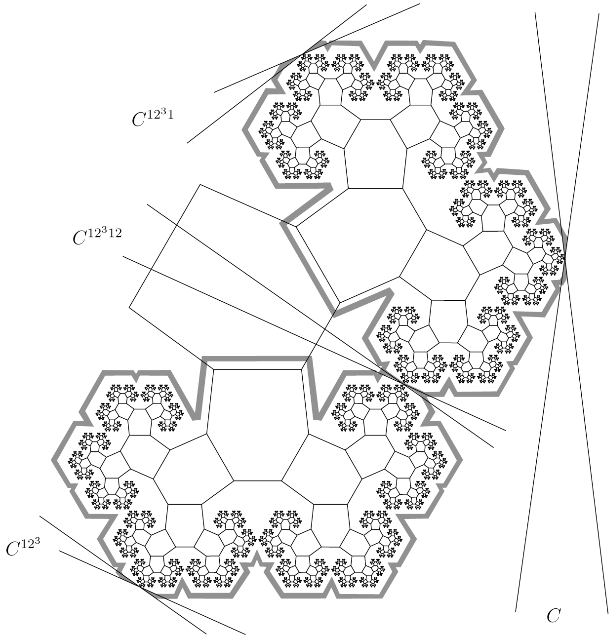

Figure 6.2. The support of $\nabla \bar{g}_\sigma$ (represented by the gray area) for $\theta = \frac{\pi}{3}$ and $\sigma = 12^{\mathfrak{m}+1}12 = 12^312$. In this case, $\mathcal{M}(\sigma) = \{\epsilon, 12^3, 12^31, 12^312\}$.

Proof. Take $n > 0$ and $\sigma \in \mathcal{A}_n$. If $n \leqslant 2$, then the result is clear. Suppose $n > 2$, we first note that $\sigma_{\restriction n-1}, \sigma \in \mathcal{M}(\sigma)$ (see Remark 6.1). Let us look for elements of $\mathcal{M}(\sigma)$ distinct from $\sigma_{\restriction n-1}$ and σ. First assume that $\sigma(n) = \sigma(n-1)$; take for example $\sigma(n) = 2$. Then, any suffix σ' of σ such that $\sigma' \in \mathcal{B}$ is of the form 12^k with $k \leqslant \mathfrak{m} + 1$. If there were two of them, then one of them would be a suffix of the other, which is impossible. Therefore, in this case, $\#\mathcal{M}(\sigma) \leqslant 3$. If $\sigma(n) \neq \sigma(n-1)$, then $(\sigma(n-1), \sigma(n)) \in \mathcal{B}$, which implies that $\sigma_{\restriction n-2} \in \mathcal{M}(\sigma)$. Let us look for a string $\sigma' \in \mathcal{M}(\sigma)$ such that $\sigma' \in \mathcal{A}_k$ with $k > 2$. Suppose for example $\sigma(n) = 2$. Then σ' must be of the form $12^{\mathfrak{m}+1}(12)^l$ or $21^{\mathfrak{m}+1}(21)^l2$, for some $l > 0$. As for the previous case, were there two such strings, one of them would be a suffix of the other, which is impossible. Hence, in this case, $\#\mathcal{M}(\sigma) \leqslant 4$. □

Proposition 6.3. *For* $\sigma \in \mathcal{A}_n$ *and* $p < 2$,

$$\|\nabla \bar{g}_\sigma\|^p_{L^p(\mathbb{R}^2)} \lesssim 2^{n\left(\frac{p}{2}+\frac{2-p}{d}\right)}. \tag{6.11}$$

Proof. First, we note that

$$\|2^{\frac{n}{2}} \nabla(\bar{g}_0 \circ f_\sigma^{-1})\|^p_{L^p(\mathbb{R}^2)} = 2^{\frac{np}{2}} a^{n(2-p)} \|\nabla \bar{g}_0\|^p_{L^p(\mathbb{R}^2)} \simeq 2^{n\left(\frac{p}{2}+\frac{2-p}{d}\right)}.$$

The other terms to consider are of the form $\|2^{\frac{n}{2}}\bar{g}_0 \circ f_\sigma^{-1}\, \nabla(\psi \circ f_\tau^{-1})\|^p_{L^p(\mathbb{R}^2)}$, where $\tau \in \mathcal{M}(\sigma) \setminus \{\sigma\}$. One has

$$\begin{aligned}\|2^{\frac{n}{2}}\bar{g}_0 \circ f_\sigma^{-1}\, \nabla(\psi \circ f_\tau^{-1})\|^p_{L^p(\mathbb{R}^2)} &\leqslant 2^{\frac{np}{2}}\|\nabla(\psi \circ f_\tau^{-1})\|^p_{L^p(f_\sigma(\operatorname{supp}\bar{g}_0))} \\ &= 2^{\frac{np}{2}} a^{k(2-p)}\|\nabla\psi\|^p_{L^p(f_{\sigma'}(\operatorname{supp}\bar{g}_0))},\end{aligned} \tag{6.12}$$

where $\sigma = \tau\sigma'$, $\tau \in \mathcal{A}_k$ and $\sigma' \in \mathcal{B}$ (see (6.8)). We show as in (6.5) that

$$\|\nabla\psi\|^p_{L^p(f_{\sigma'}(\operatorname{supp}\bar{g}_0))} \lesssim a^{(2-p)(n-k)}.$$

Together with (6.10) and (6.12), this achieves the proof since $a^d = 1/2$. □

We deduce in particular from Proposition 6.3 that $\bar{g}_\sigma \in W^{1,p}(\mathbb{R}^2)$ for $p < 2$.

6.1.2. *Geometrical results*

The following geometrical results will be crucial in the proof of Theorem A. The proofs of these results rely on simple but technical geometrical arguments and are postponed to Section 8 for the ease of the reader.

We introduce the truncated cones $S = \mathscr{C} \cap \operatorname{supp}\bar{g}_0$ and $S^\tau = f_\tau(S)$ for $\tau \in \mathcal{A}$. Define $\mathscr{S} = \bigcup_{\tau\in\mathcal{A}} S^\tau$ to be the union of these truncated cones. We also define $F = \operatorname{supp}\nabla\bar{g}_0 \setminus S$, and $F^\tau = f_\tau(F)$ for $\tau \in \mathcal{A}$. Note that, for $\sigma \in \mathcal{A}$,

$$\operatorname{supp}\nabla\bar{g}_\sigma \subset F^\sigma \cup \left(\bigcup_{\tau\in\mathcal{M}(\sigma)} S^\tau\right). \tag{6.13}$$

Proposition 6.4 below states a stronger version of condition (6.1).

Proposition 6.4. *If $\sigma, \tau \in \mathcal{A}$ and $\sigma \neq \tau$, then*

$$\operatorname{supp}\nabla\bar{g}_\sigma \cap \operatorname{supp}\nabla\bar{g}_\tau \subset \mathscr{S}. \tag{6.14}$$

Proposition 6.5 below somehow justifies the definition of the cut-off functions γ_τ^σ.

Proposition 6.5. *If $\sigma \in \mathcal{A}$ and $\tau \in \mathcal{M}(\sigma) \setminus \{\sigma\}$, then*

$$\bar{g}_\sigma = \gamma_\tau^\sigma \tilde{g}_\sigma \ \ in \ \ S^\tau. \tag{6.15}$$

Propositions 6.6 and 6.7 below describe the case when, for a given $\tau \in \mathcal{A}$, $\nabla\bar{g}_\sigma$ is not identically zero in S^τ.

Proposition 6.6. *There exists a constant $C > 0$ such that for all $\tau \in \mathcal{A}$ and $x \in S^\tau$,*

$$\#\{\sigma \in \mathcal{A} :\ \tau \in \mathcal{M}(\sigma),\ \nabla(\bar{g}_0 \circ f_\sigma^{-1})(x) \neq 0\} \leqslant C.$$

Proposition 6.7. *If $\sigma, \tau \in \mathcal{A}$ and $\tau \notin \mathcal{M}(\sigma)$, then $\nabla\bar{g}_\sigma \equiv 0$ in S^τ.*

In particular, if $\nabla\bar{g}_\sigma \not\equiv 0$ on S^τ, then τ is a prefix of σ.

6.1.3. *Proof of point* 1 *in Theorem* A

The proof will use the following discrete Hardy inequality (see [17, page 121]).

Lemma 6.8. *If $p \geqslant 1$, for any $\gamma > 0$ and $a \in (0,1)$ there exists a constant C such that, for any sequence of positive real numbers $(c_k)_{k\in\mathbb{N}}$,*

$$\sum_{n\in\mathbb{N}} a^{\gamma n}\left(\sum_{k\leqslant n} c_k\right)^p \leqslant C \sum_{n\in\mathbb{N}} a^{\gamma n} {c_n}^p. \tag{6.16}$$

Proof of point 1. Take $v \in \mathrm{JLip}\left(1-\frac{2-d}{p}, p, p, 0; \Gamma^\infty\right)$ and suppose in the first place that $\langle v\rangle_{\Gamma^\infty} = \int_{\Gamma^\infty} v\, d\mu = 0$. The function v then reads $v = \sum_n \sum_{\sigma\in\mathcal{A}_n} \beta_\sigma g_\sigma$ where the β_σ are the coefficients of v in the Haar wavelet basis of Γ^∞. We introduce the lifting $\mathcal{E}v$ of v defined on $\mathbb{R}^2$ by:

$$\mathcal{E}v = \sum_{n\in\mathbb{N}} \sum_{\sigma\in\mathcal{A}_n} \beta_\sigma \bar{g}_\sigma. \tag{6.17}$$

Recall that $\mathscr{S}$ is the union of all the truncated cones S^τ, $\tau \in \mathcal{A}$. By Proposition 6.3 and Proposition 6.4,

$$\begin{aligned}
\|\nabla(\mathcal{E}v)\|^p_{L^p(\mathbb{R}^2\setminus\mathscr{S})} &= \int_{\mathbb{R}^2\setminus\mathscr{S}} \left|\sum_{n\in\mathbb{N}} \sum_{\sigma\in\mathcal{A}_n} \beta_\sigma \nabla\bar{g}_\sigma(x)\right|^p \mathrm{d}x \\
&= \sum_{n\in\mathbb{N}} \sum_{\sigma\in\mathcal{A}_n} |\beta_\sigma|^p \int_{\operatorname{supp}\bar{g}_\sigma\setminus\mathscr{S}} |\nabla\bar{g}_\sigma(x)|^p \,\mathrm{d}x \\
&\lesssim \sum_{n\in\mathbb{N}} 2^{n\left(\frac{p}{2}+\frac{p}{d}-\frac{2}{d}\right)} \sum_{\sigma\in\mathcal{A}_n} |\beta_\sigma|^p \\
&= \|v\|^p_{\mathrm{JLip}(1-\frac{2-d}{p}, p, p, 0; \Gamma^\infty)}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|\nabla(\mathcal{E}v)\|^p_{L^p(\mathscr{S})} &= \int_{\mathscr{S}} \left|\sum_{n\in\mathbb{N}} \sum_{\sigma\in\mathcal{A}_n} \beta_\sigma \nabla\bar{g}_\sigma(x)\right|^p \mathrm{d}x \\
&= \sum_{\tau\in\mathcal{A}} \int_{S^\tau} \left|\sum_{n\in\mathbb{N}} \sum_{\sigma\in\mathcal{A}_n} \beta_\sigma \nabla\bar{g}_\sigma(x)\right|^p \mathrm{d}x.
\end{aligned}$$

Take $k \in \mathbb{N}$ and $\tau \in \mathcal{A}_k$. By Proposition 6.7, if $\sigma \in \mathcal{A}_n$ ($n \in \mathbb{N}$) is such that $\nabla\bar{g}_\sigma \not\equiv 0$ in S^τ, then $\tau \in \mathcal{M}(\sigma)$. Therefore, by Proposition 6.5, $\bar{g}_\sigma$ coincides with

$\gamma_\tau^\sigma \tilde{g}_\sigma$ in S^τ. Hence,

$$\int_{S^\tau} \left| \sum_{n\in\mathbb{N}} \sum_{\sigma\in\mathcal{A}_n} \beta_\sigma \nabla \bar{g}_\sigma(x) \right|^p \mathrm{d}x = \int_{S^\tau} \left| \sum_{n\in\mathbb{N}} \sum_{\substack{\sigma\in\mathcal{A}_n,\\ \tau\in\mathcal{M}(\sigma)}} 2^{\frac{n}{2}} \beta_\sigma \nabla\Big((\bar{g}_0 \circ f_\sigma^{-1}) \times \gamma_\tau^\sigma \Big)(x) \right|^p \mathrm{d}x$$
$$\lesssim I_1^\tau + I_2^\tau,$$

where

$$\begin{aligned} I_1^\tau &= \int_{S^\tau} \left| \sum_{n\in\mathbb{N}} \sum_{\sigma\in\mathcal{A}_n, \tau\in\mathcal{M}(\sigma)} 2^{\frac{n}{2}} \beta_\sigma \nabla(\bar{g}_0 \circ f_\sigma^{-1})(x) \right|^p \mathrm{d}x \\ I_2^\tau &= \int_{S^\tau} \left| \sum_{n\in\mathbb{N}} \sum_{\sigma\in\mathcal{A}_n, \tau\in\mathcal{M}(\sigma)} 2^{\frac{n}{2}} \beta_\sigma \bar{g}_0 \circ f_\sigma^{-1}(x) \nabla\gamma_\tau^\sigma(x) \right|^p \mathrm{d}x. \end{aligned} \tag{6.18}$$

Let us first consider I_1^τ. By Proposition 6.6, one has

$$\begin{aligned} I_1^\tau &\leqslant C^{p-1} \sum_{n\in\mathbb{N}} \sum_{\substack{\sigma\in\mathcal{A}_n\\ \tau\in\mathcal{M}(\sigma)}} 2^{\frac{np}{2}} |\beta_\sigma|^p \int_{S^\tau} |\nabla(\bar{g}_0 \circ f_\sigma^{-1})(x)|^p \mathrm{d}x \\ &\leqslant C^{p-1} \sum_{n\in\mathbb{N}} 2^{\frac{np}{2}} a^{(2-p)n} \sum_{\sigma\in\mathcal{A}_n} |\beta_\sigma|^p \int_{S^\tau} |\nabla \bar{g}_0(x)|^p \mathrm{d}x, \end{aligned} \tag{6.19}$$

where C is the constant in Proposition 6.6. Therefore,

$$\begin{aligned} \sum_{\tau\in\mathcal{A}} I_1^\tau &\lesssim \sum_{n\in\mathbb{N}} 2^{\frac{np}{2}} a^{(2-p)n} \sum_{\sigma\in\mathcal{A}_n} |\beta_\sigma|^p \sum_{\tau\in\mathcal{A}} \int_{S^\tau} |\nabla \bar{g}_0(x)|^p \mathrm{d}x \\ &\leqslant \sum_{n\in\mathbb{N}} 2^{\frac{np}{2}} a^{(2-p)n} \sum_{\sigma\in\mathcal{A}_n} |\beta_\sigma|^p \int_{\mathbb{R}^2} |\nabla \bar{g}_0(x)|^p \mathrm{d}x \\ &\lesssim \|v\|^p_{\mathrm{JLip}(1-\frac{2-d}{p}, p, p, 0; \Gamma^\infty)}. \end{aligned} \tag{6.20}$$

We are left to deal with I_2^τ. Denote $\Phi = (-\varphi_0, \varphi_0) \cup (\pi - \varphi_0, \pi + \varphi_0)$. We resort to a polar change of variables centered at the point $f_\tau(A)$ such that the vertical half-line originating at the point $f_\tau(A)$ and pointing up is given by $\{\varphi = 0\}$:

$$I_2^\tau \lesssim \int_0^\infty \int_\Phi \left| \sum_{n\in\mathbb{N}} \sum_{\sigma\in\mathcal{A}_n,\ \tau\in\mathcal{M}(\sigma)} 2^{\frac{n}{2}} \beta_\sigma \bar{g}_0 \circ f_\sigma^{-1}(r, \varphi) \right|^p r^{1-p} \mathrm{d}r\, \mathrm{d}\varphi, \tag{6.21}$$

since for $r > 0$ and $\varphi \in \Phi$, $|\nabla\gamma_\tau^\sigma(r, \varphi)| \lesssim \frac{1}{r}$. Define $R = \operatorname{diam} \operatorname{supp} \bar{g}_0$. If $\tau \in \mathcal{M}(\sigma)$, then $f_\tau(A) \in \overline{\Omega^\sigma} \subset \operatorname{supp} \bar{g}_0 \circ f_\sigma^{-1}$, by the definition of $\mathcal{M}(\sigma)$ in (6.8). Therefore, if $\operatorname{supp} \bar{g}_0 \circ f_\sigma^{-1} \cap \mathscr{C}(f_\tau(A), r) \neq \emptyset$ where $C(f_\tau(A), r)$ is the circle centered at $f_\tau(A)$ with radius r, then $r \leqslant a^n R$, *i.e.* $n \leqslant N_r$ where $N_r = \frac{\log r/R}{\log a}$. Hence

$$I_2^\tau \lesssim \int_0^{a^k R} \int_\Phi \left| \sum_{n=k}^{[N_r]} \sum_{\sigma \in \mathcal{A}_n, \tau \in \mathcal{M}(\sigma)} 2^{\frac{n}{2}} \beta_\sigma \bar{g}_0 \circ f_\sigma^{-1}(r, \varphi) \right|^p r^{1-p} \, \mathrm{d}r \, \mathrm{d}\varphi$$
$$\lesssim \int_0^{a^k R} \left(\sum_{n=k}^{[N_r]} \sum_{\substack{\sigma \in \mathcal{A}_n \\ \tau \in \mathcal{M}(\sigma)}} 2^{\frac{n}{2}} |\beta_\sigma| \right)^p r^{1-p} \, \mathrm{d}r.$$

Recall that if $\tau \in \mathcal{M}(\sigma)$, then σ is of the form $\tau\sigma'$ with $\sigma' \in \mathcal{B}$. Consequently,

$$I_2^\tau \lesssim \int_0^{a^k R} \left(\sum_{n=0}^{[N_r^k]} \sum_{\sigma' \in \mathcal{B}} 2^{\frac{n+k}{2}} |\beta_{\tau\sigma'}| \right)^p r^{1-p} \, \mathrm{d}r,$$

where $N_r^k = N_r - k = \frac{\log(r/(a^k R))}{\log a}$. Therefore, the change of variable $\rho = N_r^k$ yields:

$$I_2^\tau \lesssim \int_0^\infty a^{(\rho+k)(2-p)} \left(\sum_{n=0}^{[\rho]} \sum_{\sigma' \in \mathcal{B}} 2^{\frac{n+k}{2}} |\beta_{\tau\sigma'}| \right)^p \mathrm{d}\rho$$
$$\leqslant \sum_{m=0}^{\infty} a^{(m+k)(2-p)} \left(\sum_{n=0}^{m} \sum_{\sigma' \in \mathcal{B}} 2^{\frac{n+k}{2}} |\beta_{\tau\sigma'}| \right)^p,$$

since $\rho \mapsto a^{(\rho+k)(2-p)} \left(\sum_{n=0}^{[\rho]} \sum_{\sigma' \in \mathcal{B}} 2^{\frac{n}{2}} |\beta_{\tau\sigma'}| \right)$ is increasing. Then, by the discrete Hardy inequality stated in Lemma 6.8,

$$I_2^\tau \lesssim \sum_{m=0}^{\infty} a^{(m+k)(2-p)} 2^{\frac{(m+k)p}{2}} \sum_{\sigma' \in \mathcal{B}} |\beta_{\tau\sigma'}|^p = \sum_{m=k}^{\infty} a^{m(2-p)} 2^{\frac{mp}{2}} \sum_{\substack{\sigma \in \mathcal{A}_m \\ \tau \in \mathcal{M}(\sigma)}} |\beta_\sigma|^p. \quad (6.22)$$

Therefore, we obtain

$$\sum_{\tau \in \mathcal{A}} I_2^\tau \lesssim \sum_{k \in \mathbb{N}} \sum_{\tau \in \mathcal{A}_k} \sum_{m=k}^{\infty} a^{m(2-p)} 2^{\frac{mp}{2}} \sum_{\substack{\sigma \in \mathcal{A}_m \\ \tau \in \mathcal{M}(\sigma)}} |\beta_\sigma|^p$$
$$= \sum_{m \in \mathbb{N}} a^{m(2-p)} 2^{\frac{mp}{2}} \sum_{\sigma \in \mathcal{A}_m} \sum_{k=0}^{m} \sum_{\substack{\tau \in \mathcal{A}_k \\ \tau \in \mathcal{M}(\sigma)}} |\beta_\sigma|^p \quad (6.23)$$
$$\lesssim \sum_{m \in \mathbb{N}} a^{m(2-p)} 2^{\frac{mp}{2}} \sum_{\sigma \in \mathcal{A}_m} |\beta_\sigma|^p,$$

since $\#\mathcal{M}(\sigma) \leqslant 4$ by Lemma 6.2, which shows that $\sum_{\tau\in\mathcal{A}} I_2^\tau \lesssim \|v\|^p_{\mathrm{JLip}(1-\frac{2-d}{p},p,p,0;\Gamma^\infty)}$. Finally, if $\langle v\rangle_{\Gamma^\infty} \neq 0$, then we get the desired result by taking $\mathcal{E}v = \langle v\rangle_{\Gamma^\infty}\chi + \mathcal{E}(v - \langle v\rangle_{\Gamma^\infty})$. □

6.2. Proof of point 2

We will proceed in the same manner as we did in the proof of point 1 in Theorem A. The extensions of the Haar wavelets will have to be constructed with more care since the set $\Xi = f_1(\Gamma^\infty) \cap f_2(\Gamma^\infty)$ is now infinite.

6.2.1. *Lifting of the Haar wavelets*

We start by constructing a set $\mathscr{C}$ playing the same role as the cone constructed in the proof of point 1. Let A_u be the limit point of the string $12^{\mathrm{m}+1}(12)^\infty$ and A_l the limit point of $12^{\mathrm{m}+1}(21)^\infty$, see Figure 6.3. The points A_u and A_l are respectively the upper and lower ends of the set Ξ. As in the proof of point 1, it is not hard to show that there exists an angle $\varphi_0 \in (0,\theta)$ We introduce the points $M_1 = f_{12^{\mathrm{m}+1}12(21)^\infty}(O)$ and $M_2 = f_{12^{\mathrm{m}+1}21(12)^\infty}(O)$. Call D the diamond-shaped intersection of the vertical open half-cones with respective vertices M_1 and M_2 and with common angle φ_0, as in Figure 6.3. Call M_3 and M_4 the other two vertices of D, see Figure 6.3. It is easily checked that the set D does not intersect the ramified domain Ω. Call $D^0 = f^{-1}_{12^{\mathrm{m}+1}}(D)$ and $D^\eta = f_\eta(D^0)$ for $\eta \in \mathcal{A}$. Note that with this notation $D = D^{12^{\mathrm{m}+1}}$. Similarly, we define $M_i^\eta = f_\eta(f^{-1}_{12^{\mathrm{m}+1}}(M_i))$ for $i = 1,2$ and $\eta \in \mathcal{B}^+$; note that M_1^η and M_2^η are vertices of the diamond D^η. Write $\mathcal{B}^+ = 12^{\mathrm{m}+1}(12|21)^\star$, we also introduce the sets $\mathscr{D} = \bigcup_{\eta\in\mathcal{B}^+} D^\eta$ and

$$\mathscr{C} = \overline{\mathscr{C}_u \cup \mathscr{C}_l \cup \mathscr{D}}. \tag{6.24}$$

The set $\mathscr{C}$ corresponds to the gray area in the right part of Figure 6.3. In view of (2.14), introduce the set $\mathcal{B} \subset \mathcal{A}$ such that $\sigma \in \mathcal{B}$ if and only if one of the two following conditions is satisfied:

(i) σ is a prefix of 12^{m} or 21^{m},

(ii) σ belongs to the set $12^{\mathrm{m}+1}(12|21)^\star(1|2|\varepsilon)$ or the set $21^{\mathrm{m}+1}(12|21)^\star(1|2|\varepsilon)$, (6.25)

where the notation has been defined in Paragraph 2.1.1, see also Example 2.1. As previously, $\mathcal{B}$ is the set of finite strings σ such that $d(f_\sigma(\Omega),\Lambda) = 0$. An analogous result to (6.2) can be proved in this case (see Lemma 11 in [1] and [9]): there exists a constant $c > 0$ such that for any $\sigma \in \mathcal{A}_n$ ($n \in \mathbb{N}$) such that $\sigma \notin \mathcal{B}$,

$$d(\mathrm{conv}(\Omega^\sigma),\mathscr{C}) > ca^n. \tag{6.26}$$

We introduce a smooth compactly supported function χ in $\mathbb{R}^2$ such that:

- $\chi \equiv 1$ in a neighborhood of the closure of the ramified domain Ω;
- χ is symmetric with respect to the axis Λ.

We also introduce a function ψ in $\mathbb{R}^2$ such that:

- $\psi \equiv 1$ in $\{x_1 \leqslant 0\} \setminus (\mathscr{C}_u \cup \mathscr{C}_l)$ and $\psi \equiv 0$ in $\{x_1 > 0\} \setminus (\mathscr{C}_u \cup \mathscr{C}_l)$;
- ψ is smooth in the interior of $\mathscr{C}_u$ (respectively $\mathscr{C}_l$) and $\nabla\psi(r, \varphi) = (0, \frac{\alpha}{r})$ in $\mathrm{int}(\mathscr{C}_u)$ (respectively in $\mathrm{int}(\mathscr{C}_l)$), where (r, φ) are the polar coordinates centered at the vertex A_u of $\mathscr{C}_u$ (respectively the vertex A_l of $\mathscr{C}_l$) and α is a constant;
- ψ is continuous in $\mathbb{R}^2 \setminus [A_u A_l]$.

Finally, consider a function ζ in $\mathbb{R}^2$ such that valued in [0, 1] such that:

- $\zeta|_{[M_1 M_3)} = \zeta|_{[M_3 M_2)} \equiv 1$, and $\zeta|_{[M_2 M_4)} = \zeta|_{[M_4 M_1)} \equiv 0$;
- $\nabla\zeta(r,\varphi) = (0, \frac{\alpha}{r})$ in the triangle $M_1 M_3 M_4$ (respectively in the triangle $M_2 M_4 M_3$) where (r, φ) are the polar coordinates centered at M_1 (respectively at M_2).

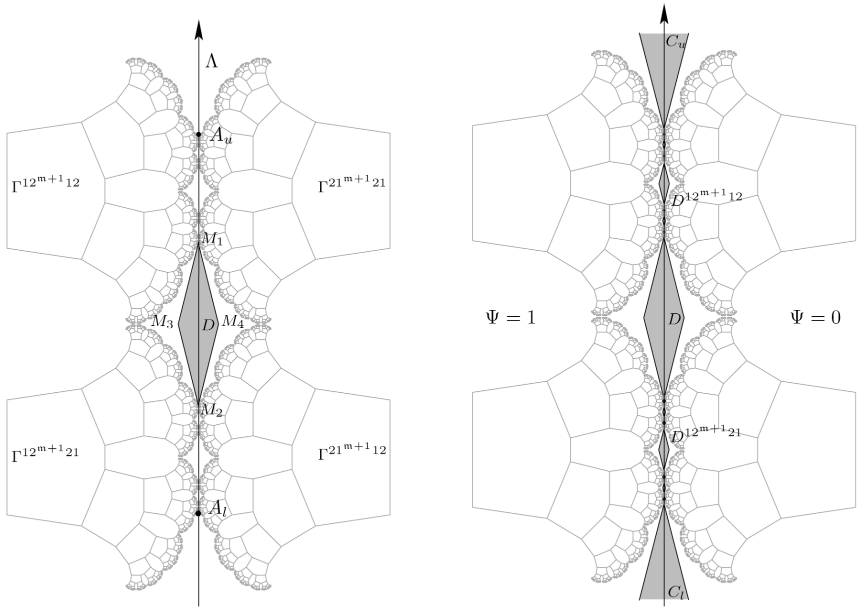

Figure 6.3. The functions ζ and Ψ in the case $\theta = \frac{\pi}{8}$ ($\mathfrak{m} = 4$). Left: construction of the function ζ, in the gray area lies the support of ζ. Right: the gray area corresponds to the support of $\nabla\Psi$.

For $\eta \in \mathcal{B}^+$, define $\zeta_\eta = \zeta \circ f_{12^{\mathfrak{m}+1}} \circ f_\eta^{-1} \in W^{1,p}(D^\eta)$. We introduce the function

$$\Psi : x = (x_1, x_2) \in \mathbb{R}^2 \ \mapsto \ \begin{cases} \zeta_\eta(x) & \text{if } x \in D^\eta, \ \eta \in \mathcal{B}^+, \\ \psi(x) & \text{if } x \in \mathscr{C}_u \cup \mathscr{C}_l, \\ \quad 1 & \text{if } x_1 \leqslant 0 \text{ and } x \notin \mathscr{C}, \\ \quad 0 & \text{if } x_1 > 0 \text{ and } x \notin \mathscr{C}, \end{cases} \tag{6.27}$$

see Figure 6.3. This definition is unambiguous since the sets D^η are pairwise disjoint. The function Ψ will play the role of the function ψ in Paragraph 6.1.1. Note that Ψ is continuous in $\mathbb{R}^2 \setminus \Xi$. We start by defining the lifting of the Haar mother wavelet by

$$\bar{g}_0 = (\chi \circ f_1^{-1})\Psi - (\chi \circ f_2^{-1})(1 - \Psi). \tag{6.28}$$

One has:

$$\begin{aligned}\sum_{\eta \in \mathcal{B}^+} \int_{D^\eta} |\nabla \zeta_\eta|^p \, \mathrm{d}x &\simeq \sum_{n \in \mathbb{N}} \sum_{\eta \in \mathcal{A}_n \cap \mathcal{B}^+} a^{n(2-p)} \int_{\mathbb{R}^2} |\nabla \zeta|^p \, \mathrm{d}x \\ &\lesssim \sum_{n \in \mathbb{N}} 2^{\frac{n}{2}} a^{n(2-p)} \|\nabla \zeta\|^p_{L^p(\mathbb{R}^2)},\end{aligned}$$

since $\#\mathcal{A}_n \cap \mathcal{B}^+ \lesssim 2^{\frac{n}{2}}$. Since $2^n a^{2n(2-p)} = 2^{n\left(1 + \frac{2(p-2)}{d}\right)}$ and $p < 2 - \frac{d}{2}$, the latter sum converges, and $\Psi \in W^{1,p}_{\mathrm{loc}}$. Since the function Ψ is continuous in $\mathbb{R}^2 \setminus \Xi$, so is $\bar{g}_0$, which implies that $\bar{g}_0 \in W^{1,p}(\mathbb{R}^2)$. We will need the cut-off functions

$$\gamma_\tau^\sigma = (\mathbb{1}_{\sigma(k+1)=1}\Psi + \mathbb{1}_{\sigma(k+1)=2}(1 - \Psi)) \circ f_\tau^{-1}, \tag{6.29}$$

for $\sigma \in \mathcal{A}$ and $\tau \in \mathcal{A}_k$ a prefix of σ, as we did in Paragraph 6.1.1. For every $\sigma \in \mathcal{A}$, we introduce the set

$$\mathcal{M}(\sigma) = \{\tau \in \mathcal{A},\ \sigma = \tau\sigma',\ \sigma' \in \mathcal{B}\}, \tag{6.30}$$

as in Paragraph 6.1.1, where $\mathcal{B}$ was defined in (6.25). Note that Lemma 6.2 is still true in that case. The liftings $\bar{g}_\sigma$ for the Haar wavelets are defined in an analogous manner as previously: for $n > 0$ and $\sigma \in \mathcal{A}_n$,

$$\bar{g}_\sigma = \Bigg(\prod_{\substack{\tau \in \mathcal{M}(\sigma) \\ \tau \neq \sigma}} \gamma_\tau^\sigma \Bigg) \tilde{g}_\sigma. \tag{6.31}$$

As in (6.11), there is a constant C such that, for all $\sigma \in \mathcal{A}$,

$$\int_{\mathbb{R}^2} |\nabla \bar{g}_\sigma|^p \, \mathrm{d}x \ \leqslant\ C\, 2^{n\left(\frac{1}{2} + \frac{p-2}{d}\right)} \int_{\mathbb{R}^2} |\nabla \bar{g}_0|^p \, \mathrm{d}x. \tag{6.32}$$

As in Paragraph 6.1.1, we introduce the sets $S = \mathscr{C} \cap \operatorname{supp} \bar{g}_0$, $\mathscr{C}^\tau = f_\tau(\mathscr{C})$ and $S^\tau = f_\tau(S)$ for $\tau \in \mathcal{A}$, $\mathscr{S} = \bigcup_{\tau \in \mathcal{A}} S^\tau$. We also write $S_u = \mathscr{C}_u \cap S$, $S_l = \mathscr{C}_l \cap S$ and $S_u^\tau = f_\tau(S_u)$, $S_l^\tau = f_\tau(S_l)$ for $\tau \in \mathcal{A}$. Finally, we define $F = \operatorname{supp} \nabla \bar{g}_0 \setminus S$, and $F^\tau = f_\tau(F)$ for $\tau \in \mathcal{A}$.

6.2.2. *Geometrical results*

The proof of point 2 in Theorem A will require a few geometrical results. A first observation is that Propositions 6.4, 6.5, 6.6 and 6.7 remain true in this case (a proof will be given in Section 8). We will also use the following result, whose proof is also postponed to Section 8.

Proposition 6.9.

1 *Take* $\eta \in \mathcal{B}^+$ *and* $\sigma \in \mathcal{A}$ *with* $\sigma(1) = \eta(1)$. *If* σ *is not a prefix of* $\eta 12(21)^\infty$ *or* $\eta 21(12)^\infty$, *then* $\tilde{g}_\sigma \equiv 0$ *in* D^η.
2 *If* $\sigma \in \mathcal{A}$ *is not a prefix of* $12^{\mathrm{m}+1}(12)^\infty$ *or* $21^{\mathrm{m}+1}(21)^\infty$ *(respectively* $12^{\mathrm{m}+1}(21)^\infty$ *or* $21^{\mathrm{m}+1}(12)^\infty$*), then* $\tilde{g}_\sigma \equiv 0$ *in* S_u *(respectively* S_l*).*

6.2.3. *Proof of point 2 in Theorem A*

The proof that $\|\nabla(\mathcal{E}v)\|^p_{L^p(\mathbb{R}^2\setminus\mathscr{S})} \lesssim \|v\|^p_{\mathrm{JLip}(1-\frac{2-d}{p},p,p;0;\Gamma^\infty)}$ is the same as in the proof of point 1. We are left to deal with $\|\nabla(\mathcal{E}v)\|^p_{L^p(\mathscr{S})}$. Since Proposition 6.7 still holds, we can still write

$$\|\nabla(\mathcal{E}v)\|^p_{L^p(\mathscr{S})} \lesssim \sum_{\tau\in\mathcal{A}} (I_1^\tau + I_2^\tau), \tag{6.33}$$

where I_1^τ and I_2^τ are as in (6.18). Since Proposition 6.6 remains true, we can deal with $\sum_{\tau\in\mathcal{A}} I_1^\tau$ as in the proof of point 1. Since $S^\tau = S_u^\tau \cup S_l^\tau \cup f_\tau(\mathscr{D})$, the integration on S^τ in I_2^τ can be decomposed into integrals on S_u^τ, S_l^τ and $f_\tau(\mathscr{D})$. The first two integrals can be dealt with exactly as in the proof of point 1. We refer to the last one as J_2^τ. For $\eta \in \mathcal{B}^+$, call $\mathcal{B}_\eta = \{\sigma \in \mathcal{A},\ \sigma \text{ is a prefix of } \eta 12(21)^\infty \text{ or } \eta 12(21)^\infty\}$. Proposition 6.9 together with an argument of self-similarity imply that $\nabla \bar{g}_\sigma \equiv 0$ on $f_\tau(D^\eta)$ if $\sigma \notin \tau\mathcal{B}_\eta$. Therefore,

$$J_2^\tau = \sum_{\eta\in\mathcal{B}^+} \int_{f_\tau(D^\eta)} \left| \sum_{n\in\mathbb{N}} \sum_{\sigma\in\mathcal{A}_n\cap\tau\mathcal{B}_\eta} 2^{\frac{n}{2}} \beta_\sigma \bar{g}_0 \circ f_\sigma^{-1}(x) \nabla\gamma_\tau^\sigma(x) \right|^p \mathrm{d}x. \tag{6.34}$$

We split the integral over $f_\tau(D^\eta)$ into two integrals over portions of cones with respective vertices $f_\tau(M_1^\eta)$ and $f_\tau(M_2^\eta)$. As in (6.21), we express them in polar coordinates centered respectively at $f_\tau(M_1^\eta)$ and $f_\tau(M_2^\eta)$. Call ℓ the length of the sides of the diamond D^0. The length of the sides of $f_\tau(D^\eta)$ is $a^{k+l}\ell \leqslant a^{k+l}R$, and we may take $r \leqslant a^{k+l}R$ in the integrals. As in the proof of point 1, we note that if $\tau \in \mathcal{M}(\sigma)$ and $\operatorname{supp} \bar{g}_0 \circ f_\sigma^{-1} \cap C(f_\tau(M_i^\tau), r) \neq \emptyset$, $i = 1, 2$, then $n \leqslant N_r$ where

$N_r = \frac{\log(r/R)}{\log a}$. Therefore,

$$
\begin{aligned}
&\left\| \sum_{n\in\mathbb{N}} \sum_{\sigma\in\mathcal{A}_n\cap\tau\mathcal{B}_\eta} 2^{\frac{n}{2}} \beta_\sigma \bar{g}_0 \circ f_\sigma^{-1} \nabla\gamma_\tau^\sigma \right\|^p_{L^p(f_\tau(D^\eta))} \\
&\lesssim \int_0^{a^{k+l}R} \left(\sum_{n=k}^{[N_r]} \sum_{\sigma\in\mathcal{A}_n\cap\tau\mathcal{B}_\eta} 2^{\frac{n}{2}} |\beta_\sigma| \right)^p r^{1-p}\,\mathrm{d}r \\
&= \int_0^{a^{k+l}R} \left(\sum_{n=0}^{[N_r]-k} \sum_{\sigma'\in\mathcal{A}_n\cap\mathcal{B}_\eta} 2^{\frac{n+k}{2}} |\beta_{\tau\sigma'}| \right)^p r^{1-p}\,\mathrm{d}r \\
&\lesssim \sum_{m=0}^{\infty} a^{(m+k+l)(2-p)} \left(\sum_{n=0}^{m} \sum_{\sigma'\in\mathcal{A}_n\cap\mathcal{B}_\eta} 2^{\frac{n+k}{2}} |\beta_{\tau\sigma'}| \right)^p \\
&\lesssim a^{l(2-p)} \sum_{m=k}^{\infty} a^{m(2-p)} 2^{\frac{mp}{2}} \sum_{\sigma\in\mathcal{A}_n\cap\tau\mathcal{B}_\eta} |\beta_\sigma|^p,
\end{aligned}
$$

where we have proceeded exactly as in the proof of point 1, using the discrete Hardy inequality stated in Lemma 6.8. Hence,

$$
\begin{aligned}
J_2^\tau &\lesssim \sum_{l=0}^{\infty} \sum_{\eta\in\mathcal{A}_l\cap\mathcal{B}^+} a^{l(2-p)} \sum_{m=k}^{\infty} a^{m(2-p)} 2^{\frac{mp}{2}} \sum_{\sigma\in\mathcal{A}_n\cap\tau\mathcal{B}_\eta} |\beta_\sigma|^p \\
&\lesssim \sum_{l=0}^{\infty} a^{l(2-p)} 2^{\frac{l}{2}} \sum_{m=k}^{\infty} a^{m(2-p)} 2^{\frac{mp}{2}} \sum_{\substack{\sigma\in\mathcal{A}_n \\ \tau\in\mathcal{M}(\sigma)}} |\beta_\sigma|^p,
\end{aligned}
$$

since $\#\mathcal{A}_l \cap \mathcal{B}^+ \lesssim 2^{\frac{l}{2}}$ and $\#\mathcal{A}_n \cap \tau\mathcal{B}_\eta = 2$. Therefore, since $a^{l(2-p)} 2^{\frac{l}{2}} = a^{l(2-\frac{d}{2}-p)}$ and $p < 2 - \frac{d}{2}$,

$$
J_2^\tau \lesssim \sum_{m=k}^{\infty} a^{m(2-p)} 2^{\frac{mp}{2}} \sum_{\substack{\sigma\in\mathcal{A}_n \\ \tau\in\mathcal{M}(\sigma)}} |\beta_\sigma|^p. \tag{6.35}
$$

We show that $\sum_{\tau\in\mathcal{A}} J_2^\tau \lesssim \|v\|^p_{\mathrm{JLip}\left(1-\frac{2-d}{p},p,p;0;\Gamma^\infty\right)}$ exactly as in the proof of point 1, see (6.23).

7. Proof of the extension theorem

First observe that, in the proof of Theorem B, it is enough to show that when $p < p^\star$, there exists a linear continuous extension operator $\mathcal{F} : W^{1,p}(\Omega) \to W^{1,p}(\mathcal{R})$, where $\mathcal{R} = \mathbb{R} \times \mathbb{R}_+$. Take $\theta \in (0, \frac{\pi}{2})$, and $p \in (1, p^\star_\theta)$. We will construct a sequence of continuous linear operators $\mathcal{F}_n : W^{1,p}(\Omega) \to W^{1,p}(\mathcal{R})$ such that for all $u \in W^{1,p}(\Omega)$,

$$(\mathcal{F}_n u)|_{Z^n} = u|_{Z^n}, \tag{7.1}$$

where $Z^n = \operatorname{int} \bigcup\{\overline{Y^\sigma},\ \sigma \in \mathcal{A}_k,\ k \leqslant n\}$. This implies that $\ell^n((\mathcal{F}_n u)|_\Omega) = \ell^n(u)$. It will be proved in Proposition 7.2 that the sequence $(\mathcal{F}_n)_n$ converges pointwise to a continuous linear operator $\mathcal{F}$, which will yield Theorem B since $\mathcal{F}u|_\Omega = u$ for all $u \in W^{1,p}(\Omega)$. An immediate consequence is that $\ell^\infty(\mathcal{F}u) = \ell^\infty(u)$.

First, we introduce an extension operator from $W^{1,p}(\Omega)$ to $W^{1,p}(\widehat{\Omega})$, where $\widehat{\Omega}$ is a larger ramified domain defined below and presented in Figure 7.1.

The domain $\widehat{\Omega}$ Take $\varepsilon < 2$ and write $P_1' = (-1-\varepsilon, 0)$, and $P_2' = (1+\varepsilon, 0)$. Define $\widehat{Y}^0$ to be the open domain inside the closed polygonal line joining the points P_1', P_2', $f_2(P_2')$, $f_2(P_1')$, $f_1(P_2')$, $f_1(P_1')$, P_1' in this order. Let $\widehat{K}^0$ be the closure of $\widehat{Y}^0$. We define the wider ramified domain $\widehat{\Omega}$ to be:

$$\widehat{\Omega} = \text{Interior}\left(\widehat{K}^0 \cup \left(\bigcup_{\sigma \in \mathcal{A}} f_\sigma(\widehat{K}^0)\right)\right), \tag{7.2}$$

see Figure 7.1. We can suppose $\varepsilon > 0$ is small enough so that $f_2(P_1')$ and $f_2(P_2')$ have positive coordinates, the domain $\widehat{Y}^0$ is convex, and Assumption 1 is satisfied. We introduce the open domains $\widehat{Y}^\sigma = f_\sigma(\widehat{Y}^0)$ for $\sigma \in \mathcal{A}$, along with their closures $\widehat{K}^\sigma$. We also write $\widehat{\Omega}^\sigma = f_\sigma(\widehat{\Omega})$ for $\sigma \in \mathcal{A}$, and $\widehat{\Omega}^n = \bigcup_{\sigma \in \mathcal{A}_n} \widehat{\Omega}^\sigma$.

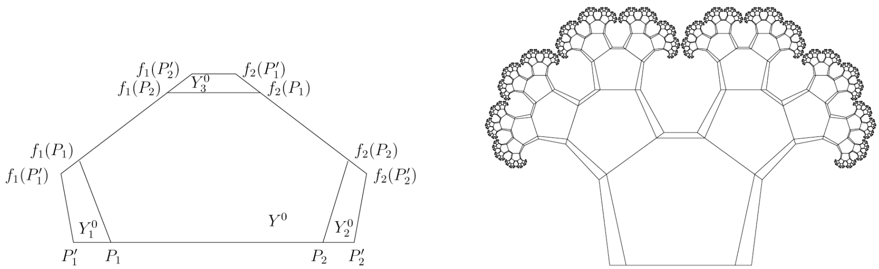

Figure 7.1. Left: First cells Y^0 and $\widehat{Y}^0$ of the ramified domains. Right: The ramified domains Ω and $\widehat{\Omega}$.

We introduce the open domain Y_1^0 inside the polygonal line joining the points P_1', P_1, $f_1(P_1)$, $f_1(P_1')$, P_1', its symmetric Y_2^0 with respect to the vertical axis Λ, and the open domain Y_3^0 inside the polygonal line joining the points $f_1(P_2)$, $f_2(P_1)$, $f_2(P_1')$, $f_1(P_2')$, $f_1(P_2)$. For $\sigma \in \mathcal{A}$ and $i = 1, 2, 3$, write $Y_i^\sigma = f_\sigma(Y_i^0)$.

Proposition 7.1. *There exists a continuous extension operator $\mathcal{G}$ from $W^{1,p}(\Omega)$ to $W^{1,p}(\widehat{\Omega})$.*

Proof. First, we introduce continuous mappings $\xi_i : Y_i^0 \to Y^0$ satisfying the following self-similarity properties:

- ◇ $\xi_1 \equiv f_1 \circ \xi_1 \circ f_1{}^{-1}$ on $f_1(P_1' P_1)$;
- ◇ $\xi_3 \equiv f_2 \circ \xi_1 \circ f_2{}^{-1}$ on $f_2(P_1' P_1)$;
- ◇ ξ_2 is the symmetric of ξ_1 with respect to the vertical axis Λ and ξ_3 is symmetric with respect to Λ.

We now define the mapping $\xi : \widehat{\Omega} \to \Omega$ by

$$\xi : x \mapsto \begin{cases} x & \text{if } x \in \Omega \\ f_\sigma \circ \xi_i \circ f_\sigma{}^{-1}(x) & \text{if } x \in Y_i^\sigma. \end{cases}$$

Note that the conditions imposed on the function ξ imply that it is unambiguously defined and continuous.

For any function $u \in W^{1,p}(\Omega)$, we put $\mathcal{G}u = u \circ \xi$. Observe that for all $\sigma \in \mathcal{A}$ and $i \in \{1, 2, 3\}$,

$$\int_{Y_i^\sigma} |\nabla \mathcal{G}u|^p \leqslant C \int_{Y^\sigma} |\nabla u|^p, \tag{7.3}$$

where the constant C is independent of i and σ. Since ξ is continuous, $\mathcal{G}u \in W^{1,p}_{\text{loc}}(\widehat{\Omega})$, which implies that $\mathcal{G}u \in W^{1,p}(\widehat{\Omega})$, and we deduce from (7.3) that $\mathcal{G}$ is continuous. □

The extension operators $\mathcal{F}_n$ Let us now construct the sequence $(\mathcal{F}_n)_n$. Introduce a smooth function χ in $\widehat{Y^0}$ valued in $[0, 1]$ such that $\chi \equiv 1$ in Y^0, the trace of χ on the segments $[P_1' f_1(P_1')]$, $[P_2' f_2(P_2')]$ and $[f_1(P_2') f_2(P_1')]$ is 0, and

$$\begin{aligned} \chi \circ f_1{}^{-1} &\equiv \chi \text{ on } f_1([P_1' P_2']), \\ \chi \circ f_2{}^{-1} &\equiv \chi \text{ on } f_2([P_1' P_2']). \end{aligned} \tag{7.4}$$

Condition (7.4) implies that a certain self-similar property is satisfied by χ. Introduce a smooth function η in $\widehat{K_0}$ with values in $[0, 1]$ such that $\eta = 1$ on $\widehat{\Gamma}^0 = [P_1' P_2']$, and $\eta = 0$ on $f_1(\widehat{\Gamma}^0) \cup f_2(\widehat{\Gamma}^0)$.

For every $n > 0$, we define a function ρ_n in $\mathcal{R}$ by

$$\rho_n = \sum_{k=0}^{n} \sum_{\sigma \in \mathcal{A}_k} \chi \circ f_\sigma{}^{-1} \mathbb{1}_{\widehat{K}^\sigma} + \sum_{\sigma \in \mathcal{A}_{n+1}} \chi\eta \circ f_\sigma{}^{-1} \mathbb{1}_{\widehat{K}^\sigma}. \tag{7.5}$$

Note that ρ_n is continuous in $\mathcal{R}$. Introduce the linear operators $\mathcal{F}_n$ on $W^{1,p}(\Omega)$ defined by

$$\mathcal{F}_n u \;=\; \rho_n \mathcal{G}u + (1 - \rho_n)\mathcal{E}\ell^\infty(u), \quad \forall u \in W^{1,p}(\Omega). \tag{7.6}$$

Condition (7.4) implies that for all $u \in W^{1,p}(\Omega)$, $\mathcal{F}_n u \in W^{1,p}_{\text{loc}}(\mathcal{R})$, and therefore $\mathcal{F}_n u \in W^{1,p}(\mathcal{R})$. Moreover, note that since $\mathcal{G}$, $\mathcal{E}$ and ℓ^∞ are continuous, so are the operators $\mathcal{F}_n$.

Proposition 7.2. *The sequence* $(\mathcal{F}_n)_{n\in\mathbb{N}}$ *converges pointwise to a linear continuous operator* $\mathcal{F} : W^{1,p}(\Omega) \to W^{1,p}(\mathcal{R})$ *such that* $(\mathcal{F}u)|_\Omega = u$ *for all* $u \in W^{1,p}(\Omega)$.

The proof of Proposition 7.2 will use the following Poincaré-Wirtinger inequality: if $u \in W^{1,p}(\widehat{\Omega})$, there exists a constant $C > 0$ such that

$$\int_{\widehat{Y}^0} |u(x) - \langle u\rangle_{\Gamma^0}|^p \, \mathrm{d}x \;\leqslant\; C \int_{\widehat{Y}^0} |\nabla u(x)|^p \, \mathrm{d}x. \tag{7.7}$$

We will also use a strengthened trace inequality proved in [5, Theorem 1.3.3]. We will in fact use a slighlty different version of this inequality, whose proof can be easily adapted to get the following result.

Theorem 7.3. *For all real number* κ *satisfying* $(2a^2)^{p-1} < \kappa < 1$, *there exists a constant* $C > 0$ *such that for all* $u \in W^{1,p}(\Omega)$,

$$\|\ell^\infty(u) - \langle u\rangle_{\Gamma^0}\|^p_{L^p(\Gamma^\infty)} \leqslant C \sum_{i\in\mathbb{N}} \kappa^i \sum_{\tau\in\mathcal{A}_i} \|\nabla u\|^p_{L^p(Y^\tau)}. \tag{7.8}$$

Proof of Proposition 7.2. Take $u \in W^{1,p}(\Omega)$. Let us prove that $(\mathcal{F}_n u)_{n\in\mathbb{N}}$ is a Cauchy sequence in $W^{1,p}(\mathcal{R})$. Take $n, m \in \mathbb{N}$ with $n < m$. We denote $\hat{u} = \mathcal{G}u$ and $\bar{u} = \mathcal{E}\ell^\infty(u)$. First, we note that

$$\begin{aligned}\int_{\mathcal{R}} |\mathcal{F}_m u - \mathcal{F}_n u|^p \, \mathrm{d}x \;&=\; \sum_{k=n+1}^{m+1} \int_{\widehat{Y}^k} |\mathcal{F}_m u - \mathcal{F}_n u|^p \, \mathrm{d}x \\ &=\; \sum_{k=n+1}^{m+1} \int_{\widehat{Y}^k} |(\rho_n - \rho_m)(\hat{u} - \bar{u})|^p \, \mathrm{d}x \\ &\leqslant\; \int_{\widehat{\Omega}^n} |\hat{u} - \bar{u}|^p \, \mathrm{d}x \;\underset{n\to\infty}{\longrightarrow}\; 0,\end{aligned} \tag{7.9}$$

since $\hat{u} - \bar{u}|_{\widehat{\Omega}} \in L^p(\widehat{\Omega})$. On the other hand,

$$\int_{\mathcal{R}} |\nabla\mathcal{F}_m u - \nabla\mathcal{F}_n u|^p \, \mathrm{d}x \;=\; \sum_{k=n+1}^{m+1} \sum_{\sigma\in\mathcal{A}_k} \int_{\widehat{Y}^\sigma} |\nabla\mathcal{F}_m u - \nabla\mathcal{F}_n u|^p \, \mathrm{d}x. \tag{7.10}$$

Take $k \in \mathbb{N}$ such that $n \leqslant k \leqslant m$ and $\sigma \in \mathcal{A}_k$. One has:

$$\begin{aligned}\int_{\widehat{Y}^\sigma} |\nabla\mathcal{F}_m u - \nabla\mathcal{F}_n u|^p \;&=\; \int_{\widehat{Y}^\sigma} |\nabla((\rho_m - \rho_n)(\hat{u} - \bar{u}))|^p \\ &\leqslant\; C\left(\int_{\widehat{Y}^\sigma} |\nabla(\hat{u} - \bar{u})|^p + a^{-kp} \int_{\widehat{Y}^\sigma} |\hat{u} - \bar{u}|^p\right),\end{aligned}$$

where C is a constant independent of σ, since $\rho_m - \rho_n = \chi \circ {f_\sigma}^{-1}$ if $k > n$, and $\rho_m - \rho_n = \chi \circ {f_\sigma}^{-1} - \chi\eta \circ {f_\sigma}^{-1}$ if $k = n$. Therefore,

$$\begin{aligned}&\int_{\mathcal{R}} |\nabla \mathcal{F}_m u - \nabla \mathcal{F}_n u|^p \\ &\lesssim \sum_{k=n+1}^{m+1} \sum_{\sigma\in\mathcal{A}_k} \int_{\widehat{Y}^\sigma} |\nabla(\hat{u} - \bar{u})|^p + \sum_{k=n+1}^{m+1} \sum_{\sigma\in\mathcal{A}_k} a^{-kp} \int_{\widehat{Y}^\sigma} |\hat{u} - \bar{u}|^p.\end{aligned} \tag{7.11}$$

To deal with the first term of the right-hand side in (7.11), we note that

$$\sum_{k=n}^{m} \sum_{\sigma\in\mathcal{A}_k} \int_{\widehat{Y}^\sigma} |\nabla(\hat{u} - \bar{u})|^p \,\mathrm{d}x \;=\; \|\nabla(\hat{u} - \bar{u})\|^p_{L^p(\widehat{\Omega}^n)} \underset{n\to\infty}{\longrightarrow} 0. \tag{7.12}$$

Therefore, we are left to consider the second term. For $\sigma \in \mathcal{A}_k$, one has

$$\sum_{k=n+1}^{m+1} \sum_{\sigma\in\mathcal{A}_k} a^{-pk} \int_{\widehat{Y}^\sigma} |\hat{u} - \bar{u}|^p \,\mathrm{d}x \;\lesssim\; S_1 + S_2,$$

where

$$S_1 = \sum_{k=n+1}^{m+1} \sum_{\sigma\in\mathcal{A}_k} a^{-pk} \int_{\widehat{Y}^\sigma} |\hat{u} - \langle u\rangle_{\Gamma^\sigma}|^p \,\mathrm{d}x$$

and

$$S_2 = \sum_{k=n+1}^{m+1} \sum_{\sigma\in\mathcal{A}_k} a^{-pk} \int_{\widehat{Y}^\sigma} |\bar{u} - \langle u\rangle_{\Gamma^\sigma}|^p \,\mathrm{d}x.$$

By (7.7), one has

$$S_1 \;\leqslant\; C \sum_{k=n+1}^{m+1} \sum_{\sigma\in\mathcal{A}_k} a^{(2-p)k} \int_{\widehat{Y}^\sigma} |\nabla\hat{u}|^p \,\mathrm{d}x \;\lesssim\; \int_{\widehat{\Omega}_n} |\nabla\hat{u}|^p \,\mathrm{d}x \underset{n\to\infty}{\longrightarrow} 0. \tag{7.13}$$

We are left to consider S_2. Since for all $\tau \in \mathcal{A}_n$ with $n > k$ we have $\bar{g_\tau}|_{\widehat{Y}^\sigma} \equiv 0$, then

$$\int_{\widehat{Y}^\sigma} |\bar{u} - \langle u\rangle_{\Gamma^\sigma}|^p \,\mathrm{d}x = \int_{\widehat{Y}^\sigma} \Big|P_0\ell^\infty(u) + \sum_{i\leqslant k} \sum_{\tau\in\mathcal{A}_i} \beta_\tau \bar{g}_\tau - \langle u\rangle_{\Gamma^\sigma}\Big|^p \,\mathrm{d}x. \tag{7.14}$$

By writing the expansions of $\ell^\infty(u \circ f_\sigma)$ and $\ell^\infty(u)$ in the Haar wavelet basis on Γ^∞, and observing that $\ell^\infty(u \circ f_\sigma) = \ell^\infty(u) \circ f_\sigma$ we observe that on $\Gamma^{\infty,\sigma}$,

$$P_0\ell^\infty(u) + \sum_{i<k} \sum_{\tau\in\mathcal{A}_i} \beta_\tau g_\tau \;=\; P_0\ell^\infty(u \circ f_\sigma). \tag{7.15}$$

If $i < k$ and $\tau \in \mathcal{A}_i$, then $\bar{g}_\tau$ is constant on $f_\tau(f_j(\Omega))$, for $j = 1, 2$, therefore $\bar{g}_\tau$ is constant on Y^σ and $\bar{g}_\tau(Y^\sigma) = g_\tau(\Gamma^{\infty,\sigma})$. Hence, the following equality holds in Y^σ:

$$P_0\ell^\infty(u) + \sum_{i<k} \sum_{\tau\in\mathcal{A}_i} \beta_\tau \bar{g}_\tau = P_0\ell^\infty(u \circ f_\sigma). \tag{7.16}$$

Therefore, combining (7.16) with (7.14), we get

$$\begin{aligned}\int_{\widehat{Y}^\sigma} |\bar{u} - \langle u\rangle_{\Gamma^\sigma}|^p \,\mathrm{d}x &\lesssim a^{2k} |P_0\ell^\infty(u \circ f_\sigma) - \langle u\rangle_{\Gamma^\sigma}|^p + \int_{\widehat{Y}^\sigma} |\beta_\sigma \bar{g}_\sigma|^p \,\mathrm{d}x &(7.17)\\ &\lesssim a^{2k}\left(\int_{\Gamma^\infty} |\ell^\infty(u\circ f_\sigma) - \langle u\circ f_\sigma\rangle_{\Gamma^0}|^p \,\mathrm{d}\mu + 2^{\frac{kp}{2}} |\beta_\sigma|^p\right). &(7.18)\end{aligned}$$

By Theorem 7.3, for all $\kappa \in](2a^2)^{p-1}, 1[$ we have

$$\begin{aligned}\int_{\Gamma^\infty} |\ell^\infty(u\circ f_\sigma) - \langle u\circ f_\sigma\rangle_{\Gamma^0}|^p \,\mathrm{d}\mu &\leqslant C \sum_{i\geqslant 0} \kappa^i \sum_{\tau\in\mathcal{A}_i} \int_{Y^\tau} |\nabla(u \circ f_\sigma)|^p \,\mathrm{d}x\\ &= C\, a^{(p-2)k} \sum_{i\geqslant 0} \kappa^i \sum_{\tau\in\mathcal{A}_i} \int_{Y^{\sigma\tau}} |\nabla u|^p \,\mathrm{d}x \qquad (7.19)\\ &= C\, a^{(p-2)k} \sum_{i\geqslant k} \kappa^{i-k} \sum_{\substack{\tau\in\mathcal{A}_i,\\ \tau_{\restriction k}=\sigma}} \int_{Y^\tau} |\nabla u|^p \,\mathrm{d}x.\end{aligned}$$

Hence,

$$\begin{aligned}&\sum_{k=n+1}^{m+1} \sum_{\sigma\in\mathcal{A}_k} a^{(2-p)k} \int_{\Gamma^\infty} |\ell^\infty(u \circ f_\sigma) - \langle u \circ f_\sigma\rangle_{\Gamma^0}|^p \,\mathrm{d}\mu\\ &\lesssim \sum_{k=n+1}^{m+1} \sum_{\sigma\in\mathcal{A}_k} \sum_{i\geqslant k} \kappa^{i-k} \sum_{\tau\in\mathcal{A}_i,\ \tau_{\restriction k}=\sigma} \int_{Y^\tau} |\nabla u|^p \,\mathrm{d}x\\ &= \sum_{i\geqslant n+1} \sum_{\tau\in\mathcal{A}_i} \sum_{k=n+1}^{\min(i,m+1)} \kappa^{i-k} \int_{Y^\tau} |\nabla u|^p \,\mathrm{d}x\\ &\lesssim \sum_{i\geqslant n} \sum_{\tau\in\mathcal{A}_i} \int_{Y^\tau} |\nabla u|^p = \int_{\Omega^n} |\nabla u|^p \,\mathrm{d}x.\end{aligned}$$

Then, (7.18) yields

$$S_2 \lesssim \int_{\Omega^n} |\nabla u|^p \,\mathrm{d}x + \sum_{k\geqslant n} \sum_{\sigma\in\mathcal{A}_k} a^{(2-p)k} 2^{\frac{kp}{2}} |\beta_\sigma|^p \underset{n\to\infty}{\longrightarrow} 0. \tag{7.20}$$

Indeed, since $\ell^\infty(u) \in \mathrm{JLip}(1-\frac{2-d}{p}, p, p, 0; \Gamma^\infty)$, $\sum_k 2^{\frac{kp}{2}} \sum_{\sigma\in\mathcal{A}_k} a^{(2-p)k} |\beta_\sigma|^p < \infty$. This proves that the sequence $(\mathcal{F}_n u)_n$ has a limit in $W^{1,p}(\mathcal{R})$. We define $\mathcal{F}u$ to be the latter limit; $\mathcal{F}$ obviously defines a linear operator from $W^{1,p}(\Omega)$ to $W^{1,p}(\mathcal{R})$. Therefore, the Banach-Steinhaus Theorem ensures that $\mathcal{F}$ is continuous. The fact that $(\mathcal{F}u)|_\Omega = u$ is a direct consequence of (7.1). $\square$

8. Proof of the geometrical results

In this section we prove the geometrical results stated in Paragraphs 6.1.2 and 6.2.2.

8.1. Proof of the geometrical results from Paragraph 6.1.2

First, we observe that we may assume that we have chosen the function χ such that

$$d_2 < c \tag{8.1}$$

$$d_1 > ad_2, \tag{8.2}$$

where $d_1 = d(\Omega, \operatorname{supp} \nabla\chi)$ and $d_2 = \sup\{d(x, \Omega),\ x \in \operatorname{supp} \chi\}$, see Figure 6.1. Condition (8.2) plays an important role in the proof that (6.1) is indeed fulfilled. Conditions (8.1) and (6.6) helps dealing with the pairwise intersections of the sets $\operatorname{supp} \nabla\bar{g}_\sigma$. Before proving the results from Section 6.1.2, we prove several geometrical lemmas.

Lemma 8.1. *If $\sigma \notin \mathcal{B}$, then*

$$\operatorname{supp} \tilde{g}_\sigma \cap \mathscr{C} = \emptyset, \tag{8.3}$$

where $\mathcal{B}$ was defined in (6.3).

Proof. We first observe that, by self-similarity and by symmetry, $\sup\{d(x, \Omega \setminus Y^0),\ x \in \operatorname{supp} \bar{g}_0\} \leqslant ad_2$, where we recall that $d_2 = \sup\{d(x, \Omega),\ x \in \operatorname{supp} \chi\}$. Therefore, by an argument of self-similarity, $\sup\{d(x, \Omega^\sigma \setminus Y^\sigma),\ x \in \operatorname{supp} \tilde{g}_\sigma\} \leqslant a^{n+1}d_2 < ca^n$, by condition (8.1). Since $\sigma \notin \mathcal{B}$, $d(\mathscr{C}, \Omega^\sigma \setminus Y^\sigma) \geqslant d(\mathscr{C}, \Omega^\sigma) > ca^n$, by (6.2), which implies the result. $\square$

Lemma 8.2. *If $\sigma \in \mathcal{A} \setminus \{\epsilon\}$ and $\sigma(1) = 1$ (respectively $\sigma(1) = 2$), then*

$$\operatorname{supp} \bar{g}_\sigma \setminus S \subset \{(x_1, x_2) \in \mathbb{R}^2,\ x_1 < 0\} \tag{8.4}$$

(respectively $\operatorname{supp} \bar{g}_\sigma \setminus S \subset \{(x_1, x_2) \in \mathbb{R}^2,\ x_1 > 0\}$*).*

Proof. Suppose for example $\sigma(1) = 1$. If $\sigma \notin \mathcal{B}$, then by Lemma 8.1, $\operatorname{supp} \bar{g}_\sigma \cap \mathscr{C} = \emptyset$. Therefore, $\operatorname{supp} \bar{g}_\sigma$ lies in the left-hand connected component of $\mathbb{R}^2 \setminus \mathscr{C}$, and (8.4) is satisfied.

If $\sigma \in \mathcal{B}$, then, by the definition of $\bar{g}_\sigma$, the function ψ is a factor of $\bar{g}_\sigma$, which implies that $\operatorname{supp} \bar{g}_\sigma \setminus \mathscr{C} \subset \{(x_1, x_2) \in \mathbb{R}^2,\ x_1 > 0\}$. Since $\operatorname{supp} \bar{g}_\sigma \subset \operatorname{supp} \bar{g}_0$, one has $\operatorname{supp} \bar{g}_\sigma \setminus S = \operatorname{supp} \bar{g}_\sigma \setminus \mathscr{C}$, hence the result. $\square$

Lemma 8.3. *If $\tau, \sigma \in \mathcal{A}$ and τ is a prefix of σ and $\tau \neq \sigma$, then*

$$F^{\tau} \cap \operatorname{supp} \bar{g}_{\sigma} = \emptyset. \tag{8.5}$$

Proof. Take $\tau \in \mathcal{A}_k$ and $\sigma \in \mathcal{A}_n$ such that τ is a prefix of σ. We first note that, by an argument of self-similarity and by symmetry, $d(\Omega \setminus Y^0, F) \geqslant ad_1$, where we recall that $d_1 = d(\Omega, \operatorname{supp} \nabla\chi)$. Therefore, $d(\Omega^{\sigma} \setminus Y^{\sigma}, F^{\tau}) \geqslant d(\Omega^{\tau} \setminus Y^{\tau}, F^{\tau}) \geqslant a^{k+1} d_1$. On the other hand, as seen in the proof of Lemma 8.1, $\sup\{d(x, \Omega^{\sigma} \setminus Y^{\sigma}),\ x \in \operatorname{supp} \bar{g}_{\sigma}\} \leqslant a^{n+1} d_2$. Since $k < n$, condition (8.2) yields

$$d(\Omega^{\sigma} \setminus Y^{\sigma}, F^{\tau}) \ > \ \sup\{d(x, \Omega^{\sigma} \setminus Y^{\sigma}),\ x \in \operatorname{supp} \bar{g}_{\sigma}\}.$$

Therefore, if $x \in \operatorname{supp} \bar{g}_{\sigma}$, then $d(x, \Omega^{\sigma} \setminus Y^{\sigma}) < d(\Omega^{\sigma} \setminus Y^{\sigma}, F^{\tau})$ and $x \notin F^{\tau}$. □

Lemma 8.4. *If $n > 0$ and $\sigma \in \mathcal{A}_n$, then, for every $k < n$,*

$$\operatorname{supp} \bar{g}_{\sigma} \subset f_{\sigma_{\restriction k}}(\operatorname{supp} \bar{g}_{\sigma'}), \tag{8.6}$$

where $\sigma = \sigma_{\restriction k} \sigma'$.

Proof. First observe that for any $\tau' \in \mathcal{M}(\sigma')$, one has $\gamma_{\tau'}^{\sigma'} = \gamma_{\eta\tau'}^{\eta\sigma'} \circ f_{\eta}^{-1}$ for all $\eta \in \mathcal{A}$. In particular, $\gamma_{\tau'}^{\sigma'} = \gamma_{\sigma_{\restriction k}\tau'}^{\sigma} \circ f_{\sigma_{\restriction k}}^{-1}$. Since for all $\tau' \in \mathcal{M}(\sigma') \setminus \{\sigma'\}$, one has $\sigma_{\restriction k}\tau' \in \mathcal{M}(\sigma) \setminus \{\sigma\}$,

$$\begin{aligned} |\bar{g}_{\sigma}| = \Bigg(\prod_{\substack{\tau \in \mathcal{M}(\sigma) \\ \tau \neq \sigma}} \gamma_{\tau}^{\sigma} \Bigg) |\tilde{g}_{\sigma}| \leqslant \Bigg(\prod_{\substack{\tau' \in \mathcal{M}(\sigma') \\ \tau' \neq \sigma'}} \gamma_{\sigma_{\restriction k}\tau'}^{\sigma} \Bigg) |\tilde{g}_{\sigma}| &= \Bigg(\prod_{\substack{\tau' \in \mathcal{M}(\sigma') \\ \tau' \neq \sigma'}} \gamma_{\tau'}^{\sigma'} \circ f_{\sigma_{\restriction k}}^{-1} \Bigg) |\tilde{g}_{\sigma}| \\ &= 2^{\frac{k}{2}} |\bar{g}_{\sigma'} \circ f_{\sigma_{\restriction k}}^{-1}|, \end{aligned}$$

which implies the desired result. □

Lemma 8.5. *If $\sigma \in \mathcal{A}$ is a non-empty string, then $S^{\sigma} \cap \mathscr{C}$. As a consequence, if $\sigma, \tau \in \mathcal{A}$ and $\sigma \neq \tau$, then $S^{\sigma} \cap S^{\tau} = \emptyset$.*

Proof. Take $\sigma \in \mathcal{A} \setminus \{\epsilon\}$, and suppose for example $\sigma(1) = 1$. The set Ω^{σ} lies in the convex set $\mathcal{D}$ defined by $\{x_1 < 0\} \setminus \mathscr{C}$. Therefore, its convex hull $\operatorname{conv}(\Omega^{\sigma})$ also lies in $\mathcal{D}$. Since condition (6.6) is fulfilled, $S^{\sigma} \subset \operatorname{conv}(\Omega^{\sigma})$, S^{σ} lies in $\mathcal{D}$, and $S^{\sigma} \cap \mathscr{C} = \emptyset$, hence the result. Now take $\sigma, \tau \in \mathcal{A}$ such that $\sigma \neq \tau$, and suppose at first that τ is a prefix of σ: write $\sigma = \tau\sigma'$, where $\sigma' \in \mathcal{A}$. Then, $S^{\sigma} \cap S^{\tau} = f_{\tau}(S^{\sigma'} \cap S) = \emptyset$ by the previous point. If none of the strings σ, τ is a prefix of the other, then there exist $\eta \in \mathcal{A}$ and non-empty strings σ', τ' such that $\sigma = \eta\sigma'$ and $\tau = \eta\tau'$ with $\sigma'(1) \neq \tau'(1)$. Therefore, $S^{\sigma'} \cap \mathscr{C} = S^{\tau'} \cap \mathscr{C} = \emptyset$ and, since the vertices of $S^{\sigma'}$ and $S^{\tau'}$ lie in opposite sides of the vertical axis Λ, $S^{\sigma'} \cap S^{\tau'} = \emptyset$. Hence, $S^{\sigma} \cap S^{\tau} = f_{\eta}(S^{\sigma'} \cap S^{\tau'}) = \emptyset$. □

Remark 8.6. In particular, if $\sigma \in \mathcal{A}$ and $\sigma(1) = 1$ (respectively $\sigma(1) = 2$), then $S^\sigma \subset \{x_1 < 0\}$ (respectively $S^\sigma \subset \{x_1 > 0\}$).

Proof of Proposition 6.4. If one of the strings σ, τ is a prefix of the other, then $F^\sigma \cap F^\tau = \emptyset$ by Lemma 8.3. Since $\operatorname{supp} \nabla \bar{g}_\sigma \setminus \mathscr{S} \subset F^\sigma$ and $\operatorname{supp} \nabla \bar{g}_\tau \setminus \mathscr{S} \subset F^\tau$ by (6.13), $(\operatorname{supp} \nabla \bar{g}_\sigma \setminus \mathscr{S}) \cap (\operatorname{supp} \nabla \bar{g}_\tau \setminus \mathscr{S}) = \emptyset$. Now suppose none of the sequences σ, τ is a prefix of the other, *i.e.* there exist $\eta \in \mathcal{A}$ and non-empty strings σ', τ' such that $\sigma = \eta\sigma'$ and $\tau = \eta\tau'$ with $\sigma'(1) \neq \tau'(1)$. By Lemma 8.2, $(\operatorname{supp} \bar{g}_{\sigma'} \setminus S) \cap (\operatorname{supp} \bar{g}_{\tau'} \setminus S) = \emptyset$, and Lemma 8.4 implies the result. □

Proof of Proposition 6.5. By the definition of $\bar{g}_\sigma$ (see (6.9)), it is enough to show that if $\tau' \in \mathcal{M}(\sigma) \setminus \{\sigma, \tau\}$, then $\gamma^\sigma_{\tau'} \equiv 1$ in $S^\tau \cap \operatorname{supp} \tilde{g}_\sigma$. First note that the function $\gamma^\sigma_\tau \mathbb{1}_{\operatorname{supp} \tilde{g}_\sigma}$ is constant outside $S^\tau \cap \operatorname{supp} \tilde{g}_\sigma$. Therefore, it is constant in $S^{\tau'}$ if $\tau' \neq \tau$. Take $\tau' \in \mathcal{M}(\sigma) \setminus \{\sigma, \tau\}$ and write $\sigma = \tau'\sigma'$. Suppose for example that $\sigma'(1) = 1$, which implies that $\gamma^\sigma_{\tau'} \equiv 1$ in $f_{\tau'}(\{x_1 < 0\}) \setminus \mathscr{C}^{\tau'}$. Since $\sigma'(1) = 1$, we deduce that $\Omega^\sigma \subset f_{\tau'}(\{x_1 < 0\})$. By the definition of $\mathcal{M}(\sigma)$ (see (6.8)), one has $f_\tau(A) \in \overline{\Omega^\sigma}$. Consequently, $f_\tau(A) \in f_{\tau'}(\{x_1 < 0\}) \setminus \mathscr{C}^{\tau'}$, since $f_{\tau'}(A) \neq f_\tau(A)$. Therefore, the connected set $S^\tau \cap \operatorname{supp} \bar{g}_\sigma$ lies in the connected component $f_{\tau'}(\{x_1 < 0\}) \setminus \mathscr{C}^{\tau'}$ of $\mathbb{R}^2 \setminus \mathscr{C}^{\tau'}$, which yields the result. □

Proof of Proposition 6.6. Take $x \in S^\tau$ and write $M_x = \{\sigma \in \mathcal{A}, \tau \in \mathcal{M}(\sigma),\ \nabla(\bar{g}_0 \circ f_\sigma^{-1})(x) \neq 0\}$. We first note that if $\sigma \in M_x$ and $\sigma \neq \tau$, then $x \in F^\sigma$ since $S^\sigma \cap S^\tau = \emptyset$ by Lemma 8.5 (see (6.13)). If $\sigma, \sigma' \in M_x$, write $\sigma = \tau\eta$ and $\sigma' = \tau\eta'$ where $\eta, \eta' \in \mathcal{B}$. Suppose that η, η' are prefixes of $12^{\mathrm{m}+1}(12)^\infty$ and distinct from ϵ. Therefore, one of the two strings η, η' is a prefix of the other. Assume for example that η' is a prefix of η. Consequently, σ' is a prefix of σ, and Lemma 8.3 implies that $\sigma = \sigma'$, since $x \in F^{\sigma'} \cap F^\sigma \subset F^{\sigma'} \cap \operatorname{supp}(\bar{g}_0 \circ f_\sigma^{-1})$. Therefore, there is at most one string $\sigma \in M_x \setminus \{\tau\}$ such that σ is a prefix of $12^{\mathrm{m}+1}(12)^\infty$. Similarly, there is at most one string $\sigma \in M_x \setminus \{\tau\}$ such that σ is a prefix of $21^{\mathrm{m}+1}(21)^\infty$. Since $\tau \in M_x$, there are at most three strings in M_x. □

Proof of Proposition 6.7. For $n, k \in \mathbb{N}$, take $\sigma \in \mathcal{A}_n$ and $\tau \in \mathcal{A}_k$.

1. First, we prove that if $n < k$, then $\nabla \bar{g}_\sigma \equiv 0$ in S^τ. For all $\eta \in \mathcal{M}(\sigma)$ with $\eta \in \mathcal{A}_m$, one has $m \leqslant n < k$, and $\eta \neq \tau$, which implies by Lemma 8.5 that $S^\tau \cap S^\eta = \emptyset$. Therefore, by (6.13), $\operatorname{supp} \nabla \bar{g}_\sigma \cap S^\tau \subset F^\sigma$. If σ is a prefix of τ, then Lemma 8.3 implies that $\operatorname{supp} \nabla \bar{g}_\sigma \cap S^\tau = \emptyset$ since $S^\tau \subset \operatorname{supp} \bar{g}_\tau$. If σ is not a prefix of τ, then there exists $m < n$ such that $\sigma = \sigma_{\restriction m}\sigma'$, $\tau = \sigma_{\restriction m}\tau'$ with $\sigma', \tau' \in \mathcal{A} \setminus \{\epsilon\}$ and $\sigma'(1) \neq \tau'(1)$. Remark 8.6 and Lemma 8.2 imply that $\operatorname{supp} \bar{g}_{\sigma'} \setminus S$ and $S^{\tau'}$ lie in opposite sides of the axis Λ. Since $S \cap S^{\tau'} = \emptyset$, $\operatorname{supp} \bar{g}_{\sigma'} \cap S^{\tau'} = \emptyset$. Therefore, $\operatorname{supp} \nabla \bar{g}_\sigma \cap S^\tau = \emptyset$, since $\operatorname{supp} \bar{g}_\sigma \subset f_{\sigma_{\restriction k}}(\operatorname{supp} \bar{g}_{\sigma'})$ by Lemma 8.4.

2. Second, we prove that if $n \geqslant k$ and $\tau \notin \mathcal{M}(\sigma)$, then $\bar{g}_\sigma \equiv 0$ on S^τ. This will achieve the proof of Proposition 6.7. Suppose $\tau \notin \mathcal{M}(\sigma)$. If τ is not a prefix of σ, then the same argument as above applies. If τ is a prefix of σ, write $\sigma = \tau\sigma'$. Since $\tau \notin \mathcal{M}(\sigma)$, $\sigma' \notin \mathcal{B}$ and $\operatorname{supp} \bar{g}_{\sigma'} \cap S = \emptyset$ by Lemma 8.1. Lemma 8.4 yields the result as above. □

8.2. Proof of the geometrical results from Paragraph 6.2

First, it is straightforward to check that, with the notation introduced in Paragraph 6.2.1 and (6.26), Lemmas 8.1, 8.2, 8.3 and 8.4 still hold in this case. We will now prove that the other results still hold as well. Note that up to taking a smaller angle φ_0 for the diamonds D^η, $\eta \in \mathcal{B}^+$, one can assume that $\sup_{x\in D} d(x,\Lambda) < ca^{\mathfrak{m}+2}$ where c is the constant from (6.26). We assume this is true in the following. Observe that a consequence of the previous assumption is that $\sup\limits_{x\in D} d(x, \operatorname{conv}(f_{12^{\mathfrak{m}+1}}(\Omega))) < ca^{\mathfrak{m}+2}$, which implies by self-similarity that

$$\sup_{x\in D^0} d(x, \operatorname{conv}(\Omega)) < c. \tag{8.7}$$

We may further assume that we have chosen the function χ such that:

- conditions (8.1) and (8.2) are satisfied, with the same notation;
- χ satisfies the following condition that replaces condition (6.6) in the proof of point 1:
$$d_2 < d(\mathscr{C} \cap D^0, \Omega), \tag{8.8}$$
where we recall that $d_2 = \sup\{d(x,\Omega),\ x \in \operatorname{supp}\chi\}$.

Condition (8.8) implies by a simple geometric argument that

$$S \cap D^0 = \emptyset. \tag{8.9}$$

Lemma 8.7. *If $\eta \notin \mathcal{B}$ and $\eta(1) = 1$ (respectively $\eta(1) = 2$), then $D^\eta \subset \{x_1 < 0\} \setminus \mathscr{C}$ (respectively $D^\eta \subset \{x_1 > 0\} \setminus \mathscr{C}$).*

Proof. Take $\eta \in \mathcal{A}_n \setminus \mathcal{B}$ and suppose $\eta(1) = 1$. By (6.26), we know that $d(\operatorname{conv}(\Omega^\eta), \mathscr{C}) > ca^n$. By (8.7), we have $\sup\limits_{x\in D^\eta} d(x, \operatorname{conv}(\Omega^\eta)) < ca^n$ which implies that $D^\eta \cap \mathscr{C} = \emptyset$. The result follows since the vertices of D^η lie in the left-hand side connected component of $\mathbb{R}^2 \setminus \mathscr{C}$. □

We now give the proof that Lemma 8.5 also remains true:

Proof. Take a string $\sigma \neq \epsilon$. The proof that $S^\sigma \cap \mathscr{C}_u = S^\sigma \cap \mathscr{C}_l = \emptyset$ is the same as in Lemma 8.5. We are left with checking that $S^\sigma \cap \mathscr{D} = \emptyset$. Take $\eta \in \mathcal{B}^+$, and suppose first that none of the strings η, σ is a prefix of the other. That is, there exist non-empty strings η', σ' and $k \geqslant 0$ such that $\eta = \eta_{\restriction k}\eta'$, $\sigma = \eta_{\restriction k}\sigma'$ and $\sigma'(1) \neq \eta'(1)$. Suppose for example that $\sigma'(1) = 1$. Then $S^{\sigma'} \subset \operatorname{conv}(\Omega^{\sigma'}) \subset \{x_1 > 0\}$. On the other hand, by Lemma 8.7, since $\eta'(1) = 2$, $D^{\eta'} \subset \{x_1 > 0\}$, and $S^\sigma \cap D^\eta = \emptyset$. We now examine the cases when one string is a prefix of the other.

– If $\eta = \sigma$, then $S^\sigma \cap D^\eta = \emptyset$ by (8.9) and by self-similarity.
– If $\sigma = \eta\sigma'$ with $\sigma' \neq \epsilon$, then $S^\sigma \subset \operatorname{conv}(\Omega^\sigma) \subset \operatorname{conv}(\Omega^{\eta\sigma'(1)})$. It is easy to see that $\operatorname{conv}(f_i(\Omega)) \cap D^0 = \emptyset$, $i = 1, 2$, from which we deduce by self-similarity that $\operatorname{conv}(\Omega^{\eta\sigma'(1)}) \cap D^\eta = \emptyset$, and $S^\sigma \cap D^\eta = \emptyset$.
– If $\eta = \sigma\eta'$ with $\eta' \neq \epsilon$, then necessarily $\eta' \notin \mathcal{B}$, which implies by Lemma 8.7 that $D^{\eta'} \cap \mathscr{C} = \emptyset$. We deduce that $D^\eta \cap S^\sigma = \emptyset$.

We may now conclude that $S^\sigma \cap S^\tau = \emptyset$ when $\sigma \neq \tau$ as in Lemma 8.5. □

Therefore, Propositions 6.4, 6.5 and 6.7 still hold, with the same proofs. We now give a proof of Proposition 6.9.

Proof of Proposition 6.9. Take $\sigma \in \mathcal{A}$ such that σ is not a prefix of $\tau\eta 12(21)^\infty$ or $\tau\eta 21(12)^\infty$. We first note that if $\sigma \notin \mathcal{B}$, then Lemma 8.1 yields the result. In the following, we assume that $\sigma \in \mathcal{B}$. By the hypothesis, σ cannot be a prefix of η. We now examine the other cases.

– We first suppose η is not a prefix of σ. By symmetry, we can assume that $\sigma(1) = \eta(1) = 1$. There exists an integer $k > 0$ such that $\eta = \eta_{\restriction k}\eta'$, $\sigma = \eta_{\restriction k}\sigma'$ with $\eta'(1) \neq \sigma'(1)$. Suppose for example $\sigma'(1) = 1$. Since $\sigma \in \mathcal{B}$, $\sigma' \notin \mathcal{B}$, and Lemma 8.1 implies that $\operatorname{supp} \tilde{g}_{\sigma'} \subset \{x_1 < 0\}$. Since $\eta' \notin \mathcal{B}$ and $\eta'(1) = 2$, the set $D^{\eta'}$ lies in $\{x_1 > 0\}$ by Lemma 8.7, which implies that $\operatorname{supp} \tilde{g}_{\sigma'} \cap D^{\eta'} = \emptyset$. We conclude by self-similarity.
– In the case where η is a prefix of σ, the hypothesis we made on σ and η imply that σ has a prefix of the form $\eta 12(21)^l 1$ or $\eta 21(12)^l 2$ with $l \geqslant 0$. Suppose for example that the former is true, and write $\sigma = \eta 12(21)^l \sigma'$. Therefore, since $\sigma' \notin \mathcal{B}$, $\operatorname{supp} \tilde{g}_\sigma \subset \{x_1 < 0\}$ by Lemma 8.1, from which we deduce that $\operatorname{supp} \tilde{g}_\sigma$ lies above the horizontal axis $f_{\eta 12(21)^l}(\Lambda)$. On the other hand, the limit point of $\eta 12(21)^\infty$ lies below this axis. Since this point is the highest vertex of the set D^η, the result follows.

The second point of the proposition can be proved with the same argument as above. □

We may now deduce from Proposition 6.9 that Proposition 6.6 also remains true in this case.

Proof. The proof is very similar to that of Proposition 6.6, we use the same notation. Take $x \in S^\tau = S_u^\tau \cup f_\tau(\mathscr{D}) \cup S_l^\tau$. By Proposition 6.9, if $x \in S_u^\tau$ or $x \in S_l^\tau$, the same proof as that of Propostion 6.9 applies, and $\# M_x \leqslant 3$.

Now suppose $x \in D^\eta$ for $\eta \in \mathcal{B}^+$. If $\sigma \in M_x$, write $\sigma = \tau\sigma'$. Then, by Proposition 6.9, σ' is a prefix of $\eta 12(21)^\infty$ or $\eta 21(12)^\infty$ if $\sigma'(1) = \eta(1)$. If $\sigma(1) \neq \eta(1)$, then the situation is symmetric. The same argument as in the proof of Proposition 6.6 applies, and there are at most 5 strings in M_x. □

References

[1] Y. ACHDOU, T. DEHEUVELS and N. TCHOU, JLip *versus Sobolev spaces on a class of self-similar fractal foliages*, J. Math. Pures Appl. **97** (2012), 142–172.

[2] Y. ACHDOU, T. DEHEUVELS and N. TCHOU, *Comparison of different definitions of traces for a class of ramified domains with self-similar fractal boundaries*, Potential Anal. **14** (2014), 354–362.

[3] Y. ACHDOU and N. TCHOU, *Neumann conditions on fractal boundaries*, Asymptot. Anal. **53** (2007), 61–82.

[4] Y. ACHDOU and N. TCHOU, *Trace results on domains with self-similar fractal boundaries*, J. Math. Pures Appl. **89** (2008), 596–623.

[5] Y. ACHDOU and N. TCHOU, *Trace theorems for a class of ramified domains with self-similar fractal boundaries*, SIAM J. Math. Anal. **42** (2010), 828–860 (electronic).

[6] G. BERGER, *Eigenvalue distribution of elliptic operators of second order with Neumann boundary conditions in a snowflake domain*, Math. Nachr. **220** (2000), 11–32.

[7] A.-P. CALDERÓN, *Lebesgue spaces of differentiable functions and distributions*, In: "Proc. Sympos. Pure Math.", Vol. IV, American Mathematical Society, Providence, R.I., 1961, 33–49.

[8] R. CAPITANELLI, *Asymptotics for mixed Dirichlet-Robin problems in irregular domains*, J. Math. Anal. Appl. **362** (2010), 450–459.

[9] T. DEHEUVELS, "Contributions to the Study of Function Spaces and PDE for a Class of Domains with Fractal Self-similar Boundary" PhD Thesis, Rennes, 2013.

[10] K. J. FALCONER, "Techniques in Fractal Geometry", John Wiley & Sons Ltd., Chichester, 1997.

[11] P. HAJŁASZ, P. KOSKELA and H. TUOMINEN, *Measure density and extendability of Sobolev functions*, Rev. Mat. Iberoam. **24** (2008), 645–669.

[12] P. HAJŁASZ, P. KOSKELA and H. TUOMINEN, *Sobolev embeddings, extensions and measure density condition*, J. Funct. Anal. **254** (2008), 1217–1234.

[13] J. E. HUTCHINSON, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713–747.

[14] P. W. JONES, *Quasiconformal mappings and extendability of functions in Sobolev spaces*, Acta Math. **147** (1981), 71–88.

[15] A. JONSSON, *Wavelets on fractals and Besov spaces*, J. Fourier Anal. Appl. **4** (1998), 329–340.

[16] A. JONSSON, *Haar wavelets of higher order on fractals and regularity of functions*, J. Math. Anal. Appl. **290** (2004), 86–104.

[17] A. JONSSON and H. WALLIN, "Function Spaces on Subsets of $\mathbb{R}^n$", Math. Rep. 2 (1), 1984.

[18] A. JONSSON and H. WALLIN, *The dual of Besov spaces on fractals*, Studia Math. **112** (1995), 285–300.

[19] J. KIGAMI, "Analysis on Fractals", Vol. 143 of Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 2001.

[20] P. KOSKELA, *Extensions and imbeddings*, J. Funct. Anal. **159** (1998), 369–383.

[21] P. KOSKELA, "Capacity Extension Domains", Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes (73), 42, 1990. Dissertation, University of Jyväskylä, Jyväskylä, 1990.

[22] M. R. LANCIA, *A transmission problem with a fractal interface*, Z. Anal. Anwend. **21** (2002), 113–133.

[23] M. R. LANCIA, *Second order transmission problems across a fractal surface*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. **27** (2003), 191–213.

[24] B. B. MANDELBROT and M. FRAME, *The canopy and shortest path in a self-contacting fractal tree*, Math. Intelligencer **21** (1999), 18–27.

[25] O. MARTIO and J. SARVAS, *Injectivity theorems in plane and space*, Ann. Acad. Sci. Fenn. Ser. A I Math. **4** (1979), 383–401.

[26] V. G. MAZ'JA, "Sobolev Spaces", Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985, translated from the Russian by T. O. Shaposhnikova.
[27] V. G. MAZ′YA and S. V. POBORCHI, "Differentiable Functions on Bad Domains", World Scientific Publishing Co., Inc., River Edge, NJ, 1997.
[28] P. A. P. MORAN, *Additive functions of intervals and Hausdorff measure*, Proc. Cambridge Philos. Soc. **42** (1946), 15–23.
[29] U. MOSCO and M. A. VIVALDI, *Variational problems with fractal layers*, Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. **27** (2003), 237–251.
[30] E. M. STEIN, "Singular Integrals and Differentiability Properties of Functions", Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.
[31] S. K. VODOP′JANOV, V. M. GOL′DŠTEĬN and T. G. LATFULLIN, *A criterion for the extension of functions of the class L_2^1 from unbounded plane domains*, Sibirsk. Mat. Zh. **20** (1979), 416–419, 464.
[32] H. WHITNEY, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. **36** (1934), 63–89.

Ećole Normale Supérieure de Rennes
Campus de Ker Lann
avenue Robert Schuman
35170 Bruz, France
thibaut.deheuvels@ens-rennes.fr