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The Lie algebra generated by locally nilpotent derivations on a Danielewski surface

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Abstract. We give a full description of the Lie algebra generated by locally nilpotent derivations (shortly LNDs) on smooth Danielewski surfaces D_p given by $xy = p(z)$. In case $\deg(p) \geq 3$ it turns out that it is not equal to the whole Lie algebra $\mathrm{VF}^{\omega}_{\mathrm{alg}}(D_p)$ of volume-preserving algebraic vector fields, thus answering a question posed by Lind and the first author. We also show an algebraic volume density property (shortly AVDP) for a certain homology plane (a homogeneous space of the form $SL_2(\mathbb{C})/N$, where N is the normalizer of the maximal torus) and a related example.

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1. Introduction

In this paper we study (using algebraic methods) the holomorphic automorphism group $\mathrm{Aut}_{\mathrm{hol}}(D_p)$ of a *Danielewski surface* of the form $D_p = \{xy = p(z)\}$. These surfaces are an object of intensive studies in affine algebraic geometry, see, *e.g.*, [4–10, 12, 20, 21] and [22],

The study of these surfaces from the complex analytic point of view started in the paper of Kaliman and Kutzschebauch [13], where they proved the so-called density property, or for short DP. This is a remarkable property, discovered in the 1990s by Andersén and Lempert [1, 2] for Euclidean spaces, that to a great extent compensates for the lack of partition of unity for holomorphic automorphisms. The terminology was later introduced by Varolin [25]: a Stein manifold X has DP if the Lie algebra generated by completely integrable holomorphic vector fields is dense (in the compact-open topology) in the space of all holomorphic vector fields on X. In the presence of DP one can construct global holomorphic automorphisms of X with prescribed local properties. More precisely, any local phase flow on a Runge

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domain in X can be approximated by global automorphisms. Needless to say that this leads to remarkable consequences (see the surveys [15,23]).

If X is equipped with a holomorphic volume form ω (*i.e.* ω is a nowhere vanishing top holomorphic differential form), then one can ask whether a similar approximation holds for automorphisms and phase flows preserving the ω, so-called *volume preserving automorphisms*. Under a mild additional assumption the answer is yes in the presence of the volume density property (VDP), which means that the Lie algebra generated by completely integrable holomorphic vector fields of ω-divergence zero is dense in the space of all holomorphic vector fields of ω-divergence zero. Danielewski surfaces carry a unique nondegenerate algebraic 2-form ω, and we will concentrate on the group $\mathrm{Aut}^{\omega}_{\mathrm{hol}}(D_p)$ of volume-preserving holomorphic automorphisms.

The following definitions are due to Varolin and Kaliman-Kutzschebauch:

Definition 1.1. We say that X has the algebraic density property (ADP) if the Lie algebra $\mathrm{Lie}_{\mathrm{alg}}(X)$ generated by the set $\mathrm{IVF}(X)$ of completely integrable algebraic vector fields coincides with the space $\mathrm{AVF}(X)$ of all algebraic vector fields on X. Similarly, in the presence of ω we can speak of the algebraic volume density property (AVDP). That means X has the AVDP if the Lie algebra $\mathrm{Lie}^{\omega}_{\mathrm{alg}}(X)$ generated by the set $\mathrm{IVF}_{\omega}(X)$ of completely integrable volume presenving algebraic vector fields coincides with the space $\mathrm{AVF}_{\omega}(X)$ of all volume preserving algebraic vector fields on X.

It is worth mentioning that ADP and AVDP imply DP and VDP respectively (where the second implication is not that obvious) and in particular all remarkable consequences for complex analysis on X.

The study of holomorphic automorphisms of Danielewski surfaces was continued by Lind and the first author in [19], where shear and overshear automorphisms were introduced, generalizing this notion introduced by Rudin and Rosay from Euclidean spaces to Danielewski surfaces. Shears are volume-preserving automorphisms whereas overshears are not. Note that the algebraic shear vector fields are (up to a coordinate change) exactly the LNDs (see Theorem 2.15). Generalizing the results of Andersén and Lempert it was proved in [19] that *on a Danielewski surface the group generated by shears and overshears is dense in the path-connected component of the group* $\mathrm{Aut}_{\mathrm{hol}}(D_p)$ *of holomorphic automorphisms, with respect to the compact-open topology.*

From the proof of DP in [13] it follows that the group generated by shears, overshears and hyperbolic automorphisms is dense in $\mathrm{Aut}_{\mathrm{hol}}(D_p)$. The point in the above result was to avoid hyperbolic automorphisms. The corresponding generalization of the Andersén–Lempert result in the volume-preserving case, namely the question whether the group generated by shears is dense in the group $\mathrm{Aut}^{\omega}_{\mathrm{hol}}(D_p)$ of volume-preserving holomorphic automorphisms with respect to the compact-open topology, remained an unsolved question (see [19, Problem 5.1]).

In the present paper we show that the answer to the “infinitesimal version” of this question is negative. We prove that the algebraic shear vector fields *do not generate* the Lie algebra $\mathrm{VF}^{\omega}_{\mathrm{alg}}(D_p)$ of algebraic-volume-preserving vector fields if the degree of the defining polynomial p is at least 3. More precisely we prove the following statement:

Corollary (3.15). *For $p \in \mathbb{C}[z]$ with degree $n \geqslant 3$ the Lie algebra generated by holomorphic shear fields is not dense in the Lie algebra of holomorphic volume-preserving vector fields.*

If the degree is 2 or 1, we prove that the algebraic shear vector fields *do generate* the Lie algebra $\mathrm{VF}_{\mathrm{alg}}^{\omega}(D_p)$ of algebraic volume preserving vector fields. If the degree is 1, the Danielewski surface is biholomorphic to $\mathbb{C}^2$ and we recover exactly the Andersén-Lempert result. Our main result is:

Theorem (3.26). *A volume-preserving vector field Θ on the Danielewski surface D_p is a Lie combination of LNDs if and only if its corresponding function f with $i_\Theta \omega = df$ is of the form (modulo a constant)*

$$f(x, y, z) = \sum_{\substack{i=1 \\ j=0}}^{k} a_{ij} x^i z^j + \sum_{\substack{i=1 \\ j=0}}^{l} b_{ij} y^i z^j + (pq)'(z)$$

for a polynomial $q \in \mathbb{C}[z]$.

In the "positive" cases of degree 1 and 2 the proof of the main theorem of the Andersén-Lempert theory implies the density of the group generated by shears in the (path-connected component of the) group $\mathrm{Aut}_{\mathrm{hol}}^{\omega}(D_p)$ of volume-preserving holomorphic automorphisms, whereas in the "negative" cases –degree ≥ 3– we cannot conclude that the group generated by shears is not dense in the group $\mathrm{Aut}_{\mathrm{hol}}^{\omega}(D_p)$ of volume-preserving holomorphic automorphisms. Here we are lacking a quantity attached to an automorphism which is zero for all shear automorphisms but nonzero for the hyperbolic automorphisms H_f whose function f is not the second derivative of a function divisible by the defining polynomial p.

The results of our paper are also interesting in connection with the following open problem formulated in [3]: *does a flexible affine algebraic manifold equipped with an algebraic volume form have the algebraic volume density property?*

Remember that an affine algebraic manifold is called flexible if the LNDs on it generate the tangent space at every point. By Proposition 2.5 this is true for D_p. Even though D_p has the volume density property the Lie algebra generated by LND in not the Lie algebra $\mathrm{VF}_{\mathrm{alg}}^{\omega}(D_p)$. The additional hyperbolic fields (algebraic $\mathbb{C}^*$-actions) are needed to get all of $\mathrm{VF}_{\mathrm{alg}}^{\omega}(D_p)$. Thus we do not have a counterexample to the above problem, but almost a counterexample: we have an example where the LNDs span the tangent space at each point and at the same time do not generate the Lie algebra of volume-preserving algebraic vector fields.

The paper is organized as follows. In Section 2 we recall some known facts for Danielewski surfaces and give certain proofs in order to make the paper self-contained. We believe that some of these proofs are new.

In Section 3 we explain how volume-preserving vector fields can be related to functions on the Danielewski surface and how this relation works with respect to the Lie bracket. This is a new method, which is afterwards used to prove our main result, the characterization of the Lie algebra generated by LNDs on a Danielewski surface.

On the way we use our method based on the duality between volume-preserving vector fields and functions to prove (a version of) the algebraic volume density property for $D = \mathrm{Sl}_2(\mathbb{C})/N$, where N is the normalizer of the maximal torus $N \cong \mathbb{C}^* \rtimes \mathbb{Z}_2$. The importance of this lies in the fact that the methods (compatible pairs of globally integrable fields) for proving AVDP recently developed by Kaliman and the first author do not work for this particular homogeneous space, as explained in [16]. We also prove AVDP for $(D \times \mathbb{C}^*)/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts diagonally. This is a good exercise, since the proof given in [16] is using very abstract methods. Comparing our calculations to that proof lets one feel the strength of the method of semi-compatible vector fields developed in [16].

2. Danielewski surface

Let $p \in \mathbb{C}[z]$ be a polynomial with simple zeros. The variety given by $D_p = \{(x, y, z) \in \mathbb{C}^3 : xy = p(z)\}$ is called a *Danielewski surface*. Since p has only simple zeros D_p is the preimage of a regular value and hence a complex manifold. Often it is useful to work in one of the two charts $\mathbb{C}^* \times \mathbb{C} \to D_p : (x, z) \mapsto \big(x, \frac{p(z)}{x}, z\big)$ or $(y, z) \mapsto \big(\frac{p(z)}{y}, y, z\big)$, which cover all the points of D_p with $x \neq 0$ or, respectively, $y \neq 0$. An important fact is that every regular function $f \in \mathbb{C}[D_p]$ can be written uniquely as

$$f(x, y, z) = \sum_{\substack{i=1\\ j=0}}^{k} a_{ij} x^i z^j + \sum_{\substack{i=1\\ j=0}}^{l} b_{ij} y^i z^j + \sum_{i=0}^{m} c_i z^i \tag{1}$$

by substituting $xy = p(z)$ successively. As proven in [14] there is an algebraic volume form ω on D_p, which is unique up to a constant. In the local charts from before it is given by $\omega = \frac{dx}{x} \wedge dz$ and $\omega = -\frac{dy}{y} \wedge dz$, respectively. Here comes the first well-known fact.

Proposition 2.1. *The Danielewski surfaces D_p are simply connected and we have $H^2(D_p, \mathbb{C}) \cong \mathbb{C}^{\deg(p)-1}$.*

Proof. It is possible to construct a strong deformation retraction onto a bouquet of $(\deg(p) - 1)$ 2-spheres connecting the zeros of p. First choose a smooth curve $\gamma : [0, 1] \to \mathbb{C}_z \subset D_p$ in the z-plane connecting the zeros of p and then retract D_p onto the spheres around the segments of the path between the zeros. Let $\rho_t : [0, 1] \times \mathbb{C}_z \to \mathbb{C}_z$ be a strong deformation retraction onto γ. We use this retraction to define the strong deformation retraction

$$R_t : D_p \to \{(x, y, z) \in D_p : \ z \in \gamma\} : (x, y, z) \mapsto \left(\frac{p(\rho_t(z))}{p(z)} x, y, \rho_t(z) \right).$$

Additionally, we define a strong deformation retraction H_t from $\{(x, y, z) \in D_p : z \in \gamma\}$ onto a bouquet of 2-spheres.

$$H_t(x, y, z) := \left(\frac{p(z)}{t|p(z)|^{1/2} \frac{y}{|y|} + (1-t)y}, t|p(z)|^{1/2} \frac{y}{|y|} + (1-t)y, z \right)$$

for $p(z) \neq 0$ and $|y| \geq |p(z)|^{1/2}$ and

$$H_t(x, y, z) := \left(t|p(z)|^{1/2} \frac{x}{|x|} + (1-t)x, \frac{p(z)}{t|p(z)|^{1/2} \frac{x}{|x|} + (1-t)x}, z \right)$$

for $p(z) \neq 0$ and $|x| \geq |p(z)|^{1/2}$. When $p(z) = 0$ then either $x = 0$ or $y = 0$ (or both). In this case choose

$$H_t(x, y, z) := (0, (1-t)y, z) \quad \text{or} \quad H_t(x, y, z) := ((1-t)x, 0, z)\,.$$

The composition of R_t and H_t is the desired strong deformation retraction from D_p to the bouquet of 2-spheres, therefore D_p is simply connected and has $H^2(D_p, \mathbb{C}) \cong \mathbb{C}^{\deg(p)-1}$. □

2.1. Vector fields on a Danielewski surface

Let us begin with two equivalent definitions of locally nilpotent derivations:

Definition 2.2. A globally integrable vector field Θ is a *locally nilpotent derivation (LND)* if its flow ψ^t is an algebraic $\mathbb{C}^+$-action, *i.e.* $t \mapsto \psi^t$ is an algebraic map. Equivalently a vector field Θ is an LND whenever for all $f \in \mathbb{C}[D_p]$ there is an integer N such that $\Theta^N(f) = \Theta \circ \ldots \circ \Theta(f) = 0$. For the equivalence of these definitions see [11, page 31]. The subgroup of $\mathrm{Aut}_{\mathrm{alg}}(D_p)$ generated by flows from LND is called the *special automorphism group* $\mathrm{SAut}_{\mathrm{alg}}(D_p)$.

Definition 2.3. The algebraic vector fields of the Danielewski surface D_p

$$SF_i^x := p'(z)x^i \frac{\partial}{\partial y} + x^{i+1} \frac{\partial}{\partial z},$$
$$SF_i^y := p'(z)y^i \frac{\partial}{\partial x} + y^{i+1} \frac{\partial}{\partial z}$$

are called *shear fields* for all $i \in \mathbb{N}_0$ and the vector fields

$$HF_f := f(z) \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right)$$

are called *hyperbolic fields* for all $f \in \mathbb{C}[z]$.

The above vector fields are globally integrable and volume-preserving, and their flows are

$$\phi_1^t : (x, y, z) \mapsto \left(x, \frac{p(z + tx^{i+1})}{x}, z + tx^{i+1} \right),$$

$$\phi_2^t : (x, y, z) \mapsto \left(\frac{p(z + ty^{i+1})}{y}, y, z + ty^{i+1} \right),$$

$$\phi_3^t : (x, y, z) \mapsto \left(e^{tf(z)}x, e^{-tf(z)}y, z \right).$$

Note that $\frac{p(z+tx^{i+1})}{x} = \frac{p(z)+tx^{i+1}(\ldots)}{x} = y + tx^i(\ldots)$. This shows that the shear fields are locally nilpotent derivations and the hyperbolic fields are not. For $t = 1$ these automorphisms are called x-(respectively y-) *shear automorphisms* (for short: *shears*), and *hyperbolic automorphisms*, respectively.

Recall the following definition from [3].

Definition 2.4. M is said to be *flexible* if the LND-vector fields span the tangent space in all points of M. For properties of flexible manifolds see [3].

Proposition 2.5 ([17]). *A Danielewski surface is flexible.*

Proof. The two following LND-vector fields span the tangent space in every point of D_p where $p'(z) \neq 0$.

$$p'(z)\frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad p'(z)\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}.$$

For the points with $p'(z) = 0$ we have to use further vector fields. Set $\alpha_k(x, y, z) = \left(x, \frac{p(z-kx)}{x}, z - kx\right)$ and look at the vector field SF_0^y conjugated with α_k (see the remark below)

$$\begin{aligned}\alpha_k^*(SF_0^y) = {} & p'(z + kx)\frac{\partial}{\partial x} \\ & + \frac{p(z + kx)p'(z) - p'(z + kx)p(z) - kxp'(z + kx)p'(z)}{x^2}\frac{\partial}{\partial y} \\ & + \left(-kp'(z + kx) + \frac{p(z + kx)}{x}\right)\frac{\partial}{\partial z}.\end{aligned}$$

Assume that p' has n zeros, then the fields $\alpha_k^*(SF_0^y)$ for $k = 1, \ldots, n$ together with the two shear fields from above will span the tangent space at any point. □

Remark 2.6. Given a vector field Θ and a holomorphic automorphism $\phi : M \to M$ then the *vector field conjugated by* ϕ is given by $(\phi^*\Theta)_p := ((D\phi^{-1})\Theta)_{\phi(p)}$. The vector field $\phi^*\Theta$ is globally integrable whenever Θ is it. Its flow is $\phi\psi^t\phi^{-1}$ where ψ^t is the flow of Θ. In particular, an LND conjugated by an algebraic automorphism is an LND again. The interior product by a k-form ω is $i_{(\phi^*\Theta)}\omega = \phi^*(i_\Theta(\phi^{-1*}\omega))$; in particular, if ω is invariant under ϕ, then $i_{(\phi^*\Theta)}\omega = \phi^*(i_\Theta\omega)$.

2.2. The (special) automorphism group

The goal of this subsection is to see that the LNDs are exactly the shear fields and the shear fields conjugated by shear automorphisms. This result is not new [22], but in order to make the paper self-contained we give a proof (which to our knowledge is new).

We begin with the description of the algebraic automorphism group $\mathrm{Aut}_{\mathrm{alg}}(D_p)$. The following theorem is due to Makar-Limanov; he stated it at the end of the paper [20] without proving it.

Theorem 2.7 ([20]). *Let* $\deg(p) \geq 3$ *and let* p *be generic in the following sense: no affine automorphism* α *of* $\mathbb{C}$ *permutes the roots of* p*. Then the group of all algebraic automorphisms* $\mathrm{Aut}_{\mathrm{alg}}(D_p)$ *of* D_p *is* $G_0 \rtimes (H \rtimes J)$ *where* $G_0 = G_x * G_y$ *is the free product of the subgroups* G_x *(respectively* G_y*) generated by the* x*-(respectively* y*-) shear automorphisms,* H *is the subgroup of algebraic hyperbolic automorphisms and* J *is the subgroup consisting of the identity and the involution* $I(x, y, z) = (y, x, z)$*.*

In the non-generic case denote by Γ *the group of the affine automorphisms* γ *of* $\mathbb{C}$ *permuting the roots of* p*,* i.e. *such that* $p \circ \gamma = a_0 \cdot p$*, where* a_0 *is a root of unity (depending on* γ*). Then* Γ *induces a group of automorphisms of* D_p*, which we denote by* $\tilde{\Gamma}$*. In this case we denote by* J *the group generated by* $\tilde{\Gamma}$ *and* I*, and we have again* $\mathrm{Aut}_{\mathrm{alg}}(D_p) = G_0 \rtimes (H \rtimes J)$ *with* G_0 *and* H *as above.*

We will give a proof using the following main theorem in [20] .

Theorem 2.8 ([20]). *Let* $\deg(p) \geq 3$ *and let* p *be generic as above. Then the group of algebraic automorphisms of* D_p *is generated by the following automorphisms:*

x-shears: $\Delta_f(x, y, z) = \left(x, \frac{p(z+xf(x))}{x}, z + xf(x)\right)$ *for* $f \in \mathbb{C}[z]$*;*
Hyperbolic rotations: $H_\lambda(x, y, z) = (\lambda x, \lambda^{-1} y, z)$ *for* $\lambda \in \mathbb{C}^*$*;*
Involution: $I(x, y, z) = (y, x, z)$*.*

Note that the y-shears are exactly the automorphisms of the form $I \Delta_f I$*.*

In the non-generic case or if $\deg(p) = 2$ *one has to add (the finite group)* $\tilde{\Gamma}$ *of automorphisms coming from symmetries* σ *of* p*:*

$\gamma(x, y, z) = (x, \mu y, \sigma(z))$*, where* $\sigma(z) = az + b$ *is such that* $(p \circ \sigma)(z) = \mu p(z)$*, for some* $\mu \in \mathbb{C}^*$*.*

Lemma 2.9. *For* $\deg(p) \geq 3$ *a nontrivial composition of* x*- and* y*- shears will never have a* z*-coordinate of the form* $az + b$*.*

Proof. Since compositions of x- (respectively y-) shears are x- (respectively y-) shears again, G_x and G_y are subgroups and we can assume that the composition is written in a reduced way (*i.e.* alternating x- and y- shears). For instance take an element $\Delta^x_{f_n} \Delta^y_{f_{n-1}} \cdots \Delta^y_{f_2} \Delta^x_{f_1}$ (the letter $\{x, y\}$ denotes whether it is an x- or a y-shear). Denote the image of $(x, y, z) = (x_0, y_0, z_0)$ after the first i shears by (x_i, y_i, z_i) *e.g.* for i odd we get

$$x_i = x_{i-1},\ y_i = \frac{p(z_{i-1} + x_{i-1} f_i(x_{i-1}))}{x_{i-1}} \text{ and } z_i = z_{i-1} + x_{i-1} f_i(x_{i-1}).$$

Since we have $y = \frac{p(z)}{x}$ the elements x_i, y_i, z_i can be seen as unique elements in $\mathbb{C}[x, x^{-1}, z]$. Therefore it makes sense to speak of the x-degree of such an element. It is enough to prove that z_i has a strictly positive x-degree for $i > 0$, and therefore is not of the form $az + b$. After applying the first shear we see that $z_1 = z + xf_1(x)$ is of positive x-degree. More precisely, it has degree $\deg(f_1) + 1$. Applying inductively the proceeding shear automorphism a term $x_i f_{i+1}(x_i)$ or $y_i f_{i+1}(y_i)$ will be added to z_i. If we can see that the x-degree of a such term is always bigger than all previous ones, then the claim is proven. Indeed the x-degree of $y_i f_{i+1}(y_i)$ is $\deg(y_i)(\deg(f_{i+1}) + 1) \geq \deg(y_i) = \deg(p)\deg(z_{i-1} + x_{i-1} f_i(x_{i-1})) - \deg(x_{i-1})$ which is by induction $\deg(p)\deg(x_{i-1})(\deg(f_i) + 1) - \deg(x_{i-1}) = \deg(x_{i-1})(\deg(p)(\deg(f_i) + 1) - 1) > \deg(x_{i-1})(\deg(f_i) + 1) = \deg(x_{i-1} f_i(x_{i-1}))$. The inequality from the second last step follows from the fact that $\deg(p) \geq 3$. The same calculation holds for $x_i f_{i+1}(x_i)$. And if the composition of shear fields starts with a y-shear, then the same calculation holds, when exchanging x and y. □

Proof of Theorem 2.7. In order to prove that $\mathrm{Aut}_{\mathrm{alg}}(D_p) = G_0 \rtimes (H \rtimes J)$ in the generic case, we need to verify several things. First we see that $\mathrm{Aut}_{\mathrm{alg}}(D_p) = G_0 \rtimes H_0$ where G_0 is the group generated by automorphisms of the form Δ_f and $I\Delta_f I$ and H_0 is generated by automorphisms H_λ and I. G_0 is indeed normal since $I\Delta_f I$ and $II\Delta_f II = \Delta_f \in G_0$ and $H_\lambda^{-1}\Delta_f H_\lambda = \Delta_{\lambda f(\lambda\cdot)} \in G_0$. Since $IH_\lambda = H_\lambda^{-1} I$ we have $h^{-1}gh \in G_0$ for all elements $h \in H_0$ and $g \in G_0$. By the theorem above the subgroups G_0 and H_0 generate $\mathrm{Aut}_{\mathrm{alg}}(D_p)$. Thus it remains to check that the intersection is trivial. All elements of H_0 fix the z-coordinate, but no nontrivial element from G_0 does so by the previous lemma. Take a look at the surjective homomorphism $G_x * G_y \to G_0$ sending a word to its interpretation in the group. It is injective since by the previous lemma the identity map cannot be written as a nontrivial composition of shear automorphisms. The subgroup H_0 generated by hyperbolic rotations and the involution is equal to $H \rtimes J$. Indeed, $IH_\lambda I = H_\lambda^{-1}$ and therefore the subgroup H generated by hyperbolic rotations is normal. Moreover, I is orientation reversing and therefore does not belong to H. The statement in the non generic case is easy to see as well. □

Here are some consequences of the theorem; remember that all of them hold just for $\deg(p) \geq 3$.

Remark 2.10. In the generic case the group of algebraic volume-preserving automorphisms is therefore $\mathrm{Aut}^{\omega}_{\mathrm{alg}}(D_p) = G_0 \rtimes H$. Indeed, shears and hyperbolic automorphism are volume-preserving and the involution is volume reversing. The (non trivial) elements of $\tilde{\Gamma}$ from the non generic case multiply the volume form by a (non zero) root of unity, so the group can be bigger since it is possible to get an order two volume-preserving automorphism of the form $I \circ \gamma$ with $\gamma \in \tilde{\Gamma}$. In this case the group of volume algebraic volume-preserving automorphisms is $G_0 \rtimes (H \rtimes \mathbb{Z}_2)$.

Proposition 2.11. *The group of special automorphisms* $\mathrm{SAut}_{\mathrm{alg}}(D_p)$ *(*i.e. *the group generated by all the algebraic* $\mathbb{C}^+$*-actions) is the group* $G_0 \cong G_x * G_y$ *generated by the shear automorphisms.*

Proof. Take any algebraic one-parameter subgroup $\psi : \mathbb{C} \to \mathrm{Aut}_{\mathrm{alg}}(D_p)$. Since we have the projection homomorphism $\mathrm{Aut}_{\mathrm{alg}}(D_p) = G_0 \rtimes (H \rtimes J) \to H \rtimes J$ we get an induced algebraic one-parameter subgroup on $H \rtimes J$ and hence on its connected component H the subgroup of hyperbolic rotations, but this subgroup has to be trivial since one-parameter subgroups in H can never be algebraic $\mathbb{C}^+$ action. Hence ψ has its image in the shear automorphisms. □

Lemma 2.12. *A smooth one-parameter subgroup* $\psi : \mathbb{C} \to G_x * G_y$ *is conjugated to a one-parameter subgroup* ψ^t *either in* G_x *or in* G_y.

In order to prove this lemma we need some facts about free groups. Recall that for two groups G and H any element g in $G * H$ has a unique reduced form with length denoted by $l(g)$.

Theorem 2.13. *A subgroup* K *of* $G * H$ *is conjugated to a subgroup of either* G *or* H *if and only if* $\sup\,(l(k); k \in K) < \infty$.

Proof. See [24, Theorem 8, page 36]. □

The following lemma is well known, see *e.g.* [18]. In order to make the paper more selfcontained we give the proof.

Lemma 2.14.

(1) *Every element in* $G * H$ *is conjugated either to an element in either* G *or* H *or to an element of even length* > 0.
(2) *Two commuting elements of* $G * H$ *with length* > 0 *have either both even or both odd length.*

Proof. (1) Whenever an element has odd length its first and last letter belongs to the same group, so after conjugating with the inverse of one of these letters either it is of even length or the length descends by 2, and we can proceed by induction. (2) Take an element a with even length n and an element b with odd length m, then either $l(ab) = m + n$ and $l(ba) < m + n$ or $l(ab) < m + n$ and $l(ba) = m + n$ and hence they cannot commute. □

Proof of Lemma 2.12. We first show that for all $z \in \mathbb{C}$ the element $\psi(z)$ is conjugated to a shear automorphism (*i.e.* is conjugated to an element of either G_x or G_y). Assume that this is not the case; then $a\psi(z)a^{-1}$ were of even length for some $a \in G_x * G_y$. Since $a\psi(z)a^{-1}$ and $a\psi(\frac{z}{n})a^{-1}$ commute $a\psi(\frac{z}{n})a^{-1}$ is also of even length. Therefore $l(a\psi(z)a^{-1}) = l((a\psi(\frac{z}{n})a^{-1})^n) > n$ for all n, which is of course a contradiction. Therefore with Lemma 2.14 we have for each z an element g_z such that $g_z^{-1}\psi(z)g_z$ is a shear automorphism. Now take an element $m + n\sqrt{2} + i(p + q\sqrt{2}) \in \mathbb{Q}[\sqrt{2}, i]$:

$$\begin{aligned}\psi(m+n\sqrt{2}+i(p+q\sqrt{2})) &= \psi(1)^m\psi(\sqrt{2})^n\psi(i)^p\psi(i\sqrt{2})^q \\ &= g_1(g_1^{-1}\psi(1)g_1)^m g_1^{-1} g_{\sqrt{2}}(g_{\sqrt{2}}^{-1}\psi(\sqrt{2})g_{\sqrt{2}})^m g_{\sqrt{2}}^{-1} \\ &\quad \cdot g_i(g_i^{-1}\psi(i)g_i)^m g_i^{-1} g_{i\sqrt{2}}(g_{i\sqrt{2}}^{-1}\psi(i\sqrt{2})g_{i\sqrt{2}})^m g_{i\sqrt{2}}^{-1}.\end{aligned}$$

Therefore the length of elements in $\psi(\mathbb{Q}[\sqrt{2}, i])$ is bounded by $2(l(g_1) + l(g_i) + l(g_{\sqrt{2}}) + l(g_{i\sqrt{2}})) + 4$ and hence $\psi(\mathbb{Q}[\sqrt{2}, i])$ is by Lemma 2.12 conjugate to a subgroup of G_x or G_y. Now, the only thing that remains, is to show that G_x and G_y are closed in $G_x * G_y$. Then we also know that

$$\psi(\mathbb{C}) = \psi(\overline{\mathbb{Q}[\sqrt{2}, i]}) \subset \overline{\psi(\mathbb{Q}[\sqrt{2}, i])}$$

is conjugate to a subgroup of G_x or G_y. To see that for instance G_x is closed we take any converging sequence of x-shears $\Delta_{f_n} \to \eta = (\eta_1, \eta_2, \eta_3)$. So we know that $(z + f_n(x))_n$ converges pointwise, hence $f_n(z)$ converges, say to $f(z)$. Now clearly $\eta_1(x, y, z) = x$ and $\eta_3(x, y, z) = z + f(x)$, since η is algebraic f is a polynomial and therefore $\eta = \Delta_f$ is an x-shear. □

Theorem 2.15 ([22]). *The LNDs of the Danielewski surface D_p for* $\deg(p) \geq 3$ *are exactly the shear fields and the shear fields conjugated by compositions of shear automorphisms.*

Proof. An algebraic $\mathbb{C}^+$-action $\psi : \mathbb{C} \to \mathrm{SAut}_{\mathrm{alg}}(D_p)$ is by Proposition 2.11 and Lemma 2.12 conjugated to a one-parameter subgroup in G_x or G_y. □

3. Lie combinations of shear fields

In this chapter we will understand which algebraic volume-preserving vector fields of the Danielewski surface can be written as a Lie combination of the shear fields 2.3. The main tool for the description will be the 1-forms $i_\Theta\omega$ for volume-preserving vector fields. Recall that the *interior product* $i_\Theta : \Omega^{k+1}(M) \to \Omega^k(M)$ is given by $i_\Theta\mu(\Theta_1, \dots, \Theta_k) := \mu(\Theta, \Theta_1, \dots, \Theta_k)$. We will also use the *Lie derivative* of a differential form μ with respect to a vector field Θ, which is given by $L_\Theta\mu = \frac{d}{dt}\psi^{t*}\mu \mid_{t=0}$ or the Cartan formula $L_\Theta\mu = (d \circ i_\Theta + i_\Theta \circ d)(\mu)$. The formula $i_{[\Theta_1,\Theta_2]}\mu = L_{\Theta_1}(i_{\Theta_2}\mu) - i_{\Theta_2}(L_{\Theta_1}\mu)$ gives a link between the interior product and the Lie derivative. Another useful formula $L_\Theta d\mu = dL_\Theta\mu$ is a direct consequence of the Cartan formula.

3.1. The Lie algebra generated by shear fields is a proper subalgebra of $\mathrm{VF}^{\omega}_{\mathrm{alg}}(D_p)$

From now on we will use the one-to-one correspondence between algebraic volume-preserving vector fields and polynomial functions modulo constants on D_p. For every volume-preserving vector field Θ holds $L_\omega(\Theta) = di_\Theta\omega + i_\Theta d\omega = 0$. Since $d\omega = 0$ the 1-form $i_\Theta\omega$ is closed and therefore exact (because D_p is simply connected by Proposition 2.1), hence if Θ is algebraic then $i_\Theta\omega = df$ for some regular $f \in \mathbb{C}[D_p]$. This defines a bijection between algebraic volume-preserving vector fields and polynomial functions modulo constants.

This correspondence is in analogy to the correspondence between symplectic vector fields and Hamiltonian functions in symplectic geometry (on simply connected symplectic manifolds). In other words, we use the structure of Poisson algebra on the functions on the manifold. This analogy is using the facts that ω is closed and non-degenerate. If we consider higher dimensional manifolds (not surfaces) the correspondence will be between volume-preserving vector fields and $n-2$ forms, see [16]. The following lemma gives the functions which corresponding to the shear fields and hyperbolic vector fields.

Lemma 3.1. *For $i \in \mathbb{N}_0$ we have*

$$i_{SF_i^x}\omega = -\frac{dx^{i+1}}{i+1}, \quad i_{SF_i^y}\omega = \frac{dy^{i+1}}{i+1}, \quad i_{HF_{z^i}}\omega = \frac{dz^{i+1}}{i+1}.$$

Proof.

$$\begin{aligned}
i_{SF_i^x}\omega(\Theta) &= \omega(SF_i^x, \Theta) = \frac{1}{x} dx \wedge dz(SF_i^x, \Theta) \\
&= \frac{1}{x}\big(dx(SF_i^x)dz(\Theta) - dx(\Theta)dz(SF_i^x)\big) \\
&= \frac{1}{x}\big(-x^{i+1}dx(\Theta)\big) = -x^i dx(\Theta) = -\frac{dx^{i+1}}{i+1}(\Theta). \\
i_{SF_i^y}\omega(\Theta) &= \omega(SF_i^y, \Theta) = -\frac{1}{y} dy \wedge dz(SF_i^y, \Theta) \\
&= -\frac{1}{y}\big(dy(SF_i^y)dz(\Theta) - dy(\Theta)dz(SF_i^y)\big) \\
&= -\frac{1}{y}\big(-y^{i+1}dy(\Theta)\big) = y^i dy(\Theta) = \frac{dy^{i+1}}{i+1}(\Theta). \\
i_{HF_{z^i}}\omega(\Theta) &= \omega(HF_{z^i}, \Theta) = \frac{1}{x} dx \wedge dz(HF_{z^i}, \Theta) \\
&= \frac{1}{x}\big(dx(HF_{z^i})dz(\Theta) - dx(\Theta)dz(HF_{z^i})\big) \\
&= \frac{1}{x}(z^i x dz) = z^i dz(\Theta) = \frac{dz^{i+1}}{i+1}(\Theta).
\end{aligned}$$

□

In general, it is not hard to see that for a given function f the corresponding vector field Θ is given by

$$\Theta = \big(p'(z) f_y + x f_z\big) \frac{\partial}{\partial x} - \big(p'(z) f_x + y f_z\big) \frac{\partial}{\partial y} + \big(y f_y - x f_x\big) \frac{\partial}{\partial z},$$

where f_x, f_y, f_z denote the partial derivatives of f. We need to know how to calculate the Lie bracket on the level of functions. An easy calculation shows the following lemma.

Lemma 3.2. *Let Θ be a volume-preserving vector field with $i_\Theta \omega = df$ and Ψ be another volume-preserving vector field. Then*

$$i_{[\Psi,\Theta]}\omega = L_\Psi(i_\Theta \omega) - i_\Theta(L_\Psi(\omega)) = L_\Psi(df) = dL_\Psi(f).$$

This lemma also allows us the compute the Lie bracket only in terms of functions (which is usually called the Poisson bracket):

$$\{f, g\} = p'(z)(f_y g_x - f_x g_y) + x(f_z g_x - f_x g_z) - y(f_z g_y - f_y g_z).$$

However we will never use this precise description.

The previous facts allow us to reprove the fact from [16] that D_p has the volume density property.

Theorem 3.3. *The Danielewski surface D_p with the volume form ω satisfies the algebraic volume density property; in fact every algebraic volume-preserving vector field is a Lie combination of shear fields and hyperbolic fields. Precisely: every volume-preserving vector field is a linear combination of the vector fields SF_i^x, SF_i^y, HF_f, $[SF_i^x, HF_f]$ and $[SF_i^y, HF_f]$ for $i \in \mathbb{N}_0$ and polynomials $f \in \mathbb{C}[z]$.*

Proof. We have to find a Lie combination A of shear fields and hyperbolic fields for every polynomial function f on D_p such that $i_A\omega = df$ holds. It is sufficient to find the corresponding Lie combination for the monomials x^i, y^i, z^i, $x^i z^j$ and $y^i z^j$ for all $i, j > 0$. The first three are already covered by Lemma 3.1. The corresponding vector fields of the last two monomials are $[SF_{i-1}^x, HF_{z^j}]$ and $[SF_{i-1}^y, HF_{z^j}]$. Indeed:

$$i_{[SF_{i-1}^x, HF_{z^j}]}\omega = dL_{SF_{i-1}^x}\left(\frac{z^{j+1}}{j+1}\right) = dx^i \frac{1}{j+1}(j+1)z^j = dx^i z^j .$$

A similarly calculation shows $i_{[SF_{i-1}^y, HF_{z^j}]}\omega = dy^i z^j$. □

We have now developed the method to show AVDP for the cases mentioned in the introduction. Let D be the quotient of $SL_2(\mathbb{C})$ by the normalizer of the maximal torus. Consider $G = SL_2(\mathbb{C})$ as a subvariety of $\mathbb{C}^4_{a_1,a_2,b_1,b_2}$ given by $a_1b_2 - a_2b_1 = 1$, *i.e.* matrices

$$A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

are elements of G. Let $T \simeq \mathbb{C}^*$ be the torus consisting of the diagonal elements and N be the normalizer of T in SL_2. That is, $N/T \simeq \mathbb{Z}_2$ where the matrix

$$A_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in N$$

generates the nontrivial coset of N/T.

Lemma 3.4. *The variety $D = G/T$ is isomorphic to the hypersurface $xy = z^2 - 1$ in $\mathbb{C}^3_{x,y,z}$ such that the $\mathbb{Z}_2$-action is given by $(x, y, z) \to (-x, -y, -z)$.*

Proof. Note that the ring of T-invariant regular functions on G is generated by $x = a_1b_1$, $y = a_2b_2$, $v = a_1b_2$, and $z = a_2b_1$ where $v = z + 1$. Hence X is isomorphic to the hypersurface $xy = z(z + 1)$ in $\mathbb{C}^3_{x,y,z,}$. After a linear isomorphism of $\mathbb{C}^3$ we get the desired form. The formula for the $\mathbb{Z}_2$-action (induced by multiplication by A_0) is also a straightforward computation. □

Definition 3.5. Let X be an affine algebraic manifold equipped with an algebraic volume form ω. Suppose a finite group Γ acts freely and algebraically on X. We say that X has the Γ-AVDP if the Lie algebra generated by Γ-invariant completely integrable volume-preserving algebraic vector fields on X is equal to the Lie algebra of all Γ-invariant volume-preserving algebraic vector fields on X.

Theorem 3.6. *The Danielewski surface D has $\mathbb{Z}_2$-AVDP.*

Proof. We proceed as in the proof of the previous theorem, The volume form ω is $\mathbb{Z}_2$ anti-invariant, *i.e.*, $\sigma^*\omega = -\omega$. Thus using the invariant globally integrable fields $SF^x_{2n}, SF^y_{2n}, HF_{2n}$ $n \geq 0$ we have to produce all anti-invariant monomials x^i, y^i, z^i for odd i and $z^i x^j$, $z^i y^j$ for $i, j \geq 1, i + j \geq 3$ and odd. The first three are again covered by Lemma 3.1 for even i.

For the other monomials we have to use the exact form of the the defining polynomial $p(z) = z^2 - 1$. We obtain the monomials $z^i x^j$ by induction on i. The monomials $z^i y^j$ are then obtained analogously.

Starting the induction with $i = 1$ consider

$$i_{[SF^y_0, SF^x_{2k}]}\omega = dSF^y_0\left(-\frac{x^{2k+1}}{2k+1}\right) = d\left(\left(2z\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right)\left(-\frac{x^{2k+1}}{2k+1}\right)\right) = -2dzx^{2k}$$

Suppose by induction hypothesis that all monomials $z^m x^n$, $m + n$ odd for $m \leq i$ are obtained. In order to produce a monomial $z^{i+1}x^j$, we use the Lie bracket of SF_0^y with the field corresponding to the monomial $z^i x^{j+1}$ (which by induction hypothesis is obtained). We obtain the polynomial

$$\begin{aligned}\left(2z\frac{\partial}{\partial x}+y\frac{\partial}{\partial z}\right)\left(z^i x^{j+1}\right) &= 2z^{i+1}(j+1)x^j + iz^{i-1}yx^{j+1}\\ &= 2z^{i+1}(j+1)x^j + iz^{i-1}(z^2-1)x^j\\ &= (2j+2+i)z^{i+1}x^j - iz^{i-1}x^j.\end{aligned}$$

The monomial $z^{i-1}x^j$ is already obtained by induction hypothesis, thus the induction step is completed.

We do not get constant functions, they are not needed since they correspond to the zero field. □

In fact the use of Lie brackets is not necessary in the previous theorem, one can show that linear span is enough.

Remark 3.7. The vector space (instead of Lie algebra) spanned by globally integrable $\mathbb{Z}_2$-invariant algebraic vector fields on D is equal to all $\mathbb{Z}_2$ invariant algebraic vector fields. Also the vector space spanned by globally integrable $\mathbb{Z}_2$-anti-invariant algebraic vector fields on D is equal to all $\mathbb{Z}_2$ anti-invariant algebraic vector fields[1].

This follows from the fact that in the above proof one uses Lie brackets of LND and maximally one other (hyperbolic) globally integrable field and the following general fact which holds on any affine algebraic manifold.

Lemma 3.8. *If Θ is an LND and Ψ a finite sum of globally integrable algebraic vector field, then the Lie bracket $[\Theta, \Psi]$ is contained in the span of globally integrable algebraic vector fields. In particular the vector space spanned by LNDs is equal to the Lie algebra generated by LNDs.*

Proof. Let ϕ_t denote the flow of Θ (which is an algebraic $\mathbb{C}$-action). Then the set $A = \{(\phi_t)^*(\Psi)\}$ is contained in a finite dimensional subspace of AVF and thus its span is closed (see Lemma 3.25). Since global integrability is preserved when applying an automorphisms, all fields in A are in the span of globally integrable fields. Moreover, the definition

$$[\Theta, \Psi] = \lim_{t\to 0}\frac{(\phi_t)^*(\Psi)-\Psi}{t}$$

shows that the bracket $[\Theta, \Psi]$ is in the closure of the span of A, thus in the span. □

Now to the other example: Let $X = D \times \mathbb{C}^*$ equipped with the volume form $\omega_0 = \omega \times \frac{d\theta}{\theta}$ and $\mathbb{Z}_2$-action generated by $(x, y, z, \theta) \mapsto (-x, -y, -z, -\theta)$. The next theorem states that X has $\mathbb{Z}_2$-AVDP, the proof technique is very close to the

[1] For the anti-invariant case one shows exactly as in the proof above that all anti-invariant fields are obtained as Lie brackets of one anti-invariant globally integrable field and invariant LND.

technique we have seen above. For a vector field Θ we again look at the corresponding form $i_\Theta \omega_0$ which is in this situation an anti-invariant closed 2-form. In order to find all those forms we need to find all anti-invariant exact 2-forms, and additionally for each cohomology class one representative.

Theorem 3.9 ([16]). *The manifold X has $\mathbb{Z}_2$-AVDP.*

Proof. The volume form ω_0 is anti-invariant. We wish to find all anti-invariant closed 2-forms α on X as $i_\chi \omega_0$ where χ is a Lie combination of invariant completely integrable fields on X. By Proposition 2.1 we have $H^2(D, \mathbb{C}) = \mathbb{C}$ and it is easy to check that the volume form ω represents the nontrivial class. By the Künneth formula and since $H^1(D, \mathbb{C}) = 0$ we have that $H^2(X, \mathbb{C})$ is isomorphic to $\mathbb{C}$ and ω (considered as a 2 -form on X) is a generator. Remark that $\omega = i_{\theta \frac{\partial}{\partial \theta}} \omega_0$. Thus subtracting the completely integrable volume-preserving invariant field $\theta \frac{\partial}{\partial \theta}$ from a given field χ we can assume that the form α is exact. It remains to construct all anti-invariant 1-forms β in the expression $d\beta = i_\chi \omega_0$ where χ is a Lie combination of invariant completely integrable fields on X. Of course we have to find all 1-forms β up to closed ones, since these correspond to the zero vector field.

Since the restrictions of the 1-forms dx, dy and dz from $\mathbb{C}^3$ to the tangent space of D generate the cotangent space of D at any point, all 1-forms on X can be written as

$$\beta = \sum_{n=-N}^{N} f_n(X)\theta^n dx + \sum_{n=-N}^{N} g_n(X)\theta^n dy + \sum_{n=-N}^{N} h_n(X)\theta^n dz + \sum_{n=-N}^{N} j_n(X)\theta^n d\theta$$

where $X = (x, y, z)$ and f_n, g_n, h_n, j_n are regular functions on D which are invariant if n is even and anti-invariant if n is odd. Of course this representation of a 1-form on X is not unique, the relation $xdy + ydx = 2zdz$ holds, but this is irrelevant for our proof.

We begin by constructing all summands of the fourth sum. First consider the case of even n. The proof is analogous to the proof of the preceding theorem. The monomial forms $x^i\theta^n d\theta$, i even, you construct by inner product of the invariant completely integrable field $\theta^{n+1} SF_{i-1}^x$ with ω_0, $y^i\theta^n d\theta$ comes from $\theta^{n+1} SF_{i-1}^y$, i even, and $z^i\theta^n d\theta$ comes from the invariant field $\theta^{n+1} HF_{i-1}$, i even. Now use inductively Lie brackets with the invariant field SF_0^y to obtain out of the form $x^i\theta^n d\theta$ the forms $x^{i-1}z\theta^n d\theta, x^{i-2}z^2\theta^n d\theta$ and so on. Thus obtaining all 1- forms $x^k z^l \theta^n d\theta$ for $k+l$ even. The forms $y^k z^l \theta^n d\theta, k+l$ even, are obtained analogously. Now consider the case n odd. Start with the monomial forms $x^i\theta^n d\theta$, i odd, you constructed by inner product of the invariant completely integrable field $\theta^{n+1} SF_{i-1}^x$ with ω_0, all the rest goes analogously. Thus we have constructed all anti-invariant 1- forms $\sum_{n=-N}^{N} j_n(X)\theta^n d\theta$, except for $j_n =$ constant, but the forms $\theta^n d\theta$ are closed and therefore corresponding to the zero field.

In order to produce the summand in the first sum we introduce the invariant globally integrable volume-preserving vector field $V = x\theta(x\partial/\partial x - y\partial/\partial y -$

$\theta\partial/\partial\theta$) and take the Lie bracket with the vector field corresponding to the 1-form $f(X)\theta^n d\theta$ (say n even and f invariant). This produces the 1-form

$$L_V(f(X)\theta^n d\theta) = V(f(X)\theta^n)d\theta - f(X)\theta^n d(x\theta^2) = (\ldots)d\theta - f(X)\theta^{n+2}dx$$

and therefore we get together with the above all 1-forms of the form $f(X)\theta^n dx$ where n is even and f invariant and similarly the ones with n odd and f anti-invariant. In the identical way we get all 1-forms $f(X)\theta^n dy$ by taking the invariant vector field $W = y\theta(x\partial/\partial x - y\partial/\partial y + \theta\partial/\partial\theta)$ instead. The invariant vector fields $xz\theta\partial/\partial\theta$, $yz\theta\partial/\partial\theta$ and $z^2\theta\partial/\partial\theta$ will help to construct all forms of the form $f(X)\theta^n dz$. Indeed, the calculations

$$L_{xz\theta\frac{\partial}{\partial\theta}}(f(X)\theta^n d\theta) = (\ldots)d\theta + (\ldots)dx + xf(X)\theta^{n+1}dz,$$
$$L_{yz\theta\frac{\partial}{\partial\theta}}(f(X)\theta^n d\theta) = (\ldots)d\theta + (\ldots)dy + yf(X)\theta^{n+1}dz,$$
$$L_{z^2\theta\frac{\partial}{\partial\theta}}(f(X)\theta^n d\theta) = (\ldots)d\theta + 2zf(X)\theta^{n+1}dz$$

show that we get all 1-forms of the form $g(X)\theta^n dz$ where $g(X)$ is either a multiple of x, y or z. Hence allowing linear combinations only the constant term $\theta^n dz$ is missing. But since the form $\theta^n dz + n\theta^{n-1}z d\theta$ is closed the corresponding vector field also corresponds to $-n\theta^{n-1}z d\theta$ and hence is already obtained. □

The question we like to investigate in the remaining part of the paper is whether the hyperbolic vector fields are needed in the proof of Theorem 3.3, or if the shear fields are enough. In the following section it is shown that the Lie algebra generated by the shear fields does not contain all the hyperbolic fields. Here are some preliminaries. The proof of the first fact is an easy consequence of the Jacobi identity.

Lemma 3.10. *Let M be a set of vector fields. Then the Lie algebra* $\mathrm{Lie}(M)$ *generated by M is spanned (as a vector space) by elements of the form* $[A_n, [\ldots[A_2, [A_1, A_0]]\ldots]]$ *with $A_i \in M$.*

In order to study which polynomials correspond to Lie combinations of shear fields it is therefore necessary to study functions of the type $i_{[A_n,[..[A_2,[A_1,A_0]]..]]}\omega$, where the A_i are shear fields.

Lemma 3.11. *Let A_i be shear fields for $0 \leqslant i \leqslant n$. Then, the polynomial f with $i_{[A_n,[..[A_2,[A_1,A_0]]..]]}\omega = df$ is of type (a) $x^j q(z)$, (b) $y^j q(z)$ or (c) $q(z)$ for some $j > 0$ and some polynomial $q \in \mathbb{C}[z]$.*

Proof. For $n = 0$ the claim holds due to Theorem 3.1.
(a) If $f = x^j q(z)$, Lemma 3.2 shows

$$i_{[SF_k^x,[A_n[..[A_2,[A_1,A_0]]..]]}\omega = dL_{SF_k^x}(x^j q(z)) = dx^{j+k+1}q'(z),$$

hence the polynomial is again of type (a). Furthermore:

$$\begin{aligned} i_{[SF_k^y,[A_n[..[A_2,[A_1,A_0]]..]]}\omega &= dL_{SF_k^y}(x^j q(z)) \\ &= dp'(z)y^k jx^{j-1}q(z) + y^{k+1}x^j q'(z) \\ &= dy^k x^{j-1}(jp'(z)q(z) + xyq'(z)) \\ &= dy^k x^{j-1}(jp'(z)q(z) + p(z)q'(z)). \end{aligned}$$

After substituting $xy = p(z)$ this polynomial is also of type (a), (b) or (c), depending on whether $k < j - 1$, $k > j - 1$ or $k = j - 1$. The other cases are treated similarly:
(b) If $f = y^j q(z)$, then

$$\begin{aligned} i_{[SF_k^x,[A_n[..[A_2,[A_1,A_0]]..]]}\omega &= dx^k y^{j-1}(jp'(z)q(z) + p(z)q'(z)), \\ i_{[SF_k^y,[A_n[..[A_2,[A_1,A_0]]..]]}\omega &= dy^{j+k+1}q'(z). \end{aligned}$$

(c) If $f = q(z)$, then

$$\begin{aligned} i_{[SF_k^x,[A_n[..[A_2,[A_1,A_0]]..]]}\omega &= dx^{k+1}q'(z), \\ i_{[SF_k^y,[A_n[..[A_2,[A_1,A_0]]..]]}\omega &= dy^{k+1}q'(z). \end{aligned}$$

□

Lemma 3.12. *If case* (c) *of Lemma* 3.11 *occurs, that is* $i_{[A_n,[..[A_2,[A_1,A_0]]..]]}\omega = df$ *for* A_i *shear fields, and in addition* $f = f(z)$ *for some polynomial in* z*, then* $f(z) = (p(z)q(z))'$ *for some polynomial* q *in* z*.*

Proof. Consider the vector field $[A_{n-1}, [..[A_2, [A_1, A_0]..]]$. Due to Lemma 3.11 exactly one of the following cases occurs:

$$i_{[A_{n-1},[..[A_2,[A_1,A_0]]..]]}\omega = \begin{cases} dx^j q(z) & \text{(a)} \\ dy^j q(z) & \text{(b)} \\ dq(z) & \text{(c)} \end{cases}$$

for some $j > 0$ and some $q \in \mathbb{C}[z]$. If $A_n = SF_k^x$ for some $k \in \mathbb{N}_0$, then together with the calculation in the proof of Lemma 3.11 one gets:

$$i_{[A_n,[A_{n-1}[..[A_2,[A_1,A_0]]..]]]}\omega = df = \begin{cases} dx^{j+k+1}q'(z) & \text{(a)} \\ dx^k y^{j-1}(jp'(z)q(z) + p(z)q'(z)) & \text{(b)} \\ dx^{k+1}q'(z) & \text{(c).} \end{cases}$$

Since f is a polynomial in z all cases except (b) with $k = j - 1$ can be excluded. Therefore $f = x^k y^{j-1}(jp'(z)q(z) + p(z)q'(z)) = p(z)^k((k + 1)p'(z)q(z) + p(z)q'(z)) = (p(z)^{k+1}q(z))'$ for some $q \in \mathbb{C}[z]$. An identical consideration works for $A_n = SF_k^y$. □

Remark 3.13. If we choose $k = j + i - 1$ instead of $k = j - 1$ for some $i \in \mathbb{N}$ in the last step, then the polynomial in the end of the calculation is $x^i(pq)' = x^i f(z)$ (respectively $y^i f(z)$). Hence if $f(z)$ corresponds to a Lie combination of shear fields, then so does the polynomial $x^i f(z)$ (respectively $y^i f(z)$). By permuting SF_i^x and SF_i^y the corresponding polynomial switches the sign and x and y get permuted, hence both $x^i f(z)$ and $y^i f(z)$ correspond to Lie combinations of shear fields.

Corollary 3.14. *If a hyperbolic vector field is a Lie combination of shear fields, then it is of the form $HF_{(pq)''}$ for some $q \in \mathbb{C}[z]$. In particular if $p \in \mathbb{C}[z]$ with degree $n \geqslant 3$, then the the hyperbolic vector fields HF_{z^i} with $i < n-2$ are not Lie combinations of shear fields.*

In addition we can make the following observation:

Corollary 3.15. *For $p \in \mathbb{C}[z]$ with degree $n \geqslant 3$ the Lie algebra generated by holomorphic shear fields is not dense in the Lie algebra of holomorphic volume-preserving vector fields.*

Proof. Formula (1) for a regular function$f$ on D_p can be viewed as a Laurent expansion of the restriction of f to the open subset $x \neq 0 \cong \mathbb{C}_x^* \times \mathbb{C}_z$ with respect to the variable $x \in \mathbb{C}^*$ with coefficients being functions of z. Analogously any holomorphic function g on D_p has such a Laurent expansion with coefficients holomorphic functions in z

$$g = \sum_{i=-\infty}^{\infty} a_i(z) x^i.$$

We have established that the regular function f corresponding under $i_\Theta \omega = df$ to an algebraic vector field Θ which is a Lie combination of algebraic shear fields satisfy the special condition $a_0(z) = (hp)'$, *i.e.*, the absolute term $a_0(z)$ (which is unique associated to Θ up to a constant) is the derivative of a function divisible by the defining polynomial p. The condition that a function g on $\mathbb{C}^* \times \mathbb{C}$ has an absolute term which is up to a constant the derivative of a function divisible by the defining polynomial p is closed in c.-o. topology. More explicitly, let $z_1, \dots, z_n$ be the distinct simple zeros of p, then the condition is equivalent to the equality of all the expressions

$$(z_j - z_1) \int_{z_1}^{z_j} \int_{|x|=1} g(x, z) \frac{dz \wedge dx}{x} \qquad j = 2, 3, \dots, n.$$

Since holomorphic shear fields are limits (in c.-o. topology) of algebraic shear fields the holomorphic function corresponding to a Lie combination of holomorphic shear fields has an absolute term of the same form. Thus for p with degree ≥ 3 the Lie algebra generated by holomorphic shear fields is contained in the closed proper subset of the Lie algebra of holomorphic volume-preserving vector fields defined by the above condition on the absolute term. □

3.2. Description of the Lie algebra generated by shear fields

After negating the question whether every volume-preserving vector field is a Lie combination of shear fields, in the this section it will be investigated which vector fields exactly are Lie combination of such ones. Concretely, all of the volume-preserving vector fields whose absolute term of the corresponding function is of the special form described in Lemma 3.12 are a Lie combination of shear fields. This proof is following the same concept developed in [19] where it was used in order to prove the fact that the shear fields and another class of (non volume-preserving) vector fields called overshear fields do generate the Lie algebra of algebraic vector fields of D_p.

Lemma 3.16. *The following equalities hold:*

$$i_{[SF_i^x, SF_i^y]}\omega = d(p^i p') \tag{3.1}$$

$$i_{[SF_0^x,[SF_0^x, SF_1^y]]}\omega = d(pp')' \tag{3.2}$$

$$i_{[SF_{i_k-1}^x,\dots[SF_{i_2-1}^x,[SF_{i_1-1}^x, HF_f]]\dots]}\omega = d(x^{i_1+\dots+i_k} f^{(k-1)}) \tag{3.3}$$

$$i_{[HF_{f_k},\dots[HF_{f_2},[SF_i^x, HF_{f_1}]]\dots]}\omega = (i+1)^{k-1} d(x^{i+1} f_1 f_2 \dots f_k). \tag{3.4}$$

Proof. The following calculations are according to Theorem 3.1 and Lemma 3.2:

$$\begin{aligned}
i_{[SF_i^x, SF_i^y]}\omega &= dL_{SF_i^x}\left(\frac{y^{i+1}}{i+1}\right) && (3.1)\\
&= dp'(z)x^i y^i \\
i_{[SF_0^x, SF_1^y]}\omega &= dyp'(z) && (3.2)\\
i_{[SF_0^x,[SF_0^x, SF_1^y]]}\omega &= dL_{SF_0^x}\big(yp'(z)\big) \\
&= dp'(z)p'(z) + xyp''(z) \\
&= d(p(z)p'(z))' \\
i_{[SF_{i_1-1}^x, HF_f]}\omega &= dx^{i_1} f && (3.3)\\
i_{[SF_{i_2-1}^x,[SF_{i_1-1}^x, HF_f]]}\omega &= dL_{SF_{i_2-1}^x}(x^{i_1} f) \\
&= x^{i_1+i_2} f' \\
i_{[SF_{i_k-1}^x,\dots[SF_{i_2-1}^x,[SF_{i_1-1}^x, HF_f]]\dots]}\omega &= dL_{SF_{i_k-1}^x}(x^{i_1+i_2+\dots+i_{k-1}} f^{(k-2)}) \\
&= d(x^{i_1+i_2+\dots+i_k} f^{(k-1)}) \\
i_{[SF_i^y, HF_{f_1}]}\omega &= dx^{i+1} f_1 && (3.4)\\
i_{[HF_{f_2},[SF_i^x, HF_{f_1}]]}\omega &= dL_{HF_{f_2}}(x^{i+1} f_1) \\
&= (i+1)x^{i+1} f_1 f_2 \\
i_{[HF_{f_k},\dots[HF_{f_2},[SF_i^x, HF_{f_1}]]\dots]}\omega &= dL_{HF_{f_k}}((i+1)^{k-2} x^{i+1} f_1 \dots f_{k-1}) \\
&= d((i+1)^{k-1} x^{i+1} f_1 \dots f_k).
\end{aligned}$$

□

Corollary 3.17. *The previous lemma shows:*

$$[SF_i^x, SF_i^y] = HF_{(p^i p')'} \tag{3.5}$$

$$\big[SF_0^x, [SF_0^x, SF_1^y]\big] = HF_{(pp')''} \tag{3.6}$$

$$\Big[SF^y_{i_1+\cdots+i_k-1}, [SF^x_{i_k-1}, \dots [SF^x_{i_1-1}, HF_f]\dots]\Big] = HF_{(p^{i_1+\cdots+i_k} f^{(k-1)})''} \tag{3.7}$$

$$\big[SF_i^y, [HF_{f_k}, \dots [SF_i^x, HF_{f_1}]\dots]\big] = HF_{(i+1)^{k-1}(p^{i+1} f_1 f_2 \cdots f_k)''}. \tag{3.8}$$

Lemma 3.18. *Let $n = \deg(p)$; then for every $q \in \mathbb{C}[z]$ the vector field $HF_{(p^n q)''}$ is a Lie combination of shear fields.*

Proof. In a first step one observes that every polynomial $x^n q$ corresponds to a Lie combination of shear fields. Truly due to Lemma 3.16(3.3) the polynomials $x^n f^{(k)}$ for $k = 0, \dots, n-1$ correspond to a Lie combination of shear fields, if HF_f were already such a combination. According to Corollary 3.17(3.5) it is possible to choose for f the polynomials $p'', (pp')', (p^2 p')', \dots$ (*i.e.* polynomials of degree $n-2$, $2n-2$, $3n-2, \dots$). Therefore after differentiating up to n times there is a polynomial for every degree. Hence they build a basis for $\mathbb{C}[z]$ and every polynomial $q \in \mathbb{C}[z]$ can be substituted in $x^n q$. After taking the Lie bracket with the shear field SF^y_{n-1} the vector field becomes $HF_{(p^n q)''}$. □

Let $n = \deg p$ and $W \subset \mathbb{C}[z]$ be a vector space with

(i)	$(p^i)'' \in W$	$\forall i \in \mathbb{N}$
(ii)	$(pp')'' \in W$	
(iii)	$(p^n q)'' \in W$	$\forall q \in \mathbb{C}[z]$
(iv)	$f_1, .., f_k \in W \Longrightarrow (pf_1 \dots f_k)'' \in W$	$\forall k \in \mathbb{N}.$

Now, we show that W contains all polynomials of the type $(pq)''$. Since the vector space of all f with HF_f a Lie combination of shear fields is a vector space with properties (i)-(iv), every vector field $HF_{(pq)''}$ would be a such combination.

In a first step it is shown that the algebra $A_W = \operatorname{span}\{f_1 \cdots f_k :\ f_i \in W,\ 1 \le i \le k \in \mathbb{N}\}$ generated by W is equal to $\mathbb{C}[z]$. Then it is allowed to substitute all polynomials in (iv) and hence the claim is proven.

Lemma 3.19. *There is no element $a \in \mathbb{C}$, such that $f(a) = 0$ for all $f \in A_W$.*

Proof. Suppose there is such an a, then $p''(a) = 0$ and $p(a)p''(a) + p'(a)^2 = 0$ ((i) with $i = 1$ and $i = 2$) would hold, and hence $p'(a) = 0$. Since p has no double zero point it follows that $p(a) \neq 0$. Due to (iii) $(p^n q)''(a) = (p^n)'' q + 2(p^n)' q' + p^n q''(a) = 0$ holds for all $q \in \mathbb{C}[z]$. The first summand vanishes due to (i), the second due to $p'(a) = 0$, therefore it remains $p^n q''(a) = 0$. So it would be true that $q''(a) = 0$ for all $q \in \mathbb{C}[z]$ what is clearly a contradiction. □

Lemma 3.20. *There is no element $a \in \mathbb{C}$ such that $f'(a) = 0$ for all $f \in A_W$.*

Proof. Suppose there is such an a. (i) with $i = 1, 2, 3$ shows that $p'''(a) = 0$, $(p^2)'''(a) = 2(p(a)p'''(a) + 3p'(a)p''(a)) = 0$ and $0 = (p^3)'''(a) = 3(p(a)^2 p'''(a) + 6p(a)p'(a)p''(a) + 2p'(a)^3)$. The second equation shows that $p'(a)p''(a) = 0$ and therefore due to the third equation we have $p'(a) = 0$ and hence $p(a) \neq 0$. Furthermore (iii) shows $(p^n z)'''(a) = (p^n)'''(z)z + 3(p^n)''(z) \mid_{z=a} = 0$ and since the first summand vanishes $(p^n)''(a) = 0$ remains. Altogether we have $(p^n q)'''(a) = ((p^n)'''q + 3(p^n)''q' + 3(p^n)'q'' + p^n q''')(a) = p(a)^n q'''(a) = 0$ or $q'''(a) = 0$ for all $q \in \mathbb{C}[z]$, what is again a contradiction. □

Lemma 3.21. *There are no elements $a \neq b \in \mathbb{C}$, such that $f(a) = f(b)$ for all $f \in A_W$.*

Proof. Suppose there are two such elements $a, b \in \mathbb{C}$. (iv) shows that $(p^n z^i)'' \mid_{z=a} = (p^n z^i)'' \mid_{z=b}$ for all $i \in \mathbb{N}_0$. Since $(p^n z^i)'' = (p^n)'' z^i + 2i(p^n)' z^{i-1} + i(i-1)p^n z^{i-2}$ one gets the system of linear equations, which summarizes the equations for $i = 0, \dots, 5$:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ a & b & 1 & 1 & 0 & 0 \\ a^2 & b^2 & 2a & 2b & 2 & 2 \\ a^3 & b^3 & 3a^2 & 3b^2 & 6a & 6b \\ a^4 & b^4 & 4a^3 & 4b^3 & 12a^2 & 12b^2 \\ a^5 & b^5 & 5a^4 & 5b^4 & 20a^3 & 20b^3 \end{pmatrix} \cdot \begin{pmatrix} (p^n)''(a) \\ -(p^n)''(b) \\ 2(p^n)'(a) \\ -2(p^n)'(b) \\ (p^n)(a) \\ -(p^n)(b) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The determinant of this matrix is $4(a-b)^9$ and therefore nonzero for $a \neq b$ and hence it is shown that the coefficient vector is the zero vector and in particular $p(a) = p(b) = 0$ and therefore $p'(a) \neq 0 \neq p'(b)$.

Due to (ii) we have $(pp')''(a) = p(a)p'''(a) + 3p'(a)p''(a) = p(b)p'''(b) + p'(b)p''(b)$ and since $p(a) = p(b) = 0$ and $p''(a) = p''(b)$ (due to (i)) $p'(a) = p'(b)$ holds. With (iv) ($k = 1$) follows $(pp'')''(a) = p(a)p''''(a) + 2p'(b)p'''(b) + p''(b)^2 = p(b)p''''(b) + 2p'(b)p'''(a) + p''(b)^2$ and hence $p'''(a) = p'''(b)$. Using (iv) inductively one gets that $p^{(l)}(a) = p^{(l)}(b)$ for all l. Indeed, a simple calculation shows that:

$$W \ni P := \Bigg(\underbrace{p(p(p\cdots(p}_{j}\, p'')''\dots)'')''\Bigg)'' = \sum_{\substack{i_1+\dots+i_{j+1}=2j+2 \\ i_1\le\dots\le i_{j+1}}} \alpha_I \cdot p^{(i_1)} \cdots p^{(i_{j+1})}$$

with $a_I \in \mathbb{N}$. After inserting a (respectively b) all summands with $i_1 = 0$ vanish due to $p(a) = p(b) = 0$. Assume that $p^{(l)}(a) = p^{(l)}(b)$ for all $l \leq j+1$, so all the summands with $i_{j+1} \leq j+1$ have on both sides of the equation $P(a) =$

$P(b)$ the same value and hence vanish as well. For this reason only the equation $\alpha_I p'(a)^j p^{(j+2)}(a) = \alpha_I p'(b)^j p^{(j+2)}(b)$ remains and it follows inductively that $p^{(l)}(a) = p^{(l)}(b)$ for all l. This is a contradiction since the $(n-1)$-st derivative of a polynomial of degree n is a polynomial of degree one with a nonzero slope. $\square$

Proposition 3.22. *The algebra A_W generated by W is equal to $\mathbb{C}[z]$.*

Proof. The previous two lemmas show that there is a $k \in \mathbb{N}$ and polynomials $q_1, \dots, q_k \in A_W$ such that the map

$$F : \mathbb{C} \to \mathbb{C}^k : \qquad z \mapsto (q_1(z), \dots, q_k(z))$$

is an injective and immersive embedding. To achieve injectivity take the ideal in $\mathbb{C}[x, y]$ generated by the polynomials $q(x) - q(y)$ with $q \in A_w$ which is finitely generated by polynomials $q_1(x) - q_1(y), \dots, q_k(x) - q_k(y)$. Now we see that there are no $c_1 \neq c_2$ such that $q_i(c_1) = q_i(c_2)$ for all i, otherwise we would have $q(c_1) = q(c_2)$ for all $q \in A_W$ which is not possible due to Lemma 3.21. To guarantee immersivity we add for each cusp singularity (finite number!) a polynomial $q \in A_W$ whose derivative does not vanish at this point (Lemma 3.20).

Now take any polynomial function g on $\mathbb{C}$ and regard it as a regular function on $\mathbb{C}$ in $\mathbb{C}^k$ (embedded by F). This function extends to a regular function G on $\mathbb{C}^n$, hence $G = a_0 + \sum_I a_I z_1^{i_1} \dots z_k^{i_k}$. So if we pull back G we get $g(z) = G(F(z)) = a_0 + \sum_I a_I q_1(z)^{i_1} \dots q_k(z)^{i_k}$ so the algebra generated by $q_1, \dots, q_k$ and constants is $\mathbb{C}[z]$. Now the algebra generated by W is $\mathbb{C}[z]$ or a subspace with codimension 1 and an ideal hence in the second case W is a principle ideal generated by a polynomial $(z - a)$. But this case cannot occur since a would be a common root of all elements of W what is impossible (Lemma 3.19). $\square$

Now we know that a hyperbolic field HF_f is a Lie combination of shear fields if and only if $f = (pq)''$ for some polynomial q. In Theorem 3.3 it was shown that every volume-preserving vector field is a linear combination of the vector fields SF_i^x, SF_i^y, HF_f, $[SF_i^x, HF_f]$ and $[SF_i^y, HF_f]$ for $i \in \mathbb{N}_0$ and polynomials $f \in \mathbb{C}[z]$. To understand which vector fields are Lie combinations of shear fields it remains to study the vector fields $[SF_i^x, HF_f]$ and $[SF_i^y, HF_f]$.

Proposition 3.23. *All the vector fields $[SF_i^x, HF_f]$ and $[SF_i^y, HF_f]$ for $i \in \mathbb{N}_0$ and polynomials $f \in \mathbb{C}[z]$ are Lie combinations of shear fields.*

Proof. Since $i_{[SF_{i-1}^x, HF_f]}\omega = dx^i f$ it suffices to see that the polynomial $x^i f(z)$ corresponds for every $i \in \mathbb{N}$ and every $f \in \mathbb{C}[z]$ to a Lie combination of shear fields. In the proof of Lemma 3.18 we saw that this is true for $i \geq n = \deg p$. So we already have every $x^n z^j$ for $j \in \mathbb{N}$. If one takes the Lie bracket with the vector field SF_0^y one gets with the calculation in the proof of Lemma 3.11 (a) the polynomial $x^i((i+1)p'(z)z^j + jp(z)z^{j-1})$ for $i = n - 1$. Since every polynomial $(p(z)z^j)'$ corresponds to a Lie combination of shear fields, so does the polynomial $x^i(p(z)z^j)' = x^i(p'(z)z^j + jp(z)z^{j-1})$ (due to Remark 3.13). After a suitable linear combination of this two polynomials it follows that $x^i p'(z) z^j$ and $x^i p(z) z^{j-1}$ correspond to a Lie combination of shear fields for all j. Therefore every $x^i f(z)$ with $f(z) \in (p) \cup (p') \subset \mathbb{C}[z]$ belongs to a Lie combination. Since p and p' have no common zeros it is true that $(p) \cup (p') = \mathbb{C}[z]$ and the claim is shown for $i = n-1$. Repeat the same procedure for $i = n-2, \dots, 1$ and the claim is shown for every $i \in \mathbb{N}$. □

Now we have to make the final step allowing not only shear fields but also LND in our Lie combination. Since LND are shears conjugated by compositions of shear automorphisms (see Theorem 2.15) the following lemma will do the job.

Lemma 3.24. *Let $\phi : D_p \to D_p$ be a shear automorphism and let Θ be a Lie combination of shear fields. Then $\phi^*\Theta$ is a Lie combination of shear fields.*

The proof of this lemma follows immediately from the following general fact.

Lemma 3.25. *Let Θ be an LND with flow ϕ_t and Ψ any algebraic vector field. Then for any fixed t the vector field $(\phi_t)^*(\Psi)$ is contained in the Lie algebra generated by Θ and Ψ.*

Proof. Since Θ is an LND the Taylor expansion of $(\phi_t)^*(\Psi)$ with respect to the variable t around $t_0 = 0$

$$(\phi_t)^*(\Psi) = \Psi + t[\Theta, \Psi] + \frac{1}{2}t^2\big[\Theta, [\Theta, \Psi]\big] + \dots + \frac{1}{n!}t^n\big[\Theta, [\Theta \dots [\Theta, \Psi]] \dots\big]$$

is a polynomial in t. This implies the claim. □

Thus we can now prove the main result.

Theorem 3.26. *A volume-preserving vector field Θ on the Danielewski surface D_p is a Lie combination of LND if and only if its corresponding function f with $i_\Theta \omega = df$ is of the form (modulo constants)*

$$f(x, y, z) = \sum_{\substack{i=1 \\ j=0}}^{k} a_{ij} x^i z^j + \sum_{\substack{i=1 \\ j=0}}^{l} b_{ij} y^i z^j + (pq)'(z)$$

for a polynomial $q \in \mathbb{C}[z]$.

Proof. By Proposition 3.22 together with Lemma 3.16 (3.4) and Proposition 3.23 the Lie algebra generated by shear fields consists exactly of those volume-preserving fields described in the theorem. By Theorem 2.15 any LND Θ is conjugated to a shear field S by an automorphism ψ which is a finite composition of shear automorphisms $\psi = \alpha_m \circ \dots \circ \alpha_1$. Thus by Lemma 3.24 $\Theta = \psi^* S = \alpha_1^*(\dots \alpha_{m-1}^*(\alpha_m^* S) \dots)$ is contained in the Lie algebra generated by shear fields. □

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