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Failure of the local-to-global property for *CD(K, N)* spaces

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Abstract. Given any $K \in \mathbb{R}$ and $N \in [1, \infty]$ we show that there exists a compact geodesic metric measure space satisfying locally the $\mathsf{CD}(0, 4)$ condition but failing to satisfy $\mathsf{CD}(K, N)$ globally. The space with this property is a suitable non-convex subset of $\mathbb{R}^2$ equipped with the l^∞-norm and the Lebesgue measure. Combining many such spaces gives a (non-compact) complete geodesic metric measure space satisfying $\mathsf{CD}(0, 4)$ locally but failing to satisfy $\mathsf{CD}(K, N)$ globally for every K and N.

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1. Introduction

A definition of Ricci curvature lower bounds in metric measure spaces was proposed by Sturm in [23,24], and independently at the same time by Lott and Villani in [16]. The definitions were in terms of convexity properties of functionals in the space of probability measures. The most relevant definition in the context of this paper is the $\mathsf{CD}(0, N)$ condition, with 0 taking the place of a lower bound on the curvature, which is usually denoted by $K \in \mathbb{R}$ in the more general definition ($K = 0$ here means non-negative Ricci curvature), and $N < \infty$ being the upper bound on the dimension of the space. The $\mathsf{CD}(0, N)$ condition on a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ requires that between any two probability measures on the space there exists at least one geodesic along which the entropy

$$\mathrm{Ent}_{N'}(\rho\mathfrak{m}) = -\int_X \rho^{1-\frac{1}{N'}} \mathrm{d}\mathfrak{m}$$

is convex for all $N' \geq N$. (See Section 2 for more details.)

Soon after the definition of $\mathsf{CD}(0, N)$ had been introduced it was noticed that $\mathbb{R}^n$ equipped with any norm and with the Lebesgue measure satisfies $\mathsf{CD}(0, n)$. See

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the end of Villani's book [25] for an outline of the proof of this fact. In particular we have:

Theorem 1.1 (Cordero-Erausquin, Sturm and Villani). *The space* $(\mathbb{R}^2, \|\cdot\|_\infty, \mathscr{L}_2)$ *satisfies* $\mathsf{CD}(0,2)$.

A problematic feature of spaces like $(\mathbb{R}^2, ||\cdot||_\infty, \mathscr{L}_2)$ is that between most of the points there exist a huge number of geodesics joining them. In particular, there are a lot of branching geodesics. This is not only a feature of normed spaces; branching geodesics are also known to exist, for example, in some positively curved $\mathsf{CD}(K,N)$ spaces, see the recent paper by Ohta [17]. Initially many results for $\mathsf{CD}(K,N)$ spaces were proven under the assumption that there are no branching geodesics. Later some of these results have been proven without such assumption (for instance local Poincaré inequalities [19,20]). In some results the general case with branching geodesics remains open.

Until now one of the basic open questions for general $\mathsf{CD}(K,N)$ spaces was the local-to-global property of the $\mathsf{CD}(K,N)$ condition. It is known that under the non branching assumption assuming $\mathsf{CD}(0,N)$ (or $\mathsf{CD}(K,\infty)$ or $\mathsf{CD}^*(K,N)$) to hold locally (*i.e.* in a neighbourhood of any point) is the same as assuming it to hold globally. For $\mathsf{CD}(K,\infty)$ this was proven by Sturm [23], for $\mathsf{CD}(0,N)$ by Villani [25], and for $CD^*(K,N)$ by Bacher and Sturm [7]. Such a property is natural to expect from an abstract notion of Ricci curvature lower bounds –after all, the classical definition is local. The notion $\mathsf{CD}^*(K,N)$ refers to the reduced curvature-dimension condition. It is not (at least a priori) as restrictive as the $\mathsf{CD}(K,N)$ condition, but it is more natural in the local-to-global questions.
In this paper we show that not even $\mathsf{CD}(0,N)$ does, in general, have the local-to-global property. The idea of our example showing that local $\mathsf{CD}(0,N)$ does not imply global $\mathsf{CD}(0,N)$ is surprisingly simple. One starts with the observation, which we already mentioned, that $(\mathbb{R}^2, ||\cdot||_\infty)$ has lots of geodesics. There are even so many geodesics that one can go around some Euclidean corners with them. Therefore we at least have domains in $\mathbb{R}^2$ that are not convex in the Euclidean sense but still (weakly) geodesically convex with the l^∞-norm. Next we observe that we can locally move two identical objects around a corner, see the left picture in Figure 1.1. This roughly means that moving measures that are approximately the same should not be a problem in view of the local $\mathsf{CD}(0,2)$ condition.

For more general sets the 45 degree angle gives the extremal case when going around a corner. See the right picture in Figure 1.1 for the extremal case. There we have to shrink the measure in the vertical direction when we move it around the corner. This suggests that we have to give up our hope on $\mathsf{CD}(0,2)$. Still the particular transport seems to satisfy $\mathsf{CD}(0,4)$, for instance. However, when we take thinner and thinner strips closer and closer to the corner we notice that the estimates do not scale properly. An obvious idea to try to correct this is to smoothen the corner, and in fact replacing the corner with a piece of a circle will do:

Example 1.2. Let $K \in \mathbb{R}$. Take X to be the closed subset of $\mathbb{R}^2$ shown in Figure 1.2. (We shall specify it more carefully in Section 3.) As the distance take $\mathsf{d}(x,y) =$

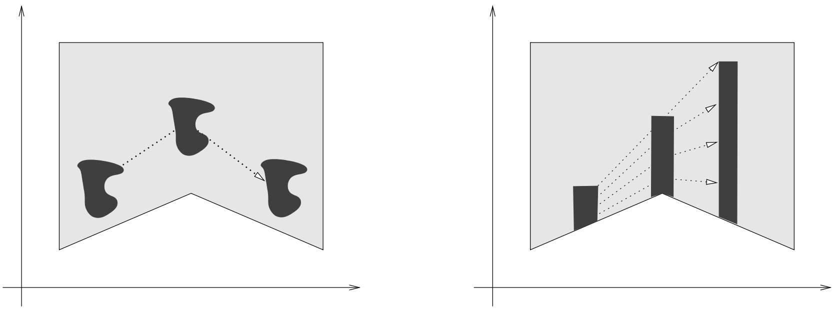

Figure 1.1. On the left we see how a measure can be easily transported around a corner when our distance is given by the l^∞-norm. On the right we have the extremal case when we go around a corner. In this case we have to squeeze the measure a bit.

$||x - y||_\infty$ and as the reference measure the restriction of the Lebesgue measure $\mathfrak{m} = \mathscr{L}_{2|X}$. The space $(X, \mathsf{d}, \mathfrak{m})$ is a compact geodesic metric measure space

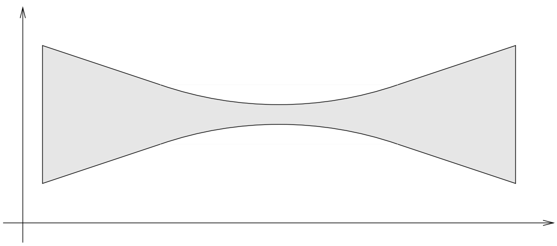

Figure 1.2. An illustration of the space X of Example 1.2 as a compact subset of $\mathbb{R}^2$ with the l^∞-norm. The space satisfies $\mathsf{CD}(0, 4)$ locally, but not globally.

satisfying $\mathsf{CD}(0, 4)$ locally, but failing to globally satisfy $\mathsf{CD}(K, \infty)$ (and hence $\mathsf{CD}(K, N)$ and $\mathsf{CD}^*(K, N)$ for any $N \geq 1$).

We note that if in Example 1.2 we were to drop either the requirement that (X, d) is complete or the requirement that it is geodesic the example would be close to trivial. However, with both of these assumptions in place, if we want to get the example as a subset of $\mathbb{R}^2$, we are forced to consider optimal transport at and near the boundary of a non-convex set. Verifying the $\mathsf{CD}(0, 4)$ condition at the boundary turned out to require some calculations.

Indeed, proving that $(X, \mathsf{d}, \mathfrak{m})$ locally satisfies $\mathsf{CD}(0, 4)$ takes most of this paper whereas the failure of global $\mathsf{CD}(K, \infty)$ follows immediately by considering optimal transport between measures with large supports on the opposite sides of the 'neck'. Gluing together infinitely many spaces of the type shown in Example 1.2 gives a (non compact) complete geodesic metric measure space satisfying $\mathsf{CD}(0, 4)$ locally but failing the global $\mathsf{CD}(K, \infty)$ for any $K \in \mathbb{R}$.

We will show that the space in Example 1.2 fails $\mathsf{CD}(K, \infty)$. In fact, not only does the space fail the $\mathsf{CD}(K, \infty)$ condition but it also fails the measure contraction property $\mathsf{MCP}(K, N)$. This is easy to see with a similar argument showing the failure of $\mathsf{CD}(K, \infty)$. The $\mathsf{MCP}(K, N)$ is a curvature-dimension condition involving optimal transports from absolutely continuous measures to Dirac masses; for the definition and basic properties see the papers by Sturm [24] and Ohta [18]. The $\mathsf{MCP}(K, N)$ is (almost) implied by the $\mathsf{CD}(K, N)$ condition: The more restrictive version of $\mathsf{MCP}(K, N)$, requiring the existence of a Markov kernel (a universal choice of geodesics) giving the $\mathsf{MCP}(K, N)$, follows from the $\mathsf{CD}(K, N)$ condition in non branching metric spaces, see [24]. The version of the $\mathsf{MCP}(K, N)$ without the Markov kernel follows from the $\mathsf{CD}(K, N)$ condition also without the non branching assumption, see [20]. The $\mathsf{MCP}(K, N)$ is known to fail the local-to-global property even in non branching metric spaces, see [24].

Although $\mathsf{CD}(K, N)$ (and $\mathsf{CD}^*(K, N)$) fails to have the local-to-global property, the more recent definition of Riemannian Ricci curvature bounds by Ambrosio, Gigli and Savaré [4] (see also [2] for some generalization and simplifications and [6, 12] for the finite dimensional definitions), $\mathsf{RCD}^*(K, N)$ for short, could still have the local-to-global property. The fact that $\mathsf{RCD}^*(K, N)$ spaces are essentially non branching and there exist optimal maps from absolutely continuous measures [14,15,22] strongly supports this conjecture.

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2. Preliminaries

In this paper the norm we mostly use is the l^∞-norm and hence we sometimes abbreviate $||(x_0, y_0) - (x_1, y_1)|| := ||(x_0, y_0) - (x_1, y_1)||_\infty := \max\{|x_0 - x_1|, |y_0 - y_1|\}$. We denote the Euclidean norm in $\mathbb{R}$ by $|\cdot|$.

2.1. Optimal mass transportation

We will give here only a few facts about optimal mass transportation. For a more detailed introduction we refer to the books by Villani [25] and by Ambrosio and Gigli [1]. We denote by $\mathscr{P}(X)$ the space of Borel probability measures on the complete and separable metric space (X, d) and by $\mathscr{P}_2(X) \subset \mathscr{P}(X)$ the subspace consisting of all the probability measures with finite second moment. Our example X is compact and thus for it we have $\mathscr{P}_2(X) = \mathscr{P}(X)$. However, in general the measures with finite second moment are considered in order to have finite W_2-distance between the measures (see below for the definition of the distance W_2).

Given two probability measures $\mu_0, \mu_1 \in \mathscr{P}(X)$ and a Borel cost function $c\colon X \times X \to [0, \infty]$ the optimal mass transportation problem is to minimize

$$\int_X c(x, y)\, \mathrm{d}\boldsymbol{\gamma}(x, y) \tag{2.1}$$

among all $\boldsymbol{\gamma} \in \mathscr{P}(X \times X)$ with $(\mathsf{p}_1)_\sharp\boldsymbol{\gamma} = \mu_0$ and $(\mathsf{p}_2)_\sharp\boldsymbol{\gamma} = \mu_1$, *i.e.* μ_0 and μ_1 as the first and the second marginal. The p_i denotes the projection to the i:th component.

In the definition of the Ricci curvature lower bounds we will use the quadratic transportation distance $W_2(\mu_0, \mu_1)$, which is given by the cost function $c(x, y) = \mathsf{d}(x, y)^2$. In other words, for $\mu_0, \mu_1 \in \mathscr{P}_2(X)$ it is defined by

$$W_2^2(\mu_0, \mu_1) = \inf_{\boldsymbol{\gamma}} \int_X \mathsf{d}^2(x, y)\, \mathrm{d}\boldsymbol{\gamma}(x, y), \tag{2.2}$$

where again the infimum is taken over all $\boldsymbol{\gamma} \in \mathscr{P}(X \times X)$ with μ_0 and μ_1 as the first and the second marginal. Assuming the space (X, d) to be geodesic, also the space $(\mathscr{P}_2(X), W_2)$ is geodesic. We denote by $\mathrm{Geo}(X)$ the space of (constant speed minimizing) geodesics on (X, d). The notation $\mathrm{e}_t : \mathrm{Geo}(X) \to X, t \in [0, 1]$ is used for the evaluation maps defined by $\mathrm{e}_t(\gamma) := \gamma_t$. A useful fact is that any geodesic $(\mu_t) \in \mathrm{Geo}(\mathscr{P}_2(X))$ can be lifted to a measure $\boldsymbol{\pi} \in \mathscr{P}(\mathrm{Geo}(X))$, so that $(\mathrm{e}_t)_\#\boldsymbol{\pi} = \mu_t$ for all $t \in [0, 1]$. Given $\mu_0, \mu_1 \in \mathscr{P}_2(X)$, we denote by $\mathrm{OptGeo}(\mu_0, \mu_1)$ the space of all $\boldsymbol{\pi} \in \mathscr{P}(\mathrm{Geo}(X))$ for which $(\mathrm{e}_0, \mathrm{e}_1)_\#\boldsymbol{\pi}$ realizes the minimum in (2.2).

A property of optimal transport plans that we will frequently use is cyclical monotonicity. It holds in a great generality, and in particular in the minimization problems we are considering in this paper. A set $\Gamma \subset X \times X$ is called c-cyclically monotone if for any $k \in \mathbb{N}$ and $(x_1, y_1), \ldots, (x_k, y_k) \in \Gamma$ we have

$$\sum_{i=1}^{k} c(x_i, y_i) \leq \sum_{i=1}^{k} c(x_i, y_{i+1})$$

with the identification $y_{k+1} = y_1$. Now, given $\mu_0, \mu_1 \in \mathscr{P}_2(X)$ and an optimal transport plan $\boldsymbol{\gamma}$ minimizing (2.1) there exists a c-cyclically monotone subset Γ with full $\boldsymbol{\gamma}$-measure.

2.2. Ricci curvature lower bounds in metric measure spaces

The definition of the full $\mathsf{CD}(K, N)$ condition is a bit complicated. In this paper we will only need the special cases with $K = 0$ and $N < \infty$, and with $N = \infty$. In the first case the definition reduces to the following:

Definition 2.1. Let $N \in [1, \infty)$. We say that a complete geodesic metric measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathsf{CD}(0, N)$ condition if for any two measures $\mu_0, \mu_1 \in \mathscr{P}(X)$ with support bounded and contained in $\mathrm{supp}(\mathfrak{m})$ there exists a measure $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu_0, \mu_1)$ such that for every $t \in [0, 1]$ and $N' \geq N$ we have

$$\mathrm{Ent}_{N'}(\rho_t\mathfrak{m}) \leq (1 - t)\mathrm{Ent}_{N'}(\rho_0\mathfrak{m}) + t\mathrm{Ent}_{N'}(\rho_1\mathfrak{m}), \tag{2.3}$$

where for any $t \in [0, 1]$ we have written $(\mathrm{e}_t)_\sharp\boldsymbol{\pi} = \rho_t\mathfrak{m} + \mu_t^s$ with $\mu_t^s \perp \mathfrak{m}$.

Recall that the entropy $\mathrm{Ent}_{N'}$ with finite N' is defined as

$$\mathrm{Ent}_{N'}(\rho\mathfrak{m}) = -\int_X \rho^{1-\frac{1}{N'}}\,\mathrm{d}\mathfrak{m}.$$

For $N' = \infty$ the definition of entropy that is used is

$$\mathrm{Ent}_\infty(\mu) = \int_X \rho \log \rho \,\mathrm{d}\mathfrak{m},$$

if $\mu = \rho\mathfrak{m}$ is absolutely continuous with respect to $\mathfrak{m}$ and $\mathrm{Ent}_\infty(\mu) = \infty$ otherwise. It is used in defining the $\mathsf{CD}(K, \infty)$ condition.

Definition 2.2. Let $K \in \mathbb{R}$. We say that a metric measure space $(X, \mathsf{d}, \mathfrak{m})$ satisfies the $\mathsf{CD}(K, \infty)$ condition if for any two measures $\mu_0, \mu_1 \in \mathscr{P}(X)$ with support bounded and contained in $\mathrm{supp}(\mathfrak{m})$ there exists a measure $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu_0, \mu_1)$ such that for every $t \in [0, 1]$ we have

$$\mathrm{Ent}_\infty(\mu_t) \le (1-t)\mathrm{Ent}_\infty(\mu_0) + t\mathrm{Ent}_\infty(\mu_1) - \frac{K}{2}t(1-t)W_2^2(\mu_0, \mu_1),$$

where we have written $\mu_t = (\mathrm{e}_t)_\sharp\boldsymbol{\pi}$.

We will show that our example fails the $\mathsf{CD}(K, \infty)$ condition but satisfies $\mathsf{CD}(0, N)$ locally. A complete geodesic metric measure space $(X, \mathsf{d}, \mathfrak{m})$ is said to satisfy $\mathsf{CD}(0, N)$ locally if for any $x \in X$ there exists a radius $r > 0$ such that for any $\mu_0, \mu_1 \in \mathscr{P}(X)$ with supports in $B(x, r)$ there is a measure $\boldsymbol{\pi} \in \mathrm{OptGeo}(\mu_0, \mu_1)$ such that for every $t \in [0, 1]$ and $N' \ge N$ we have (2.3).

Remark 2.3. There is a reduced version of the $\mathsf{CD}(K, N)$ condition denoted by $\mathsf{CD}^*(K, N)$, that was introduced in a paper by Bacher and Sturm [7]. The local-to-global properties are more natural to expect for $\mathsf{CD}^*(K, N)$ rather than for $\mathsf{CD}(K, N)$. Indeed, in the non branching case $\mathsf{CD}^*(K, N)$ condition has the local-to-global property, see [7]. For $K \ge 0$ and $N \in [1, \infty)$ we have

$$\mathsf{CD}(K, N) \Rightarrow \mathsf{CD}^*(K, N) \Rightarrow \mathsf{CD}\left(\frac{N-1}{N}K, N\right) \tag{2.4}$$

and so for $K > 0$ the $\mathsf{CD}^*(K, N)$ condition is (at least a priori) less strict than $CD(K, N)$. For the proof of this and for a more detailed discussion of the relation with $\mathsf{CD}^*(K, N)$ and $\mathsf{CD}(K, N)$ we refer to [7] (see also the papers by Cavalletti and Sturm [10] and by Cavalletti [9]). Since we show that our example fails $\mathsf{CD}(K, \infty)$, it will also fail $\mathsf{CD}(K, N)$ and $\mathsf{CD}^*(K, N)$ for all N. Also, our example satisfies $\mathsf{CD}(0, N)$ locally, and in this particular from (2.4) we see that the $\mathsf{CD}(0, N)$ condition is the same as the $\mathsf{CD}^*(0, N)$ one.

2.3. Approximate differentiability and the Jacobian equation

Given two absolutely continuous measures $\mu_0, \mu_1 \in \mathscr{P}_2(\mathbb{R}^2)$ and an optimal map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ pushing μ_0 to μ_1, our aim is to express the density ρ_1 of μ_1 using the density ρ_0 of μ_0 and the mapping T. Assuming T to be one-to-one and smooth, this expression is the standard Jacobian equation

$$\rho_1(T(x, y))J_T(x, y) = \rho_0((x, y)) \qquad \text{for } \mu_0\text{-almost every } (x, y), \tag{2.5}$$

where $J_T(x, y)$ is the absolute value of the Jacobian determinant of T. A way to relax the assumptions on T to be one-to-one and smooth is to require it to be one-to-one almost everywhere and approximately differentiable, see for instance the book by Ambrosio, Gigli and Savaré [3, Lemma 5.5.3] for a precise statement.

Recall that a mapping $f\colon U \to \mathbb{R}^m$, with $U \subset \mathbb{R}^n$ open, is called *approximately differentiable at* $x \in U$ if there exists a measurable function $\tilde{f}\colon U \to \mathbb{R}^m$ which is differentiable at x and for which

$$\lim_{r\to 0} \frac{\mathscr{L}_n\left(\left\{z \in B(x, r) \,:\, f(z) = \tilde{f}(z)\right\}\right)}{\mathscr{L}_n(B(x, r))} = 1.$$

The approximate differential of f at x is defined to be that of $\tilde{f}$ at x. Correspondingly we define the approximate partial derivatives (of the components), denoted simply by $\frac{\partial f_i}{\partial z_i}$.

Approximate differentiability for T would follow from the almost everywhere existence of approximate partial derivatives, see Federer's book [13, Theorem 3.1.4]. However, our mapping will not in general have approximate partial derivatives in all the directions. Due to the special structure of our optimal maps the following easy version will suffice. In the proposition below, and later on, we write the components of a map $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ as f_1 and f_2. In other words $f(x, y) = (f_1(x, y), f_2(x, y))$.

Proposition 2.4. *Let $\mu_0, \mu_1 \in \mathscr{P}_2(\mathbb{R}^2)$ be absolutely continuous with respect to $\mathscr{L}_2$ with densities ρ_0 and ρ_1, respectively, and let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a map such that $\mu_1 = f_\sharp\mu_0$ and f is one-to-one outside a set of measure zero. Suppose that at μ_0-almost every point $z \in \mathbb{R}^2$ there exists a set $D_z \subset \mathbb{R}^2$ for which z is a density point and $f_1(x, y) = f_1(x)$ for all $(x, y) \in D_z$,* i.e. *approximately f_1 does not depend on y. Suppose also that, when restricted to D_z, f_1 is increasing in x and $f_2(x, y)$ is increasing in y for almost every $x \in \mathbb{R}$. Then* (2.5) *holds with $J_f(x, y) = \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y}$.*

Proof. Take $z \in \mathbb{R}^2$ for which there exists D_z as in the statement of the proposition. Take a measurable $A \subset D_z$ and write $A_x = \{y \in \mathbb{R} \,:\, (x, y) \in A\}$. Because $f_1(x)$ is increasing in x and $f_2(x, y)$ is increasing in y for almost every x, f_1 is almost everywhere approximately differentiable and f_2 has an approximate partial derivative in the y-direction at almost every point. Since μ_0 and μ_1 are absolutely continuous with respect to $\mathscr{L}_2$ and f is one-to-one outside a set of measure zero, we have

$$\int_A \rho_0((\bar{x},\bar{y}))\,\mathrm{d}\mathscr{L}_2(\bar{x},\bar{y}) = \mu_0(A) = \mu_1(f(A)) = \int_{f(A)} \rho_1(\tilde{x},\tilde{y})\,\mathrm{d}\mathscr{L}_1(\tilde{y})\,\mathrm{d}\mathscr{L}_1(\tilde{x})$$

$$= \int_{-\infty}^{\infty}\int_{A_{\bar{x}}} \rho_1(f(\bar{x},\bar{y}))\frac{\partial f_2}{\partial y}(\bar{x},\bar{y})\,\mathrm{d}\mathscr{L}_1(\bar{y})\frac{\partial f_1}{\partial x}(\bar{x})\,\mathrm{d}\mathscr{L}_1(\bar{x})$$

$$= \int_A \rho_1(f(\bar{x},\bar{y}))\frac{\partial f_2}{\partial y}(\bar{x},\bar{y})\frac{\partial f_1}{\partial x}(\bar{x},\bar{y})\,\mathrm{d}\mathscr{L}_2(\bar{x},\bar{y}).$$

The claim follows from this. □

3. Details of the example

Most of this section is devoted to verifying the local $\mathsf{CD}(0,4)$ condition in Example 1.2. The plan is to use the Jacobian equation (2.5) to estimate the density along a chosen geodesic in $\mathscr{P}_2(\mathbb{R}^2)$. Before arriving at this we will first show that we have an optimal map T between two absolutely continuous measures μ_0 and μ_1, that this map is essentially one-to-one and that it can be used in a Jacobian equation. Using the optimal map T we will then select a midpoint measure whose support is still inside our domain. Here we also have to make sure that the map sending an initial point to the midpoint is essentially one-to-one. After this we verify that the midpoint measure satisfies $\mathsf{CD}(0,4)$ with respect to the endpoint measures. At the very end we will also indicate why the global $\mathsf{CD}(K,\infty)$ condition fails.

3.1. Definition of the local domain

Since Theorem 1.1 can be proven by approximating the norm $||\cdot||_\infty$ with strictly convex norms (see [25]), the $\mathsf{CD}(0,4)$ condition (in fact the $\mathsf{CD}(0,2)$ condition) holds inside any domain that is convex in the Euclidean sense. What needs to be done is to verify the $\mathsf{CD}(0,4)$ condition inside domains of the type shown in Figure 3.1.

Referring to Figure 3.1 for the notation, the width $b-a$ and the height $d+\frac{b-a}{2}-c$ of the domain E are assumed to be less than $\frac{1}{128}$. The bottom of the domain is a piece of a sphere with radius one and whose center (x_c,y_c) satisfies

$$|x-x_c| < \frac{1}{2}(y-y_c) \qquad \text{for all } (x,y)\in E. \tag{3.1}$$

Let for every $(x,y)\in[a,b]\times\mathbb{R}$ the point $(x,S(x))$ be the vertical projection to the lower (circular) boundary of E, see Figure 3.1. Notice that $S(x)$ only depends on x. Our assumption (3.1) guarantees (via K-convexity)

$$S\left(\frac{x_1+x_2}{2}\right) - \frac{S(x_1)+S(x_2)}{2} \le \frac{|x_1-x_0|^2}{2} \qquad \text{for all } x_1,x_2\in[a,b]. \tag{3.2}$$

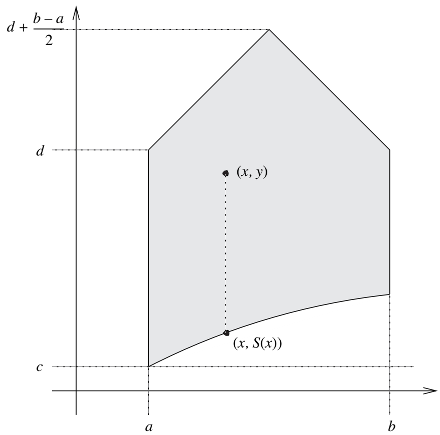

Figure 3.1. The local domain E where we verify the $\mathsf{CD}(0, 4)$ condition.

3.2. Preliminary reductions and definitions

Let us now mention two simplifications that we can always make when checking the $\mathsf{CD}(0, N)$ condition. We will return to both of them in more detail near the end of the paper when we finally prove the $\mathsf{CD}(0, 4)$ condition. The first standard reduction in checking the $\mathsf{CD}(0, N)$ condition (or more generally $\mathsf{CD}(K, N)$ or $\mathsf{CD}^*(K, N)$ condition) is to assume the measures to be absolutely continuous with respect to the reference measure. This reduction is possible because we can approximate any probability measure in the W_2-distance by an absolutely continuous measure without increasing the entropy.

The second standard simplification we make is that we only define the midpoint between any two given measures. This has been used for example by Bacher and Sturm [7] and the author [21]. We can then iterate the procedure of taking midpoints and eventually use the lower semi-continuity of the entropy to have the correct entropy bound along the whole geodesic. Notice that if $K \neq 0$ this procedure works only for the $\mathsf{CD}^*(K, N)$ condition and not for the $\mathsf{CD}(K, N)$ condition.

Let us then turn to the notation and definitions that are less standard than the ones we recalled in Section 2. Given a metric space (X, d), for $z_0, z_1 \in X$ we denote the set of all the midpoints between z_0 and z_1 by

$$\mathrm{Mid}(z_0, z_1) := \left\{ z \in X \,:\, \mathsf{d}(z_0, z) = \mathsf{d}(z_1, z) = \frac{1}{2}\mathsf{d}(z_0, z_1) \right\}.$$

We will not make the distance d visible in the notation because Mid will only be used for $(\mathbb{R}^2, ||\cdot||_\infty)$ and $(\mathscr{P}_2(\mathbb{R}^2), W_2)$, and for those no confusion should arise.

In the following we will often consider separately the part of the transport that moves more in the horizontal (or vertical) direction. In order to simplify the presentation we define the set of horizontal transportation

$$H := \left\{ ((x_0, y_0), (x_1, y_1)) \in \mathbb{R}^2 \times \mathbb{R}^2 \,:\, |x_0 - x_1| > |y_0 - y_1| \right\},$$

the set of vertical transportation

$$V := \left\{((x_0, y_0), (x_1, y_1)) \in \mathbb{R}^2 \times \mathbb{R}^2 \,:\, |x_0 - x_1| < |y_0 - y_1|\right\}$$

and the set of diagonal transportation

$$D := \left\{((x_0, y_0), (x_1, y_1)) \in \mathbb{R}^2 \times \mathbb{R}^2 \,:\, |x_0 - x_1| = |y_0 - y_1|\right\}.$$

Given any $\boldsymbol{\gamma} \in \mathscr{P}(\mathbb{R}^2 \times \mathbb{R}^2)$, the restricted measure $\boldsymbol{\gamma}_{|H}$ moves every infinitesimal mass more in the horizontal direction than the vertical, $\boldsymbol{\gamma}_{|V}$ the other way around, and $\boldsymbol{\gamma}_{|D}$ moves mass in the diagonal directions.

3.3. Selecting an optimal map

One possible way of trying to obtain the needed optimal maps could be to analyse the proof of Theorem 1.1, or the $\mathsf{CD}(0, 2)$ condition in $(\mathbb{R}^2, ||\cdot||_\infty, \mathscr{L}_2)$. However, we chose a more direct approach of first selecting a suitable optimal transport plan via three consecutive minimizations and then showing that this plan has all the desired properties. The idea behind the three minimizations is that the l^∞-norm allows locally a lot of freedom for the coordinate in which the mass is transported less. By doing extra minimization on the two directions separately after the main minimization, we will increase the monotonicity properties of the optimal transport.

The idea of using consecutive minimizations to choose a better optimal transport plan goes back to [5,11] where the existence of optimal maps from absolutely continuous measures in $\mathbb{R}^n$ for cost functions of the form $c(x, y) = ||x - y||$ was proven - first with any crystalline norm $||\cdot||$ by Ambrosio, Kirchheim and Pratelli [5] and then with any norm $||\cdot||$ by Champion and De Pascale [11]. Let us also note that the existence of an optimal map in our case with $c(x, y) = ||x - y||_\infty^2$ has been proven by Carlier, De Pascale and Santambrogio in [8]. We will prove here the existence of a specific optimal transport map using the consecutive minimizations in order to keep the paper reasonably self-contained and, more importantly, in order to guarantee that the chosen optimal plan has all the needed cyclical monotonicity properties.

Let us give the three minimizations. Suppose that $\mu_0, \mu_1 \in \mathscr{P}_2(\mathbb{R}^2)$ are given. Let $\mathrm{Opt}_1(\mu_0, \mu_1) \subset \mathscr{P}(\mathbb{R}^2 \times \mathbb{R}^2)$ be the set of those $\boldsymbol{\gamma}$ that minimize

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} ||z_1 - z_2||^2 \mathrm{d}\boldsymbol{\gamma}(z_1, z_2)$$

and satisfy $(\mathsf{p}_1)_\sharp\boldsymbol{\gamma} = \mu_0$ and $(\mathsf{p}_2)_\sharp\boldsymbol{\gamma} = \mu_1$. The set $\mathrm{Opt}_1(\mu_0, \mu_1)$ is a nonempty closed and convex subset of $\mathscr{P}(\mathbb{R}^2 \times \mathbb{R}^2)$. Next let $\mathrm{Opt}_2(\mu_0, \mu_1) \subset \mathrm{Opt}_1(\mu_0, \mu_1)$ be the set of those $\boldsymbol{\gamma}$ that minimize

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} |x_1 - x_2|^2 \mathrm{d}\boldsymbol{\gamma}((x_1, y_1), (x_2, y_2))$$

in $\mathrm{Opt}_1(\mu_0, \mu_1)$. Again $\mathrm{Opt}_2(\mu_0, \mu_1)$ is a nonempty closed and convex subset of $\mathscr{P}(\mathbb{R}^2 \times \mathbb{R}^2)$. Finally let $\mathrm{Opt}_3(\mu_0, \mu_1) \subset \mathrm{Opt}_2(\mu_0, \mu_1)$ be the set of those $\boldsymbol{\gamma}$ that minimize

$$\int_{\mathbb{R}^2\times\mathbb{R}^2} |y_1 - y_2|^2 \mathrm{d}\boldsymbol{\gamma}((x_1, y_1), (x_2, y_2))$$

in $\mathrm{Opt}_2(\mu_0, \mu_1)$. Clearly also $\mathrm{Opt}_3(\mu_0, \mu_1)$ is nonempty. We will see in Proposition 3.2 that in the case $\mu_0 \ll \mathscr{L}_2$ the set $\mathrm{Opt}_3(\mu_0, \mu_1)$ consists of only one optimal plan which is given by a map. Before this, let us list the cyclical monotonicity properties we immediately get from the three minimizations.

Lemma 3.1. *Let* $\mu_0, \mu_1 \in \mathscr{P}_2(\mathbb{R}^2)$ *and* $\boldsymbol{\gamma} \in \mathrm{Opt}_3(\mu_0, \mu_1)$. *Then there exists a set* $\Gamma \subset \mathbb{R}^2 \times \mathbb{R}^2$ *of full* $\boldsymbol{\gamma}$*-measure such that for all* $(z_1, w_1), (z_2, w_2) \in \Gamma$ *we have*

$$||z_1 - w_1||^2 + ||z_2 - w_2||^2 \leq ||z_1 - w_2||^2 + ||z_2 - w_1||^2 \tag{3.3}$$

and for all $((x_1, y_1), (x_1', y_1')), ((x_2, y_2), (x_2', y_2')) \in \Gamma$ *we have*

$$\begin{aligned} &|y_1 - y_1'|^2 + |y_2 - y_2'|^2 \leq |y_1 - y_2'|^2 + |y_2 - y_1'|^2, \\ &\textit{if } |x_1 - x_1'|^2 + |x_2 - x_2'|^2 = |x_2 - x_1'|^2 + |x_1 - x_2'|^2 \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} &|x_1 - x_1'|^2 + |x_2 - x_2'|^2 \leq |x_1 - x_2'|^2 + |x_2 - x_1'|^2, \\ &\textit{if } |y_1 - y_1'|^2 + |y_2 - y_2'|^2 = |y_2 - y_1'|^2 + |y_1 - y_2'|^2. \end{aligned} \tag{3.5}$$

Let us then prove that in the case $\mu_0 \ll \mathscr{L}_2$ the optimal plan in $\mathrm{Opt}_3(\mu_0, \mu_1)$ is given by a map. This is a fairly standard consequence of Lemma 3.1, so we present only parts of the proof to give the idea.

Proposition 3.2. *Suppose* $\mu_0 \ll \mathscr{L}_2$. *Then* $\mathrm{Opt}_3(\mu_0, \mu_1)$ *is a singleton and its only element is induced by an optimal map* T.

Proof. The fact that $\mathrm{Opt}_3(\mu_0, \mu_1)$ is a singleton follows once we know that any element in $\mathrm{Opt}_3(\mu_0, \mu_1)$ is induced by an optimal map. Indeed, if there were two different measures $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \in \mathrm{Opt}_3(\mu_0, \mu_1)$, then by convexity also $\boldsymbol{\gamma}_3 = \frac{1}{2}(\boldsymbol{\gamma}_1 + \boldsymbol{\gamma}_2) \in \mathrm{Opt}_3(\mu_0, \mu_1)$. However, the measure $\boldsymbol{\gamma}_3$ would not be induced by a map.

Suppose now that there exists $\boldsymbol{\gamma} \in \mathrm{Opt}_3(\mu_0, \mu_1)$ that is not induced by a map. Then the disintegration $\{\boldsymbol{\gamma}_z\}$ of $\boldsymbol{\gamma}$ with respect to $\mathtt{p}_1$ is not a Dirac mass for a μ_0-positive set of points $z \in \mathbb{R}^2$. Now there are several cases to check. We use different cyclical monotonicities to arrive at a contradiction in each of the cases. The different cases are:

(i) $\boldsymbol{\gamma}_z(H) > 0$ and $\boldsymbol{\gamma}_z(V) > 0$ for a μ_0-positive set of z;
(ii) $\boldsymbol{\gamma}_{z|_H}, \boldsymbol{\gamma}_{z|_V}$ or $\boldsymbol{\gamma}_{z|_D}$ is not a Dirac mass for a μ_0-positive set of z;
(iii) $\boldsymbol{\gamma}_z(D) > 0$ and $\boldsymbol{\gamma}_z(H) > 0$ (or $\boldsymbol{\gamma}_z(V) > 0$) for a μ_0-positive set of z.

The contradiction follows from all of the cases in a similar way. We will only give details in the first case. Thus assume that $\boldsymbol{\gamma}_z(H) > 0$ and $\boldsymbol{\gamma}_z(V) > 0$ for a μ_0-positive set of z. Let $\Gamma \subset \mathbb{R}^2 \times \mathbb{R}^2$ be the set from Lemma 3.1 having all the cyclical monotonicity properties. Suppose that the set

$$\{z \in \mathbb{R}^2 \,:\, \boldsymbol{\gamma}_z(H \cap \Gamma) > 0 \text{ and } \boldsymbol{\gamma}_z(V \cap \Gamma) > 0\}$$

has positive μ_0-measure. Now there exist $\delta > \epsilon > 0$ and $(x_h, y_h), (x_v, y_v) \in \mathbb{R}^2$ such that

$$||(x_h, y_h) - (x_v, y_v)|| \geq 4\delta$$

and the set

$$\begin{aligned} A = \Big\{ z \in \mathbb{R}^2 : \boldsymbol{\gamma}_z\Big(&\big\{((x_1, y_1), (x_2, y_2)) : |y_1 - y_2| < |x_1 - x_2| - \epsilon\big\} \\ &\cap \Gamma \cap \mathbb{R}^2 \times B((x_h, y_h), \delta)\Big) > 0 \\ \text{and } \boldsymbol{\gamma}_z\Big(&\big\{((x_1, y_1), (x_2, y_2)) \,:\, |x_1 - x_2| < |y_1 - y_2| - \epsilon\big\} \\ &\cap \Gamma \cap \mathbb{R}^2 \times B((x_v, y_v), \delta)\Big) > 0 \Big\} \end{aligned}$$

has positive μ_0-measure. Let $(\bar{x}, \bar{y})$ be a density point of A. By symmetry, assume $||(\bar{x}, \bar{y}) - (x_v, y_v)|| \geq 2\delta$. Because $(\bar{x}, \bar{y})$ is a density point, for some $x \in [\bar{x} - \frac{\epsilon}{2}, \bar{x} + \frac{\epsilon}{2}]$ there exist $y_1, y_2 \in [\bar{y} - \frac{\epsilon}{2}, \bar{y} + \frac{\epsilon}{2}]$, $y_1 \neq y_2$, such that $(x, y_1), (x, y_2) \in A$. We may assume that $|y_v - y_2| < |y_v - y_1|$. Let $(x_{h,2}, y_{h,2}) \in B((x_h, y_h), \delta)$ and $(x_{v,1}, y_{v,1}) \in B((x_v, y_v), \delta)$ be such that

$$((x, y_1), (x_{v,1}, y_{v,1})), ((x, y_2), (x_{h,2}, y_{h,2})) \in \Gamma,$$

$$|y_2 - y_{h,2}| < |x - x_{h,2}| - \epsilon \quad \text{and} \quad |x - x_{v,1}| < |y_1 - y_{v,1}| - \epsilon.$$

But now

$$\begin{aligned} ||(x, y_2) - (x_{v,1}, y_{v,1})||^2 &+ ||(x, y_1) - (x_{h,2}, y_{h,2})||^2 \\ &= |y_2 - y_{v,1}|^2 + |x - x_{h,2}|^2 < |y_1 - y_{v,1}|^2 + |x - x_{h,2}|^2 \\ &= ||(x, y_1) - (x_{v,1}, y_{v,1})||^2 + ||(x, y_2) - (x_{h,2}, y_{h,2})||^2 \end{aligned}$$

contradicting the cyclical monotonicity (3.3) of Γ. This proves the claim in the case (i).

In the case (ii) we argue similarly and use the cyclical monotonicities (3.3) and (3.4) if $\boldsymbol{\gamma}_z|_H$ is not Dirac, (3.3) and (3.5) if $\boldsymbol{\gamma}_z|_V$ is not, and (3.3) if $\boldsymbol{\gamma}_z|_D$ is not. In the case (iii) we use (3.4) if $\boldsymbol{\gamma}_z(D) > 0$ and $\boldsymbol{\gamma}_z(H) > 0$, and (3.5) if $\boldsymbol{\gamma}_z(D) > 0$ and $\boldsymbol{\gamma}_z(V) > 0$. $\square$

Next we list some properties of the map T in the case $\mu_0, \mu_1 \ll \mathscr{L}_2$. Recall that we use the notation T_1 and T_2 for the components of T, *i.e.* $T(x, y) = (T_1(x, y), T_2(x, y))$.

Lemma 3.3. *Let* $\mu_0 \ll \mathscr{L}_2$, T *the map from Proposition* 3.2 *and* Γ *the set from Lemma* 3.1. *Then for all* $(x, y_1), (x, y_2), (x_1, y), (x_2, y) \in F = \{(x, y) \in \mathbb{R}^2 : ((x, y), T(x, y)) \in \Gamma\}$ *we have the following.*

$$\text{If } y_1 \neq y_2 \text{ and } T_1(x, y_1) = T_1(x, y_2), \text{ then } \frac{T_2(x, y_1) - T_2(x, y_2)}{y_1 - y_2} \geq 0 \tag{3.6}$$

and

$$\text{if } x_1 \neq x_2 \text{ and } T_2(x_1, y) = T_2(x_2, y), \text{ then } \frac{T_1(x_1, y) - T_1(x_2, y)}{x_1 - x_2} \geq 0. \tag{3.7}$$

Proof. Suppose that (3.6) does not hold for some $(x, y_1), (x, y_2) \in F$. We may assume that $y_2 < y_1$ so that $T_2(x, y_1) < T_2(x, y_2)$. By the cyclical monotonicity (3.3) we have $|T_1(x, y_1) - x| \geq |T_2(x, y_1) - y_1|$ and $|T_1(x, y_2) - x| \geq |T_2(x, y_2) - y_2|$. Therefore

$$\begin{aligned} ||T(x, y_1) - (x, y_1)|| = ||T(x, y_1) - (x, y_2)|| &= ||T(x, y_2) - (x, y_2)|| \\ &= ||T(x, y_2) - (x, y_1)||. \end{aligned}$$

But now

$$|T_2(x, y_1) - y_2|^2 + |T_2(x, y_2) - y_1|^2 < |T_2(x, y_1) - y_1|^2 + |T_2(x, y_2) - y_2|^2$$

violating the cyclical monotonicity (3.4). This proves (3.6). The inequality (3.7) follows similarly from the cyclical monotonicities (3.3) and (3.5). □

In estimating the densities at the midpoints we will also need an infinitesimal version of Lemma 3.3. Recalling the discussion from Section 2.3 we would like to use a Jacobian equation

$$\rho_1(T(x, y)) J_T(x, y) = \rho_0((x, y)) \qquad \text{for } \mu_0\text{-almost every } (x, y). \tag{3.8}$$

Here a few comments are in order. As we mentioned in Section 2.3, usually in writing the Jacobian equation the mapping is assumed to be at least approximately differentiable almost everywhere. However, the optimal map T is not in general approximately differentiable. To see this, take a measurable function $f : [0,1] \to [0,1]$ that is not approximately differentiable and consider the optimal transport between the uniform measures on $[0, 1]^2$ and $\{(x + 3, y) : x \in [0, 1], y \in [f(x), f(x) + 1]\}$.

Nevertheless, because locally in H we are sending vertical lines to vertical lines by cyclical monotonicity (3.3) the first coordinate function T_1 is approximately differentiable almost everywhere. Then, because of cyclical monotonicity (3.4) the second coordinate function T_2 is approximately differentiable in the variable y for almost every x. Now, since T_1 was locally approximately constant in y, we get (3.8) in H using Proposition 2.4. Similarly we get it also in V and D.

Lemma 3.4. *Let* $\mu_0, \mu_1 \ll \mathscr{L}_2$ *and* $\mathrm{Opt}_3(\mu_0, \mu_1) = \{(\mathrm{id}, \mathrm{T})_\sharp \mu_0\}$. *Then the map* T *satisfies* μ_0*-almost everywhere*

$$\frac{\partial T_1}{\partial x} \geq 0 \text{ and } \frac{\partial T_2}{\partial y} \geq 0 \text{ if } ((x, y), T(x, y)) \in H \cup V. \tag{3.9}$$

Still μ_0*-almost everywhere we have that*

$$\begin{aligned} &T_1 \text{ is locally approximately constant in } y, \text{ if } ((x, y), T(x, y)) \in H \text{ and} \\ &T_2 \text{ is locally approximately constant in } x, \text{ if } ((x, y), T(x, y)) \in V. \end{aligned} \tag{3.10}$$

Proof. Let us start by showing that in H vertical lines are locally approximately sent to vertical lines outside a set of μ_0-measure zero. Let Γ be the cyclically monotone set from Lemma 3.1 and

$$F_H = \{(x, y) \in \mathbb{R}^2 : ((x, y), T(x, y)) \in \Gamma \cap H\}.$$

Take a density point $(x, y) \in F_H$ of F_H where T is approximately continuous. Let $A \subset F_H$ be such that (x, y) is a density point of A and $T_{|A}$ is continuous.

Let $\epsilon > 0$ be such that $|x - T_1(x, y)| > |y - T_2(x, y)| + \epsilon$ and $0 < \delta < \frac{\epsilon}{2}$ such that for every $(\bar{x}, \bar{y}) \in A$ we have

$$||(x, y) - (\bar{x}, \bar{y})|| < \delta \quad \Longrightarrow \quad ||T(x, y) - T(\bar{x}, \bar{y})|| < \frac{\epsilon}{2}.$$

Suppose that there exist $(x_1, y_1), (x_1, y_2) \in A$ such that $||(x, y) - (x_1, y_1)|| < \delta$, $||(x, y) - (x_1, y_2)|| < \delta$ and $T_1(x_1, y_1) < T_1(x_1, y_2)$. Now for any $(\bar{x}, \bar{y}) \in A$ with $\bar{x} \neq x_1$ and $||(x, y) - (\bar{x}, \bar{y})|| < \delta$ we have

$$T_1(\bar{x}, \bar{y}) \leq T_1(x_1, y_1) \qquad \text{or} \qquad T_1(\bar{x}, \bar{y}) \geq T_1(x_1, y_2)$$

by cyclical monotonicity (3.3), the choice of ϵ and δ, and the fact that $A \subset F_H$. Therefore there exist only countably many $\tilde{x} \in [x - \delta, x + \delta]$ for which $T_1(A \cap \{\tilde{x}\} \times]y - \delta, y + \delta[)$ is not an empty set or a singleton. This proves the claim that outside a set of μ_0-measure zero vertical lines are locally approximately sent to vertical lines in H.

In proving (3.9) first assume that $((x, y), T(x, y)) \in H$. Then by the cyclical monotonicity (3.3) we have $\frac{\partial T_1}{\partial x} \geq 0$. The claim $\frac{\partial T_2}{\partial y} \geq 0$ follows from (3.6) as well as the fact that in H vertical lines are locally approximately sent to vertical lines. In a similar way we can prove (3.9) assuming $((x, y), T(x, y)) \in V$.

The first claim in (3.10) is a restatement of the fact that in H vertical lines are locally approximately sent to vertical lines and the second claim follows analogously. □

3.4. Defining the midpoint

As we already saw in the Introduction (Figure 1.1) we have to deviate the midpoint of a geodesic from the Euclidean midpoint by an amount depending on the end-points of the geodesics. A geodesic going in a diagonal direction has to remain the same geodesic and a geodesic going in the horizontal direction can deviate the most.

The idea behind defining the midpoint the way we do here is that we want to keep the height of the transport right for a (vertical) $\mathsf{CD}(0, 2)$ condition. If the height is exactly the correct one for the condition between vertical strips with their base on the sphere bounding our domain, it will also be infinitesimally correct.

Naturally the correction for the midpoints needs to be done only in the horizontal part H of the transport. For the vertical part V and the diagonal part D we can use the Euclidean midpoints (who will respectively give a $\mathsf{CD}(0, 2)$ and $\mathsf{CD}(0, 1)$ condition for those parts of the transport).

The midpoint $\mu_{\frac{1}{2}}$ will be defined using the mapping $M\colon E \times E \to \mathbb{R}^2$ given by

$$M\left((x_0, y_0), (x_1, y_1)\right) = \left(\frac{x_0 + x_1}{2}, \frac{y_0 + y_1}{2}\right), \tag{3.11}$$

if $((x_0, y_0), (x_1, y_1)) \notin H$ (corresponding to the Euclidean midpoint in the non horizontal transport), and by

$$\begin{aligned} M\left((x_0, y_0), (x_1, y_1)\right) = \Bigg(\frac{x_0 + x_1}{2}, \max\Bigg\{\frac{y_0 + y_1}{2}, \frac{S(x_0) + S(x_1)}{2} + (x_0 - x_1)^2 \\ + \frac{1}{4}\left(\sqrt{y_0 - S(x_0)} + \sqrt{y_1 - S(x_1)}\right)^2\Bigg\}\Bigg), \end{aligned} \tag{3.12}$$

if $((x_0, y_0), (x_1, y_1)) \in H$ (corresponding to the vertical shrinking to satisfy the $CD(0, 2)$ condition in the horizontal transport). See Figure 3.2 for an illustration of the selection of the midpoint in this case.

The first thing to check is that M really gives midpoints. As usual, we write $M = (M_1, M_2)$.

Lemma 3.5. *$M\left((x_0, y_0), (x_1, y_1)\right) \in \mathrm{Mid}\left((x_0, y_0), (x_1, y_1)\right)$.*

Proof. We may assume $x_0 \leq x_1$. If $((x_0, y_0), (x_1, y_1)) \notin H$, the claim is obvious. Let then $((x_0, y_0), (x_1, y_1)) \in H$ so that M is given by (3.12). By symmetry, we may assume that $y_0 \leq y_1$. We have to show that

$$M_2\left((x_0, y_0), (x_1, y_1)\right) - y_0 \leq \frac{x_1 - x_0}{2} \tag{3.13}$$

and

$$y_1 - M_2\left((x_0, y_0), (x_1, y_1)\right) \leq \frac{x_1 - x_0}{2}. \tag{3.14}$$

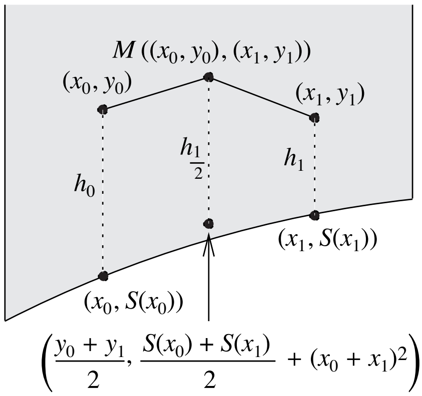

Figure 3.2. The midpoint $M((x_0, y_0), (x_1, y_1))$ for horizontal transport from (x_0, y_0) to (x_1, y_1) is defined so that below the midpoint we have sufficient room to satisfy $\mathsf{CD}(0, 2)$ vertically, *i.e.* in the illustration $h_{\frac{1}{2}}$ satisfies $\sqrt{h_{\frac{1}{2}}} = \frac{1}{2}\left(\sqrt{h_0} + \sqrt{h_1}\right)$.

Because M_2 is increasing in y_1, for verifying (3.13) it is enough to check the extreme case $y_1 - y_0 = x_1 - x_0$ (even though in this case the mapping M is defined using (3.11)). Notice that by assumption on the width and height of E we have

$$|y_1 - y_0|, |y_0 - S(x_0)| \leq \frac{1}{128} \tag{3.15}$$

and by (3.1) we have

$$|S(x_1) - S(x_0)| \leq \frac{1}{2}|x_1 - x_0| = \frac{1}{2}|y_1 - y_0|. \tag{3.16}$$

From (3.15) we have

$$64(y_0 - S(x_0)) \leq \frac{1}{2} \leq (1 - 8(y_1 - y_0))^2$$

which gives

$$\begin{aligned}
&2(y_0 - S(x_0)) + \frac{1}{2}(y_1 - y_0) - 4(y_1 - y_0)^2 \\
&= \sqrt{4(y_0 - S(x_0))^2 + \frac{1}{4}(y_1 - y_0)^2(1 - 8(y_1 - y_0))^2 + 2(y_0 - S(x_0))(y_1 - y_0)(1 - 8(y_1 - y_0))} \\
&\geq \sqrt{4(y_0 - S(x_0))^2 + 2(y_1 - y_0)(y_0 - S(x_0))}.
\end{aligned}$$

This together with (3.16) gives

$$\begin{aligned}4(y_1 - y_0)^2 &\le \left(\sqrt{y_0 - S(x_0) + \frac{1}{2}(y_1 - y_0)} - \sqrt{y_0 - S(x_0)}\right)^2 \\ &\le \left(\sqrt{y_0 - S(x_0) + (y_1 - y_0) + (S(x_0) - S(x_1))} - \sqrt{y_0 - S(x_0)}\right)^2 \\ &= \left(\sqrt{y_1 - S(x_1)} - \sqrt{y_0 - S(x_0)}\right)^2\end{aligned}$$

which is (3.13) in the extreme case $y_1 - y_0 = x_1 - x_0$.

The inequality (3.14) follows from

$$M_2\left((x_0, y_0), (x_1, y_1)\right) \le \frac{y_0 + y_1}{2}$$

and the fact that $((x_0, y_0), (x_1, y_1)) \in H$. □

The second thing to check is that the midpoints are inside our domain E.

Lemma 3.6. *The mapping M has values in E.*

Proof. Again, if $((x_0, y_0), (x_1, y_1)) \notin H$, the claim is obvious. Hence, suppose $((x_0, y_0), (x_1, y_1)) \in H$. By Lemma 3.5 we know that $M\left((x_0, y_0), (x_1, y_0)\right) \in \mathrm{Mid}\left((x_0, y_0), (x_1, y_0)\right)$. Therefore the only thing to check is that

$$M_2\left((x_0, y_0), (x_1, y_0)\right) > S\left(\frac{x_0 + x_1}{2}\right).$$

This follows from our assumptions on the domain E, more precisely from (3.2). □

3.5. Verifying the local $\mathsf{CD}(0, 4)$ condition

In order to be able to use the Jacobian equation (2.5) for the midpoints we first have to check that our mapping giving the midpoint is essentially one-to-one.

Lemma 3.7. *Let $\mu_0, \mu_1 \in \mathscr{P}(E)$ with $\mu_0, \mu_1 \ll \mathscr{L}_2$. Let T be the optimal map from Proposition 3.2. Then the map $M \circ (\mathrm{id}, T)$ is one-to-one outside a set of μ_0-measure zero.*

Proof. Let Γ be the set from Lemma 3.1. Suppose that there exist $(x_1, y_1), (x_2, y_2) \in E$ so that $((x_1, y_1), T(x_1, y_1)), ((x_2, y_2), T(x_2, y_2)) \in \Gamma$, $(x_1, y_1) \neq (x_2, y_2)$ and

$$M \circ (\mathrm{id}, T)(x_1, y_1) = M \circ (\mathrm{id}, T)(x_2, y_2). \tag{3.17}$$

First observe that by cyclical monotonicity (3.3) we have

$$\begin{aligned}||(x_1, y_1) - M \circ (\mathrm{id}, T)(x_1, y_1))|| &= ||(x_2, y_2) - M \circ (\mathrm{id}, T)(x_1, y_1))|| \\ &= ||M \circ (\mathrm{id}, T)(x_1, y_1)) - T(x_1, y_1)|| \qquad (3.18) \\ &= ||M \circ (\mathrm{id}, T)(x_1, y_1)) - T(x_2, y_2)||.\end{aligned}$$

We have three cases to check:

(i) $((x_1, y_1), T(x_1, y_1)), ((x_2, y_2), T(x_2, y_2)) \in H$;
(ii) $((x_1, y_1), T(x_1, y_1)), ((x_2, y_2), T(x_2, y_2)) \notin H$;
(iii) $((x_1, y_1), T(x_1, y_1)) \in H, ((x_2, y_2), T(x_2, y_2)) \notin H$.

In the case (i) we have for $i = 1, 2$ that

$$((x_i, y_i), M \circ (\mathrm{id}, T)(x_i, y_i)), (M \circ (\mathrm{id}, T)(x_i, y_i), T(x_i, y_i)) \notin V$$

since $M \circ (\mathrm{id}, T)(x_i, y_i)$ is a midpoint between (x_i, y_i) and $T(x_i, y_i)$. Therefore by (3.18) we have $x_1 = x_2$ and $T_1(x_1, y_1) = T_1(x_2, y_2)$. By symmetry we may assume $y_1 < y_2$. Then by Lemma 3.3 we have $T_2(x_1, y_1) < T_2(x_2, y_2)$. Since M_2 is strictly increasing in both of the y-coordinates, we have

$$M_2 \circ (\mathrm{id}, T)(x_1, y_1) < M_2 \circ (\mathrm{id}, T)(x_2, y_2)$$

contradicting the assumption (3.17).

In the case (ii) we have, similarly as in the case (i), $y_1 = y_2$ and $T_2(x_1, y_1) = T_2(x_2, y_2)$. The assumption (3.17) gives

$$\frac{x_1 + T_1(x_1, y_1)}{2} = \frac{x_2 + T_1(x_2, y_2)}{2}.$$

This implies via Lemma 3.3 that $x_1 = x_2$, which contradicts the assumption $(x_1, y_1) \neq (x_2, y_2)$.

Finally we have the case (iii). We may assume $x_1 < T_1(x_1, y_1)$. If $T_1(x_2, y_2) < T_1(x_1, y_1)$, then $((x_2, y_2), T(x_2, y_2)) \in V$ and consequently

$$((x_2, y_2), M \circ (\mathrm{id}, T)(x_1, y_1)), (M \circ (\mathrm{id}, T)(x_1, y_1), T(x_2, y_2)) \in V.$$

By cyclical monotonicity (3.3) we have

$$((x_1, y_1), M \circ (\mathrm{id}, T)(x_1, y_1)), (M \circ (\mathrm{id}, T)(x_1, y_1), T(x_1, y_1)) \notin V$$

since otherwise $M \circ (\mathrm{id}, T)(x_1, y_1))$ would not be a midpoint of (x_1, y_1) and $T(x_1, y_1)$. Now if it would happen that $((x_1, y_1), M \circ (\mathrm{id}, T)(x_1, y_1)) \in H$, then

$$\begin{aligned} ||(x_1, y_1) - T(x_2, y_2)|| &< ||(x_1, y_1) - M \circ (\mathrm{id}, T)(x_1, y_1)|| \\ &\quad + ||M \circ (\mathrm{id}, T)(x_1, y_1) - T(x_2, y_2)|| \\ &= ||(x_1, y_1) - T(x_1, y_1)|| \end{aligned}$$

and similarly if $(M \circ (\mathrm{id}, T)(x_1, y_1), T(x_1, y_1)) \in H$, then

$$\begin{aligned} ||(x_2, y_2) - T(x_1, y_1)|| &< ||(x_2, y_2) - M \circ (\mathrm{id}, T)(x_1, y_1)|| \\ &\quad + ||M \circ (\mathrm{id}, T)(x_1, y_1) - T(x_1, y_1)|| \\ &= ||(x_1, y_1) - T(x_1, y_1)||. \end{aligned}$$

Both of these contradict (3.3) via (3.18). Thus we have either $((x_1,y_1), T(x_1,y_1)) \in D$ contradicting (iii), or we have $x_1 = T_1(x_1, y_1)$ in which case again

$$||(x_1, y_1) - T(x_2, y_2)|| < ||(x_1, y_1) - T(x_1, y_1)||$$

or

$$||(x_2, y_2) - T(x_1, y_1)|| < ||(x_1, y_1) - T(x_1, y_1)||$$

contradict (3.3). On the other hand, if $T_1(x_2, y_2) = T_1(x_1, y_1)$, we have $y_2 < y_1$, $x_1 = x_2$ and $T_2(x_1, y_1) < T_2(x_2, y_2)$ contradicting (3.6). □

Now we are in a position to estimate the density of the midpoint measure. Recall the definition of Ent_N:

$$\mathrm{Ent}_N(\rho\mathfrak{m}) = -\int_X \rho^{1-\frac{1}{N}}\,\mathrm{d}\mathfrak{m}.$$

Proposition 3.8. *Let $\mu_0, \mu_1 \in \mathscr{P}(E)$ with $\mu_0, \mu_1 \ll \mathscr{L}_2$. Then for all $N \geq 4$ we have*

$$\mathrm{Ent}_N(\mu_{\frac{1}{2}}) \leq \frac{1}{2}\left(\mathrm{Ent}_N(\mu_0) + \mathrm{Ent}_N(\mu_1)\right),$$

where $\mu_{\frac{1}{2}} = (M \circ (\mathrm{id}, T))_\sharp \mu_0$ with T being the optimal map from Proposition 3.2.

Proof. We will show that for μ_0-almost every $(x, y) \in E$ we have

$$\rho_{\frac{1}{2}}\left(M((x, y), T(x, y))\right)^{-\frac{1}{4}} \geq \frac{1}{2}\left(\rho_0((x, y))^{-\frac{1}{4}} + \rho_1(T(x, y))^{-\frac{1}{4}}\right), \tag{3.19}$$

where $\mu_0 = \rho_0\mathscr{L}_2$, $\mu_1 = \rho_1\mathscr{L}_2$ and $\mu_{\frac{1}{2}} = (M \circ (\mathrm{id}, T))_\sharp\mu_0 = \rho_{\frac{1}{2}}\mathscr{L}_2$. The claim of the Proposition then follows by Hölder's inequality and integration.

By Lemma 3.7 the mapping $M \circ (\mathrm{id}, T)$ is essentially one-to-one. Our claim (3.19) will therefore follow via the Jacobian identity (2.5) if we are able to show that

$$J_{M\circ(\mathrm{id},T)}(x, y)^{\frac{1}{4}} \geq \frac{1}{2}\left(1 + J_T(x, y)^{\frac{1}{4}}\right) \tag{3.20}$$

holds μ_0-almost everywhere.

By Lemma 3.4 we have μ_0-almost everywhere in $H \cup V$ that T_1 is locally approximately constant in y, $\frac{\partial T_1}{\partial x} \geq 0$ and $\frac{\partial T_2}{\partial y} \geq 0$. Thus μ_0-almost everywhere in $H \cup V$ we can write, using Proposition 2.4,

$$J_T(x, y) = \frac{\partial T_1}{\partial x}\frac{\partial T_2}{\partial y}. \tag{3.21}$$

For the density $\rho_{\frac{1}{2}}$ we will need to estimate the Jacobian determinant of the mapping $M \circ (\mathrm{id}, T)$. Recall that the mapping is given by

$$(M \circ (\mathrm{id}, T))(x, y) = \left(\frac{x + T_1}{2}, \frac{y + T_2}{2}\right),$$

if $((x, y), T(x, y)) \notin H$, and by

$$(M \circ (\mathrm{id}, T))(x, y) = \Bigg(\frac{x + T_1}{2}, \max \Bigg\{ \frac{y + T_2}{2}, \frac{S(x) + S(T_1)}{2} + \frac{1}{4} \left(\sqrt{y - S(x)} + \sqrt{T_2 - S(T_1)} \right)^2 + (x - T_1)^2 \Bigg\} \Bigg),$$

if $((x, y), T(x, y)) \in H$.

Again by Lemma 3.4 we have μ_0-almost everywhere

$$J_{M \circ (\mathrm{id}, T)}(x, y) = \frac{1}{2} \left(1 + \frac{\partial T_1}{\partial x} \right) \cdot \frac{1}{2} \left(1 + \frac{\partial T_2}{\partial y} \right), \tag{3.22}$$

if $((x, y), T(x, y)) \in V$ or if $((x, y), T(x, y)) \in H$ and

$$\frac{y + T_2}{2} \geq \frac{S(x) + S(T_1)}{2} + \frac{1}{4} \left(\sqrt{y - S(x)} + \sqrt{T_2 - S(T_1)} \right)^2 + (x - T_1)^2, \tag{3.23}$$

and

$$J_{M \circ (\mathrm{id}, T)}(x, y) = \frac{1}{2} \left(1 + \frac{\partial T_1}{\partial x} \right) \cdot \frac{1}{4} \left(1 + \frac{\partial T_2}{\partial y} + \sqrt{\frac{T_2 - S(T_1)}{y - S(x)}} + \sqrt{\frac{y - S(x)}{T_2 - S(T_1)} \frac{\partial T_2}{\partial y}} \right), \tag{3.24}$$

if $((x, y), T(x, y)) \in H$ and (3.23) does not hold.

Let us check (3.20) in the case (3.24) holds. The case where (3.22) holds follows easily and the case $((x, y), T(x, y)) \in D$ will be considered at the end of the proof. First observe that

$$1 + \frac{\partial T_1}{\partial x} \geq \frac{1}{2} \left(1 + \sqrt{\frac{\partial T_1}{\partial x}} \right)^2$$

and

$$\sqrt{\frac{T_2 - S(T_1)}{y - S(x)}} + \sqrt{\frac{y - S(x)}{T_2 - S(T_1)} \frac{\partial T_2}{\partial y}} \geq 2 \sqrt{\frac{\partial T_2}{\partial y}}.$$

Therefore

$$J_{M \circ (\mathrm{id}, T)}(x, y) \geq \frac{1}{16} \left(1 + \sqrt{\frac{\partial T_1}{\partial x}} \right)^2 \left(1 + \sqrt{\frac{\partial T_2}{\partial y}} \right)^2.$$

Now, in order to obtain (3.20) it is then sufficient to have

$$\left(1 + \sqrt{\frac{\partial T_1}{\partial x}} \right) \left(1 + \sqrt{\frac{\partial T_2}{\partial y}} \right) \geq \left(1 + \left(\frac{\partial T_1}{\partial x} \frac{\partial T_2}{\partial y} \right)^{\frac{1}{4}} \right)^2,$$

which immediately follows from

$$\left(\left(\frac{\partial T_1}{\partial x}\right)^{\frac{1}{4}} - \left(\frac{\partial T_2}{\partial y}\right)^{\frac{1}{4}}\right)^2 \geq 0.$$

Let us then consider the case $((x, y), T(x, y)) \in D$. By changing to coordinates $\tilde{x} = \frac{1}{\sqrt{2}}(x + y)$, $\tilde{y} = \frac{1}{\sqrt{2}}(x - y)$ we may assume that either $\tilde{T}_1(\tilde{x}, \tilde{y}) - \tilde{x}$ or $\tilde{T}_2(\tilde{x}, \tilde{y}) - \tilde{y}$ (in the new coordinates) is constant. Assuming the first, we have

$$\frac{\partial \tilde{T}_1}{\partial \tilde{x}} = 1 \qquad \text{and} \qquad \frac{\partial \tilde{T}_1}{\partial \tilde{y}} = 0$$

and hence by Proposition 2.4

$$J_{\tilde{T}}(\tilde{x}, \tilde{y}) = \frac{\partial \tilde{T}_2}{\partial \tilde{y}},$$

which is non-negative μ_0-almost everywhere in D by cyclical monotonicity (3.3), and

$$J_{\tilde{M}\circ(\mathrm{id},\tilde{T})}(\tilde{x}, \tilde{y}) = \frac{1}{2}\left(1 + \frac{\partial \tilde{T}_1}{\partial \tilde{x}}\right) \cdot \frac{1}{2}\left(1 + \frac{\partial \tilde{T}_2}{\partial \tilde{y}}\right) = \frac{1}{2}\left(1 + J_{\tilde{T}}(\tilde{x}, \tilde{y})\right)$$

leading to (3.20). □

Proposition 3.8 then gives the $\mathsf{CD}(0, 4)$ condition in E. We also justify here the initial reductions.

Theorem 3.9. *The space* $(E, ||\cdot||_\infty, \mathscr{L}_2|_E)$ *satisfies* $\mathsf{CD}(0, 4)$.

Proof. We have to show that for any $\mu_0, \mu_1 \in \mathscr{P}(E)$ there exists a geodesic $(\mu_t) \subset \mathscr{P}(E)$ along which we have the estimate

$$\mathrm{Ent}_N(\mu_t) \leq (1 - t)\mathrm{Ent}_N(\mu_0) - t\mathrm{Ent}_N(\mu_1) \tag{3.25}$$

for all $N \geq 4$ and $t \in (0, 1)$.

Let us first show that we can obtain this for $t = \frac{1}{2}$. Take $\epsilon > 0$ and consider the approximated measures $\mu_{0,\epsilon} = \rho_{0,\epsilon}\mathscr{L}_2$, $\mu_{1,\epsilon} = \rho_{1,\epsilon}\mathscr{L}_2$ that are obtained from the measures μ_0 and μ_1 by setting

$$\rho_{i,\epsilon} = \frac{\mu_i\left(E \cap [n\epsilon, (n + 1)\epsilon) \times [m\epsilon, (m + 1)\epsilon)\right)}{\mathscr{L}_2\left(E \cap [n\epsilon, (n + 1)\epsilon) \times [m\epsilon, (m + 1)\epsilon)\right)}$$

on

$$E \cap [n\epsilon, (n + 1)\epsilon) \times [m\epsilon, (m + 1)\epsilon)$$

for every $n, m \in \mathbb{Z}, i = 0, 1$. If necessary, we can move the grid slightly so that

$$E \cap [n\epsilon, (n+1)\epsilon) \times [m\epsilon, (m+1)\epsilon) \neq \emptyset \\ \iff \mathscr{L}_2 \left(E \cap [n\epsilon, (n+1)\epsilon) \times [m\epsilon, (m+1)\epsilon)\right) > 0$$

for every $n, m \in \mathbb{Z}$.

Now $\mathrm{Ent}_N(\mu_{i,\epsilon}) \leq \mathrm{Ent}_N(\mu_i)$ for all $N > 1$ by Jensen's inequality, $W_2(\mu_{i,\epsilon}, \mu_i) \leq \epsilon$ and $\mu_{i,\epsilon} \ll \mathscr{L}_2$. From $\mu_{0,\epsilon}$ to $\mu_{1,\epsilon}$ there exists an optimal map T given by Proposition 3.2 and by Proposition 3.8 we get

$$\mathrm{Ent}_N(\mu_{\frac{1}{2},\epsilon}) \leq \frac{1}{2}\left(\mathrm{Ent}_N(\mu_{0,\epsilon}) + \mathrm{Ent}_N(\mu_{1,\epsilon})\right) \leq \frac{1}{2}\left(\mathrm{Ent}_N(\mu_0) + \mathrm{Ent}_N(\mu_1)\right),$$

with $\mu_{\frac{1}{2},\epsilon} = (M \circ (\mathrm{id}, T))_\sharp \mu_{0,\epsilon}$. Lemma 3.5 guarantees that $\mu_{\frac{1}{2},\epsilon} \in \mathrm{Mid}(\mu_{0,\epsilon}, \mu_{1,\epsilon})$ and Lemma 3.6 that $\mu_{\frac{1}{2},\epsilon} \in \mathscr{P}(E)$. Letting $\epsilon \downarrow 0$ along a subsequence we find a weak limit measure $\mu_{\frac{1}{2}} \in \mathrm{Mid}(\mu_0, \mu_1)$ satisfying

$$\mathrm{Ent}_N(\mu_{\frac{1}{2}}) \leq \frac{1}{2}\left(\mathrm{Ent}_N(\mu_0) + \mathrm{Ent}_N(\mu_1)\right)$$

for all $N \geq 4$ by the lower semi-continuity of the entropies Ent_N.

Now that we have (3.25) at $t = \frac{1}{2}$ we can continue by taking midpoints between μ_0 and $\mu_{\frac{1}{2}}$, and between $\mu_{\frac{1}{2}}$ and μ_1 and in this way obtain (3.25) at $t = \frac{1}{4}$ and $t = \frac{3}{4}$. Continuing iteratively we get (3.25) for a dense set of times. Finally, by the lower semi-continuity of Ent_N we have (3.25) for all t, the measures μ_t being obtained as weak limits of μ_s as $s \to t$ along the dyadic time points. □

3.6. Failure of the global $\mathsf{CD}(K, \infty)$ condition

Finally, let us show the calculation implying that the space X does not globally satisfy $\mathsf{CD}(K, N)$. Since, given any $K \in \mathbb{R}$ and $N \in [1, \infty)$, the $\mathsf{CD}(K, N)$ condition implies the $\mathsf{CD}(K, \infty)$ condition, it suffices to check the case $N = \infty$.

Theorem 3.10. *Given $K \in \mathbb{R}$, the space $(X, ||\cdot||_\infty, \mathscr{L}_2|_X)$ can be constructed in such a way that it does not satisfy $\mathsf{CD}(K, \infty)$.*

Proof. Let $A_0, A_1 \subset X$ be two rectangles that are identical (up to a translation in the horizontal direction) and are located on the opposite sides of the thin part of the space X. Define $\mu_0 = \frac{1}{\mathscr{L}_2(A_0)}\mathscr{L}_2|_{A_0}$ and $\mu_1 = \frac{1}{\mathscr{L}_2(A_1)}\mathscr{L}_2|_{A_1}$. See Figure 3.3 for an illustration. Since A_1 is a translation of A_0 in the horizontal direction by some distance l, every optimal transport between μ_0 and μ_1 transports infinitesimal measures by a constant distance l.

Suppose that the space $(X, ||\cdot||_\infty, \mathscr{L}_2|_X)$ satisfies $\mathsf{CD}(K, \infty)$. Then there exists $\mu_{\frac{1}{2}} = \rho_{\frac{1}{2}}\mathscr{L}_2 \in \mathrm{Mid}(\mu_0, \mu_1)$ satisfying

$$\mathrm{Ent}_\infty(\mu_{\frac{1}{2}}) = \int \rho_{\frac{1}{2}} \log \rho_{\frac{1}{2}} \, \mathrm{d}\mathscr{L}_2 \leq -\log \mathscr{L}_2(A_0) - \frac{K}{8}l^2.$$

On the other hand by Jensen's inequality

$$\mathrm{Ent}_\infty(\mu_{\frac{1}{2}}) \geq -\log \mathscr{L}_2(A),$$

where $A = \{x \in X \,:\, \rho_{\frac{1}{2}}(x) > 0\}$. Therefore

$$\mathscr{L}_2(A) \geq e^{\frac{K}{8}l^2}\mathscr{L}_2(A_0),$$

where the multiplicative factor $e^{\frac{K}{8}l^2}$ depends only on K and l. Therefore, by making the thin part of the space thin enough, the corresponding midpoint measure does not fit into the thin part and we have a contradiction. See again Figure 3.3. $\square$

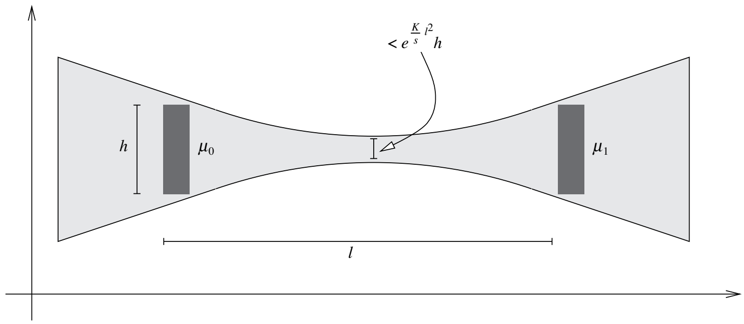

Figure 3.3. The space fails to satisfy the $\mathsf{CD}(K, \infty)$ condition. The measure μ_0 cannot be transported to μ_1 without the midpoint measure $\mu_{\frac{1}{2}}$ having too small support.

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