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A criterion for good reduction of Drinfeld modules and Anderson motives in terms of local shtukas

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Abstract. For an Anderson A-motive over a discretely valued field whose residue field has A-characteristic ε, we prove a criterion for good reduction in terms of its associated local shtuka at ε. This yields a criterion for good reduction of Drinfeld modules. Our criterion is the function-field analog of Grothendieck's [15, Proposition IX.5.13] and de Jong's [19, 2.5] criterion for good reduction of an Abelian variety over a discretely valued field with residue characteristic p in terms of its associated p-divisible group

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1. Introduction

We fix a finite field $\mathbb{F}$ with r elements and characteristic p. Let $\mathcal{C}$ be a smooth projective and geometrically irreducible curve over $\mathbb{F}$ with function field $Q = \mathbb{F}(\mathcal{C})$. Let $\infty \in \mathcal{C}$ be a closed point and let $A = \Gamma(\mathcal{C} \smallsetminus \{\infty\}, \mathcal{O}_{\mathcal{C}})$ be the $\mathbb{F}$-algebra of those rational functions on $\mathcal{C}$ which are regular outside ∞. For every $\mathbb{F}$-algebra R we let σ be the endomorphism of $A_R := A \otimes_{\mathbb{F}} R$ given by $\sigma := \mathrm{id}_A \otimes \mathrm{Frob}_{r,R} : a \otimes b \mapsto a \otimes b^r$ for $a \in A$ and $b \in R$.

Let o_L be a complete discrete valuation ring containing $\mathbb{F}$, with fraction field L, uniformizing parameter π, maximal ideal $\mathfrak{m}_L = (\pi)$ and residue field $\ell = o_L/\mathfrak{m}_L$. We assume that ℓ is a finite field extension of ℓ^p. This is equivalent to saying that ℓ has a finite p-basis over ℓ^p in the sense of [7, Section V.13, Definition 1]. It holds for example if ℓ is perfect, or if ℓ is a finitely generated field. Since every Anderson A-motive over L can be defined over a finitely generated subfield of L our restriction on ℓ is not serious. Let $c^* : A \to o_L$ be a homomorphism of $\mathbb{F}$-algebras such that the kernel of the composition $A \to o_L \twoheadrightarrow \ell$ is a *maximal*

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ideal ε in A. We say that *the residue field ℓ has finite A-characteristic ε*. We do not assume that $c^* : A \to o_L$ is injective. So L can have either generic A-characteristic $\ker c^* = (0)$ or finite A-characteristic $\ker c^* = \varepsilon$. In the following we will consider various A_{o_L}-algebras. In all of them we consider the ideal generated by $\{a \otimes 1 - 1 \otimes c^*(a) : a \in A\} \subset A_{o_L}$. By abuse of notation we denote all these ideals by $\mathfrak{J}$.

By an *Anderson A-motive over L* we mean a pair $\underline{M} = (M, F_M)$ consisting of a locally free A_L-module M of finite rank, and an injective A_L-homomorphism $F_M : \sigma^* M \to M$ where $\sigma^* M := M \otimes_{A_L, \sigma} A_L$, such that $\operatorname{coker}(F_M)$ is a finite dimensional L-vector space and is annihilated by a power of $\mathfrak{J}$. We say that $\underline{M}$ has *good reduction over o_L* if there exists a locally free A_{o_L}-module $\mathcal{M}$ and an injective A_{o_L}-homomorphism $F_{\mathcal{M}} : \sigma^* \mathcal{M} \to \mathcal{M}$ such that $(\mathcal{M}, F_{\mathcal{M}}) \otimes_{A_{o_L}} A_L \cong \underline{M}$ and $\operatorname{coker}(F_{\mathcal{M}})$ is a finite free o_L-module which is annihilated by a power of $\mathfrak{J}$. We call $\underline{\mathcal{M}} = (\mathcal{M}, F_{\mathcal{M}})$ a *good model of $\underline{M}$*. In particular if $\underline{M} = \underline{M}(\phi)$ is the Anderson A-motive associated with a Drinfeld A-module ϕ over L, then $\underline{M}$ has good reduction if and only if ϕ has good reduction; see Proposition 4.10.

Anderson A-motives are function-field analogs of Abelian varieties. For an Abelian variety $\mathcal{A}$ over a discretely valued field K with residue field of characteristic p there are criteria for good reduction in terms of local data. For a prime number $l \neq p$ the criterion of Néron-Ogg-Shavarevich [22, Section 1, Theorem 1] states that $\mathcal{A}$ has good reduction if and only if the l-adic Tate module $T_l \mathcal{A}$ of $\mathcal{A}$ is unramified as a $\operatorname{Gal}(K^{\mathrm{alg}}/K)$-representation. At the prime p the criterion of Grothendieck [15, Proposition IX.5.13] (for $\operatorname{char}(K) = 0$), respectively de Jong [19, 2.5] (for $\operatorname{char}(K) = p$) states that $\mathcal{A}$ has good reduction if and only if the Barsotti-Tate group $\mathcal{A}[p^\infty]$ has good reduction.

These criteria have function-field analogs for Anderson A-motives. The analog of the Néron-Ogg-Shavarevich-criterion was proved by Gardeyn [12, Theorem 1.1]. In this article we simultaneously prove the analog of Grothendieck's and de Jong's criterion. Here the function-field analogs of Barsotti-Tate groups are local shtukas [17, Section 2.1] which are defined as follows. Let $A_{o_L,(\varepsilon,\pi)}$ be the (ε, π)-adic completion of A_{o_L}. An (*effective*) *local shtuka at ε over o_L* is a pair $\underline{\hat{M}} = (\hat{M}, F_{\hat{M}})$ consisting of a finite free $A_{o_L,(\varepsilon,\pi)}$-module $\hat{M}$ and an injective $A_{o_L,(\varepsilon,\pi)}$-homomorphism $F_{\hat{M}} : \sigma^* \hat{M} \to \hat{M}$ such that $\operatorname{coker}(F_{\hat{M}})$ is a finite free o_L-module and is annihilated by a power of $\mathfrak{J}$. The local shtuka associated with a good model $\underline{\mathcal{M}}$ of an Anderson A-motive is $\underline{\hat{M}}(\underline{\mathcal{M}}) := \underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,(\varepsilon,\pi)}$. Strictly speaking effective local shtukas are the function field analogs of the F-crystals of Barsotti-Tate groups. The analogs of the latter are called *ε-divisible local Anderson-modules* and their category is equivalent to the category of effective local shtukas; see [18] for more details. Our analog of Grothendieck's and de Jong's reduction criterion is now the following:

Corollary 6.6. *Let $\underline{M}$ be an Anderson A-motive over L. Then $\underline{M}$ has good reduction over o_L if and only if there is an effective local shtuka $\underline{\hat{M}}$ at ε over o_L and an isomorphism $\underline{M} \otimes_{A_L} A_{o_L,(\varepsilon,\pi)}[1/\pi] \cong \underline{\hat{M}} \otimes_{A_{o_L,(\varepsilon,\pi)}} A_{o_L,(\varepsilon,\pi)}[1/\pi]$.*

(In the body of the text we prove a slightly stronger statement.) This applies in particular if $\underline{M}$ is the Anderson A-motive associated with a Drinfeld module ϕ over L to give a criterion for good reduction of ϕ in terms of its associated local shtuka. The reformulation of this criterion in terms of the ε-divisible local Anderson-module of ϕ is given in [18].

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2. The base rings

Let o_L be an equi-characteristic complete discrete valuation ring containing the finite field $\mathbb{F}$, with quotient field $L = \mathrm{Frac}(o_L)$ and residue field $\ell = o_L/\mathfrak{m}_L$, where $\mathfrak{m}_L \subset o_L$ is the maximal ideal of o_L. We assume that ℓ is a *finite* field extension of $\ell^p := \{b^p : b \in \ell\}$. We fix a uniformizer $\pi = \pi_L$ of o_L and sometimes identify o_L with $\ell[\![\pi]\!]$. Let $v = v_\pi = \mathrm{ord}_\pi(\cdot)$ be the discrete valuation on L normalized by $v(\pi) = 1$.

We assume that there is an o_L-valued point $c \in \mathcal{C}(o_L)$ such that the corresponding $\mathbb{F}$-morphism $c : \mathrm{Spec}(o_L) \to \mathcal{C}$ factors via $\mathcal{C} \smallsetminus \{\infty\} \subset \mathcal{C}$. Such a datum corresponds to a homomorphism of $\mathbb{F}$-algebras $c^* : A \to o_L$ which we call the *characteristic map*. We further assume that the closed point $V(\pi) \subset \mathrm{Spec}(o_L)$ is mapped to a closed point ε of $\mathrm{Spec}(A) \subset \mathcal{C}$. The latter is the kernel of the composition $A \to o_L \twoheadrightarrow \ell$. So, in accordance with Drinfeld's terminology [9], we call ε the *residue characteristic* or *residual characteristic place of Q*. By continuity, the characteristic map $c^* : A \to o_L$ factors through a morphism of complete discrete valuation rings $A_\varepsilon \to o_L$ where A_ε is the completion of A at the characteristic place ε. Note that $A_\varepsilon \to o_L$ is injective if c^* is injective, and factors through A/ε if c^* is not injective.

Remark 2.1. Since A is a Dedekind domain there is a power ε^m which is a principal ideal in A. We fix a generator t of ε^m and frequently use the finite flat monomorphism of $\mathbb{F}$-algebras $\iota : \mathbb{F}[z] \to A$, $z \mapsto t$.

For any $\mathbb{F}$-algebra R we abbreviate $A_R := A \otimes_{\mathbb{F}} R$. In particular, $A_{o_L} \subset A_L$ is a noetherian integral domain, and by virtue of the equality $A_\ell \cong A_{o_L}/\pi A_{o_L}$ it follows that $\pi \in o_L$ is a prime element of A_{o_L}.

Definition 2.2. Let $A_{o_L,\pi}$ (respectively, $A_{o_L,(\varepsilon,\pi)}$) be the completion of the o_L-algebra A_{o_L} for the π-adic topology (respectively, the (ε, π)-adic topology).

By Krull's Theorem ([8], III.3.2), the ring A_{o_L} is separated for both the π-adic and the (ε, π)-adic topology. The topological o_L-algebra $A_{o_L,\pi}$ is admissible in the sense of Raynaud, *i.e.* it is of topologically finite presentation and has no π-torsion. In particular, the L-algebra $A_{o_L,\pi}[1/\pi]$ is affinoid in the sense of rigid analytic geometry; see [4–6].

For example if $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}}$ and $A = \mathbb{F}[z]$ then we have $A_{o_L} = o_L[z]$ and correspondingly $A_L = L[z]$. Let us specify that $\varepsilon = z\mathbb{F}[z]$. Our choice of a uniformizer π gives rise to an identification $o_L = \ell[\![\pi]\!]$. Consequently $o_L[\![z]\!] = \ell[\![\pi]\!][\![z]\!] = \ell[\![\pi, z]\!] = A_{o_L,(\varepsilon,\pi)}$. On the other hand, the π-adic completion of $o_L[z]$ equals $o_L\langle z\rangle := \{\sum_{i=0}^{\infty} b_i z^i : v(b_i) \to \infty (i \to \infty)\}$, and since $L\langle z\rangle = o_L\langle z\rangle \otimes_{o_L} L$, we may view $A_{o_L,\pi}[1/\pi]$ as a replacement, for general $\mathcal{C}$, of the Tate algebra $L\langle z\rangle$ of strictly convergent power series in one indeterminate z over L, which serves as coordinate ring for the one-dimensional affinoid unit ball in rigid analytic geometry.

There is a natural embedding $A_L \to A_{o_L,\pi}[1/\pi]$ which, for general $\mathcal{C}$, replaces the completion homomorphism $L[z] \to L\langle z\rangle$, and which itself can be regarded as a completion map with respect to the L-algebra norm-topology on the *reduced* affinoid L-algebra $A_{o_L,\pi}[1/\pi]$ and its restriction on A_L; see [4, Section 1.4, Proposition 19]. Note that the canonical homomorphism $A_{o_L} \to A_{o_L,(\varepsilon,\pi)}$ factors uniquely via $A_{o_L,\pi}$, where the induced map $A_{o_L,\pi} \to A_{o_L,(\varepsilon,\pi)}$ identifies $A_{o_L,(\varepsilon,\pi)}$ with the $(\varepsilon,\pi)A_{o_L,\pi}$-adic completion of $A_{o_L,\pi}$. Since $A_{o_L,\pi}$ is a regular integral domain, it is $(\varepsilon,\pi)A_{o_L,\pi}$-adically separated by Krull's theorem and $A_{o_L,\pi} \to A_{o_L,(\varepsilon,\pi)}$ is injective and flat.

Recall that there is a finite flat monomorphism of $\mathbb{F}$-algebras $\iota : \mathbb{F}[z] \to A$ which identifies the indeterminate z with the generator $t \in A$ of ε^m chosen in Remark 2.1. The o_L-algebra homomorphism $\iota \otimes \mathrm{id} : o_L[z] \to A_{o_L}$, $\sum_\nu a_\nu z^\nu \mapsto \sum_\nu t^\nu \otimes a_\nu$, is finite flat, so that we obtain finite flat maps

$$o_L\langle z\rangle \to A_{o_L,\pi}, \quad L\langle z\rangle \to A_{o_L,\pi}[1/\pi], \quad o_L[\![z]\!] \to A_{o_L,(t,\pi)}, \quad \ell[z] \to A_\ell. \tag{2.1}$$

Here the (t,π)-adic completion $A_{o_L,(t,\pi)}$ of A_{o_L} equals $A_{o_L,(\varepsilon,\pi)}$ since $(\varepsilon,\pi)^m \subset (\varepsilon^m,\pi) = (t,\pi)$ in A_{o_L}.

Lemma 2.3. *If $A_{o_L,\varepsilon}$ denotes the ε-adic completion of A_{o_L}, the canonical map $A_{o_L,\varepsilon} \to A_{o_L,(\varepsilon,\pi)}$ is an isomorphism.* □

3. Frobenius modules

The r-Frobenius $\mathrm{Frob}_r : o_L \to o_L$, $x \mapsto x^r$, gives rise to an endomorphism

$$\sigma = \mathrm{id}_A \otimes \mathrm{Frob}_r : A_{o_L} \to A_{o_L}, \quad a \otimes x \mapsto a \otimes x^r,$$

which extends to give a map $\mathrm{id}_A \otimes \mathrm{Frob}_{r,L} : A_L \to A_L$ again denoted by σ. On the other hand, reducing mod π gives $\bar{\sigma} = \mathrm{id}_A \otimes \mathrm{Frob}_{r,\ell} : A_\ell \to A_\ell$. The latter is a finite flat endomorphism of the Dedekind domain A_ℓ, because ℓ is finite over ℓ^p. The map $\sigma : A_{o_L} \to A_{o_L}$ is π-adically and (ε,π)-adically continuous and therefore extends to give endomorphisms $A_{o_L,\pi} \to A_{o_L,\pi}$ and $A_{o_L,(\varepsilon,\pi)} \to A_{o_L,(\varepsilon,\pi)}$, again denoted by σ.

Lemma 3.1. *In the commutative diagram*

$$\begin{array}{ccccc} A_{o_L} & \longrightarrow & A_{o_L,\pi} & \longrightarrow & A_{o_L,(\varepsilon,\pi)} \\ \sigma\downarrow & & \sigma\downarrow & & \sigma\downarrow \\ A_{o_L} & \longrightarrow & A_{o_L,\pi} & \longrightarrow & A_{o_L,(\varepsilon,\pi)} \end{array}$$

both squares are co-Cartesian, and the vertical arrows are finite flat.

We let the proof be preceded by the following:

Remark 3.2. Via the identification $o_L = \ell[\![\pi]\!]$, the r-Frobenius $\mathrm{Frob}_{r,o_L} : o_L \to o_L$ is mirrored by the map $\ell[\![\pi]\!] \to \ell[\![\pi]\!]$, $\sum_{\nu=0}^{\infty} a_\nu \pi^\nu \mapsto \sum_{\nu=0}^{\infty} a_\nu^r \pi^{r\nu}$. Choosing an ℓ^r-basis of ℓ and lifting it to a subset W of o_L, this implies $(\mathrm{Frob}_{r,o_L})_* o_L = \bigoplus_{i=0}^{r-1} \bigoplus_{w\in W} o_L\, w\pi^i$, so that $\mathrm{Frob}_{r,o_L} : o_L \to o_L$ is finite flat.

Proof of Lemma 3.1. By base change the remark implies that $\sigma = \mathrm{id}_A \otimes \mathrm{Frob}_{r,o_L} : A_{o_L} \to A_{o_L}$ is finite flat, and that $A_{o_L} \otimes_{\sigma, A_{o_L}} A_{o_L,\pi}$ is a finite flat $A_{o_L,\pi}$-module and hence equals the π-adic completion of the A_{o_L}-module $\sigma_* A_{o_L}$. If we let $\mathfrak{a} = \sigma(\pi A_{o_L}) A_{o_L} = \pi^r A_{o_L}$ and $\mathfrak{b} = \pi A_{o_L}$, we get $\mathfrak{b}^r = \mathfrak{a} \subset \mathfrak{b}$. Consequently, by [10, Lemma 7.14], the inverse systems $(A_{o_L}/\mathfrak{a}^n)_n$ and $(A_{o_L}/\mathfrak{b}^n)_n$ give the same limit, which shows that the square on the left is co-Cartesian, and that $\sigma : A_{o_L,\pi} \to A_{o_L,\pi}$ is finite flat. Similarly, we have $\sigma(\varepsilon,\pi)A_{o_L} = (\varepsilon,\pi^r) \subset (\varepsilon,\pi)$ as well as $(\varepsilon,\pi)^r \subset (\varepsilon,\pi^r)$, which proves that the displayed diagram qualifies $A_{o_L,(\varepsilon,\pi)}$ as tensor product $A_{o_L,(\varepsilon,\pi)} \otimes_{A_{o_L},\sigma} A_{o_L}$, and that $\sigma : A_{o_L,(\varepsilon,\pi)} \to A_{o_L,(\varepsilon,\pi)}$ is finite flat. □

Finally, note that the embedding of o_L-algebras $\iota \otimes \mathrm{id} : o_L[z] \to A_{o_L}$ commutes with $\sigma : A_{o_L} \to A_{o_L}$ and the r-Frobenius lift of $o_L[z]$, given by $o_L[z] \to o_L[z]$, $\sum_\nu a_\nu z^\nu \mapsto \sum_\nu a_\nu^r z^\nu$. Consequently, also the embeddings from (2.1) are Frobenius-equivariant.

Let B be an o_L-algebra together with a ring endomorphism $\sigma : B \to B$ such that σ and $\mathrm{Frob}_{r,o_L} : o_L \to o_L$ are compatible with the structure map $o_L \to B$. For example, B could be any of the base rings considered above.

Definition 3.3. We define the category $\mathrm{FMod}(B)$ of *Frobenius B-modules* (or simply *F-modules* over B) as follows:

- An object of $\mathrm{FMod}(B)$ is a pair $\underline{M} = (M, F)$ consisting of a B-module M which is locally free of finite rank, together with an *injective* B-linear map $F = F_M : \sigma^* M \to M$, where $\sigma^* M := M \otimes_{B,\sigma} B$.
- A *morphism* of Frobenius B-modules $(M, F_M) \to (N, F_N)$ is a B-linear map $\phi : M \to N$ between the underlying B-modules such that ϕ is *F-equivariant*, *i.e.* such that $\phi \circ F_M = F_N \circ \sigma^*\phi$. It is called an *isomorphism* if ϕ is an isomorphism of the underlying B-modules.

Let B' be a flat B-algebra together with a ring endomorphism $\sigma : B' \to B'$ extending the Frobenius lift of B, as explained before. Then the exact functor $\cdot \otimes_B B'$ from B-modules to B'-modules yields a functor $\mathrm{FMod}(B) \to \mathrm{FMod}(B')$. If the structure map $B \to B'$ is, in addition, injective then the induced functor on $\mathrm{FMod}(B)$ is faithful since, given a map $f : M \to N$ of finite projective B-modules, restricting its image $f \otimes \mathrm{id} : M \otimes_B B' \to N \otimes_B B'$ to M gives back f. In particular, we obtain a natural commutative diagram of categories and faithful functors

$$\begin{array}{ccccc} \mathrm{FMod}(A_{o_L}) & \longrightarrow & \mathrm{FMod}(A_{o_L,\pi}) & \longrightarrow & \mathrm{FMod}(A_{o_L,(\varepsilon,\pi)}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{FMod}(A_L) & \longrightarrow & \mathrm{FMod}(A_{o_L,\pi}[1/\pi]) & \longrightarrow & \mathrm{FMod}(A_{o_L,(\varepsilon,\pi)}[1/\pi]) \end{array}$$

Slightly abusing notation, we agree to write $\underline{M} \otimes_B B'$ for $(M \otimes_B B', F_M \otimes \mathrm{id}_{B'})$, whenever $\underline{M} = (M, F_M)$.

4. Anderson motives

Let $\mathfrak{J} \subset A_{o_L}$ be the ideal generated by $a \otimes 1 - 1 \otimes c^*(a)$ for all $a \in A$. For example, if $\mathcal{C} = \mathbb{P}^1_{\mathbb{F}}$ and $A = \mathbb{F}[z]$, then $\mathfrak{J} = (z - \zeta) \subset o_L[z]$ where $\zeta = c^*(z)$. Note that the convention introduced in Remark 2.1 that $(z) = \varepsilon^m$ implies $\zeta \in \mathfrak{m}_L$. So $\zeta = 0$ if c^* is not injective. By abuse of notation we denote the ideal generated by $\mathfrak{J}$ in any A_{o_L}-algebra again by $\mathfrak{J}$. We consider the following variant of Anderson's [1] t-motives.

Definition 4.1. An *Anderson A-motive over L* is an object $\underline{M} = (M, F_M) \in \mathrm{FMod}(A_L)$ such that $\mathrm{coker}(F_M)$ is a finite-dimensional L-vector space and is annihilated by a power of $\mathfrak{J}$. A *morphism* of Anderson A-motives is defined as a morphism inside $\mathrm{FMod}(A_L)$.

Since $\mathrm{Spec}(A_L)$ is of finite type over L, one can consider its rigid analytification $\mathrm{Spec}(A_L)^{\mathrm{an}}$; see [4, 5, 11]. In accordance with [2], we denote this rigid analytic L-space by $\mathfrak{A}(\infty)$. On the other hand, the formal completion of the o_L-scheme $X = \mathrm{Spec}(A_{o_L})$ along its special fiber $V(\pi)$ leads to the formal o_L-scheme $\mathfrak{X} = \mathrm{Spf}(A_{o_L,\pi})$; see [14, $\mathrm{I}_{\mathrm{new}}$, I.10.8.3]. Its associated rigid analytic space $\mathfrak{X}_{\mathrm{rig}}$ ([4, 11]) is given by the affinoid L-space $\mathfrak{A}(1) := \mathrm{Sp}(A_{o_L,\pi}[1/\pi])$. This space can be regarded as the unit disc of the rigid analytic space $\mathfrak{A}(\infty)$ as it corresponds to "radius of convergence 1", hence the notation.

We study the following instance of rigid analytic τ-sheaves over $A_{o_L,\pi}[1/\pi]$, in the sense of [2].

Definition 4.2. An *analytic Anderson $A(1)$-motive* over L is an object $\underline{M} = (M, F_M) \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ such that $\mathrm{coker}(F_M)$ is a finite-dimensional L-vector space and is annihilated by a power of $\mathfrak{J}$. A *morphism* of analytic Anderson $A(1)$-motives is defined as a morphism in the category $\mathrm{FMod}(A_{o_L,\pi}[1/\pi])$.

Here the prefix "$A(1)$-" indicates that we are considering an analytic variant of Anderson A-motives over the rigid analytic "unit disc" $\mathfrak{A}(1)$ in $\mathrm{Spec}(A_L)$.

Proposition 4.3. *The natural functor* $\mathrm{FMod}(A_L) \to \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$, $\underline{M} \mapsto \underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ *restricts to a functor* (*Anderson A-motives over L*) $\to$ (*analytic Anderson $A(1)$-motives over L*). □

Definition 4.4. (a) Let $\underline{M}_L \in \mathrm{FMod}(A_L)$ be an F-module over A_L. A *model* of $\underline{M}_L$ is a pair $(\underline{\mathcal{M}}, \alpha)$ consisting of an object $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$ and an isomorphism $\alpha : \underline{M}_L \xrightarrow{\sim} \underline{\mathcal{M}} \otimes_{A_{o_L}} A_L$ inside $\mathrm{FMod}(A_L)$.
(b) Let $\underline{M}_L \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ be an F-module over $A_{o_L,\pi}[1/\pi]$. A *(formal) model* of $\underline{M}_L$ is a pair $(\underline{\mathcal{M}}, \alpha)$ consisting of an object $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L,\pi})$ and an isomorphism $\alpha : \underline{M}_L \xrightarrow{\sim} \underline{\mathcal{M}} \otimes_{A_{o_L,\pi}} A_{o_L,\pi}[1/\pi]$ inside $\mathrm{FMod}(A_{o_L,\pi}[1/\pi])$.
(c) In both cases a *morphism* of models $\beta : (\underline{\mathcal{M}}, \alpha) \to (\underline{\mathcal{M}}', \alpha')$ is an isomorphism $\beta : \underline{\mathcal{M}} \xrightarrow{\sim} \underline{\mathcal{M}}'$ of F-modules satisfying $\alpha' = \beta[1/\pi] \circ \alpha$. In particular the sets $\mathrm{Hom}\big((\underline{\mathcal{M}}, \alpha), (\underline{\mathcal{M}}', \alpha')\big)$ contain at most one element.
We will sometimes drop the α from the notation and simply speak of $\underline{\mathcal{M}}$ as a model of $\underline{M}_L$.

For every $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$, respectively $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L,\pi})$ we can consider the reduction $\underline{\mathcal{M}} \otimes_{A_{o_L}} A_\ell$, respectively $\underline{\mathcal{M}} \otimes_{A_{o_L,\pi}} A_\ell$. Note, however, that this does *not* induce a functor from $\mathrm{FMod}(A_{o_L})$, respectively $\mathrm{FMod}(A_{o_L,\pi})$ to $\mathrm{FMod}(A_\ell)$, since the induced F-map need not be injective. This circumstance lies at the origin of our study of good models:

Definition 4.5. Let $\underline{\mathcal{M}}$ be a model of an F-module $\underline{M}_L$ over A_L, respectively over $A_{o_L,\pi}[1/\pi]$. Then $\underline{\mathcal{M}}$ is called a *good* model if $\underline{\mathcal{M}}/\pi\underline{\mathcal{M}}$ is an F-module over A_ℓ, *i.e.* if the induced A_ℓ-linear map

$$\bar{\sigma}^*(\mathcal{M}/\pi\mathcal{M}) = (\mathcal{M}/\pi\mathcal{M}) \otimes_{A_\ell,\bar{\sigma}} A_\ell \to \mathcal{M}/\pi\mathcal{M}$$

is injective.

If $\underline{M}_L$ is an (analytic) Anderson motive there is an alternative notion of good reduction as follows.

Definition 4.6. Let $\underline{\mathcal{M}}$ be a model of an Anderson A-motive $\underline{M}_L$, respectively of an analytic Anderson $A(1)$-motive $\underline{M}_L$. Then $\underline{\mathcal{M}}$ is called a *good model in the strong sense* if $\mathrm{coker}(F_{\mathcal{M}})$ is a finite free o_L-module and is annihilated by $\mathfrak{J}^d$, for some $d \geq 0$. In this case we also say that $\underline{\mathcal{M}}$ has *good reduction over o_L*.

Theorem 4.7. *Let $\underline{\mathcal{M}}$ be a model of an Anderson A-motive, respectively of an analytic Anderson $A(1)$-motive $\underline{M}_L$. Then $\underline{\mathcal{M}}$ is a good model in the weak sense of Definition 4.5 if and only if it is a good model in the strong sense of Definition* 4.6.

Proof. Since $\sigma^*\mathcal{M}$ is locally free over A_{o_L}, respectively over $A_{o_L,\pi}$, the natural map $\sigma^*\mathcal{M} \to \sigma^* M_L$ is injective and hence $F_{\mathcal{M}}$ is injective because F_{M_L} is. We obtain a short eqact sequence

$$0 \longrightarrow \sigma^*\mathcal{M} \xrightarrow{F_{\mathcal{M}}} \mathcal{M} \longrightarrow \mathrm{coker}(F_{\mathcal{M}}) \longrightarrow 0\,. \tag{4.1}$$

Let $\underline{\mathcal{M}}$ be a good model in the strong sense. Tensoring the short exact sequence (4.1) with ℓ over o_L and using that $\mathrm{coker}(F_{\mathcal{M}})$ is supposed to be free over o_L shows that the induced A_ℓ-linear map $\bar{\sigma}^*(\mathcal{M}/\pi\mathcal{M}) \to \mathcal{M}/\pi\mathcal{M}$ remains injective. So $\underline{\mathcal{M}}$ is a good model in the weak sense.

Conversely suppose that $\underline{\mathcal{M}}$ is a good model in the weak sense. This time tensoring (4.1) with ℓ over o_L yields

$$0 \longrightarrow \mathrm{Tor}_1^{o_L}\big(\mathrm{coker}F_{\mathcal{M}}, \ell\big) \longrightarrow \sigma^*\mathcal{M} \otimes_{o_L} \ell \xrightarrow{F_{\mathcal{M}}\otimes \mathrm{id}_\ell} \mathcal{M} \otimes_{o_L} \ell \longrightarrow \mathrm{coker}(F_{\mathcal{M}}) \otimes_{o_L} \ell \longrightarrow 0\,.$$

By assumption $F_{\mathcal{M}} \otimes \mathrm{id}_\ell$ is injective, and so $0 = \mathrm{Tor}_1^{o_L}\big(\mathrm{coker}F_{\mathcal{M}}, \ell\big) = \{x \in \mathrm{coker}(F_{\mathcal{M}}) : \pi x = 0\}$ and $\mathrm{coker}(F_{\mathcal{M}})$ is flat over o_L by [10, Corollary 6.3]. This implies $\mathrm{coker}(F_{\mathcal{M}}) \hookrightarrow \mathrm{coker}(F_{\mathcal{M}}) \otimes_{o_L} L = \mathrm{coker}(F_{M_L})$. Since $\mathrm{coker}(F_{M_L})$ is annihilated by $\mathfrak{J}^d$ for some d, the same is true for $\mathrm{coker}(F_{\mathcal{M}})$ which therefore is a finitely generated $A_{o_L}/\mathfrak{J}^d$-module, respectively $A_{o_L,\pi}/\mathfrak{J}^d$-module, and a fortiori a finitely generated o_L-module. Being flat, $\mathrm{coker}(F_{\mathcal{M}})$ is a finite free o_L-module. Thus $\mathcal{M}$ is a good model in the strong sense. □

Remark 4.8. In [13] Gardeyn develops a theory of semi-stable reduction of analytic Anderson $A(1)$-motives $\underline{M}_L$. He shows that after replacing L by a finite separable extension, $\underline{M}_L$ has a model $\underline{\mathcal{M}}$ such that the reduction $F_{\mathcal{M}} \otimes \mathrm{id}_\ell$ is not nilpotent [13, Proposition 3.3]. If $\overline{\underline{\mathcal{M}}}' \subset \underline{\mathcal{M}}/\pi\underline{\mathcal{M}}$ is the maximal Frobenius A_ℓ-submodule with injective $F_{\overline{\mathcal{M}}'}$, he further shows that the support of $\mathrm{coker}(F_{\overline{\mathcal{M}}'})$ is a finite set $S \subset \mathrm{Spec}\, A_\ell$. After removing S from $\mathfrak{A}(1) := \mathrm{Sp}(A_{o_L,\pi}[1/\pi])$ one can lift $\overline{\underline{\mathcal{M}}}'$ to an F-submodule $\underline{\mathcal{M}}' \subset \underline{\mathcal{M}}|_{\mathfrak{A}(1)\smallsetminus S}$ which has good reduction in the weak sense of Definition 4.5; see [13, Theorem 4.7]. As one sees from the following example, it is false in general that S is the zero locus of $\mathfrak{J}$ in $\mathrm{Spec}\, A_\ell$ and so we cannot expect that $\underline{\mathcal{M}}'$ has good reduction in the strong sense of Definition 4.6.

Let $A = \mathbb{F}[z]$ and $\zeta = c^*(z) \in \mathfrak{m}_L$. Then $\mathfrak{J} = (z-\zeta)$. Let $\mathcal{M} = o_L\langle z\rangle^{\oplus 2}$ and $F_{\mathcal{M}} = \begin{pmatrix} 0 & \pi(z-\zeta) \\ \pi & z-1 \end{pmatrix}$. Then $\underline{\mathcal{M}} = (\mathcal{M}, F_{\mathcal{M}})$ is a model of the analytic Anderson $A(1)$-motive $\underline{M}_L := \underline{\mathcal{M}} \otimes_{o_L} L$. The reduction $\underline{\mathcal{M}}/\pi\underline{\mathcal{M}} = \big(\ell[z]^{\oplus 2}, \begin{pmatrix} 0 & 0 \\ 0 & z-1 \end{pmatrix}\big)$ contains the maximal Frobenius A_ℓ-submodule $\overline{\underline{\mathcal{M}}}' = \ell[z] \cdot \binom{0}{1}$, whose Frobenius is $F_{\overline{\mathcal{M}}'} = z-1$. So $S = V(z-1) \neq V(z) = V(\mathfrak{J})$.

Proposition 4.9. *If $\underline{M}_L$ is an Anderson A-Motive over L having a (good) model $\underline{\mathcal{M}}$ then its analytification $\underline{M}_L \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ is an analytic Anderson $A(1)$-motive having the (good) model $\underline{\widehat{\mathcal{M}}} := \underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi}$ and the reduction $\underline{\widehat{\mathcal{M}}}/\pi\underline{\widehat{\mathcal{M}}}$ of $\underline{\widehat{\mathcal{M}}}$ is canonically isomorphic to the reduction $\underline{\mathcal{M}}/\pi\underline{\mathcal{M}}$ of $\underline{\mathcal{M}}$.*

Proof. The statement without the properties of being a good model is obvious. From the isomorphism $\underline{\widehat{\mathcal{M}}}/\pi\underline{\widehat{\mathcal{M}}} \xrightarrow{\sim} \underline{\mathcal{M}}/\pi\underline{\mathcal{M}}$ it follows that $\underline{\mathcal{M}}$ is a good model in the sense of Definition 4.5 if and only if $\underline{\widehat{\mathcal{M}}}$ is a good model in the sense of Definition 4.5. □

Let us also mention the following result of Gardeyn on good reduction of Drinfeld A-modules.

Proposition 4.10. *Let $\phi : A \to L[\tau]$ be a Drinfeld A-module over L; see* [9] *or* [21]. *Let $\underline{M} = \underline{M}(\phi)$ be the associated Anderson A-motive; see* [1, Section 4.1] *or* [12, Section 8.1]. *Then the following are equivalent:*

(i) *ϕ has good reduction over o_L,* i.e. *ϕ is isomorphic over L to a Drinfeld A-module $\psi : A \to L[\tau]$ satisfying $\psi(A) \subset o_L[\tau]$ such that the reduction $\overline{\psi} : A \to o_L[\tau] \twoheadrightarrow \ell[\tau]$ is a Drinfeld A-module over ℓ of the same rank as ψ and ϕ;*

(ii) *$\underline{M}$ has good reduction over o_L in the weak and strong senses of Definitions* 4.6 *and* 4.5.

Proof. Gardeyn [12, Theorem 8.1] proved that ϕ has good reduction over o_L if and only if $\underline{M}$ has a good model in the weak sense. So the proposition follows from Theorem 4.7. □

5. Local shtukas and analytic Anderson motives

Anderson A-motives can be viewed as function-field analogs of Abelian varieties. Barsotti-Tate groups, which can be associated with Abelian varieties over $\mathbb{Z}_p$-schemes, have effective local shtukas as function-field analogs.

Definition 5.1. An (*effective*) *local shtuka at ε over o_L* is an object $\underline{\hat{M}} = (\hat{M}, F_{\hat{M}}) \in \mathrm{FMod}(A_{o_L,(\varepsilon,\pi)})$ such that $\mathrm{coker}(F_{\hat{M}})$ is a finite free o_L-module and is annihilated by a power of $\mathfrak{J}$.

Remark 5.2. If the residue field $\mathbb{F}_\varepsilon = A/\varepsilon$ of ε is larger than $\mathbb{F}$, *i.e.*, if the degree $d_\varepsilon := [\mathbb{F}_\varepsilon : \mathbb{F}] > 1$, the ring $A_{o_L,(\varepsilon,\pi)}$ is not an integral domain but a product $A_{o_L,(\varepsilon,\pi)} = \prod_{i\in\mathbb{Z}/d_\varepsilon\mathbb{Z}} A_{o_L,(\varepsilon,\pi)}/\mathfrak{a}_i$ of integral domains. To describe this product decomposition, note that $A_{o_L,(\varepsilon,\pi)} = \varprojlim_n A_{o_L}/\varepsilon^n = \varprojlim_n (A/\varepsilon^n) \otimes_{\mathbb{F}} o_L = A_\varepsilon \widehat{\otimes}_{\mathbb{F}} o_L$. By Cohen's structure theorem $A_\varepsilon \cong \mathbb{F}_\varepsilon[\![z_\varepsilon]\!]$ for a uniformizer z_ε of A at ε. Then $\mathfrak{a}_i = (\alpha \otimes 1 - 1 \otimes c^*(\alpha)^{r^i} : \alpha \in \mathbb{F}_\varepsilon \subset A_\varepsilon)$, where we use that $c^* : A \to o_L$ factors through $c^* : A_\varepsilon \to o_L$. The factors $A_{o_L,(\varepsilon,\pi)}/\mathfrak{a}_i$ are isomorphic to $o_L[\![z_\varepsilon]\!]$ and hence are integral domains. They are cyclically permuted by σ because $\sigma(\mathfrak{a}_i) = \mathfrak{a}_{i+1}$. By [3, Proposition 8.8] the functor $(\hat{M}, F_{\hat{M}}) \mapsto (\hat{M}/\mathfrak{a}_0\hat{M}, (F_{\hat{M}})^{d_\varepsilon})$ is an equivalence between the category of effective local shtukas at ε over o_L as in Definition 5.1 and the category of pairs $(\hat{M}_0, \widetilde{F}_{\hat{M}})$ where $\hat{M}_0$ is a free module of finite rank over $A_{o_L,(\varepsilon,\pi)}/\mathfrak{a}_0$ and $\widetilde{F}_{\hat{M}} : (\sigma^{d_\varepsilon})^*\hat{M}_0 \to \hat{M}_0$ is injective with $\mathrm{coker}(\widetilde{F}_{\hat{M}})$ being a finite free o_L-module. In [16, 17] these pairs $(\hat{M}_0, \widetilde{F}_{\hat{M}})$ are called (*effective*) *local shtukas*.

The following criterion for good reduction of analytic Anderson $A(1)$-motives can be regarded as a *good-reduction Local-Global Principle at the characteristic place*.

Theorem 5.3. *Let* $\underline{M}_L = (M_L, F_{M_L})$ *be an analytic Anderson* $A(1)$*-motive over* L *such that* $\operatorname{coker}(F_{M_L})$ *is annihilated by* $\mathfrak{J}^d$ *for some* d*. Then the following assertions are equivalent:*

(i) $\underline{M}_L$ *admits a good model in the strong sense of Definition* 4.6*;*
(ii) *There is an effective local shtuka* $\underline{\hat{M}} = (\hat{M}, F_{\hat{M}})$ *at* ε *over* o_L *such that* $\operatorname{coker}(F_{\hat{M}})$ *is annihilated by* $\mathfrak{J}^d$*, and an isomorphism* $\underline{M}_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi] \cong \underline{\hat{M}} \otimes_{A_{o_L,(\varepsilon,\pi)}} A_{o_L,(\varepsilon,\pi)}[1/\pi]$ *in* $\operatorname{FMod}(A_{o_L,(\varepsilon,\pi)}[1/\pi])$.

Proof. 1. In order to show that (ii) implies (i), we let $f : M_L \otimes A_{o_L,(\varepsilon,\pi)}[1/\pi] \xrightarrow{\sim} \hat{M} \otimes A_{o_L,(\varepsilon,\pi)}[1/\pi] =: \hat{M}[1/\pi]$ be an F-equivariant isomorphism of $A_{o_L,(\varepsilon,\pi)}[1/\pi]$-modules as in (ii). We have canonical F-equivariant $A_{o_L,\pi}$-linear maps

$$i : M_L \to M_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi], \qquad j : \hat{M} \to \hat{M}[1/\pi]$$

where i (respectively, j) is injective since M_L (respectively, $\hat{M}$) is flat. Consider the $A_{o_L,\pi}$-module $\mathcal{M} = \operatorname{im}(i) \cap f^{-1}(\operatorname{im}(j))$. We will show that $\mathcal{M}$ is a good model of $\underline{M}_L$. The inclusion $\mathcal{M} \hookrightarrow M_L$ gives rise to an $A_{o_L,\pi}[1/\pi]$-linear embedding $\mathcal{M}[1/\pi] \hookrightarrow M_L[1/\pi] \cong M_L$, which is in fact an isomorphism, because if $m \in M_L$ there is an $s \geq 0$ such that $\pi^s f(m \otimes 1) \in \operatorname{im}(j)$, *i.e.* $\pi^s m \in \mathcal{M}$.

2. In order to show that $\mathcal{M}$ is a finitely generated $A_{o_L,\pi}$-module we use the embedding $\iota : \mathbb{F}[z] \to A$ from Remark 2.1 and the induced maps $L\langle z\rangle \to A_{o_L,\pi}[1/\pi]$ and $o_L[\![z]\!] \to A_{o_L,(\varepsilon,\pi)}$ from (2.1). Let $(e_1, \ldots, e_m)$ be a basis of M_L over the principal ideal domain $L\langle z\rangle$. Furthermore, let $(d_1, \ldots, d_n)$ be a basis for $\hat{M}$ over the local ring $o_L[\![z]\!]$. Note that the basis $(e_1, \ldots, e_m)$ gives rise to an isomorphism $M_L \otimes_{L\langle z\rangle} o_L[\![z]\!][1/\pi] \cong o_L[\![z]\!][1/\pi]^{\oplus m}$. For every $\nu = 1, \ldots, n$ we consider $f^{-1}(d_\nu)$ and regard it as an element of the right-hand side of this isomorphism. We choose $N \geq 0$ big enough, such that $f^{-1}(\pi^N d_\nu) \in o_L[\![z]\!]^{\oplus m}$ for all ν, say

$$f^{-1}(\pi^N d_\nu) = (\rho_{\nu,1}, \ldots, \rho_{\nu,m})$$

where $\rho_{\nu,\mu} \in o_L[\![z]\!]$. Now let $x \in \mathcal{M}$. Via f we obtain $f(x) = \sum_\nu \lambda_\nu d_\nu$ in $\hat{M}$, with suitable $\lambda_\nu \in o_L[\![z]\!]$. Consequently $f(\pi^N x) = \sum_\nu \lambda_\nu (\pi^N d_\nu)$, so that the image of $\pi^N x$ in $o_L[\![z]\!]^{\oplus m}$ satisfies $\pi^N x = \sum_\mu (\sum_\nu \lambda_\nu \rho_{\nu,\mu}) e_\mu$. The appearing scalars $h_\mu = \sum_\nu \lambda_\nu \rho_{\nu,\mu}$ have, in fact, to be elements of $L\langle z\rangle \cap o_L[\![z]\!] = o_L\langle z\rangle$. Inside M_L we may write $x = \pi^{-N}\pi^N x = \sum_\mu h_\mu \pi^{-N} e_\mu$, so that we may conclude

$$\mathcal{M} \subset \sum_\mu o_L\langle z\rangle \pi^{-N} e_\mu.$$

Being a submodule of a finitely generated module over a noetherian ring, $\mathcal{M}$ has to be a finitely generated $o_L\langle z\rangle$-module and hence a finitely generated $A_{o_L,\pi}$-module.

3. We claim that $\mathcal{M}/\pi\mathcal{M}$ is torsion-free and hence free over $\ell[z]$, because it is finitely generated. Let $x \in \mathcal{M}$, and let $\lambda \in o_L\langle z\rangle$ be such that $\lambda \notin \pi o_L\langle z\rangle$ and $\lambda x \in$

$\pi\mathcal{M}$, say $\lambda x = \pi y$ for some $y \in \mathcal{M}$. In order to prove that $\mathcal{M}/\pi\mathcal{M}$ is torsion-free we must show that $x \in \pi\mathcal{M}$. First suppose that $\lambda \in o_L\langle z\rangle \cap o_L[\![z]\!]^\times$. We consider $\pi^{-1}x \in M_L$. In fact, this element lies in $\mathcal{M}$, since we have $f(\pi^{-1}x) = \lambda^{-1}f(y) \in \hat{M}$. Consequently $x = \pi(\pi^{-1}x) \in \pi\mathcal{M}$.

Let us next assume that $\lambda = z^n$ and show that $z^n x \in \pi\mathcal{M}$ implies $x \in \pi\mathcal{M}$ for any $n \geq 0$. By induction, it suffices to consider the case $n = 1$. So suppose $zx \in \pi\mathcal{M}$, say $zx = \pi y$. Let $f(x) = \sum_\nu \beta_\nu d_\nu$, where $(d_1, ..., d_n)$ is the finite $o_L[\![z]\!]$-basis of $\hat{M}$ fixed before. The relation $zx = \pi y$ implies that $\pi \mid z\beta_\nu$ for every index ν, so that $\pi \mid \beta_\nu$ for every ν. Therefore $\pi^{-1}x \in M_L$ necessarily maps via f to an element of $\hat{M}$, *i.e.* $x \in \pi\mathcal{M}$.

Finally we treat the case for general $\lambda = \sum_s \lambda_s z^s$ and suppose that $\lambda \notin o_L[\![z]\!]^\times$, that is $\pi \mid \lambda_0$. This means we find $\lambda' \in o_L[z]$ and $\lambda'' \in o_L\langle z\rangle \cap o_L[\![z]\!]^\times$ such that $\lambda = \pi\lambda' + z^N\lambda''$ for some $N \geq 1$. We have $\pi y = \lambda x = \pi\lambda' x + z^N\lambda'' x$. In particular $z^N\lambda'' x = \pi(y - \lambda' x) \in \pi\mathcal{M}$ and by the above $\lambda'' x \in \pi\mathcal{M}$ and $x \in \pi\mathcal{M}$.

Thus we have proved that $\mathcal{M}/\pi\mathcal{M}$ is free over $\ell[z]$. It follows that $\mathcal{M}/\pi\mathcal{M}$ is locally free of finite rank over A_ℓ.

4. We claim that $\mathcal{M}$ is locally free of finite rank over $A_{o_L,\pi}$. Since it is finitely generated it only remains to show that $\mathcal{M}$ is flat over $A_{o_L,\pi}$. Since $A_{o_L,\pi}$ is π-adically complete and separated, $\pi A_{o_L,\pi}$ is contained in the Jacobson radical $\mathrm{j}(A_{o_L,\pi})$ by [20, Theorem 8.2], and the $A_{o_L,\pi}$-module $\mathcal{M}$ is finitely generated, so that $\mathcal{M}$ is π-adically *ideally Hausdorff* in the sense of [8, III.5.1]. In the preceding step we have shown that $\mathcal{M}/\pi\mathcal{M}$ is flat over $A_\ell \cong A_{o_L,\pi}/\pi A_{o_L,\pi}$, and we know that $\mathcal{M}$ has no π-torsion, so that the canonical map $\pi A_{o_L,\pi} \otimes_{A_{o_L,\pi}} \mathcal{M} \to \pi\mathcal{M}$ is an isomorphism. Therefore, by Bourbaki's Flatness Criterion [8, Section III.5.2, Théorème 1(iii)], we may conclude that $\mathcal{M}$ is indeed flat over $A_{o_L,\pi}$.

5. We note that $\sigma^*\mathcal{M} = \sigma^*\mathrm{im}(i) \cap (\sigma^* f)^{-1}(\sigma^*\mathrm{im}(j))$ because the functor σ^* is exact by Lemma 3.1. By the F-equivariance of f we obtain a Frobenius $F_\mathcal{M} : \sigma^*\mathcal{M} \to \mathcal{M}$. It is injective because F_{M_L} is. We set $\underline{\mathcal{M}} := (\mathcal{M}, F_\mathcal{M})$.

6. Next we claim that $\mathfrak{J}^d\mathrm{coker}(F_\mathcal{M}) = 0$. Let $x = \sum_\nu h_\nu m_\nu \in \mathfrak{J}^d\mathcal{M}$ where $h_\nu \in \mathfrak{J}^d$ and $m_\nu \in \mathcal{M}$. Since $\mathrm{coker}(F_{M_L})$ is annihilated by $\mathfrak{J}^d$, there is a (unique) $y \in \sigma^* M_L$ such that $x = \sum_\nu h_\nu m_\nu = F_{M_L}(y)$. We have to show that $y \in \sigma^*\mathcal{M} = \sigma^*\mathrm{im}(i) \cap (\sigma^* f)^{-1}(\sigma^*\mathrm{im}(j))$. So it remains to see that $(\sigma^* f)(y) \in \mathrm{im}(\sigma^* j)$. Indeed, inside $\hat{M}[1/\pi]$ we have $f(x) = f(F_{M_L}(y)) = F_{\hat{M}}((\sigma^* f)(y))$. On the other hand, the linearity of f and j gives that $f(x) = \sum_\nu h_\nu f(m_\nu \otimes 1) = j(y')$ for some $y' \in \mathfrak{J}^d\hat{M} \subset \mathrm{im}(F_{\hat{M}})$, say $y' = F_{\hat{M}}(y'')$ for a $y'' \in \sigma^*\hat{M}$. Thus $f(x) = F_{\hat{M}}((\sigma^* j)(y''))$. So finally, since $F_{\hat{M}} : \sigma^*\hat{M}[1/\pi] \to \hat{M}[1/\pi]$ is injective, we obtain that $(\sigma^* f)(y) = (\sigma^* j)(y'')$, as desired.

7. Finally we show that the kernel V of $\overline{F} : \sigma^*(\mathcal{M}/\pi\mathcal{M}) \to \mathcal{M}/\pi\mathcal{M}$ is trivial. This implies that $\underline{\mathcal{M}}$ is a good model of $\underline{M}_L$ in the weak sense of Definition 4.5, which is enough by Theorem 4.7.

We have already shown that $\mathfrak{J}^d\mathcal{M} \subset \mathrm{im}(F_\mathcal{M})$. Since $(z - \zeta) \in \mathfrak{J}$ for $\zeta := c^*(z) \in o_L$ we have a chain of $o_L\langle z\rangle$-modules $(z - \zeta)^d\mathcal{M} \subset \mathrm{im}(F_\mathcal{M}) \subset \mathcal{M}$. The element $\zeta \in o_L$ is zero mod π, and we obtain

$$z^d(\mathcal{M}/\pi\mathcal{M}) \subset \mathrm{im}(\overline{F}) \subset \mathcal{M}/\pi\mathcal{M}. \tag{5.1}$$

We know that $\mathcal{M}/\pi\mathcal{M}$ is finite free over $\ell[z]$. Therefore the middle term $W := \mathrm{im}(\overline{F})$ in the latter chain has full rank inside $\mathcal{M}/\pi\mathcal{M}$. Finally, taking ranks in the (split) short exact sequence of finite free $\ell[z]$-modules

$$0 \to V \to \sigma^*(\mathcal{M}/\pi\mathcal{M}) \xrightarrow{\overline{F}} W \to 0$$

accomplishes the proof that V indeed is trivial.

8. Conversely, in order to show that (i) implies (ii), suppose that $(\underline{\mathcal{M}}, \alpha)$ is a good model of $\underline{M}_L$. We define

$$\underline{\hat{M}} = \underline{\mathcal{M}} \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)},$$

i.e. $\underline{\hat{M}}$ equals the completion of $\underline{\mathcal{M}}$ for the $(\varepsilon, \pi)A_{o_L,\pi}$-adic topology. It is clear that the F-equivariant isomorphism $\alpha : M_L \xrightarrow{\sim} \mathcal{M}[1/\pi]$ of $A_{o_L,\pi}[1/\pi]$-modules gives rise to a natural F-equivariant $A_{o_L,(\varepsilon,\pi)}[1/\pi]$-linear isomorphism

$$M_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi] \cong \hat{M}[1/\pi].$$

We claim that $\underline{\hat{M}}$ is a local shtuka. Indeed, by base change, $\hat{M}$ is again locally free of finite rank. Furthermore, since the completion map $A_{o_L,\pi} \to A_{o_L,(\varepsilon,\pi)}$ is Frobenius-equivariant and flat, we obtain an injective map $\hat{M} \otimes_{(A_{o_L,(\varepsilon,\pi)}),\sigma} A_{o_L,(\varepsilon,\pi)} \to \hat{M}$. Let C' be its cokernel, and let $C = \mathrm{coker}(F_{\mathcal{M}})$, *i.e.* $C' \cong C \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)}$. Since C is annihilated by $\mathfrak{J}^d$ the module C' equals C and it is finite free over o_L. Thus $\underline{\hat{M}}$ is an effective local shtuka over o_L. □

Remark 5.4. Steps 1-4 in the previous proof suggest that there is an equivalence of categories

$$\mathcal{F} : \left\{ \begin{array}{l} \text{finite locally free} \\ A_{o_L,\pi}\text{-modules } \mathcal{M} \end{array} \right\}$$

$$\xrightarrow{\sim} \left\{ \begin{array}{l} \text{triples } (M_L, \hat{M}, f) \text{ consisting of} \\ \bullet \text{ a finite locally free } A_{o_L,\pi}[1/\pi]\text{-module } M_L, \\ \bullet \text{ a finite locally free } A_{o_L,(\varepsilon,\pi)}\text{-module } \hat{M}, \text{ and} \\ \bullet \text{ an isomorphism of } A_{o_L,(\varepsilon,\pi)}[1/\pi]\text{-modules} \\ \quad f : M_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi] \xrightarrow{\sim} \hat{M} \otimes_{A_{o_L,(\varepsilon,\pi)}} A_{o_L,(\varepsilon,\pi)}[1/\pi] \end{array} \right\}$$

$$\mathcal{M} \longmapsto \big(\mathcal{M} \otimes_{A_{o_L,\pi}} A_{o_L,\pi}[1/\pi],\ \mathcal{M} \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)},\ \mathrm{id}_{\mathcal{M} \otimes A_{o_L,(\varepsilon,\pi)}[1/\pi]}\big),$$

where on the right a morphism $\underline{h} = (h_L, \hat{h}) : (M_L, \hat{M}, f) \to (M'_L, \hat{M}', f')$ consists of a morphism $h_L : M_L \to M'_L$ and a morphism $\hat{h} : \hat{M} \to \hat{M}'$ such that $f' \circ (h_L \otimes \mathrm{id}_{A_{o_L,(\varepsilon,\pi)}[1/\pi]}) = (\hat{h} \otimes \mathrm{id}_{A_{o_L,(\varepsilon,\pi)}[1/\pi]}) \circ f$.

However, *this is false* as can be seen from the following example, where we take $A = \mathbb{F}[z]$. We choose an element $a \in \ell[\![z]\!] \subset \ell(\!(z)\!)$ such that $a \notin \ell(z)$, and we let $\Delta = \left(\begin{smallmatrix} 1 & \pi^{-1}a \\ 0 & \pi^{-1} \end{smallmatrix}\right)$. Set $M_L = L\langle z\rangle^{\oplus 2}$, $\hat{M} = \Delta \cdot o_L[\![z]\!]^{\oplus 2}$ and $f = \mathrm{id}_{o_L[\![z]\!][1/\pi]^2}$. Then $\Delta^{-1} = \left(\begin{smallmatrix} 1 & -a \\ 0 & \pi \end{smallmatrix}\right) \in o_L[\![z]\!]^{2\times 2}$ and

$$o_L[\![z]\!]^{\oplus 2} \;=\; \Delta \cdot \Delta^{-1} o_L[\![z]\!]^{\oplus 2} \;\subset\; \hat{M} \;\subset\; \pi^{-1} o_L[\![z]\!]^{\oplus 2}\,.$$

If there were a finite free $A_{o_L,\pi}$-module $\mathcal{M}$ with $(h_L, \hat{h}) : \mathcal{F}(\mathcal{M}) \xrightarrow{\sim} (M_L, \hat{M}, f)$, then it had to satisfy $\mathcal{M} \cong M_L \cap \hat{M}$ with h_L and $\hat{h}$ induced from the inclusions $M_L \cap \hat{M} \subset M_L$ and $M_L \cap \hat{M} \subset \hat{M}$. So we may take directly $\mathcal{M} := M_L \cap \hat{M}$. It satisfies $o_L\langle z\rangle^{\oplus 2} \subset \mathcal{M} \subset \pi^{-1} o_L\langle z\rangle^{\oplus 2}$. We claim that, in fact, the first inclusion is an equality. Namely let $\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right) = \left(\begin{smallmatrix} \pi^{-1}v_0 + v' \\ \pi^{-1}w_0 + w' \end{smallmatrix}\right) \in \mathcal{M}$ with $v_0, w_0 \in \ell[z]$ and $v', w' \in o_L\langle z\rangle$. Then $\Delta^{-1}\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right) = \left(\begin{smallmatrix} \pi^{-1}v_0 + v' - \pi^{-1}aw_0 - aw' \\ w_0 + \pi w' \end{smallmatrix}\right) \in o_L[\![z]\!]^{\oplus 2}$. This implies $v_0 = aw_0$ in $\ell[\![z]\!]$. If $w_0 \neq 0$ we get $a = v_0/w_0 \in \ell(z)$ in contradiction to our assumption. So $w_0 = v_0 = 0$ and $\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right) \in o_L\langle z\rangle^{\oplus 2}$. This proves our claim that $\mathcal{M} = o_L\langle z\rangle^{\oplus 2}$. We conclude that $\mathcal{F}(\mathcal{M}) \ncong (M_L, \hat{M}, f)$ and $\mathcal{F}$ is not an equivalence of categories.

After this example the following result is even more surprising.

Corollary 5.5. *Let $\underline{M}_L$ be an analytic Anderson $A(1)$-motive over L. Then there is an equivalence of categories*

$$\left\{\begin{array}{l} \textit{good models } (\underline{\mathcal{M}}, \alpha) \textit{ of } \underline{M}_L \textit{ in the} \\ \textit{sense of Definitions } 4.6 \textit{ and } 4.5 \end{array}\right\}$$

$$\xleftrightarrow{\;\sim\;} \left\{\begin{array}{l} \textit{pairs } (\underline{\hat{M}}, f) \textit{ consisting of} \\ \bullet \textit{ a local shtuka } \underline{\hat{M}} \textit{ at } \varepsilon \textit{ over } o_L, \textit{ and} \\ \bullet \textit{ an isomorphism in } \mathrm{FMod}(A_{o_L,(\varepsilon,\pi)}[1/\pi]) \\ \quad f : \underline{M}_L \otimes A_{o_L,(\varepsilon,\pi)}[1/\pi] \xrightarrow{\sim} \underline{\hat{M}}[1/\pi] \end{array}\right\}$$

$$(\underline{\mathcal{M}}, \alpha) \longmapsto (\underline{\mathcal{M}}, \alpha) \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)}\,,$$

where on the right-hand side a morphism *of pairs $\hat{\beta} : (\underline{\hat{M}}, f) \xrightarrow{\sim} (\underline{\hat{M}}', f')$ is defined to be an isomorphism of local shtukas $\hat{\beta} : \underline{\hat{M}} \xrightarrow{\sim} \underline{\hat{M}}'$ satisfying $f' = \hat{\beta} \circ f$.*

Proof. Suppose that $(\underline{\mathcal{M}}, \alpha)$ is a good model of $\underline{M}_L$. In the proof of 5.3 we have seen that its completion $\underline{\hat{\mathcal{M}}} := \underline{\mathcal{M}} \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)}$ is a local shtuka at ε. The F-equivariant isomorphism $\alpha : M_L \xrightarrow{\sim} \mathcal{M}[1/\pi]$ of $A_{o_L,\pi}[1/\pi]$-modules induces the isomorphism

$$f := \alpha \otimes \mathrm{id}_{A_{o_L,(\varepsilon,\pi)}[1/\pi]} : M_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi]$$
$$\xrightarrow{\sim} \hat{\mathcal{M}} \otimes_{A_{o_L,(\varepsilon,\pi)}} A_{o_L,(\varepsilon,\pi)}[1/\pi]$$

which is F-equivariant, and satisfies $\mathcal{M} = f(M_L) \cap \hat{\mathcal{M}}$, because $A_{o_L,\pi} = A_{o_L,\pi}[1/\pi] \cap A_{o_L,(\varepsilon,\pi)}$.

To see that this functor is fully faithful let $(\underline{\mathcal{M}}, \alpha)$ and $(\underline{\mathcal{M}}', \alpha')$ be good models of $\underline{M}_L$ and let $\hat{\beta} : (\underline{\hat{\mathcal{M}}}, f) := (\underline{\mathcal{M}}, \alpha) \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)} \xrightarrow{\sim} (\underline{\hat{\mathcal{M}}}', f') := (\underline{\mathcal{M}}', \alpha') \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)}$ be an isomorphism. This means $f' = \hat{\beta} \circ f$. Applying $\mathcal{M} = f(M_L) \cap \hat{\mathcal{M}}$ and $\mathcal{M}' = f'(M_L) \cap \hat{\mathcal{M}}'$ we see that $\hat{\beta}(\mathcal{M}) = \mathcal{M}'$. Therefore $\beta := \hat{\beta}|_{\mathcal{M}} : \mathcal{M} \xrightarrow{\sim} \mathcal{M}'$ is the desired isomorphism satisfying $\beta \otimes \mathrm{id}_{A_{o_L,(\varepsilon,\pi)}} = \hat{\beta}$. This implies $\alpha' = \beta \circ \alpha$ and the F-equivariance of β, and hence $\beta : (\underline{\mathcal{M}}, \alpha) \xrightarrow{\sim} (\underline{\mathcal{M}}', \alpha')$.

To prove essential surjectivity, let a local shtuka $\underline{\hat{M}}$ together with an isomorphism $f : \underline{M}_L \otimes_{A_{o_L,\pi}[1/\pi]} A_{o_L,(\varepsilon,\pi)}[1/\pi] \xrightarrow{\sim} \underline{\hat{M}}[1/\pi]$ be given. It remains to show that the $(\varepsilon, \pi)A_{o_L,\pi}$-adic completion $\underline{\hat{\mathcal{M}}} := \underline{\mathcal{M}} \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)}$ of the good model $\mathcal{M} = M_L \cap f^{-1}(\hat{M})$ gained in the proof of 5.3 gives back $\underline{\hat{M}}$. Then we take α as the canonical isomorphism $\mathrm{id} : \mathcal{M} \otimes_{A_{o_L,\pi}} A_{o_L,\pi}[1/\pi] \xrightarrow{\sim} M_L$. By construction of $\underline{\mathcal{M}}$, the map f restricts to an embedding $\mathcal{M} \hookrightarrow \hat{M}$, which in turn induces an F-equivariant and $A_{o_L,(\varepsilon,\pi)}$-linear map $\psi := f|_{\hat{\mathcal{M}}} : \hat{\mathcal{M}} \to \hat{M}$, which becomes an isomorphism after inverting π. Our aim is to show that already the map ψ is an isomorphism $(\underline{\mathcal{M}}, \mathrm{id}) \otimes_{A_{o_L,\pi}} A_{o_L,(\varepsilon,\pi)} \xrightarrow{\sim} (\underline{\hat{M}}, f)$. According to Remark 5.4 we have to use the Frobenius morphisms $F_{\hat{\mathcal{M}}}$ and $F_{\hat{M}}$ in an essential way.

We know that $\mathcal{M}$ is finite free over $o_L\langle z\rangle$ and that $\mathrm{rk}_{o_L[\![z]\!]}(\hat{\mathcal{M}}) = \mathrm{rk}_{o_L[\![z]\!]}(\hat{M}) =: s$. We fix an $o_L[\![z]\!]$-basis $\mathfrak{B}$ (respectively, $\mathfrak{C}$) of $\hat{\mathcal{M}}$ (respectively, of $\hat{M}$) and let $\mathbf{A} = {}_{\mathfrak{C}}[\psi]_{\mathfrak{B}} \in o_L[\![z]\!]^{s\times s}$ be the matrix which describes ψ with respect to $\mathfrak{B}$ and $\mathfrak{C}$. Likewise, we let

$$\mathbf{T} = {}_{\mathfrak{B}}[F_{\hat{\mathcal{M}}}]_{\sigma^*\mathfrak{B}}, \qquad \mathbf{T}' = {}_{\mathfrak{C}}[F_{\hat{M}}]_{\sigma^*\mathfrak{C}}$$

be the matrices corresponding to $F_{\hat{\mathcal{M}}}$ and $F_{\hat{M}}$, so that $\mathbf{AT} = \mathbf{T}'\sigma(\mathbf{A})$ by virtue of the F-equivariance of ψ. In order to see that ψ is an isomorphism, we need to show that $\det(\mathbf{A})$ is a unit in $o_L[\![z]\!]$. To begin with, an elementary application of the Weierstraß Division Theorem for $o_L[\![z]\!]$ ([8, VII.3.8.5]) shows that the kernel of the epimorphism $o_L[\![z]\!] \to o_L$, $z \mapsto \zeta$, is generated by $z - \zeta$, so that the latter is a prime element of $o_L[\![z]\!]$. Furthermore, recall that $o_L[\![z]\!]$, being a regular local ring, is factorial ([20], 20.3). We know that $\underline{\hat{\mathcal{M}}}$ is a local shtuka, so that $F_{\hat{\mathcal{M}}}$ becomes an isomorphism after inverting $z - \zeta$ which means that $\det(\mathbf{T})^{-1}$ lies in $o_L[\![z]\!][\frac{1}{z-\zeta}]$. Say we have a relation $(z - \zeta)^e = \det(\mathbf{T})u$ in $o_L[\![z]\!]$, for some $e \geq 0$ and some $u \in o_L[\![z]\!]$. By a comparison of powers of $z - \zeta$, we may assume that u is not divisible by $z-\zeta$. In this equation there is only one prime element of $o_L[\![z]\!]$ occurring on both sides, which, by factoriality, implies that u has to be a unit in $o_L[\![z]\!]$. Let $(z-\zeta)^{e'} = \det(\mathbf{T}')u'$ be the corresponding relation for the local shtuka $\hat{M}$, with a unit $u' \in o_L[\![z]\!]^\times$ and some suitable $e' \geq 0$. Since $\hat{\mathcal{M}} \to \hat{M}$ becomes an isomorphism after inverting π, we see that $\det(\mathbf{A}) \in o_L[\![z]\!][1/\pi]^\times$. Note that the natural reduction-mod-z map $o_L[\![z]\!] \to o_L, h \mapsto h(0)$, induces an epimorphism of Abelian groups $o_L[\![z]\!][\frac{1}{\pi}]^\times \to L^\times$, so that the absolute term $\delta := \det(\mathbf{A})(0)$ of $\det(\mathbf{A})$ lies in $L^\times$. By virtue of the relations derived above, the equation $\det(\mathbf{A})\det(\mathbf{T}) = \det(\mathbf{T}')\sigma(\det(\mathbf{A}))$ yields

$$\det(\mathbf{A})u^{-1}(z-\zeta)^e = u'^{-1}(z-\zeta)^{e'}\sigma(\det(\mathbf{A}))$$

which modulo z gives $\delta^{q-1} = \frac{u'(0)}{u(0)}(-\zeta)^{e-e'}$ in $L^\times$. Suppose for a moment that $e = e'$. In this case it follows at once that δ is a unit in o_L, so that $\det(\mathbf{A})$ is a unit in $o_L[\![z]\!]$. Therefore it remains to verify that our assumption $e = e'$ is justified. This can be seen as follows: The reduction-mod-π map $o_L[\![z]\!] \to \ell[\![z]\!]$ is an epimorphism with kernel $\pi o_L[\![z]\!]$, and via applying the functor $\cdot \otimes_{o_L[\![z]\!]} \ell[\![z]\!]$ to $F_{\hat{M}} : \sigma^*\hat{M} \to \hat{M}$ we obtain a commutative diagram

$$\begin{array}{ccc}
\sigma^*\hat{M} = \hat{M} \otimes_{o_L[\![z]\!],\sigma} o_L[\![z]\!] & \longrightarrow & \hat{M} \\
\downarrow & & \downarrow \\
\bar{\sigma}^*\hat{M}/\pi\hat{M} = \hat{M}/\pi\hat{M} \otimes_{\ell[\![z]\!],\bar{\sigma}} \ell[\![z]\!] & \longrightarrow & \hat{M}/\pi\hat{M}
\end{array}$$

where in the upper row (respectively, the bottom row) both modules are finite free of the same rank over $o_L[\![z]\!]$ (respectively, over $\ell[\![z]\!]$) and the arrow is given by $F_{\hat{M}}$ (respectively, by $\bar{F} = F_{\hat{M}} \otimes \mathrm{id}_{\ell[\![z]\!]}$). The reduced matrix $\overline{\mathbf{T}'} \in \ell[\![z]\!]^{s\times s}$ describes the map $\bar{F}$ with respect to the $\ell[\![z]\!]$-bases $\overline{\sigma^*\mathfrak{C}} = \bar{\sigma}^*\bar{\mathfrak{C}}$ of $\bar{\sigma}^*\hat{M}/\pi\hat{M}$ and $\bar{\mathfrak{C}}$ of $\hat{M}/\pi\hat{M}$ respectively, and from what we have seen before, we derive the relation $\det(\overline{\mathbf{T}'})\overline{u'} = z^{e'}$, *i.e.* $e' = \mathrm{ord}_z(\det(\overline{\mathbf{T}'}))$, the latter being true since $\overline{u'} \in \ell[\![z]\!]^\times$. In particular we have $\det(\overline{\mathbf{T}'}) \in \ell[\![z]\!] - \{0\}$. A similar observation for the local shtuka $\hat{\mathcal{M}}$ instead of $\hat{M}$ shows that $e = \mathrm{ord}_z(\det(\overline{\mathbf{T}}))$. Let $C = \mathrm{coker}(F_{\hat{\mathcal{M}}})$ and $C' = \mathrm{coker}(F_{\hat{M}})$. Multiplication with the matrix $\overline{\mathbf{T}'}$ gives rise to a finite presentation $\ell[\![z]\!]^s \to \ell[\![z]\!]^s \to C'/\pi C' \to 0$. Taking determinants in an equation of the form $\mathbf{S}_1\overline{\mathbf{T}'}\mathbf{S}_2 = \mathrm{Diag}(a_1, ..., a_d, 0, 0, ..., 0)$, where $\mathbf{S}_1, \mathbf{S}_2 \in \mathrm{Gl}_s(\ell[\![z]\!])$ are suitable matrices such that $a_1, ..., a_d \in \ell[\![z]\!] - \{0\}$ are the elementary divisors of $\overline{\mathbf{T}'}$ (see [7], VII.4.5.1), yields that necessarily $d = s$, so that $C'/\pi C'$ is a torsion $\ell[\![z]\!]$-module and

$$C'/\pi C' \cong \ell[\![z]\!]/a_1\ell[\![z]\!] \oplus ... \oplus \ell[\![z]\!]/a_s\ell[\![z]\!] \cong \ell^{n_1} \oplus ... \oplus \ell^{n_s}$$

where $n_j = \mathrm{ord}_z(a_j)$ and $\sum_j n_j = e'$, *i.e.* $e' = \mathrm{ord}_z(\det(\overline{\mathbf{T}'})) = \mathrm{rk}_\ell(C'/\pi C') = \mathrm{rk}_{o_L}(C')$, the latter equation being valid since $C'/\pi C' \cong C' \otimes_{o_L[\![z]\!]} \ell[\![z]\!]$. Finally, imitating this argument for the local shtuka $\hat{\mathcal{M}}$ yields that $e = \mathrm{ord}_z(\det(\overline{\mathbf{T}})) = \mathrm{rk}_\ell(C/\pi C) = \mathrm{rk}_{o_L}(C)$. So it remains to show that $\mathrm{rk}_{o_L}(C) = \mathrm{rk}_{o_L}(C')$. Indeed, we know that $\psi : \hat{\mathcal{M}} \to \hat{M}$ gives back f in the generic fiber, which means that ψ is an isomorphism after inverting π. Therefore, inverting π in the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \sigma^*(\hat{\mathcal{M}}) & \longrightarrow & \hat{\mathcal{M}} & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma^*\psi} & & \downarrow{\scriptstyle \psi} & & \downarrow & & \\
0 & \longrightarrow & \sigma^*\hat{M} & \longrightarrow & \hat{M} & \longrightarrow & C' & \longrightarrow & 0
\end{array}$$

exhibits $(\sigma^*\psi)[1/\pi] = \sigma^*(\psi[1/\pi])$ and $\psi[1/\pi]$ as $o_L[\![z]\!][1/\pi]$-linear isomorphisms, so that the Snake Lemma yields $C'[1/\pi] \cong C[1/\pi]$, and we obtain $\mathrm{rk}_{o_L}(C') = \dim_L(C'[1/\pi]) = \dim_L(C[1/\pi]) = \mathrm{rk}_{o_L}(C)$, as desired. □

6. The reduction criterion for Anderson motives

Definition 6.1. (a) Let $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$. Following Gardeyn [13], $\underline{\mathcal{M}}$ is called A_{o_L}*-maximal* if for every $\underline{\mathcal{N}} \in \mathrm{FMod}(A_{o_L})$ the canonical map

$$\mathrm{Hom}_{\mathrm{FMod}(A_{o_L})}(\underline{\mathcal{N}}, \underline{\mathcal{M}}) \to \mathrm{Hom}_{\mathrm{FMod}(A_L)}(\underline{\mathcal{N}}[1/\pi], \underline{\mathcal{M}}[1/\pi])$$

is surjective (and hence bijective).

(b) An object $\underline{\mathcal{M}}' \in \mathrm{FMod}(A_{o_L,\pi})$ is called $A_{o_L,\pi}$*-maximal* if for every $\underline{\mathcal{N}}' \in \mathrm{FMod}(A_{o_L,\pi})$ the canonical map

$$\mathrm{Hom}_{\mathrm{FMod}(A_{o_L,\pi})}(\underline{\mathcal{N}}', \underline{\mathcal{M}}') \to \mathrm{Hom}_{\mathrm{FMod}(A_{o_L,\pi}[1/\pi])}(\underline{\mathcal{N}}'[1/\pi], \underline{\mathcal{M}}'[1/\pi])$$

is surjective (and hence bijective).

(c) Let $\underline{M} \in \mathrm{FMod}(A_L)$. An object $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$ is called an A_{o_L}*-maximal model* for $\underline{M}$ if $\underline{\mathcal{M}}[1/\pi] \cong \underline{M}$ inside $\mathrm{FMod}(A_L)$ (*i.e.* $\underline{\mathcal{M}}$ is a *model* for $\underline{M}$) and if $\underline{\mathcal{M}}$ is A_{o_L}-maximal. Correspondingly, given $\underline{M}' \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$, an object $\underline{\mathcal{M}}' \in \mathrm{FMod}(A_{o_L,\pi})$ is called an $A_{o_L,\pi}$*-maximal model* for $\underline{M}'$ if $\underline{\mathcal{M}}'[1/\pi] \cong \underline{M}'$ inside $\mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ and if $\underline{\mathcal{M}}'$ is $A_{o_L,\pi}$-maximal.

The existence of (A_{o_L}- and $A_{o_L,\pi}$-)maximal models has been established in [13].

Proposition 6.2 ([13, Proposition 2.13]). *Let* $\underline{M} \in \mathrm{FMod}(A_L)$. *Then the following assertions hold:*

(i) $\underline{M}$ *admits an* A_{o_L}*-maximal model, which is unique up to unique isomorphism;*
(ii) *If a model* $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$ *of* $\underline{M}$ *is good in the weak sense of Definition* 4.5*, then it is* A_{o_L}*-maximal.*

The next proposition is a variant of Gardeyn's theory of maximal models.

Proposition 6.3. *The following assertions hold:*

(i) *Every* $\underline{M} \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ *admits a maximal model, which is unique up to unique isomorphism;*
(ii) *If* $\underline{M} \in \mathrm{FMod}(A_L)$ *is given and if* $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$ *is an* A_{o_L}*-maximal model of* $\underline{M}$ *then* $\underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi} \in \mathrm{FMod}(A_{o_L,\pi})$ *is an* $A_{o_L,\pi}$*-maximal model of* $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi] \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$*;*
(iii) *Let* $\underline{M} \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ *and let* $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L,\pi})$ *be a model of* $\underline{M}$*. If* $\underline{\mathcal{M}}$ *is a good model in the weak sense of Definition* 4.5*, then it is* $A_{o_L,\pi}$*-maximal.*

Proof. For (i) (respectively (ii); respectively (iii)), see [13], 3.3(i) (respectively 3.4(i); respectively 2.13(ii)). Note that strictly speaking Gardeyn proves these statements for the rings $\Gamma(\mathfrak{A}(\infty), \mathcal{O}_{\mathfrak{A}(\infty)})$ instead of $A_{o_L,\pi}[1/\pi]$ and $\Gamma(\mathfrak{A}(\infty), \mathcal{O}_{\mathfrak{A}(\infty)}) \cap A_{o_L,\pi}$ instead of $A_{o_L,\pi}$. His arguments carry over literally to our rings. □

We may conclude:

Proposition 6.4. *In the weak sense of Definition* 4.5 *a Frobenius* A_L*-module* $\underline{M}$ *admits a good model over* A_{o_L} *if and only if* $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi] \in \mathrm{FMod}(A_{o_L,\pi}[1/\pi])$ *admits a good model over* $A_{o_L,\pi}$*. If this is the case, the functor* $(\underline{\mathcal{M}}, \alpha) \mapsto (\underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi},\ \alpha \otimes \mathrm{id}_{A_{o_L,\pi}[1/\pi]})$ *is an equivalence of categories between the good models of* $\underline{M}$ *and the good models of* $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$*.*

Proof. First suppose that $\underline{M}$ admits a good model $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$. It follows that $\underline{\mathcal{M}}$ is an A_{o_L}-maximal model of $\underline{M}$. Furthermore, its image $\underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi}$ inside $\mathrm{FMod}(A_{o_L,\pi})$ is an $A_{o_L,\pi}$-maximal model of $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$. Since the reduction of $\underline{\mathcal{M}}$ is canonically isomorphic to the reduction of $\underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi}$ by Proposition 4.9, it follows that the latter is a good model.

Conversely, suppose that $\underline{M} \otimes_{A_L} A_{o_L,\pi}[1/\pi]$ admits a good model $\widehat{\underline{\mathcal{M}}} \in \mathrm{FMod}(A_{o_L,\pi})$. Necessarily $\widehat{\underline{\mathcal{M}}}$ is a maximal model by Proposition 6.3(iii). We know that there is an A_{o_L}-maximal model $\underline{\mathcal{M}} \in \mathrm{FMod}(A_{o_L})$ of $\underline{M}$ such that $\underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi} \cong \widehat{\underline{\mathcal{M}}}$, and that the reduction of $\widehat{\underline{\mathcal{M}}}$ is canonically isomorphic to the reduction of $\underline{\mathcal{M}}$ by Propositions 6.2, 6.3(ii) and 4.9. Since $\widehat{\underline{\mathcal{M}}}$ is a good model, so is $\underline{\mathcal{M}}$. This proves the first statement and it also proves essential surjectivity of the functor.

To prove full faithfulness let $(\underline{\mathcal{M}}, \alpha)$ and $(\underline{\mathcal{M}}', \alpha')$ be good models of $\underline{M}$ and let $\hat{\beta} : \underline{\mathcal{M}} \otimes_{A_{o_L}} A_{o_L,\pi} \xrightarrow{\sim} \underline{\mathcal{M}}' \otimes_{A_{o_L}} A_{o_L,\pi}$ be an isomorphism in $\mathrm{FMod}(A_{o_L,\pi})$ satisfying $\alpha' \otimes \mathrm{id} = \hat{\beta} \circ (\alpha \otimes \mathrm{id})$. Since $A_{o_L} = A_L \cap A_{o_L,\pi}$ inside $A_{o_L,\pi}[1/\pi]$, we can recover $\mathcal{M}$ as $\mathcal{M} = \alpha(M) \cap \mathcal{M} \otimes_{A_{o_L}} A_{o_L,\pi}$. This implies $\hat{\beta}(\mathcal{M}) = \mathcal{M}'$ and $\beta := \hat{\beta}|_{\mathcal{M}}$ is the desired isomorphism $\beta : \underline{\mathcal{M}} \xrightarrow{\sim} \underline{\mathcal{M}}$ with $\alpha' = \beta \circ \alpha$. This proves full faithfulness. □

For Anderson A-motives Proposition 6.4 and Theorem 4.7 imply the following:

Corollary 6.5. *Let $\underline{M}$ be an Anderson A-motive over L. Then in the strong sense of Definition* 4.6*, $\underline{M}$ admits a good model $\underline{\mathcal{M}}$ if and only if the associated analytic Anderson $A(1)$-motive $\underline{M}\otimes_{A_L} A_{o_L,\pi}[1/\pi]$ admits a good model $\underline{\mathcal{M}}'$. If this is the case, the functor $(\underline{\mathcal{M}},\alpha)\mapsto(\underline{\mathcal{M}}\otimes_{A_{o_L}} A_{o_L,\pi},\ \alpha\otimes \mathrm{id}_{A_{o_L,\pi}[1/\pi]})$ is an equivalence of categories between the good models of $\underline{M}$ and the good models of $\underline{M}\otimes_{A_L} A_{o_L,\pi}[1/\pi]$.*

This corollary together with Theorem 5.3 and Corollary 5.5 implies the following criterion for good reduction of Anderson A-motives, which can be regarded as an analog of the reduction criteria for Abelian varieties of Grothendieck [15, Proposition IX.5.13] and de Jong [19, 2.5].

Corollary 6.6. *Let $\underline{M}$ be an Anderson A-motive over L such that* $\mathrm{coker}(F_{\underline{M}})$ *is annihilated by $\mathfrak{J}^d$ for some d. Then the following assertions are equivalent:*

(i) *$\underline{M}$ admits a good model $(\underline{\mathcal{M}},\alpha)$ in the strong sense of Definition 4.6,* i.e. *there is an object $\underline{\mathcal{M}}\in$* $\mathrm{FMod}(A_{o_L})$ *such that* $\mathrm{coker}(F_{\underline{\mathcal{M}}})$ *is a finite free o_L-module and is annihilated by $\mathfrak{J}^d$, together with an isomorphism $\alpha:\underline{M}\xrightarrow{\sim}\underline{\mathcal{M}}[1/\pi]$ inside* $\mathrm{FMod}(A_L)$*;*

(ii) *There is an effective local shtuka $\underline{\hat{M}}$ at ε over o_L such that* $\mathrm{coker}(F_{\underline{\hat{M}}})$ *is annihilated by $\mathfrak{J}^d$, and an isomorphism $\underline{M}\otimes_{A_L} A_{o_L,(\varepsilon,\pi)}[1/\pi]\cong\underline{\hat{M}}[1/\pi]$ inside* $\mathrm{FMod}(A_{o_L,(\varepsilon,\pi)}[1/\pi])$.

Moreover, there is an equivalence of categories

$$\left\{\begin{array}{l}\textit{good models } (\underline{\mathcal{M}},\alpha) \textit{ of } \underline{M} \textit{ in the}\\ \textit{sense of Definitions } 4.6 \textit{ and } 4.5\end{array}\right\}$$

$$\xleftrightarrow{\sim}\left\{\begin{array}{l}\textit{pairs } (\underline{\hat{M}},f) \textit{ consisting of}\\ \bullet\ \textit{a local shtuka } \underline{\hat{M}} \textit{ at } \varepsilon \textit{ over } o_L, \textit{ and}\\ \bullet\ \textit{an isomorphism in } \mathrm{FMod}(A_{o_L,(\varepsilon,\pi)}[1/\pi])\\ \quad f:\underline{M}\otimes_{A_L} A_{o_L,(\varepsilon,\pi)}[1/\pi]\xrightarrow{\sim}\underline{\hat{M}}[1/\pi]\end{array}\right\}$$

$$(\underline{\mathcal{M}},\alpha)\longmapsto(\underline{\mathcal{M}},\alpha)\otimes_{A_{o_L}} A_{o_L,(\varepsilon,\pi)}\,,$$

where on the right-hand side a morphism *of pairs $\hat{\beta}:(\underline{\hat{M}},f)\xrightarrow{\sim}(\underline{\hat{M}}',f')$ is defined to be an isomorphism of local shtukas $\hat{\beta}:\underline{\hat{M}}\xrightarrow{\sim}\underline{\hat{M}}'$ satisfying $f'=\hat{\beta}\circ f$.* □

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