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Complex geodesics in convex tube domains

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Abstract. We describe all the complex geodesics in convex tube domains. In the case where the base of a convex tube domain does not contain any real line, the obtained description involves the notion of boundary measure of a holomorphic map and it is expressed in the language of real Borel measures on the unit circle. Applying our result, we calculate all complex geodesics in convex tube domains with unbounded base covering a special class of Reinhardt domains.

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1. Introduction

Among the most important objects in complex analysis connected with the theory of holomorphically invariant functions there are two extremal ones: the Carathéodory pseudodistance and the Lempert function. For a domain $D \subset \mathbb{C}^n$, the *Carathéodory pseudodistance* $c_D : D \times D \to [0, \infty)$ is defined as

$$c_D(z, w) = \sup \{\rho(f(z), f(w)) : f \in \mathcal{O}(D, \mathbb{D})\}$$

and the *Lempert function* $\ell_D : D \times D \to [0, \infty)$ as

$$\ell_D(z, w) = \inf \{\rho(0, \sigma) : \exists f \in \mathcal{O}(\mathbb{D}, D), \sigma \in (0, 1) : f(0) = z, f(\sigma) = w\},$$

where $\mathbb{D}$ is the unit disc in $\mathbb{C}$ and ρ is the Poincaré distance on $\mathbb{D}$. These objects are closely related to the notion of complex geodesics, especially in convex domains. A holomorphic map $\varphi : \mathbb{D} \to D$ is called a *complex geodesic* for D if it admits a *left inverse*, *i.e.* a holomorphic function $f : D \to \mathbb{D}$ such that $f \circ \varphi$ is the identity of the unit disc. Equivalently, φ is a complex geodesic if and only if it is an isometry between $\mathbb{D}$ equipped with the Poincaré distance and D equipped with the Carathéodory pseudodistance. If points $z, w \in D$ lie in the image of a complex geodesic, then $c_D(z, w) = \ell_D(z, w)$. One can also consider "geodesics" with respect to other holomorphically invariant pseudodistances, *e.g.* holomorphic

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isometries from $\mathbb{D}$ with the distance ρ to D with the Kobayashi pseudodistance, etc. However, by the famous Lempert theorem (see [8] or [5, Chapter 8]), in convex domains –which are the subject of the research in this paper– all these notions coincide. The Lempert theorem states that if $D \subset \mathbb{C}^n$ is a taut convex domain then for any pair of points $z, w \in D$ there exists a complex geodesic passing through them, and for any point $z \in D$ and any non-zero vector $v \in \mathbb{C}^n$ there exists a complex geodesic f such that $f(0) = z$ and $f'(0)$ is parallel to v over the field $\mathbb{C}$. From the Lempert theorem it follows that if $D \subset \mathbb{C}^n$ is a convex domain, then $c_D \equiv \ell_D$. Recall that a domain $D \subset \mathbb{C}^n$ is *taut* if every sequence $(f_n)_{n=1}^{\infty} \subset \mathcal{O}(\mathbb{D}, D)$ is compactly divergent or admits a subsequence convergent to a map $f \in \mathcal{O}(\mathbb{D}, D)$. If D is convex, then D is taut if and only if it contains no complex affine lines (see *e.g.* [2, Theorem 1.1]).

A problem of describing and calculating formulas for all complex geodesics of a given (not necessarily convex) domain is a fundamental subject of research in complex analysis. In this paper we focus on this problem in the case of convex tube domains. A domain $D \subset \mathbb{C}^n$ is called a *tube domain* if D is of the form $\Omega + i\mathbb{R}^n$ for some domain $\Omega \subset \mathbb{R}^n$ called the *base* of D. Throughout this paper we denote the base of D by $\operatorname{Re} D$. We present equivalent conditions for a holomorphic map $\varphi : \mathbb{D} \to D$ to be a complex geodesic and we calculate direct formulas for complex geodesics in some classes of convex tube domains.

Results on complex geodesics obtained for tube domains may be applied, for example, to Reinhardt domains. A domain $G \subset \mathbb{C}^n$ is called a *Reinhardt domain* if it is n-circled, *i.e.* $(\lambda_1 z_1, \dots, \lambda_n z_n) \in G$ for every $(z_1, \dots, z_n) \in G$ and $\lambda_1, \dots, \lambda_n \in \mathbb{T}$. Here $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$. Any Reinhardt domain contained in $(\mathbb{C}_*)^n$, where $\mathbb{C}_*$ is the punctured plane, admits a natural covering by a tube domain via the mapping $(z_1, \dots, z_n) \mapsto (e^{z_1}, \dots, e^{z_n})$. What is more, the pseudoconvex Reinhardt domains contained in $(\mathbb{C}_*)^n$ are exactly those which are covered by convex tubes. Complex geodesics of tube domains are useful in calculating ℓ_G-extremal discs in Reinhardt domains. Recall that a map $f \in \mathcal{O}(\mathbb{D}, G)$ is called an ℓ_G-extremal disc for the points $z, w \in G$ if it "realizes" the infimum in the definition of $\ell_G(z, w)$, *i.e.* if there exists $\sigma \in (0, 1)$ such that $f(0) = z$, $f(\sigma) = w$ and $\ell_G(z, w) = \rho(0, \sigma)$. It is known that if $G \subset \mathbb{C}^n$ is a Reinhardt domain and $D \subset \mathbb{C}^n$ is a tube which covers $G \cap (\mathbb{C}_*)^n$ then any ℓ_G-extremal disc which does not intersect the axes (*i.e.* that maps $\mathbb{D}$ into $G \cap (\mathbb{C}_*)^n$) can be lifted to a complex geodesic in D. If we know formulas of all complex geodesics in D, then, given such an ℓ_G-extremal disc $\psi = (\psi_1, \dots, \psi_n)$, we get that ψ is of the form $\psi_j = e^{\varphi_j}$ for a complex geodesic $\varphi = (\varphi_1, \dots, \varphi_n)$ in D, so we know a formula for ψ.

We shall see that for our purposes it is possible to restrict our considerations to taut convex tubes, which, in view of the Lempert theorem, admit complex geodesics passing through any pair of points. In fact, any convex tube in $\mathbb{C}^n$ is linearly biholomorphic to a cartesian product of a taut convex tube and some $\mathbb{C}^k$ (see Observation 2.4). A convex tube is taut if and only if it contains no complex affine lines or, equivalently, if its base contains no real affine lines. On such a tube the Carathédory distance becomes a distance. If $D \subset \mathbb{C}^n$ is a taut convex tube domain and $\varphi = (\varphi_1, \dots, \varphi_n) : \mathbb{D} \to D$ is a holomorphic map, then *radial limits*

$\varphi^*(\lambda) = \lim_{r\to 1^-} \varphi(r\lambda)$ of φ exist for almost every $\lambda \in \mathbb{T}$ (with respect to the Lebesgue measure $\mathcal{L}^{\mathbb{T}}$ on $\mathbb{T}$) and, what is very important, φ admits a *boundary measure*, *i.e.* a unique n-tuple $\mu = (\mu_1, \dots, \mu_n)$ of (finite) real Borel measures on $\mathbb{T}$ such that the following *Poisson formula* holds:

$$\varphi(\lambda) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\zeta + \lambda}{\zeta - \lambda} \, d\mu(\zeta) + i \operatorname{Im} \varphi(0), \ \lambda \in \mathbb{D}$$

(by the integral with respect to the tuple $\mu = (\mu_1, \dots, \mu_n)$ of measures we just mean the tuple of integrals with respect to $\mu_1, \dots, \mu_n$). Although μ is not a measure, but an n-tuple of measures, we call it the boundary measure for the map φ. The reader may find all necessary details in Section 2.

In Section 3 of this paper we characterise complex geodesics in taut convex tubes in $\mathbb{C}^n$. A starting point for us is the characterisation for bounded convex domains presented in [9] and [5, Subsection 8.2] (and also discussed in the recent paper [7]). It essentially uses boundedness of domains and it does not hold for tubes, even in the simpliest case of a left half-plane in $\mathbb{C}$. In this paper we prove the following result, which gives an equivalent condition for a holomorphic map to be a complex geodesic:

Theorem 1.1. *Let $D \subset \mathbb{C}^n$ be a taut convex tube domain and let $\varphi : \mathbb{D} \to D$ be a holomorphic map. Then φ is a complex geodesic for D if and only if there exists a mapping $h : \mathbb{C} \to \mathbb{C}^n$ of the form $h(\lambda) = \bar{a}\lambda^2 + b\lambda + a$ with some $a \in \mathbb{C}^n$ and $b \in \mathbb{R}^n$, such that*:

(i) $\operatorname{Re}\left[\bar{\lambda} h(\lambda) \bullet (z - \varphi^*(\lambda))\right] < 0$ *for all $z \in D$ and a.e. $\lambda \in \mathbb{T}$,*

(ii) $\operatorname{Re}\left[h(\lambda) \bullet \frac{\varphi(0) - \varphi(\lambda)}{\lambda}\right] < 0$ *for every $\lambda \in \mathbb{D}_*$.*

Moreover, if the base of D is bounded, then condition (ii) *can be omitted.*

Here $\bullet$ is the standard dot product in $\mathbb{C}^n$, *i.e.* $z \bullet w = \sum_{j=1}^n z_j w_j$. If the base of D is bounded, then the description becomes almost the same as in the case of bounded convex domain. In that situation, it is not very hard to get nice integral formulas for every complex geodesics for D. It can be done as, for instance, in Example 4.6, where we consider the tube domain with the base equal to the unit ball in $\mathbb{R}^2$. In the example we calculate real parts of almost all radial limits φ^* of φ, using only condition (i), which says that for almost every λ the vector $\bar{\lambda} h(\lambda) \in \mathbb{R}^n$ is "outward" from $\operatorname{Re} D$ at the boundary point $\operatorname{Re} \varphi^*(\lambda)$. Next, we obtain formula for φ using Poisson formula, because –as the map $\operatorname{Re} \varphi$ is **bounded**– the boundary measure of φ is just the measure $\operatorname{Re} \varphi^*(\lambda) \, d\mathcal{L}^{\mathbb{T}}(\lambda)$.

The situation becomes much more complicated if the base of D is unbounded, even if it is bounded "from the right-hand side", *i.e.* $D \subset (-\infty, a)^n + i\mathbb{R}^n$ for some $a \in \mathbb{R}$. It turns out that generally many of the geodesics for such domains have boundary measures absolutely different from $\operatorname{Re} \varphi^*(\lambda) \, d\mathcal{L}^{\mathbb{T}}(\lambda)$, and even if we calculate the radial limits of φ using (i), we cannot say much about φ itself. Even in the simpliest case where D is a left half-plane in $\mathbb{C}$, complex geodesics have boundary measures of the form $\alpha\delta_{\lambda_0}$ for some $\alpha < 0$ and $\lambda_0 \in \mathbb{T}$, while the real parts of their radial limits vanish $\mathcal{L}^{\mathbb{T}}$-almost everywhere on $\mathbb{T}$. Here δ_{λ_0} is the Dirac delta at λ_0. If we want to calculate geodesics using Theorem 1.1, we must use condition (ii), which seems to be rather hard to do.

Our main idea was to replace the conditions from Theorem 1.1 by one condition, expressed in the language of measure theory, which is similar to (i), but uses the boundary measure μ of φ instead of its radial limits. This approach has two great advantages. The first one is that if we calculate μ, then via Poisson formula we obtain φ. The second one is that, as we shall see, for some classes of domains we can calculate μ in a quite similar way as the radial limits, that is, using "shape properties" of the boundary of Re D. The main result of this paper is the following:

Theorem 1.2. *Let $D \subset \mathbb{C}^n$ be a taut convex tube domain and let $\varphi : \mathbb{D} \to D$ be a holomorphic map with the boundary measure μ. Then φ is a complex geodesic for D if and only if there exists a map $h : \mathbb{C} \to \mathbb{C}^n$ of the form $h(\lambda) = \bar{a}\lambda^2 + b\lambda + a$ with some $a \in \mathbb{C}^n$ and $b \in \mathbb{R}^n$, such that $h \not\equiv 0$ and the measure*

$$\bar{\lambda}h(\lambda) \bullet \Big(\operatorname{Re} z\, d\mathcal{L}^{\mathbb{T}}(\lambda) - d\mu(\lambda)\Big)$$

is negative for every $z \in D$.

The expression $\bar{\lambda}h(\lambda) \bullet (\operatorname{Re} z\, d\mathcal{L}^{\mathbb{T}}(\lambda) - d\mu(\lambda))$ is just the real measure

$$\sum_{j=1}^{n} \bar{\lambda}h_j(\lambda)\Big(\operatorname{Re} z_j\, d\mathcal{L}^{\mathbb{T}}(\lambda) - d\mu_j(\lambda)\Big)$$

on the unit circle $\mathbb{T}$. Notation and all necessary properties of considered objects are explained in Section 2. In Section 4 we present how to use Theorem 1.2 to calculate complex geodesics in some classes of convex tube domains. We focus mainly on tubes with unbounded base, where the situation is more interesting. In Example 4.5 we derive direct formulas for complex geodesics in convex tube domains in $\mathbb{C}^2$, which are finite intersections of $\mathbb{H}^2_-$ and tubes of the form

$$\Big\{(z_1, z_2) \in \mathbb{C}^2 : p\operatorname{Re} z_1 + q\operatorname{Re} z_2 < \alpha\Big\}$$

for some $p, q > 0$ and $\alpha < 0$. Such domains cover finite intersections of Reinhardt domains of the form

$$\Big\{(z_1, z_2) \in \mathbb{D}^2 : 0 < |z_1|^p|z_2|^q < \alpha\Big\}$$

for $p, q > 0$ and $\alpha \in (0, 1)$.

2. Preliminaries

Let us begin with some notation: $\mathbb{D}$ is the unit disc in $\mathbb{C}$, $\mathbb{T}$ is the circle $\partial\mathbb{D}$, $\mathbb{H}_-$ is the left half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$, $\mathbb{S}$ is the strip $\{z \in \mathbb{C} : \operatorname{Re} z \in (0, 1)\}$, $\mathbb{D}_*$ is the punctured unit disc, *i.e.* the set $\mathbb{D} \setminus \{0\}$, $\mathcal{L}^{\mathbb{T}}$ is the Lebesgue measure on $\mathbb{T}$, and δ_{λ_0} is the Dirac delta at a point $\lambda_0 \in \mathbb{T}$. By T_c, $c \in \mathbb{D}$, we denote the automorphism $\lambda \mapsto \frac{\lambda - c}{1-\bar{c}\lambda}$ of $\mathbb{D}$. For $z, w \in \mathbb{C}^n$, $\langle z, w\rangle$ is the standard hermitian inner product in $\mathbb{C}^n$, and $z \bullet w$ is the dot product, *i.e.* $z \bullet w = \langle z, \bar{w}\rangle$. Vectors from $\mathbb{C}^n$ are identified with vertical matrices $n \times 1$, and hence $z \bullet w = z^T \cdot w$, where for a matrix A the symbol A^T denotes the transpose of A and $\cdot$ is the standard matrix multiplication. For a holomorphic map $\varphi : \mathbb{D} \to \mathbb{C}^n$, by $\varphi^*(\lambda)$ we denote the radial limit $\lim_{r\to 1^-} \varphi(r\lambda)$ of φ at a point $\lambda \in \mathbb{T}$, whenever it exists. Finally, $\mathcal{C}(\mathbb{T})$ is the space of all complex continuous functions on $\mathbb{T}$, equipped with the supremum norm, and H^p, $p \in [1, \infty]$, is the Hardy space on the unit disc.

Definition 2.1. Let $D \subset \mathbb{C}^n$ be a convex domain and let $\varphi : \mathbb{D} \to D$ be a holomorphic map. We call φ a *complex geodesic* for D if there exists a holomorphic function $f : D \to \mathbb{D}$ such that $f \circ \varphi = \mathrm{id}_{\mathbb{D}}$. In this situation we call f a *left inverse* of φ.

Definition 2.2. We say that a domain $D \subset \mathbb{C}^n$ is *taut* if every sequence $(f_n)_{n=1}^\infty \subset \mathcal{O}(\mathbb{D}, D)$ is compactly divergent or has a subsequence convergent locally uniformly on $\mathbb{D}$ to a map $f \in \mathcal{O}(\mathbb{D}, D)$.

Complex geodesics are holomorphic isometries from $\mathbb{D}$ equipped with the Poincaré distance to D equipped with the Carathéodory pseudodistance. If D is a taut convex domain, then the Carathéodory pseudodistance becomes a distance, and by the Lempert theorem, for any pair of points in D there exists a complex geodesic passing through them.

Definition 2.3. We say that a domain $D \subset \mathbb{C}^n$ is a convex tube if $D = \Omega + i\mathbb{R}^n$ for some convex domain $\Omega \subset \mathbb{R}^n$. In that situation we call Ω the *base* of D and we denote it by $\operatorname{Re} D$.

Observation 2.4. Let $D \subset \mathbb{C}^n$ be a convex tube domain. Then there exist a number $k \in \{0, \dots, n\}$, a convex tube $G \subset \mathbb{H}_-^k$ and a complex affine isomorphism Φ of $\mathbb{C}^n$ such that $\Phi(\mathbb{R}^n) = \mathbb{R}^n$ and $\Phi(D) = G \times \mathbb{C}^{n-k}$.

Moreover, a holomorphic mapping $\varphi : \mathbb{D} \to D$ is a complex geodesic for D if and only if $(\Phi_1, \dots, \Phi_k) \circ \varphi$ is a complex geodesic for G.

Sketch of the proof. It suffices to show the following claim: there exist $k \in \{0, \dots, n\}$, a convex domain $U \subset (-\infty, 0)^k$ and a real affine isomorphism Ψ of $\mathbb{R}^n$ such that $\Psi(\operatorname{Re} D) = U \times \mathbb{R}^{n-k}$. The number k is chosen such that $n - k$ is equal to the maximal dimension of a real affine subspace contained in $\operatorname{Re} D$. It may be assumed that $\operatorname{Re} D = V \times \mathbb{R}^{n-k}$ for a convex domain $V \subset \mathbb{R}^k$ containing no real affine lines. Now one can proceed similarly as in the proof of [2, Proposition 3.5]. $\square$

The condition $\Phi(\mathbb{R}^n) = \mathbb{R}^n$ in Observation 2.4 implies that Φ maps tube domains on tube domains. In view of the above observation, it is enough to restrict our considerations to taut convex tubes. It follows from [2, Theorem 1.1] that a convex tube is taut if and only if its base contains no real affine lines. If D is a taut convex tube, then $k = n$ and $\Phi(D) \subset \mathbb{H}^n_-$. The last inclusion allows us to conclude that if $\varphi : \mathbb{D} \to D$ is a holomorphic map, then the radial limits $\varphi^*(\lambda) = \lim_{r\to 1^-} \varphi(r\lambda)$ exist end belong to $\overline{D}$ for $\mathcal{L}^{\mathbb{T}}$-almost every $\lambda \in \mathbb{T}$. As we will see, φ admits also a boundary measure. All details are presented below.

Now we recall some facts connected with complex measures on the unit circle. Below, we consider only Borel measures on $\mathbb{T}$, so we usually omit the word "Borel". It is known that any finite positive measure on $\mathbb{T}$ is regular in the sense that the measure of any Borel subset $A \subset \mathbb{T}$ may be approximated by the measures of both compact subsets and open supersets of A. Hence, any complex measure on $\mathbb{T}$ is regular, *i.e.* its variation is a regular measure. In view of the Riesz representation theorem, complex measures on $\mathbb{T}$ may be identified with continuous linear functionals on $\mathcal{C}(\mathbb{T})$. Thus, a complex measure ν on $\mathbb{T}$ is uniquely determined by the sequence of integrals $\int_{\mathbb{T}} \zeta^k \, d\nu(\zeta)$, $k \in \mathbb{Z}$. In this paper we usually use real measures. By a real measure we just mean a complex measure with values in $\mathbb{R}$.

We use common shortcut "a.e." for phrase "almost everywhere" or "almost every". If not stated otherwise, it is meant *with respect to the Lebesgue measure* (usually on $\mathbb{T}$).

We use the symbols $\langle \cdot, \cdot \rangle$ and $\bullet$ also for measures and functions. For example, if μ is a tuple $(\mu_1, \ldots, \mu_n)$ of complex measures and $v = (v_1, \ldots, v_n)$ is a vector or a bounded Borel-measurable mapping on $\mathbb{T}$, then $\langle d\mu, v \rangle$ is the measure $\sum_{j=1}^n \bar{v}_j d\mu_j$, and $v \bullet d\mu$ is the measure $\sum_{j=1}^n v_j d\mu_j$, etc. The fact that a real measure ν is positive (respectively negative, null) is shortly denoted by $\nu \geq 0$ (respectively $\nu \leq 0$, $\nu = 0$).

We introduce the family

$$\mathcal{M} := \{f_\mu + i\alpha : \mu \text{ is a real measure on } \mathbb{T}, \quad \alpha \in \mathbb{R}\},$$

where $f_\mu : \mathbb{D} \to \mathbb{C}$ is the holomorphic function given by

$$f_\mu(\lambda) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\zeta + \lambda}{\zeta - \lambda} d\mu(\zeta), \quad \lambda \in \mathbb{D}. \tag{2.1}$$

It is a classical result (see *e.g.* [6, page 10]) that the measures $\operatorname{Re} f_\mu(r\lambda) d\mathcal{L}^{\mathbb{T}}(\lambda)$ tend weakly-* (as continuous linear functionals on $\mathcal{C}(\mathbb{T})$) to μ when $r \to 1^-$, *i.e.* for every $u \in \mathcal{C}(\mathbb{T})$ there is

$$\int_{\mathbb{T}} u(\lambda) \operatorname{Re} f_\mu(r\lambda) d\mathcal{L}^{\mathbb{T}}(\lambda) \to \int_{\mathbb{T}} u(\lambda) d\mu(\lambda) \text{ as } r \to 1^-.$$

In particular, μ is uniquely determined by f_μ. Thus, any $f \in \mathcal{M}$ has a unique decomposition $f = f_\mu + i\alpha$, and μ is then called the *boundary measure* of f. From (2.1) it follows that if a real measure μ on $\mathbb{T}$ and $f \in \mathcal{O}(\mathbb{D}, \mathbb{C})$ are such that

$$\operatorname{Re} f(\lambda) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1 - |\lambda|^2}{|\zeta - \lambda|^2} d\mu(\zeta), \quad \lambda \in \mathbb{D}, \tag{2.2}$$

then $f \in \mathcal{M}$ and μ is the boundary measure for f. The right-hand side of (2.2) is just the real part of the right-hand side of (2.1). It follows from (2.2) and from weak-* limit property that if $f \in \mathcal{M}$, then $\operatorname{Re} f \geq 0$ on $\mathbb{D}$ if and only if its boundary measure is positive.

If a holomorphic function $f : \mathbb{D} \to \mathbb{C}$ is of Hardy class H^1, then it belongs to $\mathcal{M}$, its boundary measure is just $\operatorname{Re} f^* d\mathcal{L}^{\mathbb{T}}$ and when $r \to 1^-$, the functions $\lambda \mapsto f(r\lambda)$ tend to f^* in the L^1 norm with respect to the measure $\mathcal{L}^{\mathbb{T}}$ (see *e.g.* [6, page 35]). Such a situation occurs *e.g.* when the real part of f is bounded (see [6, page 87]). However, we emphasize that generally the boundary measure of a function $f \in \mathcal{M}$ is not equal to $\operatorname{Re} f^* d\mathcal{L}^{\mathbb{T}}$ and formula (2.2) does not hold for f with its radial limits, *i.e.* with the measure $\operatorname{Re} f^* d\mathcal{L}^{\mathbb{T}}$. For example, when $f(\lambda) = \frac{1+\lambda}{1-\lambda}$, we have $\operatorname{Re} f^*(\lambda) = 0$ for a.e. $\lambda \in \mathbb{T}$, while $\mu = 2\pi\delta_1$.

By $\mathcal{M}^n$ we denote the set of all mappings $\varphi = (\varphi_1, \ldots, \varphi_n) \in \mathcal{O}(\mathbb{D}, \mathbb{C}^n)$ with $\varphi_1, \ldots, \varphi_n \in \mathcal{M}$. By the boundary measure of φ we mean the n-tuple $\mu = (\mu_1, \ldots, \mu_n)$ of measures with each μ_j being the boundary measure of φ_j. We write the Poisson formula for mappings in the same way as for functions, *i.e.*

$$\varphi(\lambda) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{\zeta + \lambda}{\zeta - \lambda} d\mu(\zeta) + i \operatorname{Im} \varphi(0), \quad \lambda \in \mathbb{D}, \tag{2.3}$$

where the integral with respect to the tuple $\mu = (\mu_1, \ldots, \mu_n)$ is just the tuple of integrals with respect to $\mu_1, \ldots, \mu_n$.

A weak-* limit argument and uniqueness of boundary measure show that if V is a real $m \times n$ matrix, $b \in \mathbb{R}^m$ and $\varphi \in \mathcal{M}^n$, then the map $\lambda \mapsto V \cdot \varphi(\lambda) + b$ belongs to $\mathcal{M}^m$ and its boundary measure is just $V \cdot \mu + b\, d\mathcal{L}^{\mathbb{T}}$.

The Herglotz representation theorem (see *e.g.* [6, page 5]) states that any $f \in \mathcal{O}(\mathbb{D}, \mathbb{C})$ with non-negative real part belongs to $\mathcal{M}$. As a consequence, the family $\mathcal{O}(\mathbb{D}, \mathbb{H}^n_-)$ is contained in $\mathcal{M}^n$.

Observation 2.4 and the remarks made in last two paragraphs lead us to the following important fact:

Observation 2.5. Let $D \subset \mathbb{C}^n$ be a taut convex tube domain and let $\varphi : \mathbb{D} \to D$ be a holomorphic map. Then $\varphi \in \mathcal{M}^n$, so φ admits a boundary measure.

Proof. Let Φ be as in Observation 2.4. Write $\Phi(z) = V \cdot z + b$ for a square matrix V and a vector $b \in \mathbb{C}^n$. As $\Phi(\mathbb{R}^n) = \mathbb{R}^n$, the matrix V and the vector b have real entries. Set $\widetilde{\varphi}(\lambda) := V \cdot \varphi(\lambda) + b$. We have $\Phi(D) \subset \mathbb{H}^n_-$, so $\widetilde{\varphi}(\mathbb{D}) \subset \mathbb{H}^n_-$. In view of the Herglotz representation theorem, the map $\widetilde{\varphi}$ belongs to $\mathcal{M}^n$ and it admits a boundary measure $\widetilde{\mu}$. Put $\mu := (\mu_1, \ldots, \mu_n) := V^{-1} \cdot (\widetilde{\mu} - b\, d\mathcal{L}^{\mathbb{T}})$. One can check that for each $j = 1, \ldots, n$ the equality (2.2) holds for φ_j and μ_j. □

3. Characterisation of complex geodesics

In this section we focus on proving Theorems 1.1 and 1.2. We start showing Theorem 1.1. Next we state Lemma 3.7 which allows us to derive Theorem 1.2 from Theorem 1.1. As we shall see, Lemma 3.7 is an essential part of this section.

The first result of this section provides us with sufficient conditions for a map to be a geodesic:

Proposition 3.1. *Let $D \subset \mathbb{C}^n$ be a taut convex tube domain, and let $\varphi : \mathbb{D} \to D$ be a holomorphic map. Suppose that there exists a mapping $h : \mathbb{C} \to \mathbb{C}^n$ of the form $h(\lambda) = \bar{a}\lambda^2 + b\lambda + a$ with some $a \in \mathbb{C}^n$, $b \in \mathbb{R}^n$, such that:*

(i) $\operatorname{Re}\left[\bar{\lambda}h(\lambda) \bullet (z - \varphi^*(\lambda))\right] < 0$ *for all $z \in D$ and a.e. $\lambda \in \mathbb{T}$,*
(ii) $\operatorname{Re}\left[h(\lambda) \bullet \frac{\varphi(0)-\varphi(\lambda)}{\lambda}\right] < 0$ *for every $\lambda \in \mathbb{D}_*$.*

Then φ is a complex geodesic for D.

As it was mentioned in Section 2, the above sentence "a.e. $\lambda \in \mathbb{T}$" is meant with respect to the Lebesgue measure. Let us also clarify, that the sentence "for all $z \in D$ and a.e. $\lambda \in \mathbb{T}$" in (i) means "for all $(z, \lambda) \in D \times A$, where $A \subset \mathbb{T}$ is some Borel subset of full $\mathcal{L}^{\mathbb{T}}$ measure". In other words, this "almost everywhere" in "a.e. $\lambda \in \mathbb{T}$" does not depend on z. Similar sentences in further theorems shall be meant in the same way.

Remark 3.2. If $\operatorname{Re} D$ is bounded, then condition (ii) in Proposition 3.1 may be omitted. It follows from the fact that $\operatorname{Re}\varphi$ is bounded and hence the maps φ and $\lambda \mapsto h(\lambda) \bullet \frac{\varphi(0)-\varphi(\lambda)}{\lambda}$ are of class H^1 (with h being a polynomial), so maximum principle can be applied to deduce (ii) from (i) for $z = \varphi(0)$.

Remark 3.3. Generally, condition (i) in Proposition 3.1 turns out to be necessary (see Proposition 3.6) but not sufficient for φ to be a complex geodesic for D. For example, take $D = \mathbb{H}_-$ and $\varphi(\lambda) = \frac{\lambda^2+1}{\lambda^2-1}$. One can check that φ satisfies (i) with $h(\lambda) = \lambda$, but clearly it is not a complex geodesic for D and it does not satisfy (ii). It means that in general (more precisely: for D with unbounded base) one cannot deduce condition (ii) from (i) (with $z = \varphi(0)$) using maximum principle for harmonic functions, because the function in (ii) is not necessarily bounded from above.

Proposition 3.1 is a consequence of the following general lemma. It is worth to point out that the lemma works for any domain D in $\mathbb{C}^n$, not necessarily tube.

Lemma 3.4. *Let $D \subset \mathbb{C}^n$ be a domain and let $\varphi : \mathbb{D} \to D$ be a holomorphic map. Suppose that there exists a map $h \in \mathcal{O}(\mathbb{D}, \mathbb{C}^n)$ such that*

$$\operatorname{Re}\left[h(0) \bullet \varphi'(0)\right] \neq 0$$

and for every $z \in D$ the function $\psi_z \in \mathcal{O}(\mathbb{D}, \mathbb{C})$ defined as

$$\psi_z(\lambda) := \frac{\varphi(0) - \varphi(\lambda)}{\lambda} \bullet h(\lambda) + \frac{h(\lambda) - h(0)}{\lambda} \bullet (z - \varphi(0)) + \lambda\, \overline{h(0) \bullet (z - \varphi(0))}$$

satisfies

$$\operatorname{Re}\psi_z(\lambda) \leq 0, \ \ \lambda \in \mathbb{D}.$$

Then the map φ admits a left inverse on D.

One can check that if $\lambda \in \mathbb{T}$ is such that $\varphi^*(\lambda)$ and $h^*(\lambda)$ exist, then $\psi_z^*(\lambda)$ exists for every $z \in D$ and there holds

$$\operatorname{Re} \psi_z^*(\lambda) = \operatorname{Re} \left[\bar{\lambda} h^*(\lambda) \bullet (z - \varphi^*(\lambda))\right], \ z \in D. \tag{3.1}$$

Observe that Proposition 3.1 follows directly from Lemma 3.4. Indeed, if D, φ, h are as in the proposition, then almost all radial limits of φ exist and using (3.1) we get

$$\operatorname{Re} \psi_z^*(\lambda) < 0 \text{ for all } z \in D \text{ and a.e. } \lambda \in \mathbb{T}.$$

Assumption (ii) and the fact that h is a polynomial imply that every $\operatorname{Re} \psi_z$ is bounded from above. Thus, maximum principle gives $\operatorname{Re} \psi_z(\lambda) \leq 0$ for all $z \in D$ and $\lambda \in \mathbb{D}$. The assumptions of Lemma 3.4 are fulfilled and the conclusion of Proposition 3.1 follows.

Remark 3.5. One can check that if ψ_z are as in Lemma 3.4, then

$$\operatorname{Re} \psi_z(\lambda) \leq 0 \text{ for all } (z, \lambda) \in D \times \mathbb{D}$$

if and only if the following two conditions holds:

(i) $\operatorname{Re} \psi_z$ is bounded from above for every $z \in D$,
(ii) $\operatorname{Re} \psi_z^*(\lambda) \leq 0$ for all $z \in D$ and a.e. $\lambda \in \mathbb{T}$.

Applying (3.1) we see that the above two conditions "correspond" to conditions (ii) and (i) of Proposition 3.1. The reason for which Lemma 3.4 is not formulated in the same way as Proposition 3.1 is to avoid using radial limits of φ, as they do not necessarily exist. From the assumptions of Lemma 3.4 it follows only that there exist radial limits of the maps h and $\lambda \mapsto h(\lambda) \bullet \varphi(\lambda)$. One can deduce it from the fact that almost all radial limits of every ψ_z exist, which follows from the assumption that $\psi_z(\mathbb{D}) \subset \mathbb{H}_-$ for each $z \in D$.

To show Lemma 3.4 we follow the proof of [5, Lemma 8.2.2]. The latter is similar to Lemma 3.4, but it works for bounded domains and it is stated in a slightly different form. In the proof of [5, Lemma 8.2.2] a version of maximum principle for harmonic functions is applied a few times. It causes some troubles if we do not assume boundedness of D, because knowing only that a function $u : \mathbb{D} \to \mathbb{R}$ is harmonic on $\mathbb{D}$ and $u^* < 0$ a.e. on $\mathbb{T}$, generally we cannot conclude that $u < 0$ in $\mathbb{D}$. It turns out that the assumption $\operatorname{Re} \psi_z(\lambda) \leq 0$ allows us to avoid this problem and argue as in the proof of [5, Lemma 8.2.2].

Proof of Lemma 3.4. For $\epsilon \geq 0$ define

$$\begin{aligned} \Phi_\epsilon(z, \lambda) &= (z - \varphi(\lambda)) \bullet h(\lambda) - \epsilon\lambda, \ z \in \mathbb{C}^n, \lambda \in \mathbb{D}, \\ \Psi_\epsilon(z, \lambda) &= \frac{1}{\lambda} \Phi_\epsilon(z, \lambda), \ z \in \mathbb{C}^n, \lambda \in \mathbb{D}_*. \end{aligned}$$

The function $\lambda \mapsto \Psi_\epsilon(\varphi(0), \lambda)$ extends holomorphically through 0. There holds

$$\Psi_\epsilon(\varphi(0), \lambda) = \psi_{\varphi(0)}(\lambda) - \epsilon, \ \lambda \in \mathbb{D}, \epsilon \geq 0.$$

This implies

$$\operatorname{Re}\Psi_\epsilon(\varphi(0),\lambda) < 0,\ \lambda\in\mathbb{D}, \epsilon > 0. \tag{3.2}$$

Moreover, as $\operatorname{Re}\psi_{\varphi(0)}(0) = -\operatorname{Re}\left[h(0)\bullet\varphi'(0)\right] \neq 0$ and $\operatorname{Re}\psi_{\varphi(0)} \leq 0$ on $\mathbb{D}$, maximum principle gives $\operatorname{Re}\psi_{\varphi(0)} < 0$ on $\mathbb{D}$. Hence

$$\operatorname{Re}\Psi_0(\varphi(0),\lambda) < 0,\ \lambda\in\mathbb{D}. \tag{3.3}$$

From (3.2) and (3.3) it follows that for every $\epsilon \geq 0$ the point 0 is the only root of the function $\lambda \mapsto \Phi_\epsilon(\varphi(0),\lambda)$ on $\mathbb{D}$ and it is a simple root.

Assume for a moment that

$$\text{there exists } f \in \mathcal{O}(D,\mathbb{D}) \text{ such that } \Phi_0(z, f(z)) = 0,\ z\in D. \tag{3.4}$$

We claim that f is a left inverse for φ. There holds $f(\varphi(0)) = 0$, because 0 is the only root of $\lambda\mapsto\Phi_0(\varphi(0),\lambda)$ on $\mathbb{D}$. Set

$$\Gamma_1 = \{(z, f(z)) : z\in D\}, \qquad \Gamma_2 = \{(\varphi(\lambda),\lambda) : \lambda\in\mathbb{D}\}.$$

It is clear that $\Gamma_1 \subset \Phi_0^{-1}(0)$ and $\Gamma_2 \subset \Phi_0^{-1}(0)$. We have $\Phi_0(\varphi(0), 0) = 0$ and, by inequality (3.3), $\frac{\partial\Phi_0}{\partial\lambda}(\varphi(0),0) = \Psi_0(\varphi(0),0) \neq 0$. Thus, the implicit mapping theorem gives that there exists an open neighbourhood $U \subset D\times\mathbb{D}$ of $(\varphi(0),0)$ such that $U\cap\Phi_0^{-1}(0)$ is equal to the graph of some holomorphic function of the variable z defined near $\varphi(0)$ which maps $\varphi(0)$ to 0. Since $(\varphi(0),0)\in\Gamma_1$, shrinking U if necessary we get $U\cap\Phi_0^{-1}(0) = U\cap\Gamma_1$. Therefore $U\cap\Gamma_2 \subset U\cap\Gamma_1$, so $(\varphi(\lambda),\lambda)\in U\cap\Gamma_1$ for λ near 0. This implies $f(\varphi(\lambda)) = \lambda$ for λ near 0 and hence on the whole $\mathbb{D}$.

It remains to prove (3.4), and for this it suffices to show that

$$\text{for any } \epsilon > 0 \text{ there is } f_\epsilon \in \mathcal{O}(D,\mathbb{D}) \text{ such that } \Phi_\epsilon(z, f_\epsilon(z)) = 0,\ z\in D. \tag{3.5}$$

Indeed, using the Montel theorem we choose a sequence $(f_{\epsilon_k})_k$ ($\epsilon_k \to 0$ as $k\to\infty$) convergent to a holomorphic function $f : D\to\mathbb{C}$. As 0 is the only root of $\lambda\mapsto\Phi_\epsilon(\varphi(0),\lambda)$, we get $f_\epsilon(\varphi(0)) = 0$ and hence $f(D)\subset\mathbb{D}$, which allows us to derive (3.4).

The statement (3.5) follows from the following claim:

$$\begin{array}{l}\text{for every } \epsilon > 0 \text{ and } K \subset\subset D \text{ there exists } r\in(0,1) \text{ such that}\\ \operatorname{Re}\Psi_\epsilon(z,\lambda) < 0 \text{ for } z\in K, |\lambda|\in[r,1).\end{array} \tag{3.6}$$

Indeed, assume (3.6) and fix $\epsilon > 0$. Let $z\in D$ and let $r = r(\epsilon, z)$ be taken as above for $K = \{z\}$. The function $\lambda\mapsto\Phi_\epsilon(z,\lambda)$ has no roots in $\mathbb{D}\setminus r\mathbb{D}$, because $\operatorname{Re}\Psi_\epsilon(z,\lambda) < 0$ for $|\lambda|\in[r,1)$. Moreover,

$$\frac{1}{2\pi i}\int_{r\mathbb{T}} \frac{\frac{\partial\Phi_\epsilon}{\partial\lambda}(z,\lambda)}{\Phi_\epsilon(z,\lambda)}d\lambda = 1 + \frac{1}{2\pi i}\int_{r\mathbb{T}} \frac{\frac{\partial\Psi_\epsilon}{\partial\lambda}(z,\lambda)}{\Psi_\epsilon(z,\lambda)}d\lambda = 1. \tag{3.7}$$

The last integral is just the index at 0 of the curve $s \mapsto \Psi_\epsilon(z, re^{is})$, equal to 0 by (3.6). In view of (3.7), the function $\lambda \mapsto \Phi_\epsilon(z, \lambda)$ has only one root in $\mathbb{D}$ (counting with multiplicities). Denote this root by $f_\epsilon(z)$. The function $f_\epsilon : D \to \mathbb{D}$ satisfies $\Phi_\epsilon(z, f_\epsilon(z)) = 0$ for every $z \in D$. We need only to show that it is holomorphic.

Fix $K \subset\subset D$ and let $r = r(\epsilon, K)$ be as in (3.6). Again, $\lambda \mapsto \Phi_\epsilon(z, \lambda)$ has no roots in $\mathbb{D} \setminus r\mathbb{D}$ for $z \in K$, so $f_\epsilon(K) \subset r\mathbb{D}$. As $f_\epsilon(z)$ is the only root of $\lambda \mapsto \Phi_\epsilon(z, \lambda)$ and it belongs to $r\mathbb{D}$, we have the formula

$$f_\epsilon(z) = \frac{1}{2\pi i} \int_{r\mathbb{T}} \lambda \frac{\frac{\partial \Phi_\epsilon}{\partial \lambda}(z, \lambda)}{\Phi_\epsilon(z, \lambda)} d\lambda, \ z \in K, \tag{3.8}$$

which implies that f_ϵ is holomorphic in the interior of K. As K is arbitrary, we obtain $f_\epsilon \in \mathcal{O}(D, \mathbb{D})$.

It remains to show (3.6). Fix $\epsilon > 0$ and $K \subset\subset D$. For $z \in K$ and $\lambda \in \mathbb{D}_*$ we have

$$\operatorname{Re} \Psi_\epsilon(z, \lambda) = \operatorname{Re} \psi_z(\lambda) + \operatorname{Re} \left[\frac{1}{\lambda} h(0) \bullet (z - \varphi(0)) - \lambda \overline{h(0) \bullet (z - \varphi(0))} \right] - \epsilon.$$

The second term of the right-hand side tends uniformly (with respect to $z \in K$) to 0 as $|\lambda| \to 1$, and the first term is non-positive. This gives (3.6) and finishes the proof. □

Now we state necessary conditions for a map φ to be a complex geodesic:

Proposition 3.6. *Let $D \subset \mathbb{C}^n$ be a taut convex tube domain, let $\varphi : \mathbb{D} \to D$ be a complex geodesic and let $f : D \to \mathbb{D}$ be a left inverse for φ. Define*

$$h(\lambda) := \left(\frac{\partial f}{\partial z_1}(\varphi(\lambda)), \dots, \frac{\partial f}{\partial z_n}(\varphi(\lambda)) \right), \quad \lambda \in \mathbb{D}.$$

Then:

(i) $h(\lambda) = \bar{a}\lambda^2 + b\lambda + a$ *for some* $a \in \mathbb{C}^n$, $b \in \mathbb{R}^n$, *and* $h \not\equiv 0$,

(ii) $\operatorname{Re} \left[\bar{\lambda} h(\lambda) \bullet (z - \varphi^*(\lambda)) \right] < 0$ *for all* $z \in D$ *and a.e.* $\lambda \in \mathbb{T}$,

(iii) $\operatorname{Re} \left[h(\lambda) \bullet \frac{\varphi(0) - \varphi(\lambda)}{\lambda} \right] < 0$ *for every* $\lambda \in \mathbb{D}_*$.

Propositions 3.1 and 3.6 immediately give Theorem 1.1. In the proof of Proposition 3.6 we strongly use the assumption that D is a convex tube domain.

Proof of Proposition 3.6. Differentiating the equality $f(\varphi(\lambda)) = \lambda$ we get

$$h(\lambda) \bullet \varphi'(\lambda) = 1, \ \lambda \in \mathbb{D}. \tag{3.9}$$

In particular, $h \not\equiv 0$.

For $z \in D$ and $t \in [0,1]$ define

$$f_{z,t}(\lambda) := f((1-t)\varphi(\lambda) + tz), \ \ \lambda \in \mathbb{D}.$$

We have $f_{z,t} \in \mathcal{O}(\mathbb{D}, \mathbb{D})$ and $f_{z,0}(\lambda) = \lambda$. One can also check that

$$\left.\frac{d|f_{z,t}(\lambda)|^2}{dt}\right|_{t=0} = 2\,\mathrm{Re}\left[\bar{\lambda} h(\lambda) \bullet (z - \varphi(\lambda))\right]. \tag{3.10}$$

On the other hand, since $f_{z,t} \in \mathcal{O}(\mathbb{D}, \mathbb{D})$, [1, Lemma 1.2.4] gives

$$\frac{1 - |f_{z,t}(\lambda)|}{1 - |\lambda|} \geq \frac{1 - |f_{z,t}(0)|}{1 + |f_{z,t}(0)|}.$$

We get the inequality

$$|f_{z,t}(\lambda)| - |\lambda| \leq \frac{2|f_{z,t}(0)|}{1 + |f_{z,t}(0)|}(1 - |\lambda|).$$

Therefore

$$\frac{|f_{z,t}(\lambda)|^2 - |\lambda|^2}{t} \leq 2\,\frac{|f_{z,t}(\lambda)| - |\lambda|}{t} \leq \frac{4\left|\frac{1}{t} f_{z,t}(0)\right|}{1 + |f_{z,t}(0)|}(1 - |\lambda|).$$

Taking the limit for t tending to 0, we obtain

$$\left.\frac{d|f_{z,t}(\lambda)|^2}{dt}\right|_{t=0} \leq 4(1-|\lambda|)\left|\left.\frac{df_{z,t}(0)}{dt}\right|_{t=0}\right| \leq 4(1-|\lambda|)\,|h(0) \bullet (z - \varphi(0))|. \tag{3.11}$$

In summary, from (3.10) and (3.11) we get the following important inequality:

$$\mathrm{Re}\left[\bar{\lambda} h(\lambda) \bullet (z - \varphi(\lambda))\right] \leq 2(1 - |\lambda|)\,|h(0) \bullet (z - \varphi(0))|, \ \ \lambda \in \mathbb{D}, z \in D. \tag{3.12}$$

Putting $z = \varphi(0)$ in (3.12) and dividing this inequality by $|\lambda|^2$ we obtain the weak inequality in (iii). The strong one follows from maximum principle for the harmonic function

$$\mathbb{D} \ni \lambda \mapsto \mathrm{Re}\left[h(\lambda) \bullet \frac{\varphi(0) - \varphi(\lambda)}{\lambda}\right].$$

It is bounded from above, because we already proved the weak inequality in (iii), and it is non-constant, because its value at $\lambda = 0$ equals to $-\mathrm{Re}\left[h(0) \bullet \varphi'(0)\right] = -1$, by (3.9). Condition (iii) is proved.

Putting $z = \varphi(0) + ise_j$ in (3.12), where $j \in \{1, \dots, n\}$, $s \in \mathbb{R}$ and $e_1, \dots, e_n$ is the usual canonical base of $\mathbb{C}^n$, we get

$$\mathrm{Re}\left[\bar{\lambda} h(\lambda) \bullet (\varphi(0) - \varphi(\lambda))\right] \leq \mathrm{Im}\left(\bar{\lambda} h_j(\lambda)\right) s + 2(1 - |\lambda|)\,|h_j(0)||s|.$$

Hence, for fixed $\lambda \in \mathbb{D}$ the function of variable s on the right-hand side is bounded from below. One can check that this implies

$$\left|\operatorname{Im}\left(\bar{\lambda} h_j(\lambda)\right)\right| \leq 2(1-|\lambda|)\,|h_j(0)|, \ \lambda \in \mathbb{D}. \tag{3.13}$$

Writing $h_j(\lambda) = h_j(0) + \lambda g_j(\lambda)$ we obtain

$$|\lambda|^2 \left|\operatorname{Im} g_j(\lambda)\right| - \left|\operatorname{Im}\left(\bar{\lambda} h_j(0)\right)\right| \leq 2\,(1-|\lambda|)\left|h_j(0)\right|.$$

From this it follows that $\operatorname{Im} g_j$ is bounded on the annulus $\mathbb{D} \setminus \frac{1}{2}\mathbb{D}$. By maximum principle, $\operatorname{Im} g_j$ is bounded on whole $\mathbb{D}$, so g_j and h_j are of class H^1. In particular, almost all radial limits of g_j and h_j exist. Taking limit for λ tending radially to $\mathbb{T}$ in (3.13), we obtain $\operatorname{Im}(\bar{\lambda} h_j^*(\lambda)) = 0$ a.e. on $\mathbb{T}$. This implies

$$\operatorname{Im}\left(g_j^*(\lambda) - \overline{h_j(0)}\lambda\right) = \operatorname{Im}\left(\bar{\lambda} h_j^*(\lambda)\right) = 0 \ \text{ for a.e. } \lambda \in \mathbb{T}.$$

The function $\lambda \mapsto i(g_j(\lambda) - \overline{h_j(0)}\lambda)$ is of class H^1 and real parts of its radial limits vanish a.e. on $\mathbb{T}$, so its boundary measure is a null measure. Therefore $g_j(\lambda) - \overline{h_j(0)}\lambda$ is equal to some real constant b_j. This gives (i) and allows us to extend h to the whole $\mathbb{C}$.

Taking limit for λ tending radially to $\mathbb{T}$ in (3.12), we get the weak inequality in (ii). The strong inequality follows from the fact that for a.e. $\lambda \in \mathbb{T}$ the mapping

$$D \ni z \mapsto \operatorname{Re}\left[\bar{\lambda} h(\lambda) \bullet (z - \varphi^*(\lambda))\right] \in \mathbb{R}$$

is affine over $\mathbb{R}$, non-constant, and hence open. $\square$

Note that we can obtain the statement (i) in Proposition 3.6 immediately, using more general results (see [3, Theorem 3]).

We are ready to prove Theorem 1.2 which will follow, as announced, from Theorem 1.1 and the following important:

Lemma 3.7. *Let $D \subset \mathbb{C}^n$ be a taut convex tube, let $\varphi : \mathbb{D} \to D$ be a holomorphic map with the boundary measure μ, and let $h(\lambda) = \bar{a}\lambda^2 + b\lambda + a$, $\lambda \in \mathbb{D}$, for some $a \in \mathbb{C}^n$, $b \in \mathbb{R}^n$, with $h \not\equiv 0$. Then*

$$\text{the measure } \bar{\lambda} h(\lambda) \bullet \left(\operatorname{Re} z\, d\mathcal{L}^{\mathbb{T}}(\lambda) - d\mu(\lambda)\right) \text{ is negative for every } z \in D \tag{m}$$

if and only if the following two conditions holds:

(i) $\operatorname{Re}\left[\bar{\lambda} h(\lambda) \bullet (z - \varphi^*(\lambda))\right] < 0$ *for all $z \in D$ and a.e. $\lambda \in \mathbb{T}$,*
(ii) $\operatorname{Re}\left[h(\lambda) \bullet \frac{\varphi(0)-\varphi(\lambda)}{\lambda}\right] < 0$ *for every $\lambda \in \mathbb{D}_*$.*

All measures in (m) are regular and real. Let us also note that in view of this lemma, the function h in Theorem 1.2 is the same h as in Theorem 1.1. We shall use this fact in Section 4.

Proof of Lemma 3.7. We start with showing that conditions (i) and (ii) are both satisfied if and only if

$$\operatorname{Re}\psi_z(\lambda) \le 0 \text{ for all } \lambda \in \mathbb{D}, z \in D, \tag{3.14}$$

where $\psi_z : \mathbb{D} \to \mathbb{C}$ is the holomorphic function defined as

$$\psi_z(\lambda) = \frac{\varphi(0) - \varphi(\lambda)}{\lambda} \bullet h(\lambda) + \frac{h(\lambda) - h(0)}{\lambda} \bullet (z - \varphi(0)) + \lambda\, \overline{h(0) \bullet (z - \varphi(0))} \tag{3.15}$$

for $z \in D$, $\lambda \in \mathbb{D}_*$ (and extended holomorphically through 0). The functions ψ_z are defined in the same way as in Lemma 3.4. As D is a taut convex tube, almost all radial limits of φ exist. What is more, whenever $\varphi^*(\lambda)$ exists, there also exists $\psi_z^*(\lambda)$ for every $z \in D$. There holds

$$\operatorname{Re}\psi_z^*(\lambda) = \operatorname{Re}\left[\bar{\lambda}h(\lambda) \bullet (z - \varphi^*(\lambda))\right] \text{ for all } z \in D \text{ and a.e. } \lambda \in \mathbb{T}. \tag{3.16}$$

Now, if (i) and (ii) hold, then $\operatorname{Re}\psi_z$ is bounded from above and $\operatorname{Re}\psi_z^* < 0$ a.e. on $\mathbb{T}$, so the maximum principle gives (3.14). On the other hand, (3.14) allows us to derive the weak inequalities in (i) and (ii). As $h \not\equiv 0$, the strong inequality in (i) follows from the fact that the map $z \mapsto \operatorname{Re}\left[\bar{\lambda}h(\lambda) \bullet (z - \varphi^*(\lambda))\right]$ is open for a.e. $\lambda \in \mathbb{T}$. The strong inequality in (ii) is a consequence of maximum principle, because by (i) with $z = \varphi(0)$, the function $\lambda \mapsto \operatorname{Re}\left[h(\lambda) \bullet \frac{\varphi(0)-\varphi(\lambda)}{\lambda}\right]$ is not identically equal to 0.

Let ν_z denote the measure in condition (m), that is

$$\nu_z = \bar{\lambda}h(\lambda) \bullet \left(\operatorname{Re} z\, d\mathcal{L}^{\mathbb{T}}(\lambda) - d\mu(\lambda)\right).$$

To finish the proof, it suffices to show that conditions (3.14) and (m) are equivalent, and for this it is enough to prove that $\psi_z \in \mathcal{M}$ and ν_z is the boundary measure of ψ_z.

We claim that formula (2.2) holds for $\operatorname{Re}\psi_z$ and ν_z, *i.e.*

$$\operatorname{Re}\psi_z(\lambda) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{1 - |\lambda|^2}{|\zeta - \lambda|^2} d\nu_z(\zeta), \quad \lambda \in \mathbb{D}, z \in D. \tag{3.17}$$

In view of the definition of $\mathcal{M}$, this will complete the proof. Fix $z \in D$. Set

$$\begin{aligned}
\nu_{z,1} &= \bar{\lambda}h(\lambda) \bullet \left(\operatorname{Re}\varphi(0) d\mathcal{L}^{\mathbb{T}}(\lambda) - d\mu(\lambda)\right),\\
\nu_{z,2} &= \operatorname{Re}\left[\bar{\lambda}\,(h(\lambda) - h(0)) \bullet (z - \varphi(0))\right] d\mathcal{L}^{\mathbb{T}}(\lambda),\\
\nu_{z,3} &= \operatorname{Re}\left[\bar{\lambda}h(0) \bullet (z - \varphi(0))\right] d\mathcal{L}^{\mathbb{T}}(\lambda).
\end{aligned}$$

Every $\nu_{z,k}$ is a real measure on $\mathbb{T}$. Using the fact that $\bar{\lambda}h(\lambda) \in \mathbb{R}^n$ for $\lambda \in \mathbb{T}$, we can write ν_z as the sum $\nu_z = \nu_{z,1} + \nu_{z,2} + \nu_{z,3}$. On the other hand, the function ψ_z

equals to sum of three functions, as in (3.15): $\psi_z = \psi_{z,1} + \psi_{z,2} + \psi_{z,3}$, where

$$\begin{aligned}\psi_{z,1}(\lambda) &= \frac{\varphi(0)-\varphi(\lambda)}{\lambda} \bullet h(\lambda),\\ \psi_{z,2}(\lambda) &= \frac{h(\lambda)-h(0)}{\lambda} \bullet (z-\varphi(0)),\\ \psi_{z,3}(\lambda) &= \lambda\, \overline{h(0)\bullet(z-\varphi(0))}.\end{aligned}$$

To get (3.17) it suffices to show that the terms of the sum for ν_z "correspond" to the terms of the sum for ψ_z, *i.e.* that for $k = 1, 2, 3$ there holds

$$\operatorname{Re}\psi_{z,k}(\lambda) = \frac{1}{2\pi}\int_{\mathbb{T}} \frac{1-|\lambda|^2}{|\zeta-\lambda|^2} d\nu_{z,k}(\zeta),\ \ \lambda\in\mathbb{D}, z\in D. \tag{3.18}$$

We have

$$\nu_{z,2} = \operatorname{Re}\left[\frac{h(\lambda)-h(0)}{\lambda}\bullet(z-\varphi(0))\right] d\mathcal{L}^{\mathbb{T}}(\lambda)$$

and

$$\nu_{z,3} = \operatorname{Re}\left[\lambda\,\overline{h(0)\bullet(z-\varphi(0))}\right] d\mathcal{L}^{\mathbb{T}}(\lambda).$$

Therefore, it follows directly from classical Poisson formula for functions which are harmonic in a neighbourhood of $\overline{\mathbb{D}}$ that (3.18) holds for $k = 2, 3$. To finish the proof it remains to show it for $k = 1$.

Observe the following fact: if $u \in \mathcal{C}(\mathbb{T})$, σ and σ_r ($r \in (0,1)$) are real measures on $\mathbb{T}$ such that σ_r tend weakly-* to σ when $r \to 1^-$, then the measures $u d\sigma_r$ tend weakly-* to $u d\sigma$.

We apply this fact in the following way. Write $\varphi = (\varphi_1, \dots, \varphi_n)$, $\mu = (\mu_1, \dots, \mu_n)$ and $h = (h_1, \dots, h_n)$. Since $\operatorname{Re}\varphi_j(r\lambda)d\mathcal{L}^{\mathbb{T}}(\lambda)$ tend weakly-* to μ_j and $\lambda \mapsto \bar\lambda h_j(\lambda)$ is continuous on $\mathbb{T}$, the measures $\bar\lambda h_j(\lambda)\operatorname{Re}\varphi_j(r\lambda)d\mathcal{L}^{\mathbb{T}}(\lambda)$ tend to $\bar\lambda h_j(\lambda)d\mu_j(\lambda)$. This implies

$$\bar\lambda h_j(\lambda)\operatorname{Re}\left(\varphi_j(0)-\varphi_j(r\lambda)\right)d\mathcal{L}^{\mathbb{T}}(\lambda) \longrightarrow \bar\lambda h_j(\lambda)\left(\operatorname{Re}\varphi_j(0)d\mathcal{L}^{\mathbb{T}}(\lambda) - d\mu_j(\lambda)\right)$$

for $j = 1, \dots, n$. Taking the sum over $j = 1, \dots, n$ and using the fact that $\bar\lambda h_j(\lambda) \in \mathbb{R}$, we obtain

$$\nu_{z,1} = \lim_{r\to 1^-}\operatorname{Re}\left[\frac{\varphi(0)-\varphi(r\lambda)}{\lambda}\bullet h(\lambda)\right] d\mathcal{L}^{\mathbb{T}}(\lambda).$$

For $r \in (0,1)$ the function $\lambda \mapsto \operatorname{Re}\left[\frac{\varphi(0)-\varphi(r\lambda)}{\lambda}\bullet h(\lambda)\right]$ is harmonic in a neighbourhood of $\overline{\mathbb{D}}$, so for fixed $\lambda \in \mathbb{D}$ the classical Poisson formula gives

$$\frac{1}{2\pi}\int_{\mathbb{T}}\frac{1-|\lambda|^2}{|\zeta-\lambda|^2}\operatorname{Re}\left[\frac{\varphi(0)-\varphi(r\zeta)}{\zeta}\bullet h(\zeta)\right]d\mathcal{L}^{\mathbb{T}}(\zeta) = \operatorname{Re}\left[\frac{\varphi(0)-\varphi(r\lambda)}{\lambda}\bullet h(\lambda)\right].$$

The last expression converges to $\operatorname{Re}\psi_{z,1}(\lambda)$ when $r \to 1^-$ and hence (3.18) holds for $k = 1$. The proof is complete. □

4. Calculating complex geodesics

In this section we focus on deriving formulas for complex geodesics in convex tubes in $\mathbb{C}^2$ covering finite intersections of Reinhardt domains of the form

$$\left\{(z_1, z_2) \in \mathbb{D}^2 : \ 0 < |z_1|^p |z_2|^q < \alpha\right\}$$

with some $p, q > 0$ and $\alpha \in (0, 1)$ (Example 4.5). For this, we state two lemmas which partially describe boundary measures of geodesics in some special situations (Lemmas 4.2 and 4.3) and which are applied afterwards to calculate all complex geodesics in the example.

Before we start analysing the examples, we make a few useful remarks. Below we assume that $D \subset \mathbb{C}^n$ is a taut convex tube domain.

If $\varphi : \mathbb{D} \to D$ is a complex geodesic and $\lambda \in \mathbb{T}$ is such that $\bar{\lambda}h(\lambda) \neq 0$ and the inequality $\operatorname{Re}\left[\bar{\lambda}h(\lambda) \bullet (z - \varphi^*(\lambda))\right] < 0$ holds for all $z \in D$, then $\varphi^*(\lambda) \in \partial D$ and the vector $\bar{\lambda}h(\lambda)$ is outward from D at $\varphi^*(\lambda)$ (and hence it is outward from $\operatorname{Re} D$ at $\operatorname{Re}\varphi^*(\lambda)$, as $\bar{\lambda}h(\lambda) \in \mathbb{R}^n$). This observation is helpful in deriving some information about h and φ, or even in deriving a formula for φ in the case of bounded base of D, as, for instance, in Example 4.6. However, it turns out that it is not sufficient if the base of D is unbounded.

If φ is a complex geodesic for D, then, by Lemma 3.7, the mapping h from Theorem 1.2 satisfies the conclusion of Theorem 1.1, and vice versa (what is more, h may be chosen as in Proposition 3.6). In particular, given a map h as in Theorem 1.2 we can apply for it the conclusions made in the previous paragraph. Let us note that a map h satisfying the conclusion of Theorem 1.2 need not to be unique.

Observe the following fact: given a finite positive measure ν on $\mathbb{T}$ and a non-negative continuous function u on $\mathbb{T}$ with $u^{-1}(\{0\}) = \{\lambda_1, \dots, \lambda_m\}$, if $u d\nu$ is a null measure, then $\nu = \sum_{j=1}^m \alpha_j \delta_{\lambda_j}$ for some constants $\alpha_1, \dots, \alpha_m \geq 0$.

Recall that by [5, Lemma 8.4.6], if $h \in \mathcal{O}(\mathbb{D}, \mathbb{C})$ is of class H^1 and such that $\bar{\lambda}h^*(\lambda) > 0$ for a.e. $\lambda \in \mathbb{T}$, then h is of the form $c(\lambda - d)(1 - \bar{d}\lambda)$ for some $d \in \overline{\mathbb{D}}$, $c > 0$. In particular, $\bar{\lambda}h(\lambda) = c|\lambda - d|^2$ for $\lambda \in \mathbb{T}$ and the function h has at most one zero on $\mathbb{T}$ (counting without multiplicities). By the observation we made above, if ν is a finite negative measure on $\mathbb{T}$ such that the measure $\bar{\lambda}h(\lambda)d\nu(\lambda)$ is null, then $\nu = \alpha\delta_{\lambda_0}$ for some $\alpha \leq 0$, $\lambda_0 \in \mathbb{T}$, with $\alpha h(\lambda_0) = 0$ (we take $\lambda_0 = d$ if $d \in \mathbb{T}$, otherwise ν is null and we put $\alpha = 0$ with an arbitrary λ_0). We shall quite often use this fact.

Let as also note that if for some $p, v \in \mathbb{R}^n$ the inequality $\langle \operatorname{Re} z - p, v\rangle < 0$ holds for all $z \in D$ and $\varphi : \mathbb{D} \to D$ is a holomorphic map with the boundary measure μ, then a similar inequality holds for measures: $\langle d\mu - p\, d\mathcal{L}^{\mathbb{T}}, v\rangle \leq 0$. This is an immediate consequence of the fact that this measure is equal to the weak-* limit of the negative measures $\langle \operatorname{Re}\varphi(r\lambda) - p, v\rangle\, d\mathcal{L}^{\mathbb{T}}(\lambda)$, when $r \to 1^-$. In particular, if $\operatorname{Re} D \subset (-\infty, 0)^n$, then $\mu_1, \dots, \mu_n \leq 0$.

Example 4.1. This example is just an introduction. We present an application of Theorem 1.2 for calculating complex geodesics in the simpliest n-dimensional convex tube with unbounded base. A modification of the argument below let us derive Lemma 4.2, which plays a key role in Example 4.5.

A map $\varphi \in \mathcal{M}^n$ with the boundary measure $\mu = (\mu_1, \ldots, \mu_n)$ is a complex geodesic for the domain $\mathbb{H}^n_-$ if and only if $\mu_{j_0} = \alpha\delta_{\lambda_0}$ for some $j_0 \in \{1, \ldots, n\}$, $\alpha < 0, \lambda_0 \in \mathbb{T}$.

Indeed, assume that φ is a geodesic for $\mathbb{H}^n_-$ and let $h = (h_1, \ldots, h_n)$ be as in Theorem 1.2, *i.e.* $h \not\equiv 0$ and

$$\bar{\lambda}h(\lambda) \bullet \Big(\operatorname{Re} z\, d\mathcal{L}^{\mathbb{T}}(\lambda) - d\mu(\lambda)\Big) \leq 0, \;\; z \in \mathbb{H}^n_-.$$

Taking the limit for z tending to 0 we obtain $\bar{\lambda}h(\lambda) \bullet d\mu(\lambda) \geq 0$. On the other hand, $\bar{\lambda}h_j(\lambda) \geq 0$ on $\mathbb{T}$, because h is continuous on $\mathbb{T}$ and $\bar{\lambda}h(\lambda)$ is outward from $\mathbb{H}^n_-$ for a.e. $\lambda \in \mathbb{T}$, and $\mu_j \leq 0$, as $\operatorname{Re}\varphi_j < 0$ on $\mathbb{D}$. This implies $\bar{\lambda}h(\lambda) \bullet d\mu(\lambda) \leq 0$, and finally:

$$\bar{\lambda}h_1(\lambda)d\mu_1(\lambda) + \ldots + \bar{\lambda}h_n(\lambda)d\mu_n(\lambda) = 0.$$

Since all terms of the above sum are negative measures, we have $\bar{\lambda}h_j(\lambda)d\mu_j(\lambda) = 0$ for every $j = 1, \ldots, n$. There exists j_0 such that $h_{j_0} \not\equiv 0$, and as μ_j is non-null for every j (because $\operatorname{Re}\varphi_j \not\equiv 0$), the function h_{j_0} must admit a root λ_0 on $\mathbb{T}$. Hence, we have $\mu_{j_0} = \alpha\delta_{\lambda_0}$ for some $\alpha < 0$. In view of (2.3), the map φ_{j_0} is given by the formula

$$\varphi_{j_0}(\lambda) = \frac{\alpha}{2\pi}\frac{\lambda_0 + \lambda}{\lambda_0 - \lambda} + i\beta, \;\; \lambda \in \mathbb{D},$$

for some real constant β, which is a well-known expression for a complex geodesic of $\mathbb{H}^n_-$.

Lemma 4.2. *Let $D \subset \mathbb{C}^n$ be a taut convex tube and let $p \in \partial\operatorname{Re} D$. Define*

$$V := \left\{v \in \mathbb{R}^n : \langle \operatorname{Re} z - p, v\rangle < 0, \; z \in D\right\}.$$

Let $\varphi : \mathbb{D} \to D$ be a complex geodesic for D with the boundary measure μ and let h be as in Theorem 1.2. *Put $A := \left\{\lambda \in \mathbb{T} : \bar{\lambda}h(\lambda) \in \operatorname{int} V\right\}$. Then $\chi_A\, d\mu = p\chi_A\, d\mathcal{L}^{\mathbb{T}}$ and* $\operatorname{Re}\varphi^*(\lambda) = p$ *for every* $\lambda \in A$.

In the situation of the above lemma, if $\operatorname{int} V \neq \varnothing$, then we can say that $\operatorname{Re} D$ has a "vertex" at the point p. The aim of the lemma is to handle the situation where $\operatorname{Re}\varphi^*$ sends some λ's to that vertex. To detect some (not all) of these λ's we analyse the behaviour of the function h instead of analysing behaviour of $\operatorname{Re}\varphi^*$. All λ's detected in this way form the set A (this is the reason for which in the definition of A there is $\operatorname{int} V$, not V itself – for λ such that $\bar{\lambda}h(\lambda) \in V \setminus \operatorname{int} V$ it is possible that $\operatorname{Re}\varphi^*(\lambda) \neq p$). This approach allows us to state not only that $\operatorname{Re}\varphi^*(\lambda) = p$ for $\lambda \in A$, but much more: the boundary measure of φ is equal to $p\, d\mathcal{L}^{\mathbb{T}}$ on the set A. This lemma plays a key role in Example 4.5.

If the set $\operatorname{int} V$ is not empty, then it is an open, convex, infinite cone with the vertex at 0. In the case $n = 2$ one can find two vectors $v_1, v_2 \in \mathbb{R}^n$ such that $\operatorname{int} V$

consists of those $v \in \mathbb{R}^n$ which lies "between" v_1 and v_2, *i.e.* $\operatorname{int} V = \{v \in \mathbb{R}^n : \det[v_1, v], \det[v, v_2] > 0\}$.

In the definition of A the set $\operatorname{int} V$ cannot be replaced by V, because then the equality $\chi_A\, d\mu = p\chi_A\, d\mathcal{L}^{\mathbb{T}}$ does not hold any longer. For example, take $D = \mathbb{H}^2_-$, $p = (0,0)$, and let φ be given by the measure $-(\delta_1, \delta_1 + \delta_{-1})$, that is $\varphi(\lambda) = \frac{1}{2\pi}\left(\frac{\lambda+1}{\lambda-1}, \frac{\lambda+1}{\lambda-1} + \frac{\lambda-1}{\lambda+1}\right)$. The map φ is clearly a geodesic for D and one can check that if $h = (h_1, h_2)$ is as in Theorem 1.2, then $h_1(1) = 0$ and $h_2 \equiv 0$ (because $\bar{\lambda} h_2(\lambda) \geq 0$ on $\mathbb{T}$ and h has roots on $\mathbb{T}$ at 1 and -1). As $V = [0,\infty)^2 \setminus \{(0,0)\}$, we have $\{\lambda \in \mathbb{T} : \bar{\lambda} h(\lambda) \in V\} = \mathbb{T} \setminus \{1\}$, while the measure μ is clearly not equal to $(0,0)$ on $\mathbb{T} \setminus \{1\}$.

Proof. We may assume that $\operatorname{int} V \neq \varnothing$. For linearly independent vectors $v_1, \ldots, v_n \in \operatorname{int} V$ set

$$Q_{v_1,\ldots,v_n} := \{\alpha_1 v_1 + \ldots + \alpha_n v_n : \alpha_1, \ldots, \alpha_n > 0\}.$$

One can check that the sets $Q_{v_1,\ldots,v_n}$ form an open covering of $\operatorname{int} V$, and hence it suffices to show the conclusion with the set $\operatorname{int} V$ replaced by $Q_{v_1,\ldots,v_n}$.

Fix $v_1, \ldots, v_n \in \operatorname{int} V$ linearly independent, and let W be a non-singular, real $n \times n$ matrix with rows $v_1, \ldots, v_n$. Set $Q := Q_{v_1,\ldots,v_n}$, $B := \{\lambda \in \mathbb{T} : \bar{\lambda} h(\lambda) \in Q\}$. We are going to show that $\chi_B\, d\mu = p\chi_B\, d\mathcal{L}^{\mathbb{T}}$ and $\operatorname{Re} \varphi^* = p$ on B. Let

$$\widetilde{\mu} := \left(\widetilde{\mu}_1, \ldots, \widetilde{\mu}_n\right) := W \cdot \left(d\mu - p\, d\mathcal{L}^{\mathbb{T}}\right).$$

As $\widetilde{\mu}_j = \langle d\mu - p\, d\mathcal{L}^{\mathbb{T}}, v_j\rangle$, the measures $\widetilde{\mu}_j$ are negative. The mapping

$$\widetilde{h}(\lambda) := \left(\widetilde{h}_1(\lambda), \ldots, \widetilde{h}_n(\lambda)\right) := \left(W^{-1}\right)^T \cdot h(\lambda),\ \lambda \in \mathbb{C},$$

satisfies $\bar{\lambda}\widetilde{h}(\lambda) \in (0,\infty)^n$ for $\lambda \in B$, because $(W^T)^{-1} \cdot (\alpha_1 v_1 + \ldots + \alpha_n v_n) = (\alpha_1, \ldots, \alpha_n)$. Thus

$$\chi_B(\lambda)\, \bar{\lambda}\widetilde{h}(\lambda) \bullet d\widetilde{\mu}(\lambda) \leq 0.$$

By the definition of $\widetilde{\mu}$ and $\widetilde{h}$, there is

$$\bar{\lambda}\widetilde{h}(\lambda) \bullet \left(W \cdot (\operatorname{Re} z - p)\, d\mathcal{L}^{\mathbb{T}}(\lambda) - d\widetilde{\mu}(\lambda)\right) = \bar{\lambda} h(\lambda) \bullet \left(\operatorname{Re} z\, d\mathcal{L}^{\mathbb{T}}(\lambda) - d\mu(\lambda)\right),\ z \in D,$$

so the measure $\chi_B(\lambda)\, \bar{\lambda}\widetilde{h}(\lambda) \bullet (W \cdot (\operatorname{Re} z - p)\, d\mathcal{L}^{\mathbb{T}}(\lambda) - d\widetilde{\mu}(\lambda))$ is negative for every $z \in D$. Taking limit for z tending to p, we obtain

$$\chi_B(\lambda)\, \bar{\lambda}\widetilde{h}(\lambda) \bullet d\widetilde{\mu}(\lambda) \geq 0.$$

In summary, the measure $\chi_B(\lambda)\, \bar{\lambda}\widetilde{h}(\lambda) \bullet d\widetilde{\mu}(\lambda)$ is null. As it is the sum of the negative measures $\chi_B(\lambda)\, \bar{\lambda}\widetilde{h}_j(\lambda) d\widetilde{\mu}_j(\lambda)$, all of them are null, and hence all $\chi_B d\widetilde{\mu}_j$ are also null. Therefore

$$\chi_B\, d\mu = W^{-1} \cdot \chi_B\, d\widetilde{\mu} + p\chi_B\, d\mathcal{L}^{\mathbb{T}} = p\chi_B\, d\mathcal{L}^{\mathbb{T}},$$

so the first part is proved.

For the second, applying Poisson formula to $\varphi - p$, we have

$$\operatorname{Re}\varphi(\lambda) - p = \frac{1}{2\pi}\int_{\mathbb{T}\setminus B} \frac{1-|\lambda|^2}{|\zeta-\lambda|^2} d(\mu - p\mathcal{L}^{\mathbb{T}})(\zeta), \ \lambda \in \mathbb{D},$$

so $\operatorname{Re}\varphi(r\lambda) \to p$ as $r \to 1^-$, for any $\lambda \in B$. □

Lemma 4.3. *Let $D \subset \mathbb{C}^n$ be a taut convex tube and let V be a real $m \times n$ matrix with linearly independent rows $v_1, \dots, v_m \in \mathbb{R}^n$, $m \geq 1$, such that the domain*

$$\widetilde{D} := \{V \cdot z : z \in D\}$$

is a taut convex tube in $\mathbb{C}^m$.

(i) *Let $\varphi : \mathbb{D} \to D$ be a complex geodesic for D, and let h be as in Theorem 1.2. If $\bar{\lambda}h(\lambda) \in \operatorname{span}_{\mathbb{R}}\{v_1, \dots, v_m\}$ for every $\lambda \in \mathbb{T}$, then the mapping $\widetilde{\varphi} : \lambda \mapsto V \cdot \varphi(\lambda)$ is a complex geodesic for $\widetilde{D}$.*
(ii) *If for a holomorphic map $\varphi : \mathbb{D} \to D$ the mapping $\widetilde{\varphi} : \lambda \mapsto V \cdot \varphi(\lambda)$ is a complex geodesic for $\widetilde{D}$, then φ is a complex geodesic for D.*

This lemma allows us to "decrease" the dimension n, when we are trying to find a formula for φ, provided that the functions $h_1, \dots, h_n$ are linearly dependent. In such situation, if we know formulas for geodesics in $\widetilde{D}$ (*e.g.* for $m = 1$, because then $\widetilde{D}$ is a strip or a half-plane in $\mathbb{C}$), then by (i) we obtain some information about φ, and by (ii) we conclude that it may be hard to get something more if we have no additional knowledge. We use this lemma in Example 4.5, where the set $\partial\operatorname{Re} D$ consists of segments and half-lines.

The situation when $h_1, \dots, h_n$ are linearly dependent occurs *e.g.* when for some proper affine subspace W of $\mathbb{R}^n$ there is $\operatorname{Re}\varphi^*(\lambda) \in \operatorname{int}_W(W \cap \partial\operatorname{Re} D)$ on the set of positive $\mathcal{L}^{\mathbb{T}}$ measure, because then the vectors $\bar{\lambda}h(\lambda)$ are orthogonal to W on the set of positive measure and hence on whole $\mathbb{T}$ (by the identity principle). Here int_W denotes the interior with respect to W. If the set $\operatorname{int}_W(W \cap \partial\operatorname{Re} D)$ is non-empty, then $W \cap \operatorname{Re} D = \varnothing$.

Note that in the situation as in (ii) the map φ admits in fact a left inverse defined on the convex tube domain $\{z \in \mathbb{C}^n : V \cdot z \in \widetilde{D}\}$, which may be larger than D and not taut (its base may contain real lines).

Proof. We prove the first part. The matrix V^T may be viewed as a complex linear isomorphism from $\mathbb{C}^m$ to $\operatorname{span}_{\mathbb{C}}\{v_1, \dots, v_m\}$. The mapping $\widetilde{h} : \mathbb{C} \to \mathbb{C}^m$ defined as $\widetilde{h}(\lambda) = \left(V^T\right)^{-1} \cdot h(\lambda)$ is of the form $\bar{a}\lambda^2 + b\lambda + a$ (with some $a \in \mathbb{C}^m, b \in \mathbb{R}^m$) and it satisfies $\widetilde{h}(\lambda)^T \cdot V = h(\lambda)^T$ and $\widetilde{h} \not\equiv 0$. We are going to apply Theorem 1.2 for $\widetilde{\varphi}, \widetilde{D}, \widetilde{h}$. A weak-* limit argument shows that the boundary measure $\widetilde{\mu}$ of $\widetilde{\varphi}$ equals $V \cdot d\mu$. For any $z \in D$ there is

$$\begin{aligned}\bar{\lambda}\widetilde{h}(\lambda) \bullet \Big(V \cdot \operatorname{Re} z \, d\mathcal{L}^{\mathbb{T}}(\lambda) - d\widetilde{\mu}(\lambda)\Big) &= \bar{\lambda}\widetilde{h}(\lambda)^T \cdot V \cdot \Big(\operatorname{Re} z \, d\mathcal{L}^{\mathbb{T}}(\lambda) - d\mu(\lambda)\Big) \\ &= \bar{\lambda}h(\lambda) \bullet \Big(\operatorname{Re} z \, d\mathcal{L}^{\mathbb{T}}(\lambda) - d\mu(\lambda)\Big).\end{aligned}$$

The last measure is negative.

To prove the second part it suffices to observe that if $f : \widetilde{D} \to \mathbb{D}$ is a left inverse for $\widetilde{\varphi}$, then the map $z \mapsto f(V \cdot z)$, defined on the domain $\{z \in \mathbb{C}^n : V \cdot z \in \widetilde{D}\} \supset D$, is a left inverse for φ. □

Example 4.4. A map $\varphi \in \mathcal{M}$ with the boundary measure μ is a complex geodesic for the strip $\mathbb{S}$ if and only if

$$\mu = \chi_{\{\lambda\in\mathbb{T}:\,\bar{\lambda}h(\lambda)>0\}}\, d\mathcal{L}^{\mathbb{T}} \tag{4.1}$$

for some function $h : \lambda \mapsto \bar{a}\lambda^2 + b\lambda + a$ with $a \in \mathbb{C}, b \in \mathbb{R}, |b| < 2|a|$ (the last condition is equivalent to $\mathcal{L}^{\mathbb{T}}(\{\lambda \in \mathbb{T} : \bar{\lambda}h(\lambda) > 0\}) \in (0, 2\pi)$).

Indeed, assume that φ is a complex geodesic for $\mathbb{S}$ and let h be as in Theorem 1.2. The vector $\bar{\lambda}h(\lambda)$ is outward from $\operatorname{Re}\mathbb{S} = (0, 1)$ at $\operatorname{Re}\varphi^*(\lambda) \in \partial\operatorname{Re}\mathbb{S} = \{0, 1\}$ for a.e. $\lambda \in \mathbb{T}$, so $\operatorname{Re}\varphi^*(\lambda) = 1$ when $\bar{\lambda}h(\lambda) > 0$, and $\operatorname{Re}\varphi^*(\lambda) = 0$ when $\bar{\lambda}h(\lambda) < 0$ (for a.e. λ). Thus $\mu = \operatorname{Re}\varphi^* d\mathcal{L}^{\mathbb{T}} = \chi_{\{\lambda\in\mathbb{T}:\,\bar{\lambda}h(\lambda)>0\}}\, d\mathcal{L}^{\mathbb{T}}$. As φ is non-constant, there is $0 \neq \mu \neq \mathcal{L}^{\mathbb{T}}$, and hence $\mathcal{L}^{\mathbb{T}}(\{\lambda \in \mathbb{T} : \bar{\lambda}h(\lambda) > 0\}) \in (0, 2\pi)$.

Of course, formulas for geodesics in $\mathbb{S}$ are well-known, and it is good to write explicitly the formula for the map induced by the boundary measure (4.1) (we shall need it in Example 4.5). Take h as above. The mapping

$$\tau(\lambda) := -\frac{i}{\pi}\log\left(i\,\frac{1+\lambda}{1-\lambda}\right), \tag{4.2}$$

where log denotes the branch of the logarithm with the argument in $[0, 2\pi)$, is a biholomorphism from $\mathbb{D}$ to $\mathbb{S}$. It extends continuously to $\overline{\mathbb{D}} \setminus \{-1, 1\}$, and it sends the arc $\{\lambda \in \mathbb{T} : \operatorname{Im}\lambda > 0\}$ to the line $1 + i\,\mathbb{R}$ and the arc $\{\lambda \in \mathbb{T} : \operatorname{Im}\lambda < 0\}$ to the line $i\,\mathbb{R}$. Put $c = \frac{-b}{2|a|+\sqrt{4|a|^2-b^2}}$. One can check that $\operatorname{Im}(iT_c(\frac{\bar{a}}{|a|}\lambda)) \in \bar{\lambda}h(\lambda)\,(0, \infty)$ for every $\lambda \in \mathbb{T}$. This implies that the map $\lambda \mapsto \tau\left(iT_c\left(\frac{\bar{a}}{|a|}\lambda\right)\right)$ sends the arc $\{\lambda \in \mathbb{T} : \bar{\lambda}h(\lambda) > 0\}$ to the line $1 + i\,\mathbb{R}$ and the arc $\{\lambda \in \mathbb{T} : \bar{\lambda}h(\lambda) < 0\}$ to the line $i\,\mathbb{R}$. It is of class H^1, so its boundary measure equals $\chi_{\{\lambda\in\mathbb{T}:\,\bar{\lambda}h(\lambda)>0\}}\, d\mathcal{L}^{\mathbb{T}}$. Since its value at 0 is real, we obtain the equality

$$\varphi_h(\lambda) := \frac{1}{2\pi}\int_{\{\zeta\in\mathbb{T}:\,\bar{\zeta}h(\zeta)>0\}} \frac{\zeta+\lambda}{\zeta-\lambda}\, d\mathcal{L}^{\mathbb{T}}(\zeta) = \tau\left(iT_c\left(\tfrac{\bar{a}}{|a|}\lambda\right)\right),\ \lambda \in \mathbb{D}. \tag{4.3}$$

We shall use it in Example 4.5. Let us recall that the above equality holds for every $h : \mathbb{C} \to \mathbb{C}$ of the form $\bar{a}\lambda^2+b\lambda+a$, $a \in \mathbb{C}, b \in \mathbb{R}$, with $|b| < 2|a|$, or equivalently: $\mathcal{L}^{\mathbb{T}}(\{\lambda \in \mathbb{T} : \bar{\lambda}h(\lambda) > 0\}) \in (0, 2\pi)$.

Example 4.5. Consider a convex tube domain D with $\operatorname{Re} D$ contained in $(-\infty, 0)^2$ and $\partial\operatorname{Re} D$ being a sum of a horizontal half-line contained in $(-\infty, 0] \times \{0\}$, a vertical half-line contained in $\{0\} \times [-\infty, 0)$, and some finite number of segments. More formally: let

$$D := \left\{z \in \mathbb{C}^2 : \left\langle \operatorname{Re} z - p_j, v_j \right\rangle < 0 \text{ for } j = 1, \ldots, m\right\},$$

where $m \geq 2$, $v_1, \dots, v_m \in [0,\infty)^2$, $p_0, \dots, p_m \in (-\infty, 0]^2$, $v_j = (v_{j,1}, v_{j,2})$, $p_j = (p_{j,1}, p_{j,2})$ are such that:

- $0 = p_{0,1} = p_{1,1} > p_{2,1} > \dots > p_{m-1,1} > p_{m,1}$,
- $0 = p_{m,2} = p_{m-1,2} > p_{m-2,2} > \dots > p_{1,2} > p_{0,2}$,
- $\det\big[v_j, v_{j+1}\big] > 0$ for $j = 1, \dots, m-1$,
- $\langle p_{j+1} - p_j, v_{j+1}\rangle = 0$ for $j = 0, \dots, m-1$

(the points p_0 and p_m play only a supporting role). The base of D is shown in Figure 4.1. By the assumptions we have:

- $\langle \operatorname{Re} z - p_j, v_{j+1}\rangle < 0$ for $z \in D$, $j = 0, \dots, m-1$,
- $v_{1,1} > 0$, $v_{1,2} = 0$, $v_{m,1} = 0$, $v_{m,2} > 0$,
- $\partial \operatorname{Re} D = \{0\} \times (-\infty, p_{1,2}] \cup \bigcup_{j=1}^{m-2} [p_j, p_{j+1}] \cup (-\infty, p_{m-1,1}] \times \{0\}$.

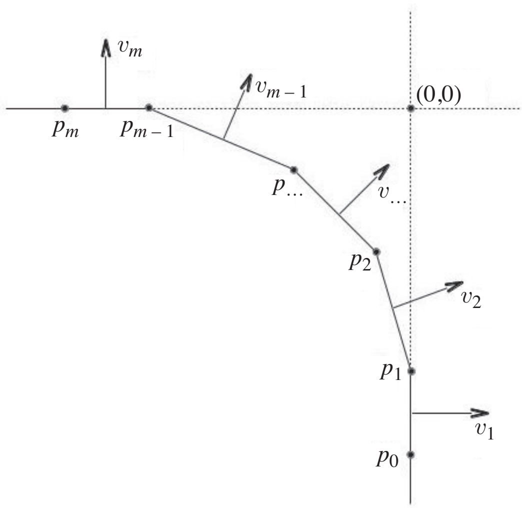

Figure 4.1. The base of D.

Let $\varphi \in \mathcal{O}(\mathbb{D}, D)$ be a complex geodesic and let $\mu = (\mu_1, \mu_2)$ be its boundary measure. Choose h as in Theorem 1.2, *i.e.* $h(\lambda) = \bar{a}\lambda^2 + b\lambda + a$ with $a = (a_1, a_2) \in \mathbb{C}^2$, $b = (b_1, b_2) \in \mathbb{R}^2$, $h = (h_1, h_2)$, $h \not\equiv 0$, such that

$$\bar{\lambda} h(\lambda) \bullet \Big(\operatorname{Re} z \, d\mathcal{L}^{\mathbb{T}}(\lambda) - d\mu(\lambda)\Big) \leq 0, \ \ z \in D. \tag{4.4}$$

For a.e. $\lambda \in \mathbb{T}$ the vector $\bar{\lambda} h(\lambda)$ is outward from $\operatorname{Re} D$ at the boundary point $\operatorname{Re} \varphi^*(\lambda)$, so

$$\bar{\lambda} h_l(\lambda) \geq 0, \ \ \lambda \in \mathbb{T}, l = 1, 2.$$

Set

$$\begin{aligned} A_j &:= \left\{\lambda \in \mathbb{T} : \det\left[\bar\lambda h(\lambda), v_j\right] < 0 < \det\left[\bar\lambda h(\lambda), v_{j+1}\right]\right\},\ j = 1, \dots, m-1, \\ B_j &:= \left\{\lambda \in \mathbb{T} : \det\left[\bar\lambda h(\lambda), v_j\right] = 0\right\},\ j = 1, \dots, m, \\ B &:= \bigcup_{j=1}^{m} B_j. \end{aligned}$$

The sets $A_1, \dots, A_{m-1}, B$ are pairwise disjoint and we have

$$B \cup \bigcup_{j=1}^{m-1} A_j = \mathbb{T}$$

because every non-zero vector from $[0, \infty)^2$ lies "between" some v_j and v_{j+1}, or is parallel to some v_j.

If $\mathcal{L}^{\mathbb{T}}(B) > 0$ then for some $j_0 \in \{1, \dots, m\}$ there is $\mathcal{L}^{\mathbb{T}}(B_{j_0}) > 0$ and the identity principle gives $B_{j_0} = \mathbb{T}$. Applying part (i) of Lemma 4.3 to the 1×2 matrix with the row v_{j_0} we get that $\langle \varphi(\cdot) - p_{j_0}, v_{j_0}\rangle$ is a geodesic for $\mathbb{H}_-$. In view of part (ii) of that lemma, the condition obtained is sufficient for φ to be a complex geodesic, so there is nothing more to do in this case.

Consider the situation where $\mathcal{L}^{\mathbb{T}}(B) = 0$; the set B is then finite and $v_{j,2}h_1 - v_{j,1}h_2 \not\equiv 0$, which in particular gives $h_1 \not\equiv 0, h_2 \not\equiv 0$. By equation (4.4) we get that the measure $\chi_B(\lambda)\,\bar\lambda h(\lambda) \bullet d\mu(\lambda)$ is positive ($\chi_B\, d\mathcal{L}^{\mathbb{T}}$ is null). Since $\bar\lambda h_l(\lambda) \geq 0$ on $\mathbb{T}$ and $\mu_l \leq 0$, we have $\chi_B(\lambda)\bar\lambda h_l(\lambda) d\mu_l(\lambda) \leq 0$ $(l = 1, 2)$. Hence, the measure $\chi_B(\lambda)\,\bar\lambda h(\lambda) \bullet d\mu(\lambda)$ is negative and in summary it is null. As it is equal to sum of the negative measures $\chi_B(\lambda)\bar\lambda h_1(\lambda) d\mu_1(\lambda)$ and $\chi_B(\lambda)\bar\lambda h_2(\lambda) d\mu_2(\lambda)$, both of them are null. Each h_l has at most one root on $\mathbb{T}$ (counting without multiplicities), so $\chi_B\, d\mu_l = \alpha_l \delta_{\lambda_l}$ for some $\lambda_l \in \mathbb{T}, \alpha_l \leq 0$, with $\alpha_l h_l(\lambda_l) = 0$. Applying Lemma 4.2 to D, p_j, φ and h (the set A in the lemma is here exactly the set A_j) we obtain

$$\chi_{A_j} d\mu = p_j \chi_{A_j} d\mathcal{L}^{\mathbb{T}},\ j = 1, \dots, m-1.$$

Therefore

$$\mu_l = \sum_{j=1}^{m-1} p_{j,l} \chi_{A_j} d\mathcal{L}^{\mathbb{T}} + \alpha_l \delta_{\lambda_l},\ l = 1, 2, \tag{4.5}$$

because $\mu_l = \sum_{j=1}^{m-1} \chi_{A_j} d\mu_l + \chi_B d\mu_l$. At this point, using (4.5) and the Poisson formula we can express the map φ as an integral with parameters a, b, α_1, α_2, and up to an imaginary constant. In fact, it is possible to derive a direct formula for it using the mappings φ_h defined in equation (4.3) in Example 4.4. To this end, let

$$C_j := \left\{\lambda \in \mathbb{T} : \det\left[\bar\lambda h(\lambda), v_j\right] < 0\right\},\ j = 1, \dots, m. \tag{4.6}$$

We have $C_1 \supset C_2 \supset \dots \supset C_m$. The set $(C_j \setminus C_{j+1}) \setminus A_j \subset B$ is of zero Lebesgue measure and $A_j \subset C_j \setminus C_{j+1}$, so $\chi_{A_j} d\mathcal{L}^{\mathbb{T}} = \chi_{C_j} d\mathcal{L}^{\mathbb{T}} - \chi_{C_{j+1}} d\mathcal{L}^{\mathbb{T}}$. Moreover,

$\mathcal{L}^{\mathbb{T}}(C_1) = 2\pi$ and $\mathcal{L}^{\mathbb{T}}(C_m) = 0$, because $C_1 = \{\lambda \in \mathbb{T} : \bar{\lambda} h_2(\lambda) > 0\}$ and $C_m = \{\lambda \in \mathbb{T} : \bar{\lambda} h_1(\lambda) < 0\} = \varnothing$. Thus, formula (4.5) may be written as

$$\mu_l = p_{1,l} d\mathcal{L}^{\mathbb{T}} + \sum_{j=2}^{m-1} \big(p_{j,l} - p_{j-1,l}\big)\chi_{C_j} d\mathcal{L}^{\mathbb{T}} + \alpha_l \delta_{\lambda_l},\ l = 1, 2. \tag{4.7}$$

The measures $\chi_{C_j}\, d\mathcal{L}^{\mathbb{T}}$ induces complex geodesics in $\mathbb{S}$, provided that $\mathcal{L}^{\mathbb{T}}(C_j) \in (0, 2\pi)$, because

$$C_j = \big\{\lambda \in \mathbb{T} :\ \bar{\lambda}\left(v_{j,1} h_2(\lambda) - v_{j,2} h_1(\lambda)\right) > 0\big\}$$

(see Example 4.4 for details). Therefore, it is fine to remove from the sum (4.7) those j's which do not satisfy this condition. Thus, set

$$k_1 := \max\left\{j \geq 1 :\ \mathcal{L}^{\mathbb{T}}\big(C_j\big) = 2\pi\right\},\ k_2 := \min\left\{j \leq m :\ \mathcal{L}^{\mathbb{T}}(C_j) = 0\right\}. \tag{4.8}$$

There is $1 \leq k_1 < k_2 \leq m$. By (4.7) we obtain

$$\mu_l = p_{k_1,l} d\mathcal{L}^{\mathbb{T}} + \sum_{j=k_1+1}^{k_2-1} \big(p_{j,l} - p_{j-1,l}\big)\chi_{C_j} d\mathcal{L}^{\mathbb{T}} + \alpha_l \delta_{\lambda_l},\ l = 1, 2 \tag{4.9}$$

(note that it is possible that the above sum is empty, *i.e.* that $k_1 + 1 > k_2 - 1$). Now, for $j \in \{k_1 + 1, \dots, k_2 - 1\}$ we have $\mathcal{L}^{\mathbb{T}}(C_j) \in (0, 2\pi)$, and the Poisson formula allows us to derive the following formula for φ:

$$\varphi_l(\lambda) = p_{k_1,l} + \sum_{j=k_1+1}^{k_2-1} \big(p_{j,l} - p_{j-1,l}\big)\varphi_{v_{j,1}h_2 - v_{j,2}h_1}(\lambda) + \frac{\alpha_l}{2\pi}\,\frac{\lambda_l + \lambda}{\lambda_l - \lambda} + i\beta_l,\quad l = 1, 2, \tag{4.10}$$

where β_1, β_2 are some real constants and $\varphi_{v_{j,1}h_2 - v_{j,2}h_1}$ are as in (4.3), *i.e.*

$$\varphi_{v_{j,1}h_2 - v_{j,2}h_1}(\lambda) = \tau\left(iT_{c_j}\left(\frac{\overline{v_{j,1}a_2 - v_{j,2}a_1}}{|v_{j,1}a_2 - v_{j,2}a_1|}\lambda\right)\right),\ \lambda \in \mathbb{D},$$

with $\tau(\lambda) = -\frac{i}{\pi}\log\left(i\,\frac{1+\lambda}{1-\lambda}\right)$ and

$$c_j = \frac{-\big(v_{j,1}b_2 - v_{j,2}b_1\big)}{2\big|v_{j,1}a_2 - v_{j,2}a_1\big| + \sqrt{4\big|v_{j,1}a_2 - v_{j,2}a_1\big|^2 - \big(v_{j,1}b_2 - v_{j,2}b_1\big)^2}}$$

(note that for $j = k_1 + 1, \dots, k_2 - 1$ there is $|v_{j,1}b_2 - v_{j,2}b_1| < 2|v_{j,1}a_2 - v_{j,2}a_1|$, because $\mathcal{L}^{\mathbb{T}}(C_j) \in (0, 2\pi)$, and hence c_j and $\varphi_{v_{j,1}h_2 - v_{j,2}h_1}$ are well-defined).

In summary, a holomorphic map $\varphi : \mathbb{D} \to \mathbb{C}^2$ is a complex geodesic for the domain D if and only if at least one of the following conditions holds:

(i) $\varphi(\mathbb{D}) \subset D$ and for some $j \in \{1 \dots m\}$ the map $\lambda \mapsto \langle \varphi(\lambda) - p_j, v_j \rangle$ is a complex geodesic for $\mathbb{H}_-$, or

(ii) $\varphi(\mathbb{D}) \subset D$ and the map φ is of the form (4.10) with some $\lambda_1, \lambda_2 \in \mathbb{T}$, $\alpha_1, \alpha_2 \le 0, \beta_1, \beta_2 \in \mathbb{R}$, and a map $h = (h_1, h_2)$ of the form $\bar{a}\lambda^2 + b\lambda + a$ with $a = (a_1, a_2) \in \mathbb{C}^2$, $b = (b_1, b_2) \in \mathbb{R}^2$, such that $\bar{\lambda}h_1(\lambda), \bar{\lambda}h_2(\lambda) \ge 0$ on $\mathbb{T}$, $\alpha_1 h_1(\lambda_1) = \alpha_2 h_2(\lambda_2) = 0$, $v_{j,1}h_2 - v_{j,2}h_1 \not\equiv 0$ for any $j = 1 \dots, m$, where k_1, k_2 are given by (4.8) with C_j given by (4.6).

So far, we have proved only that if φ is a complex geodesic for D, then it satisfies one of the above conditions. We are going to show the opposite implication now. Take a holomorphic map $\varphi : \mathbb{D} \to \mathbb{C}^2$. If φ satisfies (i), then Lemma 4.3 does the job, so consider the situation as in (ii). As $\varphi(\mathbb{D}) \subset D$, clearly φ admits a boundary measure $\mu = (\mu_1, \mu_2)$. Then (4.10) holds, which gives (4.9) and hence (4.7). As $v_{j,1}h_2 - v_{j,2}h_1 \not\equiv 0$ for any j, the set B is of $\mathcal{L}^{\mathbb{T}}$ measure 0, so $\chi_{A_j} = \chi_{C_j} - \chi_{C_{j+1}}$ a.e. on $\mathbb{T}$ (with respect to $\mathcal{L}^{\mathbb{T}}$). Thus, (4.7) implies (4.5). From the equality (4.5) it follows that

$$\chi_{A_j} d\mu_l = p_{j,l}\chi_{A_j} d\mathcal{L}^{\mathbb{T}} \text{ and } \chi_B d\mu_l = \alpha_l \delta_{\lambda_l} \quad \text{for } j = 1, \dots, m-1, \; l = 1, 2. \tag{4.11}$$

Indeed, since $\mathbb{T}$ is equal to sum of the pairwise disjoint sets $A_1, \dots, A_{m-1}, B$, the first statement is obvious, and for the second observe that if $\alpha_l = 0$, then we are done, and if $\alpha_l < 0$, then $h_l(\lambda_l) = 0$, so $\lambda_l \notin A_j$ for any j and hence $\lambda_l \in B$.

If we show that for every set $E \in \{A_1, \dots, A_{m-1}, B\}$ and every point $z \in D$ the measure

$$\bar{\lambda}h(\lambda) \bullet \Big(\operatorname{Re} z\, \chi_E(\lambda)\, d\mathcal{L}^{\mathbb{T}}(\lambda) - \chi_E(\lambda)\, d\mu(\lambda)\Big)$$

is negative, then we are done via Theorem 1.2.

If $E = B$, then $\chi_E\, d\mathcal{L}^{\mathbb{T}}$ is a null measure and as $\bar{\lambda}h_l(\lambda)\alpha_l\, d\delta_{\lambda_l}(\lambda) = 0$, $l = 1, 2$, by (4.11) the measure $\bar{\lambda}h(\lambda) \bullet \chi_E(\lambda)\, d\mu(\lambda)$ is also null.

If $E = A_j$ for some $j = 1, \dots, m-1$, then by (4.11) we need to show that the measure $\bar{\lambda}h(\lambda) \bullet (\operatorname{Re} z - p_j)\, \chi_{A_j}(\lambda)\, d\mathcal{L}^{\mathbb{T}}(\lambda)$ is negative for every $z \in D$. But if $\lambda \in A_j$, then the vector $\bar{\lambda}h(\lambda)$ lies “between” v_j and v_{j+1}, so $\bar{\lambda}h(\lambda) = \gamma_1 v_j + \gamma_2 v_{j+1}$ for some $\gamma_1, \gamma_2 \ge 0$ and hence $\bar{\lambda}h(\lambda) \bullet (\operatorname{Re} z - p_j) \le 0$.

Therefore, we proved that complex geodesics for D are exactly the mappings of the form (i) or (ii).

At the end we present a simple example of convex tube domain with bounded base. Here, the condition with radial limits (Theorem 1.1) suffices to obtain a direct formula for the real part of a geodesic φ, as its boundary measure is just $\operatorname{Re} \varphi^* d\mathcal{L}^{\mathbb{T}}$.

Example 4.6. Let

$$D = \Big\{(z_1, z_2) \in \mathbb{C}^2 : \; (\operatorname{Re} z_1)^2 + (\operatorname{Re} z_2)^2 < 1\Big\}.$$

Let $\varphi : \mathbb{D} \to D$ be a complex geodesic and let h be as in Theorem 1.1. For a.e. $\lambda \in \mathbb{T}$ the vector $\bar{\lambda} h(\lambda)$ is a normal vector to $\partial \mathrm{Re}\, D$ at the point $\mathrm{Re}\,\varphi^*(\lambda) \in \partial \mathrm{Re}\, D$, so $\bar{\lambda} h(\lambda) \in [0, \infty)\,\mathrm{Re}\,\varphi^*(\lambda)$. As $\|\mathrm{Re}\,\varphi^*(\lambda)\| = 1$ (we mean the euclidean norm), we get

$$\mathrm{Re}\,\varphi^*(\lambda) = \frac{\bar{\lambda} h(\lambda)}{\|\bar{\lambda} h(\lambda)\|} \quad \text{for a.e. } \lambda \in \mathbb{T}.$$

The map h is of the form $\bar{a}\lambda^2 + 2b\lambda + a$ with $a \in \mathbb{C}^n$, $b \in \mathbb{R}^n$, $(a, b) \neq (0, 0)$, so

$$\mathrm{Re}\,\varphi^*(\lambda) = \frac{\mathrm{Re}\,(\bar{a}\lambda) + b}{\|\mathrm{Re}\,(\bar{a}\lambda) + b\|}, \quad \text{a.e. } \lambda \in \mathbb{T}. \tag{4.12}$$

As the boundary measure of φ equals $\mathrm{Re}\,\varphi^* d\mathcal{L}^{\mathbb{T}}$, we can derive an integral formula for φ using Poisson formula.

On the other hand, by a similar argument one can show that any $\varphi \in \mathcal{O}(\mathbb{D}, D)$ satisfying (4.12) with some $a \in \mathbb{C}^n$, $b \in \mathbb{R}^n$, $(a, b) \neq (0, 0)$, is a complex geodesic for D.

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