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Borromean surgery equivalence of spin 3-manifolds with boundary

EVA CONTRERAS AND KAZUO HABIRO

Abstract. Matveev introduced Borromean surgery on 3-manifolds, and proved that the equivalence relation on closed, oriented 3-manifolds generated by Borromean surgery is characterized by the first homology group and the torsion linking pairing. Massuyeau generalized this result to closed, spin 3-manifolds, and the second author to compact, oriented 3-manifolds with boundary.

In this paper we give a partial generalization of these results to compact, spin 3-manifolds with boundary.

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1. Introduction

Matveev [5] introduced an equivalence relation on 3-manifolds generated by *Borromean surgery*. This surgery transformation removes a genus 3 handlebody from a 3-manifold and glues it back in a nontrivial, but homologically trivial way. Thus, Borromean surgeries preserve the homology groups of 3-manifolds, and moreover the torsion linking pairings. Matveev gave the following characterization of this equivalence relation.

Theorem 1.1 (Matveev [5]). *Two closed, oriented* 3*-manifolds M and M' are related by a sequence of Borromean surgeries if and only if there is an isomorphism $f\colon H_1(M;\mathbb{Z}) \to H_1(M';\mathbb{Z})$ inducing isomorphism on the torsion linking pairings.*

Massuyeau [4] showed that Borromean surgery induces a natural correspondence on spin structures, and thus can be regarded as a surgery move on spin 3-manifolds. He generalized Theorem 1.1 as follows.

Theorem 1.2 (Massuyeau [4]). *Two closed spin* 3*-manifolds M and M' are related by a sequence of Borromean surgeries if and only if there is an isomorphism $f\colon H_1(M;\mathbb{Z}) \to H_1(M';\mathbb{Z})$ inducing isomorphism on the torsion linking pairings, and the Rochlin invariants of M and M' are congruent modulo* 8.

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In a paper in preparation [3], the second author generalizes Matveev's theorem to compact 3-manifolds with boundary (see Theorem 2.2 below).

In the present paper, we attempt to generalize the above results to compact spin 3-manifolds with boundary.

After defining the necessary ingredients in Sections 2 and 3, our main result is stated in Theorem 3.6.

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2. Y-surgeries on 3-manifolds

Unless otherwise specified, we will make the following assumptions in the rest of the paper. All manifolds are compact and oriented. Moreover, all 3-manifolds are connected. All homeomorphisms are orientation-preserving. The (co)homology groups with unspecified coefficient group are assumed to be with coefficients in $\mathbb{Z}$.

2.1. Y-surgeries and Y-equivalence

Borromean surgery is equivalent to Y-surgery used in the theory of finite type 3-manifold invariants in the sense of Goussarov and the second author [1,2].

A *Y-clasper* in a 3-manifold M is a connected surface (of genus 0, with 4 boundary components) embedded in M, which is decomposed into one disk, three bands and three annuli as depicted in Figure 2.1.

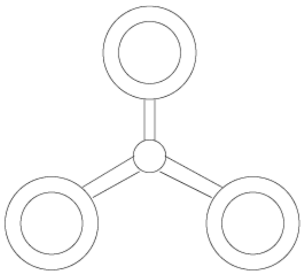

Figure 2.1. A Y-clasper.

We associate to a Y-clasper G in M a 6-component framed link L_G contained in a regular neighborhood of G in M as depicted in Figure 2.2. Surgery along the Y-clasper G is defined to be surgery along the framed link L_G. The result M_{L_G} from M of surgery along L_G is called the result of *surgery along the Y-clasper G* and is denoted by M_G.

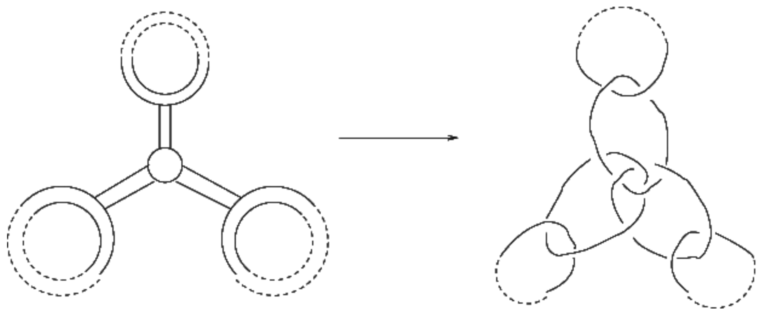

Figure 2.2. How to replace a Y-clasper with a 6-component framed link. Here the framings of the three inner components are zero and the framings of the three outer components are determined by the annuli in the Y-clasper.

By *Y-surgery* we mean surgery along a Y-clasper. Thus, we say that a 3-manifold M' is obtained from another 3-manifold M by a Y-surgery if there is a Y-clasper G in M such that the result of surgery, M_G, is homeomorphic to M'. It is well-known that this relation is symmetric, *i.e.*, if M' is obtained from M by a Y-surgery then, conversely, M can be obtained from M' by a Y-surgery.

The *Y-equivalence* is the equivalence relation on 3-manifolds generated by Y-surgeries.

2.2. Σ-bordered 3-manifolds

Throughout the paper, we fix a closed surface Σ, which may have an arbitrary finite number of components. In this paper, we consider 3-manifolds whose boundaries are parameterized by Σ.

A *Σ-bordered* 3-manifold is a pair (M, ϕ) of a compact, connected 3-manifold M and a homeomorphism $\phi\colon \Sigma \xrightarrow{\cong} \partial M$.

Two Σ-bordered 3-manifolds (M, ϕ) and (M', ϕ') are said to be *homeomorphic* if there is a homeomorphism $\Phi\colon M \xrightarrow{\cong} M'$ such that $(\Phi|_{\partial M}) \circ \phi = \phi'$.

2.3. Y-equivalence for Σ-bordered 3-manifolds

The notions of Y-surgery and Y-equivalence extend to Σ-bordered 3-manifolds in a natural way.

For a Σ-bordered 3-manifold $(M, \phi\colon \Sigma \xrightarrow{\cong} \partial M)$ and a Y-clasper G in M, the result of surgery M_G has an obvious boundary parameterization $\phi_G\colon \Sigma \xrightarrow{\cong} \partial M_G$ induced by ϕ. Thus surgery along a Y-clasper G in a Σ-bordered 3-manifold (M, ϕ) yields a Σ-bordered 3-manifold $(M, \phi)_G := (M_G, \phi_G)$. Two Σ-bordered 3-manifolds (M, ϕ) and (M', ϕ') are said to be related by a Y-surgery if there is a Y-clasper G in M such that $(M, \phi)_G$ is homeomorphic to (M', ϕ'). The Y-equivalence on Σ-bordered 3-manifolds is generated by Y-surgeries.

The following well known characterization of the Y-equivalence is useful.

Lemma 2.1. *Two Σ-bordered 3-manifolds (M,ϕ) and (M',ϕ') are Y-equivalent if and only if there are finitely many, mutually disjoint Y-claspers $G_1,\dots,G_n$ $(n\geq 0)$ in M such that the result of surgery, $(M,\phi)_{G_1,\dots,G_n}$ is homeomorphic to (M',ϕ').*

2.4. Homology isomorphisms between compact 3-manifolds

Let (M,ϕ) and (M',ϕ') be Σ-bordered 3-manifolds. Set

$$\delta := \phi'\circ\phi^{-1}\colon \partial M \xrightarrow{\cong} \partial M'.$$

A *homology isomorphism*[1] from (M,ϕ) to (M',ϕ') (or a homology isomorphism from M to M' along δ) is an isomorphism $f=(f_i,\overline{f_i})_{i=0,1,2,3}$ of the homology exact sequences of pairs $(M,\partial M)$ and $(M',\partial M')$

$$\begin{array}{ccccccccc}
\cdots \to & H_i(\partial M) & \to & H_i(M) & \to & H_i(M,\partial M) & \to & H_{i-1}(\partial M) & \to \cdots \\
 & \downarrow{\scriptstyle \delta_*} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle \overline{f}_i} & & \downarrow{\scriptstyle \delta_*} & \\
\cdots \to & H_i(\partial M') & \to & H_i(M') & \to & H_i(M',\partial M') & \to & H_{i-1}(\partial M') & \to \cdots
\end{array}$$

satisfying the following properties:

(i) $f_0([pt])=[pt]$;

(ii) f_i and $\overline{f_i}$ are compatible with the intersection forms, *i.e.*, for $i=0,1,2,3$, the square commutes:

$$\begin{array}{ccc}
H_i(M)\times H_{3-i}(M,\partial M) & \xrightarrow{\langle,\rangle_M} & \mathbb{Z} \\
\downarrow{\scriptstyle f_i\times \overline{f}_{3-i}} & & \| \\
H_i(M')\times H_{3-i}(M',\partial M') & \xrightarrow{\langle,\rangle_{M'}} & \mathbb{Z}\,;
\end{array}$$

here $\langle,\rangle_M$ and $\langle,\rangle_{M'}$ denote the intersection forms;

(iii) f_1 and $\overline{f_1}$ are compatible with the torsion linking pairings, *i.e.*, the square commutes:

$$\begin{array}{ccc}
\operatorname{Tors} H_1(M)\times \operatorname{Tors} H_1(M,\partial M) & \xrightarrow{\tau_M} & \mathbb{Q}/\mathbb{Z} \\
\downarrow{\scriptstyle \operatorname{Tors} f_1\times \operatorname{Tors}\overline{f}_1} & & \| \\
\operatorname{Tors} H_1(M')\times \operatorname{Tors} H_1(M',\partial M') & \xrightarrow{\tau_{M'}} & \mathbb{Q}/\mathbb{Z};
\end{array}$$

here Tors denotes torsion part, and τ_M denotes the torsion linking pairing of M.

[1] In [3], this is called "full enhanced homology isomorphism". In this paper, we call it "homology isomorphism" for simplicity.

The classification of compact 3-manifolds up to Y-equivalence is given by the following result.

Theorem 2.2 ([3]). *Let Σ be a closed surface, and let (M, ϕ) and (M', ϕ') be two Σ-bordered 3-manifolds. Then the following conditions are equivalent:*

(1) *(M, ϕ) and (M', ϕ') are Y-equivalent;*
(2) *There is a homology isomorphism from (M, ϕ) to (M', ϕ').*

For closed 3-manifolds, Theorem 2.2 is equivalent to Matveev's theorem (Theorem 1.1).

3. Y-surgery on spin 3-manifolds

3.1. Spin structures

For an oriented manifold M with vanishing second Stiefel-Whitney class, let $\mathrm{Spin}(M)$ denote the set of spin structures on M.

It is well known that $\mathrm{Spin}(M)$ is affine over $H^1(M;\mathbb{Z}_2)$, *i.e.*, acted by $H^1(M;\mathbb{Z}_2)$ freely and transitively

$$\mathrm{Spin}(M) \times H^1\big(M;\mathbb{Z}_2\big) \to \mathrm{Spin}(M), \quad (s, c) \mapsto s + c.$$

An embedding $f\colon M' \hookrightarrow M$ of a manifold M' into M induces a map

$$i^*\colon \mathrm{Spin}(M) \to \mathrm{Spin}(M').$$

If i is an inclusion map, $i^*(s)$, for $s \in \mathrm{Spin}(M)$, is denoted also by $s|_{M'}$.

3.2. Y-surgery and spin structures

Let G be a Y-clasper in a 3-manifold M. Let $N(G)$ be a regular neighborhood of G in M. Note that the result of surgery, M_G, can be identified with the manifold

$$(M \setminus \mathrm{int}\, N(G)) \cup_{\partial N(G)} N(G)_G$$

obtained by gluing $M \setminus \mathrm{int}\, N(G)$ with $N(G)_G$ along $\partial N(G)$.

As is proved by Massuyeau [4], for a spin structure $s \in \mathrm{Spin}(M)$, there is a unique spin structure s_G on M_G such that

$$s_G|_{M\setminus \mathrm{int}\, N(G)} = s|_{M\setminus \mathrm{int}\, N(G)}.$$

This gives a bijection

$$\mathrm{Spin}(M) \overset{\cong}{\to} \mathrm{Spin}\big(M_G\big), \quad s \longmapsto s_G.$$

The spin 3-manifold (M_G, s_G) is called the result of surgery on the spin 3-manifold (M, s) along G.

As in Section 2.1, the *Y-equivalence* on spin 3-manifolds is the equivalence relation generated by Y-surgery.

3.3. Twisting a spin structure along an orientable surface

Let (M, s) be a spin 3-manifold possibly with boundary, and let T be an orientable surface properly embedded in M. Then we can *twist* the spin structure s along T. More precisely, we can define a new spin structure

$$s * T = s + [T]^! \in \mathrm{Spin}(M),$$

where $[T]^! \in H^1(M; \mathbb{Z}_2)$ is the Poincaré dual of $[T] \in H_2(M, \partial M; \mathbb{Z}_2)$. (One can consider similar operation when T is non-orientable, but we do not need it in this paper.)

Note that twisting along a closed surface preserves the restriction of the spin structure to the boundary.

Proposition 3.1. *If T is a closed, orientable surface in a spin 3-manifold (M, s), then $(M, s * T)$ is Y-equivalent to (M, s).*

Proof. We may assume that T is connected, since the general case follows from this special case.

Take a bicollar neighborhood $T \times [-1, 2] \subset M$. Set $T_2 = T \times \{2\} \subset M$. Let c be a simple closed curve in T bounding a disk in T. Let A denote a bicollar neighborhood of c in T. Let D and T' be the two components of $T \setminus \mathrm{int}\, A$, where D is a disk. Set

$$V_0 = A \times [-1, 1], \quad V_1 = (A \cup D) \times [-1, 1], \quad V_2 = (A \cup T') \times [-1, 1],$$
$$M_i = M \setminus \mathrm{int}\, V_i, \quad i = 0, 1, 2.$$

Note that $M_1, M_2 \subset M_0$. For $i = 0, 1, 2$, set $s_i = s|_{M_i} \in \mathrm{Spin}(M_i)$.

Let $K = (c, +1)$ denote the framed knot in M whose underlying knot is c and the framing is $+1$. Let M_K denote the result of surgery along K, which may be regarded as the manifold $M_0 \cup_\partial (V_0)_K$ obtained from M_0 and the result of surgery $(V_0)_K$ by gluing along their boundaries in the natural way. We may regard M_0, M_1 and M_2 as submanifolds of M_K.

Note that V_1 and $(V_1)_K$ are 3-balls. Hence there is a unique spin structure $s_K \in \mathrm{Spin}(M_K)$ such that $(s_K)|_{M_1} = s_1$. We have the spin homeomorphism $(M, s) \cong (M_K, s_K)$.

We have

$$s_K|_{M_0} = s_0 * D = s_0 * T_2.$$

Hence we have

$$s_K|_{M_2} = s_2 * T_2.$$

It suffices to prove that (M_K, s_K) is Y-equivalent to $(M, s * T_2) = (M, s * T)$. Since the framed knot K is null-homologous in V_2 and $+1$-framed, $(V_2)_K$ is Y-equivalent to V_2 in a way respecting the boundary [5]. This Y-equivalence extends to Y-equivalence of M_K and M. This Y-equivalence implies the desired Y-equivalence of (M_K, s_K) and $(M, s * T_2)$ since we have

$$s_K \,|_{M_2} = s_2 * T_2 = \big(s * T_2\big)\big|_{M_2},$$

and since the maps

$$\mathrm{Spin}\,\big(M_K\big) \to \mathrm{Spin}\,\big(M_2\big), \quad \mathrm{Spin}(M) \to \mathrm{Spin}\,\big(M_2\big)$$

induced by inclusions are injective. □

3.4. (Σ, s_Σ)-bordered spin 3-manifolds

We fix a spin structure $s_\Sigma \in \mathrm{Spin}(\Sigma)$. In the following we consider Y-equivalence of spin 3-manifolds with boundary parameterized by the spin surface (Σ, s_Σ).

A (Σ, s_Σ)-*bordered spin* 3-*manifold* is a triple (M, ϕ, s) consisting of a Σ-bordered 3-manifold (M, ϕ) and a spin structure $s \in \mathrm{Spin}(M)$ such that $\phi^*(s) = s_\Sigma$.

Clearly, surgery along a Y-clasper in M preserves the spin structure on the boundary of M. Hence a Y-surgery on a (Σ, s_Σ)-bordered spin 3-manifold yields another (Σ, s_Σ)-bordered spin 3-manifold.

3.5. Gluing of (Σ, s_Σ)-bordered spin 3-manifolds

Let (M, ϕ, s) and (M', ϕ', s') be two (Σ, s_Σ)-bordered spin 3-manifolds. Let $M'' = (-M) \cup_{\phi,\phi'} M'$ be the closed 3-manifold obtained from $-M$ (the orientation reversal of M) and M' by gluing their boundaries along $\phi' \circ \phi^{-1}$.

By a *gluing* of s and s', we mean a spin structure $s'' \in \mathrm{Spin}(M'')$ satisfying

$$s''\big|_{-M} = s, \quad s''\big|_{M'} = s'.$$

If Σ is empty or connected, then s'' is uniquely determined by s and s'. Otherwise, s'' is not unique.

The spin manifold (M'', s'') is called a *gluing* of (M, ϕ, s) and (M', ϕ', s').

Proposition 3.2. *All the gluings of two* (Σ, s_Σ)*-bordered spin* 3*-manifolds* (M, ϕ, s) *and* (M', ϕ', s') *are mutually* Y*-equivalent.*

Proof. If Σ has at most one boundary component, then there is nothing to prove since there is only one gluing of (M, ϕ, s) and (M', ϕ', s').

Suppose Σ has components $\Sigma_1, \ldots, \Sigma_n$ with $n \geq 2$. For $i = 2, \ldots, n$, choose a framed knot K_i in $M'' = (-M) \cup_{\phi,\phi'} M'$ which transversely intersects each of Σ_1 and Σ_i by exactly one point and is disjoint from the other components of Σ. There are 2^{n-1} gluings $s''_{\epsilon_2,\ldots,\epsilon_n} \in \mathrm{Spin}(M'')$ of s and s' for $\epsilon_2, \ldots, \epsilon_n \in \{0, 1\}$, where for

$i = 2, \dots, n$ the framed knot K_i is even framed with respect to $s''_{\epsilon_2,\dots,\epsilon_n}$ if $\epsilon_i = 0$, and odd framed otherwise. Moreover, we have

$$s''_{\epsilon_2,\dots,\epsilon_n} = s''_{0,\dots,0} * \left(\bigcup_{2 \le i \le n,\ \epsilon_i = 1} \Sigma_i \right).$$

Hence, by Proposition 3.1, (M'', s''_1) and (M'', s''_2) are Y-equivalent. □

3.6. Rochlin invariant mod 8 of pairs of (Σ, s_Σ)-bordered spin 3-manifolds

Let (M, ϕ, s) and (M', ϕ', s') be two (Σ, s_Σ)-bordered spin 3-manifolds. Set

$$R_8\Big((M, \phi, s), \big(M', \phi', s'\big)\Big) := \Big(R\big(M'', s''\big) \mod 8\Big) \in \mathbb{Z}_8, \tag{3.1}$$

where $M'' = (-M) \cup_{\phi,\phi'} M'$ as before and $s'' \in \mathrm{Spin}(M'')$ is any gluing of s and s'. Proposition 3.2 and Theorem 1.2 imply that (3.1) is well defined.

Lemma 3.3. *The invariant $R_8((M, \phi, s), (M', \phi', s'))$ depends only on the Y-equivalence classes of (M, ϕ, s) and (M', ϕ', s').*

Proof. Suppose that (M_1, ϕ_1, s_1) is Y-equivalent to (M_2,ϕ_2,s_2) and that (M'_1,ϕ'_1,s'_1) is Y-equivalent to (M'_2, ϕ'_2, s'_2). Consider gluings (M''_i, s''_i) of (M_i, ϕ_i, s_i) and (M'_i, ϕ'_i, s'_i) for $i = 1, 2$. Then (M''_1, s''_1) and (M''_2, s''_2) are Y-equivalent. Hence we have

$$\begin{aligned} R_8\Big(\big(M_1, \phi_1, s_1\big), \big(M'_1, \phi'_1, s'_1\big)\Big) &= \Big(R\big(M''_1, s''_1\big) \mod 8\Big) \\ &= \Big(R\big(M''_2, s''_2\big) \mod 8\Big) = R_8\Big(\big(M_2, \phi_2, s_2\big), \big(M'_2, \phi'_2, s'_2\big)\Big). \end{aligned}$$

□

3.7. Main results

Now we state the main result of the present paper, which gives a characterization of Y-equivalence of (Σ, s_Σ)-bordered spin 3-manifolds in terms of homology isomorphism and the Rochlin invariant mod 8.

Conjecture 3.4. Let (M, ϕ, s) and (M', ϕ', s') be two (Σ, s_Σ)-bordered spin 3-manifolds. Then the following conditions are equivalent.

(1) (M, ϕ, s) and (M', ϕ', s') are Y-equivalent.
(2) There is a homology isomorphism from (M, ϕ) to (M', ϕ'), and we have

$$R_8\Big((M, \phi, s), \big(M', \phi', s'\big)\Big) = 0 \pmod 8.$$

It follows from Theorem 2.2 that Conjecture 3.4 is equivalent to the following.

Conjecture 3.5. Let (M, ϕ, s) and (M', ϕ', s') be two (Σ, s_Σ)-bordered spin 3-manifolds. Then the following conditions are equivalent.

(1) (M, ϕ, s) and (M', ϕ', s') are Y-equivalent.
(2) (M, ϕ) and (M', ϕ') are Y-equivalent, and we have

$$R_8\Big((M, \phi, s), \big(M', \phi', s'\big)\Big) = 0 \pmod 8.$$

The following theorem says that Conjecture 3.5 holds when $H_1(M; \mathbb{Z})$ has no 2-torsion. The proof of this result does not use definitions and results given in [3], which is not available when we are writing the present paper.

Theorem 3.6. *In the setting of Conjecture* 3.5*,* (1) *implies* (2)*. Moreover, if* $H_1(M; \mathbb{Z})$ *has no* 2*-torsion, then*

(2′) (M, ϕ) *and* (M', ϕ') *are* Y*-equivalent*

implies (1).

4. Proof of Theorem 3.6

4.1. Proof of (1) ⇒ (2)

Suppose that (1) of Theorem 3.5 holds. Then, clearly, (M, ϕ) and (M', ϕ') are Y-equivalent. We have to prove $R(M'', s'') \equiv 0 \pmod 8$, where (M'', s'') is a gluing of (M, ϕ, s) and (M', ϕ', s').

Since (M, ϕ, s) and (M', ϕ', s') are Y-equivalent, Lemma 3.3 implies that (M'', s'') is Y-equivalent to a gluing (M''_0, s''_0) of (M, s) and itself.

Consider the 4-manifold C which is the quotient of the cylinder $M \times [0, 1]$ by the equivalence relation $(x, t) \sim (x, t')$ for $x \in \partial M$ and $t, t' \in [0, 1]$. Then we may naturally identify M''_0 with ∂C. The 4-manifold C has a spin structure s_C induced by the spin structure $s \times s_{[0,1]} \in \mathrm{Spin}(M \times [0, 1])$, where $s_{[0,1]}$ is the unique spin structure of $[0, 1]$. We have

$$R\big(C, s_C\big) \equiv R\big(M \times [0, 1], s \times s_{[0,1]}\big) \equiv \sigma(M \times [0, 1]) = 0 \pmod{16}.$$

Since both s''_0 and s_C are gluings of (M, s) and itself, Proposition 3.2 implies that (M''_0, s''_0) and (C, s_C) are Y-equivalent. Hence, by Theorem 1.2, we have

$$R\big(M'', s''\big) \equiv R\big(M''_0, s''_0\big) \equiv R\big(C, s_C\big) \equiv 0 \pmod 8.$$

4.2. Proof of (2′) ⇒ (1) when $H_1(M; \mathbb{Z})$ has no 2-torsion

We assume that $H_1(M; \mathbb{Z})$ has no 2-torsion.

We divide the proof into three cases:

- M is a $\mathbb{Z}_2$-homology handlebody, *i.e.*, ∂M is connected and $H_1(M, \partial M; \mathbb{Z}_2) = 0$;
- M has non-empty boundary;
- M is closed.

4.2.1. *Case where M is a $\mathbb{Z}_2$-homology handlebody*

Since $\mathrm{Spin}(M) \overset{\phi^*}{\to} \mathrm{Spin}(\Sigma)$ and $\mathrm{Spin}(M') \overset{(\phi')^*}{\to} \mathrm{Spin}(\Sigma)$ are injective, Y-equivalence of (M, ϕ) and (M', ϕ') implies Y-equivalence of (M, ϕ, s) and (M', ϕ', s').

4.2.2. *Case where ∂M is non-empty*

We will use the following result.

Lemma 4.1. *Let M be a 3-manifold with boundary such that $H_1(M; \mathbb{Z})$ has no 2-torsion. Then M can be obtained from a $\mathbb{Z}_2$-homology handlebody V by attaching 2-handles $h_1, \dots, h_n$ (with $n \geq 0$) along simple closed curves $c_1, \dots, c_n$ in ∂V in such a way that each c_i is null-homologous (over $\mathbb{Z}$) in V.*

Proof. M can be obtained from a solid torus V' of genus g by attaching some 2-handles along simple closed curves $c'_1, \dots, c'_k$ in $\partial V'$. After finitely many handle-slides, we can assume the following.

- There is a basis $x_1, \dots, x_g$ of $H_1(V'; \mathbb{Z})$ such that we have

$$[c_i] = \sum_{j=1}^{g} a_{i,j} x_j$$

for $i = 1, \dots, k$, where the matrix $(a_{i,j})$ is diagonal (but not necessarily square), in the sense that $a_{i,j} = \delta_{i,j} d_i$.

Clearly, $H_1(M; \mathbb{Z})$ is isomorphic to $\bigoplus_{i=1}^{k} \mathbb{Z}_{d_i}$. By the assumption that $H_1(M; \mathbb{Z})$ has no 2-torsion, each d_i is either odd or 0.

We may assume that, for some n, we have $d_1 = \cdots = d_n = 0$ and $d_{n+1}, \dots, d_k$ are odd. The union $V := V' \cup h'_{n+1} \cup \cdots \cup h'_k$ is a $\mathbb{Z}_2$-homology handlebody. Setting $c_i = c'_i$, $h_i = h'_i$ for $i = 1, \dots, n$, we have the result. □

Let M be obtained as above from a $\mathbb{Z}_2$-homology handlebody V by attaching 2-handles $h_1, \dots, h_n$ along disjoint simple closed curves $c_1, \dots, c_n \subset \partial V$, $n = \operatorname{rank} H_2(M; \mathbb{Z}) \geq 0$, such that c_i is null-homologous in M and such that $\partial M \setminus (c_1 \cup \cdots \cup c_n)$ is connected.

The proof is by induction on n. The case $n = 0$ is proved in Section 4.2.1. Suppose $n > 0$.

Let $N = h_n = D^2 \times [0, 1] \subset M$ be one of the 2-handles. Set

$$A = \partial D^2 \times [0, 1] \subset \partial N,$$
$$B = D^2 \times \{0, 1\} \subset \partial N,$$
$$M_0 := \overline{M \setminus N} = V \cup h_1 \cup \cdots \cup h_{n-1} \subset M.$$

Thus, $M = M_0 \cup_A N$ is obtained from a 3-manifold M_0 by attaching N along an annulus $A \subset \partial M_0$.

Since (M,ϕ) and (M',ϕ') are Y-equivalent, it follows from Lemma 2.1 that there exists a disjoint family $\mathcal{G}$ of Y-claspers in M and a homeomorphism

$$\Psi\colon \big(M_{\mathcal{G}},\phi_{\mathcal{G}}\big) \overset{\cong}{\to} \big(M',\phi'\big).$$

By isotoping $\mathcal{G}$ if necessary, we may assume that $\mathcal{G}$ is contained in the interior of M_0.

Set $\Sigma_0 := (\Sigma \setminus \mathrm{int}(\phi^{-1}(B))) \cup A$. Then we have a Σ_0-bordered 3-manifold (M_0,ϕ_0) where $\phi_0\colon \Sigma_0 \overset{\cong}{\to} \partial M_0$ is obtained by gluing $\phi|_{\Sigma\setminus\mathrm{int}(\phi^{-1}(B))}$ and id_A.

Set

$$M_0' := \Psi\Big(\big(M_0\big)_{\mathcal{G}}\Big) = \overline{M'\setminus\Psi(N)} \subset M'.$$

We have a Σ_0-bordered 3-manifold (M_0',ϕ_0'), where $\phi_0'\colon \Sigma_0 \overset{\cong}{\to} \partial M_0'$ is obtained by gluing $\phi'|_{\Sigma\setminus\mathrm{int}(\phi^{-1}(B))}$ and $\Psi|_A\colon A \overset{\cong}{\to} \Psi(A)$.

We have a homeomorphism of Σ_0-bordered 3-manifolds

$$\Psi_0 := \Psi|_{M_0}\colon \Big(\big(M_0\big)_{\mathcal{G}},\big(\phi_0\big)_{\mathcal{G}}\Big) \overset{\cong}{\to} \big(M_0',\phi_0'\big).$$

Set $s_{\Sigma_0} = (\phi_0)^*(s|_{M_0}) \in \mathrm{Spin}(\Sigma_0)$ and $s'_{\Sigma_0} = (\phi_0')^*(s'|_{M_0'}) \in \mathrm{Spin}(\Sigma_0)$. Note that $s_{\Sigma_0}|_{\Sigma\setminus\mathrm{int}(\phi^{-1}(B))} = s'_{\Sigma_0}|_{\Sigma\setminus\mathrm{int}(\phi^{-1}(B))}$. Hence we have either

$$s_{\Sigma_0} = s'_{\Sigma_0} \tag{4.1}$$

or

$$s'_{\Sigma_0} = s_{\Sigma_0} + [a]^! \quad \text{and} \quad s_{\Sigma_0} \neq s'_{\Sigma_0}, \tag{4.2}$$

where $a = c_n = \partial D^2 \times \{1/2\} \subset A$ is the core of the annulus A, and $[a]^! \in H^1(\Sigma_0;\mathbb{Z}_2)$ is the Poincaré dual to $[a] \in H_1(\Sigma_0;\mathbb{Z}_2)$.

Claim 4.2. We may assume (4.1).

Proof. If a is separating in Σ_0, then we have (4.1).

Suppose that a is non-separating in Σ_0, and that we have (4.2). Since a is null-homologous in $\partial V \subset M_0$, it is so also in M_0'. Therefore, there is a connected, oriented surface T_0' properly embedded in M_0' such that $\partial T_0' = a$. Set $D' = \Psi(D^2 \times \{1/2\})$, and $T' = T_0' \cup D'$, which is a connected, oriented, closed surface in M'.

Set $\hat{s}' := s' * T' \in \mathrm{Spin}(M')$ and $\hat{s}'_{\Sigma_0} = (\phi_0')^*(\hat{s}'|_{M_0'}) \in \mathrm{Spin}(\Sigma_0)$. By Proposition 3.1, it follows that (M',s') and $(M',\hat{s}')$ are Y-equivalent. Thus, we may replace the spin manifold (M',s') with $(M',\hat{s}')$. We have

$$\hat{s}'_{\Sigma_0} = \big(\phi_0'\big)^*\Big(\big(s' * T'\big)|_{M_0'}\Big) = \big(\phi_0'\big)^*\big(s'\big) + [a]^! = s'_{\Sigma_0} + [a]^! = s_{\Sigma_0}.$$

Hence, we have only to consider the case where (4.1) holds. □

We assume (4.1). Set $s_0 = s|_{M_0} \in \mathrm{Spin}(M_0)$ and $s'_0 = s'|_{M'_0} \in \mathrm{Spin}(M'_0)$. Then (M_0, ϕ_0, s_0) and (M'_0, ϕ'_0, s'_0) are (Σ_0, s_{Σ_0})-bordered spin 3-manifolds.

We can use the induction hypothesis to deduce that (M_0,ϕ_0,s_0) and (M'_0,ϕ'_0,s'_0) are Y-equivalent, and hence so are (M, ϕ, s) and (M', ϕ', s').

4.2.3. *Case where M is closed*

This case is a special case of Theorem 1.2.

Alternatively, this case easily follows from the previous case by considering the punctures $M \setminus \mathrm{int}\, B^3$ and $M' \setminus \mathrm{int}\, B^3$.

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Department of Mathematics
University of California, Berkeley
970 Evans Hall
Berkeley, CA 94720-3840, USA
eva.contreras@berkeley.edu

Research Institute for Mathematical Sciences
Kyoto University
Kyoto 606-8502, Japan
habiro@kurims.kyoto-u.ac.jp