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# A Bernstein-type result for the minimal surface equation

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**Abstract.** We prove the following Bernstein-type theorem: if $u$ is an entire solution to the minimal surface equation, such that $N-1$ partial derivatives $\frac{\partial u}{\partial x_j}$ are bounded on one side (not necessarily the same), then $u$ is an affine function. Its proof relies *only* on the Harnack inequality on minimal surfaces proved in [4] thus, besides its novelty, our theorem also provides a new and self-contained proof of celebrated results of Moser and of Bombieri and Giusti.

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## 1. Introduction and main results

In this short article we are concerned with a Bernstein-type theorem for solutions to the minimal surface equation

$$-\text{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=0 \qquad \text{in} \qquad \mathbb{R}^N, \qquad N \geq 2. \tag{1.1}$$

The classical Bernstein Theorem [2,7] asserts that the affine functions are the only solutions of (1.1) in $\mathbb{R}^2$. This result has been generalized to $\mathbb{R}^3$ by E. De Giorgi [5], to $\mathbb{R}^4$ by J.F. Almgren [1] and, up to dimension $N=7$, by J. Simons [9]. On the other hand, E. Bombieri, E. De Giorgi and E. Giusti [3] proved the existence of a non-affine solution of the minimal surface equation (1.1) for any $N \geq 8$. Nevertheless, J. Moser [8] was able to prove that, if $\nabla u$ is bounded on $\mathbb{R}^N$, then $u$ must be again an affine function, and this for every dimension $N \geq 2$. Later, E. Bombieri and E. Giusti [4] generalized Moser's result by assuming that *only* $N-1$ partial derivatives of $u$ are bounded on $\mathbb{R}^N$, $N \geq 2$. To prove their result, the Authors of [4] demonstrate a Harnack inequality for uniformly elliptic equations on minimal surfaces (oriented boundary of least area) and then they use it to show that, if

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$N-1$ partial derivatives of $u$ are bounded on $\mathbb{R}^N$, then $u$ has bounded gradient on $\mathbb{R}^N$, and they conclude by invoking the result of Moser. Our main theorem (see Theorem 1.1 below) provides a further extension of the above results. Its proof relies *only* on the Harnack inequality on minimal surfaces proved in [4] thus, besides its novelty, it also provides a new and self-contained proof of the celebrated results of Moser and of Bombieri and Giusti. We believe that this is another interesting feature of our work.

Our main result is stated in the following theorem.

**Theorem 1.1.** *Assume $N \geq 2$. Let $u$ be a solution of the minimal surface equation* (1.1) *such that $N-1$ partial derivatives $\frac{\partial u}{\partial x_j}$ are bounded on one side* (*not necessarily the same*). *Then $u$ is an affine function.*

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## 2. Auxiliary results and proofs

To prove our results we briefly recall some standard notations and some well-known facts concerning the solutions of the minimal surface equation (1.1) (*cf.* [4], [6]). For a given solution $u$ of equation (1.1), we denote by $S$ the minimal graph $x_{N+1} = u(x)$ over $\mathbb{R}^N$ (*i.e.*, the complete smooth area minimizing hypersurface without boundary $S \subset \mathbb{R}^{N+1}$, given by the graph of $u$ over the entire $\mathbb{R}^N$). Then the (upward pointing) unit normal to $S$ at a point $(x, u(x))$ is $\nu = (\nu_1, \dots, \nu_{N+1}) = \frac{(-\nabla u(x), 1)}{\sqrt{1+|\nabla u(x)|^2}}$ and we can define the tangential derivatives $\delta_k$ by

$$\delta_k := \frac{\partial}{\partial x_k} - \nu_k \sum_{h=1}^{N+1} \nu_h \frac{\partial}{\partial x_h} \qquad \forall\, k = 1, \dots, N+1. \tag{2.1}$$

Moreover the functions $\nu_h$ satisfy the equation

$$\sum_{k=1}^{N+1} \delta_k \delta_k \nu_h + c^2 \nu_h = 0 \qquad \text{on} \quad S, \qquad \forall\, h = 1, \dots, N+1 \tag{2.2}$$

where $c^2 := \sum_{j,k=1}^{N+1} (\delta_j \nu_k)^2$ denotes the sum of the squares of the principal curvatures of the hypersurface $S$ at the point $(x, u(x))$. Therefore, for any vector $a := (a_1, \dots, a_{N+1}) \in \mathbb{R}^{N+1}$, the function $(a \cdot \nu) = \sum_{j=1}^{N+1} a_j \nu_j$ also solves

$$\sum_{k=1}^{N+1} \delta_k \delta_k (a \cdot \nu) + c^2 (a \cdot \nu) = 0 \qquad \text{on} \quad S. \tag{2.3}$$

**Lemma 2.1.** *Assume $N \geq 2$ and let $S$ be a minimal graph $x_{N+1} = u(x)$ over $\mathbb{R}^N$. If $v > 0$ and $w$ are smooth solutions of the equation* (2.3) *on $S$, then the smooth function $\theta := \arctan\left(\frac{w}{v}\right) \in L^\infty(S)$ solves the equation*

$$\sum_{k=1}^{N+1} \delta_k \left[ (v^2 + w^2)\delta_k\theta \right] = 0 \qquad \text{on} \quad S. \tag{2.4}$$

*Proof.* Consider the smooth complex-valued function $z := v + iw$. Since $v > 0$ everywhere, we have that $z = \rho e^{i\theta}$ on $S$ and

$$\sum_{k=1}^{N+1} \delta_k\delta_k z + c^2 z = 0 \qquad \text{on} \quad S, \tag{2.5}$$

where $\rho := \sqrt{v^2 + w^2} > 0$ everywhere on $S$. Hence, by definition of $\delta_k$ we get

$$\begin{aligned}
0 =& \sum_{k=1}^{N+1} \delta_k\delta_k \left(\rho e^{i\theta}\right) + c^2\rho e^{i\theta} = \sum_{k=1}^{N+1} \delta_k \left(e^{i\theta}\delta_k\rho + i\rho e^{i\theta}\delta_k\theta\right) + c^2\rho e^{i\theta} \\
=& \sum_{k=1}^{N+1} e^{i\theta}\delta_k\delta_k\rho + ie^{i\theta}\delta_k\theta\delta_k\rho + i\rho e^{i\theta}\delta_k\delta_k\theta + i\left(e^{i\theta}\delta_k\rho + i\rho e^{i\theta}\delta_k\theta\right)\delta_k\theta + c^2\rho e^{i\theta} \\
=& \sum_{k=1}^{N+1} e^{i\theta}\delta_k\delta_k\rho - \rho e^{i\theta}\delta_k\theta\delta_k\theta + ie^{i\theta}\left(\rho\delta_k\delta_k\theta + 2\delta_k\rho\delta_k\theta\right) + c^2\rho e^{i\theta} \qquad \text{on} \quad S.
\end{aligned}$$

Hence

$$0 = \sum_{k=1}^{N+1} \delta_k\delta_k\rho - \rho\delta_k\theta\delta_k\theta + i\left(\rho\delta_k\delta_k\theta + 2\delta_k\rho\delta_k\theta\right) + c^2\rho \qquad \text{on} \quad S$$

and taking the imaginary part of the latter identity we obtain

$$0 = \sum_{k=1}^{N+1} \rho\delta_k\delta_k\theta + 2\delta_k\rho\delta_k\theta = \frac{1}{\rho}\sum_{k=1}^{N+1} \delta_k\left[\rho^2\delta_k\theta\right] \qquad \text{on} \quad S$$

which immediately implies (2.4). □

Now we are in position to prove our main result.

*Proof of Theorem* 1.1. We divide the proof into three steps.

*Step* 1. *Every partial derivative of $u$ is bounded on one side.*

By assumption there exists an integer $n \in \{1, \ldots, N\}$ such that for every integer $j \in \{1, \ldots, N\} \setminus \{n\} := J$, the partial derivative $\frac{\partial u}{\partial x_j}$ is bounded on one side. We set $A := \{\alpha \in J : \frac{\partial u}{\partial x_\alpha}$ is bounded from below$\}$ and $B := \{\beta \in J : \frac{\partial u}{\partial x_\beta}$ is bounded from above$\}$. Hence

$$\forall\, \alpha \in A \qquad \exists\, c_\alpha > 0 \quad : \quad \frac{\partial u}{\partial x_\alpha} + c_\alpha > 1 \qquad \text{on} \quad \mathbb{R}^{\mathrm{N}}, \tag{2.6}$$

$$\forall\, \beta \in B \qquad \exists\, c_\beta > 0 \quad : \quad c_\beta - \frac{\partial u}{\partial x_\beta} > 1 \qquad \text{on} \quad \mathbb{R}^{\mathrm{N}}. \tag{2.7}$$

Now we observe that

$$|\nabla u|^2 = \left(\frac{\partial u}{\partial x_n}\right)^2 + \sum_{\alpha\in A}\left(\frac{\partial u}{\partial x_\alpha}\right)^2 + \sum_{\beta\in B}\left(\frac{\partial u}{\partial x_\beta}\right)^2 \tag{2.8}$$

$$= \left(\frac{\partial u}{\partial x_n}\right)^2 + \sum_{\alpha\in A}\left(\frac{\partial u}{\partial x_\alpha} + c_\alpha - c_\alpha\right)^2 + \sum_{\beta\in B}\left(c_\beta - \frac{\partial u}{\partial x_\beta} - c_\beta\right)^2 \tag{2.9}$$

$$= \left(\frac{\partial u}{\partial x_n}\right)^2 + \sum_{\alpha\in A}\left(\frac{\partial u}{\partial x_\alpha} + c_\alpha\right)^2 + \sum_{\alpha\in A} c_\alpha^2 - 2\sum_{\alpha\in A} c_\alpha\left(\frac{\partial u}{\partial x_\alpha} + c_\alpha\right) \tag{2.10}$$

$$+ \sum_{\beta\in B}\left(c_\beta - \frac{\partial u}{\partial x_\beta}\right)^2 + \sum_{\beta\in B} c_\beta^2 - 2\sum_{\beta\in B} c_\beta\left(c_\beta - \frac{\partial u}{\partial x_\beta}\right) \tag{2.11}$$

$$\leq \left(\frac{\partial u}{\partial x_n}\right)^2 + \sum_{\alpha\in A}\left(\frac{\partial u}{\partial x_\alpha} + c_\alpha\right)^2 + \sum_{\beta\in B}\left(c_\beta - \frac{\partial u}{\partial x_\beta}\right)^2 + \sum_{j\in J} c_j^2 \tag{2.12}$$

$$\leq \left(\frac{\partial u}{\partial x_n}\right)^2 + \left[\sum_{\alpha\in A}\left(\frac{\partial u}{\partial x_\alpha} + c_\alpha\right) + \sum_{\beta\in B}\left(c_\beta - \frac{\partial u}{\partial x_\beta}\right)\right]^2 + \sum_{j\in J} c_j^2 \tag{2.13}$$

where in the latter we have used (2.6) and (2.7).

Now we set $\xi := \sum_{\alpha\in A} e_\alpha - \sum_{\beta\in B} e_\beta \in \mathbb{R}^N$, $k_1 := \sum_{j\in J} c_j^2 > 0$, $k_2 := \sum_{j\in J} c_j > 0$, where $\{e_1, \dots, e_N\}$ denotes the canonical basis of $\mathbb{R}^N$ and we rewrite (2.13) as

$$\left(\frac{\partial u}{\partial x_n}\right)^2 + \left(\nabla u \cdot \xi + k_2\right)^2 + k_1 \qquad \text{on} \quad \mathbb{R}^{\mathrm{N}} \tag{2.14}$$

and observe that

$$\nabla u \cdot \xi + k_2 > 1 \qquad \text{on} \quad \mathbb{R}^{\mathrm{N}}, \tag{2.15}$$

again by (2.6) and (2.7).

Combining (2.8)-(2.14) and (2.15) we find

$$1 + |\nabla u|^2 \leq \left(\frac{\partial u}{\partial x_n}\right)^2 + (2 + k_1)\left(\nabla u \cdot \xi + k_2\right)^2 \tag{2.16}$$

$$\leq (2 + k_1)\left[\left(\frac{\partial u}{\partial x_n}\right)^2 + \left(\nabla u \cdot \xi + k_2\right)^2\right]. \tag{2.17}$$

Set $\chi := (-e_n, 0) \in \mathbb{R}^{N+1}$, $\tau := (-\xi, k_2) \in \mathbb{R}^{N+1}$ and consider the functions $w := \frac{\frac{\partial u}{\partial x_n}}{\sqrt{1+|\nabla u|^2}} = (\chi \cdot \nu)$ and $v := \frac{\nabla u \cdot \xi + k_2}{\sqrt{1+|\nabla u|^2}} = (\tau \cdot \nu) > 0$. Since $v > 0$ and $w$ are solutions of the equation (2.3), an application of Lemma 2.1 implies that $\theta := \arctan\left(\frac{w}{v}\right) \in L^\infty(S)$ solves the equation

$$\sum_{k=1}^{N+1} \delta_k \left[ \left(v^2 + w^2\right) \delta_k \theta \right] = 0 \qquad \text{on} \quad S. \tag{2.18}$$

Thanks to (2.16)-(2.17) we see that the above equation (2.18) is uniformly elliptic on $S$. Indeed, from (2.16)-(2.17) we get

$$\frac{1 + |\nabla u|^2}{2 + k_1} \leq \left[ \left( \frac{\partial u}{\partial x_n} \right)^2 + \left( \nabla u \cdot \xi + k_2 \right)^2 \right] \leq 2\left(N + k_2^2\right) \left[1 + |\nabla u|^2\right] \tag{2.19}$$

which implies

$$\frac{1}{2 + k_1} \leq v^2 + w^2 \leq 2\left(N + k_2^2\right) \qquad \text{on} \qquad S. \tag{2.20}$$

Thus $\theta$ must be constant, by an application of the Harnack inequality proved by Bombieri and Giusti (*cf.* [4, Theorem 5]), *i.e.*, $w = \lambda v$ on $S$, for some $\lambda \in \mathbb{R}$. The latter immediately implies that $\frac{\partial u}{\partial x_n}$ has a sign. In particular, all the partial derivatives of $u$ are bounded on one side.

*Step* 2. *For every unit vector $\eta \in \mathbb{R}^N$ the directional derivative $\frac{\partial u}{\partial \eta}$ has sign, that is, one and only one of the following assertions holds*:

(i) $\frac{\partial u}{\partial \eta}(x) = 0 \quad \forall x \in \mathbb{R}^N$,

(ii) $\frac{\partial u}{\partial \eta}(x) > 0 \quad \forall x \in \mathbb{R}^N$,

(iii) $\frac{\partial u}{\partial \eta}(x) < 0 \quad \forall x \in \mathbb{R}^N$.

Let $\sigma$ be any unit vector of $\mathbb{R}^N$ and set $I := \{1, \dots, N\}$, $A := \{\alpha \in I : \frac{\partial u}{\partial x_\alpha}$ is bounded from below$\}$ and $B := \{\beta \in I : \frac{\partial u}{\partial x_\beta}$ is bounded from above$\}$. Hence

$$\forall \alpha \in A \qquad \exists c_\alpha > 0 \quad : \quad \frac{\partial u}{\partial x_\alpha} + c_\alpha > 1 \qquad \text{on} \quad \mathbb{R}^{\mathrm{N}}, \tag{2.21}$$

$$\forall \beta \in B \qquad \exists c_\beta > 0 \quad : \quad c_\beta - \frac{\partial u}{\partial x_\beta} > 1 \qquad \text{on} \quad \mathbb{R}^{\mathrm{N}}, \tag{2.22}$$

and proceeding as before we obtain

$$\left(\frac{\partial u}{\partial \sigma}\right)^2 \le \left|\nabla u\right|^2 \le \left[\sum_{\alpha\in A}\left(\frac{\partial u}{\partial x_\alpha}+c_\alpha\right)+\sum_{\beta\in B}\left(c_\beta-\frac{\partial u}{\partial x_\beta}\right)\right]^2+\sum_{j\in I}c_j^2 \quad (2.23)$$

$$= (\nabla u\cdot\xi+k_4)^2+k_3 \qquad \text{on} \quad \mathbb{R}^{\mathrm{N}} \quad (2.24)$$

and

$$\nabla u\cdot\xi+k_4>1 \qquad \text{on} \quad \mathbb{R}^{\mathrm{N}}, \quad (2.25)$$

where $\xi := \sum_{\alpha\in A} e_\alpha - \sum_{\beta\in B} e_\beta \in \mathbb{R}^N$, $k_3 := \sum_{j=1}^N c_j^2 > 0$, $k_4 := \sum_{j=1}^N c_j > 0$.

We notice that $\xi$, $k_3$ and $k_4$ are independent of the unit vector $\sigma$ and let $\{\eta, \sigma_2, \dots, \sigma_N\}$ be an orthonormal basis of $\mathbb{R}^N$. From (2.23)-(2.24) we get

$$\begin{aligned}1+|\nabla u|^2 &= 1+\left(\frac{\partial u}{\partial \eta}\right)^2+\sum_{j=2}^N\left(\frac{\partial u}{\partial \sigma_j}\right)^2 \le 1+\left(\frac{\partial u}{\partial \eta}\right)^2\\ &\quad +(N-1)\left[\left(\nabla u\cdot\xi+k_4\right)^2+k_3\right]\end{aligned} \quad (2.26)$$

and using (2.25) in the latter we immediately infer that

$$1+|\nabla u|^2 \le \left(N+(N-1)k_3\right)\left[\left(\frac{\partial u}{\partial \eta}\right)^2+\left(\nabla u\cdot\xi+k_4\right)^2\right] \quad (2.27)$$

$$\le 3\left(N+(N-1)k_3\right)\left(N+k_4^2\right)\left[1+\left|\nabla u\right|^2\right]. \quad (2.28)$$

Setting $\chi := (-\eta, 0) \in \mathbb{R}^{N+1}$, $\tau := (-\xi, k_4) \in \mathbb{R}^{N+1}$, $w := \frac{\frac{\partial u}{\partial \eta}}{\sqrt{1+|\nabla u|^2}} = (\chi\cdot\nu)$ and $v := \frac{\nabla u\cdot\xi+k_4}{\sqrt{1+|\nabla u|^2}} = (\tau\cdot\nu) > 0$, and applying Lemma 2.1 as before, we see that the function $\theta := \arctan\left(\frac{w}{v}\right) \in L^\infty(S)$ solves the equation (2.4), which is again unifomly elliptic on $S$ in view of the above (2.27)-(2.28). It follows that $\theta$ is constant, which implies that the directional derivative $\frac{\partial u}{\partial \eta}$ has a sign.

*Step* 3. *End of the proof.*

Either $u$ is constant, and in this case we are done, or there exists $x_0 \in \mathbb{R}^N$ such that $\nabla u(x_0) \ne 0$. In the latter case there are $N-1$ unit vectors of $\mathbb{R}^N$, denoted by $\sigma_1, \dots, \sigma_{N-1}$, which are orthogonal to $\nabla u(x_0)$, *i.e.*, such that

$$0 = \nabla u(x_0)\cdot\sigma_j = \frac{\partial u}{\partial \sigma_j}(x_0) \qquad \forall j = 1, \dots, N-1. \quad (2.29)$$

By the previous step, we must have

$$\frac{\partial u}{\partial \sigma_j}(x) \equiv 0 \qquad \text{on} \quad \mathbb{R}^N, \qquad \forall\, j = 1, \dots, N-1, \quad (2.30)$$

thus $u(x) = h(\tau \cdot x)$, where $\tau = \frac{\nabla u(x_0)}{|\nabla u(x_0)|}$ and $h = h(t)$ is a non constant solution of the ODE $-\left(\frac{h'}{\sqrt{1+|h'|^2}}\right)' = 0$ on $\mathbb{R}$. A direct integration of the latter gives $h(t) = at + b, a \neq 0$. Thus $u$ is an affine function. □

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