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Vortex dynamics for the two-dimensional non-homogeneous Gross-Pitaevskii equation

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Abstract. We derive the asymptotical dynamical law for Ginzburg-Landau vortices in an inhomogeneous background density under the Schrödinger dynamics, when the Ginzburg-Landau parameter goes to zero. New ingredients include lower bounds and approximations across the vortex cores.

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1. Introduction

We are interested in the two-dimensional Gross-Pitaevskii equation

$$i\partial_t u - \Delta u + \frac{1}{\varepsilon^2}\left(V(x) + |u|^2\right)u = 0 \tag{GP}$$

for $u : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{C}$, where $0 < \varepsilon \ll 1$ and $V : \mathbb{R}^2 \to \mathbb{R}^+$ is a smooth potential such that

$$V(x) \to +\infty \text{ as } |x| \to +\infty.$$

The Gross-Pitaevskii equation is a widely used model to describe the dynamics of a Bose-Einstein condensate in a trapping potential V. The equation on $\mathbb{R}^2$ arises via dimension reduction from 3 dimensions; this has been justified for particular choices of V in [1] for example.

Equation (GP) is Hamiltonian, with Hamiltonian given by

$$\mathcal{E}_{\varepsilon,V}(u) = \int_{\mathbb{R}^2} \frac{|\nabla u|^2}{2} + \frac{1}{\varepsilon^2}\left(V(x)\frac{|u|^2}{2} + \frac{|u|^4}{4}\right).$$

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Another quantity which is preserved by the flow associated with (GP) is the total mass M, given by

$$M(u) = \int_{\mathbb{R}^2} |u|^2.$$

For each $m > 0$, there exists[1] at least one positive ground state $\eta \equiv \eta_{\varepsilon,m} : \mathbb{R}^2 \to \mathbb{R}^+$ of total mass equal to m. By definition, a ground state η realizes the infimum

$$\mathcal{E}_{\varepsilon,V}(\eta) = \inf\left\{\mathcal{E}_{\varepsilon,V}(g),\ g \in H^1\left(\mathbb{R}^2,\mathbb{C}\right),\ M(g) = m\right\},$$

and satisfies the Euler-Lagrange equation

$$-\Delta\eta + \frac{1}{\varepsilon^2}\left(V + \eta^2\right)\eta = \frac{1}{\varepsilon^2}\lambda\eta,$$

where we write the Lagrange multiplier as $\frac{1}{\varepsilon^2}\lambda$ for some $\lambda \equiv \lambda_{\varepsilon,m}$.

In the limit $\varepsilon \to 0$, we have

$$\eta^2 \to \rho_{TF} \qquad \text{in } L^2\left(\mathbb{R}^2\right),$$

where the function ρ_{TF}, known as the *Thomas-Fermi profile* in the physics literature, is given by $\rho_{TF}(x) := (\lambda_0 - V)^+(x)$ where the number λ_0 is uniquely determined by the mass condition

$$\int_{\mathbb{R}^2} (\lambda_0 - V(x))^+ \, dx = m.$$

We will study the behaviour of solutions of (GP) which correspond, in a sense to be made precise later, to perturbations of the ground state η by a finite number of quantized vortices, each carrying a single quantum of vorticity. Our goal is to prove that these vortices persist, and to describe their evolution in time.

We will show that to leading order the vortices do not interact, and that each one evolves (in a renormalized time scale) by the orthogonal gradient flow for the function $\log \rho_{TF}$, with a sign depending on the winding number of the given vortex. More precisely, let

$$\Omega_{TF} := \left\{x : \rho_{TF}(x) > 0\right\}$$

be the interior of the limiting support[2] of the ground state, let $\{b_i^0\}_{i=1}^l$ be distinct points in Ω_{TF}, and let $d_1, \ldots, d_l \in \{-1, +1\}$. For each $i \in \{1, \cdots, l\}$, we denote by $b_i(t)$ the solution of the ordinary differential equation

$$\dot{b}_i(t) = d_i \frac{\nabla^\perp \rho_{TF}}{\rho_{TF}}\left(b_i(t)\right), \tag{1.1}$$

where $\nabla^\perp = \left(-\partial_{x_2}, \partial_{x_1}\right)$, with initial datum $b_i(0) = b_i^0$.

[1] We refer to Section 9 for the details on a number of statements regarding the ground states which we state without justification in this introduction.

[2] Note that we do *not* assume that Ω_{TF} is simply connected or that its boundary is smooth.

Theorem 1.1. *Let* $(u_\varepsilon^0)_{\varepsilon>0}$ *be a family of initial data for* (GP) *such that*

$$M\left(u_\varepsilon^0\right) = m,$$

$$\mathcal{E}_{\varepsilon,V}\left(u_\varepsilon^0\right) \le \mathcal{E}_{\varepsilon,V}(\eta) + \pi \sum_{i=1}^{l} \rho_{TF}\left(b_i^0\right) |\log \varepsilon| + o\left(|\log \varepsilon|\right),$$

and

$$\operatorname{curl}\left(\frac{j(u_\varepsilon^0)}{\rho_{TF}}\right) \longrightarrow 2\pi \sum_{i=1}^{l} d_i \delta_{b_i^0} \quad in \quad W_{\text{loc}}^{-1,1}(\Omega_{TF}),$$

as $\varepsilon \to 0$*. Then, as long as the points* $\{b_i(t)\}_{i=1}^l$ *remain distinct,*

$$\operatorname{curl}\left(\frac{j(u_\varepsilon^t)}{\rho_{TF}}\right) \longrightarrow 2\pi \sum_{i=1}^{l} d_i \delta_{b_i(t)} \quad in \quad W_{\text{loc}}^{-1,1}(\Omega_{TF}),$$

as $\varepsilon \to 0$*, where* $u_\varepsilon^t := u_\varepsilon(\cdot, t|\log \varepsilon|)$*.*

Here, $j(u_\varepsilon^t) := (iu_\varepsilon^t, \nabla u_\varepsilon^t)$ where $(z, w) := \operatorname{Im}(z\bar{w})$. Therefore,

$$\frac{1}{2}\operatorname{curl}\left(\frac{j(u_\varepsilon^t)}{\rho_{TF}}\right) = \frac{1}{2}\operatorname{curl} j\left(\frac{u_\varepsilon^t}{\sqrt{\rho_{TF}}}\right) = J\left(\frac{u_\varepsilon^t}{\sqrt{\rho_{TF}}}\right)$$

is the Jacobian determinant of $u_\varepsilon^t/\sqrt{\rho_{TF}}$. It is widely recognized, in the present regime for the Ginzburg-Landau energy, that the notion of a vortex of winding number d_i located at the point $b_i(t)$ is appropriately described by the presence of the term $2\pi d_i \delta_{b_i(t)}$ in the limit of the vorticity field $\operatorname{curl} j\left(u_\varepsilon^t/\sqrt{\rho_{TF}}\right)$.

Remark 1.2. Note that the ordinary differential equations (1.1) are decoupled. Also, since $\rho_{TF}(b_i(t)) = \rho_{TF}(b_i^0)$ for any $t \in \mathbb{R}$ the points $\{b_i(t)\}_{i=1}^l$ remain distinct for all times unless two of them are located on the same level line of ρ_{TF} and have opposite circulations.

Results of this sort in the homogeneous case $\eta \equiv 1$ were first proved in the late 1990s, see [5,6], and have subsequently been developed by a number of authors, see for example [2,11,15]. The point of this paper is thus to understand the effect of the inhomogeneity on the dynamical law for the vortices.

We remark that a number of authors have studied questions about vortex dynamics in inhomogeneous backgrounds for parabolic equations [12,14], or more recently [19] for a quite general class of equations of mixed parabolic-Schrödinger type. The case of pure Schrödinger dynamics presents distinct difficulties and as far as we know has not been treated until now.

The sequel of this introduction is devoted to the presentation of the strategy leading to Theorem 1.1. We notice that will actually prove a result (Theorem 1.3

below) which is stronger in two respects than Theorem 1.1: first it describes the dynamics of vortices at small but fixed value of ε, rather than asymptotically as $\varepsilon \to 0$ in Theorem 1.1, and second it applies to a broader class of inhomogeneous equations (see (NHG) below) where η need not necessarily be the profile of a ground state.

1.1. Perturbation equation and Theorem 1.3

For the class of initial data which we consider in Theorem 1.1, it is convenient to rewrite the corresponding solutions of (GP) in the form

$$u(x,t) = \eta(x)w(x,t) \tag{1.2}$$

and to study the evolution equation for w. One easily checks that if u is a solution to (GP), then w solves

$$i\eta^2\partial_t w - \operatorname{div}\left(\eta^2\nabla w\right) + \frac{1}{\varepsilon^2}\eta^4\left(|w|^2-1\right)w = -\frac{\lambda}{\varepsilon^2}\eta^2 w.$$

In particular, the change of phase and time scale

$$v(x,t) = \exp\left(i\frac{\lambda}{\varepsilon^2}\frac{t}{|\log\varepsilon|}\right) w\left(x, \frac{t}{|\log\varepsilon|}\right)$$

leads to the equation

$$i|\log\varepsilon|\eta^2\partial_t v - \operatorname{div}\left(\eta^2\nabla v\right) + \frac{1}{\varepsilon^2}\eta^4(|v|^2-1)v = 0 \tag{NHG}$$

for v. Note that the change of time scale is related to the fact that the phenomenon which we wish to describe, namely vortex motion, arises in times of order one in that new time scale (see the definition of u_ε^t in the statement of Theorem 1.1).

Our analysis will henceforth focus on equation (NHG). Equation (NHG), like (GP), is Hamiltonian, with Hamiltonian given by the weighted Ginzburg-Landau energy

$$E_{\varepsilon,\eta}(v) \equiv \int_{\mathbb{R}^2} e_{\varepsilon,\eta}(v) = \int_{\mathbb{R}^2} \eta^2\frac{|\nabla v|^2}{2} + \eta^4\frac{\left(|v|^2-1\right)^2}{4\varepsilon^2}. \tag{1.3}$$

As a matter of fact, using the Euler-Lagrange equation for η, one realizes that

$$\mathcal{E}_{\varepsilon,V}(u) = \mathcal{E}_{\varepsilon,V}(\eta) + E_{\varepsilon,\eta}(v) + \frac{\lambda}{2\varepsilon^2}\Big(M(u) - M(\eta)\Big). \tag{1.4}$$

In the sequel, we enlarge our framework and consider equation (NHG) where $\eta : \mathbb{R}^2 \to \mathbb{R}$ is any smooth positive function such that the corresponding Cauchy problem is globally well-posed for initial data in $H^1(\mathbb{R}^2, \eta\, dx)$ and such that the corresponding solutions can be approximated by smooth solutions[3]. In particular, under those assumptions the energy $E_{\varepsilon,\eta}$ is preserved along the flow of (NHG).

[3] This can be verified for a wide variety of weight functions η, but we wish not consider that discussion here since we already know by means of the change of unknown (1.2) that it satisfied when η is a ground state.

Let $\varepsilon > 0$ and let $\Omega \subset \mathbb{R}^2$ be a bounded open set. Let $\{a_i^0\}_{i=1}^l$ be distinct points in Ω, and let $d_1, \dots, d_l \in \{-1, +1\}$. For each $i \in \{1, \cdots, l\}$, we denote by $a_i(t)$ the solution, as long as it does not reach $\partial\Omega$, of the ordinary differential equation

$$\dot{a}_i(t) = d_i \nabla^{\perp} \log \eta^2 (a_i(t)), \tag{1.5}$$

where $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$, with initial datum $a_i(0) = a_i^0$.

We assume that $\eta_{\min} := \inf_{x\in\Omega} \eta(x) > 0$, and we fix a time $T_{\rm col} > 0$ such that

$$\rho_{\min} := \min_{t\in[0,T_{\rm col}]} \min \left\{ \left\{ \frac{1}{2} d(a_i(t), a_j(t)) \right\}_{i\neq j} \cup \{d(a_i(t), \partial\Omega)\}_i \cup \{1\} \right\} > 0. \tag{1.6}$$

Finally, we consider a finite energy solution v of (NHG), we set $v^t := v(\cdot, t)$ and we define, for $t \in [0, T_{\rm col}]$,

$$r_a^t := \|Jv^t - \pi \sum_{i=1}^{l} d_i \delta_{a_i(t)}\|_{W^{-1,1}(\Omega)}, \tag{1.7}$$

and

$$\Sigma^t := \frac{E_{\varepsilon,\eta}(v^t)}{|\log\varepsilon|} - \pi \sum_{i=1}^{l} \eta^2(a_i(t)). \tag{1.8}$$

We will deduce Theorem 1.1 from:

Theorem 1.3. *There exist positive constants ε_0, γ_0 and C_0, depending only on l, $\eta_{\min}$, $\rho_{\min}$, and $\|\nabla\eta^2\|_{L^\infty(\Omega)}$, such that if $0 < \varepsilon \le \varepsilon_0$ and if $\Sigma^0 + r_a^0 \le \gamma_0$, then*

$$r_a^t \le r_a^0 + \left(\Sigma^0 + r_a^0 + \frac{\log|\log\varepsilon|}{|\log\varepsilon|} \right) \left(e^{C_0 t} - 1 \right) + C_0 \varepsilon^{\frac{1}{2}}, \tag{1.9}$$

as long as $t \le T_{\rm col}$ and $\Sigma^0 + r_a^t(t) \le \gamma_0$.

Remark 1.4. i) As we shall discuss in Section 1.2 below, the quantity r_a^t, which is a sort of discrepancy measure, can be thought of as measuring the distances between the "actual vortex locations" and the desired vortex locations. The quantity Σ^t, multiplied by $|\log\varepsilon|$, corresponds to the excess of energy of the solution v with respect to an energy minimizing field possessing the vortices at the points $a_i(t)$. Notice that since $E_{\varepsilon,\eta}$ is preserved by the flow for v and η^2 is preserved by the flow for the $a_i's$, we have $\Sigma^t \equiv \Sigma^0, \forall\, t \in [0, T]$.

ii) Theorem 1.3 is interesting for initial data such that $\Sigma^0 + r_a^0$ is small. The existence of such data is standard. For example, if we fix $f : [0, \infty) \to [0, 1]$ such that $f' \ge 0$, $f(0) = 0$, and $f(s) \to 1$ as $s \to \infty$, then for the initial data

$$v^0(z) := \prod_{i=1}^{l} f\left(\frac{|z - a_i|}{\varepsilon}\right) \left(\frac{z - a_i^0}{|z - a_i^0|}\right)^{d_i}, \qquad x := (x_1, x_2) \cong z = x_1 + ix_2,$$

one can check that $\Sigma^0 \le C|\log\varepsilon|^{-1}, r_a^0 \le C\varepsilon$. In any case, (1.9) contains the error term in $\log|\log\varepsilon|/|\log\varepsilon|$ which implies that (1.9) only yield the inequality $\Sigma^0 + r_a^t \le \gamma_0$ for times at most of order $\log|\log\varepsilon|$.

iii) One could supplement the claims of Theorem 1.3 with closeness estimate for $j(v)$ to a reference field j_* of very simple form. This would follow from an application of Corollary 5.2 below; however at the level of approximation which we have adopted here it is only meaningful in a neighborhood of size $o(1/|\log\varepsilon|)$ of the vortex core.

1.2. Elements in the proofs

Under the conditions that will prevail throughout most of this paper, we will be able to identify points $\xi_1^t, \dots \xi_l^t$ and a number r_ξ^t such that

$$\left\| Jv^t - \pi\sum_{i=1}^{l} d_i\delta_{\xi_i^t}\right\|_{W^{-1,1}(\Omega)} \le r_\xi^t \le \varepsilon^{1/2} \ll r_a^t. \tag{1.10}$$

This is expressed in Proposition 4.1 below, and entitles us to think of $\xi_i^t, i = 1, \dots, l$ as being the "actual locations of the vortices" in v^t, up to precision of order $\le r_\xi^t$. Admitting this interpretation, then basic facts about the $W^{-1,1}$ norm, recalled in Section 2, imply that

$$r_a^t = \frac{1}{\pi}(1+o(1))\sum_{i=1}^{l}\left|\xi_i^t - a_i^t\right| \tag{1.11}$$

is essentially the aggregate distance between the actual vortex locations and the desired vortex locations, as remarked above.

Heuristic considerations also suggest that if v^t is a function with vortices at points $\xi_1^t, \dots, \xi_l^t$ (or more precisely, if (1.10) holds), then

$$E_{\varepsilon,\eta}(v^t) \gtrapprox \pi|\log\varepsilon|\big(1-o(1)\big)\sum_{i=1}^{l}\eta^2\big(\xi_i^t\big). \tag{1.12}$$

Hence $E_{\varepsilon,\eta}(v^t) - \pi|\log\varepsilon|\sum_{i=1}^{l}\eta^2(\xi_i^t)$ corresponds to energy that is not committed to the vortices, and this energy in principle can cause difficulties for our analysis. From (1.10), (1.11), we have

$$\begin{aligned} E_{\varepsilon,\eta}(v^t) - \pi|\log\varepsilon|\sum_{i=1}^{l}\eta^2(\xi_i^t) &\le |\log\varepsilon|\left(\Sigma^t + \frac{1}{\pi}(1+o(1))\left\|\nabla\eta^2\right\|_\infty r_a^t\right) \\ &\le |\log\varepsilon|\left(\Sigma^0 + Cr_a^t\right). \end{aligned} \tag{1.13}$$

For our analysis, it suffices to use estimates in the spirit of (1.13) that are a little weaker than those suggested in (1.13), these are established in Proposition 3.1. We expect from (1.12) and (1.13) that control of r_a^t should yield a good deal of information about v^t. This is expressed in Proposition 5.1, where we compare $j(v^t)$ to a reference field j_*^t. An important feature of that approximation is that it holds across the vortex core.

In order to control the evolution in time of r_a^t, we rely on some evolution equations satisfied by smooth solutions of (NHG). Conservation of energy is a consequence of the identity

$$\partial_t e_{\varepsilon,\eta}(v) = \operatorname{div}\big(\eta^2(\nabla v, v_t)\big), \tag{1.14}$$

and the canonical equation for conservation of mass can be written

$$\frac{|\log\varepsilon|}{2}\partial_t\big(\eta^2(|v|^2-1)\big) = \nabla\cdot\big(\eta^2 j(v)\big). \tag{1.15}$$

The vorticity Jv satisfies an evolution equation that it is convenient to write in integral form:

$$\begin{aligned}\frac{d}{dt}\int_\Omega \varphi Jv \\ = \frac{1}{|\log\varepsilon|}\int_\Omega\left(\epsilon_{lj}\varphi_{x_l}\frac{\eta^2_{x_k}}{\eta^2}\left[v_{x_j}\cdot v_{x_k} + \delta_{jk}\frac{\eta^2}{\varepsilon^2}\big(|v|^2-1\big)^2\right] + \epsilon_{lj}\varphi_{x_kx_l}v_{x_j}\cdot v_{x_k}\right)\end{aligned} \tag{1.16}$$

where φ is any smooth, compactly supported test function and ε_{lj} is the usual antisymmetric tensor. This follows from the fact that $Jv = \frac{1}{2}\operatorname{curl} j(v)$ together with the equation for the evolution of $j(v)$, which is obtained from (NHG) after multiplying by ∇v and rewriting the result.

Identity (1.16) is central to our analysis of vortex dynamics, as in previous works [2, 5, 6, 11, 15] on the homogeneous case (for which of course (1.16) still holds, with $\eta \equiv 1$). Under the conditions that Jv is approximately a measure of the form $\pi\sum_{i=1}^l d_i\delta_{\xi_i(t)}$, where $\xi_i(t)$ are the vortex locations and $d_i \in \{\pm1\}$ their signs, one expects the left-hand side of (1.16) to satisfy

$$\begin{aligned}\frac{d}{dt}\int_\Omega\varphi Jv &\approx \frac{d}{dt}\int_\Omega\varphi\left(\pi\sum d_i\delta_{\xi_i(t)}\right) \approx \frac{d}{dt}\left(\pi\sum d_i\varphi\big(\xi_i(t)\big)\right)\\ &= \pi\sum d_i\nabla\varphi(\xi_i(t))\cdot\dot{\xi}_i(t).\end{aligned}$$

Assuming that this holds, then to understand the vortex velocities $\dot{\xi}_i$, it only suffices to understand the right-hand side of (1.16). It turns out that it also suffices to consider test functions φ that are linear near each vortex. For such test functions, in the homogeneous case $\nabla\eta^2 \equiv 0$, the integrand on the right-hand side of (1.16) is supported away from the vortex locations, and one is thus able to control vortex dynamics by controlling terms of the form $v_{x_i} \cdot v_{x_j}$ away from the vortex cores. This argument is a key feature of all existing work on vortex dynamics in the homogeneous case.

When $\nabla\eta^2 \neq 0$, it becomes necessary to control terms like $v_{x_i} \cdot v_{x_j}$ across the vortex cores. Carrying this out, in particular relying on the approximation given by Proposition 5.1, is the main new point in our analysis. Once this is established, the whole argument is completed by using a Gronwall type argument on a quantity related to r_a^t, namely $\Sigma^0 + g(r_a^t)$, where the function gb is defined in (3.1). This demonstrates in particular that the new information found in Proposition 5.1 is strong enough to close the estimates and conclude the proof.

2. A useful lemma

We frequently use the $W^{-1,1}$ norm. The specific convention we use is in our definition is

$$\|\mu\|_{W^{-1,1}(\Omega)} := \sup\left\{\langle\mu,\varphi\rangle \;:\; \varphi \in W_0^{1,\infty}(\Omega), \max\left\{\|\varphi\|_\infty, \|\nabla\varphi\|_\infty\right\} \leq 1\right\}.$$

In this paper, we will only use this norm on measures or more regular objects, although of course it is well-defined for a somewhat larger class of distributions.

The following lemma, which we will use numerous times, is an easy special case of classical results (see [3] for example).

Lemma 2.1. *Suppose that Ω is an open subset of $\mathbb{R}^n$, and that $\{a_i\}_{i=1}^l$ are distinct points in Ω. Define $\rho_a := \min\{\{\frac{1}{2}|a_i - a_j|\}_{i\neq j} \cup \{d(a_i, \partial\Omega)\}_i \cup \{1\}\}$. Given any points $\{\xi_i\}_{i=1}^l$ in Ω and $\{d_i\}_{i=1}^l \in \{\pm 1\}^l$, if*

$$\left\|\sum_{i=1}^l d_i \delta\left(a_i - \xi_i\right)\right\|_{W^{-1,1}(\Omega)} \leq \frac{1}{4}\rho_a, \tag{2.1}$$

then (after possibly relabelling the points $\{\xi_i\}_{i=1}^l$),

$$\left\|\sum_{i=1}^l d_i \delta\big(a_i - \xi_i\big)\right\|_{W^{-1,1}(\Omega)} = \sum_{i=1}^l \big|a_i - \xi_i\big|. \tag{2.2}$$

In the remainder of this paper, we will always tacitly assume that under the conditions of the lemma, the points ξ_i are labelled so that the conclusion holds.

We give a short proof for the reader's convenience.

Proof. For $i = 1, \ldots, l$, define $\varphi_i(x) := d_i(\frac{1}{2}\rho_a - |x - a_i|)^+$.

Then $\max(\|\varphi_i\|_\infty, \|\nabla\varphi_i\|_\infty) = \max(\frac{1}{2}\rho_a, 1) = 1$, for every i, so

$$\begin{aligned}\left\|\sum_{i=1}^{l} d_i\delta\,(a_i - \xi_i)\right\|_{W^{-1,1}(\Omega)} &\geq \left\langle \sum_{j=1}^{l} d_j\left(\delta_{a_j} - \delta_{\xi_j}\right), \varphi_i\right\rangle \\ &= \frac{\rho_a}{2} - \sum_j d_i d_j \left(\frac{\rho_a}{2} - |\xi_j - a_i|\right)^+.\end{aligned}$$

Then (2.1) implies that $\{\xi_j\}_{j=1}^l \cap B(a_i, \rho_a/2)$ is nonempty for every i. Since $\{B(a_i, \rho_a/2)\}_{i=1}^l$ are pairwise disjoint, it follows (after possibly reindexing) that $\{\xi_j\}_{j=1}^l \cap B(a_i, \rho_a/2) = \{\xi_i\}$ for all i. Now let $\varphi = \sum_i \varphi_i$. The functions $\{\varphi_i\}$ have disjoint support, so $\max(\|\varphi\|_\infty, \|\nabla\varphi\|_\infty) = 1$, and thus $\|\sum_{i=1}^l d_i\delta(a_i - \xi_i)\|_{W^{-1,1}(\Omega)} \geq \langle\sum_{i=1}^l d_i(\delta_{a_i} - \delta_{\xi_i}), \varphi\rangle = \sum_{i=1}^l |a_i - \xi_i|$. On the other hand, if ψ is any compactly supported function such that $\max(\|\psi\|_\infty, \|\nabla\psi\|_\infty) \leq 1$, then

$$\left\langle \sum_{i=1}^{l} d_i\left(\delta_{a_i} - \delta_{\xi_i}\right), \psi\right\rangle \leq \sum_{i=1}^{l} \left|\psi(a_i) - \psi(\xi_i)\right| \leq \sum_{i=1}^{l} \left|a_i - \xi_i\right|.$$

Hence $\|\sum_{i=1}^l d_i\delta(a_i - \xi_i)\|_{W^{-1,1}(\Omega)} \leq \sum_{i=1}^l |a_i - \xi_i|$. □

3. Relating weighted and unweighted energy

In this section, we relate the weighted and unweighted energy under some localization assumptions on the Jacobian. For a measurable subset $A \subset \mathbb{R}^2$ and $v \in \dot{H}^1(A, \mathbb{C})$ we set

$$E_{\varepsilon,\eta}(v; A) := \int_A e_{\varepsilon,\eta}(v) \quad \text{and} \quad E_\varepsilon(v; A) := E_{\varepsilon,1}(v, A).$$

Define the function g on $\mathbb{R}^+$ by

$$g(x) = \begin{cases} x + \dfrac{|\log x|}{|\log\varepsilon|} & \text{if } x > \dfrac{1}{|\log\varepsilon|} \\ \dfrac{1 + \log|\log\varepsilon|}{|\log\varepsilon|} & \text{otherwise.} \end{cases} \tag{3.1}$$

We have

Proposition 3.1. *Let* $\Omega \subset \mathbb{R}^2$ *an open set,* $\{a_i\}_{i=1}^l$ *distinct points in* Ω, $\{d_i\}_{i=1}^l \in \{\pm 1\}$, *and* $\eta : \Omega \to \mathbb{R}$ *a positive Lipschitz function such that* $\inf_\Omega \eta =: \eta_{\min} > 0$.

Set $\rho_a := \min\{\{\frac{1}{2}d(a_i, a_j)\}_{i\neq j} \cup \{d(a_i, \partial\Omega)\}_i \cup \{1\}\}$, *and let* $\varepsilon \leq \exp(-\frac{8}{\rho_a})$ *and* $v \in \dot{H}^1(\Omega, \mathbb{C})$ *be such that*

$$\Sigma_a := \left(\frac{E_{\varepsilon,\eta}(v)}{|\log\varepsilon|} - \pi\sum_{i=1}^{l}\eta^2(a_i)\right)^+ < +\infty. \tag{3.2}$$

Assume also that

$$r_a := \left\| Jv - \pi\sum_{i=1}^{l} d_i\delta_{a_i}\right\|_{W^{-1,1}(\Omega)} \leq \frac{\rho_a}{8}. \tag{3.3}$$

Then there exists a constant C, *depending only on* l, $\|\nabla\eta^2\|_\infty$ *and* $\eta_{\min}$, *such that*

$$\begin{aligned} \frac{E_{\tilde{\varepsilon}}(v; B(a_i, R))}{|\log\tilde{\varepsilon}|} &\leq \pi + C\big(\Sigma_a + g(r_a)\big) \qquad \text{for } i = 1, \dots, l \\ \frac{E_{\tilde{\varepsilon}}(v; \Omega\setminus\cup_{i=1}^{l}B(a_i, R))}{|\log\tilde{\varepsilon}|} &\leq C\big(\Sigma_a + g(r_a)\big) \end{aligned} \tag{3.4}$$

where $R = 4\max(r_a, |\log\varepsilon|^{-1}) \leq \frac{\rho_a}{4}$ *and* $\tilde{\varepsilon} := \frac{\varepsilon}{\eta_{\min}}$, *and the function* g *is defined in* (3.1).

Proof. Let $r \in [r_a, \frac{\rho_a}{8}]$ be a number that will be fixed later. Then the balls $\{B(a_i, 4r)\}_{i=1}^{l}$ are disjoint and contained in Ω. Let $i \in \{1, \cdots, l\}$; by monotonicity of the $W^{-1,1}$ norms with respect to the domain, we deduce from (3.3) that

$$\|Jv - \pi d_i\delta_{a_i}\|_{W^{-1,1}(B(a_i,4r))} \leq r_a \leq r.$$

It follows from the lower bounds estimates of Jerrard [8] or Sandier [18] that

$$E_\delta\big(v, B(a_i, 4r)\big) \geq \pi\log\frac{4r}{\delta} - K_1, \tag{3.5}$$

for every $\delta > 0$, where K_1 is a universal constant. We next write

$$\begin{aligned} E_{\varepsilon,\eta}(v, B(a_i, 4r)) &= \int_{B(a_i,4r)} \eta^2(x)\frac{|\nabla v|^2}{2} + \eta^4(x)\frac{(|v|^2-1)^2}{4\varepsilon^2} \\ &\geq \int_{B(a_i,4r)} \eta^2(x)\left[\frac{|\nabla v|^2}{2} + \frac{(|v|^2-1)^2}{4\left(\frac{\varepsilon}{\eta_{\min}}\right)^2}\right] \\ &\geq \left(\inf_{x\in B(a_i,4r)}\eta^2(x)\right) E_{\tilde{\varepsilon}}\big(v, B(a_i, 4r)\big). \end{aligned} \tag{3.6}$$

Therefore, from (3.5) with the choice $\delta = \tilde\varepsilon$, and noting that $|\log r| \geq \log|\frac{\rho_a}{8}| \geq \log 8 \geq 1$, we obtain

$$E_{\varepsilon,\eta}\big(v, B(a_i, 4r)\big) \geq \eta^2(a_i)\pi|\log\varepsilon| - K_2\left(r|\log\varepsilon| + |\log r|\right), \tag{3.7}$$

where K_2 depends only on $\|\nabla\eta^2\|_\infty$ and $\eta_{\min}$.

On the other hand, we deduce from (3.2) and (3.7) that

$$\begin{aligned} E_{\varepsilon,\eta}\big(v, B(a_i, 4r)\big) &\leq E_{\varepsilon,\eta}(v,\Omega) - \sum_{j\neq i} E_{\varepsilon,\eta}\big(v, B(a_j, 4r)\big) \\ &\leq \pi\eta^2(a_i)|\log\varepsilon| + \Sigma_a|\log\varepsilon| + (l-1)K_2\left(r|\log\varepsilon| + |\log r|\right). \end{aligned} \tag{3.8}$$

Hence, going back to (3.6) we obtain

$$\begin{aligned} E_{\tilde\varepsilon}\big(v, B(a_i, 4r)\big) &\leq \frac{1}{\big(\inf_{x\in B(a_i,4r)}\eta^2(x)\big)} E_{\varepsilon,\eta}\big(v, B(a_i, 4r)\big) \\ &\leq \pi|\log\tilde\varepsilon| + K_3\big(\Sigma_a|\log\varepsilon| + r|\log\varepsilon| + |\log r|\big), \end{aligned} \tag{3.9}$$

where K_3 depends only on l, $\|\nabla\eta^2\|_\infty$ and $\eta_{\min}$.

Concerning the energy outside the balls $B(a_i, 4r)$, we have from (3.2) and (3.7)

$$\begin{aligned} E_{\varepsilon,\eta}(v, \Omega\setminus\cup_i B(a_i, 4r)) &= E_{\varepsilon,\eta}\big(v, \Omega\big) - \sum_i E_{\varepsilon,\eta}\big(v, B(a_j, 4r)\big) \\ &\leq \Sigma_a|\log\varepsilon| + lK_2\left(r|\log\varepsilon| + |\log r|\right). \end{aligned} \tag{3.10}$$

Hence,

$$\begin{aligned} E_{\tilde\varepsilon}(v, \Omega\setminus\cup_i B(a_i, 4r)) &\leq \frac{1}{\inf\eta^2} E_{\varepsilon,\eta}\big(v, \Omega\setminus\cup_i B(a_i, 4r)\big) \\ &\leq K_4\big(\Sigma_a|\log\varepsilon| + r|\log\varepsilon| + |\log r|\big), \end{aligned} \tag{3.11}$$

where K_4 depends only on l, $\|\nabla\eta^2\|_\infty$ and $\eta_{\min}$.

The function $r \mapsto r + |\log r|/|\log\varepsilon|$ is minimized taking $r := \max\big(r_a, \frac{1}{|\log\varepsilon|}\big)$, in which case $r \leq \frac{\rho_a}{8}$ by assumption on r_a and ε. The conclusions (3.4) follow with the choice $C := \max(K_3, K_4)$. □

Remark 3.2. If we define $\tilde\Sigma_a := \frac{E_{\varepsilon,\eta}(v)}{|\log\varepsilon|} - \pi\sum_{i=1}^{l}\eta^2(a_i)$, then (3.7) implies that

$$\tilde\Sigma_a \geq \sum_i\left(\frac{E_{\varepsilon,\eta}\big(v, B(a_j, 4r)\big)}{|\log\varepsilon|} - \pi\eta^2(a_i)\right) \geq -lK_2\left(r + \frac{|\log r|}{|\log\varepsilon|}\right)$$

for every $r \in [r_a, \frac{\rho_a}{8}]$. Choosing $r = \max(r_a, \frac{1}{|\log\varepsilon|})$ as above, we find that $\tilde\Sigma_a \geq -lK_2 g(r_a)$. In particular, $\Sigma_a = (\tilde\Sigma_a)^+ \leq \tilde\Sigma_a + 2lK_2 g(r_a)$. So all our estimates remain true if we replace $C(\Sigma_a + g(r_a))$ by $C(\tilde\Sigma_a + (2lK_2 + 1)g(r_a))$.

4. Improved localization for Jacobians

In this section, we prove that if the Jacobian of a function v is known to be sufficiently localized, then, provided the excess energy of v with respect to the points of localization is not to big, the localisation is actually potentially much stronger. A result in the same spirit was obtained by Jerrard and Spirn in [10] for the Ginzburg-Landau functional without a weight. Our proof here below makes a direct use of Theorem 1.1 and Theorem 1.2′ in [10] by relating the weighted and unweighted Ginzburg-Landau energies according to Section 3.

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, open set, $\{a_i\}_{i=1}^l$ distinct points in Ω, $\{d_i\}_{i=1}^l \in \{\pm 1\}$, and $\eta : \Omega \to \mathbb{R}$ a positive Lipschitz function such that $\inf_\Omega \eta =: \eta_{\min} > 0$. Set $\rho_a = \min\big\{\{\frac{1}{2}d(a_i,a_j)\}_{i\neq j} \cup \{d(a_i,\partial\Omega)\}_i \cup \{1\}\big\}$. Let $\varepsilon \le \exp(-\frac{8}{\rho_a})$ and let $v \in \dot{H}^1(\Omega,\mathbb{C})$ be such that*

$$\Sigma_a := \left(\frac{E_{\varepsilon,\eta}(v)}{|\log\varepsilon|} - \pi \sum_{i=1}^l \eta^2(a_i) \right)^+ < +\infty. \tag{4.1}$$

Also, assume that

$$r_a = \| Jv - \pi \sum_{i=1}^l d_i \delta_{a_i} \|_{W^{-1,1}(\Omega)} \le \frac{\rho_a}{16}. \tag{4.2}$$

Then there exists $C_1 \ge 1$, depending only on a lower bound for ρ_a and $\eta_{\min}$ and on an upper bound for l and $\|\nabla\eta^2\|_\infty$, and for each $i \in \{1,\cdots,l\}$ there exists a point $\xi_i \in B(a_i, 2r_a)$, such that

$$\begin{aligned}\left\| Jv - \pi \sum_{i=1}^l d_i \delta_{\xi_i} \right\|_{W^{-1,1}(\Omega)} &\le r_\xi \equiv r_\xi(\Sigma_a, r_a) \\ &\equiv \varepsilon \exp\big(C_1\big(\Sigma_a + g(r_a)\big)|\log\varepsilon|\big)\end{aligned} \tag{4.3}$$

where g is defined in Proposition 3.1.

Remark 4.2. Note that Lemma 2.1 and (4.2), (4.3) imply that

$$\sum_{i=1}^l \big|a_i - \xi_i\big| \le \frac{1}{\pi}(r_a + r_\xi). \tag{4.4}$$

Remark 4.3. Since $g(r) \ge \frac{\log|\log\varepsilon|}{|\log\varepsilon|}$ for every r, our requirement that $C_1 \ge 1$ implies that

$$r_\xi \ge \varepsilon|\log\varepsilon|. \tag{4.5}$$

As mentioned, the proof of Proposition 4.1 relies very heavily on estimates from [10]. Following the proof, we discuss some small adjustments we have made in employing these estimates here. Also, from here upon in many places we will denote by C constants whose actual value may change from line to line but which could eventually be given a common value depending only on l, $\rho_{\min}$, $\eta_{\min}$ and $\|\nabla\eta^2\|_\infty$.

Proof. Since $\varepsilon \le \exp(-\frac{8}{\rho_a})$, our assumptions imply that the hypotheses of Proposition 3.1 are verified. Then, since $B(a_i, \frac{\rho_a}{2}) \subset B(a_i, R) \cup (\Omega \setminus \cup_{i=1}^l B(a_i, R))$ for any $R < \frac{\rho_a}{2}$, and recalling that $g(r) \ge \frac{\log|\log\varepsilon|}{|\log\varepsilon|}$ for all r, we deduce from (3.4) that

$$\frac{E_{\tilde\varepsilon}\big(v; B\big(a_i, \frac{\rho_a}{2}\big)\big)}{\log(\frac{\rho_a}{2\tilde\varepsilon})} \le \frac{E_{\tilde\varepsilon}\big(v; B\big(a_i, \frac{\rho_a}{2}\big)\big)}{|\log\tilde\varepsilon|}\left(1 + 2\frac{|\log\frac{\rho_a}{2}|}{|\log\tilde\varepsilon|}\right) \le \pi + C\big(\Sigma_a + g(r_a)\big) \tag{4.6}$$

for $i = 1, \dots, l$, and similarly (3.4) implies that

$$\frac{E_{\tilde\varepsilon}\big(v; \Omega \setminus \cup_{i=1}^l B\big(a_i, \frac{\rho_a}{4}\big)\big)}{|\log\tilde\varepsilon|} \le C\big(\Sigma_a + g(r_a)\big). \tag{4.7}$$

According to Theorem 1.2′ in [10], it follows from (4.2) and (4.6) that for every $i \in \{1, \dots, l\}$, there exists some $\xi_i \in B(a_i, 2r_a)$ such that

$$\big\| Jv - \pi d_i \delta_{\xi_i} \big\|_{W^{-1,1}(B(a_i, \frac{\rho_a}{2}))} \le C\,\tilde\varepsilon \exp\big[C\big(\Sigma_a + g(r_a)\big)|\log\varepsilon|\big]. \tag{4.8}$$

In addition, Theorem 1.1 in [10] implies that if V is any bounded, open subset of Ω then

$$\|Jv\|_{W^{-1,1}(V)} \le C\,\tilde\varepsilon\, E_{\tilde\varepsilon}(v, V)\exp\left(\frac{E_{\tilde\varepsilon}(v, V)}{\pi}\right) \le C\,\tilde\varepsilon\, \exp\big(E_{\tilde\varepsilon}(v, V)\big). \tag{4.9}$$

In particular, this and (4.7) imply that

$$\|Jv\|_{W^{-1,1}(\Omega\setminus\cup_{i=1}^l B(a_i \frac{\rho_a}{4}))} \le C\,\tilde\varepsilon \exp\big[C\big(\Sigma_a + g(r_a)\big)|\log\varepsilon|\big]. \tag{4.10}$$

Now for $i \in \{1, \dots, l\}$, let $\chi_i \in C_c^\infty(B(a_i, \frac{\rho_a}{2}))$ be functions such that $\chi_i = 1$ on $B(a_i \frac{\rho_a}{4})$, $0 \le \chi_i \le 1$, and $\|\nabla\chi_i\|_\infty \le C\rho_a^{-1}$. Also, let $\chi_0 = 1 - \sum_{i=1}^l \chi_i$. Then for any $\varphi \in C_0^\infty(\Omega)$, such that $\|\varphi\|_{W^{1,\infty}(\Omega)} \le 1$,

$$\begin{aligned}
\left\langle \varphi, Jv - \pi\sum_{i=1}^l d_i\delta_{\xi_i} \right\rangle &= \sum_{j=0}^l \left\langle \chi_j\varphi, Jv - \pi\sum_{i=1}^l d_i\delta_{\xi_i}\right\rangle \\
&= \langle \chi_0\varphi, Jv\rangle + \sum_{i=1}^l \big\langle \chi_i\varphi, Jv - \pi d_i\delta_{\xi_i}\big\rangle \\
&\le \sum_{i=1}^l \big\|\chi_i\varphi\big\|_{W^{1,\infty}} C\varepsilon\exp\big[C(\Sigma_a + g(r_a))|\log\varepsilon|\big]
\end{aligned}$$

where we have used (4.8) for $i = 1, \dots, l$ and (4.10) for $i = 0$. Thus

$$\left\langle \varphi, Jv - \pi \sum_{i=1}^{l} d_i \delta_{\xi_i} \right\rangle \leq \frac{C}{\rho_a} \varepsilon \exp\left[C\big(\Sigma_a + g(r_a)\big)|\log \varepsilon|\right]$$
$$\leq \varepsilon \exp\left[C_1\big(\Sigma_a + g(r_a)\big)|\log \varepsilon|\right]$$

for a suitable C_1, depending on the lower bound ρ_0 for ρ_a as well as $l, \eta_{\min}$, $\|\nabla \eta^2\|_\infty$. This implies (4.3). □

To facilitate comparison between some facts that we have used above and the precise statements in [10], we make the following remarks.

First, we have used some estimates in cruder but simpler forms than they appear in [10]. For example, on the right-hand side of (4.8), we have replaced an expressions of the form $(C + K_0)^2 \exp(\frac{K_0}{\pi})$ from [10], where here we take $K_0 = C(\Sigma_a + g(r_a))|\log \varepsilon|$, by the simpler expression $C \exp(K_0)$. We have also used the fact that $K_0 = C(\Sigma_a + g(r_a))|\log \varepsilon| \geq \log|\log \varepsilon|$ to allow us to absorb some lower-order terms from [10].

Second, estimates in [10] are stated in terms of a slightly different norm, $\|\mu\|_{\dot{W}^{-1,1}(V)} := \sup\{\langle \mu, \phi \rangle : \phi \in C_c^\infty(V), \|\nabla \phi\|_\infty \leq 1\}$. This does not cause any problems for us, since clearly $\|\mu\|_{W^{-1,1}(V)} \leq \|\mu\|_{\dot{W}^{-1,1}(V)}$.

Finally, the estimate corresponding to (4.9) in [10] is a special case of a more general result, and as stated there requires the additional assumption that $\frac{E_{\tilde{\varepsilon}}(v,V)}{|\log \tilde{\varepsilon}|} < \pi$. However, since $\|Jv\|_{W^{-1,1}(V)} \leq \|Jv\|_{L^1(V)} \leq 2E_{\tilde{\varepsilon}}(v; V)$, it is clear that (4.9) is still true if $\frac{E_{\tilde{\varepsilon}}(v,V)}{|\log \tilde{\varepsilon}|} \geq \pi$.

Remark 4.4. If Ω is simply connected, then we can alternately argue by citing a result from [11] to obtain an estimate of the form (4.3) with C_1 *independent of* ρ_a, at the rather small expense of having to replace $\frac{\rho_a}{16}$ on the right-hand side of (4.2) by some smaller quantity depending on l as well as ρ_a. This is in principle useful if one wants to consider large numbers of vortices. The relevant result (Theorem 3) from [11] is proved using facts from [10], as in the proof above, but combining estimates on the balls and away from the balls in a more careful way, to avoid introducing the factors of ρ_a^{-1} that arise from the cutoff functions that we have employed here.

The proof of Theorem 3 from [11] can surely be adapted to yield a similar result without the assumption that Ω be simply connected, but since the proof is slightly complicated, we prefer not to tinker with it here.

5. Across the core approximation by reference field

In this section we prove:

Proposition 5.1. *Let* $\Omega \subset \mathbb{R}^2$ *be an open set,* $\{\xi_i\}_{i=1}^l$ *distinct points in* Ω, $\{d_i\}_{i=1}^l \in \{\pm 1\}$, *and* $\eta : \Omega \to \mathbb{R}$ *a positive Lipschitz function such that* $\inf_\Omega \eta =: \eta_{\min} > 0$.

Set $\rho_\xi = \min\left\{\{\frac{1}{2}d(\xi_i,\xi_j)\}_{i\neq j} \cup \{d(\xi_i,\partial\Omega)\}_i \cup \{1\}\right\}$. *Let* $\varepsilon \leq \exp(-\frac{8}{\rho_\xi})$ *and let* $v \in \dot{H}^1(\Omega,\mathbb{C})$ *be such that*

$$\Sigma_\xi = \left(\frac{E_{\varepsilon,\eta}(v)}{|\log\varepsilon|} - \pi\sum_{i=1}^{l}\eta^2(\xi_i)\right)^+ < +\infty \tag{5.1}$$

and

$$\left\| Jv - \pi\sum_{i=1}^{l} d_i\delta_{\xi_i}\right\|_{W^{-1,1}(\Omega)} \leq r_\xi \equiv \varepsilon\exp(K|\log\varepsilon|) = \varepsilon^{1-K} \tag{5.2}$$

for some $K \leq \frac{1}{2}$.

Define $j_* = j_*(\{\xi_i\}, r_\xi)$ *in* Ω *by*

$$j_*(x) = \begin{cases} d_i\dfrac{(x-\xi_i)^\perp}{\max(r_\xi,|x-\xi_i|)^2} & \text{if } x \in B\left(\xi_i,\dfrac{1}{|\log\varepsilon|}\right) \\ 0 & \text{if } x \in \Omega\setminus\cup_{i=1}^{l}B\left(\xi_i,\dfrac{1}{|\log\varepsilon|}\right) \end{cases}$$

where $(y_1,y_2)^\perp := (-y_2,y_1)$. *Then*

$$E_{\varepsilon,\eta}(|v|) + \frac{1}{2}\int_\Omega \eta^2\left|\frac{j(v)}{|v|} - j_*\right|^2 \leq (C\Sigma_\xi + K)|\log\varepsilon| + C\log|\log\varepsilon|, \tag{5.3}$$

and

$$\|\nabla\times(j(v)-j_*)\|_{W^{-1,1}} \leq C_3 r_\xi. \tag{5.4}$$

where the constant C *depends only on* l, $\|\nabla\eta^2\|_\infty$ *and* $\eta_{\min}$.

Since $K \leq \frac{1}{2}$, the assumption that $\varepsilon < \exp(-8/\rho_\xi)$ implies that $r_\xi < \frac{1}{|\log\varepsilon|} < \frac{\rho_\xi}{8}$. In particular, the balls $B(\xi_i,|\log\varepsilon|^{-1})$, $i = 1,\ldots,l$ are pairwise disjoint and contained in Ω.

Proof. We will use more than once the fact that

$$|\nabla v|^2 = |\nabla|v||^2 + \left|\frac{j(v)|}{|v|}\right|^2. \tag{5.5}$$

Step 1: verification of (5.4). A direct calculation, using the definition of j_*, shows that for any smooth φ,

$$\left\langle\varphi,\nabla\times j_* - 2\pi\sum d_i\delta_{\xi_i}\right\rangle = \sum_{i=1}^{l}\frac{2}{r_\xi^2}\int_{B(\xi_i,r_\xi)} d_i(\varphi(x)-\varphi(\xi_i)) \leq C\,l\|\nabla\varphi\|_\infty r_\xi.$$

Thus $\|\nabla\times j_* - 2\pi\sum d_i\delta_{\xi_i}\|_{W^{-1,1}(\Omega)} \leq Cr_\xi$. Recalling that $Jv = \frac{1}{2}\nabla\times j(v)$, we deduce (5.4) from this estimate and our assumption (5.2).

It remains to prove (5.3).

Step 2: decomposing the energy. Note that our assumptions (5.1), (5.2) about the points $\{\xi_i\}_{i=1}^l$ are exactly the same as the hypotheses (3.2), (3.3) about the points $\{a_i\}_{i=1}^l$ in Proposition 3.1, except that here we impose an additional smallness condition on r_ξ. Thus estimates from Proposition 3.1 are all available here. In particular, recalling (3.10) with the choice $r = \max(r_\xi, \frac{1}{|\log\varepsilon|}) = \frac{1}{|\log\varepsilon|}$, we see that

$$E_{\varepsilon,\eta}\left(v, \Omega \setminus \cup_i B\left(\xi_i, \frac{4}{|\log\varepsilon|}\right)\right) \le C(\Sigma_\xi |\log\varepsilon| + \log|\log\varepsilon|).$$

In view of (5.5), and noting that j_* is supported in $\cup_i B(\xi_i, \frac{1}{|\log\varepsilon|})$ to prove (5.3) it therefore suffices to show that

$$E_{\varepsilon,\eta}\left(|v|, B\left(\xi_i, \frac{4}{|\log\varepsilon|}\right)\right) + \frac{1}{2}\int_{B(\xi_i, \frac{4}{|\log\varepsilon|})} \eta^2 \left|\frac{j(v)}{|v|} - j_*\right|^2 \le (C\Sigma_\xi + K)|\log\varepsilon| + C\log|\log\varepsilon| \tag{5.6}$$

for $i = 1, \dots, l$. Toward this end, note that

$$\begin{aligned}
|\log\varepsilon|\left[\pi\eta^2(\xi_i) + \Sigma_\xi + C\frac{\log|\log\varepsilon|}{|\log\varepsilon|}\right] &\overset{(3.8)}{\ge} E_{\varepsilon,\eta}\left(v, B\left(\xi_i, \frac{4}{|\log\varepsilon|}\right)\right) \\
&\overset{(5.5)}{=} E_{\varepsilon,\eta}\left(|v|, B\left(\xi_i, \frac{4}{|\log\varepsilon|}\right)\right) \\
&\quad + \frac{1}{2}\int_{B(\xi_i, \frac{4}{|\log\varepsilon|})} \eta^2 \left|\frac{j(v)}{|v|} - j_*\right|^2 \\
&\quad + \frac{1}{2}\int_{B(\xi_i, \frac{4}{|\log\varepsilon|})} \eta^2 |j_*|^2 \\
&\quad + \int_{B(\xi_i, \frac{4}{|\log\varepsilon|})} \eta^2 \left(\frac{j(v)}{|v|} - j_*\right) \cdot j_*.
\end{aligned}$$

Using the explicit form of j_* and of r_ξ,

$$\begin{aligned}
\frac{1}{2}\int_{B(\xi_i, \frac{4}{|\log\varepsilon|})} \eta^2 |j_*|^2 &\ge \frac{1}{2} \min_{B(\xi_i, 4|\log\varepsilon|^{-1})} \eta^2 \int_{B(\xi_i, \frac{4}{|\log\varepsilon|})} |j_*|^2 \\
&\ge \left(\eta^2(\xi_i) - C\|\nabla\eta^2\|_\infty |\log\varepsilon|^{-1}\right)\pi \log\frac{|\log\varepsilon|^{-1}}{r_\xi} \\
&= \left(\pi\eta^2(\xi_i) - C\frac{\log|\log\varepsilon|}{|\log\varepsilon|}\right)|\log\varepsilon|(1-K).
\end{aligned}$$

By combining the previous two inequalities and rearranging, we see that to prove (5.6), it suffices to check that

$$\left|\int_{B(\xi_i,\frac{4}{|\log\varepsilon|})} \eta^2\Big(\frac{j(v)}{|v|} - j_*\Big)\cdot j_*\right| \le C\big(\Sigma_\xi |\log\varepsilon| + \log|\log\varepsilon|\big). \tag{5.7}$$

Step 3: proof of (5.7)**.**

First note that

$$\begin{aligned}\int_{B(\xi_i,\frac{4}{|\log\varepsilon|})} \eta^2\left(\frac{j(v)}{|v|} - j_*\right)\cdot j_* &= \int_{B(\xi_i,\frac{4}{|\log\varepsilon|})} \big(\eta^2(x) - \eta^2(\xi_i)\big)\left(\frac{j(v)}{|v|} - j_*\right)\cdot j_* \\ &\quad + \eta^2(\xi_i)\int_{B(\xi_i,\frac{4}{|\log\varepsilon|})} \frac{j(v)}{|v|}\cdot j_*\ (1-|v|) \\ &\quad + \eta^2(\xi_i)\int_{B(\xi_i,\frac{4}{|\log\varepsilon|})} \big(j(v) - j_*\big)\cdot j_*\,.\end{aligned}$$

We estimate the three terms on the right-hand side in turn. First,

$$\begin{aligned}&\left|\int_{B(\xi_i,\frac{4}{|\log\varepsilon|})} \Big(\eta^2(x) - \eta^2(\xi_i)\Big)\left(\frac{j(v)}{|v|} - j_*\right)\cdot j_*\right| \\ &\le \frac{C}{|\log\varepsilon|}\|\nabla\eta^2\|_\infty \Big(\|\nabla v\|_2^2 + \|j_*\|_2^2\Big) \\ &\le C\left[\frac{E_{\varepsilon,\eta}(v)}{|\log\varepsilon|} + \Big(1 - K + |\log\varepsilon|^{-1}\Big)\right] \\ &\le C\left(\Sigma_\xi + \log|\log\varepsilon|\right),\end{aligned}$$

where we have used the fact that $|\log\varepsilon|^{-1}E_{\varepsilon,\eta}(v) \overset{(5.1)}{\le} C(\Sigma_\xi + l\pi\|\eta^2\|_\infty) \le C(\Sigma_\xi + \log|\log\varepsilon|)$.

Next,

$$\begin{aligned}&\eta^2(\xi_i)\left|\int_{B(\xi_i,\frac{4}{|\log\varepsilon|})} \frac{j(v)}{|v|}\cdot j_*\ (1-|v|)\right| \\ &\le C\|j_*\|_\infty \int_{B(\xi_i,\frac{4}{|\log\varepsilon|})} \frac{\varepsilon}{2}|\nabla v|^2 + \frac{1}{2\varepsilon}\big(|v|^2 - 1\big)^2 \\ &\le C r_\xi^{-1}\varepsilon E_{\varepsilon,\eta}(v) \\ &\le C\big(\Sigma_\xi + \log|\log\varepsilon|\big),\end{aligned}$$

using the lower bound (4.5) for r_ξ and arguing as above.

To estimate the final term, note that $j_* = \nabla^\perp h$, for

$$h(x) := \begin{cases} d_i \left[\dfrac{1}{2r_\xi^2}\big(|x-\xi_i|^2 - 1\big) + \log\big(r_\xi|\log\varepsilon|\big)\right] & \text{if } |x-\xi_i| \le r_\xi \\ d_i \log\big(|x-\xi_i|\,|\log\varepsilon|\big) & \text{if } r_\xi \le |x-\xi_i| \le |\log\varepsilon|^{-1} \\ 0 & \text{if } x \notin \cup_i B\big(\xi_i, |\log\varepsilon|^{-1}\big). \end{cases}$$

Thus, we can integrate by parts to find that

$$\begin{aligned} \left| \int_{B(\xi_i, \frac{4}{|\log\varepsilon|})} (j(v) - j_*)\cdot j_* \right| &= \left| \int_{B(\xi_i, \frac{4}{|\log\varepsilon|})} h\ \nabla \times \big(j(v) - j_*\big) \right| \\ &\le \max\big(\|h\|_\infty, \|\nabla h\|_\infty\big)\ \|j(v) - j_*\|_{W^{-1,1}} \\ &\le C, \end{aligned}$$

after using (5.4) and noting that $\|\nabla h\|_\infty = \|j\|_\infty = r_\xi^{-1}$. This completes the proof. □

It follows from the definitions (4.1) and (5.1) of Σ_ξ and Σ_a, together with (4.4), that

$$\Sigma_\xi \le \Sigma_a + Cg(r_a)$$

for C depending only on $\|\nabla\eta^2\|_\infty$. Combining this with Propositions 4.1 and 5.1, we immediately obtain

Corollary 5.2. *Under the assumptions of Proposition* 4.1

$$E_{\varepsilon,\eta}(|v|) + \frac{1}{2}\int_\Omega \eta^2 \left| \frac{j(v)}{|v|} - j_* \right|^2 \le C\big(\Sigma_a + g(r_a)\big)|\log\varepsilon|, \tag{5.8}$$

where $j_* = j_*(\{\xi_i\}, r_\xi)$, the points $\{\xi_i\}_{i=1}^l$ are given by Proposition 4.1*, and the constant C depends only on l, ρ_a, $\|\nabla\eta^2\|_\infty$ and $\eta_{\min}$.*

6. Small time upper bound on the speed of vortices

Let C_1 be the constant given by Proposition 4.1 corresponding to the lower bound $\rho_{\min}$ (as defined in (1.6)) for ρ_a. Let also $\varepsilon \le \exp\big(-\frac{8}{\rho_{\min}}\big)$. Then the conclusions of Proposition 4.1, applied to v^t with this choice of constants, are available to us for all $0 \le t \le T_{\text{col}}$. Since the conclusions of Proposition 4.1 remain true if we increase C_1, we may assume that

$$\frac{1}{C_1} \le \frac{\rho_{\min}}{8}, \tag{6.1}$$

which we do in the sequel. We define the stopping time

$$T_{\mathrm{loc}} = \sup\left\{t \le T_{\mathrm{col}}\ ;\ \Sigma^0 + g(r_a^s) \le \frac{1}{2C_1},\ \forall\, 0 \le s \le t\right\}.$$

Since the function g satisfies $g(r) \ge r$ on $\mathbb{R}^+$, for $t \le T_{\mathrm{loc}}$ we have $r_a^t \le \frac{1}{2C_1} \le \frac{\rho_{\min}}{16}$. In particular, we may apply Proposition 4.1 to v^t, $\{a_i(t)\}_{i=1}^l$ and $\{d_i\}_{i=1}^l$, which yields points $\{\xi_i(t)\}$ such that

$$\left\|Jv^t - \pi\sum_{i=1}^l d_i\delta_{\xi_i(t)}\right\|_{W^{-1,1}(\Omega)} \le r_\xi^t \equiv r_\xi\left(\Sigma_a^t, r_a^t\right) \equiv \varepsilon \exp\!\big(C_1\big(\Sigma_a^t + g\left(r_a^t\right)\big)\,|\log\varepsilon|\big), \quad (6.2)$$

where[4]

$$\Sigma_a^t = \left(\frac{E_{\varepsilon,\eta}(v^t)}{|\log\varepsilon|} - \pi\sum_{i=1}^l \eta^2\left(a_i(t)\right)\right)^+.$$

Since $t \mapsto v^t|_\Omega$ is continuous in $H^1(\Omega)$, it is clear that $t \mapsto Jv^t$ is continuous as a function from $\mathbb{R}$ into $W^{-1,1}(\Omega)$, and hence we can choose $\{\xi_i(t)\}$ to be piecewise constant, and in particular measurable, as functions of t. Since $E_{\varepsilon,\eta}$ is preserved by the flow for v and $\eta^2(a_i)$ is preserved by the flow for the a_i's, we have $\Sigma_a^t \equiv \Sigma^0$. Note in particular that $r_\xi^t \le \sqrt{\varepsilon}$ for $t < T_{\mathrm{loc}}$.

Proposition 6.1. *There exist positive constants τ_0, ε_0 and C, depending only on l, $\rho_{\min}$, $\eta_{\min}$ and $\|\nabla\eta^2\|_\infty$, such that $\varepsilon_0 \le \exp(-\frac{8}{\rho_{\min}})$ and if $0 < \varepsilon < \varepsilon_0$ and*

$$\Sigma^0 + g(r_a^t) \le \frac{1}{4C_1}$$

for some $t \le T_{\mathrm{loc}}$, then $T_{\mathrm{loc}} \ge t + \tau_0$ and

$$\|Jv^s - Jv^t\|_{W^{-1,1}(\Omega)} \le C\left(|t-s| + r_\xi^t\right), \quad (6.3)$$

$$r_\xi^s \le r_\xi^t + C|\log\varepsilon|\varepsilon^{1/2}\left(|s-t| + r_\xi^t\right), \quad (6.4)$$

$$\left\{a_i(s), \xi_i(s)\right\} \subset B\left(a_i(t), \frac{\rho_{\min}}{4}\right), \qquad i = 1, \dots, l \quad (6.5)$$

for every $t \le s \le t + \tau_0$.

[4] Proposition 4.1 actually uses a version of surplus energy for which the weighted energy $E_{\varepsilon,\eta}$ is restricted to Ω. Since the energy density and the weight are non-negative, our definition of surplus Σ_a^t here, integrating on the whole $\mathbb{R}^2$, yields a larger number, and is therefore compatible with the claim of the proposition.

Proof. For the ease of notation in the present proof, $\|\cdot\|$ is understood to mean $W^{-1,1}(\Omega)$ while $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^2$.

Step 1. Let $t \le s \le \min\{T_{\mathrm{loc}}, t+\tau_0\}$, for τ_0 to be fixed below. We first use the fact that Jv^s, Jv^t are well-approximated by sums of point masses to show that $\|Jv^s - Jv^t\|$ can be estimated by computing $\langle Jv^s - Jv^t, \varphi\rangle$ for a specific test function φ with certain good properties (in particular, bounds on *second* derivatives of φ). Toward this end, note that

$$\begin{aligned}\|Jv^s - Jv^t\| &\le \left\| Jv^s - \pi\sum_{i=1}^{l} d_i\delta_{\xi_i(s)}\right\| + \left\| Jv^t - \pi\sum_{i=1}^{l} d_i\delta_{\xi_i(t)}\right\| \\ &\quad + \left\|\pi\sum_{i=1}^{l} d_i\left(\delta_{\xi_i(s)} - \delta_{\xi_i(t)}\right)\right\| \\ &\le r_\xi^s + r_\xi^t + \pi\sum_{i=1}^{l}|\xi_i(s)-\xi_i(t)|\,.\end{aligned} \tag{6.6}$$

We now fix τ_0, depending only on $\|\nabla\eta^2\|_\infty$, $\eta_{\min}$ and $\rho_{\min}$, such that if $t \le s \le t+\tau_0$, we have $|a_i(s)-a_i(t)| \le \frac{\rho_{\min}}{8}$ for all $i \in \{1,\cdots,l\}$. By Proposition 4.1, the choice of T_{loc}, and Lemma 2.1, for every $\tau \le T_{\mathrm{loc}}$ we have $|a_i(\tau)-\xi_i(\tau)| \le 2r_a^\tau \le \frac{\rho_{\min}}{8}$. By the triangle inequality, it follows that $\xi_i(s) \in B(a_i(t), \frac{\rho_{\min}}{4})$ for all $t \le s \le \min(t+\tau_0, T_{\mathrm{loc}})$ and $i \in \{1,\cdots,l\}$. Let

$$\varphi(x) = \sum_{i=1}^{l} d_i \frac{(x-a_i(t))\cdot(\xi_i(s)-\xi_i(t))}{|\xi_i(s)-\xi_i(t)|}\chi\Big(|x-a_i(t)|\Big), \tag{6.7}$$

where $\chi \in \mathcal{C}^\infty(\mathbb{R}^+,[0,1])$ is such that $\chi \equiv 1$ on $[0,\rho_{\min}/4]$, $\chi \equiv 0$ on $[\frac{\rho_{\min}}{2},+\infty)$. By construction and the definition of $\rho_{\min}$, we have $\varphi \in \mathcal{D}(\Omega)$ and it follows that

$$\begin{aligned}\pi\sum_{i=1}^{l}|\xi_i(s)-\xi_i(t)| &= \left\langle \pi\sum_{i=1}^{l} d_i\left(\delta_{\xi_i(s)}-\delta_{\xi_i(t)}\right), \varphi\right\rangle \\ &\le \left(r_\xi^t + r_\xi^s\right)\|\varphi\|_{W^{1,\infty}} + \left\langle Jv^s - Jv^t, \varphi\right\rangle.\end{aligned}$$

Combining this with (6.6), we conclude that

$$\|Jv^s - Jv^t\| \le C(r_\xi^s + r_\xi^t) + \langle Jv^s - Jv^t, \varphi\rangle, \quad \text{with } \|\varphi\|_{W^{2,\infty}} \le C\rho_{\min}^{-2}. \tag{6.8}$$

Step 2. We now deduce from (6.8), together with (1.16), the fact that $\Sigma^0 \le \frac{1}{4C_1}$, and conservation of energy, that

$$\begin{aligned}\|Jv^s - Jv^t\| &\le \left(r_\xi^t + r_\xi^s\right)\|\varphi\|_{W^{1,\infty}} + (s-t)\sup_{\tau\in[t,s]}\left\|\frac{d}{d\tau}Jv^\tau\right\|_{W^{-2,1}(\Omega)}\|\varphi\|_{W^{2,\infty}} \\ &\le C\left(r_\xi^t + r_\xi^s + |t-s|\right),\end{aligned} \tag{6.9}$$

for $t \le s \le \min(t+\tau_0, T_{\mathrm{loc}})$, where C depends only on l, $\rho_{\min}$, $\eta_{\min}$ and $\|\nabla\eta^2\|_\infty$.

Step 3. It remains to estimate r_ξ^s and to show that $t + \tau_0 \le T_{\rm loc}$. For that purpose, since $r_\xi^s = r_\xi(\Sigma^0, r_a^s)$, we first use (6.9) to compute

$$\begin{aligned} r_a^s &= \left\| Jv^s - \pi \sum_{i=1}^{l} d_i \delta_{a_i(s)} \right\| \\ &\le \left\| Jv^s - Jv^t \right\| + \left\| Jv^t - \pi \sum_{i=1}^{l} d_i \delta_{a_i(t)} \right\| + \left\| \pi \sum_{i=1}^{l} d_i \left(\delta_{a_i(s)} - \delta_{a_i(t)} \right) \right\| \\ &\le r_a^t + C\left(|t-s| + r_\xi^t + r_\xi^s \right). \end{aligned} \tag{6.10}$$

Next, since $s \le T_{\rm loc}$,

$$\begin{aligned} r_\xi^s &= \varepsilon \exp\left(C_1 \left[\Sigma^0 + g\left(r_a^s\right)\right] |\log\varepsilon| \right) \\ &\le r_\xi^t + C_1 |\log\varepsilon| \varepsilon^{\frac12} \left\| g' \right\|_\infty \left(r_a^s - r_a^t\right)^+ \\ &\le r_\xi^t + \frac{1}{2C}\left(r_a^s - r_a^t\right)^+, \end{aligned} \tag{6.11}$$

provided we assume, and this is again no loss of generality, that $C|\log\varepsilon_0|\varepsilon_0^{\frac12} \le \frac{1}{2C}$ for the same constant C as in (6.10). Combining (6.10) with (6.11) we obtain

$$r_a^s - r_a^t \le C\big(|t-s| + r_\xi^t\big). \tag{6.12}$$

Going back to (6.11), this yields the desired estimate of r_ξ^s:

$$r_\xi^s \le r_\xi^t + C|\log\varepsilon|\varepsilon^{1/2}\big(|s-t| + r_\xi^t\big).$$

Then going back to (6.9),

$$\|Jv^s - Jv^t\| \le C\left(|t-s| + r_\xi^t\right)$$

for $t \le s \le \min(t+\tau_0, T_{\rm loc})$. Finally, by assumption we have $\Sigma^0 + g(r_a^t) \le 1/(4C_1)$ so that by (6.12) and the fact that $g' \le 1$,

$$\Sigma^0 + g\left(r_a^s\right) \le 1/\left(4C_1\right) + C\left(|t-s| + r_\xi^t\right) \le 1/\left(4C_1\right) + C\left(\tau_0 + \varepsilon^{\frac12}\right) \le 1/\left(3C_1\right),$$

provided we assume, and this is no loss of generality, that $C(\tau_0 + \varepsilon_0^{\frac12}) \le 1/(12C_1)$. It follows that $\min(t + \tau_0, T_{\rm loc}) = t + \tau_0$, and the proof is complete. □

7. Control of the discrepancy

In this section, we prove a discrete differential inequality for the quantity r_a^t. More precisely, we will prove:

Proposition 7.1. *There exist positive constants ε_0 and C_0, depending only on l, $\rho_{\min}$, $\eta_{\min}$ and $\|\nabla\eta^2\|_\infty$, such that $\varepsilon_0 \le \exp(-\frac{8}{\rho_{\min}})$ and if $0 < \varepsilon < \varepsilon_0$ and*

$$\Sigma^0 + g\left(r_a^t\right) \le \frac{1}{4C_1} \tag{7.1}$$

for some $t \le T_{\mathrm{loc}}$, then

$$\frac{r_a^T - r_a^t}{T-t} \le C_0\left(\Sigma^0 + g\left(r_a^t\right)\right)$$

where $T = t + \frac{(r_\xi^t)^2}{\varepsilon} \le T_{\mathrm{loc}}$.

This is the main estimate in the proof of Theorem 1.3.

Proof. We first require the constant ε_0 to be smaller than the one appearing in the statement of Proposition 6.1. As in the proof of Proposition 6.1, we will write simply $\|\cdot\|$ to denote the $W^{-1,1}(\Omega)$ norm. Note that the condition (7.1) states exactly that

$$r_t^\xi \le \varepsilon^{3/4} \tag{7.2}$$

and then the definition of T and (6.4) yield

$$r_\xi^s \le 2r_\xi^t \qquad \text{for all } s \in [t,T] \tag{7.3}$$

if C is large enough and ε_0 small enough, which we henceforth take to be the case. Moreover, from (6.12), we see that $r_a^s \le r_a^t + C(T - t + r_\xi^t)$ for all $s \in [t,T]$, and then the choice of T and the definition of g imply that

$$g\left(r_a^s\right) \le 2g\left(r_a^t\right) \qquad \text{for all } s \in [t,T]. \tag{7.4}$$

1. First note that

$$\begin{aligned} r_a^T - r_a^t &= \left\| Jv^T - \pi\sum_{i=1}^l d_i\delta_{a_i(T)}\right\| - \left\| Jv^t - \pi\sum_{i=1}^l d_i\delta_{a_i(t)}\right\| \\ &\le \pi\sum_{i=1}^l \left(|\xi_i(T) - a_i(T)| - |\xi_i(t) - a_i(t)|\right) + r_\xi^T + r_\xi^t \qquad (7.5)\\ &\le \pi\sum_{i=1}^l \nu_i\cdot\left(\xi_i(T) - \xi_i(t) + a_i(t) - a_i(T)\right) + r_\xi^T + r_\xi^t \end{aligned}$$

for $\nu_i = \frac{\xi_i(T)-a_i(T)}{|\xi_i(T)-a_i(T)|}$ (unless $\xi_i(T) - a_i(T) = 0$, in which case ν_i can be any unit vector). We now define

$$\varphi(x) = \sum_i d_i \nu_i \cdot (x - a_i(t))\, \chi\left(|x - a_i(t)|\right)$$

for $\chi \in C^\infty(\mathbb{R}^+, [0, 1])$ such that $\chi \equiv 1$ on $[0, \frac{1}{2}\rho_{\min}]$ and $\chi \equiv 0$ on $(\rho_{\min}, \infty)$. It follows from (6.5) that (since ${d_i}^2 = 1$ for all i)

$$\begin{aligned}\pi \sum_{i=1}^{l} \nu_i \cdot \big(\xi_i(T) - \xi_i(t) + a_i(t) - a_i(T)\big) \\ = \pi \sum_{i=1}^{l} d_i \Big[\varphi(\xi_i(T)) - \varphi(\xi_i(t)) - \varphi(a_i(T)) + \varphi(a_i(t))\Big],\end{aligned}$$

so that (7.5) and the definition of r_ξ^T imply that

$$\begin{aligned} r_a^T - r_a^t \le \langle \varphi, Jv^T - Jv^t\rangle \;-\; \pi \sum_{i=1}^{l} d_i \Big[\varphi(a_i(T)) - \varphi(a_i(t))\Big] \\ +\; C(r_\xi^T + r_\xi^t). \end{aligned} \tag{7.6}$$

2. The remainder of the proof is devoted to an estimate of $\langle \varphi, Jv^T - Jv^t\rangle$. First, using (1.16),

$$\begin{aligned}\Big\langle \varphi, Jv^T - Jv^t \Big\rangle &= \int_t^T \frac{\partial}{\partial s} \left\langle \varphi, Jv^s \right\rangle ds \\ &= \frac{1}{|\log \varepsilon|} \int_t^T \int_\Omega \left(\epsilon_{lj} \varphi_{x_l} \eta_{x_j}^2 \frac{\left(|v_\varepsilon|^2 - 1\right)^2}{4\varepsilon^2} + \epsilon_{lj} \varphi_{x_k x_l} v_{\varepsilon,x_j} \cdot v_{\varepsilon,x_k} \right) \\ &\quad + \int_t^T \int_\Omega \epsilon_{lj} \varphi_{x_l} \frac{\eta_{x_k}^2}{\eta^2} \frac{v_{\varepsilon,x_j} \cdot v_{\varepsilon,x_k}}{|\log \varepsilon|}. \end{aligned} \tag{7.7}$$

We immediately see from (5.8) that

$$\left| \frac{1}{|\log \varepsilon|} \int_\Omega \left(\epsilon_{lj} \varphi_{x_l} \eta_{x_j}^2 \frac{\left(|v|^2 - 1\right)^2}{4\varepsilon^2} \right) \right| \le C \left(\Sigma^0 + g\left(r_a^s\right) \right) \tag{7.8}$$

for every $s \in [t, T]$. Moreover, it follows from (6.5) that $B(\xi_i(s); 4|\log \varepsilon|^{-1}) \subset B(a_i(t), \frac{1}{2}\rho_{\min})$ if ε_0 is small enough, and the definition of φ implies that $\varphi_{x_i x_j} = 0$

in $\cup_{i=1}^{l} B(a_i(t), \frac{1}{2}\rho_{\min})$, so (3.4) implies that

$$\begin{aligned}\frac{1}{|\log \varepsilon|}\int_\Omega \left|\epsilon_{lj}\varphi_{x_k x_l} v_{\varepsilon,x_j}\cdot v_{\varepsilon,x_k}\right| &\le C\frac{1}{|\log\varepsilon|}E_\varepsilon\left(v;\Omega\setminus \cup B\left(\xi_i, 4|\log\varepsilon|^{-1}\right)\right)\\ &\le C\left(\Sigma^0 + g\left(r_\xi^s\right)\right) \qquad (7.9)\\ &\le C\left(\Sigma^0+g\left(r_a^s\right)\right)\end{aligned}$$

for every $s \in [t, T]$.

3. We now decompose the remaining term in (7.7). For every $s \in [t, T]$, let[5] $j_*^s = j_*(\{\xi_i(s)\}, r_\xi^t)$ be the approximation to $j(v^s)$ obtained in Proposition 5.1. Note that

$$v_{\varepsilon,x_j}\cdot v_{\varepsilon,x_k} = |v_\varepsilon|_{x_j}\,|v_\varepsilon|_{x_k} + \frac{j(v)_j}{|v|}\,\frac{j(v)_k}{|v|}$$

where for example $j(v)_j = (iv, \partial_{x_j} v)$ denotes the jth component of $j(v)$. Thus, adding and subtracting j_*^s in various places, and writing ψ_{jk} as an abbreviation for $\epsilon_{lj}\varphi_{x_l}\frac{\eta^2_{x_k}}{\eta^2}$, we have for every $s\in[t,T]$,

$$\begin{aligned}\int_\Omega \psi_{jk}\frac{v_{\varepsilon,x_j}\cdot v_{\varepsilon,x_k}}{|\log\varepsilon|} &= \int_\Omega \psi_{jk}\frac{\left(j_*^s\right)_j\left(j_*^s\right)_k}{|\log\varepsilon|}\\ &\quad+\int_\Omega\psi_{jk}\frac{1}{|\log\varepsilon|}\left[\left(\frac{j(v)}{|v|}-j_*^s\right)_j(j_*^s)_k + \left(\frac{j(v)}{|v|}-j_*^s\right)_k (j_*^s)_j\right] \qquad (7.10)\\ &\quad+\int_\Omega\psi_{jk}\frac{1}{|\log\varepsilon|}\left[|v_\varepsilon|_{x_j}\,|v_\varepsilon|_{x_k} + \left(\frac{j(v)}{|v|}-j_*^s\right)_j\left(\frac{j(v)}{|v|}-j_*^s\right)_k\right].\end{aligned}$$

We immediately dispense with the easiest terms by using (5.8) to see that

$$\int_\Omega\psi_{jk}\frac{1}{|\log\varepsilon|}\left[|v_\varepsilon|_{x_j}\,|v_\varepsilon|_{x_k}+\left(\frac{j(v)}{|v|}-j_*^s\right)_j\left(\frac{j(v)}{|v|}-j_*^s\right)_k\right]\le C\left(\Sigma^0+g\left(r_a^s\right)\right) \qquad (7.11)$$

for every $s\in[t,T]$.

4. We next consider the first term on the right-hand side of (7.10), which is the term that yields the dominant contribution. Since j_*^s is supported in $\cup_i B(\xi_i(s), |\log\varepsilon|^{-1})$, clearly

$$\int_\Omega \psi_{jk}\frac{\left(j_*^s\right)_j\left(j_*^s\right)_k}{|\log\varepsilon|} = \sum_{i=1}^{l}\int_{B(\xi_i(s),|\log\varepsilon|^{-1})}\psi_{jk}\frac{\left(j_*^s\right)_j\left(j_*^s\right)_k}{|\log\varepsilon|}.$$

[5] Note that the regularization scale r_ξ^t is fixed for $s\in[t,T]$.

For each $i = 1, \ldots, l$, if $x \in B(\xi_i(s), |\log\varepsilon|^{-1})$, then $|x - a_i(s)| \le |\log\varepsilon|^{-1} + r_a^s + r_\xi^s$, by (4.4), so for every $s \in [t, T]$,

$$\left| \int_{B(\xi_i(s),|\log\varepsilon|^{-1})} \big(\psi_{jk}(x) - \psi_{jk}(a_i(s))\big) \frac{\big(j_*^s\big)_j \big(j_*^s\big)_k}{|\log\varepsilon|} \right|$$

$$\le \|\nabla\psi_{jk}\|_\infty \left(|\log\varepsilon|^{-1} + r_a^s + r_\xi^s\right) \frac{\|j_*^s\|_2^2}{|\log\varepsilon|}$$

$$\le C\left(|\log\varepsilon|^{-1} + r_a^s + r_\xi^s\right),$$

using the explicit form of j_*^s, which (together with the definition (6.2) of r_ξ^s) also implies that

$$\begin{aligned}\int_{B(\xi_i(s),|\log\varepsilon|^{-1})} \frac{\big(j_*^s\big)_j \big(j_*^s\big)_k}{|\log\varepsilon|} &= \frac{\pi}{|\log\varepsilon|}\delta_{jk}\left(\log\frac{1}{r_\xi^s} - \log|\log\varepsilon| + \frac{1}{4}\right) \\ &= \pi\delta_{jk}\left(1 - C_1\left(\Sigma^0 + g(r_a^s)\right)\right) + O\left(\frac{\log|\log\varepsilon|}{|\log\varepsilon|}\right).\end{aligned}$$

Combining the above computations and recalling that $g(r) \ge \max\left(r, \frac{\log|\log\varepsilon|}{|\log\varepsilon|}\right)$ for all r and that $g(r_a^s) \ge r_\xi^s$ for $s \le T_{\mathrm{loc}}$, we conclude that

$$\begin{aligned}\int_\Omega \psi_{jk}\frac{\big(j_*^s\big)_j \big(j_*^s\big)_k}{|\log\varepsilon|} &= \pi\sum_i \psi_{kk}(a_i(s)) + O\Big(C_1\left(\Sigma^0 + g\left(r_a^s\right)\right)\Big) \\ &= \pi\frac{d}{ds}\left(\sum_i d_i\varphi(a_i(s))\right) + O\Big(C_1\left(\Sigma^0 + g\left(r_a^s\right)\right)\Big).\end{aligned} \tag{7.12}$$

In the last line we have used the definition $\psi_{kk} = \epsilon_{lk}\varphi_{x_l}\partial_{x_k}(\log\eta^2) = \nabla\varphi\cdot\nabla^\perp(\log\eta^2)$ together with the ordinary differential equation (1.5) satisfied by the points $a_i(\cdot)$.

5. Combining (7.6), (7.7), (7.8), (7.9), (7.10), (7.11), and (7.12), and recalling (7.3), (7.4), we find that

$$\begin{aligned}r_a^T - r_a^t \le C(T-t)\big(\Sigma^0 + g(r_a^t)\big) \;&+\; Cr_\xi^t \\ &+ \int_t^T \int_\Omega \frac{\psi_{jk}}{|\log\varepsilon|}\left(\frac{j(v)}{|v|} - j_*^s\right)_j \big(j_*^s\big)_k \,dx\,ds \\ &+ \int_t^T \int_\Omega \frac{\psi_{jk}}{|\log\varepsilon|}\left(\frac{j(v)}{|v|} - j_*^s\right)_k \big(j_*^s\big)_j \,dx\,ds.\end{aligned} \tag{7.13}$$

We now begin to control the integrals on the right-hand side above. We will consider only the first one, since the estimate of the second one is identical. First,

$$\begin{aligned}\int_t^T \int_\Omega \frac{\psi_{jk}}{|\log\varepsilon|}\left(\frac{j(v)}{|v|}-j_*^s\right)_j (j_*^s)_k \,dx\,ds \\ = \int_\Omega \frac{\psi_{jk}}{|\log\varepsilon|}(j_*^t)_k \int_t^T \left(\frac{j(v)}{|v|}-j_*^s\right)_j ds\,dx \\ + \int_\Omega\int_t^T \frac{\psi_{jk}}{|\log\varepsilon|}(j_*^s-j_*^t)_k\left(\frac{j(v)}{|v|}-j_*^s\right)_j ds\,dx\,.\end{aligned}\tag{7.14}$$

We claim that

$$\int_\Omega\int_t^T \frac{\psi_{jk}}{|\log\varepsilon|}(j_*^s-j_*^t)_k\left(\frac{j(v)}{|v|}-j_*^s\right)_j dsdx \le C(T-t)\left(\Sigma^0+g\left(r_a^t\right)\right).\tag{7.15}$$

Using the Cauchy-Schwarz inequality, (5.8), and (7.4), we see that it suffices to prove that

$$\int_\Omega \left|j_*^s-j_*^t\right|^2\,dx \;\le\; \left(\Sigma^0+g\left(r_a^t\right)\right)|\log\varepsilon| \qquad \text{for every } s\in[t,T].$$

Toward this end, we fix some such s, and we introduce the notation

$$\bar\xi_i := \frac{1}{2}\left(\xi_i^t+\xi_i^s\right),\qquad \sigma := r_\xi^s + r_\xi^t + \sum_{i=1}^l |\xi_i(t)-\xi_i(s)|\,.$$

Our choice of T and (6.3) imply that $\sigma \le C\frac{(r_\xi^t)^2}{\varepsilon}$. Writing $B_i := B(a_i(t), \frac{\rho_{\min}}{2})$, we deduce from (6.5) and the support properties of j_* that

$$\int_\Omega \left|j_*^t-j_*^s\right|^2\,dx \;=\sum_{i=1}^l \int_{B_i}\left|j_*^t-j_*^s\right|^2\,dx.$$

For each i, $B(\bar\xi_i,\sigma)\subset B(\xi_i(t),2\sigma)\cap B(\xi_i(s),2\sigma)$, so by an explicit computation, and recalling (7.3) and the definition (6.2) of r_ξ^t, we find that

$$\begin{aligned}\int_{B(\bar\xi_i,\sigma)}\left|j_*^t-j_*^s\right|^2\,dx &\le 2\int_{B(\xi_i(t),2\sigma)}\left|j_*^t\right|^2\,dx + 2\int_{B(\xi_i(s),2\sigma)}\left|j_*^s\right|^2\,dx \\ &\le 2\log\left(\frac{2\sigma}{r_\xi^t}\right)+2\log\left(\frac{2\sigma}{r_\xi^s}\right)+C \\ &\le C\log\left(\frac{r_\xi^t}{\varepsilon}\right) \\ &\le C\left(\Sigma^0+g\left(r_a^t\right)\right)|\log\varepsilon|.\end{aligned}$$

Next, on $B(\bar\xi_i, \frac{1}{2|\log\varepsilon|})$, the definitions imply that both j^s and j^t are nonzero, and in fact

$$|j_*^t(x) - j_*^s(x)|^2 = \frac{|\xi_i(t) - \xi_i(s)|^2}{|x - \xi(t)|^2 |x - \xi_i(s)|^2}.$$

Since $|\xi_i(\tau) - \bar\xi_i| \le \frac{\sigma}{2}$ for $\tau = t, s$, it follows that

$$\left|j_*^t(x) - j_*^s(x)\right|^2 \le \frac{4\sigma^2}{|x - \bar\xi_i|^4} \quad \text{on } B\left(\bar\xi_i, \frac{1}{2|\log\varepsilon|}\right) \setminus B\left(\bar\xi_i, \sigma\right)$$

and hence that

$$\int_{B(\bar\xi_i, \frac{1}{2|\log\varepsilon|}) \setminus B(\bar\xi_i, \sigma)} \left|j_*^t - j_*^s\right|^2 \, dx \le C.$$

Finally,

$$\int_{B_i \setminus B(\bar\xi_i, \frac{1}{2|\log\varepsilon|})} \left|j_*^t - j_*^s\right|^2 \, dx \le 2 \int_{B_i \setminus B(\xi_i(t), \frac{1}{4|\log\varepsilon|})} \left|j_*^t\right|^2 \, dx + 2 \int_{B_i \setminus B(\xi_i(s), \frac{1}{4|\log\varepsilon|})} \left|j_*^s\right|^2 \, dx \le C.$$

We deduce (7.15) by combining the previous inequalities.

6. We now consider the first term on the right-hand side of (7.14). Clearly

$$\begin{aligned} \int_\Omega \frac{\psi_{jk}}{|\log\varepsilon|} \left(j_*^t\right)_k \int_t^T \left(\frac{j(v)}{|v|} - j_*^s\right)_j ds\, dx &= \sum_{i=1}^{l} \int_{B_i} \int_t^T \frac{\psi_{jk}}{|\log\varepsilon|} (j_*^t)_k \left(\frac{j(v)}{|v|} - j(v)\right)_j ds\, dx \\ &\quad + \sum_{i=1}^{l} \int_{B_i} \int_t^T \frac{\psi_{jk}}{|\log\varepsilon|} (j_*^t)_k \left(j(v) - j_*^s\right)_j ds\, dx. \end{aligned} \tag{7.16}$$

By elementary estimates,

$$\left|\frac{j(v)}{|v|} - j(v)\right| = \frac{|j(v)|}{|v|} \left||v| - 1\right| \le \frac{\varepsilon}{2} |\nabla v|^2 + \frac{1}{2\varepsilon} \left(|v|^2 - 1\right)^2,$$

and from the definitions, and recalling (4.5), we see that $\|j_*^t\|_\infty \le (r_\xi^t)^{-1} \le (\varepsilon|\log\varepsilon|)^{-1}$. Thus for every i,

$$\begin{aligned} \left|\int_{B_i} \int_t^T \frac{\psi_{jk}}{|\log\varepsilon|} \left(j_*^t\right)_k \cdot \left(\frac{j(v)}{|v|} - j(v)\right) ds\, dx\right| &\le C \|j_*^t\|_\infty \varepsilon (T - t) \frac{E_{\varepsilon,\eta}(v)}{|\log\varepsilon|} \\ &\le \frac{C}{|\log\varepsilon|} (T - t) \left(\Sigma^0 + C\right) \\ &\le C (T - t) \left(\Sigma^0 + g\left(r_a^t\right)\right), \end{aligned} \tag{7.17}$$

since

$$\frac{E_{\varepsilon,\eta}(v)}{|\log\varepsilon|} \le \Sigma^0 + \pi\sum_{i=1}^{l}\eta^2\left(a_i(t)\right) \le \Sigma^0 + C\left(l, \|\eta\|_\infty\right). \tag{7.18}$$

7. Now fix some $i \in \{1,\ldots,l\}$ and let $\tilde\chi^i \in C_c^\infty(B(a_i(t), \frac{3}{4}\rho_{\min}))$ be a function such that $\tilde\chi^i = 1$ on B_i. Then for every $s \in [t, T]$,

$$\tilde\chi_i\left(j(v) - j_*^s\right) = \nabla f^s + \frac{1}{\eta^2}\nabla^\perp g^s \qquad \text{in } B\left(a_i(t), \frac{3}{4}\rho_{\min}\right) \tag{7.19}$$

for f^s and g^s, real-valued functions on $B(a_i(t), \frac{3}{4}\rho_{\min})$, solving

$$\begin{aligned} \nabla\cdot(\eta^2\nabla f^s) &= \nabla\cdot\left(\tilde\chi^i\eta^2\left(j(v) - j_*^s\right)\right) && \text{in } B\left(a_i(t), \frac{3}{4}\rho_{\min}\right), \\ \nu\cdot\nabla f^s &= 0 && \text{on } \partial B\left(a_i(t), \frac{3}{4}\rho_{\min}\right), \end{aligned} \tag{7.20}$$

and

$$\begin{aligned} -\nabla\cdot\left(\frac{\nabla g^s}{\eta^2}\right) &= \nabla\times\left(\tilde\chi^i\left(j(v) - j_*^s\right)\right) && \text{in } B\left(a_i(t), \frac{3}{4}\rho_{\min}\right), \\ g^s &= 0 && \text{on } \partial B\left(a_i(t), \frac{3}{4}\rho_{\min}\right). \end{aligned} \tag{7.21}$$

Indeed, if we let f^s be a solution of (7.20), then $\eta^2(\tilde\chi_i(j(v) - j_*^s) - \nabla f^s)$ is divergence-free and hence can be written as $\nabla^\perp g^s$ on $B(a_i(t), \frac{3}{4}\rho_{\min})$, so that (7.19) holds. Then it follows from (7.20) that g^s satisfies the equation in (7.21), and that the boundary condition is satisfied after adding a constant to g^s.

Thus

$$\begin{aligned} \int_{B_i}\int_t^T &\frac{\psi_{jk}}{|\log\varepsilon|}\left(j_*^t\right)_k\left(j(v) - j_*^s\right)_j\,ds\,dx \\ &= \int_{B_i}\frac{\psi_{jk}}{|\log\varepsilon|}\left(j_*^t\right)_k\left(\nabla F + \frac{\nabla^\perp G}{\eta^2}\right)_j\,dx \end{aligned} \tag{7.22}$$

for

$$F(x) = \int_t^T f^s(x)\,ds, \qquad G(x) = \int_t^T g^s(x)\,ds.$$

We write $F = F_1 + \cdots + F_4$, where

$$\nabla\cdot(\eta^2\nabla F_m) = A_m \;\text{ in } B\left(a_i(t), \frac{3}{4}\rho_{\min}\right), \quad \nu\cdot\nabla F_m = 0 \;\text{ in } \partial B\left(a_i(t), \frac{3}{4}\rho_{\min}\right),$$

for

$$A_1 = \tilde{\chi}^i \int_t^T \nabla \cdot \left(\eta^2 j(v)\right) ds$$
$$A_2 = -\tilde{\chi}^i \int_t^T \nabla \cdot \left(\eta^2 j_*^s\right) ds$$
$$A_3 = \int_t^T \eta^2 \nabla \tilde{\chi}^i \cdot \frac{j(v)}{|v|} (|v| - 1)\ ds$$
$$A_4 = \int_t^T \eta^2 \nabla \tilde{\chi}^i \cdot \left(\frac{j(v)}{|v|} - j_*^s\right) ds.$$

Using the continuity equation (1.15) — this is a key point in our argument — and (7.18), we note that

$$\begin{aligned}\|A_1\|_{L^2} &= |\log \varepsilon| \left\| \tilde{\chi}^i \eta^2 \left(|v|^2 - 1\right)\Big|_t^T \right\|_{L^2} \\ &\leq C\varepsilon |\log \varepsilon| \left(E_{\varepsilon,\eta}\left(v^T\right) + E_{\varepsilon,\eta}\left(v^t\right)\right) \\ &\leq C\varepsilon |\log \varepsilon|^3 \left(\Sigma^0 + g(r_a^t)\right)\end{aligned}$$

since $g(r) \geq \frac{1+\log|\log\varepsilon|}{|\log\varepsilon|}$ for all r. Next, the definition implies that $\nabla \cdot j_*^s = 0$ for every s and that $\| j_*^s\|_{L^p} \leq C_p |\log\varepsilon|^{1-\frac{2}{p}}$ for every $p < 2$, so

$$\|A_2\|_{L^p} \leq \|\tilde{\chi}_i\|_{L^\infty} (T-t) \sup_{s\in[t,T]} \left\|\nabla(\eta^2) \cdot j_*^s\right\|_{L^p} \leq C(T-t)|\log\varepsilon|^{1-\frac{2}{p}} \quad \text{for } p<2.$$

Very much as in (7.17), we can check that

$$\|A_3\|_{L^1} \leq C(T-t) \sup_{s\in[t,T]} \left\| \frac{j(v)}{|v|}(|v|-1)\right\|_{L^1} \leq C(T-t)\varepsilon|\log\varepsilon|^2 \left(\Sigma^0 + g\left(r_a^t\right)\right),$$

and it follows from (5.8) and (7.4) that

$$\|A_4\|_{L^2} \ \leq\ C(T-t)\Big(|\log\varepsilon| \left(\Sigma^0 + g(r_a^t)\right)\Big)^{1/2}.$$

Clearly, for any $q_1, \ldots, q_4 \in [1, \infty]$,

$$\int_{B_i} \frac{\psi_{jk}}{|\log\varepsilon|} \left(j_*^t\right)_k \cdot (\nabla F)_j \, dx \leq \frac{C}{|\log\varepsilon|} \sum_{m=1}^4 \|j_*\|_{q_m} \|\nabla F_m\|_{q_m'}$$

where $\frac{1}{q_m}+\frac{1}{q_m'}=1$. Using elliptic estimates and Sobolev embedding theorems, and taking $q_1=\frac{4}{3}$,

$$\frac{1}{|\log\varepsilon|}\|j_*\|_{\frac{4}{3}}\|\nabla F_1\|_4 \le \frac{C}{|\log\varepsilon|}\|j_*\|_{\frac{4}{3}}\|A_1\|_2$$
$$\le C\varepsilon|\log\varepsilon|^{\frac{3}{2}}\left(\Sigma^0+g\left(r_a^t\right)\right) \le C(T-t)\left(\Sigma^0+g\left(r_a^t\right)\right).$$

The last inequality follows from the choice of T and (4.5), which imply in particular that $T-t\ge\varepsilon|\log\varepsilon|^2$. Similarly, taking $q_4=4/3$,

$$\frac{1}{|\log\varepsilon|}\|j_*\|_{\frac{4}{3}}\|\nabla F_4\|_4 \le \frac{C}{|\log\varepsilon|^{\frac{3}{2}}}\|A_4\|_2 \le C\frac{(T-t)}{|\log\varepsilon|}\left(\Sigma^0+g\left(r_a^t\right)\right)^{1/2}$$
$$\le C(T-t)\left(\Sigma^0+g\left(r_a^t\right)\right),$$

since $|\log\varepsilon|^{-1}\le g(r_a^t)$. For any $q_2\in(1,2)$, taking $p_2<2$ such that $p_2^*=q_2'$, so that $\frac{1}{p_2}=\frac{3}{2}-\frac{1}{q_2}$, we find from our estimate of A_2 that

$$\frac{1}{|\log\varepsilon|}\|j_*\|_{q_2}\|\nabla F_2\|_{q_2'} \le \frac{C}{|\log\varepsilon|}\|j_*\|_{q_2}\|A_2\|_{p_2}$$
$$\le C(T-t)|\log\varepsilon|^{-2} \le C(T-t)g\left(r_a^t\right).$$

And, recalling by Stampacchia's estimate that for any $p\in[1,2)$ there exists C_p such that $\|\nabla F_3\|_p\le C_p\|A_3\|_1$, we compute, choosing $q_3=3$ for concreteness,

$$\frac{1}{|\log\varepsilon|}\|j_*\|_3\|\nabla F_3\|_{\frac{3}{2}} \le \|j_*\|_3\|A_3\|_1 \le C(T-t)(r_\xi^t)^{-\frac{1}{3}}\varepsilon|\log\varepsilon|\left(\Sigma^0+g\left(r_a^t\right)\right)$$
$$\le (T-t)\left(\varepsilon|\log\varepsilon|\right)^{2/3}\left(\Sigma^0+g\left(r_a^t\right)\right)$$

again using the fact that $r_\xi^t\ge\varepsilon|\log\varepsilon|$ for all t, see (4.5). Combining the above, we find that for every $i\in\{1,\dots,l\}$ and $0<\varepsilon<\varepsilon_0$ with ε_0 sufficiently small,

$$\int_{B_i}\frac{\psi_{jk}}{|\log\varepsilon|}\left(j_*^t\right)_k\cdot(\nabla F)_j\,dx \le C(T-t)\left(\Sigma^0+g\left(r_a^t\right)\right). \tag{7.23}$$

8. Next,

$$\int_{B_i}\frac{\psi_{jk}}{|\log\varepsilon|}(j_*^t)_k\cdot\frac{(\nabla^\perp G)_j}{\eta^2}\,dx = \int_{B_i}\frac{\psi_{jk}}{\eta^2|\log\varepsilon|}(j_*^t)_k\cdot\nabla^\perp\left(G_1+G_2+G_3\right)_j\,dx$$

for G_m solving

$$-\nabla\cdot\left(\frac{\nabla G_m}{\eta^2}\right)=A_m' \quad \text{in } B\left(a_i(t),\frac{3}{4}\rho_{\min}\right), \quad g=0 \text{ on } \partial B\left(a_i(t),\frac{3}{4}\rho_{\min}\right),$$

with

$$A_1' := \int_t^T \tilde{\chi}^i \nabla \times \left(j(v) - j_*^s \right) \, ds,$$

$$A_2' := \int_t^T \nabla^\perp \tilde{\chi}^i \cdot j(v) \left(1 - \frac{1}{|v|} \right) \, ds,$$

$$A_3' := \int_t^T \nabla^\perp \tilde{\chi}^i \cdot \left(\frac{j(v)}{|v|} - j_*^s \right) \, ds.$$

The terms containing G_2 and G_3 are estimated exactly as the terms containing F_3 and F_4 in Step 7 above, leading to

$$\int_{B_i} \frac{\psi_{jk}}{\eta^2 |\log \varepsilon|} \left(j_*^t \right)_k \; \nabla^\perp \left(G_2 + G_3 \right)_j \, dx \; \le \; C(T-t) \left(\Sigma^0 + g\left(r_a^t \right) \right).$$

For the remaining term, we invoke the interpolation inequality

$$\left\| A_1' \right\|_{W^{-1,p}} \le C \left\| A_1' \right\|_{W^{-1,1}}^{\theta} \left\| A_1' \right\|_{L^1}^{1-\theta} \tag{7.24}$$

for $p \in (1,2)$ and θ such that $\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{2}$ (see e.g. [20] Theorem 2.4.1 combined with Sobolev embedding theorem). To estimate the $W^{-1,1}$ norm, we fix $\zeta \in C_c^\infty(\Omega)$, and we compute

$$\begin{aligned} \langle \zeta, A_1' \rangle = \int_t^T \left\langle \tilde{\chi}^i \zeta, \nabla \times \left(j(v) - j_*^s \right) \right\rangle \le \int_t^T \left\| \tilde{\chi}^i \zeta \right\|_{W^{1,\infty}} \left\| \nabla \times \left(j(v) - j_*^s \right) \right\|_{W^{-1,1}} ds \\ \le C(T-t) r_\xi^t \, \|\zeta\|_{W^{1,\infty}} \end{aligned}$$

using (5.4) and (7.3). Thus

$$\left\| A_1' \right\|_{W^{-1,1}} \le C(T-t) r_\xi^t. \tag{7.25}$$

Also, for every $s \in [t, T]$,

$$\left\| \nabla \times (j(v) - j_*^s) \right\|_{L^1} \le \|2Jv\|_{L^1} + \left\| \nabla \times j_*^s \right\|_{L^1} \le C E_{\varepsilon,\eta}(v) + 2\pi l.$$

Estimating $E_{\varepsilon,\eta}$ as usual by $C|\log\varepsilon|(\Sigma^0 + g(r_a^t))$, integrating the last inequality from t to T, and combining it with (7.25) and (7.24), we obtain

$$\|A_1'\|_{W^{-1,p}} \le C(T-t)(r_\xi^t)^\theta \left(C|\log\varepsilon| \left(\Sigma^0 + g\left(r_a^t \right) \right) \right)^{1-\theta}.$$

Then using Hölder's inequality and (again) the fact that $\|j_*^s\|_{p'} \le C(r_\xi^s)^{\frac{2}{p'}-1}$ for $p' > 2$,

$$\int_{B_i} \frac{\psi_{jk}}{\eta^2|\log\varepsilon|}(j_*^t)_k \cdot \nabla^\perp (G_1)_j \, dx$$
$$\le \frac{C}{|\log\varepsilon|}(T-t)(r_\xi^t)^{\theta+\frac{2}{p'}-1}\Big(C|\log\varepsilon|\Big(\Sigma^0 + g\left(r_a^t\right)\Big)\Big)^{1-\theta}$$
$$\le C(T-t)|\log\varepsilon|^{-\theta}\Big(\Sigma^0 + g\left(r_a^t\right)\Big)^{1-\theta}$$
$$\le C(T-t)\Big(\Sigma^0 + g\left(r_a^t\right)\Big),$$

since it turns out that $\theta + \frac{2}{p'} - 1 = 0$, and noting that $|\log\varepsilon|^{-1} \le g(r_a^t)$ for all t. Assembling these estimates, we find that

$$\int_{B_i} \frac{\psi_{jk}}{|\log\varepsilon|}(j_*^t)_k \Big(\nabla^\perp G\Big)_j \, dx \le C(T-t)\Big(\Sigma^0 + g\left(r_a^t\right)\Big).$$

Now by combining this with (7.13), (7.14), (7.15), (7.17), (7.23), we finally obtain

$$r_a^T - r_a^t \le C(T-t)\Big(\Sigma^0 + g\left(r_a^t\right)\Big). \tag{7.26}$$

□

8. Proof of Theorem 1.3

Our main result is a straightforward corollary of the discrepancy estimate proved in the previous section.

Proof of Theorem 1.3. Let Y denote the solution of the ordinary differential equation

$$\dot{Y}(t) = C_0\Big(\Sigma^0 + g(Y(t))\Big), \qquad Y(0) = r_a^0,$$

where g is the function defined in (3.1), and let $\{Y_n\}_{n=0}^\infty$ be a discrete approximation to $Y(\cdot)$ obtained via an Euler approximation implicit in the statement of Proposition 7.1. Thus, we define

$$Y_0 = r_a^0, \qquad Y_{n+1} = Y_n + (t_{n+1} - t_n)\, C_0\Big(\Sigma^0 + g(Y_n)\Big),$$

$$t_{n+1} := t_n + \frac{\left(r_\xi^n\right)^2}{\varepsilon}$$

where

$$r_\xi^n := r_\xi(\Sigma^0, Y_n) = \varepsilon \exp\Big(C_0\Big(\Sigma^0 + g\left(Y^n\right)\Big)\Big)|\log\varepsilon|).$$

Since the function $f(Y) := C_0\left(\Sigma^0 + g(Y)\right)$ is convex, a forward Euler approximation to the solution of the equation $Y' = f(Y)$ is always less than or equal to the actual solution, and it follows that $Y_n \leq Y(t_n)$ for all t. Then repeated application of Proposition 7.1 shows that

$$r_a^{t_n} \leq Y_n \leq Y(t_n) \text{ for every } n \text{ such that } t_n \leq T_{\rm col} \text{ and } \Sigma^0 + g(Y_n) \leq \tfrac{1}{4C_1}.$$

Given an arbitrary $t \in (0, T_{\rm col}]$ such that $\Sigma^0 + g(Y(t)) \leq \frac{1}{4C_1}$, there exists some n such that $t \in [t_n, t_{n+1}]$ and $r_a^{t_n} \leq Y(t_n)$. Then by Proposition 6.1, see in particular (6.10), as well as (7.3),

$$r_a^t \leq r_a^{t_n} + C\left((t_{n+1} - t_n) + r_\xi^{t_n}\right) \leq Y(t) + C\varepsilon^{1/2}, \tag{8.1}$$

since the bound $\Sigma^0 + g(Y(t_n)) \leq \frac{1}{4C_1}$ guarantees that $r_\xi^{t_n} \leq \varepsilon^{3/4}$ and hence that $t_{n+1} - t_n \leq \varepsilon^{1/2}$. It remains to bound the function Y from above. For that purpose, we notice that since $g(y) \leq y + \log|\log\varepsilon|/|\log\varepsilon|$ for every $y \geq 0$, we have $Y(t) \leq \tilde{Y}(t)$ where $\tilde{Y}$ is the solution of the ordinary differential equation

$$\dot{\tilde{Y}}(t) = C_0\left(\Sigma^0 + \frac{\log|\log\varepsilon|}{|\log\varepsilon|} + \tilde{Y}(t)\right), \qquad \tilde{Y}(0) = r_a^0.$$

The solution of the latter is explicitly given by

$$\tilde{Y}(t) = r_a^0 + \left(\Sigma^0 + r_a^0 + \frac{\log|\log\varepsilon|}{|\log\varepsilon|}\right)\left(e^{C_0 t} - 1\right),$$

and the conclusion therefore follows from (8.1), increasing the value of C_0 to the value of C in (8.1) if necessary. □

9. Some properties of the ground state

In this section we briefly recall some facts about minimizers of the functional[6]

$$\mathcal{E}_{\varepsilon,V}(u) = \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{2} + \frac{1}{2\varepsilon^2}\left(V(x)|u|^2 + \frac{1}{2}|u|^4\right)dx \tag{9.1}$$

in the space

$$\mathcal{H}_m := \left\{u \in H^1\left(\mathbb{R}^N;\mathbb{C}\right) : \int_{\mathbb{R}^N} V|u|^2 < \infty, \int_{\mathbb{R}^N} |u|^2 = m\right\} \tag{9.2}$$

[6] Note that we make no restriction on the dimension N here.

where $V : \mathbb{R}^N \to [0, \infty)$ is a smooth function such that $V(x) \to \infty$ as $|x| \to \infty$, and $m > 0$ is a parameter.

For every positive ε, m, the existence of a function $\eta_{\varepsilon,m} : \mathbb{R}^N \to (0, \infty)$ minimizing $\mathcal{E}_{\varepsilon,V}$ in $\mathcal{H}_m$ is standard, and follows easily from the growth of V (which implies that the L^2 constraint is preserved for weak limits of sequences with equibounded energy) together with the strong maximum principle and the fact that $\mathcal{E}_{\varepsilon,V}(|u|) \le \mathcal{E}_{\varepsilon,V}(u)$ for all u.

In the introduction, we already introduced the unique number λ_0 such that

$$\int_{\mathbb{R}^N} (\lambda_0 - V)^+ dx = m,$$

and we have denoted by $\rho_{TF} := (\lambda_0 - V)^+$ the Thomas-Fermi profile associated to V and m. We also note $w := (\lambda_0 - V)^-$. We will prove:

Proposition 9.1. *Let $\eta = \eta_{\varepsilon.m} \in \mathcal{H}_m$ be a positive minimizer of $\mathcal{E}_{\varepsilon,V}$ in $\mathcal{H}_m$. Then*

$$\left\| \eta^2 - \rho_{TF} \right\|_{L^2(\mathbb{R}^N)} \le C\varepsilon^{2/3}. \tag{9.3}$$

Moreover, for any $K \subset\subset \Omega_{TF} := \{x \in \mathbb{R}^N : \rho_{TF}(x) > 0\}$, there exists a constant $C = C(m, V, K)$ such that

$$\left\| \eta^2 - \rho_{TF} \right\|_{L^\infty(K)} \le C\varepsilon^{2/3}, \qquad \left\| \nabla\eta^2 \right\|_{L^\infty(K)} \le C. \tag{9.4}$$

This is quite standard, and is proved for particular potentials V in [7] for example. We include a complete proof, since the references we know all impose slightly more restrictive conditions than we consider here (for example, symmetry conditions, or the assumption that λ_0 is a regular value of V).

Proof. It suffices to prove the result for $\varepsilon \le \varepsilon_0$, for some $\varepsilon_0 > 0$.

1. First, as is standard, for $u \in \mathcal{H}_m$ we rewrite

$$\begin{aligned}\mathcal{E}_{\varepsilon,V}(u) = &\int_{\mathbb{R}^N} \left[\frac{|\nabla u|^2}{2} + \frac{1}{4\varepsilon^2} \left(|u|^2 - \rho_{TF} \right)^2 + \frac{1}{2\varepsilon^2} w|u|^2 \right] dx \\ &+ \frac{1}{\varepsilon^2} \left(\lambda_0 \frac{m}{2} - \frac{1}{4} \int_{\mathbb{R}^N} \rho_{TF}^2 \right) \\ =:& \mathcal{E}_{\varepsilon,\rho_{TF}}(u) + C_1(\varepsilon, m).\end{aligned}$$

Thus, it is clear that a function minimizes $\mathcal{E}_{\varepsilon,V}$ in $\mathcal{H}_m$ if and only if it minimizes $\mathcal{E}_{\varepsilon,\rho_{TF}}$ in $\mathcal{H}_m$.

2. Next we claim that

$$\inf_{\mathcal{H}_m} \mathcal{E}_{\varepsilon,\rho_{TF}} \le C\varepsilon^{-2/3}. \tag{9.5}$$

Note that this immediately implies (9.3). We verify (9.5) by choosing $U_\varepsilon := c_\varepsilon f_\varepsilon(\sqrt{\rho}_{TF})$, where

$$f_\varepsilon(s) = \begin{cases} \varepsilon^{-\alpha} s^2 & \text{if } s \le \varepsilon^\alpha \\ s & \text{if } s \ge \varepsilon^\alpha, \end{cases}$$

where c_ε is chosen so that $U_\varepsilon \in \mathcal{H}_m$. Then straightforward estimates very much like those in [7], for example, show that $\mathcal{E}_{\varepsilon,\eta}(U_\varepsilon) \le C(\varepsilon^{-\alpha} + \varepsilon^{2\alpha-2})$, and (9.5) follows by taking $\alpha = 2/3$. (This crude estimate has the advantage of holding for *every* $m > 0$, so that we do not require λ_0 to be a regular value of V. If λ_0 is a regular value, then a variant of the same construction shows that $\inf_{\mathcal{H}_m} \mathcal{E}_{\varepsilon,\rho_{TF}} \le C|\log\varepsilon|$.)

3. Since $V - \lambda_0 = (V-\lambda_0)^+ - (V-\lambda_0)^- = w - \rho_{TF}$, we may write the variational equation satisfied by η in the form

$$-\Delta\eta + \frac{1}{\varepsilon^2}\left(\eta^2 - \rho_{TF} + w\right)\eta = \frac{1}{\varepsilon^2}(\lambda_\varepsilon - \lambda_0)\,\eta,$$

where $\frac{1}{\varepsilon^2}\lambda_\varepsilon$ is a Lagrange multiplier. Multiplying by η and integrating, and using the fact that $\eta \in \mathcal{H}_m$, we find that

$$\frac{m}{\varepsilon^2}(\lambda_\varepsilon - \lambda_0) = \int_{\mathbb{R}^2} |\nabla\eta|^2 + \frac{1}{\varepsilon^2}\left[w\eta^2 + \left(\eta^2 - \rho_{TF}\right)^2 + \left(\eta^2 - \rho_{TF}\right)\rho_{TF}\right].$$

It follows that

$$\frac{m}{\varepsilon^2}(\lambda_\varepsilon - \lambda_0) \le 4\mathcal{E}_{\varepsilon,\rho_{TF}}(\eta) + \frac{1}{\varepsilon^2}\left\|\rho_{TF}\right\|_{L^2(\mathbb{R}^N)}\left\|\eta^2 - \rho_{TF}\right\|_{L^2(\mathbb{R}^N)} \le C\varepsilon^{-4/3} \quad (9.6)$$

by (9.5) and (9.3).

4. Now let $\rho_{TF,\varepsilon} := (\lambda_\varepsilon - V)^+$. It follows from (9.6) that

$$\|\rho_{TF,\varepsilon} - \rho_{TF}\|_{L^\infty(\mathbb{R}^N)} = |\lambda_\varepsilon - \lambda_0| \le C\varepsilon^{2/3}, \quad (9.7)$$

so that $K \subset\subset \Omega_\varepsilon := \{x \in \mathbb{R}^N : \rho_{TF,\varepsilon} > 0\}$ if $\varepsilon > 0$ is sufficiently small, which we henceforth take to be the case. Note also that

$$-\Delta\eta + \frac{1}{\varepsilon^2}\left(\eta^2 - \rho_{TF,\varepsilon}\right)\eta = 0 \qquad \text{in } \Omega_\varepsilon. \quad (9.8)$$

Now fix some $r \le \frac{1}{2}\operatorname{dist}(K, \partial\Omega_{TF})$. In view of (9.7), and since V is C^2, there exists $a, k > 0$ and $\varepsilon_0 > 0$ such that

$$\rho_{TF,\varepsilon} > a^2 \quad\text{and}\quad \left|\Delta\sqrt{\rho}_{TF,\varepsilon}\right| \le k \text{ whenever } 0 < \varepsilon \le \varepsilon_0. \quad (9.9)$$

For any $x \in K$ and $b \in (0, a)$, define

$$\zeta_{x,b}(y) = \zeta(y) = b\left(\frac{|y-x|^2}{r^2} - 1\right)^2$$

in $B(x, r)$. Then for $b \in (0, \frac{a}{2})$,

$$-\Delta\zeta + \frac{1}{\varepsilon^2}\left(\zeta^2 - \rho_{TF,\varepsilon}\right)\zeta \;\le\; -\Delta\zeta - \frac{3a^2}{4\varepsilon^2}\zeta \;<\; 0 \qquad \text{in } B(x,r) \tag{9.10}$$

whenever ε is sufficiently small. It follows that $\eta \ge \zeta_{x,b}$ in $B(x, r)$ for every $b \in (0, \frac{a}{2})$, as otherwise we could find some $b_0 \in (0, \frac{a}{2})$ such that $\min_{B(x,r)}(\eta - \zeta_{x,b_0}) = 0$. Since $\eta > 0$, the minimum would have to be attained in the interior of $B(x, r)$, and this is impossible in view of (9.8) and (9.10).

It follows that

$$\eta(y) \ge \frac{9a}{32} =: \alpha \text{ in } B(x, r/2). \tag{9.11}$$

Note also that $\|\eta\|_{L^\infty(\mathbb{R}^N)} \le \|\sqrt{\rho_{TF,\varepsilon}}\|_{L^\infty(\mathbb{R}^N)}$, since otherwise $\tilde\eta := \min(\eta, \|\sqrt{\rho_{TF,\varepsilon}}\|_\infty)$ would satisfy $\mathcal{E}_{\varepsilon,\rho_{TF}}(\tilde\eta) < \mathcal{E}_{\varepsilon,\rho_{TF}}(\eta)$, contradicting the minimality of η.

5. Now write $\theta := \eta - \sqrt{\rho_{TF,\varepsilon}}$. Then

$$-\Delta\theta + a_\varepsilon(x)\theta = \Delta\sqrt{\rho_{TF,\varepsilon}} \qquad \text{for } a_\varepsilon(x) = \frac{1}{\varepsilon^2}\left(\theta + 2\sqrt{\rho_{TF,\varepsilon}}\right)\left(\theta + \sqrt{\rho_{TF,\varepsilon}}\right) \overset{(9.11)}{\ge} \frac{\alpha^2}{\varepsilon^2}$$

in $B(x, r/2)$, and $|\theta| \le 2\|\sqrt{\rho}_{TF,\varepsilon}\|_{L^\infty(\mathbb{R}^N)}$ on $B(x, r/2)$. Now for $y \in B(x, r/2)$ define

$$\Theta_\varepsilon(y) := \frac{k}{\alpha^2}\varepsilon^2 + 2\left\|\sqrt{\rho_{TF,\varepsilon}}\right\|_{L^\infty(\mathbb{R}^N)} \exp\left[\frac{\alpha}{r\varepsilon}\left(\frac{|y-x|^2}{2} - \frac{r^2}{8}\right)\right]$$

where k is the bound for $\|\Delta\sqrt{\rho_{TF,\varepsilon}}\|_\infty$ found in (9.9). Then $\Theta \ge \theta$ on $\partial B(x, r/2)$, and there exists $\varepsilon_0 > 0$ such that

$$(-\Delta + a_\varepsilon)\Theta \;\ge\; k \;\ge\; (-\Delta + a_\varepsilon)\theta \;\text{ in }\; B(x, r/2), \qquad \text{if } 0 < \varepsilon < \varepsilon_0.$$

It follows that $\Theta \ge \theta$ in $B(x, r/2)$, and similarly $-\Theta \ge -\theta$ in $B(x, r/2)$. Thus

$$\left|\eta - \sqrt{\rho_\varepsilon}\right| \le C\varepsilon^2 \quad \text{ on } B(x, r/4). \tag{9.12}$$

6. Returning to (9.8), we see that

$$-\Delta\eta + b_\varepsilon\eta = 0 \quad \text{ in } B(x, r/4), \text{ for } b_\varepsilon = \frac{1}{\varepsilon^2}\left(\eta^2 - \rho_\varepsilon\right),$$

and (9.12) implies that $\|b_\varepsilon\|_{L^\infty(B(x,r/4)} \le C$ independent of $\varepsilon \in (0, \varepsilon_0)$ and $x \in K$. Since we already know that $\|\eta\|_{L^\infty(\mathbb{R}^N)} \le C$, we conclude from standard elliptic regularity that $\|\nabla\eta\|_{L^\infty(B(x,r/8))} \le C$. Also, it follows from (9.7) and (9.12) that $\|\eta^2 - \rho\|_{L^\infty(K)} \le C\varepsilon^{2/3}$, so we have proved (9.4). □

10. Proof of Theorem 1.1

In view of (1.4), Theorem 1.1 is a direct consequence of Theorem 1.3 combined with Proposition 9.1 and the continuity of the solution of an initial value problem with respect to the nonlinearity.

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