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# $\mathcal{C}^\infty$-hypoellipticity and extension of *CR* functions

MAURO NACINOVICH AND EGMONT PORTEN

**Abstract.** Let $M$ be a *CR* submanifold of a complex manifold $X$. The main result of this article is to show that *CR*-hypoellipticity at $p_0 \in M$ is necessary and sufficient for holomorphic extension of all germs at $p_0$ of *CR* functions on $M$ to an ambient neighborhood of $p_0$ in $X$. As an application, we obtain that *CR*-hypoellipticity implies the existence of global generic embeddings and prove holomorphic extension for a large class of *CR* manifolds satisfying a higher order Levi pseudoconcavity condition. We also obtain results on the relationship of holomorphic wedge-extension and the $\mathcal{C}^\infty$-wave front set for *CR* distributions.

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## Introduction

The main result of this paper is the proof of the equivalence, for a smooth embedded *CR* manifold $M$, of *CR*-hypoellipticity and the holomorphic extension property. *CR*-hypoellipticity means that germs of distribution solutions to the tangential Cauchy-Riemann equations are $\mathcal{C}^\infty$-smooth, while the holomorphic extension property means that germs of $\mathcal{C}^\infty$-smooth solutions to the tangential Cauchy-Riemann equations are restrictions of germs of holomorphic functions in a complex ambient space. In particular, we obtain the holomorphic extension property for essentially pseudoconcave manifolds (see [2,20]).

Both *CR*-hypoellipticity and holomorphic extendability imply minimality. Thus our main result can be restated by saying that *CR*-hypoellipticity and holomorphic extendability are equivalent at minimal points.

Despite several contributions, the problem of finding a geometric characterization for the holomorphic extension property is still wide open, even for real analytic hypersurfaces. The interest of our main result is that it establishes a link between holomorphic extension and $\mathcal{C}^\infty$ regularity, a central and better understood topic in PDE theory. We illustrate this point of view by recalling in Section 5 the weak pseudoconcavity assumptions of [2], which generalize the *essential pseudoconcav-*

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*ity* of [20], and insure *CR*-hypoellipticity, and illustrating in Section 6, by some examples, how this approach leads to the proof of the holomorphic extension property for manifolds with highly degenerate Levi forms. Extension theorems had been obtained before under stronger non-degeneracy assumptions on the Levi form (see *e.g.* [9,14,31]), or for *CR* manifolds satisfying a third order pseudoconcavity condition (see [3]).

An interesting consequence is the existence and uniqueness result for the generic local embeddings of $M$ of Theorem 3.7 and Corollary 3.8, where we also deal with non-generic local embeddings (this topic recently got some consideration; view *e.g.* [17]). An equivalent formulation is that, when $M$ is *CR*-hypoelliptic and locally embeddable at all points, its *CR* structure completely determines its hypo-analytic structure (see [36]).

We also point out that our result applies to give concrete applications for the Siegel-type theorems proved in [21,22] about the transcendence degree of the fields of *CR*-meromorphic functions.

We notice that, by [7], minimality is a necessary condition for $CR$-hypoellipticity, and that some sort of pseudoconcavity is also necessary, as holomorphic extension does not hold *e.g.* when $M$ lies in the boundary of a domain of holomorphy.

In general, when germs of *CR* functions on a generically embedded *CR* manifold $M \hookrightarrow X$ fail to holomorphically extend to a full neighborhood $\mathcal{U}$ of $p_0$ in $X$, we can instead look for open subsets $\mathcal{W}$ of $X$ for which $M \cap \partial\mathcal{W}$ is a neighborhood of $p_0$ in $M$. A fundamental result of Tumanov [37] states that holomorphic local *wedge* extension is valid if $M$ is *minimal* at $p_0$. By [7], this condition is also necessary. However, the known proofs of local holomorphic wedge extension merely yield existence, but no explicit information on its shape. The analytic or hypo-analytic wave front sets tautologically give the directions of holomorphic extension. We conjecture that, in analogy with Theorem 1.4, the union of the $\mathcal{C}^\infty$ wave front sets of all germs of *CR* distributions and that of their hypo-analytic wave front sets coincide. The results of Section 4 give some first partial results in this direction.

Let us shortly describe the contents of the paper. In Section 1 we set notation, make precise the notion of *CR*-hypoellipticity, and formulate our main result (Theorem 1.4). Section 2 and Section 3 are devoted to its proof. In Section 3 we also establish various equivalences of the extension property and show that this leads to the uniqueness of the generic *CR* embeddings. Section 4 contains our result about wedge extension and the common $\mathcal{C}^\infty$ wave front set of germs of *CR* distributions. In Section 5 we review the subellipticity result of [2], obtaining a substantial amelioration of the extension result of [3]. In Section 6 we give some examples.

Finally, we want to thank the referee for his comments, which helped us to improve and make more understandable the exposition, and especially for outlining a serious gap in the original proof of Theorem 4.6, that has been filled by utilizing some results of [11,15].

## 1. Statement of the main Theorem

Let $M$ be an abstract smooth *CR* manifold, of *CR* dimension $m$ and *CR* codimension $d$. The *CR* structure on $M$ is defined by the datum of a $\mathcal{C}^{\infty}$-smooth $m$-dimensional subbundle $T^{0,1}M$ of its complexified tangent bundle $\mathbb{C}TM$, with

$$T^{0,1}M \cap \overline{T^{0,1}M} = \underline{0} \quad \text{and} \quad \left[\Gamma\big(M, T^{0,1}M\big), \Gamma\big(M, T^{0,1}M\big)\right] \subset \Gamma\big(M, T^{0,1}M\big).$$

We say that the *CR* manifold $M$ is *real-analytic* when, in addition, $T^{0,1}M$ is a real-analytic subbundle of $\mathbb{C}TM$ for a given real-analytic structure of $M$.

For $U^{\text{open}} \subset M$ and a real $a \geq 0$, let $\mathcal{C}^a(U)$ be the space of complex valued functions on $U$, which are Hölder continuous of exponent $a - [a]$ with all their derivatives up to order $[a]$. For $a > 0$, we set $\mathcal{C}^{-a}(U)$ for the space of distributions $u$ on $U$ which have bounds in terms of continuous seminorms in $\mathcal{C}^a(U)$. We write $\mathcal{C}^{\infty}(U)$ and $\mathcal{C}^{-\infty}(U)$ for the spaces of smooth functions and of distributions on $U$, respectively.

For each $a \in [-\infty, \infty]$ we denote by $\mathcal{O}^a_M(U)$ the set of $\mathcal{C}^a$-smooth solutions on $U$ to the tangential Cauchy-Riemann equations:

$$\mathcal{O}^a_M(U) = \left\{u \in \mathcal{C}^a(U) \mid Zu = 0,\ \forall Z \in \Gamma\big(U, T^{0,1}M\big)\right\}.$$

When $a < 1$ the homogeneous Cauchy-Riemann equations are understood to be satisfied in the weak sense, *i.e.* $u \in \mathcal{O}^a_M(U)$ if

$$u \in \mathcal{C}^a(U) \quad \text{and} \quad \int u\, Z'\phi\, d\mu = 0, \quad \forall \phi \in \mathcal{C}^{\infty}_0(U),\ \forall Z \in \Gamma\big(U, T^{0,1}M\big),$$

where $\mu$ is a positive measure with smooth density on $M$ and the formal adjoint $Z'$ of $Z \in \Gamma(U, T^{0,1}M)$ is defined by

$$\int Zv\, \phi\, d\mu = \int v\, Z'\phi\, d\mu, \quad \forall v, \phi \in \mathcal{C}^{\infty}_0(U).$$

The assignments $U^{\text{open}} \to \mathcal{O}^a_M(U)$ define sheaves of germs. We denote by $\mathcal{O}^a_{M,(p_0)}$ the stalk at $p_0 \in M$. Note that $\mathcal{O}^{-\infty}_{M,(p_0)} = \bigcup_{a \in \mathbb{R}} \mathcal{O}^a_{M,(p_0)}$. Likewise, if $M$ is real-analytic, we denote by $\mathcal{O}^{\omega}_M$ the sheaf of germs of real-analytic *CR* functions on $M$. When $M$ is a complex manifold we drop the superscript $a$, because the sheaves corresponding to different exponents coincide by the regularity theorem for holomorphic functions.

**Definition 1.1.** We say that $M$ is *CR-hypoelliptic* at $p_0 \in M$ if $\mathcal{O}^{-\infty}_{M,(p_0)} = \mathcal{O}^{\infty}_{M,(p_0)}$. A real-analytic $M$ is *CR-analytic-hypoelliptic* at $p_0 \in M$ if $\mathcal{O}^{-\infty}_{M,(p_0)} = \mathcal{O}^{\omega}_{M,(p_0)}$.

A *local CR-embedding* of $M$ at $p_0$ is the datum of $\mathcal{C}^{\infty}$-smooth solutions $z_1, \ldots, z_{\nu}$ to the homogeneous tangential Cauchy-Riemann equations on a neighborhood $U$ of $p_0$ in $M$ such that the map $p \mapsto (z_1(p), \ldots, z_{\nu}(p))$ is a smooth embedding $U \hookrightarrow \mathbb{C}^\nu$. We have $\nu \geq m + d = n$, and when we have equality we say that the local *CR*-embedding is *generic*. From any local *CR*-embedding we can obtain a generic local *CR*-embedding of a smaller neighborhood of $p_0$, by choosing any subset $z_{i_1}, \dots, z_{i_n}$ of $z_1, \dots, z_\nu$ with $dz_{i_1}(p_0) \wedge \cdots \wedge dz_{i_n}(p_0) \neq 0$.

We will consider in the sequel different instances of the local holomorphic extension property:

**Definition 1.2.** Let $-\infty \leq a \leq \infty$. We say that:

($\mathrm{H}_a^*$) $M$ has the *weak holomorphic extension property for* $\mathcal{O}_M^a$ at $p_0$ if for every germ $u \in \mathcal{O}^a_{M,(p_0)}$ there are an open neighborhood $U_u$ of $p_0$ in $M$, a generic local *CR*-embedding $\phi_u : U_u \hookrightarrow \mathbb{C}^n$, and a germ of holomorphic function $\tilde{u} \in \mathcal{O}_{\mathbb{C}^n,(\phi(p_0))}$ such that $\phi_u^*(\tilde{u}) = u$;

($\mathrm{H}_a$) $M$ has the *strong holomorphic extension property for* $\mathcal{O}_M^a$ at $p_0$ if there are an open neighborhood $U$ of $p_0$ in $M$ and a generic local *CR*-embedding $\phi : U \hookrightarrow \mathbb{C}^n$, such that, for every germ of smooth *CR* function $u \in \mathcal{O}^a_{M,(p_0)}$ there is a germ of holomorphic function $\tilde{u} \in \mathcal{O}_{\mathbb{C}^n,(\phi(p_0))}$ such that $\phi^*(\tilde{u}) = u$.

We drop the clause "at $p_0$" when the property holds at all points of $M$.

For $-\infty \leq a \leq b \leq \infty$ we have the obvious implications

$$\begin{array}{ccccc} & & (\mathrm{H}_a^*) & & \\ & \nearrow & & \searrow & \\ (\mathrm{H}_a) & & & & (\mathrm{H}_b^*). \\ & \searrow & & \nearrow & \\ & & (\mathrm{H}_b) & & \end{array}$$

We shall prove in Section 3 that actually $\mathrm{H}_a \Leftrightarrow \mathrm{H}_a^* \Leftrightarrow \mathrm{H}_b \Rightarrow \mathrm{H}_b^*$ for all $-\infty \leq a < b \leq \infty$, leading to a nicer and simpler notion of local holomorphic extension.

Note that the validity of any of the properties ($\mathrm{H}_a$), ($\mathrm{H}_a^*$) includes local *CR* embeddability at $p_0$, as the constants are *CR* functions.

**Remark 1.3.** In [7, Theorem 2, Corollary 1] it is shown that, for a generic *CR* submanifold $M$ of a complex manifold $X$, minimality is not only a sufficient, but also a necessary condition for wedge extendability of all smooth *CR* functions. The argument in [7] only works in the case of a fixed *CR* embedding of $M$. Thus, in particular, although $\mathrm{H}_\infty$ implies minimality at $p_0$, the sufficiency of the weaker condition $\mathrm{H}_\infty^*$ for minimality remains an open question.

**Theorem 1.4 (Main Theorem).** *Let $M$ be a CR manifold, locally CR-embeddable at $p_0 \in M$, and $a, b \in [-\infty, \infty]$, with $a < \infty$. Then the following are equivalent:*

(1) *$M$ is* CR *hypoelliptic at $p_0$;*
(2) *$M$ has the weak extension property* ($\mathrm{H}_a^*$) *at $p_0$;*
(3) *$M$ has the strong extension property* ($\mathrm{H}_b$) *at $p_0$.*

The proof of the main theorem will be done in the next two sections. First we will prove in Section 2 that *CR*-hypoellipticity is equivalent to ($\mathrm{H}_{-\infty}$) and ($\mathrm{H}^*_{-\infty}$). This gives in particular that ($\mathrm{H}_{-\infty}$) $\Leftrightarrow$ ($\mathrm{H}^*_{-\infty}$) and shows that *CR*-hypoellipticity implies ($\mathrm{H}_a$) and ($\mathrm{H}^*_a$) for all $-\infty \le a \le \infty$. In Section 3 we will complete the proof by deriving the equivalence ($\mathrm{H}_{-\infty}$) $\Leftrightarrow$ ($\mathrm{H}^*_a$), for all $-\infty \le a < \infty$, which implies that all different ($\mathrm{H}^*_a$) and ($\mathrm{H}_b$), with $a, b \in [-\infty, \infty]$ and $a < \infty$, are equivalent.

We conclude this section by stating a straightforward consequence of Theorem 1.4 for the real-analytic case. Since real-analytic *CR* manifolds are locally *CR*-embeddable (see [5]), and holomorphic functions are real-analytic, we obtain:

**Corollary 1.5.** *Assume that $M$ is a real-analytic CR-manifold, and let $p_0 \in M$. Then the following are equivalent:*

(1) *$M$ is CR-hypoelliptic at $p_0$;*
(2) *$M$ is CR-analytic-hypoelliptic at $p_0$;*
(3) $\mathcal{O}^\infty_{M,(p_0)} = \mathcal{O}^\omega_{M,(p_0)}$.

Condition (3) means that all smooth solutions to the homogeneous tangential Cauchy-Riemann equations on a neighborhood of $p_0$ are real-analytic at $p_0$.

## 2. *CR*-hypoellipticity and holomorphic extendability of *CR* distributions

In this section we prove that *CR*-hypoellipticity is equivalent to ($\mathrm{H}_{-\infty}$) and ($\mathrm{H}^*_{-\infty}$). Since the implications ($\mathrm{H}_{-\infty}$) $\Rightarrow$ ($\mathrm{H}^*_{-\infty}$) $\Rightarrow$ *CR*-hypoellipticity are trivial, it suffices to show that, if $M$ is locally embeddable at $p_0$, then *CR*-hypoellipticity implies ($\mathrm{H}_{-\infty}$) at $p_0$. We consistently keep the notation of Section 1. In particular, $M$ is a $\mathcal{C}^\infty$-smooth *CR* manifold, of *CR* dimension $m$ and *CR* codimension $d$.

Since we are dealing with local properties, we can as well assume that $M$ is a generic *CR* submanifold of an open subset $\Omega$ of $\mathbb{C}^n$, with $n = m + d$, that $p_0 = 0$, and that the holomorphic coordinates of $\mathbb{C}^n$ have been chosen in such a way that $M$ is the graph

$$y' = h\big(x', z''\big) \tag{2.1}$$

of a smooth map $h : V \to \mathbb{R}^d$, with $h(0) = 0, dh(0) = 0$, for an open neighborhood $V$ of 0 in $\mathbb{R}^d \times \mathbb{C}^m$. Here $z = (z', z'') \in \mathbb{C}^d \times \mathbb{C}^m$, and $z' = x' + iy', z'' = x'' + iy''$ with $x', y' \in \mathbb{R}^d, x'', y'' \in \mathbb{R}^m$.

Every open wedge attached to $M$ near 0 contains an open wedge $\mathcal{W}$ which, in the chosen coordinates, is described by

$$\mathcal{W} = \big\{z + \big(iy', 0\big) \mid z \in E,\ y' \in \mathrm{C}\big\}, \tag{2.2}$$

for an open set $E = \mathrm{Edge}(\mathcal{W}) \subset M$ and a truncated open cone $\mathrm{C} \subset \mathbb{R}^d$, with vertex at the origin. The wedge $\mathcal{W}$ is foliated by the translates $E_{y'} = \{z+(iy', 0) : z \in E\}$, $y' \in \mathrm{C}$, of the edge, approaching $E_0 = E$, for $y \to 0$. Recall that $f \in \mathcal{O}_{\mathbb{C}^n}(\mathcal{W})$

attains the weak boundary value $f^* \in \mathcal{O}_M^{-\infty}(E)$ along $E$ if, for every test function $\phi \in \mathcal{C}_0^\infty(V)$, we have

$$\lim_{\substack{y' \in C \\ y' \to 0}} \int f\big(x' + ih\big(x', z''\big) + iy', z''\big)\phi\big(x', z''\big)\, dm_{d+2m} = f^*[\phi]. \tag{2.3}$$

Here $dm_{d+2m}$ denotes the standard Lebesgue measure on $\mathbb{R}^d \times \mathbb{C}^m$. A function $f \in \mathcal{O}_{\mathbb{C}^n}(\mathcal{W})$ has polynomial growth along $E$ if, for every compact $K \Subset E$, there are an integer $N_K \geq 0$ and a constant $a_K > 0$ such that

$$\limsup_{\substack{y' \in C \\ y' \to 0}} \big|y'\big|^{N_K} \big|f\big(z' + iy', z''\big)\big| \leq a_K, \ \forall z = \big(z', z''\big) \in K, \ \forall y' \in C. \tag{2.4}$$

Holomorphic functions of polynomial growth attain unique distribution boundary values on $E$, which weakly satisfy the homogeneous tangential *CR* equations.

**Proof that *CR*-hypoellipticity implies** $(\mathrm{H}_{-\infty})$**. I**$^{\text{st}}$ **part**

We assume *CR*-hypoellipticity at 0 and we want to show that it implies holomorphic extension of *CR* distributions defined on a neighborhood of 0 in $M$ to full neighborhoods of 0 in $\mathbb{C}^n$.

First we note that $M$ is minimal at 0. Indeed, otherwise, $M$ contains a germ of proper *CR* submanifold $N$ through 0, of the same *CR* dimension $m$, and *CR* codimension $d' < d$, which would be the support of the germ of a non smooth *CR*-distribution (see [7,34]).

We will argue by contradiction. We begin by taking an open neighborhood $U$ of 0 in $M$ for which we assume there is a $u \in \mathcal{O}_M^{-\infty}(U)$ that cannot be holomorphically extended to a full neighborhood of 0 in $\mathbb{C}^n$.

By the approximation theorem [36, Theorem II.3.2] we know that there is an open neighborhood $U'$ of 0 in $U$ such that for all $w \in \mathcal{O}_M^{-\infty}(U)$, $w|_{U'}$ is the limit, in the sense of distributions, of the restrictions to $U'$ of a sequence of holomorphic polynomials. By the minimality of $M$ at 0, we can find an open neighborhood $E$ of 0 in $U'$ and an open wedge $\mathcal{W}$, of the form (2.2), such that the restrictions to $E$ of all elements of $\mathcal{O}_M^{-\infty}(U)$ are boundary values of holomorphic functions on $\mathcal{W}$ (see [34,37]).

**Runge Hull**

To continue our argument, we need first to review some facts on open Runge subsets of $\mathbb{C}^n$ (see *e.g.* [25, Section 2.7]).

For a compact $K \subset \mathbb{C}^n$, let

$$\tilde{K} = \big\{z \in \mathbb{C}^n \mid |f(z)| \leq \sup_{\zeta \in K} |f(\zeta)|, \ \forall f \in \mathcal{O}_{\mathbb{C}^n}\big(\mathbb{C}^n\big)\big\}$$

denote its polynomial hull. An open $U \subset \mathbb{C}^n$ is Runge if $\tilde{K} \subset U$ for every compact $K \subset U$. For an arbitrary open $U \subset \mathbb{C}^n$, let us define its *Runge hull* by

$$\tilde{U} = \bigcup_{\substack{K \text{ compact} \\ K \subset U}} \mathring{\tilde{K}},$$

where we used $\mathring{A}$ for the set of interior points of a subset $A$ of $\mathbb{C}^n$.

**Lemma 2.1.** *For every open $U \subset \mathbb{C}^n$ the set $\tilde{U}$ is open and Runge. If $f \in \mathcal{O}_{\mathbb{C}^n}(U)$ can be uniformly approximated by entire functions on compact subsets of $U$, then $f$ admits a unique holomorphic extension to $\tilde{U}$. If $U$ is connected, then $\tilde{U}$ is also connected.*

Note that, since envelopes of holomorphy may be multi-sheeted, there may be functions $f \in \mathcal{O}_{\mathbb{C}^n}(U)$ which have no holomorphic extension to $\tilde{U}$.

*Proof.* Let $K$ be compact in $\tilde{U}$. By the maximum modulus principle we have

$$\mathring{\tilde{K}} = \operatorname{int}\left( \bigcap_{f \in \mathbb{C}[z_1,\dots,z_n] \setminus \mathbb{C}} \left\{ z \in \mathbb{C}^n \mid |f(z)| < \sup_{\zeta \in K} |f(\zeta)| \right\} \right).$$

We claim that $\mathring{\tilde{K}}$ is Runge. Indeed, each $V_f = \{z \in \mathbb{C}^n \mid |f(z)| < \sup_{\zeta \in K} |f(\zeta)|\}$ is polynomially convex and hence Runge. To verify that the interior part of the intersection of the $V_f$'s is still Runge, we can use the distance functions $\delta_{\complement V_f}(z) = \inf_{\zeta \notin V_f} |z - \zeta|$. Since $V_f$ is Runge, we have $\inf_{z \in \tilde{F}} \delta(z) = \inf_{z \in F} \delta(z)$ for all compact $F \subset V_f$. Thus, if $F$ is any compact subset of the interior of $\mathring{\tilde{K}}$, we have

$$\begin{aligned} \inf_{z \in \tilde{F}} \delta_{\complement \mathring{\tilde{K}}}(z) &= \inf_{f \in \mathbb{C}[z_1,\dots,z_n] \setminus \mathbb{C}} \inf_{z \in \tilde{F}} \delta_{\complement V_f}(z) \\ &= \inf_{f \in \mathbb{C}[z_1,\dots,z_n] \setminus \mathbb{C}} \inf_{z \in F} \delta_{\complement V_f}(z) = \inf_{z \in F} \delta_{\complement \mathring{\tilde{K}}}(z) > 0, \end{aligned}$$

showing that also $\tilde{F} \subset \mathring{\tilde{K}}$.

Let $\{K_\nu\}$ be any sequence of compact subsets of $U$ with $K_\nu \Subset \mathring{K}_{\nu+1}$ and $\bigcup_{\nu=1}^{\infty} K_\nu = U$. Then $\tilde{U} = \bigcup_{\nu=1}^{\infty} \mathring{\tilde{K}}_\nu$ is Runge, because it is the union of an increasing sequence of Runge open sets.

If $f \in \mathcal{O}(U)$ can be uniformly approximated by entire functions on compact subsets of $U$, there is a sequence $\{f_\nu\} \subset \mathbb{C}[z_1, \dots, z_n]$ of polynomials approximating $f$ on all compact subsets of $U$. By construction, the sequence $\{f_\nu\}$ also converges on compact subsets of $\tilde{U}$ to a holomorphic function $\tilde{f} \in \mathcal{O}_{\mathbb{C}^n}(\tilde{U})$, with $\tilde{f} = f$ on $U$.

Assume now that $U$ is connected. Clearly $\tilde{U}$ is the smallest Runge open set which contains $U$. Then $\tilde{U}$ is connected because all connected components of a Runge open set are Runge domains and hence $\tilde{U}$ coincides with the unique connected component of $\tilde{U}$ containing $U$. □

**Proof that *CR*-hypoellipticity implies** ($\mathrm{H}_{-\infty}$)**.** II$^{nd}$ **part**

Let $\mathcal{V} = \tilde{\mathcal{W}}$ be the Runge hull of $\mathcal{W}$. The *CR* distribution $u \in \mathcal{O}_M^{-\infty}(U)$ is, on the edge $E$, the boundary value of a holomorphic function $\tilde{u}$ on $\mathcal{W}$. By the continuous dependence of the extension on its boundary value, $\tilde{u}$ can be uniformly approximated by holomorphic polynomials on compact subsets of $\mathcal{W}$. Then, by Lemma 2.1, the function $\tilde{u}$ has a holomorphic extension to $\mathcal{V}$, that for simplicity we shall still denote by $\tilde{u}$.

By the uniqueness of the limit and of the boundary value, it follows that, if $0 \in \mathcal{V}$, then $u$ holomorphically extends to a neighborhood of 0 in $\mathbb{C}^n$. Thus, to find a contradiction, it will suffice to show that, if $0 \notin \mathcal{V}$, and therefore $0 \in \partial\mathcal{V}$, then $M$ is not *CR*-hypoelliptic at 0.

We shall consistently use in the following the notation $\delta_A(z) = \inf\{|z - \zeta| \mid \zeta \in A\}$ for the distance function from a subset $A$ of $\mathbb{C}^n$.

For each nonnegative integer $k$, let

$$\mathcal{O}^{(k)}_{\mathbb{C}^n}(\mathcal{V}) = \left\{ f \in \mathcal{O}_{\mathbb{C}^n}(\mathcal{V})) \mid \ \delta^k_{\complement\mathcal{V}} \cdot f \ \text{ is bounded on } \mathcal{V} \right\}$$

be the space of holomorphic functions on $\mathcal{V}$, with $k$-polynomial growth along $\partial\mathcal{V}$. It is a Banach space with the norm $\|f\|_{\mathcal{O}^{(k)}_{\mathbb{C}^n}(\mathcal{V})} = \sup_{p\in\mathcal{V}} |\delta^k_{\complement\mathcal{V}}(p) f(p)|$.

**Lemma 2.2.** *Let $\mathcal{W}'$ be any open wedge of the form* (2.2)*, contained in $\mathcal{W}$. Assume that $0 \in \partial\mathcal{V}$. Then there is a sequence $\{p_j\}_{j=1,2,\dots}$ in $\mathcal{W}'$, with $p_j \to 0$, and a function $f \in \mathcal{O}^{(2n+1)}_{\mathbb{C}^n}(\mathcal{V})$ such that $|f(p_j)| \to \infty$.*

In the proof of this lemma, we will use the following result, which is a particular case of [26, Proposition 2.5.4]:

**Lemma 2.3.** *There is a constant $C > 0$, only depending on $\mathcal{V}$, such that*

$$\forall p \in \mathcal{V}, \ \ \exists f_p \in \mathcal{O}^{(2n+1)}_{\mathbb{C}^n}(\mathcal{V}) \ \ \textit{with} \ \ f_p(p) = 1 \ \ \textit{and} \ \ \|f_p\|_{\mathcal{O}^{(2n+1)}_{\mathbb{C}^n}(\mathcal{V})} \le C\delta_{\complement\mathcal{V}}(p).$$

*Proof of Lemma* 2.2. We construct by recurrence a sequence $\{p_j\}_{j>0}$ of points in $\mathcal{W}'$, and $\{f_j\}_{j\ge0}$ of functions in $\mathcal{O}^{(2n+1)}_{\mathbb{C}^n}(\mathcal{V})$, with $f_0 = 0$, which satisfy, for all integers $j > 0$,

**(a)** $p_j \in \mathcal{W}', |p_j| < \frac{1}{j}$,
**(b)** $|f_j(p_j)| \ge j$,
**(c)** $\|f_j - f_{j-1}\|_{\mathcal{O}^{(2n+1)}_{\mathbb{C}^n}(\mathcal{V})} \le 2^{1-j}$, and
**(d)** $\sup_{\mathcal{K}_{\delta_{j-1}}} |f_j - f_{j-1}| \le 2^{1-j}$,

where we set $\delta_j = \delta_{\complement\mathcal{V}}(p_j)$ and $\mathcal{K}_r = \{p \in \mathcal{V} : \delta_{\complement\mathcal{V}}(p) \ge r\}$. Note that $\{\mathcal{K}_r\}_{r>0}$ is a decreasing family of closed sets, and that $\mathcal{V} = \bigcup_{r>0} \mathcal{K}_r$. Set $B_0(r) = \{|z| < r\} \subset \mathbb{C}^n$.

Conditions **(a)**-**(d)** are satisfied with $f_1 \equiv 1$ and any choice of $p_1 \in \mathcal{W}' \cap B_0(1)$. This is our first recursive step. Assume that, for some $k > 1$, we already found $p_1, \dots, p_{k-1}$ and $f_1, \dots, f_{k-1} \in \mathcal{O}_{\mathbb{C}^n}^{(2n+1)}(\mathcal{V})$ satisfying **(a)**-**(d)** for $1 \le j \le k-1$. Choose $p_k \in \mathcal{W}' \cap B_0(1/k)$ such that $\delta_k \le \delta_{k-1}^{2n+1}/(k2^k C)$. If $|f_{k-1}(p_k)| \ge k$ holds, the choice $f_k = f_{k-1}$ obviously satisfies **(a)**-**(d)** for $j = k$. Otherwise, we utilize Lemma 2.3 to pick a function $f_{p_k}$ corresponding to the choice $p = p_k$, and set $f_k = f_{k-1} + k\tau f_{p_k}$, with $\tau = 1$ if $f_{k-1}(p_k) = 0$ and $\tau = f_{k-1}(p_k)/|f_{k-1}(p_k)|$ otherwise.

This yields **(a)** and **(b)** for $j = k$.

We get **(c)** for $j = k$ from the estimate

$$\|f_k - f_{k-1}\|_{\mathcal{O}_{\mathbb{C}^n}^{(2n+1)}(\mathcal{V})} = k\|f_{p_k}\|_{\mathcal{O}_{\mathbb{C}^n}^{(2n+1)}(\mathcal{V})} \le kC\delta_k \le 2^{-k}.$$

Let us show that, with this choice of $f_k$, **(d)** is also satisfied.

From $\|f_{p_k}\|_{\mathcal{O}_{\mathbb{C}^n}^{(2n+1)}(\mathcal{V})} \le C\delta_k$ we obtain that

$$\left|f_{p_k}(p)\right| \le C\delta_k/\delta_{\partial\mathcal{V}}^{2n+1}(p) \le C\delta_k/\delta_{k-1}^{2n+1} \le 2^{-k}/k$$

holds for $p \in \mathcal{K}_{\delta_{k-1}}$, which shows that

$$\sup_{\mathcal{K}_{\delta_{k-1}}} \left|f_k - f_{k-1}\right| = k \sup_{\mathcal{K}_{\delta_{k-1}}} \left|f_{p_k}\right| \le 2^{-k}.$$

This completes the proof of the recursive step. Because of **(c)**, the sequence $\{f_j\}$ converges in $\mathcal{O}_{\mathbb{C}^n}^{(2n+1)}(\mathcal{V})$. By **(b)** and **(d)** the limit $f \in \mathcal{O}_{\mathbb{C}^n}^{(2n+1)}(\mathcal{V})$ satisfies $|f(p_j)| \ge j-1$ for all $j$. This completes the proof of Lemma 2.2. □

Since $\mathcal{W} \subset \mathcal{V}$, we have $\delta_{\complement\mathcal{W}}(p) \le \delta_{\complement\mathcal{V}}(p)$ for all $p \in \mathcal{W}$. Moreover, for a sufficiently large constant $C_1 > 0$, there is a cone $\mathcal{W}' \subset \mathcal{W}$, with edge $E$, such that $\delta_E(p) \le C_1\delta_{\partial\mathcal{W}}(p)$ for $p \in \mathcal{W}'$. The function $f$ obtained in Lemma 2.2 from a sequence $\{p_j\}$ in $\mathcal{W}'$ is holomorphic and has polynomial growth while approaching the edge $E$ within $\mathcal{W}'$. In particular, $f|_{\mathcal{W}'}$ has a boundary value $f^*$, which is a *CR* distribution on $E$. By [10, Lemma 7.2.6], $f$ is continuous up to the edge near every point in $E$ near which $f^*$ happens to be continuous. Hence $f^*$ is not continuous on a neighborhood of 0, because $f$ is unbounded on a sequence in $\mathcal{W}'$ which converges to 0. This completes the proof of the equivalence of *CR*-hypoellipticity and $(\mathrm{H}_{-\infty})$, $(\mathrm{H}^*_{-\infty})$.

## 3. On the holomorphic extension property

In this section we prove that the holomorphic extension properties $(\mathrm{H}_a)$ and $(\mathrm{H}^*_b)$, with $-\infty \le a \le \infty$ and $-\infty \le b < \infty$ are all equivalent and that, when they are valid, all local generic *CR* embeddings are related by changes of holomorphic coordinates. To that end, it will be convenient to consider also, in Theorem 3.1 below, holomorphic extensions related to non generic *CR* embeddings.

Let us fix the notation. Let $M$ be a *CR* submanifold, of *CR* dimension $m$ and *CR* codimension $d$, of a $\nu$-dimensional complex manifold $X$. This means that $M$ is a smooth real submanifold of $X$ and $T^{0,1}M = T^{0,1}X \cap \mathbb{C}TM$.

Let $p_0 \in M$ and let $(G; z_1, \dots, z_\nu)$ be any coordinate neighborhood in $X$, centered at $p_0$. When $\nu = n = m + d$, the embedding $M \hookrightarrow X$ is generic and $(G; z_1, \dots, z_n)$ provides a generic *CR*-embedding of $M \cap G$ into an open neighborhood of 0 in $\mathbb{C}^n$. If $\nu > n$, we can reorder the coordinates in such a way that the restrictions of $dz_1(p), \dots, dz_n(p)$ are linearly independent in $\mathbb{C}T^*_pM$, for $p$ in an open neighborhood of $p_0$ in $M \cap G$. Then the map $\phi : p \mapsto \phi(p) = (z_1(p), \dots, z_n(p))$ yields a generic *CR*-embedding of a neighborhood $U$ of $p_0$ in $M \cap G$ into an open neighborhood $\Omega$ of 0 in $\mathbb{C}^n$. Denote by $\phi_M : U \to \Omega$ the restriction of $\phi$ to $U \subset M$. We can also assume that $(U, t_1, \dots, t_{m+n})$, with $t_i = \operatorname{Re} z_i$ for $1 \le i \le n$, $t_i = \operatorname{Im} z_{i-n}$ for $n < i \le m+n$ is a smooth real chart on $M$.

With this notation, we have:

**Theorem 3.1.** *The following are equivalent:*

(1) *the restriction map* $\mathcal{O}_{X,(p_0)} \to \mathcal{O}^{\infty}_{M,(p_0)}$ *is onto*;
(2) *the restriction map* $\mathcal{O}_{X,(p_0)} \to \mathcal{O}^{-\infty}_{M,(p_0)}$ *is onto*;
(3) *the map* $\phi^*_M : \mathcal{O}_{\mathbb{C}^n,(p_0)} \to \mathcal{O}^{\infty}_{M,(p_0)}$ *is an isomorphism*;
(4) *the map* $\phi^*_M : \mathcal{O}_{\mathbb{C}^n,(p_0)} \to \mathcal{O}^{-\infty}_{M,(p_0)}$ *is an isomorphism.*

*Proof*. The implications (2)⇒(1) and (4)⇒(3) are obvious and both (2) and (4) imply *CR*-hypoellipticity at $p_0$. Moreover, since $\phi^*_M$ is the composition of $\phi^* : \mathcal{O}_{\mathbb{C}^n,(p_0)} \to \mathcal{O}_{X,(p_0)}$ and the restriction maps $\mathcal{O}_{X,(p_0)} \to \mathcal{O}^{\pm\infty}_{M,(p_0)}$, we also obtain that (3)⇒(1) and (4)⇒(2).

Thus it suffices to prove that (1)⇒(4). To that end, we will show that (1) implies CR-hypoellipticity and then use the fact, proved in Section 2, that *CR*-hypoellipticity implies $(\mathrm{H}_{-\infty})$. In the proof, to simplify the notation, we reduce, as we can, to the case where $X$ is an open neighborhood of 0 in $\mathbb{C}^\nu$ and $p_0 = 0$. With $U \subset M$, $\Omega \subset \mathbb{C}^n$, $\phi_M$ as above, we can also assume for simplicity that $U = M$, so that $\phi_M(M) = \tilde{M}$ is a generic *CR* submanifold of an open neighborhood $\Omega$ of 0 in $\mathbb{C}^n$.

We need to show that any *CR*-distribution, defined on an open neighborhood $W$ of 0 in $M$, is equivalent to a smooth function, on some possibly smaller neighborhood $W'$ of 0 in $M$.

Fix an open neighborhood $W$ of 0 in $M$ and let $V$ be another open neighborhood of 0 in $M$, whose dependence on $W$ will be made precise later. Denote by $B_r = \{z \in \mathbb{C}^\nu \mid \ |z| < r\}$ the open ball in $\mathbb{C}^\nu$ of radius $r > 0$, centered at 0. By using the approximation theorem of Baouendi and Trèves of [8] in the stronger formulation of [36, Theorem II.2.1], we can choose a sufficiently small $r > 0$ in such a way that $B_r \cap M \Subset V$ and, for every integer $\ell \geq 0$, all $u \in \mathcal{O}^\ell_M(V)$ can be approximated, uniformly with all their derivatives up to order $\ell$, on $B_r \cap M$, by holomorphic polynomials of $z_1, \ldots, z_n$.

Next we consider, for every positive integer $\kappa$, the set

$$\mathbb{F}_\kappa = \left\{(u, v) \in \mathcal{O}^\infty_M(B_r \cap M) \times \mathcal{O}_{\mathbb{C}^\nu}(B_{r/2^\kappa}) \,\middle|\, u = v \text{ on } M \cap B_{r/2^k}\right\}.$$

It is a closed subspace of the product $\mathcal{O}^\infty_M(B_r \cap M) \times \mathcal{O}_{\mathbb{C}^\nu}(B_{r/2^\kappa})$, endowed with its standard Fréchet topology, and hence a Fréchet space. The projections into the first coordinate restrict to continuous linear maps $\pi_k : \mathbb{F}_k \to \mathcal{O}^\infty_M(B_r \cap M)$. By Assumption (1), $\bigcup_k \pi_k(\mathbb{F}_k) = \mathcal{O}^\infty_M(V)$. Hence some $\pi_\kappa(\mathbb{F}_\kappa)$ is of the second Baire category. Then $\pi_\kappa : \mathbb{F}_\kappa \to \mathcal{O}^\infty_M(V)$ is surjective and open by the Banach-Schauder theorem. In particular, the image of

$$\{(u, v) \in \mathbb{F}_\kappa \mid \sup_{B_{r/2^{k+1}}} |v| < 1\}$$

contains an open neighborhood of 0 in $\mathcal{O}^\infty_M(V)$. Thus we get:

$$\begin{cases} \exists C > 0,\ \ell \in \mathbb{Z}_+,\ K \Subset B_r \cap M \quad \text{such that} \\ \forall u \in \mathcal{O}^\infty_M(B_r \cap M)\ \ \exists\, \tilde{u} \in \mathcal{O}_{\mathbb{C}^\nu}(B_{2^{-\kappa} r}) \\ \text{with } \ \tilde{u}|_{M \cap B_{r/2^\kappa}} = u|_{M \cap B_{r/2^\kappa}}, \ \text{ and } \ \sup\limits_{B_{r/2^{\kappa+1}}} |\tilde{u}| \leq C\, \|u\|_{\ell, K}, \end{cases} \tag{3.1}$$

where $\|u\|_{\ell,K} = \sup_K \sup_{|\alpha| \leq \ell} \left|\frac{\partial^{|\alpha|} u}{\partial t^\alpha}\right|$.

From (3.1) we obtain that every $u \in \mathcal{O}^\ell_M(V)$ is $\mathcal{C}^\infty$-smooth on $B_{r/2^{\kappa+1}} \cap M$. Let indeed $\{f_\mu\}$ be a sequence of polynomials in $\mathbb{C}[z_1, \ldots, z_n]$ which approximates $u$ uniformly with all derivatives, up to order $\ell$, on $B_r \cap M$. For each $\mu$ we can find $v_\mu \in \mathcal{O}_{\mathbb{C}^\nu}(B_{r/2^\kappa})$ with $v_\mu = f_\mu$ on $B_{r/2^\kappa} \cap M$ and $\|v_\mu\|_{0, B_{r/2^{\kappa+1}}} \leq C \|f_\mu\|_{\ell, B_r \cap M}$. Then the sequence $\{v_\mu\}$ is uniformly bounded on $B_{r/2^{\kappa+1}}$ and, by Montel's Theorem, after passing to a subsequence, we may assume that the sequence $\{v_\mu|_{B_{r/2^{\kappa+1}}}\}$ converges to a $\tilde{u} \in \mathcal{O}_{\mathbb{C}^\nu}(B_{r/2^{\kappa+1}})$, which agrees with $u$ on $B_{r/2^{\kappa+1}} \cap M$. Since $\tilde{u}$ is $\mathcal{C}^\infty$-smooth on $B_{r/2^{\kappa+1}}$, by taking its restriction to $B_{r/2^{\kappa+1}} \cap M$ we find that $u$ is $\mathcal{C}^\infty$-smooth on $B_{r/2^{\kappa+1}} \cap M$.

To show that all *CR* distributions defined on $W$ are smooth on a neighborhood of 0 in $M$, we will reduce to the previous argument by a regularization technique. This process is intrinsic on $M$, so that we are allowed to work on the projection $\widetilde{M} \subset \Omega \subset \mathbb{C}^n$. Following [8], we construct a second order linear elliptic partial differential operator on $M$, with $\mathcal{C}^\infty$ coefficients, in the following way. We may assume that $dz_1, \ldots, dz_n, d\bar{z}_1, \ldots, d\bar{z}_m$ define a maximal set of independent differentials on $M$. We obtain a set of commuting smooth complex vector fields

$$L_1, \ldots, L_n, Z_1, \ldots, Z_m$$

on $M$ by requiring that

$$L_i z_j = \delta_{i,j},\ L_i \bar{z}_k = 0,\ Z_h z_j = 0,\ Z_h \bar{z}_k = \delta_{h,k},\quad \text{for } 1 \leq i, j \leq n, 1 \leq h, k \leq m.$$

Then, for a large $c \in \mathbb{R}$,

$$\Delta_{L,cZ} = \sum_{i=1}^{n} L_i^2 + c^2 \sum_{h=1}^{m} Z_h^2 \tag{3.2}$$

is elliptic on a neighborhood $U'$ of 0 in $\widetilde{M}$. As a special case of a general theorem proved in [36, Theorem II.5.2], we have the following:

**Lemma 3.2.** *Let $W$ be any open neighborhood of 0 in $M$. Then we can find an open neighborhood $V$ of 0 in $W$, with $V \Subset W \cap U'$, such that for every nonnegative integer $\ell$ and for every $u \in \mathcal{O}^{-\infty}_{\widetilde{M}}(W)$ there is $w \in \mathcal{O}^{\ell}_{\widetilde{M}}(V)$ and an integer $k \geq 0$ such that*

$$u|_V = \Delta^k_{L,cZ} w. \tag{3.3}$$

Now we can conclude. Having fixed $W$, we choose the open neighborhood $V$ above in such a way that $V \Subset W \cap U'$ and $W$, $V$ satisfy the statement of Lemma 3.2. If $u \in \mathcal{O}^{-\infty}_M(W)$, then, by the first part of the proof, a solution $w$ of (3.3) is $\mathcal{C}^\infty$-smooth on a neighborhood of 0 in $M$ and therefore also $u$ is $\mathcal{C}^\infty$-smooth on a neighborhood of 0 in $M$. This completes the proof of Theorem 3.1. ☐

**Remark 3.3.** Fix any $\ell$ with $-\infty < \ell < +\infty$. Clearly (1) and (2) are also equivalent to the fact that the restriction map $\mathcal{O}_{X,(p_0)} \to \mathcal{O}^{\ell}_{M,(p_0)}$ is onto.

We also explicitly state, as a corollary, the equivalence of the different notions of holomorphic extendability.

**Corollary 3.4.** *The following are equivalent:*

(1) *There is an $a \in [-\infty, \infty)$ such that* ($\mathrm{H}^*_a$) *holds at $p_0$;*
(2) *There is an $a \in [-\infty, \infty]$ such that* ($\mathrm{H}_a$) *holds at $p_0$;*
(3) ($\mathrm{H}^*_a$) *holds at $p_0$ for all $a \in [-\infty, \infty]$;*
(4) ($\mathrm{H}_a$) *holds at $p_0$ for all $a \in [-\infty, \infty]$.*

**Definition 3.5.** We say that $M$ has the *holomorphic extension property* (H) at $p_0$ if any of the equivalent condition of Corollary 3.4 is valid.

As a corollary of Lemma 3.2, we also state the following regularity result, which will be useful to apply [2] to obtain holomorphic extension.

**Corollary 3.6.** *Let $M$ be a CR submanifold of a complex manifold $X$, $p_0 \in M$ and assume that, for some $\ell$, with $-\infty \leq \ell < \infty$, all germs in $\mathcal{O}^{\ell}_{M,(p_0)}$ are actually in $\mathcal{O}^{\infty}_{M,(p_0)}$. Then $M$ is CR-hypoelliptic at $p_0$, i.e. $\mathcal{O}^{-\infty}_{M,(p_0)} = \mathcal{O}^{\infty}_{M,(p_0)}$.*

Next we turn to investigate *CR*-embeddings.

**Theorem 3.7.** *Let $M$ be a CR submanifold of a complex manifold $X$. If $M$ has the holomorphic extension property at all points, then we can find a complex submanifold $Y$ of $X$, containing $M$, such that the embedding $M \hookrightarrow Y$ is generic.*

*The submanifold $Y$ is essentially unique, in the sense that, if $Y'$ is another complex submanifold of $X$ in which $M$ embeds generically, then $Y \cap Y'$ is still a complex submanifold of $X$ containing $M$ as a generic CR submanifold.*

*Proof.* We begin by constructing a local embedding near a chosen point $p_0 \in M$. We use the notation set up at the beginning of the section.

We can assume that $M \cap G$ is, in the coordinates $z_1, \dots, z_\nu$, a graph

$$z_j = Z_j(q), \; j = n+1, \dots, \nu,$$

where $q$ ranges in a neighborhood of the origin in $\widetilde{M}_G = \phi(M \cap G)$, on which the $Z_j$'s are smooth *CR* functions. By (3.1) of Theorem 3.1 the $Z_j$'s uniquely extend to holomorphic functions $\tilde{Z}_j$, defined on an ambient neighborhood $\omega_0$ of 0 in $\mathbb{C}^n$. Then

$$Y_{p_0} = \{p \in G \mid \phi(p) \in \omega_0, \; z_i(p) = \tilde{Z}_i(z_1(p), \dots, z_n(p)), \; n < i \le \nu\}$$

is a complex submanifold of $X$ in which an open neighborhood of $p_0$ in $M$ generically embeds.

The submanifold $Y$ will be constructed by patching together, using the arguments of [5], the local $Y_p$ constructed in this way about different points $p$ of $M$. To finish the proof, we only need to show that a different choice of the coordinates about $p_0$ yields the same germ of complex manifold $(Y_{p_0}, p_0)$ at $p_0$.

Let $(E; \zeta_1, \dots, \zeta_\nu)$ be another holomorphic chart at $p_0$ in $X$. We rearrange the indices in such a way that $d\zeta_1, \dots, d\zeta_n$ are linearly independent on an open neighborhood of $p_0$ in $M$. We write for simplicity $z = (z_1, \dots, z_\nu)$, $z' = (z_1, \dots, z_n)$, $z'' = (z_{n+1}, \dots, z_\nu)$ and $\zeta = (\zeta_1, \dots, \zeta_\nu)$, $\zeta' = (\zeta_1, \dots, \zeta_n)$, $\zeta'' = (\zeta_{n+1}, \dots, \zeta_\nu)$. We have, on an open neighborhood $G_0$ of $p_0$ in $G \cap E$,

$$z = \Theta(\zeta) \text{ and } \zeta = \Xi(z),$$

for functions $\Theta = (\Theta_1, \dots, \Theta_\nu) = (\Theta', \Theta'')$ and $\Xi = (\Xi_1, \dots, \Xi_\nu) = (\Xi', \Xi'')$ which are holomorphic on an open neighborhood of 0 in $\mathbb{C}^\nu$.

Repeating the preceding construction in the new coordinates, we obtain a generic *CR* embedding of an open neighborhood of $p_0$ in $M$ into a complex submanifold $Y'_{p_0}$ of $X$, which is described, with obvious notation, by

$$Y'_{p_0} = \{p \in E \mid \zeta'(p) \in \eta_0, \; \zeta_i(p) = \tilde{\Lambda}_i(\zeta'(p)), \; n < i \le \nu\},$$

where $\eta_0$ is an open neighborhood of 0 in $\mathbb{C}^n$ and the $\tilde{\Lambda}_i$'s are holomorphic on $\eta_0$.

To show that $(Y'_{p_0}, p_0) = (Y_{p_0}, p_0)$ we observe that

$$Y'_{p_0} \cap G_0 = \{p \in G_0 \mid z_i = \Theta_i(\zeta'(p), \tilde{\Lambda}(p)), \; 1 \le i \le \nu\},$$

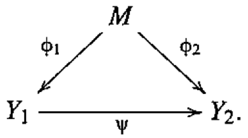
where $\tilde{\Lambda} = (\tilde{\Lambda}_{n+1}, \ldots, \tilde{\Lambda}_{\nu})$. Then we notice that, by (3.1) of Theorem 3.1,

$$\big(z'(p), \tilde{Z}_i\big(z'(p)\big)\big) = \Theta\big(\zeta'(p), \tilde{\Lambda}(p)\big)$$

for $\phi(p)$ on an open neighborhood of 0 in $\mathbb{C}^n$, because this is true for $p$ on an open neighborhood of $p_0$ in $M$. This shows that $Y_{p_0}$ and $Y'_{p_0}$ define the same germ of $n$-dimensional complex submanifold at $p_0$. The proof is complete. □

**Corollary 3.8.** *Let $M$ be a CR manifold of CR dimension $m$ and CR codimension $d$, and $n = m+d$. Assume that $M$ is locally CR-embeddable and CR-hypoelliptic at all points. Then $M$ admits a global smooth generic CR-embedding $M \hookrightarrow X$ into an $n$-dimensional complex manifold $X$.*

*This embedding is essentially unique. Indeed, assume $M$ is CR-hypoelliptic, that $X_1$, $X_2$ are complex manifolds and that there are generic CR embeddings $\phi_i : M \hookrightarrow X_i$, $i = 1, 2$. Then we can find open neighborhoods $Y_i$ of $\phi_i(M)$ in $X_i$ and a biholomorphism $\psi : Y_1 \to Y_2$ that fit into a commutative diagram*




(3.4)

*Proof.* The first part of the corollary is proved by using the abstract construction of [5] to patch together the different generic local *CR*-embeddings.

Let us turn to the proof of uniqueness. We consider the product manifold $X = X_1 \times X_2$ and the *CR*-embedding $\Phi : M \ni p \to (\phi_1(p), \phi_2(p)) \in X$. Denote by $\pi_i : X \to X_i$ the canonical projection of the Cartesian product onto its $X_i$-factor. By Theorem 3.7, the image $\Phi(M)$ of $M$ is a generic *CR* submanifold of a complex submanifold $Y$ of $X$, which, after substituting to $Y$ an open neighborhood of $\Phi(M)$ in $Y$, can be taken to be the graph of a biholomorphic $\psi : Y_1 = \pi_1(Y) \to Y_2 = \pi_2(Y)$, that fits into the commutative diagram (3.4). □

## 4. Holomorphic wedge extension and the $\mathcal{C}^\infty$ wave front set

Theorem 1.4 relates holomorphic extension to a full neighborhood to $\mathcal{C}^\infty$-regularity. Here we make some remarks on the relationship between holomorphic wedge extension and the $\mathcal{C}^\infty$ wave front set. It is known that the directions of wedge extension are nicely reflected by the *analytic* wave front set, which provides information on the extension of any *individual CR* distribution. We will prove below a result which relates the local wedge of *simultaneous* extension of all the elements of $\mathcal{O}^{\infty}_{M,(p_0)}$ to the wave front sets of the elements of $\mathcal{O}^{-\infty}_{M,(p_0)}$.

We need to introduce some notation. Let $M$ be a $\mathcal{C}^\infty$-smooth *CR* manifold, of *CR* dimension $m$ and *CR* codimension $d$. We denote by $HM$ the subbundle of

$TM$ consisting of the real parts of vectors in $T^{0,1}M$ and by $J_M : HM \to HM$ the anti-involution which associates to $X \in H_pM$ the unique $J_MX \in H_pM$ such that $X + iJX \in T^{0,1}M$. Let $\dot{T}^*M$ be the cotangent bundle of $M$ minus its zero section and $\mathcal{D}'(U)$, for $U^{\text{open}} \subset M$, the space of complex valued distributions in $U$. The wave front set[1] of $u$ in $\mathcal{D}'(U)$ is a closed conic subset $\text{WF}(u)$ of $\dot{T}^*U$. When $u \in \mathcal{O}_M^{-\infty}(U)$, its wave front set is contained in the *characteristic bundle*

$$H^0M = \{\xi \in T^*M \mid \xi(X) = 0,\ \forall X \in H_{\pi(\xi)}M\}$$

for the tangential Cauchy-Riemann equations on $M$.

The following is a simple consequence of [11,15].

**Theorem 4.1.** *Assume that the CR manifold $M$, of CR dimension $m$ and CR codimension $d$, is a smooth real hypersurface in a CR manifold $N$, of CR dimension $m+1$ and CR codimension $d-1$. We also assume that $M$ divides $N$ into two open submanifolds $N^+$, $N^-$, with $N^+ \cup N^- = N \setminus M$, and $\bar{N}^+ \cap \bar{N}^- = M$.*

*Let $f \in \mathcal{O}^1_N(N^+)$ be such that, for every compact $K$ in $N$, there are constants $C_K, \ell_K > 0$ such that*

$$|f(p)| \le C_K \operatorname{dist}(p, M)^{-\ell_K}, \quad \forall p \in N^+ \cap K. \tag{4.1}$$

*Then $f$ admits a boundary value $u \in \mathcal{D}'(M)$, which is a CR distribution on $M$.*

*Let $p_0 \in M$ and $X_{p_0}$ a nonzero tangent vector in $H_{p_0}N$ pointing into*[2] *$N^+$, with $J_NX_{p_0} \in T_{p_0}M$. Then*

$$\text{WF}(u) \cap T^*_{p_0}M \subset \{\xi \in H^0_{p_0}M \mid \langle \xi, J_NX_{p_0}\rangle \le 0\}. \tag{4.2}$$

**Remark 4.2.**

- **(a)** In (4.1) we can use any distance which is locally quasi-isometric to a Riemannian distance on $N$.
- **(b)** The boundary value $u$ of $f$ can be locally described in the following way. If $\psi : U \ni p \to (t(p), s(p)) \in \mathbb{R}^1 \times \mathbb{R}^{2m+d}$ is a coordinate chart in $N$, defined on an open subset $U$ of $N$ and with $\psi(U) = \mathbb{R}^1 \times \mathbb{R}^{2m+d}$, such that $M \cap U = \{t = 0\}$ and $N^+ \cap U = \{t > 0\}$, then, consistently with (2.3),

  $$\langle u \circ s, \phi\rangle = \lim_{t\to 0^+}\langle f_t, , \phi\rangle, \quad \forall \phi \in \mathcal{D}^{(2m+d)}(\mathbb{R}^{2m+d}),$$

  where $\mathcal{D}^{(2m+d)}(\mathbb{R}^{2m+d})$ is the space of smooth complex valued top degree alternated forms with compact support in $\mathbb{R}^{2m+d}$, and we indicate by $f_t$ the complex valued $\mathcal{C}^1$ function $s \to f \circ \psi^{-1}(t, s)$ on $\mathbb{R}^{2m+d}$.

[1] For basic definitions and a thorough introduction to this topic we refer to [23].

[2] This means that $X_{p_0} \notin TM$ and there is a smooth curve $\gamma : [0, 1] \to N$ with $\gamma(0) = p_0$, $\gamma(t) \in N^+$ for $0 < t \le 1$ and $\dot{\gamma}(0) = X_{p_0}$.

**(c)** The vector $X_{p_0}$ in (4.2) exists and is uniquely determined modulo $H_{p_0}M$ and multiplication by a positive scalar.
**(d)** The analogue of (4.2) for the hypo-analytic wave front set $WF_{ha}(u)$ is stated in [35].

*Proof.* The intersection $TM \cap HN$ is a real vector bundle of rank $2m+1$ on $M$, which contains $HM$ as a subbundle of hyperplanes, and $X_{p_0}$ is a vector of the form $-J_N Y_{p_0}$ for a $Y_{p_0} \in H_{p_0}N \setminus H_{p_0}M$. We can find an open neighborhood $\omega$ of $p_0$ in $M$ and a section $Y \in \Gamma(\omega, TM \cap HN)$ such that $-J_N Y_p$ points into $N_+$ for every $p \in \omega$. We can extend $Y$ to a section $\tilde{Y} \in \Gamma(U, HN)$, defined on an open neighborhood $U$ of $p_0$ in $N$. After shrinking, we can assume that on $U$ a coordinate patch is defined as in point **(b)** of Remark 4.2, with $J_N \tilde{Y} = \frac{\partial}{\partial t}$. Then a multiple of $J_N \tilde{Y} - i\tilde{Y} \in \Gamma(U, T^{0,1}N)$ has the form

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{2m+d} a_i(t,s)\frac{\partial}{\partial s_i}.$$

An $f \in \mathcal{O}^1_N(N^+)$ satisfies in particular the equation $Lf = 0$ on $N_+ \cap U$. Thus, if (4.1) is also satisfied, its boundary value $u \in \mathcal{O}^{-\infty}_M(M \cap U)$ is well defined by [11, Lemma 1.2]. The inclusion (4.2) is then a consequence of [11, Theorem 2.1]. □

Then we have:

**Proposition 4.3.** *Let $M$, $N$ be as in the statement of Theorem* 4.1. *Let $p_0 \in M$, and assume that $N$ is locally CR embeddable at $p_0$. Assume that*

$$\begin{cases} \forall \omega \text{ open in } M \text{ with } p_0 \in \omega \text{ and } \forall f \in \mathcal{O}^\infty_M(\omega),\ \exists\, V \text{ open in } N, \text{ with} \\ p_0 \in V \text{ and } \exists \tilde{f} \in \mathcal{O}^1_N(N^+ \cap V) \cap \mathcal{C}^0(\bar{N}^+ \cap V), \text{ so that } \tilde{f} = f \text{ on } \omega \cap V. \end{cases} \tag{4.3}$$

*Let $X_{p_0}$ be a nonzero tangent vector in $H_{p_0}N$ pointing into $N^+$, with $J_N X_{p_0} \in T_{p_0}M$. Then*

$$\mathrm{WF}(u) \cap T^*_p M \subset \left\{\xi \in H^0_p M \mid \langle \xi, J_N X_{p_0}\rangle \le 0\right\}, \quad \forall u \in \mathcal{O}^{-\infty}_{M,(p_0)}. \tag{4.4}$$

*Proof.* Using a functional analysis argument and the approximation theorem of Baouendi and Trèves, we obtain, as in the proof of Theorem 3.1, that, having fixed an open neighborhood $\omega$ of $p_0$ in $M$, there exist a fixed open neighborhood $V$ of $p_0$ in $N$ and an integer $\ell \ge 0$ such that

$$\forall f \in \mathcal{O}^\ell_M(\omega)\ \exists \tilde{f} \in \mathcal{O}^1_N(N^+ \cap V) \cap \mathcal{C}^0(\bar{N}^+ \cap V) \text{ so that } \tilde{f} = f \text{ on } \omega \cap V. \tag{4.5}$$

Then we obtain (4.4) by using Lemma 3.2: if $U$ is an open neighborhood of $p_0$ in $M$ and $u \in \mathcal{O}^{-\infty}_M(U)$, then the restriction of $u$ to a neighborhood $U' \Subset U$ belongs to $\mathcal{O}^a_M(U')$ for some integer $a$. Using Lemma 3.2, we can write $u|_\omega =$

$\Delta^k_{L,cZ} f$, where $\Delta_{L,cZ}$ was defined in (3.2), $\omega$ is an open neighborhood of $p_0$ in $U'$ and $f \in \mathcal{O}^\ell_M(\omega)$. By (4.5) and Theorem 4.1 we have $\mathrm{WF}(f) \cap T^*_{p_0}M \subset \{\xi \in H^0_p M \mid \langle \xi, J_N X\rangle < 0\}$ and (4.4) follows because $\Delta_{L,cZ}$ is elliptic and hence $\mathrm{WF}(u|_\omega) \subset \mathrm{WF}(f)$. □

Let us consider now a more general situation. Namely, we assume that $M$ is a generic *CR* submanifold, of *CR* dimension $m$ and *CR* codimension $d$, of a *CR* manifold $Q$ of *CR* dimension $\mu$ and *CR* codimension $\kappa$, with $m < \mu$, $d > \kappa$ and $m+d = \mu+\kappa = n$. Let $\Omega$ be an open subset of $Q$. If $p \in \partial\Omega$, we say that $v \in T_p Q$ points into $\Omega$ if, for every $\gamma \in \mathcal{C}^1([0,1], Q)$ with $\gamma(0) = p$ and $\dot\gamma(0) = v$, there is a $0 < \delta < 1$ such that $\gamma(t) \in \Omega$ for $0 < t < \delta$. Assume now that $M \subset \partial\Omega$. We consider the subset of the pullback $TQ|_M$ on $M$ of the tangent bundle of $Q$ consisting of the vectors that point inside $\Omega$ and denote by $\dot T_\Omega M$ its interior part in $TQ|_M$. We set:

$$\begin{aligned} \dot T_\Omega M &= \mathrm{int}\{v \in TQ|_M \mid v \text{ points into } \Omega\},\\ \dot T_{\Omega,p_0} M &= \dot T_\Omega M \cap T_{p_0} Q,\\ H_{\Omega,p_0} M &= \left\{X \in T_{p_0}M \cap H_{p_0}Q \mid J_Q X \in \dot T_{\Omega,p_0}M\right\},\end{aligned}$$

for $p_0 \in M$. We note that $\dot T_{\Omega,p_0}M$ and $H_{\Omega,p_0}M$ are cones, and that $H_{\Omega,p_0}M \subset T_{p_0}M \cap H_{p_0}Q$.

As a consequence of Proposition 4.3 we obtain:

**Corollary 4.4.** *Let $M$, $Q$, $\Omega$ be as above. Let $p_0 \in M$ and assume that*

$$\begin{cases} \forall U \text{ open in } M \text{ with } p_0 \in U \text{ and } \forall f \in \mathcal{O}^\infty_M(U),\ \exists V \text{ open in } Q, \text{ with}\\ p_0 \in V \text{ and } \exists \tilde f \in \mathcal{O}^1_N(\Omega\cap V) \cap \mathcal{C}^0((\Omega\cup M)\cap V) \text{ with } \tilde f = f \text{ on } U \cap V. \end{cases} \tag{4.6}$$

*Then*

$$\mathrm{WF}(u) \cap T^*_{p_0}M \subset \left\{\xi \in H^0_{p_0}M \mid \langle \xi, X\rangle \geq 0,\ \forall X \in H_{\Omega,p_0}M\right\},\ \ \forall u \in \mathcal{O}^{-\infty}_{M,(p_0)}. \tag{4.7}$$

*Proof.* It suffices indeed to apply Proposition 4.3 to *CR* submanifolds $N$ of *CR* dimension $m+1$ and *CR* codimension $d-1$ of a neighborhood of $p_0$ in $Q$, with $N \setminus M = N^+ \cup N^-$, $N^+$ and $N^-$ connected and open in $N$, $N^+ \subset N \cap \Omega$, and $\bar N^+ \cap \bar N^-$ containing an open neighborhood of $p_0$ in $M$. □

Let $M$ be locally *CR* embeddable and minimal at $p_0$. Fix a local generic *CR* embedding $\psi : U \to \Omega \subset \mathbb{C}^n$ of an open neighborhood $U$ of $p_0$ in $M$ into an open neighborhood of 0 in $\mathbb{C}^n$, with $\psi(p_0) = 0$ and $\psi(U)$ having the form (2.1). We know from [34,37] that there is an open neighborhood $\omega$ of $p_0$ in $U$ such that, for every *CR* distribution $u \in \mathcal{O}^{-\infty}_M(U)$, the restriction $u|_\omega$ is the pullback of the boundary value of a holomorphic function, defined on an open wedge $\mathcal{W} \subset \Omega$, with edge $E = \psi(\omega)$, that can be chosen in the form (2.2). We note that, in the corresponding coordinates, we have

$$\begin{cases} H^0_{p_0}M = \langle dx_1(0), \dots, dx_d(0)\rangle \simeq \mathbb{R}^d, \\ H_{\mathcal{W},p_0}M = H_{p_0}M \oplus \left\{ \sum_{i=1}^{d} a_i \left(\frac{\partial}{\partial x_1}\right)_{x=0} \;\middle|\; \sum_{i=1}^{d} a_i e_i \in C \right\}, \end{cases} \tag{4.8}$$

where $e_1, \dots, e_d$ is the canonical basis of $\mathbb{R}^d$.

Then, from Corollary 4.4 we obtain:

**Corollary 4.5.** *Assume that $M$ is minimal at $p_0$ and that, for every $u \in \mathcal{O}^{-\infty}_M(U)$ there is an open neighborhood $V$ of $p_0$ in $\mathbb{C}^n$ such that, for $E = \psi(M \cap V)$, the restriction $u|_{M\cap V}$ is the pullback of the the boundary value of some $\tilde{u} \in \mathcal{O}_{\mathbb{C}^n}(\mathcal{W} \cap V)$, for the wedge $\mathcal{W}$ described by* (2.2). *Then*

$$\mathrm{WF}(u) \cap H^0_{p_0}M \subset \left\{\xi \mid \xi \in C^0\right\}. \tag{4.9}$$

*Here we used the identification $H^0_{p_0}M \simeq \mathbb{R}^d$ of* (4.8) *and $C^0 = \{\xi \mid \langle \xi, v\rangle \leq 0,\ \forall v \in C\}$ is the polar cone of $C$.*

In the last part of this section, we shall discuss the relationship of $\mathcal{C}^\infty$ wave front set and minimality. For each $p \in M$ we denote by $\mathfrak{O}_M(p)$ its *CR* orbit in $M$, *i.e.* the set of points in $M$ that can be linked with $p$ by a piecewise smooth curve with velocities in $HM$. A fundamental result of Sussmann ( [33]) tells us that each *CR* orbit $\mathfrak{O}_M(p)$ is a smooth *CR* submanifold of the same *CR* dimension of $M$. If $U$ is an open neighborhood of $p$ in $M$ we can consider the orbit $\mathfrak{O}_U(p)$, computed by restricting to piecewise smooth $HM$-curves with support in $U$. Clearly, if $U$ and $V$ are open subsets of $M$ with $V \subset U$, then $\mathfrak{O}_V(p) \subset \mathfrak{O}_U(p)$ for all $p \in V$. For a fixed $p$, the family of *CR* orbits $\mathfrak{O}_U(p)$, indexed by the filter of its open neighborhoods, uniquely defines a *germ* of *CR* manifold $\mathfrak{O}_{M,\mathrm{loc}}(p)$, which is called its *local CR orbit*. Tumanov's theorem in [37] yields local holomorphic extension to open wedges if $\mathfrak{O}_U(p_0)$ is open (see also [7,27,29,30,34]), so that Corollary 4.5 applies to this case. More generally, the dimension of $\mathfrak{O}_U(p_0)$ is related to the maximal number of linearly independent directions of *CR* extension (*cf.* [38]).

Given an open subset $U$ of $M$ and a distribution $u \in \mathcal{D}'(U)$, it is convenient, to state the next theorem, to introduce the notation $\overline{\mathrm{WF}}(u)$ for the union of its $\mathcal{C}^\infty$ wave front set and the zero section of $T^*U$.

We prove the following:

**Theorem 4.6.** *Let $M$ be a smooth CR submanifold, of CR dimension $m$ and CR codimension $d$, of a complex manifold $X$, and $p_0 \in M$. Then the following are equivalent*

1. $\dim_{\mathbb{R}} \mathfrak{O}_{M,\mathrm{loc}}(p_0) = 2m + k$ $(0 \leq k \leq d)$;
2. *there is a CR distribution $u$, defined on an open neighborhood $U$ of $p_0$, such that $\overline{\mathrm{WF}}(u) \cap T^*_{p_0}M$ contains a $(d-k)$-dimensional $\mathbb{R}$-linear subspace, and $k$ is the smallest integer with this property.*

*Assume that* (1) *holds true and that* $\mathfrak{O}_{M,\mathrm{loc}}(p_0)$ *does not have the holomorphic extension property* (H) *at* $p_0$. *Then there exists a CR distribution u, defined on an open neighborhood U of* $p_0$, *such that* $\overline{\mathrm{WF}}(u) \cap T^*_{p_0}M$ *properly contains a* $(d-k)$-*dimensional* $\mathbb{R}$-*linear subspace.*

**Remark 4.7.** Using Theorem 4.6, Tumanov's theorem (see [37]) can be restated by saying that all *CR* functions defined on any fixed neighborhood of $p_0$ admit a holomorphic extension to an open wedge with edge containing $p_0$ if and only if no *CR* distribution $u$ has a $\overline{\mathrm{WF}}(u)$ which contains a real line of $T^*_{p_0}M$. Theorem 4.6 can be considered a generalization of that result to the non minimal case.

*Proof.* Since (1) and (2) are local statements, we can assume in the proof that $M$ is a generic *CR* submanifold of an open subset $\Omega$ of $\mathbb{C}^n$.

Let $\dim_{\mathbb{R}} \mathfrak{O}_{M,\mathrm{loc}}(p_0) = 2m + k$. Fix an open neighborhood $U$ of $p_0$ in $M$. By Tumanov's theorem [37], there are generic *CR* manifolds with boundary $M_1^+, \dots, M_k^+$ in $\mathbb{C}^n$, of dimension $2m+d+1$, whose boundary contain an open neighborhood $E$ of $p_0$ in $M$, and such that every continuous *CR* function $u$ on $U$ uniquely extends to each $M_j^+$ as a *CR* function, continuous up to the boundary. These $M_j$ can be chosen so that there are smooth real vector fields $X_1, \dots, X_k \in \Gamma(E, TM)$, with $X_1(p), \dots, X_k(p)$ linearly independent modulo $H_pM$ at all points $p \in E$, and with $JX_j(p)$ pointing into $M_j^+$ for all $p \in E$ and $j = 1, \dots, k$. According to [27, Lemma 10], we can assume that the $M_j^+$ are $\mathcal{C}^\infty$-smooth up to $M$. Thus each $M_j$ can be slightly enlarged to a $\mathcal{C}^\infty$-smooth open manifold $\widetilde{M}_j$, containing $E$. Thus, by Proposition 4.3 we have

$$WF(u) \cap T^*_{p_0}M \subset \left\{\xi \in H^0M \mid \xi(X_j) \geq 0\right\}, \tag{4.10}$$

and hence $\overline{WF}(u)$ cannot contain any real linear subspace of dimension larger than $d-k$.

On the other hand, assume that there are on $M$ an open neighborhood $U$ and a *CR* submanifold $E$ through $p_0$ of $U$, having the same *CR* dimension $m$ of $M$. By taking $U$ small, we can find a nonzero *CR* distribution on $U$ carried by $E$.

Indeed: When $E$ is open, there is nothing to prove. If $E$ has a smaller dimension, we fix a positive measure $\mu$ with smooth density on $E$. A construction in [7] yields a function $v$ which is $\mathcal{C}^\infty$-smooth in a neighborhood of $p_0$ in $E$, with $v(p_0) = 1$, and such that

$$T_E[\phi] = \int_E v\phi \, d\mu, \qquad \phi \in \mathcal{D}(U), \tag{4.11}$$

is a *CR* distribution on a possibly smaller neighborhood $U$ of $p_0$ in $M$. In this case $\mathrm{WF}(u) \cap T^*_{p_0}M = (T_{p_0}E)^\perp$. This completes the proof of the implication (1)$\Rightarrow$(2). The argument also shows that, if there is a *CR* distribution $u$, defined on a neighborhood $U$ of $p_0$, such that $\overline{\mathrm{WF}}(u) \cap T^*_{p_0}M$ contains an $\ell$-dimensional

$\mathbb{R}$-subspace, then $\dim_{\mathbb{R}} \mathfrak{O}_{M,\mathrm{loc}}(p_0) \le 2m+d-\ell$. Thus we obtain also the opposite implication (2)⇒(1).

Let us turn to the proof of the last statement of Theorem 4.6. If $\mathfrak{O}_{M,\mathrm{loc}}(p_0)$ is open, it is a consequence of Theorem 1.4, because a distribution $u$ with $\mathrm{WF}(u) \cap T^*_{p_0}M = \emptyset$ is smooth near $p_0$. If $\mathfrak{O}_{M,\mathrm{loc}}(p_0)$ is lower-dimensional, let $E$ be a *CR* submanifold of an open neighborhood $U$ of $p_0$ in $M$ with $(E, p_0) = \mathfrak{O}_{M,\mathrm{loc}}(p_0)$. After shrinking, we can assume that there is a *CR* isomorphism $\pi : E \to E'$, where $E'$ is a *CR* submanifold of an open subset $\Omega'$ of $\mathbb{C}^{n'}$, with $n' = n + k < n$. By choosing holomorphic coordinates as at the beginning of Section 3, we may assume that $\pi$ is induced by the projection of $\mathbb{C}^n$ onto the complex subspace $\mathbb{C}^{n'}$ of the first $n'$ coordinates $z_1, \dots, z_{n'}$. The Baouendi-Trèves approximation theorem says that there is a measure $\mu'$ on $E'$, with a smooth density on $E'$, such that any *CR* distribution $S$ on $E'$ can be approximated by polynomials $Q_j(z_1, \dots, z_{n'})$, in the sense that

$$\int_{E'} Q_j \phi \, d\mu' \to S[\phi], \quad \forall \phi \in \mathcal{D}\big(U'\big), \tag{4.12}$$

holds on an appropriate neighborhood $U' \Subset E'$ of $0 = \pi(p_0)$. We can choose $\mu = \pi^*\mu'$ in (4.11).

To complete the proof, we will use the following lemma:

**Lemma 4.8.** *There is a neighborhood $U \subset M$ of $p_0$ such that for any CR distribution $u$ on $E'$ the formula*

$$T_u[\phi] = u\big[(v\phi) \circ \pi^{-1}\big], \quad \forall \phi \in \mathcal{D}(U), \tag{4.13}$$

*defines a CR distribution $T_u$ on $U$ with support contained in $E \cap U$.*

*Proof of Lemma* 4.8. Let $\{Q_j = Q_j(z_1, \dots, z_{n'})\}$ be a sequence of holomorphic polynomials, approximating $u$ on some neighborhood $U'$ of 0 in $E'$, as in (4.12). Since $\mu = \pi^*\mu'$, the distributions $Q_j T_E : \phi \mapsto \int_E Q_j v\phi \, d\mu$ approximate the distribution in (4.13), provided we take $\phi$ with compact support in an open neighborhood $U$ of $p_0$ in $M$, with $U \cap E \Subset \pi^{-1}(U')$. Being the products of a *CR* distribution by the restriction to $U$ of holomorphic functions, the $Q_j T_E$ are *CR* distributions on $U$, and therefore also their limit in the sense of distributions is a *CR* distribution on $U$. This completes the proof of the lemma. □

*End of the proof of Theorem* 4.6 Since $E'$ does not have the extension property, by Theorem 1.4 there is a *CR* distribution $u$ with $\mathrm{WF}_{E'}(u) \cap T^*_0 E' \neq \emptyset$. It remains to check that $\mathrm{WF}(T_u)$ has the desired properties.

For this purpose, we introduce smooth coordinates $(s_1, \dots, s_{2m+k}, t_1, \dots, t_\ell)$, $\ell = d - k$, centered at $p_0$, such that $E = \{t_1 = 0, \dots, t_\ell = 0\}$. The distribution $T_u$ is a tensor product

$$T_u = (vgu^*) \otimes \delta_t,$$

where $u^*$ is the pullback of $u$ on $E$, and $\delta_t$ is the Dirac delta in the $t$-variables and $g$ is a smooth nonvanishing function such that $d\mu' = g \, ds_1 \dots ds_{2m+k}$. Since $v(p_0) =$

1, we can assume after shrinking that $v \neq 0$ on $U$. Then $\mathrm{WF}(u^* v g) = \mathrm{WF}(u^*)$ and the general rule to compute the wave front set of a tensor product [23, Theorem 8.2.9] yields

$$\mathrm{WF}(T_{\tilde{u}}) \cap T^*_{p_0} M = \left(\overline{\mathrm{WF}}_E(u^*) \times \langle dt_1, \dots, dt_\ell \rangle\right) \setminus \{(0,0)\}. \tag{4.14}$$

The proof is complete. □

Theorem 4.6 is closely related to continuity properties of *CR* measures. We say that a *CR* manifold $M$ has the *Riesz property at* $p_0 \in M$ if every *CR* measure $\mu$, defined on a neighborhood $U$ of $p_0$ in $M$, is absolutely continuous with respect to the Lebesgue measure on a neighborhood $\omega$ of $p_0$ in $U$.

As a consequence of Theorem 4.6 we reobtain a result of Chirka and Rea [16].

**Theorem 4.9.** *Let $M$ be a CR manifold, locally CR embeddable at $p_0 \in M$. Then $M$ has the Riesz property at $p_0$ if and only if it is minimal at $p_0$.*

*Proof.* If $M$ is not minimal at $p_0$, the existence of *CR* measures as in (4.11) excludes the Riesz property. If $M$ is minimal at $p_0$, and $\mu$ is a *CR* measure defined on an open neighborhood $U$ of $p_0$ in $M$, it follows from Theorem 4.6 that there is an open neighborhood $\omega$ of $p_0$ in $U$ such that $\overline{\mathrm{WF}}(\mu) \cap T^*\omega$ does not contain any real line, because, for $\mu \in \mathcal{O}_M^{-\infty}(U)$, the set of points $p \in U$ such that $\overline{\mathrm{WF}}(\mu) \cap T^*_p M$ contains a real line is closed in $U$. Then [15, Theorem 1.4] implies that $\mu|_\omega$ is absolutely continuous with respect to Lebesgue measure. □

For recent generalizations of the F. and M. Riesz theorem to the solutions of more general differential operators we refer to [11–13, 15].

## 5. Some subellipticity conditions

In this section we recall some results of [2] that are relevant for our applications. In the following, $M$ is an abstract *CR* manifold, $\mathscr{Z}(M) = \Gamma(M, T^{0,1}M)$ is the distribution of complex vector fields of type (0, 1) on $M$, and $\mathscr{H}(M) = \Gamma(M, HM)$ the distribution of the *real* vector fields which are real parts of elements of $\mathscr{Z}(M)$.

### 5.1. The system $\Theta(M)$

**Definition 5.1.** Set

$$\Theta(M) = \left\{ Z \in \mathscr{Z}(M) \;\middle|\; \begin{array}{c} \exists r \geq 0,\ \exists Z_1, \dots, Z_r \in \mathscr{Z}(M),\ \text{so that} \\ i[Z, \bar{Z}] + i \sum_{j=1}^{r} [Z_j, \bar{Z}_j] \in \mathscr{H}(M) \end{array} \right\}. \tag{5.1}$$

We denote by $\mathscr{A}(M)$ the Lie subalgebra of $\mathfrak{X}(M)$ generated by the real parts of vectors in $\Theta(M)$. If $\mathscr{H}'(M) = \{\mathrm{Re}\, Z \mid Z \in \Theta(M)\}$,

$$\mathscr{A}(M) = \mathscr{H}'(M) + [\mathscr{H}'(M), \mathscr{H}'(M)] + [\mathscr{H}'(M), [\mathscr{H}'(M), \mathscr{H}'(M)]] + \cdots$$

We showed in [2, Lemma 2.5] that:

**Proposition 5.2.** *With the notation introduced above, $\Theta(M)$ is a left $\mathcal{C}^\infty(M)$-submodule of $\mathfrak{X}^{\mathbb{C}}(M)$. For every $Z \in \Theta(M)$ and every relatively compact open subset $U$ of $M$ there are a finite set $Z_1, \dots, Z_r$ of vector fields in $\mathscr{Z}(M)$ and a constant $C > 0$ such that*

$$\|\bar{Z}u\|_0^2 \leq C \left( \|u\|_0^2 + \sum_{i=1}^{r} \|Z_i u\|_0^2 \right), \quad \forall u \in \mathcal{C}_0^\infty(U).$$

Hence, by [2, Corollary 1.15], we obtain:

**Theorem 5.3.** *Let $\mathscr{M}(M)$ be the $\mathscr{A}(M)$-Lie submodule of $\mathfrak{X}(M)$ generated by $\mathscr{H}(M)$:*

$$\begin{aligned} \mathscr{M}(M) = \mathscr{H}(M) &+ \left[\mathscr{A}_{\mathscr{Z}}(M), \mathscr{H}(M)\right] \\ &+ \left[\mathscr{A}_{\mathscr{Z}}(M), \left[\mathscr{A}_{\mathscr{Z}}(M), \mathscr{H}(M)\right]\right] + \cdots \end{aligned} \tag{5.2}$$

*If*

$$\left\{X_{p_0} \mid X \in \mathscr{M}(M)\right\} = T_{p_0}M, \tag{5.3}$$

*then the system $\mathscr{Z}(M)$ is subelliptic at $p_0$. This means that there exists an open neighborhood $U$ of $p_0$ in $M$, vector fields $Z_1, \dots, Z_m \in \mathscr{Z}(M)$, and constants $C, \varepsilon > 0$ such that*

$$\|u\|_\varepsilon^2 \leq C \left( \|u\|_0^2 + \sum_{i=1}^{m} \|Z_i u\|_0^2 \right), \quad \forall u \in \mathcal{C}_0^\infty(U). \tag{5.4}$$

### 5.2. The system $\mathscr{K}(M)$

Under a certain constant rank assumption on $\mathscr{Z}(M)$, we can give a more explicit description of $\Theta(M)$.

**Definition 5.4.** The *characteristic bundle* $H^0M$ of $\mathscr{Z}(M)$ is the set of the *real* covectors $\xi$ with $\langle Z, \xi\rangle = 0$ for all $Z \in \mathscr{Z}(M)$.

The *scalar Levi form* at $\xi \in H^0_pM$ is the Hermitian symmetric form

$$\mathfrak{L}_\xi(Z_1, \bar{Z}_2) = i\xi\big(\left[Z_1, \bar{Z}_2\right]\big) \quad \text{for} \quad Z_1, Z_2 \in \mathscr{Z}(M). \tag{5.5}$$

The value of the right hand side of (5.5) only depends on the values $Z_1(p), Z_2(p)$ of $Z_1, Z_2$ at the base point $p = \pi(\xi)$. Thus (5.5) is a Hermitian symmetric form on $T_p^{0,1}M$. Set:

$$H^{\oplus}M = \left\{\xi \in H^0M \mid \mathcal{L}_\xi \geq 0\right\}, \tag{5.6}$$

$$\mathscr{K}(M) = \left\{Z \in \mathscr{Z}(M) \mid \mathcal{L}_\xi(Z, \bar{Z}) = 0, \ \forall \xi \in H^{\oplus}M\right\}, \tag{5.7}$$

$$\mathrm{K}M = \dot{\bigcup_{p\in M}} \mathrm{K}_pM \text{ with } \mathrm{K}_pM = \left\{Z_p \mid Z \in \mathbb{Z}(M)\right\}. \tag{5.8}$$

We have (see [2, Proposition 2.13])

**Proposition 5.5.** *$\mathscr{K}(M)$ is a left $\mathcal{C}^\infty(M)$ submodule of $\Theta(M)$. Assume in addition that $H^\oplus M$ and $\mathrm{K}M$ are smooth vector bundles on $M$. Then*

$$\mathscr{K}(M) = \Theta(M). \tag{5.9}$$

## 5.3. Hypoellipticity

Subelliptic estimates imply regularity. We have indeed (see [2, Theorem 4.1], [20, Theorem 4.3]):

**Theorem 5.6.** *Let $M$ be a $\nu$-dimensional smooth manifold. Let $U$ be an open subset of $M$, and $Z_1, \dots, Z_m$ complex vector fields on $U$ such that, for some positive constants $C, \epsilon > 0$ (5.4) is valid. If $u \in L^2_{\text{loc}}(U)$, $a_i \in L^\infty_{\text{loc}}(U)$, $f_i \in L^2_{\text{loc}}(U)$ for $i = 1, \dots, n$ satisfy*

$$Z_i u + a_i u = f_i, \quad \text{for } i = 1, \dots, m \quad \text{on } U, \tag{5.10}$$

*then:*

(1) *$u \in W^\epsilon_{\text{loc}}(U)$;*
(2) *if $0 < s \le \frac{m}{2}$, $a_i \in \mathcal{C}^s(U)$ and $f_i \in W^s_{\text{loc}}(U)$, then $u \in W^{s+\epsilon}_{\text{loc}}(U)$;*
(3) *if $s > \frac{m}{2}$, $a_i \in W^s_{\text{loc}}(U)$ and $f_i \in W^s_{\text{loc}}(U)$, then $u \in W^{s+\epsilon}_{\text{loc}}(U)$;*
(4) *in particular, if $a_i \in \mathcal{C}^\infty(U)$, $f_i \in W^s_{\text{loc}}(U)$, then $u \in W^{s+\epsilon}_{\text{loc}}(U)$.*

*Here we indicate by $W^s_{\text{loc}}(U)$ the $L^2$-Sobolev space of order $s$.*

Then we obtain from Lemma 3.6:

**Corollary 5.7.** *If (5.3) holds true, then $\mathfrak{O}^{-\infty}_{M,(p_0)} = \mathfrak{O}^{\infty}_{M,(p_0)}$.*

## 5.4. Trace concave *CR* manifolds

We elaborate the above results for a class of *CR* manifolds for which we obtain a *geometric* characterization of the extension property. A *CR* manifold $M$ is said to be *trace concave at $p \in M$* if for every $\xi \in H^0_p M$ the directional Levi form $\mathfrak{L}_\xi$ is either indefinite or identically zero. A trace concave *CR* manifold is a *CR* manifold which is trace concave at every point. We set

$$\mathscr{G}_1(M) = \mathscr{H}(M),$$

and define inductively for $k \ge 1$

$$\mathscr{G}_{k+1}(M) = \mathscr{G}_k(M) + [\mathscr{H}(M), \mathscr{G}_k(M)].$$

Moreover we set $G_{k,p}M = \{Z(p) : Z \in \mathscr{G}_k(M)\}$. Trace concavity implies that $H^\oplus_p M$ is the annihilator of $G_{2,p}M$. The following result gives a geometric characterization of the extension property under a constant rank condition.

**Theorem 5.8.** *Let $M$ be a trace concave CR manifold, locally CR-embeddable at $p_0 \in M$. Assume that* $\dim G_{k,p} M$ *is constant for all $k$. Then $M$ has the holomorphic extension property at $p_0$ if and only if* $\dim G_{k,p_0} M = \dim M$ *holds for $k$ sufficiently large.*

Note that this result substantially strengthens the main theorem of [3].

*Proof.* The condition is sufficient by Theorems 5.3, 5.6 and 1.4. If $\dim G_{k,p_0} M < \dim M$ for all $k$, these dimensions eventually stabilize and the Frobenius theorem implies that the *CR* orbit through $p_0$ is lower-dimensional. In the same way as at the beginning of the proof of Theorem 1.4, it follows that the holomorphic extension property fails at $p_0$. Hence the condition is necessary. □

In particular we see that for the *CR* manifolds of Theorem 5.8, extension to open wedges and extension to full neighborhoods are equivalent. Several examples of such manifolds arise in the study of homogeneous *CR* manifolds, as outlined in the next section.

## 6. Examples

A large class of examples of *CR* submanifolds of complex manifolds is provided by the orbits of the real forms in complex flag manifolds. We recall that a complex flag manifold is a compact homogeneous space $X$ of a semisimple complex Lie group $\mathbf{G}$. The isotropy of a point of $X$ is a *parabolic* subgroup $\mathbf{Q}$ of $\mathbf{G}$, *i.e.* a closed connected subgroup whose Lie algebra $\mathfrak{q}$ contains a maximal solvable Lie subalgebra $\mathfrak{b}$ of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$. If $\mathbf{G}_0$ is a *real form* of $\mathbf{G}$, *i.e.* a connected real Lie subgroup of $\mathbf{G}_0$ with Lie algebra $\mathfrak{g}_0$ such that $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$, then $\mathbf{G}_0$ has finitely many orbits in $X$. In particular, there are open orbits and a minimal orbit $M$ which is compact (see [39]). The structures of the orbits only depend on the Lie algebras involved, and are therefore completely determined by the pairs $(\mathfrak{g}_0, \mathfrak{q})$, which are called *CR algebras*, consisting of the Lie algebra of the real form $\mathbf{G}_0$ and of the Lie algebra of the parabolic subgroup $\mathbf{Q}$.

The embedding of $M$ in $X$ defines a *CR* structure on $M$. The minimal orbits are classified by their *cross-marked Satake diagrams*. A complete list of these diagrams is given *e.g.* in the appendix to [4]. Many properties of the minimal orbits are read off from these diagrams: minimality is equivalent to the fact that the corresponding CR *algebra* $(\mathfrak{g}_0, \mathfrak{q})$ is fundamental and is described by [4, Theorem 9.3]. In [4, Section 13] all *essentially pseudoconcave* minimal orbits are classified in terms of their associated diagrams. Since essential pseudoconcavity (see [20]) implies (5.3), all these orbits are at every point *CR*-hypoelliptic and therefore have the holomorphic extension property by Theorem1.4. Globally defined *CR* functions on this class of *CR* manifolds and their properties were considered in [1].

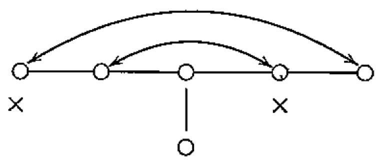
We give below some more explicit examples to illustrate this application.

Let $X$ be the complex flag manifold consisting of the flags

$$\ell_1 \subset \ell_3 \subset \cdots \subset \ell_{2k-1} \subset \ell_{2k+2} \subset \cdots \subset \ell_{4k-2} \subset \mathbb{C}^{4k},$$

where $k$ is an integer $\geq 2$ and $\ell_i$ is a $\mathbb{C}$-linear subspace of dimension $i$ of $\mathbb{C}^{4k}$. Let $M$ be the minimal orbit for the action of the group $\mathbf{SU}(2k, 2k)$ of complex $4k \times 4k$ matrices that leave invariant a Hermitian symmetric form of signature $(2k, 2k)$. Then $M$ has *CR* dimension $2k$ and *CR* codimension $8k^2 - 6k - 1$ and we need $2k$ commutators of $\mathscr{H}(M)$ to span $TM$ (these numbers were computed in [28]). However, $M$ is minimal and essentially pseudoconcave and therefore is *CR*-hypoelliptic and has the holomorphic extension property at all points.

Another example is the minimal orbit of the special group $\mathbf{G}_0$ of type $E_6III$ corresponding to the cross-marked Satake diagram



It corresponds to a *CR* manifold of *CR* dimension 4 and *CR* codimension 25, with 6 commutations needed to span $TM$ from $\mathscr{H}(M)$. This is also essentially pseudoconcave and therefore is *CR*-hypoelliptic and has the holomorphic extension property at each point.

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Dipartimento di Matematica
II Università di Roma "Tor Vergata"
Via della Ricerca Scientifica
00133 Roma, Italy
nacinovi@mat.uniroma2.it

Department of Mathematics
Mid Sweden University
85170 Sundsvall, Sweden
Egmont.Porten@miun.se