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Deformations of constant mean curvature surfaces preserving symmetries and the Hopf differential

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Abstract. We define certain deformations between minimal and non-minimal constant mean curvature (CMC) surfaces in Euclidean space $\mathbb{E}^3$ which preserve the Hopf differential. We prove that, given a CMC H surface f, either minimal or not, and a fixed basepoint z_0 on this surface, there is a naturally defined family f_h, for all $h \in \mathbb{R}$, of CMC h surfaces that are tangent to f at z_0, and which have the same Hopf differential. Given the classical Weierstrass data for a minimal surface, we give an explicit formula for the generalized Weierstrass data for the non-minimal surfaces f_h, and vice versa. As an application, we use this to give a well-defined dressing action on the class of minimal surfaces. In addition, we show that symmetries of certain types associated with the basepoint are preserved under the deformation, and this gives a canonical choice of basepoint for surfaces with symmetries. We use this to define new examples of non-minimal CMC surfaces naturally associated to known minimal surfaces with symmetries.

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1. Introduction

Let $\Sigma \subset \mathbb{C}$ be a simply connected domain. The classical Weierstrass representation for minimal surfaces states that given a pair $(\mathrm{d}\omega,\ \nu)$, where $\mathrm{d}\omega = \mu(z)\mathrm{d}z$ is a holomorphic 1-form and ν a meromorphic function on Σ, and appropriate orders of vanishing, then the formula

$$f = 2\Re \int_{z_0}^{z} f_z \,\mathrm{d}z, \qquad f_z \,\mathrm{d}z = \left(\left(1-\nu^2\right) e_1 - i\left(1+\nu^2\right) e_2 - 2\nu\, e_3\right) \mathrm{d}\omega,$$

gives a minimal surface $f : \Sigma \to \mathbb{R}^3$. Conversely, all minimal immersions of Σ can be obtained this way. This representation is one of the major tools in the study of minimal surface theory.

For non-minimal constant mean curvature (CMC) surfaces, an infinite dimensional analogue to the Weierstrass representation was given in the 1990's by

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Dorfmeister, Pedit and Wu in [8]. If we take the above 1-form $f_z\,\mathrm{d}z$ as the Weierstrass data for a minimal surface, then the analogous coordinate-independent data for a non-minimal surface is the *normalized potential* $\hat{\eta}$, which can be expressed in local coordinates as

$$\hat{\eta} = \begin{pmatrix} 0 & -\frac{H}{2}a(z) \\ \frac{Q(z)}{a(z)} & 0 \end{pmatrix} \lambda^{-1}\mathrm{d}z, \tag{1.1}$$

where Q is a holomorphic function, and a is meromorphic (with appropriate orders of vanishing), and λ is an $\mathbb{S}^1$-parameter. The holomorphic bilinear form $Q\,\mathrm{d}z^2$ is called the Hopf differential and is well-defined independently of coordinates. The surface is obtained by integrating $\hat{\eta}$, performing a loop group decomposition, and applying the Sym-Bobenko formula, a simple formula involving the factor $1/H$. The loop group decomposition is non-trivial to write down explicitly in practice, which means that it is more difficult to use this representation to construct or study CMC surfaces when compared with minimal surfaces.

The 1-forms $f_z\mathrm{d}z$ and $\hat{\eta}$ are unique given a choice of basepoint z_0 on Σ. Although one cannot substitute $H = 0$ directly into the Sym-Bobenko formula, it is a plausible guess that taking the limit as H tends to zero in the potential (1.1) might lead to a minimal surface. The main result of this article is more useful than that, because it includes the converse:

Theorem 1.1. *Let $\Sigma \subset \mathbb{C}$ be a contractible domain.*

(1) *Let $f : \Sigma \to \mathbb{E}^3$ be a conformally immersed minimal surface, and a basepoint $z_0 \in \Sigma$ fixed. Then there is a canonical family of conformally immersed CMC h surfaces $f_h : \Sigma \to \mathbb{E}^3$, all with the same Hopf differential as f, such that $f = f_0$ and all of the maps f_h, together with their tangent planes, agree at the point z_0. The family depends real analytically on h. If $(\mu\mathrm{d}z, \nu)$ are the classical Weierstrass data for f, with coordinates chosen so that $\mu(z_0) = 1$ and $\nu(z_0) = 0$, then the normalized potential for f_h is*

$$\hat{\eta}_h = \begin{pmatrix} 0 & -h\mu \\ -\nu_z & 0 \end{pmatrix} \lambda^{-1}\mathrm{d}z. \tag{1.2}$$

(2) *Conversely, let H be any non-zero real number and $f_H : \Sigma \to \mathbb{E}^3$ be a CMC H immersion. For a given basepoint z_0 let*

$$\hat{\eta}_H = \begin{pmatrix} 0 & -\frac{H}{2}a \\ \frac{Q}{a} & 0 \end{pmatrix} \lambda^{-1}\mathrm{d}z, \tag{1.3}$$

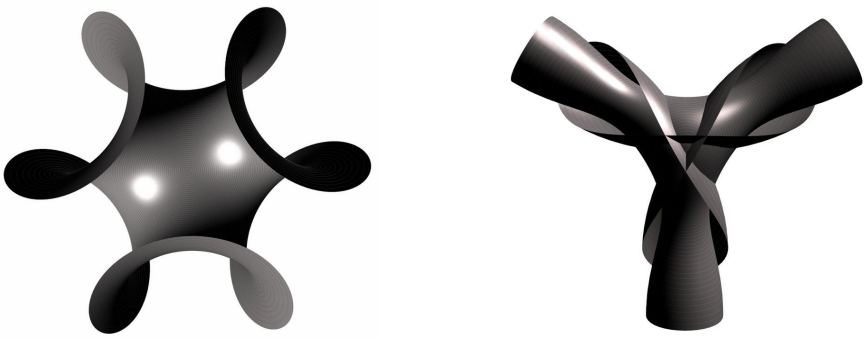
be the associated normalized potential. For any meromorphic function g, let $\mathrm{Ord}(g(z))$ denote the order of vanishing of g at the point z. Let Σ^* be the open dense subset of Σ on which the following conditions are satisfied:

(a) *the function a is holomorphic;*
(b) *at any zero of a we have $\mathrm{Ord}(Q) = (\mathrm{Ord}(a) - 2)/2$.*

Then the surface $f_H\big|_{\Sigma^*} : \Sigma^* \to \mathbb{R}$ *is part of a family* $f_h : \Sigma^* \to \mathbb{E}^3$*, of CMC immersions, for all* $h \in \mathbb{R}$*, and the normalized potential for* f_h *is given by substituting* h *for* H *in* (1.3)*. If coordinates are chosen such that* $a(z_0)$ *is a real number, then the minimal surface* $f_0 : \Sigma^* \to \mathbb{E}^3$ *has the classical Weierstrass data*

$$\mu(z)\mathrm{d}z = \frac{a(z)}{2}\mathrm{d}z, \qquad \nu = -\int_{z_0}^{z} \frac{Q(\tau)}{a(\tau)}\mathrm{d}\tau.$$

The proof of this theorem is given in Section 4.3. In item (2), the map f_0 is defined on the whole of Σ, but may have branch points at poles or zeros of a. Note also that, if the basepoint z_0 is changed to a different basepoint $\tilde{z}_0$, then the resulting family $\tilde{f}_h$ is not the same family as f_h. We also remark that if, in item (1), the data are given such that $\nu(z_0) \neq 0$, then there is an alternative, more general, formula for $\hat{\eta}_h$ given below at (4.13).



$H = 10^{-8}$, $H = 1$

Figure 1.1. Left: Almost minimal version of Enneper's surface of order 2. Right: 3-legged Mr. Bubble, or Smyth surface.

Given a CMC surface or minimal surface, although the associated family depends on the choice of the basepoint, there is sometimes a natural such choice, and therefore a canonical family associated. For example, Enneper's surface of order k (See Example 5.4 below) has a finite order rotational symmetry about a central point. Such a symmetry is preserved under the deformation if the basepoint is chosen to be this central point, and therefore, making this choice, there is one natural family of CMC surfaces that have this symmetry and which includes Enneper's surface at $h = 0$. As we will see below, these turn out to be Smyth surfaces, studied in [12]. Solutions, computed numerically for $H = 10^{-8}$ and $H = 1$, are shown in Figure 1.1.

1.1. The dressing action on minimal surfaces

The dressing action is a group action on the space of solutions, which generally exists for any integrable system represented by maps into loop groups. It was introduced into the study of harmonic maps by Uhlenbeck [13]. See Wu [14] for a description for the case of CMC surfaces. Dressing can be described as an action on the normalized potential $\hat{\eta}$. A minimal surface also has a normalized potential, but, unlike in the non-minimal case, the correspondence is not bijective: there are many minimal surfaces with the same normalized potential. In [7], the dressing action is defined, via these normalized potentials, on minimal surfaces, giving an action on the set of equivalence classes of minimal surfaces with the same potential. In Section 6, we use the analysis of Wu [14] to determine the class of dressing elements that are independent of h, in Theorem 6.1. A corollary of this, together with Theorem 1.1, is that the dressing action defined in [7] in fact gives a well-defined group action on the space of minimal immersions. This is Theorem 6.3.

1.2. Properties preserved under the deformation

The classical Weierstrass data for many minimal surfaces is known. Therefore, Theorem 1.1 can easily be used to construct examples of non-minimal CMC surfaces, with the generalized Weierstrass data given explicitly by (1.2). Clearly, it is of interest to know what properties are preserved as the mean curvature h varies.

Global topological properties are not preserved: it is true that any minimal surface can be represented by Weierstrass data on a contractible domain (the universal cover); and the same holds for any non-minimal CMC surface other than the round sphere. However, in general, any closing properties of the surface will be lost as h varies.

Because the Hopf differential is preserved, it follows (see Remark 4.4) that, not only umbilic points, but also principal curves in the coordinate domain are preserved under the deformation. Moreover, (see Remark 4.5), the values of the principal curvatures at the basepoint are given explicitly by $\kappa_{\pm}(z_0) = \pm\kappa_0 + h$, where $\pm\kappa_0$ are the principal curvatures at z_0 for the minimal surface in the family. This gives a local picture of the deformation around the basepoint.

In Section 7 we investigate surfaces with symmetries. We consider surfaces which have a reflection symmetry about a plane and surfaces with a finite order rotational symmetry about a point in the surface. In the first case, if the plane of symmetry contains the basepoint, then we show that the surface has such a symmetry if and only if coordinates can be chosen such that the Weierstrass data are real-valued along the real line. In the second case, if the rotation point is the basepoint z_0, then we show that the surface has the symmetry if and only if the Weierstrass data have Laurent expansions including only certain powers of z. Consequently, Theore 7.4 states that such symmetries are preserved under this deformation, provided the basepoint is chosen appropriately.

2. The loop group formulation and DPW method

In this section we summarize well known facts about CMC surfaces and their construction via integrable systems methods. The notation and conventions are the same as those used in [2], where more details and references can be found.

2.1. The loop group characterization of CMC maps

Let Σ be a contractible Riemann surface, and suppose $f : \Sigma \to \mathbb{E}^3$ is a conformal immersion with mean curvature H. Choosing conformal coordinates $z = x + iy$, a function $u : \Sigma \to \mathbb{R}$ is defined by the expression $\mathrm{d}s^2 = 4e^{2u}(\mathrm{d}x^2 + \mathrm{d}y^2)$ for the induced metric. The matrices for the first and second fundamental forms I and II, with respect to the coordinates x,y are then:

$$I = \begin{pmatrix} 4e^{2u} & 0 \\ 0 & 4e^{2u} \end{pmatrix}, \qquad II = \begin{pmatrix} 4He^{2u} + Q + \bar{Q} & i(Q - \bar{Q}) \\ i(Q - \bar{Q}) & 4He^{2u} - (Q + \bar{Q}) \end{pmatrix}, \tag{2.1}$$

where $H := e^{-2u}\langle f_{xx} + f_{yy}, N\rangle/8$ is the mean curvature, and $Q := \langle N, f_{zz}\rangle$. The differential 2-form $Q\,\mathrm{d}z^2$ is called the *Hopf differential*.

The Lie algebra $\mathfrak{su}(2)$, is identified with $\mathbb{E}^3$ via the following basis, which is orthonormal with respect to the inner product $\langle X, Y\rangle = -\mathrm{Trace}(XY)/2$:

$$e_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Given a choice of unit normal N, the *coordinate frame* $F : \Sigma \to SU(2)$ is uniquely determined (up to sign) by the conditions

$$\mathrm{Ad}_F e_1 = \frac{f_x}{|f_x|}, \quad \mathrm{Ad}_F e_2 = \frac{f_y}{|f_y|}, \quad \mathrm{Ad}_F e_3 = N. \tag{2.2}$$

Differentiating the expressions $f_z = e^u \mathrm{Ad}_F(e_1 - ie_2)$ and $f_{\bar{z}} = e^u \mathrm{Ad}_F(e_1 + ie_2)$, one obtains the following expression for the connection coefficients $U := F^{-1}F_z$ and $V := F^{-1}F_{\bar{z}}$:

$$U = \frac{1}{2}\begin{pmatrix} u_z & -2He^u \\ Qe^{-u} & -u_z \end{pmatrix}, \qquad V = \frac{1}{2}\begin{pmatrix} -u_{\bar{z}} & -\bar{Q}e^{-u} \\ 2He^u & u_{\bar{z}} \end{pmatrix}. \tag{2.3}$$

If H is constant, we can extend the frame F to a loop group valued map $\hat{F}$ as follows: first extend the Maurer-Cartan form of F to a loop-algebra valued 1-form $\hat{\alpha} := \hat{U}\mathrm{d}z + \hat{V}\mathrm{d}\bar{z}$, where

$$\hat{U} = \frac{1}{2}\begin{pmatrix} u_z & -2He^u\lambda^{-1} \\ Qe^{-u}\lambda^{-1} & -u_z \end{pmatrix}, \qquad \hat{V} = \frac{1}{2}\begin{pmatrix} -u_{\bar{z}} & -\bar{Q}e^{-u}\lambda \\ 2He^u\lambda & u_{\bar{z}} \end{pmatrix}. \tag{2.4}$$

The 1-form $\hat{\alpha}$ satisfies the Maurer-Cartan equation $\mathrm{d}\hat{\alpha}+\hat{\alpha}\wedge\hat{\alpha}=0$ for all $\lambda\in\mathbb{C}\setminus\{0\}$ if and only if the mean curvature H is constant or, equivalently, the Hopf differential is holomorphic.

Now fix a basepoint $z_0\in\Sigma$ and set $E_0:=F(z_0)$. We extend the initial condition E_0 to a twisted loop $\hat{E}_0$ by the formula

$$\hat{E}_0=\begin{pmatrix} A_0 & \lambda B_0 \\ -\lambda^{-1}\bar{B}_0 & \bar{A}_0 \end{pmatrix}, \qquad \text{where } E_0=\begin{pmatrix} A_0 & B_0 \\ -\bar{B}_0 & \bar{A}_0 \end{pmatrix}. \tag{2.5}$$

Now integrating $\hat{\alpha}$ with the initial condition $\hat{F}(z_0)=\hat{E}_0$, one obtains the *extended frame* $\hat{F}:\Sigma\to\Lambda G_\sigma$, a map into the twisted group of loops in $G:=SU(2)$.

If we denote, for a $\Lambda G^{\mathbb{C}}$-valued map $\hat{X}$, the corresponding map into the group $G^{\mathbb{C}}$, obtained by evaluating at the loop value $\lambda=1$, by $X=\hat{X}|_{\lambda=1}$, then the above notation for F and $\hat{F}$ is consistent.

If h is any nonzero real number, and $\lambda_0\in\mathbb{S}^1$, the Sym-Bobenko formula is:

$$\mathscr{S}_{h,\lambda_0}(\hat{F}):=-\frac{1}{2h}\left(2i\lambda\partial_\lambda\hat{F}\,\hat{F}^{-1}\;+\;\hat{F}e_3\hat{F}^{-1}-e_3\right)\Big|_{\lambda=\lambda_0}. \tag{2.6}$$

Note that if $\hat{F}:\Sigma\to\Lambda G_\sigma$ is a smooth map, then $\mathscr{S}_{h,\lambda}(\hat{F}):\Sigma\to\mathfrak{su}(2)=\mathbb{E}^3$ is also smooth. We will mainly use the formula for the case $\lambda_0=1$, and therefore use the notation $\mathscr{S}_h(\hat{F})=\mathscr{S}_{h,1}(\hat{F})$.

If $\hat{F}$ is an extended coordinate frame for a CMC H surface, and $H\neq 0$, then f is retrieved by the formula

$$f(z)=\mathscr{S}_H\left(\hat{F}(z)\right)+f(z_0). \tag{2.7}$$

Moreover, the Sym-Bobenko formula is invariant under right multiplication by a diagonal unitary matrix valued function. This corresponds to a change of $SU(2)$ frame for the Gauss map. Hence there is a well defined lift $[\hat{F}]:\Sigma\to\Lambda G_\sigma/K$, where $K=G_\sigma$ is the diagonal subgroup, of any CMC H immersion f of a simply connected surface, independent of coordinates and choice of frame; and, if $H\neq 0$, the formula $\mathscr{S}_H([\hat{F}])$ is well defined and gives f up to a translation. The case $H=0$ will be discussed below.

Slightly more generally, define an *admissible frame* to be any smooth map $\hat{F}$, from a Riemann surface Σ into ΛG_σ with the property that the Maurer-Cartan form $\hat{F}^{-1}\mathrm{d}\hat{F}$ is a Laurent polynomial of the form $\hat{\alpha}=\alpha_{-1}\lambda^{-1}+\alpha_0+\alpha_1\lambda$ where the (0, 1) part of α_{-1} is zero. The reality condition and twisting on ΛG_σ mean we can write

$$\hat{\alpha}=A\lambda^{-1}\mathrm{d}z+\alpha_0+\bar{A}\lambda\mathrm{d}\bar{z},$$

where A is an off-diagonal $\mathfrak{su}(2)$-valued function and α_0 is a diagonal $\mathfrak{su}(2)$-valued 1-form. The admissible frame is *regular* if the upper right component A_{12} is non-vanishing. For $H\neq 0$, the extended coordinate frame described above is a regular admissible frame.

Finally, for *any* regular admissible frame $\hat{F}$ and any value of $(h, \lambda) \in \mathbb{R}^* \times \mathbb{S}^1$, the map $f = \mathscr{S}_{h,\lambda}(\hat{F})$ is a conformal CMC h immersion into $\mathbb{R}^3$.

2.2. The DPW construction

Let $\Lambda G_\sigma^{\mathbb{C}}$ denote the group of twisted loops in $G^{\mathbb{C}} = SL(2, \mathbb{C})$ and $\Lambda^+ G_\sigma^{\mathbb{C}}$ and $\Lambda^- G_\sigma^{\mathbb{C}}$ the subgroups of loops which extend holomorphically to the unit disc and the exterior disc $\{\lambda \mid |\lambda| > 1\}$ in the Riemann sphere respectively. For the purpose of normalizations, we also use the subgroups

$$\Lambda_*^- G_\sigma^{\mathbb{C}} := \left\{ B \in \Lambda^- G_\sigma^{\mathbb{C}} \;\middle|\; B(\infty) = I \right\},$$
$$\Lambda_P^+ G_\sigma^{\mathbb{C}} := \left\{ B \in \Lambda^+ G_\sigma^{\mathbb{C}} \;\middle|\; B(0) = \operatorname{diag}(\rho, \rho^{-1}),\ \rho \in \mathbb{R},\ \rho > 0 \right\}.$$

The *Birkhoff decomposition* [11] states that any g in a certain open dense subset of $\Lambda G_\sigma^{\mathbb{C}}$ (called the *big cell*) has a unique factorization

$$g = g_- g_+, \qquad g_- \in \Lambda_*^- G_\sigma^{\mathbb{C}},\ \ g_+ \in \Lambda^+ G_\sigma^{\mathbb{C}}. \tag{2.8}$$

The *Iwasawa decomposition* [11] states that any g in $\Lambda G_\sigma^{\mathbb{C}}$ can be uniquely expressed as a product

$$g = FB, \qquad F \in \Lambda G_\sigma,\ \ B \in \Lambda_P^+ G_\sigma^{\mathbb{C}}. \tag{2.9}$$

In both decompositions, the factors on the right hand side depend real analytically on g. If one takes $g_- \in \Lambda^- G_\sigma^{\mathbb{C}}$ instead of $\Lambda_*^- G_\sigma^{\mathbb{C}}$, and $B \in \Lambda^+ G_\sigma^{\mathbb{C}}$ instead of in $\Lambda_P^+ G_\sigma^{\mathbb{C}}$, then the factors in the decompositions are only unique up to a middle term which is a constant loop.

A brief version of the DPW method (see [6, 8]) states the following: let $\hat{F} : \Sigma \to \Lambda G_\sigma$ be an extended frame for a non-minimal CMC H immersion $f : \Sigma \to \mathbb{E}^3$, where Σ is a contractible Riemann surface. Assume $\hat{F}(z_0) = \hat{E}_0$, of the form (2.5), at some fixed basepoint z_0. The coordinate frame $\hat{F}$ is uniquely determined by z_0 and E_0. Hence a unique meromorphic map $\hat{\Phi} : \Sigma \to \Lambda G_\sigma^{\mathbb{C}}$ is defined by the normalized Birkhoff decomposition, performed pointwise over the pre-image $\Sigma^\circ := \{z \in \Sigma \mid \hat{E}_0^{-1} \hat{F}(z) \in \Lambda_*^- G_\sigma^{\mathbb{C}} \cdot \Lambda^+ G_\sigma^{\mathbb{C}}\}$ of the big cell:

$$\hat{E}_0^{-1} \hat{F} = \hat{\Phi} \hat{G}_+, \qquad \hat{\Phi}(z) \in \Lambda_*^- G_\sigma^{\mathbb{C}}, \qquad \hat{G}_+(z) \in \Lambda^+ G_\sigma^{\mathbb{C}}.$$

For the rest of this section and the next, we take $\hat{E}_0 = I$, to simplify the expressions. This has no effect on the geometry - a change of $\hat{E}_0$ amounts to an isometry of the ambient space $\mathbb{E}^3$.

Note that it is simple to check that $\hat{\Phi}$ is holomorphic on Σ°, and it is shown in [8] that this map has only poles at the boundary of this open dense set. Moreover, in conformal coordinates $z = x + iy$, the Maurer-Cartan form of $\hat{\Phi}$ has the form:

$$\hat{\eta} = \hat{\Phi}^{-1} \mathrm{d}\hat{\Phi} = \begin{pmatrix} 0 & -\frac{H}{2} a \\ \frac{Q}{a} & 0 \end{pmatrix} \lambda^{-1} \mathrm{d}z,$$

where $a(z)$ is meromorphic and $Q(z)\mathrm{d}z^2$ is the (holomorphic) Hopf differential of f. The 1-form $\hat{\eta}$ is called a *normalized potential* for f, and $\hat{\Phi}$ is called a *normalized meromorphic frame*. An explicit formula for the normalized potential, in terms of the metric and Hopf differential, is given by Wu in [15].

Conversely, given a pair of functions (a, Q), with a meromorphic and Q holomorphic, if the zeros and poles have certain sufficient (and necessary) conditions, then the formula above for $\hat{\eta}$ is a meromorphic potential for a CMC surface [3]. The surface is uniquely determined by $\hat{\eta}$ and the basepoint z_0. The most straightforward condition on the poles and zeros of $\hat{\eta}$ is that a is holomorphic and non-vanishing, but in general one has:

Theorem 2.1 ([3]). *Let* $\mathrm{Ord}(Q)$ *denote the vanishing order of* Q *at a point. Then necessary and sufficient conditions for* $\hat{\eta}$ *to correspond to a smooth surface around the point are:*

(1) *If* $\mathrm{Ord}(a) < 0$ *then* $\mathrm{Ord}(a) = -2$ *or, for some integer* $r \geq 1$, *either*

$$\mathrm{Ord}(Q) = \frac{-\mathrm{Ord}(a)}{2r} - 2, \quad \text{or} \quad \mathrm{Ord}(Q) = \frac{-\mathrm{Ord}(a) - 2}{2r} - 2;$$

(2) *If* $\mathrm{Ord}(a) > 0$, *then, for some integer* $r \geq 1$,

$$\mathrm{Ord}(Q) = \frac{\mathrm{Ord}(a)}{2r} - 2, \quad \text{or} \quad \mathrm{Ord}(Q) = \frac{\mathrm{Ord}(a) + 2}{2r} - 2.$$

Note that the above conditions also ensure that the potential is meromorphically integrable, because they rule out the possibility that $\hat{\eta}$ has a pole of order 1. We further remark that if a has a zero and $\mathrm{Ord}(Q) \geq \mathrm{Ord}(a)$, so that $\hat{\eta}$ is holomorphic at the point, then the surface has a branch point.

If $\hat{\eta}$ is *holomorphic*, then a frame $\hat{F}$ is recovered as follows: solve the equation $\mathrm{d}\hat{\Phi} = \hat{\Phi}\hat{\eta}$, with $\hat{\Phi}(z_0) = I$. For each z perform the unique Iwasawa decomposition

$$\hat{\Phi} = \hat{F}\,\hat{B}_+, \qquad \hat{F}(z) \in \Lambda G_\sigma, \qquad \hat{B}_+(z) \in \Lambda_P^+ G_\sigma^{\mathbb{C}}.$$

Then $\hat{F}$ is an extended frame for a CMC H surface $f = \mathscr{S}_H(\hat{F})$. This is not the coordinate frame for f in general (assuming a is non-vanishing so that f is immersed), but represents the same map $[\hat{F}] : \Sigma \to \Lambda G_\sigma / K$, and therefore the same surface.

If $\hat{\eta}$ has poles, then one can prove that the surface f is also immersed at a pole of $\hat{\eta}$. To do this, one needs to perform dressing first (see [3]).

3. From CMC surfaces to minimal surfaces

Let Σ be a contractible Riemann surface, and $f_H : \Sigma \to \mathbb{E}^3$ a conformal immersion of constant mean curvature $H \neq 0$. Choose a base point z_0, and coordinates for $\mathbb{E}^3$

so that the coordinate frame satisfies $F(z_0) = E_0 = I$. There is associated a unique normalized potential $\hat{\eta}$, as described above. Let us now consider H as a real parameter, and write

$$\hat{\eta}_h = \hat{\Phi}^{-1}\mathrm{d}\hat{\Phi} = \begin{pmatrix} 0 & -\frac{h}{2}a \\ \frac{Q}{a} & 0 \end{pmatrix} \lambda^{-1}\mathrm{d}z,$$

and denote by $\hat{\Phi}_h$ the associated normalized meromorphic frame with $\hat{\Phi}_h(z_0) = I$.

The conditions on the zeros and poles of the potential for it to correspond to a smooth surface (Theorem 2.1) do not depend on h, but only on a and Q. These conditions are necessarily satisfied, since $\hat{\eta}_H$ came from an immersed CMC surface. Hence, there is a smooth CMC immersion $f_h : \Sigma \to \mathbb{E}^3$ corresponding to $\hat{\eta}_h$ for every $h \neq 0$. Given the choice of basepoint z_0, the initial conditions $\hat{F}_{C,h}(z_0) = I$ for the coordinate frame, and $f_h(z_0) = f_H(z_0)$ for the surface, f_h is uniquely determined by the formula

$$f_h(z) = \mathscr{S}_h\left(\hat{F}_{C,h}(z)\right) + f_H(z_0).$$

If we restrict to the preimage of the big cell, we also have a unique Birkhoff decomposition

$$\hat{F}_{C,h} = \hat{\Phi}_h \hat{G}_{h,+}, \qquad \hat{G}_{h,+} \in \Lambda^+ G_\sigma^{\mathbb{C}}.$$

Thus we have a family $f_h : \Sigma \to \mathbb{E}^3$ of immersed surfaces of constant mean curvature h, all with the same Hopf differential, $Q\mathrm{d}z^2$, and such that f_h and f_H, together with their tangent planes, agree at z_0. Moreover, the family f_h depends real analytically on h, because h appears analytically in the data and all the operations performed to obtain f_h preserve this property. We now show that this family includes the value $h = 0$:

Theorem 3.1. *Let f_H be as above, with extended coordinate frame $\hat{F}_C$. Let Σ° denote the pre-image under $\hat{F}_C$ of the big cell,* i.e. *the open dense set*

$$\Sigma^\circ := \left\{ z \in \Sigma \mid \hat{F}_C(z) \in \Lambda_*^- G_\sigma^{\mathbb{C}} \cdot \Lambda^+ G_\sigma^{\mathbb{C}} \right\}.$$

(1) *The map $\mathscr{F} : \Sigma \times \mathbb{R}^* \to \mathbb{E}^3$, given by $\mathscr{F}(z, h) = f_h(z)$, extends to a real analytic map $\Sigma \times \mathbb{R} \to \mathbb{E}^3$.*
(2) *The map $f_0\big|_{\Sigma^\circ} : \Sigma^\circ \to \mathbb{E}^3$ given by restricting $\mathscr{F}(z, 0)$ to Σ° is a conformally immersed minimal surface with Hopf differential $Q\mathrm{d}z^2$ and metric given by*

$$\mathrm{d}s^2 = \left(1 + |g|^2\right)^2 |a|^2 \left(\mathrm{d}x^2 + \mathrm{d}y^2\right), \qquad g(z) = \int_{z_0}^{z} \frac{Q(\tau)}{a(\tau)}\mathrm{d}\tau. \tag{3.1}$$

The map f_0, together with its tangent plane, agrees with f_H at z_0.

Proof.
Item 1: The idea of the argument is to get two expressions for $f_h(z)$: one that is very explicit, but only defined on an open dense set; and another that is not so explicit, but defined everywhere.

We give the argument first on Σ°, which is an open dense set, and then extend to Σ. On Σ° we can use, instead of the coordinate frame, the unique smooth frame $\hat{F}_h$ given by the following Iwasawa decomposition

$$\hat{\Phi}_h = \hat{F}_h \hat{B}_{h,+}, \quad \hat{B}_{h,+}(z) \in \Lambda_P^+ G_\sigma^{\mathbb{C}}. \tag{3.2}$$

Note that $\mathscr{S}_h(\hat{F}_h)$ has the same value whether one uses this frame or the coordinate frame. This frame has the advantage that it can be computed explicitly at $h = 0$, where the potential

$$\hat{\eta}_0 = \begin{pmatrix} 0 & 0 \\ \frac{Q}{a} & 0 \end{pmatrix} \lambda^{-1} \mathrm{d}z$$

can be integrated to obtain

$$\hat{\Phi}_0 = \begin{pmatrix} 1 & 0 \\ \lambda^{-1} g & 1 \end{pmatrix}, \quad g(z) = \int_{z_0}^{z} \frac{Q(\tau)}{a(\tau)} \mathrm{d}\tau.$$

The Iwasawa decomposition (3.2) is

$$\hat{\Phi}_0 = \hat{F}_0 \hat{B}_+, \quad \hat{F}_0 = \frac{1}{\sqrt{1+|g|^2}} \begin{pmatrix} 1 & -\lambda \bar{g} \\ \lambda^{-1} g & 1 \end{pmatrix}, \quad \hat{B}_+(z) \in \Lambda_P^+ G_\sigma^{\mathbb{C}}. \tag{3.3}$$

Now f_h, which is real analytic in h on $\mathbb{R}^*$, is given by

$$\begin{aligned} f_h &= \mathscr{S}_h\left(\hat{F}_h\right) + f_H(z_0) \\ &= -\frac{1}{2h} \left(2i\lambda \frac{\partial \hat{F}_h}{\partial \lambda} \hat{F}_h^{-1} + \mathrm{Ad}_{\hat{F}_h} e_3 - e_3 \right)\Bigg|_{\lambda=1} + f_H(z_0). \end{aligned}$$

$f_H(z_0)$ is constant, so we only need consider the first term. By construction, the expression inside the parentheses is analytic for all h, and therefore has an expansion in h, around $h = 0$, given by

$$\left(2i\lambda \frac{\partial \hat{F}_h}{\partial \lambda} \hat{F}_h^{-1} + \mathrm{Ad}_{\hat{F}_h} e_3 - e_3 \right)\Bigg|_{\lambda=1} = C_0 + O(h).$$

Analyticity of f_h at $h = 0$ will follow if we can show that $C_0 = 0$. But $\hat{F}_h = \hat{F}_0 + O(h)$, and so

$$C_0 = \left(2i\lambda \frac{\partial \hat{F}_0}{\partial \lambda} \hat{F}_0^{-1} + \mathrm{Ad}_{\hat{F}_0} e_3 - e_3 \right)\Bigg|_{\lambda=1}. \tag{3.4}$$

It is easy to verify that this expression is zero for any loop $\hat{F}_0$ in ΛG_σ of the form

$$\begin{pmatrix} A & -\lambda\bar{B} \\ \lambda^{-1}B & \bar{A} \end{pmatrix},$$

and $\hat{F}_0$, given at (3.3), is indeed of this form. Thus, $C_0 = 0$.

Finally we must consider points z on the boundary of Σ°, that is, points where $\hat{\eta}$ has a pole. In this case let us consider the extended coordinate frame $\hat{F}_{C,h}$. For $h \neq 0$, this frame is well defined on the whole of Σ, because it is constructed from the coordinate frame of a smooth surface. Moreover, it is analytic in all parameters. On the pre-image of the big cell, Σ°, we have the relation

$$\hat{F}_{C,h} = \hat{F}_h \begin{pmatrix} \mu & 0 \\ 0 & \bar{\mu} \end{pmatrix},$$

for some unitary function μ, constant in λ. Hence, for $z \in \Sigma^\circ$, one obtains:

$$\begin{aligned} \left(2i\lambda\partial_\lambda \hat{F}_{C,h}\, \hat{F}_{C,h}^{-1} + \mathrm{Ad}_{\hat{F}_{C,h}} e_3 - e_3\right)\Big|_{\lambda=1} &= \left(2i\lambda \frac{\partial \hat{F}_h}{\partial\lambda}\, \hat{F}_h^{-1} + \mathrm{Ad}_{\hat{F}_h} e_3 - e_3\right)\Bigg|_{\lambda=1} \\ &= C_0 + O(h). \end{aligned}$$

We have already shown that $C_0(z) = 0$ for $z \in \Sigma^\circ$. Since C_0 is continuous (in fact analytic) in z, and Σ° is open and dense, it must vanish everywhere. By a similar argument to that given above on Σ°, applied now to the expression involving $\hat{F}_{C,h}$ on Σ, it follows that, for all z, the map $f_h(z)$ is analytic in h at $h = 0$.

Item 2: First note that the extended coordinate frame $\hat{F}_{C,h}$, for $h \neq 0$, is in the big cell for all $z \in \Sigma^\circ$. Otherwise, $\hat{\eta}_h$ would have a pole on Σ°: but this condition is independent of h, and $\hat{\eta}_H$ has no poles on this set. Now, Birkhoff decomposing $\hat{F}_{C,h}$ pointwise we have $\hat{F}_{C,h} = \hat{\Phi}_h \hat{H}_{+,h}$, for a unique real-analytic $\Lambda^+ G_\sigma^{\mathbb{C}}$-valued map $\hat{H}_{+,h}$, which has a Fourier expansion in λ of the form

$$\hat{H}_{+,h} = \begin{pmatrix} \rho_h^{-1} & 0 \\ 0 & \rho_h \end{pmatrix} + O(\lambda).$$

It follows that $\hat{U}_h := \hat{F}_{C,h}^{-1}(\hat{F}_{C,h})_z$ is of the form

$$\hat{U}_h = \begin{pmatrix} * & -\rho_h^2 \dfrac{ha}{2}\lambda^{-1} \\ \rho_h^{-2}\dfrac{Q}{a}\lambda^{-1} & * \end{pmatrix}.$$

Differentiating the formula $f_h(z) = \mathscr{S}_h(\hat{F}_{C,h}(z)) + f(z_0)$, we obtain

$$\frac{\mathrm{d}f_h}{\mathrm{d}z} = \frac{1}{2} a\rho_h^2 \mathrm{Ad}_{F_{C,h}}(e_1 - ie_2).$$

Similarly, since the reality condition on the loop group means that $\hat{V}_h := \hat{F}_{C,h}^{-1}(\hat{F}C, h)_{\bar{z}} = -\overline{\hat{U}_h}^t$ one computes $\frac{\mathrm{d}f_h}{\mathrm{d}\bar{z}} = \frac{1}{2}a\rho_h^2 \mathrm{Ad}_{F_{C,h}}(e_1 + ie_2)$, so that

$$\frac{\mathrm{d}f_h}{\mathrm{d}x} = a\rho_h^2 \mathrm{Ad}_{F_{C,h}}(e_1), \qquad \frac{\mathrm{d}f_h}{\mathrm{d}y} = a\rho_h^2 \mathrm{Ad}_{F_{C,h}}(e_2).$$

We want to take the limit as $h \to 0$. For this consider the normalized frame $\hat{F}_h$, given by (3.2). This has the Birkhoff decomposition $\hat{F}_h = \hat{E}_0 \hat{\Phi}_h \hat{B}_{h,+}^{-1}$, with the Fourier expansion

$$\hat{B}_{h,+}^{-1} = \begin{pmatrix} \tilde{\rho}_h^{-1} & 0 \\ 0 & \tilde{\rho}_h \end{pmatrix} + O(\lambda), \qquad \tilde{\rho}_h(z) \in \mathbb{R}_{>0}.$$

Since the factor $\hat{X} \in \Lambda G_\sigma$ of any Iwasawa decomposition $\hat{\Phi}_h = \hat{X}\hat{B}_+$ is unique up to right multiplication by a constant (in λ) matrix, the values $\hat{B}_{h,+}^{-1}(z)$ and $\hat{H}_{h,+}(z)$ are necessarily related by left multiplication by such a matrix, and we have

$$\rho(z) = \tilde{\rho}(z)e^{i\theta_h(z)}, \qquad \theta_h(z) \in \mathbb{R}.$$

Thus we have

$$\left\| \frac{\partial f_h}{\partial x} \right\| = \left\| \frac{\partial f_h}{\partial y} \right\| = \tilde{\rho}_h^2 |a|.$$

Now by the explicit Iwasawa decomposition (3.3) of $\hat{\Phi}_0$, we have, at $h = 0$,

$$\tilde{\rho}_0 = \sqrt{1 + |g|^2}, \qquad g(z) = \int_{z_0}^{z} \frac{Q(\tau)}{a(\tau)} \mathrm{d}\tau.$$

It follows that

$$\left\| \frac{\partial f_0}{\partial x} \right\| = \left\| \frac{\partial f_0}{\partial y} \right\| = (1 + |g|^2)|a|. \tag{3.5}$$

Together with the fact that $\partial_x f_h$ and $\partial_y f_h$ are orthogonal for all $h \neq 0$, this implies that f_0 is conformally immersed on Σ°, with metric given by (3.1). The fact that the mean curvature is zero and the Hopf differential is $Q\mathrm{d}z^2$ follow by continuity with respect to the parameter h. □

Remark 3.2. The above proof shows, in fact, that f_0 is conformally immersed on the set where $(1 + |g|^2)|a|$ is finite and non-vanishing. This set includes Σ°, but would be larger in general.

4. From minimal surfaces to CMC surfaces

The connection between the loop group formulation for CMC surfaces and the classical Weierstrass representation for minimal surfaces was investigated by Dorfmeister, Pedit and Toda [7]. More details of the setup used here can be found (with slightly different conventions) in that reference.

4.1. The classical Weierstrass representation in terms of the $SU(2)$-frame

Choosing the symmetric space representation $\mathbb{S}^2 = SU(2)/\mathbb{S}^1$, where $\mathbb{S}^1$ is the diagonal subgroup, and the complex structure given by

$$\mathfrak{p}^{\mathbb{C}} = \mathbb{C}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus \mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = T_0^{(1,0)}\mathbb{S}^2 \oplus T_0^{(0,1)}\mathbb{S}^2,$$

a map $g : \Sigma \to \mathbb{S}^2$ is holomorphic if and only if any lift F into $SU(2)$ satisfies $F^{-1}\mathrm{d}F = \alpha'_{\mathfrak{p}} + \alpha_{\mathfrak{k}} + \alpha''_{\mathfrak{p}}$, with $\alpha'_{\mathfrak{p}}$ a $T_0^{(1,0)}\mathbb{S}^2$-valued 1-form. Hence the expression (2.3) shows that a conformally immersed surface in $\mathbb{E}^3$ is minimal if and only if its Gauss map is holomorphic.

The classical Weierstrass representation for the surface is obtained as follows: if F_C is the coordinate frame defined by (2.2), we can write

$$F_C = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}, \qquad F_C^{-1}\partial_z F_C = \frac{1}{2}\begin{pmatrix} u_z & 0 \\ Qe^{-u} & -u_z \end{pmatrix}.$$

We deduce from this that

$$A = e^{-u/2}s, \quad B = e^{-u/2}\bar{r},$$

for a pair of holomorphic functions s and r which satisfy

$$e^u = s\bar{s} + r\bar{r}, \qquad Q = 2(rs_z - sr_z).$$

Since F_C is the coordinate frame, we have $f_z = e^u \mathrm{Ad}_F(e_1 - ie_2)$, which works out to

$$f_z = \left(s^2 - r^2\right) e_1 - i\left(s^2 + r^2\right) e_2 - 2sr\, e_3, \tag{4.1}$$

and the Weierstrass representation is

$$f = 2\Re \int_{z_0}^{z} f_z \mathrm{d}z. \tag{4.2}$$

The commonly used Weierstrass representation states that, given a holomorphic function μ and a meromorphic function ν on Σ, such that $\mu\nu^2$ is holomorphic, then a minimal surface $f : \Sigma \to \mathbb{R}^3$ is given by the above integral, with

$$f_z = \mu\left(1 - \nu^2\right) e_1 - i\mu\left(1 + \nu^2\right) e_2 - 2\mu\nu\, e_3. \tag{4.3}$$

(The conventions used here are chosen here for convenience.) The surface is regular at points where either

(1) $\mathrm{Ord}(\mu) = 0$ and $\mathrm{Ord}(\nu) \geq 0$, or
(2) $0 \leq \mathrm{Ord}(\mu) = -2\mathrm{Ord}(\nu)$.

Comparing this with our data, we have: $s^2 = \mu, r^2 = \mu\nu^2, sr = \mu\nu$. Thus, given classical Weierstrass data μ and ν, the coordinate frame above is well defined on the set Σ: the function μ has zeros only of even order and, up to an irrelevant sign, we can solve for

$$s = \sqrt{\mu}, \qquad r = \nu\sqrt{\mu}, \tag{4.4}$$

and the metric and Hopf differential are given by

$$e^u = |\mu|(1 + |\nu|^2), \qquad Q = -2\mu\nu_z. \tag{4.5}$$

Note that s and r are both holomorphic, and so no meromorphic functions are used in this alternative representation.

4.2. The loop group frames for a minimal surface

Comparing the extended coordinate frame $\hat{F}_C$ of a CMC surface with the normalized potential $\hat{\eta}$, one deduces that the surface is minimal if and only if the normalized potential is of a simple form:

$$\hat{\eta} = \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} \lambda^{-1} \mathrm{d}z,$$

where p is some meromorphic function.

For the coordinate frame F_C derived above, the extended frame $\hat{F}_C$, defined by (2.4), has the simple expression:

$$\hat{F}_C = e^{-u/2} \begin{pmatrix} s & \lambda\bar{r} \\ -\lambda^{-1} r & \bar{s} \end{pmatrix} = \frac{1}{\sqrt{|\mu|(1+|\nu|^2)}} \begin{pmatrix} \sqrt{\mu} & \lambda\overline{\nu\sqrt{\mu}} \\ -\lambda^{-1}\nu\sqrt{\mu} & \overline{\sqrt{\mu}} \end{pmatrix}. \tag{4.6}$$

On the other hand, as we showed for $\hat{\eta}_0$ in the proof of Theorem 3.1, we can, on the set Σ° on which p has no poles, integrate $\hat{\eta}$ explicitly, with the initial condition $\hat{\Phi}(z_0) = I$, and perform an Iwasawa decomposition to obtain another frame

$$\hat{F} = \hat{E}_0 \frac{1}{\sqrt{1+|q|^2}} \begin{pmatrix} 1 & -\lambda\bar{q} \\ \lambda^{-1} q & 1 \end{pmatrix}, \qquad \hat{E}_0 = \hat{F}_C(z_0) = \begin{pmatrix} A_0 & B_0\lambda \\ -\bar{B}_0\lambda^{-1} & \bar{A}_0 \end{pmatrix},$$

where $q(z) = \int_{z_0}^{z} p(\tau)\mathrm{d}\tau$. The integral is well defined because any poles of p, on Σ, are assumed to be of even order. This amounts to:

$$\hat{F} = \frac{1}{\sqrt{1+|q|^2}} \begin{pmatrix} A_0 + B_0 q & \lambda(-A_0\bar{q} + B_0) \\ \lambda^{-1}(\bar{A}_0 q - \bar{B}_0) & \bar{A}_0 + \bar{B}_0\bar{q} \end{pmatrix}. \tag{4.7}$$

The frame $\hat{F}$ differs from $\hat{F}_C$ by right multiplication by a map into $\mathbb{S}^1$. In other words

$$\hat{F}_C = \hat{F} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

As pointed out previously (at (3.4)), we have $\mathscr{S}_h(\hat{F}) = 0$ for the type of loop given at (4.7), so a minimal surface is not likely to be obtained from its extended frame by a variant of the Sym-Bobenko formula. This is consistent with the fact that, unlike a non-minimal CMC surface, a minimal surface is not determined by its Gauss map – one needs to know the Hopf differential as well.

Comparing $\hat{F}_C$ and $\hat{F}$, we see that the ratio $\nu/(-1)$ is the same as $(\bar{A}_0 q - \bar{B}_0)/(A_0 + B_0 q)$ which is solved to get

$$q = \frac{\bar{B}_0 - A_0\nu}{\bar{A}_0 + B_0\nu}, \tag{4.8}$$

and hence a formula for $p = q_z$ in terms of μ and ν. We summarize this as:

Proposition 4.1. *Let $\Sigma \subset \mathbb{C}$ be a simply connected domain, and $f : \Sigma \to \mathbb{R}^3$ a minimal immersion with classical Weierstrass data μ and ν. Choose any basepoint $z_0 \in \Sigma$. Let $\hat{F}_C$ be the extended coordinate frame given by* (4.6), *and* (2.2), *denoting the initial data for $F_C(z_0)$ by*

$$E_0 = \begin{pmatrix} A_0 & B_0 \\ -\bar{B}_0 & \bar{A}_0 \end{pmatrix}, \quad A_0 = \frac{\sqrt{\mu_0}}{\sqrt{|\mu_0|(|\nu_0|^2+1)}}, \quad B_0 = \frac{\overline{\nu_0\sqrt{\mu_0}}}{\sqrt{|\mu_0|(|\nu_0|^2+1)}}. \tag{4.9}$$

Then the Hopf differential and normalized potential for $\hat{F}$ are given by

$$Q = -2\mu\nu_z, \quad \hat{\eta} = \begin{pmatrix} 0 & 0 \\ p\lambda^{-1} & 0 \end{pmatrix} \mathrm{d}z, \quad p = -\frac{\nu_z}{(\bar{A}_0 + B_0\nu)^2}. \tag{4.10}$$

Conversely: Let Q be a holomorphic function and p a meromorphic function on Σ, such that:

(1) $\mathrm{Ord}(p) \neq -1$;
(2) *Q/p is holomorphic, and we have* $\mathrm{Ord}(Q)(z) \geq -\mathrm{Ord}(p)(z) - 2$ *at any pole z of p.*

Let $z_0 \in \Sigma \setminus \{\text{poles of } p\} \cup \{\text{zeros of } Q/p\}$. Set

$$q = \int_{z_0}^{z} p(\tau)\mathrm{d}\tau, \qquad \bar{A}_0 := \left(\frac{Q(z_0)}{p(z_0)}\frac{\bar{p}(z_0)}{\bar{Q}(z_0)}\right)^{1/4}$$

and

$$\nu := -\bar{A}_0^2 q, \qquad \mu := \frac{Q}{2p}\bar{A}_0^{-2}.$$

Then μ and ν are the Weierstrass data for a unique minimal surface (possibly with branch points) $f : \Sigma \to \mathbb{R}^3$, given by the Weierstrass formulae (4.2) *and* (4.3). *The*

surface is regular at any point where either

(i) $\mathrm{Ord}(Q) = \mathrm{Ord}(p)$, *or*
(ii) $\mathrm{Ord}(Q) = -\mathrm{Ord}(p) - 2$, *where* $\mathrm{Ord}(p) \leq -2$.

The Hopf differential is given by $Q\mathrm{d}z^2$, *the coordinate frame for* f *has initial condition*

$$F(z_0) = \begin{pmatrix} A_0 & 0 \\ 0 & \bar{A}_0 \end{pmatrix}, \tag{4.11}$$

and the normalized potential for $\hat{F}$ *is given by*

$$\begin{pmatrix} 0 & 0 \\ p\lambda^{-1} & 0 \end{pmatrix} \mathrm{d}z.$$

Proof. The formulae at (4.10) for Q and p in terms of μ and ν have already been derived, since p is obtained by differentiating the formula (4.8) for q.

For the converse, first note that q is well defined because of the assumption (4.1) on the poles of p. Next, ν is meromorphic, μ is holomorphic and the assumption (4.1) that Q/p is holomorphic implies that, if p has a zero at a point then $\mathrm{Ord}(\mu\nu^2) = \mathrm{Ord}(-Qq^2/(2p)) = \mathrm{Ord}(Q) + \mathrm{Ord}(p) + 2 > 0$ at this point. On the other hand if p has a pole then the assumption on the orders of vanishing implies that $\mathrm{Ord}(\mu\nu^2) \geq 0$. Thus $\mu\nu^2$ is holomorphic. Hence μ and ν can be taken as the Weierstrass data for a minimal surface $f : \Sigma \to \mathbb{R}^3$. The surface is regular at points where $\mathrm{Ord}(\mu) = 0$ and $\mathrm{Ord}(\nu) \geq 0$, or $\mathrm{Ord}(\mu) = -2\mathrm{Ord}(\nu) \geq 0$, which translates to the conditions (i) and (ii).

Finally, we have $\nu_0 = \nu(z_0) = 0$, and $\mu_0 = \mu(z_0) = Q(z_0)/(2\bar{A}_0^2 p(z_0))$, and the initial condition E_0 from (4.9) is that stated at (4.11). By the first part of the Theorem, the corresponding normalized potential and the Hopf differential for f are given by the formula at (4.10), as

$$\hat{\eta} = \begin{pmatrix} 0 & 0 \\ \tilde{p}\lambda^{-1} & 0 \end{pmatrix} \mathrm{d}z, \quad \tilde{p} = -\frac{\nu_z}{(\bar{A}_0 + B_0\nu)^2} = -\frac{-\bar{A}_0^2 p}{\bar{A}_0^2} = p,$$

and $\tilde{Q} = -2\mu\nu_z = (Q/(p))\bar{A}_0^{-2}(q_z\bar{A}_0^2) = Q$. □

4.3. Proof of Theorem 1.1

Consider the meromorphic data $(\hat{\eta}_0, Q)$, where

$$\hat{\eta}_0 = \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} \lambda^{-1}\mathrm{d}z,$$

associated, by Proposition 4.1, to a minimal immersion $f : \Sigma \to \mathbb{E}^3$. The data is unique given a basepoint z_0. After a translation and rotation of $\mathbb{E}^3$, and a simple change of conformal coordinates, we may assume that

$$\mu(z_0) = 1, \qquad \nu(z_0) = 0,$$

so that $\hat{E}_0 = I$. The function p then simplifies to $-\nu_z$ and the function $a = Q/p$ is then, by (4.10) reduced to

$$a := Q/p = 2\mu.$$

This is holomorphic, and so we can define a normalized meromorphic potential

$$\hat{\eta}_h = \begin{pmatrix} 0 & -\frac{hQ}{2p} \\ p & 0 \end{pmatrix} \lambda^{-1} \mathrm{d}z = \begin{pmatrix} 0 & -h\mu \\ -\nu_z & 0 \end{pmatrix} \lambda^{-1} \mathrm{d}z, \tag{4.12}$$

for any real value of h.

According to Theorem 3.1, there is a continuous family of CMC h surfaces f_h associated to $\hat{\eta}_h$, which includes a minimal surface f_0. We now show that f_0 is the original surface f:

Theorem 4.2. *Let f, $\hat{\eta}_h$ and z_0 be as above. Then, for every $h \neq 0$, the 1-form $\hat{\eta}_h$ is the normalized potential for a unique immersed CMC h surface $f_h : \Sigma \to \mathbb{E}^3$, obtained via the DPW construction with initial condition $\hat{E}_0 = I$. The map $f_0 : \Sigma \to \mathbb{E}^3$, obtained from Theorem 3.1 as $f_0(z) = \mathscr{F}(z, 0)$, is identical with f.*

Proof. To see that the CMC-h surface corresponding to $\hat{\eta}_h$ is smooth we can use the conditions in Theorem 2.1: here $a = 2\mu$ and $Q = -2\mu\nu_z$. By the assumptions on μ and ν, the function a has no poles, and a has a zero if and only if it is of order $n = 2k$ and the function ν has a zero of order k. In this case, we have $\mathrm{Ord}(Q) = k - 1 = \frac{n+2}{2} - 2$, which satisfies the second condition of Theorem 2.1, with $r = 1$.

Finally, we must show that $f_0 = f$. We know from Theorem 3.1 that $f_0\big|_{\Sigma^\circ}$ is an immersed minimal surface, and from equation (3.1) the metric is given by $e^u = \frac{1}{2}|f_x| = \frac{1}{2}(1 + |g|^2)|a|$. In the present situation, $a = 2\mu$ and $g = q = -\nu$, and so we have

$$e^u = |\mu| \left(1 + |\nu|^2\right).$$

This is the same as the formula at (4.5) for the metric of f, and so f and f_0 have the same metric on Σ°. They also have the same Gauss map and Hopf differential. Hence they are the same surface up to an isometry. Finally, it follows from the choice of initial condition $F(z_0)$ for both surfaces, that both maps satisfy $f(z_0) = 0$ and $f_z(z_0) = e_1 - ie_2$. Hence $f\big|_{\Sigma^\circ} = f\big|_{\Sigma^\circ}$. Since both maps are real analytic, and Σ° is dense, they are identical. □

We can now finish the proof of Theorem 1.1:

Proof. The first item is Theorem 4.2. The second item follows from Theorem 3.1 for $h \neq 0$ and, for $h = 0$, from the converse part of Proposition 4.1: suppose that conditions (a) and (b) of Theorem 1.1 are satisfied. Since $a(z_0)$ is assumed real, the constant A_0 is equal to 1. We first need to check that conditions (1) and (2) of the proposition are satisfied. Here $p = Q/a$, and it follows from the conditions on the vanishing orders of Q and a in Theorem 2.1 that this cannot have a pole of order 1,

giving condition (1). Condition (4.1) is equivalent to our assumptions 1.1 and 1.1 on the orders of vanishing of Q and a. Thus, f_0 is a, possibly branched, minimal immersion, with the given Weierstrass data μ and ν. Finally, the regularity conditions (i) and (ii) of Proposition 4.1 are equivalent to condition (b) of the theorem, and so $f_0 : \Sigma^* \to \mathbb{E}^3$ is immersed. □

Remark 4.3. If the classical Weierstrass data μ and ν are not given such that $\mu(z_0) = 1$ and $\nu(z_0) = 0$: using the general formulae for Q and p given at (4.10), one obtains in the place of (4.12) the formula:

$$\begin{aligned} \hat{\eta}_h &= \begin{pmatrix} 0 & -h\mu\Gamma_0(\bar{\nu}_0\nu+1)^2 \\ -\dfrac{\nu_z}{\Gamma_0(\bar{\nu}_0\nu+1)^2} & 0 \end{pmatrix} \lambda^{-1}\mathrm{d}z, \\ \Gamma_0 &:= \frac{\bar{\mu}_0}{|\mu_0|(|\nu_0|^2+1)}, \end{aligned} \tag{4.13}$$

where $\nu_0 = \nu(z_0)$ and $\mu_0 = (\mu(z_0))$.

Remark 4.4. Considering the expressions at (2.1) for the first and second fundamental forms, the fact that the Hopf differential is the same for every h shows that not only umbilics, but also principal curves, are preserved under the deformation. The Weingarten matrix for f_H is

$$W_H = I^{-1}II = \begin{pmatrix} H + \frac{1}{2}e^{-2u}\,\Re Q & -\frac{1}{2}e^{-2u}\,\Im Q \\ -\frac{1}{2}e^{-2u}\,\Im Q & H - \frac{1}{2}e^{-2u}\,\Re Q \end{pmatrix}.$$

At an umbilic point there are no principal directions. In a neighbourhood of a point which is not umbilic we can always assume that coordinates are chosen so that Q is real; that is, the coordinates are isothermic (conformal principal coordinates) and the Weingarten matrix is diagonal. Since Q is constant with respect to h under the deformation, these coordinates are also isothermic for f_h, for every h.

Remark 4.5. The expression for the Weingarten matrix W_h also shows what happens geometrically around the basepoint z_0 as h varies. Assuming Q is real, the principal curvatures are the eigenvalues: $\kappa_\pm = h \pm \frac{1}{2}e^{-2u}Q$. Comparing the coordinate frame $\hat{F}_{C,h}$ with the meromorphic frame $\hat{\Phi}_h$, and using their relationship via the Iwasawa decomposition $\hat{\Phi}_h = \hat{F}\hat{B}_+$, where $\hat{B}_+(z_0) = I$, we have that $e^{u(z_0)} = a(z_0)/2$, independent of h. Thus the principal curvatures for f_h at the basepoint are given by

$$\kappa_\pm(z_0) = \pm C_0 + h, \qquad C_0 := \frac{1}{2}e^{-2u(z_0)}Q(z_0).$$

Assuming the basepoint is not umbilic so that we can take $C_0 > 0$, we see that the surface is negatively curved at the basepoint for $h \in (-C_0, C_0)$, flat at z_0 for $h = \pm C_0$ and positively curved for $|h| > C_0$. As h grows large, the surface becomes spherical around the point z_0.

5. Examples

5.1. Non-minimal CMC surfaces associated to well-known minimal surfaces

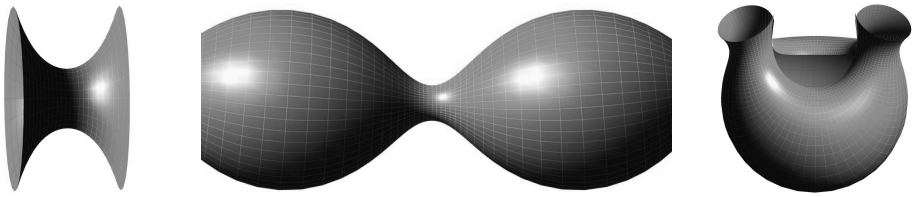
As mentioned in the introduction, Theorem 1.1 gives a means to define non-minimal CMC surfaces from known minimal surfaces. In general one has the question of which basepoint to choose, as different choices will result in different non-minimal surfaces. We consider here some examples where there is a canonical choice of basepoint: we will show in the next section that if a CMC surface f_H has a reflective symmetry with respect to a plane in $\mathbb{R}^3$ then, if the basepoint is chosen to be some point on the intersection of this plane with the surface, the associated surfaces f_h, for $h \in \mathbb{R}$ also have the same symmetry. A similar statement holds for finite order rotational symmetries about some axis.

In the examples below, the basepoints are chosen for the following reasons: for the sphere, all points are the same. For the catenoid, the circle of smallest radius lies in a plane of symmetry of the surface - choosing a basepoint on this circle will result in non-minimal CMC surfaces with the same planar symmetry, and all points on the circle are the same geometrically. For the helicoid, any point on the central axis is geometrically the same, and such a point is a natural choice. Enneper's surface of order k has a finite order rotational symmetry about an axis through the point $z_0 = 0$, and so this is a natural choice of basepoint to produce CMC surfaces with the same symmetry.

Example 5.1. The simplest minimal surface is the plane, which has Weierstrass data $\mu = \mu_0$, $\nu = \nu_0$, where μ_0 and ν_0 are non-zero constants. Using the formula (4.13), we obtain the potential

$$\hat{\eta} = \begin{pmatrix} 0 & -h|\mu_0|(1+|\nu_0|^2) \\ 0 & 0 \end{pmatrix} \lambda^{-1} \mathrm{d}z.$$

This is the potential for a once-punctured round sphere of radius $1/h$. Note that in this example f_0 is complete, but f_h is not complete for $h \neq 0$. More precisely, f_0 has a planar end as $|z| \to \infty$, whilst f_h has a finite limit.



$H = 10^{-10}$ $H = 0.1$ $H = 10$

Figure 5.1. Various CMC H surfaces in the catenoid family, with basepoint on the catenoid chosen on the parallel of smallest radius.

Example 5.2. *The Catenoid:* taking the Weierstrass data $\mu = -e^{-z}/2$ and $\nu = -e^{z}$ on $\mathbb{C}^2$ gives a covering of the catenoid. We compute the potentials for the associated non-minimal CMC surfaces. The function ν is never zero, so we use the formula (4.13) with basepoint $z_0 = 0$. Here $\nu_0 = -1$, $\mu_0 = -1/2$ and the formula (4.13) gives

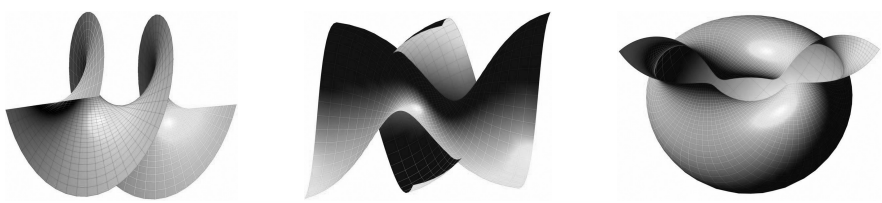
$$\hat{\eta} = \begin{pmatrix} 0 & -h\frac{1}{4}e^{-z}(e^{z}+1)^2 \\ -2\,e^{z}(e^{z}+1)^{-2} & 0 \end{pmatrix} \lambda^{-1}\mathrm{d}z, \qquad z_0 = 0.$$

Some examples are plotted in Figure 5.1. The surface with $H = 0.1$ looks rather like an unduloid, but it does not close up. The surface, around the basepoint z_0, shrinks to a tiny sphere as H grows large.

Example 5.3. *The Helicoid:* The helicoid is obtained by multiplying μ by i in the data for the catenoid. Thus the potentials for the associated surfaces are

$$\hat{\eta} = \begin{pmatrix} 0 & \frac{ih}{4}e^{-z}(e^{z}+1)^2 \\ 2e^{z}(e^{z}+1)^{-2} & 0 \end{pmatrix} \lambda^{-1}\mathrm{d}z, \qquad z_0 = 0.$$

Plots for different values of H are shown in Figure 5.2. Three plots of the "almost minimal" surface with $H = 0.001$ are shown in Figure 5.3. On the large scale it looks like a chain of spheres, although it does not close up. The "sphere" shown is of radius 1000. The third image is at the center of the second image at close range.



$H = 10^{-10}$, $H = 0.1$ $H = 5$

Figure 5.2. CMC H surfaces in the helicoid family with basepoint chosen on the central axis.

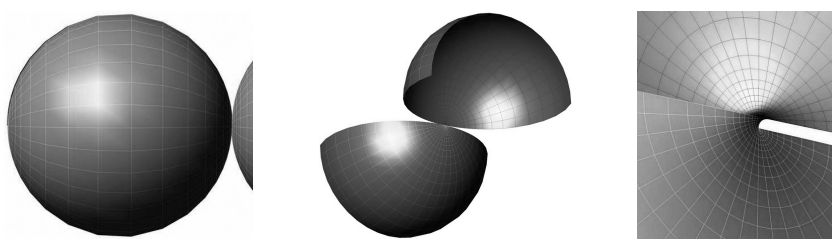


Figure 5.3. Several plots of the helicoidal surface of CMC $H = 0.001$.

Example 5.4. *Enneper's surface:* Enneper's surface of order $k \geq 1$ is given by $\mu = 1, \nu = z^k$ on $\mathbb{C}$. This gives associated CMC-h surfaces with potentials

$$\hat{\eta} = \begin{pmatrix} 0 & h \\ kz^{k-1} & 0 \end{pmatrix} \lambda^{-1} \mathrm{d}z, \qquad z_0 = 0.$$

These potentials are known: for the case $k = 1$ and $h = 1$ we obtain a round cylinder. However the surfaces do not close up in general. The other cases are known as Smyth surfaces – defined by B. Smyth in [12]) – or $(k+1)$-legged Mister Bubbles. See Figure 1.1.

6. The dressing action

The dressing action is an action by $\Lambda_P^+ G_\sigma^{\mathbb{C}}$ on the space of CMC immersions from a simply connected domain $\Sigma \subset \mathbb{C}$ into $\mathbb{E}^3$. A description of the action can be found in [14]. The action is defined as follows: for a loop $h_+ \in \Lambda_P^+ G_\sigma^{\mathbb{C}}$, and a CMC immersion with extended frame $\hat{F}$, the pointwise Iwasawa decomposition

$$h_+ \hat{F}(z) = \widetilde{F}(z) \widetilde{G}_+(z), \qquad \widetilde{F}(z) \in \Lambda G_\sigma, \quad \widetilde{G}_+(z) \in \Lambda_P^+ G_\sigma^{\mathbb{C}},$$

gives an extended frame $\widetilde{F}$ for a new CMC surface. Note that this can be equivalently defined by the Iwasawa decomposition $h_+ \hat{\Phi}(z) = \widetilde{F}(z) \widetilde{C}_+(z)$, where $\hat{\Phi}$ is a meromorphic extended frame for f, to get the same extended frame $\widetilde{F}$.

It is a question of interest whether two CMC surfaces are in the same dressing orbit: for example all CMC tori are in the dressing orbit of the cylinder [9].

Note that if $\tilde{f} = (h_+)^\# f$ is obtained from f by dressing by h_+, and $\widetilde{\Phi}$ and $\hat{\Phi}$ are their respective normalized meromorphic frames, with basepoint z_0, then writing the normalized Birkhoff decompositions $\hat{F} = \hat{\Phi} \hat{B}_+$ and $\widetilde{F} = \widetilde{\Phi} \widetilde{B}_+$, and substituting in the above relation $\widetilde{F} = h_+ \hat{F} \widetilde{B}_+^{-1}$, we obtain the relation

$$\widetilde{\Phi} = h_+ \hat{\Phi} \hat{W}_+, \qquad \hat{W}_+(z) \in \Lambda_P^+ G_\sigma^{\mathbb{C}}, \quad \hat{W}_+(z_0) = h_+^{-1}.$$

where $\hat{W}_+ = \hat{B}_+ \widetilde{G}_+^{-1} \widetilde{B}_+^{-1}$. If we denote by Σ^* the open dense set on which both $\widetilde{\Phi}$ and $\hat{\Phi}$ have no poles, then the above formula shows that $\hat{W}_+ : \Sigma^* \to \Lambda^+ G_\sigma^{\mathbb{C}}$ is holomorphic.

Conversely, given f and $\hat{\Phi}$ as above, any holomorphic map $\hat{W}_+$ from a neighbourhood U of z_0 into $\Lambda_P^+ G_\sigma^{\mathbb{C}}$ corresponds to a dressing by the element $h_+ = \hat{W}_+(z_0)^{-1}$. The new solution has meromorphic frame $\widetilde{\Phi} = \hat{W}_+(z_0)^{-1} \hat{\Phi} \hat{W}_+$. At the level of potentials, the new surface has the potential

$$\hat{\eta} \# \hat{W}_+ := \hat{W}_+^{-1} \hat{\eta} \hat{W}_+ + \hat{W}_+^{-1} \mathrm{d} \hat{W}_+. \tag{6.1}$$

Thus if $\hat{\eta}_H = \begin{pmatrix} 0 & -\frac{H}{2}a \\ \frac{Q}{a} & 0 \end{pmatrix} \lambda^{-1} \mathrm{d}z$, we have

$$\widetilde{\eta}_H = \hat{\eta} \# \hat{W}_+ = \begin{pmatrix} 0 & -\frac{H}{2}a\rho^2 \\ \frac{Q}{a\rho^2} & 0 \end{pmatrix} \lambda^{-1} \mathrm{d}z, \qquad \hat{W}_+ = \begin{pmatrix} \rho^{-1} & 0 \\ 0 & \rho \end{pmatrix} + o(\lambda).$$

In other words the data $(a(z), Q(z))$ are dressed to $(\rho^2(z)a(z), Q(z))$, where ρ is some meromorphic function. In particular the Hopf differential is the same for both surfaces.

We say that $\hat{\eta}$ and $\widetilde{\eta}$ are *dressing equivalent* if (6.1) holds for some meromorphic function $\hat{W}_+$, and *formally* dressing equivalent if the relation holds for some formal power series $\hat{W}_+$.

If we take the potentials $\hat{\eta}_H$ and $\widetilde{\eta}_H$ above, and let H vary, then, for $\hat{\eta}_h$ and $\widetilde{\eta}_h$ to be dressing equivalent, the gauge $\hat{W}_+$ will also depend on h. Thus, even if $\hat{\eta}_H$ is dressing equivalent to $\widetilde{\eta}_H$ at some value H of h, it is not automatic that the whole families of surfaces are dressing equivalent.

Theorem 6.1. *Let $\hat{\eta}_h$ and $\widetilde{\eta}_h$ be two normalized meromorphic potentials with meromorphic data (a, Q) and $(\tilde{a}, Q)$, and let z_0 be the associated basepoint. Suppose that either $Q(z_0) \neq 0$ or z_0 is a simple root of Q. Then:*

(1) *The potentials $\hat{\eta}_h$ and $\widetilde{\eta}_h$ are formally dressing equivalent for every $h \neq 0$. The gauge $\hat{W}_{+,h}$ can be computed locally around z_0.*
(2) *The map $\hat{W}_{+,h}$ extends to $h = 0$ if and only if it is constant in h and has the form*

$$\hat{W}_+ = \begin{pmatrix} a_0 & b_1\lambda \\ 0 & a_0^{-1} \end{pmatrix}, \tag{6.2}$$

$$a_0 = \sqrt{\frac{a}{\tilde{a}}}, \qquad b_1 = \frac{\tilde{a}}{Q}\frac{\mathrm{d}}{\mathrm{d}z}\sqrt{\frac{a}{\tilde{a}}}, \quad \text{where } \frac{\mathrm{d}b_1}{\mathrm{d}z} = 0. \tag{6.3}$$

Proof. Wu [14] has studied the general problem of dressing two normalized potentials $\hat{\eta}$ and $\widetilde{\eta}$ into each other. Formally, it is enough to solve the equations (5.11)-(5.16) in [14] for the map $\hat{W}_+$. (The symbols E, p and q used in [14] correspond to ours via $E = -\frac{h}{2}Q$, $p = -\frac{h}{2}a$, and $q = -\frac{h}{2}\tilde{a}$.) Writing

$$\hat{W}_{+,h}(z) = \begin{pmatrix} \sum_{k=0}^{\infty} a_{2k}(z)\,\lambda^{2k} & \sum_{k=0}^{\infty} b_{2k+1}(z)\,\lambda^{2k+1} \\ \sum_{k=0}^{\infty} c_{2k+1}(z)\,\lambda^{2k+1} & \sum_{k=0}^{\infty} d_{2k}(z)\,\lambda^{2k} \end{pmatrix},$$

the equations (also correcting two typographic errors in [14]) to be solved are:

$$a_0 = d_0^{-1} = \sqrt{\frac{a}{\tilde{a}}}, \tag{6.4}$$

$$2b_n' + b_n\left(Q' - \left(\frac{a'}{a} + \frac{\tilde{a}'}{\tilde{a}}\right)Q\right) = \left(a_{n-1}'' - a_{n-1}'\frac{a'}{a}\right)\tilde{a}, \qquad (n \geq 1), \tag{6.5}$$

$$c_n = \frac{2}{h}\left(-\frac{b_n Q}{a\tilde{a}} + \frac{a_{n-1}'}{a}\right), \qquad (n \geq 1), \tag{6.6}$$

$$a_2 = \frac{1}{2}a_0 b_1 c_1 - \frac{b_1'}{h\tilde{a}}, \qquad d_2 = \frac{1}{2}d_0 b_1 c_1 + \frac{b_1'}{ha}, \tag{6.7}$$

$$a_n = \frac{1}{2}a_0 \sum_{j=0}^{n/2-1} b_{2j+1}c_{n-2j-1} - \frac{1}{2}a_0 \sum_{j=1}^{n/2-1} a_{2j}d_{n-2j} - \frac{b_{n-1}'}{h\tilde{a}}, \quad (n \geq 4), \tag{6.8}$$

$$d_n = \frac{1}{2}d_0 \sum_{j=0}^{n/2-1} b_{2j+1}c_{n-2j-1} - \frac{1}{2}d_0 \sum_{j=1}^{n/2-1} a_{2j}d_{n-2j} - \frac{b_{n-1}'}{ha}, \quad (n \geq 4). \tag{6.9}$$

As discussed in [14], Theorem 5.17, there is always a solution $\hat{W}_{+,h}$ for any $h \neq 0$, which takes care of item 1 of this theorem.

We now consider item 2. Assume that the solution extends to $h = 0$. We argue by induction on n that:

$$\begin{aligned} &a_n \text{ is independent of } h \text{ for all } n\,, && a_n = d_n = 0 \quad \text{for } n > 0,\\ &c_n = 0 \quad \text{for all } n, && b_n = 0 \quad \text{for } n > 1. \end{aligned}$$

Clearly, our hypothesis holds for $n = 0$, by equation (6.4). Assume now that the hypothesis holds for $n - 1$, and consider n. If n is odd, we need to consider only b_n and c_n. By equation (6.5), b_n is independent of h, because the right hand side of the equation has this property (the inductive hypothesis). But then the expression $-\frac{2b_n Q}{a\tilde{a}} + \frac{2a_{n-1}'}{a}$ on the right hand side of equation (6.6) is also independent of h, and therefore, if c_n is defined for $h = 0$, the equation (6.6) becomes

$$c_n = 0 \quad \text{and} \quad b_n = \frac{a_{n-1}'\tilde{a}}{Q}. \tag{6.10}$$

If $n = 1$ we obtain the formula at (6.3) for b_1, and if $n \geq 3$ we obtain, using the inductive hypothesis on a_{n-1}, that $b_n = 0$. That deals with the odd case.

Now if n is even, we are considering a_n and d_n. We first discuss a_n. If $n = 2$ we have, from (6.7), that $a_2 = \frac{1}{2}a_0 b_1 c_1 - \frac{b_1'}{h\tilde{a}}$. We know that $c_1 = 0$, and that $b_1'/\tilde{a}$ is independent of h. Hence we must have

$$b_1' = a_2 = 0. \tag{6.11}$$

If $n \geq 4$, we use equation (6.8), which reduces, by the inductive hypothesis to

$$a_n = 0.$$

The argument for d_n is identical, using (6.9).

Thus we have proved the formula (6.2) for $\hat{W}_+$ by induction. The formulae at (6.3) are given at (6.4), (6.10) and (6.11). This proves the "only if" direction of item 2, and the "if" direction is just the observation that the stated conditions for $\hat{W}_+$ at (6.3) do give a solution to Wu's equations (6.4)-(6.9). □

Remark 6.2. Note that, again using Theorem 5.17 of [14], an analogous statement to Theorem 6.1 holds for the case that Q has an umbilic of order greater or equal to 2. The only difference is that, in addition to the conditions at (6.3), one also requires all the dressing invariants (of which there are many for higher order umbilics) must be satisfied by both potentials. The dressing invariants are independent of h (see Lemma 3.2 of [14]), and so there is no further argument needed.

6.1. Dressing minimal surfaces

The dressing action on minimal surfaces, defined in [7] in the same way as for non-minimal surfaces, via the meromorphic potential, is an action on equivalence classes of minimal surfaces, all of which have the same normalized potential. For the minimal case, there is not a unique minimal surface related to a normalized potential. Theorem 6.1 defines a natural class of dressing elements $h_+ \in \Lambda_P^+ G_\sigma^{\mathbb{C}}$, namely given by

$$h_+ = \hat{W}_+(z_0)^{-1} = \begin{pmatrix} a_0(z_0)^{-1} & -b_1(z_0)\lambda \\ 0 & a_0(z_0) \end{pmatrix}, \tag{6.12}$$

where $\hat{W}_+$ is as in the theorem, which can be used to define dressing as an action on the class of minimal surfaces. But, in [7], it is also shown (in Lemma 5.1) that when dressing normalized potentials for minimal surfaces, one can, without loss of generality, assume that the dressing element h_+ is indeed of the form (6.12), any other coefficients being irrelevant. Thus, what we have really established here is the following:

Theorem 6.3. *Let $\Sigma \subset \mathbb{C}$ be a simply connected domain with basepoint z_0. The dressing action, defined in* [7] *on normalized potentials of minimal surfaces, gives a well-defined group action on the space of minimal immersions $f : \Sigma \to \mathbb{R}^3$.*

Note that, given an element h_+ of the above form, the dressing action on a minimal immersion can be calculated using the classical Weierstrass data $(\mu \mathrm{d}z, \nu)$, the formulae in Theorem 1.1 for a and Q, and solving the equations at (6.3) for $\tilde{\mu}$ and $\tilde{\nu}$. If f has an umbilic of order greater than 1 at z_0, one expects some complications in this procedure.

7. Surfaces with symmetries

In this section we consider minimal and non-minimal CMC surfaces with symmetries that are either reflections about a plane or finite order rotations about a point contained in the surface. Symmetries of CMC surfaces have been studied by Dorfmeister and Haak in [4] and [5], and a more general derivation of the potentials that correspond to symmetries can be found in those references.

Definition 7.1. Let $f : \Sigma \to \mathbb{R}^3$ be a conformal immersion of a contractible domain.

(1) We say that f has a *reflective symmetry* with respect to a plane $\Pi \subset \mathbb{R}^3$ if there exist conformal coordinates z for Σ such that Σ is symmetric about the real axis in these coordinates, that is $\overline{\Sigma} = \Sigma$, and such that:

$$f(\bar{z}) = R_\Pi(f(z)), \qquad \forall z \in \Sigma,$$

where R_Π is the reflection about the plane Π.

(2) Let n be a positive integer. We say that f has a *fixed-point rotational symmetry of order n and axis l* if there exist conformal coordinates z for Σ such that $e^{i\theta}\Sigma = \Sigma$, where $\theta = 2\pi/n$, and such that:

$$f(e^{i\theta}z) = R_l f(z), \qquad \forall z \in \Sigma, \tag{7.1}$$

where R_l is the rotation of angle θ about the line l.

We will first show how these symmetries are reflected in the Weierstrass data. For the rest of this section we always consider a contractible domain $\Sigma \subset \mathbb{C}$, with base point $z_0 = 0$, and a conformal CMC H immersion $f : \Sigma \to \mathbb{E}^3$. If $H \neq 0$, we let $\hat{\eta}_H = \text{off-diag}(-\frac{H}{2}a, p)\lambda^{-1}\mathrm{d}z$ be the associated normalized potential, where $p = Q/a$. If $H = 0$, we let (μ, ν) be the classical Weierstrass data, chosen such that $\mu(0) = 1$ and $\nu(0) = 0$.

7.1. Reflections about a plane

Observe that, since the components of the matrices e_1 and e_3 are pure imaginary, whilst the components of e_2 are real, the reflection about the plane $e_1 \wedge e_3$ in $\mathbb{R}^3 = \mathfrak{su}(2)$ is given by

$$X \mapsto -\bar{X}.$$

The next lemma shows that a CMC surface is symmetric about the plane $e_1 \wedge e_3$ if and only if coordinates and basepoint can be chosen such that the Weierstrass data are real along the real line.

Lemma 7.2. *Suppose Σ is symmetric about the real axis, that is $\overline{\Sigma} = \Sigma$, and let $f : \Sigma \to \mathbb{E}^3$ be a CMC immersion with associated data as above. Then*

$$f(\bar{z}) = -\overline{f(z)}, \qquad \forall\, z \in \Sigma, \tag{7.2}$$

if and only if the associated Weierstrass data satisfy the condition

$$\eta_H(z) = \overline{\eta_H(\bar{z})}, \qquad \forall\, z \in \Sigma, \qquad \text{if } H \neq 0, \tag{7.3}$$

$$(\mu(z), \nu(z)) = \overline{(\mu(\bar{z}), \nu(\bar{z}))}, \qquad \text{if } H = 0. \tag{7.4}$$

Proof. First suppose that $f(\bar{z}) = -\overline{f(z)}$. It follows that $f(\mathbb{R} \cap \Sigma) \subset \operatorname{span}\{e_1, e_3\}$, and so, for real values of z, we have f_x parallel to the plane $e_1 \wedge e_3$. It also follows from the symmetry that the surface is perpendicular to the plane. This means that f_y must be parallel to e_2, for real z. Using this fact in the definition of the coordinate frame (2.2), with initial condition $E_0 = I$, we find that, along $\mathbb{R}$, the frame is $SO(2)$-valued:

$$F(x, 0) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad \theta(x, 0) \in \mathbb{R}.$$

Case $H \neq 0$. The above expression for $F(x, 0)$ means that, for real z, the extended coordinate frame $\hat{F}$ takes values in $\Lambda G_{\sigma\rho}$, the fixed point subgroup with respect to the involution ρ given by $(\rho\gamma)(\lambda) := \overline{\gamma(\bar{\lambda})}$, for $\gamma \in \Lambda G$. The involution ρ is of the first kind, which means that the group $\Lambda G^{\mathbb{C}}_{\sigma\rho}$ is Birkhoff decomposable (see [1]), which simply means that both the factors in the normalized Birkhoff decomposition

$$\hat{F}(x, 0) = \hat{F}_-(x, 0)\hat{F}_+(x, 0),$$

take values in $\Lambda G^{\mathbb{C}}_{\sigma\rho}$. Now the normalized potential is $\hat{\eta} = \hat{F}_-^{-1}\mathrm{d}\hat{F}_-$, and the reality condition ρ, which is valid along the real axis, amounts to saying that the functions $a(x, 0)$ and $p(x, 0)$ are real valued. Since Σ is connected, and the functions are meromorphic, this is equivalent to the condition (7.3).

Conversely, suppose that $\eta_H(z) = \overline{\eta_H(\bar{z})}$ for all z. It follows that $\hat{\Phi}(\bar{z}) = \overline{\hat{\Phi}(z)}$ for all $z \in \Sigma$. Let $\hat{F}$ be the unique frame obtained by the pointwise Iwasawa decomposition

$$\hat{\Phi} = \hat{F}\,\hat{B}_+, \qquad \hat{F}(z) \in \Lambda G_\sigma, \quad \hat{B}_+(z) \in \Lambda^+_P G^{\mathbb{C}}_\sigma.$$

The corresponding decomposition for $\bar{\hat{\Phi}}$ is just the conjugate of this: $\bar{\hat{\Phi}} = \bar{\hat{F}}\,\overline{\hat{B}_+}$. From this, the above symmetry of $\hat{\Phi}$, and the uniqueness of the Iwasawa decomposition, we conclude that

$$\hat{F}(\bar{z}) = \overline{\hat{F}(z)}.$$

Using this condition in the Sym-Bobenko formula

$$f := -\frac{1}{2H}\left(2i\lambda\partial_\lambda\hat{F}\,\hat{F}^{-1} + \hat{F}e_3\hat{F}^{-1} - e_3\right)\Big|_{\lambda=1},$$

we immediately obtain that $f(\bar{z}) = -\overline{f(z)}$.

Case $H = 0$. Here the coordinate frame is

$$F_C = e^{-u/2}\begin{pmatrix} s & \bar{r} \\ -r & \bar{s} \end{pmatrix},$$

and it follows from the above form for $F(x, 0)$ that the functions s and r are real valued along the real line. Since they are holomorphic, this means $s(z) = \overline{s(\bar{z})}$ and $r(z) = \overline{r(\bar{z})}$ for all z, and hence $\mu = s^2$ and $\nu = sr/\mu$ have the same property.

Conversely, if μ and ν have the symmetry given at (7.4), then the potential $\hat{\eta}_h = \text{off-diag}(-h\mu, -\nu_z)\lambda^{-1}\mathrm{d}z$ has the symmetry (7.3). Hence, by the case explained above for $H \neq 0$, the non-minimal CMC h surface, $f_h : \Sigma \to \mathbb{E}^3$, associated to this potential, satisfies $f_h(\bar{z}) = -\overline{f_h(z)}$ for all z, and for every $h \neq 0$. By Theorem 1.1, we have that this family depends continuously on h, and that $f_0 = f$. By continuity, f also has the symmetry. □

7.2. Fixed-point rotational symmetries

In Definition 7.1 for rotational symmetries, little generality is lost by taking l to be the oriented x_3-axis, and in our identification of $\mathbb{E}^3 = \mathfrak{su}(2)$ the symmetry condition (7.1) amounts to:

$$f(e^{i\theta}z) = \mathrm{Ad}_T f(z), \qquad T = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}, \qquad \theta = \frac{2\pi}{n}. \tag{7.5}$$

This condition is reflected in the Weierstrass data for a CMC surface as follows:

Lemma 7.3. *Suppose that the domain Σ is rotationally symmetric, specifically $e^{i\theta}\Sigma = \Sigma$, where $\theta = 2\pi/n$. Let $f : \Sigma \to \mathbb{E}^3$ be a conformal CMC immersion with associated data as above. Then*

(1) *If $H \neq 0$, then f has the rotational symmetry* (7.5) *if and only if, for all $z \in \Sigma$, we have*

$$a(e^{i\theta}z) = a(z),$$
$$p(e^{i\theta}z) = e^{-2i\theta}p(z).$$

(2) *If $H = 0$, then f has the rotational symmetry* (7.5) *if and only if, for all $z \in \Sigma$, we have*

$$\mu(e^{i\theta}z) = \mu(z),$$
$$\nu(e^{i\theta}z) = e^{-i\theta}\nu(z).$$

Proof. Given the symmetry condition (7.5), set $g(z) = f(e^{i\theta}z) = \mathrm{Ad}_T f(z)$. Calculating the coordinate frame for g, one obtains that the coordinate frame for f, defined by (2.2), satisfies

$$F_C(e^{i\theta}z) = \mathrm{Ad}_T F_C(z), \tag{7.6}$$

for all z, and this condition is also valid for the extended coordinate frame $\hat{F}_C$. Conversely, given a CMC H surface with extended frame $\hat{F}_C$, the Sym-Bobenko formula shows that the symmetry (7.6) implies that f has the symmetry (7.5). This argument is also valid for the case $H = 0$, taking the limit $H \to 0$ in the Sym-Bobenko formula. Thus, in either case, the condition (7.5) is equivalent with the same condition for $\hat{F}_C$.

Case $H \neq 0$. The meromorphic extended frame is obtained by the normalized Birkhoff decomposition $\hat{F}_C(z) = \hat{\Phi}(z)\hat{G}_+(z)$, and the condition (7.6) gives us

$$\begin{aligned}\hat{F}_C(e^{i\theta}z) &= \hat{\Phi}(e^{i\theta}z)\,\hat{G}_+(e^{i\theta}z)\\ &= T\hat{\Phi}(z)\,\hat{G}_+(z)T^{-1}\\ &= \mathrm{Ad}_T\hat{\Phi}(z)\mathrm{Ad}_T G_+(z).\end{aligned}$$

The uniqueness of the $\Lambda_*^- G_\sigma^{\mathbb{C}}$ factor in the normalized Birkhoff decomposition implies that

$$\hat{\Phi}(e^{i\theta}z) = \mathrm{Ad}_T\hat{\Phi}(z).$$

Differentiating, one obtains

$$\hat{\Phi}^{-1}\hat{\Phi}_z\Big|_{e^{i\theta}z} = e^{-i\theta}\mathrm{Ad}_T\left(\hat{\Phi}^{-1}\hat{\Phi}_z\right)\Big|_z, \tag{7.7}$$

which is to say

$$\begin{pmatrix} 0 & -\frac{H}{2}a(e^{i\theta}z)\\ p(e^{i\theta}z) & 0\end{pmatrix}\lambda^{-1}\mathrm{d}z = \begin{pmatrix} 0 & -\frac{H}{2}a(z)\\ e^{-2i\theta}p(z) & 0\end{pmatrix}\lambda^{-1}\mathrm{d}z.$$

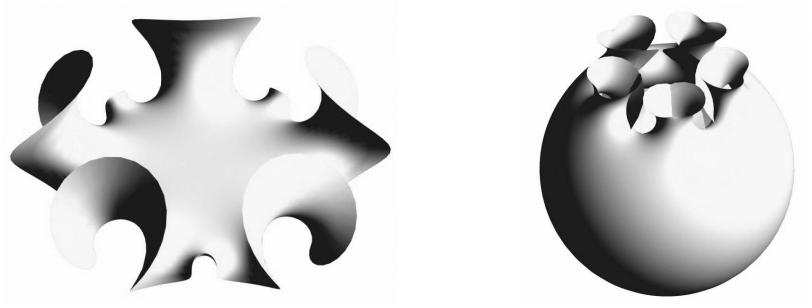
Thus a and p satisfy the conditions stated in the theorem. Conversely, integrating a potential $\hat{\eta}$ with the symmetry given at (7.7), all steps of the above argument can be reversed to conclude that the corresponding CMC surface f has the required symmetry.

Case $H = 0$. For the minimal case, the coordinate frame has the form

$$F_C = e^{-u/2}\begin{pmatrix} s & \bar{r}\\ -r & \bar{s}\end{pmatrix},$$

where $\mu = s^2$ and $\mu\nu = sr$. If the surface has the symmetry (7.5) then the metric is invariant under $z \mapsto e^{i\theta}z$, and hence $u(e^{i\theta}z) = u(z)$. Thus the symmetry (7.6) of F_C reduces to: $s(e^{i\theta}z) = s(z)$ and $r(e^{i\theta}z) = e^{-i\theta}r(z)$. This implies the stated conditions for μ and ν.

Conversely, given a minimal surface with Weierstrass data satisfying the given symmetry conditions, the associated non-minimal surface, with basepoint $z_0 = 0$, given in Theorem 1.1, has normalized potential $\hat{\eta}_h = \text{off-diag}(-ha/2, p) = \text{off-diag}(-h\mu, -\nu_z)$. The symmetry assumptions on μ and ν give $a(e^{i\theta}z) = a(z)$ and $p(e^{i\theta}z) = e^{-2i\theta}p(z)$. Hence this is the potential for a CMC h surface satisfying the symmetry (7.5). By continuity, the symmetry also holds at $h = 0$. □



$H = 10^{-8}$, $H = 2$

Figure 7.1. Two surfaces in the same family, which has an order 5 rotational symmetry.

7.3. Preservation of symmetries under deformations

A Euclidean motion of the ambient space will always bring a line l to the x_3-axis, or a plane Π to the plane $e_1 \wedge e_3$. Furthermore, the symmetry conditions of the Weierstrass data are independent of h: in particular, for both symmetries, the Weierstrass data for a minimal surface satisfy the symmetry condition if and only if the generalized Weierstrass data for the associated non-minimal surfaces do also. Hence the results of the previous two lemmas have the following corollary:

Theorem 7.4. *Let H be any real number. Let $f_H : \Sigma \to \mathbb{E}^3$ be a conformal CMC H immersion, and let f_h, for $h \in \mathbb{R}$, be the family associated by Theorem* 1.1, *with basepoint $z_0 = 0$. Then f_H satisfies one of the symmetries at Definition* 7.1 *if and only if f_h satisfies the same symmetry for all h.*

7.4. Examples with rotational symmetries

The conditions on a and p in Lemma 7.3 are equivalent to the following Laurent expansions for $a(z)$ and $p(z)$:

$$a(z) = \sum_j a_{nj}\, z^{nj}, \qquad p(z) = \sum_j p_{nj-2}\, z^{nj-2}.$$

Example 7.5. As an example with an order 5 rotational symmetry, we numerically computed solutions corresponding to the potential

$$\hat{\eta} = \begin{pmatrix} 0 & -\frac{h}{2}(5.1 + 1.5\, z^5 + 0.35\, z^{10}) \\ 1.25\, z^3 + 4.15\, z^8 & 0 \end{pmatrix} \lambda^{-1} \mathrm{d}z.$$

The images of discs around the coordinate origin for the surfaces corresponding to $h = 10^{-8}$ and $h = 2$ are shown at Figure 7.1.

Example 7.6. Kusner's surface with $p = 3$, defined in [10], has Weierstrass data

$$\mu = \frac{i(\sqrt{5}z^3 + 1)^2}{(z^6 + \sqrt{5}z^3 - 1)^2}, \qquad \nu = \frac{z^2(z^3 - \sqrt{5})}{\sqrt{5}z^3 + 1}.$$

It is proved in [10] that this minimal surface is complete, non-orientable, has finite total curvature -10π, has 3 embedded flat ends, and contains 3 straight lines which lie in a plane. The dihedral group of order 6 acts by reflections around these lines.

The corresponding potential, with basepoint $z_0 = 0$, is

$$\hat{\eta} = \begin{pmatrix} 0 & -h\frac{(\sqrt{5}z^3+1)^2}{(z^6+\sqrt{5}z^3-1)^2} \\ -\frac{2\sqrt{5}iz(z^6+\sqrt{5}z^3-1)}{(z^3+1)^2} & 0 \end{pmatrix} \lambda^{-1}\mathrm{d}z.$$

The cases $H = 10^{-9}$ and $H = 1$ are shown in Figure 7.2.

$H = 10^{-9}$ $H = 1$ $H = 1$

Figure 7.2. Left: A disc around the symmetry point in an almost minimal version of Kusner's surface with $p = 3$. Right two images: The CMC 1 version.

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