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Bilipschitz embedding of Grushin plane in $\mathbb{R}^3$

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Abstract. The Grushin plane is bilipschitz homeomorphic to a quasiplane in $\mathbb{R}^3$.

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1. Grushin plane

The Grushin plane $\mathbb{G}$ is the space $\mathbb{R}^2$ endowed with the vector fields

$$X_1 = \partial_{x_1} \quad \text{and} \quad X_2 = x_1\, \partial_{x_2}.$$

Outside the singular line $x_1 = 0$, the Grushin metric is the Riemannian metric

$$ds^2 = dx_1{}^2 + x_1{}^{-2}\, dx_2{}^2,$$

which makes X_1, X_2 an orthonormal basis for the tangent space at each point not on the singular line. The vector field $[X_1, X_2] = \partial_{x_2}$ is added to the singular line; and the Grushin metric extended across $x_1 = 0$ is the Carnot-Carathéodory metric

$$d_{\mathbb{G}}(p,q) = \inf_{\gamma} \int_0^1 \sqrt{x_1'(t)^2 + \frac{x_2'(t)^2}{x_1(t)^2}}\, dt \qquad \forall\ p, q \in \mathbb{R}^2,$$

where the infimum is taken over all paths $\gamma\colon [0,1] \to \mathbb{G}$ connecting $\gamma(0) = p$ to $\gamma(1) = q$, that are absolutely continuous with respect to the Euclidean metric. On the singular line, $d_{\mathbb{G}}((0,x_2),(0,y_2)) \simeq \sqrt{|x_2 - y_2|}$.

The Grushin plane is in some sense one of the simplest singular sub-Riemannian manifolds. For geodesics in $\mathbb{G}$ and properties of $\mathbb{G}$, see Bellaĉhe [2].

The Grushin balls $B_{\mathbb{G}}(x,r)$ can be described in terms of Euclidean rectangles

$$R(x,r) = \big[x_1 - r, x_1 + r\big] \times \big[x_2 - r(|x_1| + r), x_2 + r(|x_1| + r)\big].$$

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Precisely, there is a constant $C > 1$ such that $R(x, C^{-1}r) \subset B_{\mathbb{G}}(x, r) \subset R(x, Cr)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $r > 0$; see Franchi and Lanconelli [5], also [4]. Therefore outside the line $x_1 = 0$, the Grushin plane has a Whitney-type decomposition

$$\mathbb{G} \setminus \{x_1 = 0\} = \bigcup_{k\in\mathbb{Z}} \bigcup_{m\in\mathbb{Z}} \Big(\Big[7^{k-1}, 7^k\Big] \times \Big[m49^k, (m+1)49^k\Big] \cup \Big[-7^k, -7^{k-1}\Big] \times \Big[m49^k, (m+1)49^k\Big] \Big).$$

Meyerson [6] has shown that the Grushin plane is quasisymmetrically equivalent to the Euclidean plane by the map $(x_1, x_2) \mapsto (x_1|x_1|, x_2)$. Seo [9] has proved a general theorem on bilipschitz embeddable metric spaces, from which it follows that the Grushin plane can be bilipschitzly embedded into some Euclidean space.

Since the number of Grushin balls of radius $\varepsilon > 0$ needed to cover the unit ball $B_{\mathbb{G}}((0,0), 1)$ has magnitude $\asymp \varepsilon^{-2} \log \frac{1}{\varepsilon}$ for small ε [2], the Grushin plane can not be bilipschitzly embedded into $\mathbb{R}^2$. On the other hand it is relatively simple to embed $\mathbb{G}$ bilipschitzly into $\mathbb{R}^4$.

We prove a sharp embedding theorem for the Grushin plane.

Theorem 1.1. *The Grushin plane $\mathbb{G}$ is bilipschitz homeomorphic to a quasiplane in $\mathbb{R}^3$.*

The embedded singular line in $\mathbb{R}^3$ is necessarily a very regular snowflake curve Γ of Hausdorff dimension 2. The goal is to place, in a bounded Euclidean neighborhood of a subarc of the embedded Γ, 49^k wrinkled Whitney 2-cells of diameter comparable to 7^{-k} for all $k \geq 0$.

In contrast, Semmes [8] has observed that the first Heisenberg group $\mathbb{H}$, when equipped with its Carnot metric, can not be bilipschitzly embedded into any Euclidean space $\mathbb{R}^n$. Semmes' observation is based on a deep theorem of Pansu [7].

A *quasiplane* in $\mathbb{R}^3$ is the image of the hyperplane $\mathbb{R}^2$ in $\mathbb{R}^3$ under a global quasiconformal homeomorphism of $\mathbb{R}^3$.

A sense-preserving homeomorphism $f\colon D \to D'$ between domains in $\mathbb{R}^n$, $n \geq 2$, is said to be *quasiconformal* if

$$\limsup_{r\to 0} \frac{\max\{|f(y) - f(x)|\colon |y - x| = r\}}{\min\{|f(y) - f(x)|\colon |y - x| = r\}} \leq H < \infty$$

for all $x \in D$ and some H independent of x. This is the so-called metric definition of quasiconformal maps. For the connection between this and the geometrical or the analytic definition of quasiconformal maps, see the book by Väisälä [10].

A mapping $f\colon (X, d_X) \to (Y, d_Y)$ between metric spaces is said to be *L-bilipschitz* if

$$d_X(x, y)/L \leq d_Y(f(x), f(y)) \leq L\, d_X(x, y)$$

for all $x, y \in X$ and some constant $L \geq 1$; and a mapping is a *(λ, L)-quasisimilarity* if

$$\frac{\lambda}{L} d_X(x, y) \leq d_Y(f(x), f(y)) \leq \lambda\, L\, d_X(x, y)$$

for all $x, y \in X$ and some constants $L \geq 1$ and $\lambda > 0$. Clearly $(1, L)$-quasisimilarities are the L-bilipschitz mappings, $(\lambda, 1)$-quasisimilarities are similarities with scaling factor λ, and 1-similarities are isometries.

2. Building components

The construction in this section is largely based on an example of Assouad in [1] and an example of Bonk and Heinonen in [3]. The book-keeping below is needed in the eventual bilipschitz estimates.

2.1. Blocks

Let $\mathcal{C}$ be the cube $[-\frac{7}{2}, \frac{7}{2}] \times [-\frac{7}{2}, \frac{7}{2}] \times [0, 7]$, I be the (directed) line segment from $(0, 0, 0)$ to $(0, 0, 7)$, and

$$Q_I = [-3, 3] \times [-3, 3] \times [0, 7],$$

a rectangular cube in $\mathcal{C}$ that meets $\partial\mathcal{C}$ in two squares. We call Q_I an *I-block*, squares $Q_I \cap \{x_3 = 0\}$ and $Q_I \cap \{x_3 = 7\}$ the *entrance* and the *exit* of Q_I respectively, and ∂Q_I minus the union of the interiors of the entrance and the exit the *side* sQ_I of Q_I.

Let L be the (directed) line segment from $(0, 0, 0)$ to $(0, 0, \frac{7}{2})$ followed by the (directed) line segment from $(0, 0, \frac{7}{2})$ and $(0, \frac{7}{2}, \frac{7}{2})$, and

$$Q_L = \Big([-3, 3] \times [-3, 3] \times \Big[0, \tfrac{13}{2}\Big]\Big) \cup \Big([-3, 3] \times \Big[-3, \tfrac{7}{2}\Big] \times \Big[\tfrac{1}{2}, \tfrac{13}{2}\Big]\Big)$$

be a union of two rectangular cubes in $\mathcal{C}$ that meets $\mathcal{C}$ in two squares. We call Q_L an *L-block*, squares $Q_L \cap \{x_3 = 0\}$ and $Q_L \cap \{x_2 = \frac{7}{2}\}$ the *entrance* and the *exit* of Q_L respectively, and the boundary ∂Q_L minus the union of the interiors of the entrance and the exit the *side* sQ_L of Q_L.

We write Q for Q_I or Q_L when the type of the block is inessential. Images of I, L, Q_I, or Q_L under similarities are again called I-segments, L-segments, I-blocks, or L-blocks; they have a naturally inherited direction. When h is a similarity map and ℓ is either the I-segment $h(I)$ or the L-segment $h(L)$, we write Q_ℓ for $h(Q_I)$ or $h(Q_L)$, and call Q_ℓ the *block associated with the segment* ℓ.

2.2. Cores

There exists a simple polygonal path J_I in Q_I going from $(0, 0, 0)$ to $(0, 0, 7)$, which is symmetric with respect to the plane $x_3 = \frac{7}{2}$ and has the following properties:

(1) J_I is unknotted in Q_I, in the sense that there is a homeomorphism of Q_I onto itself which is the identity map on ∂Q_I and maps J_I onto the line segment from $(0, 0, 0)$ to $(0, 0, 7)$.

(2) J_I consists of a total of 49 I-segments and L-segments, $\ell_1, \dots, \ell_{49}$, of unit length, with mutually disjoint interiors, and labeled according to their order in J_I. Moreover, segments ℓ_1, ℓ_{49} and at least one other are I-segments.

(3) All blocks Q_{ℓ_m} are contained in the interior of Q_I with the exception that $Q_{\ell_1} \cap \partial Q_I$ is the entrance of Q_{ℓ_1} and $Q_{\ell_{49}} \cap \partial Q_I$ is the exit of $Q_{\ell_{49}}$. Only blocks associated to the consecutive segments meet, and they meet in such a way that $Q_{\ell_m} \cap Q_{\ell_{m+1}}$ is the exit of Q_{ℓ_m} and the entrance of $Q_{\ell_{m+1}}$. Finally, the union $\bigcup_1^{49} Q_{\ell_m}$ of the blocks is homeomorphic to a cube.

We call $\kappa_{Q_I} := \bigcup_1^{49} Q_{\ell_m}$ the *core of* Q_I. The entrance, the exit and the side of the core are canonically defined. For simplicity, we will write Q_m for Q_{ℓ_m}.

Similarly, there is a simple polygonal path J_L in Q_L going from $(0, 0, 0)$ to $(0, \frac{7}{2}, \frac{7}{2})$, which is symmetric with respect to the plane $x_3 = \frac{7}{2} - x_2$ and satisfies the analogues of (1), (2) and (3) adapted for Q_L. Here, only segments ℓ_1 and ℓ_{49} are required to be I-segments. The *core* κ_{Q_J} *of* Q_J, and the entrance, the exit and the side of the core are defined analogously. (Examples of paths J_I and J_L are given in the appendix.)

Symmetry of the paths is imposed to simplify the argument below; it is not essential. There is nothing special about the numbers 7 and 49 either, except that $\frac{\log 49}{\log 7}$ is the Hausdorff dimension of the singular line in $\mathbb{G}$.

2.3. Edges

Block Q_I has four edges on its side, namely, $\{(3, 3, t) : 0 \le t \le 7\}$, $\{(-3, 3, t) : 0 \le t \le 7\}$, $\{(-3, -3, t) : 0 \le t \le 7\}$, and $\{(3, -3, t) : 0 \le t \le 7\}$; only one will be labeled

$$e_0 = \{(3, 3, t) : 0 \le t \le 7\}.$$

Block Q_L has four edges on its side; they are labeled as

$$\begin{aligned}
e_1 &= \left\{(3, 3, t) : 0 \le t \le \tfrac{1}{2}\right\} \cup \left\{\left(3, t, \tfrac{1}{2}\right) : 3 \le t \le \tfrac{7}{2}\right\},\\
e_2 &= \left\{(-3, 3, t) : 0 \le t \le \tfrac{1}{2}\right\} \cup \left\{\left(-3, t, \tfrac{1}{2}\right) : 3 \le t \le \tfrac{7}{2}\right\},\\
e_3 &= \left\{(-3, -3, t) : 0 \le t \le \tfrac{13}{2}\right\} \cup \left\{\left(-3, t, \tfrac{13}{2}\right) : -3 \le t \le \tfrac{7}{2}\right\},\\
e_4 &= \left\{(3, -3, t) : 0 \le t \le \tfrac{13}{2}\right\} \cup \left\{\left(3, t, \tfrac{13}{2}\right) : -3 \le t \le \tfrac{7}{2}\right\}.
\end{aligned}$$

We define *edge paths* along core κ_{Q_L} as follows. An edge path initiates at a vertex p of κ_{Q_L} on the plane $x_3 = 0$, it moves along the edge of the first block in κ_{Q_L} that starts at p. When the path reaches the end point q of that edge, it continues along the edge of the second block in κ_{Q_L} that starts at q, and so on. The path stops when it reaches the plane $x_2 = \frac{7}{2}$. Along the way, the path is marked by the edges it passes through.

Corresponding to each starting point p, there is a unique edge path. Four choices of the starting point result in *four mutually disjoint edge paths* w_1, w_2, w_3, and w_4, each of which is symmetric with respect to the plane $x_3 = \frac{7}{2} - x_2$. Edge paths are labeled so that w_1, w_2, w_3, and w_4 start at $(\frac{3}{7}, \frac{3}{7}, 0)$, $(-\frac{3}{7}, \frac{3}{7}, 0)$, $(-\frac{3}{7}, -\frac{3}{7}, 0)$, and $(\frac{3}{7}, -\frac{3}{7}, 0)$ respectively, and end at $(\frac{3}{7}, \frac{7}{2}, -\frac{3}{7} + \frac{7}{2})$, $(-\frac{3}{7}, \frac{7}{2}, -\frac{3}{7} + \frac{7}{2})$, $(-\frac{3}{7}, \frac{7}{2}, \frac{3}{7} + \frac{7}{2})$, and $(\frac{3}{7}, \frac{7}{2}, \frac{3}{7} + \frac{7}{2})$ respectively. Note in particular that for every $i = 1, 2, 3$, or 4, the origin $(0, 0, 0)$, the starting point of w_i, and the starting point of edge e_i are collinear, and that the same can be said about the point $(0, \frac{7}{2}, \frac{7}{2})$ and the terminal points of w_i and e_i. We call $w_i \cap Q_m$ the *marked edge of Q_m derived from data (Q_L, e_i)*, for $m = 1, \ldots, 49$.

There are four edge paths along the core κ_{Q_I} as well. We label the *edge path* going from $(\frac{3}{7}, \frac{3}{7}, 0)$ to $(\frac{3}{7}, \frac{3}{7}, 7)$ by w_0, and call $w_0 \cap Q_m$ the *marked edge of Q_m derived from data (Q_I, e_0)* for $m = 1, \ldots, 49$.

2.4. Tubes

Consider *tubes*

$$\tau_{Q_I} = \overline{Q_I \setminus \kappa_{Q_I}} \qquad \text{and} \qquad \tau_{Q_L} = \overline{Q_L \setminus \kappa_{Q_L}}$$

obtained by removing the cores from the blocks. The entrance and the exit of τ_{Q_I} are canonically defined; they are congruent to the rectangular annulus

$$A = ([-3, 3] \times [-3, 3]) \setminus \left(\left(-\tfrac{3}{7}, \tfrac{3}{7}\right) \times \left(-\tfrac{3}{7}, \tfrac{3}{7}\right)\right) \subset \mathbb{R}^2 \times \{0\}.$$

The remaining part of $\partial \tau_{Q_I}$ is composed of the side sQ_I of block Q_I and the side $s\kappa_{Q_I}$ of core κ_{Q_I}. The boundary of τ_{Q_L} can be similarly partitioned.

2.5. Bilipschitz maps between tubes

Let $\mathsf{Q} = \{x \in \mathbb{R}^2 : |x| \leq 1\} \times [0, 1]$ be a round block in $\mathbb{R}^3$ having a core $\mathsf{k} = \{x \in \mathbb{R}^2 : |x| \leq \frac{1}{49}\} \times [0, 1]$, that is composed of 49 congruent blocks $\mathsf{Q}_m = \{x \in \mathbb{R}^2 : |x| \leq \frac{1}{49}\} \times [\frac{m-1}{49}, \frac{m}{49}]$, $m = 1, \ldots, 49$. Denote by $\mathsf{t} = \overline{\mathsf{Q} \setminus \mathsf{k}}$ the round tube $\{x \in \mathbb{R}^2 : \frac{1}{49} \leq |x| \leq 1\} \times [0, 1]$, and by $\mathsf{t}_m = \{x \in \mathbb{R}^2 : \frac{1}{49^2} \leq |x| \leq \frac{1}{49}\} \times [\frac{m-1}{49}, \frac{m}{49}]$, $m = 1, \ldots, 49$, 49 congruent tubes.

Denote by $\mathsf{e} = \{(1, 0, t) : 0 \leq t \leq 1\}$ an edge of Q, and by $\mathsf{w} = \{(\frac{1}{49}, 0, t) : 0 \leq t \leq 1\}$ an edge path along k; let $\mathsf{A} = \{x \in \mathbb{R}^2 : \frac{1}{49} \leq |x| \leq 1\}$ be the annulus in $\mathbb{R}^2 \times \{0\}$ that is congruent to the entrance and the exit of the tube t.

Define an isometric involution in each of Q_I, Q_L, and Q, by the reflection with respect to the planes $x_3 = \frac{7}{2}$, $x_3 = \frac{7}{2} - x_2$, or $x_3 = \frac{1}{2}$ respectively.

Let ζ_m be the similarity map in $\mathbb{R}^3$ with

$$\zeta_m : (\mathsf{Q}, \mathsf{e}) \to \left(\mathsf{Q}_m, \mathsf{w} \cap \mathsf{Q}_m\right),$$

and ϱ_α be the rotation of $\mathbb{R}^2$ about the origin by an angle α.

Let $\phi\colon \mathsf{A} \to A$ be the sense-preserving homeomorphism from the annulus A onto the rectangular annulus A, that maps each radial segment in A linearly onto a radial segment in A having the same argument. Let $\varphi = \phi \circ \varrho_{\pi/4}$. Note that φ maps the points $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$ to the points $(3,3)$, $(-3,3)$, $(-3,-3)$, $(3,-3)$, respectively.

Let $\theta_0\colon (s\mathsf{Q}, \mathsf{e}) \to (sQ_I, e_0)$ be the homeomorphism

$$\theta_0(x_1, x_2, x_3) = \big(\varphi(x_1, x_2), x_3\big);$$

fix, for each $i \in \{1,2,3,4\}$, a bilipschitz homeomorphism $\theta_i\colon (s\mathsf{Q}, \mathsf{e}) \to (sQ_L, e_i)$ that satisfies

$$\theta_i(x_1, x_2, 0) = \big(\varphi \circ \varrho_{(i-1)\pi/2}(x_1, x_2), 0\big)$$

and intertwines the involutions in Q and Q_L.

Given an integer $i \in \{0,1,2,3,4\}$, set $Q = Q_I$ when $i = 0$, and $Q = Q_L$ when $i \neq 0$. We will define, as in [3], a collection of five basic bilipschitz homeomorphisms $\Theta_i\colon (\mathsf{t}, \mathsf{e}, \mathsf{w}) \to (\tau_Q, e_i, w_i)$, $i = 0, \dots, 4$.

We first define a bilipschitz homeomorphism $\vartheta_i\colon (\partial\mathsf{t}, \mathsf{e}, \mathsf{w}) \to (\partial\tau_Q, e_i, w_i)$ on the boundary following these steps:

(i) on the outer side of t, mapping $\vartheta_i | s\mathsf{Q} = \theta_i\colon (s\mathsf{Q}, \mathsf{e}) \to (sQ, e_i)$;

(ii) the restriction of ϑ_i to the entrance (or the exit) of t is φ modulo an isometry;

(iii) associated to each block Q_m in the core κ_Q, there is a marked edge $\varepsilon(Q_m) = w_i \cap Q_m$ derived from (Q, e_i), hence there exist a unique $Q(Q_m) \in \{Q_I, Q_L\}$, a unique $\iota(Q_m) \in \{0,1,2,3,4\}$, and a similarity map $\sigma_m\colon (Q(Q_m), e_{\iota(Q_m)}) \to (Q_m, \varepsilon(Q_m))$. (To ease the notations, the dependency of i is not recorded.) The inner side of t is $s\mathsf{k}$; the mapping $\vartheta_i | s\mathsf{k}\colon (s\mathsf{k}, \mathsf{w}) \to (s\kappa_Q, w_i)$ is defined by gluing together the maps $\vartheta_i | s\mathsf{Q}_m = \sigma_m \circ \theta_{\iota(Q_m)} \circ \zeta_m^{-1}$, $1 \le m \le 49$. The gluing is well-defined because the union of marked edges of Q_m's is the edge path w_i.

During this process, each block Q_m inherits from $(Q(Q_m), e_{\iota(Q_m)})$ a core and a marked edge on each of the 49 blocks in this core, via the similarity map σ_m.

Since J_I (or J_L) is unknotted in Q and κ_Q is a regular neighborhood of J_I (or J_L) in Q, there is a (bilipschitz) homeomorphism $\Phi\colon (Q, \kappa_Q, e_i) \to (\mathsf{Q}, \mathsf{k}, \mathsf{e})$ that agrees with ϑ_i^{-1} on $\partial Q \setminus \kappa_Q$ and respects the given involutions. Since w_i ($\subset \partial\kappa_Q$) is fixed by the involution in Q, $\Phi(w_i)$ ($\subset \mathsf{Q}$) is symmetric with respect to the plane $x_3 = \frac{1}{2}$. Therefore $\Phi(w_i)$ can be straightened so that $\Phi(w_i) = \mathsf{w}$. The composition $\Phi \circ \vartheta_i\colon (\partial\mathsf{t}, \mathsf{e}, \mathsf{w}) \to (\partial\mathsf{t}, \mathsf{e}, \mathsf{w})$, which is identity on $\partial\mathsf{t} \setminus \mathsf{k}$, clearly has a bilipschitz extension $\Psi\colon \mathsf{t} \to \mathsf{t}$. Therefore $\vartheta_i | \partial\mathsf{t}$ has a bilipschitz extension

$$\Theta_i = \Phi^{-1} \circ \Psi\colon (\mathsf{t}, \mathsf{e}, \mathsf{w}) \to \big(\tau_Q, e_i, w_i\big)$$

between tubes.

We now have five basic bilipschitz homeomorphisms $\Theta_i\colon (\mathsf{t}, \mathsf{e}, \mathsf{w}) \to (\tau_Q, e_i, w_i)$, $i = 0, \ldots, 4$, at our disposal. Uniqueness has been emphasized throughout their construction to ensure the forward and backward iteration processes below are self-similar.

2.6. Quasiconformal maps between blocks

We define a quasiconformal map f from block Q onto block Q_I with data (Q_I, e_0). Set

$$\mathsf{K}_{-k} = \left\{x \in \mathbb{R}^2\colon |x| \le 49^{-k}\right\} \times [0, 1] \text{ and } \mathsf{T}_{-k} = \overline{\mathsf{K}_{-k} \setminus \mathsf{K}_{-k-1}},$$

for all $k \ge 0$. Then

$$\mathsf{Q} = \left\{(0, 0, t)\colon 0 \le t \le 1\right\} \cup \bigcup_{k\ge 0} \mathsf{T}_{-k}.$$

Set also $Q = Q_I$, $K_0 = Q$, $K_{-1} = \kappa_Q$, and $T_0 = \overline{K_0 \setminus K_{-1}} = \tau_Q$; note that $\mathsf{T}_0 = \mathsf{t}$, and let

$$f\big|\mathsf{T}_0 = \Theta_0\colon \mathsf{T}_0 \to T_0.$$

For every $m \in [1, 49]$, let $\varepsilon(Q_m)$ be the marked edge on Q_m derived from (Q_I, e_0), and $\sigma_m\colon (Q(Q_m), e_{\iota(Q_m)}) \to (Q_m, \varepsilon(Q_m))$ be the similarity used in constructing Θ_0. The similarity σ_m induces naturally a core $\kappa_m (= \kappa_{Q_m})$, consequently a tube $\tau_m = \overline{Q_m \setminus \kappa_m}$ to each block Q_m in K_{-1}.

Set $K_{-2} = \cup_m \kappa_m$ and $T_{-1} = \overline{K_{-1} \setminus K_{-2}} = \cup_m \tau_{Q_m}$. Since $\mathsf{T}_{-1} = \cup_m \mathsf{t}_m$, the mapping $f|\mathsf{T}_{-1}\colon \mathsf{T}_{-1} \to T_{-1}$ will be defined by gluing together homeomorphisms

$$f\big|\mathsf{t}_m = \sigma_m \circ \Theta_{\iota(Q_m)} \circ \zeta_m^{-1}\colon \mathsf{t}_m \to \tau_{Q_m}.$$

Because the union W_{-1} of marked edges $\varepsilon(Q_m)$ is an edge path along κ_Q going from $(\frac{3}{7}, \frac{3}{7}, 0)$ to $(\frac{3}{7}, \frac{3}{7}, 7)$, and the restrictions of $f|\mathsf{t}_m$ to the entrance and to the exit of t_m are essentially identical (modulo isometries) for all m, we conclude that the gluing, therefore the homeomorphism $f|\mathsf{T}_{-1}$, is well-defined. We now have the extension

$$f\colon \mathsf{T}_0 \cup \mathsf{T}_{-1} \to T_0 \cup T_{-1}.$$

Before starting the next step, we write $\varepsilon_m, \iota_m, Q(m)$ in place of $\varepsilon(Q_m)$, $\iota(Q_m)$, $Q(Q_m)$ to simplify the notation, and replace the index m in the previous step by m_1. For each $m_1 \in [1, 49]$, the process of defining $f|\mathsf{t}_{m_1}$ has uniquely defined a core κ_{m_1,m_2}, a tube τ_{m_1,m_2}, a marked edge ε_{m_1,m_2}, a block $Q(m_1, m_2) \in \{Q_I, Q_L\}$, a number $\iota_{m_1,m_2} \in \{0,1,2,3,4\}$, and a similarity map $\sigma_{m_1,m_2}\colon (Q(m_1, m_2), e_{\iota_{m_1,m_2}}) \to (Q_{m_1,m_2}, \varepsilon_{m_1,m_2})$, associated to each of the 49 $(1 \le m_2 \le 49)$ blocks Q_{m_1,m_2} in the core κ_{m_1}.

The union W_{-2} of these 49^2 marked edges is an edge path along K_{-2} from $(\frac{3}{7^2}, \frac{3}{7^2}, 0)$ to $(\frac{3}{7^2}, \frac{3}{7^2}, 7)$, and the union K_{-3} of the cores of these 49^2 new blocks is a topological cube. Set $T_{-2} = \overline{K_{-2} \setminus K_{-3}}$. We now extend $f\colon \mathsf{T}_0 \to \mathsf{T}_{-1} \cup \mathsf{T}_{-2} \to T_0 \cup T_{-1} \cup T_{-2}$ by gluing together homeomorphisms

$$f|\mathsf{t}_{m_1,m_2} = \sigma_{m_1,m_2} \circ \Theta_{\iota_{m_1,m_2}} \circ \zeta_{m_2}^{-1} \circ \zeta_{m_1}^{-1}\colon \mathsf{t}_{m_1,m_2} \to \tau_{m_1,m_2}.$$

We observe, after a moment's reflection, the self-similar property on I-blocks: whenever Q_{m_1} is an I-block,

$$f|\mathsf{t}_{m_1,m_2} = \sigma_{m_1} \circ f|\mathsf{t}_{m_2} \circ \zeta_{m_1}^{-1}|\mathsf{t}_{m_1,m_2}.$$

Continue this process inductively in a self-similar manner, we arrive at a homeomorphism f from $\mathsf{Q} \setminus \{(0,0,t)\colon 0 \le t \le 1\}$ onto $Q_I \setminus \gamma$, where γ is the snowflake arc

$$\gamma = \bigcap_{k=1}^{\infty} K_{-k}.$$

In view of the scaling in the domain and in the target, f is a $(7^{-k}, C)$-quasisimilarity on each of the 49^k tubes in T_{-k}, for some constant $C > 1$, therefore the mapping $f\colon \mathsf{Q} \setminus \{(0,0,t)\colon 0 \le t \le 1\} \to Q_I \setminus \gamma$ is quasiconformal.

By a theorem of Väisälä on removable sets [10], f can be extended to be quasiconformal from Q onto Q_I.

3. Quasiconformal homeomorphism of $\mathbb{R}^3$

3.1. Quasiconformal extension to $\mathbb{R}^3$

Mapping $f\colon \mathsf{Q} \to Q_I$ will be extended to a quasiconformal homeomorphism of $\mathbb{R}^3$ by backward iteration.

We begin with Q_I and a fixed I-block $Q_{m'}$ in its core with $m' \ne 1, 49$.

Let $\zeta = \zeta_{m'}$ be the similarity in $\mathbb{R}^3$ that maps (Q, e) to $(\mathsf{Q}_{m'}, \mathsf{w} \cap \mathsf{Q}_{m'})$, and $\sigma = \sigma_{m'}$ be the similarity in $\mathbb{R}^3$ that maps (Q_I, e_0) to $(Q_{m'}, w_0 \cap Q_{m'})$ used in constructing Θ_0. Note that ζ has a scaling factor $1/49$ and σ has a scaling factor $1/7$.

Because $m' \ne 1, 49$, the space $\mathbb{R}^3$ is the union of an increasing sequence of I-blocks

$$\mathbb{R}^3 = \bigcup_{k \ge 0} \sigma^{-k} Q_I,$$

and also can be expressed as the union of an increasing sequence of round blocks

$$\mathbb{R}^3 = \bigcup_{k \ge 0} \zeta^{-k} \mathsf{Q}.$$

Observe that these unions are proper subsets of $\mathbb{R}^3$ when $m' = 1$ or 49.

Define homeomorphisms $F_k\colon \zeta^{-k}\mathsf{Q} \to \sigma^{-k}Q_I$, $k \geq 0$, by

$$F_k = \sigma^{-k} \circ f \circ \zeta^k.$$

In view of the self-similar property on I-blocks, we have $f \circ \zeta|\mathsf{Q} = \sigma \circ f$. Therefore, $F_k|\mathsf{Q} = \sigma^{-k} \circ f \circ \zeta^k|\mathsf{Q} = f$ for all $k \geq 0$, and

$$F_{k'}|\zeta^{-k}\mathsf{Q} = F_k \quad \text{for all } k' \geq k \geq 0.$$

The limiting map $F = \lim_{k\to\infty} F_k\colon \mathbb{R}^3 \to \mathbb{R}^3$ is well-defined, homeomorphic and quasiconformal.

3.2. A snowflake curve and a quasiplane

The snowflake arc $\gamma = \bigcap_{k=1}^{\infty} K_{-k}$ is in fact $F(\{(0,0,t)\colon 0 \leq t \leq 1\})$. Through backward iterations we get an infinite snowflake curve

$$\Gamma = \lim_{k\to\infty} \sigma^{-k}\gamma = F(\{(0,0)\} \times \mathbb{R}).$$

The plane $\mathsf{P} = \mathbb{R} \times \{0\} \times \mathbb{R}$ in $\mathbb{R}^3$ has a decomposition

$$\begin{aligned}\mathsf{P} &= (\{(0,0)\} \times \mathbb{R}) \cup \bigcup_{k\in\mathbb{Z}}\bigcup_{m\in\mathbb{Z}} \Big(\big[49^{k-1}, 49^k\big] \times \{0\} \times \big[m49^k, (m+1)49^k\big] \\ &\qquad\qquad \cup\big[-49^k, -49^{k-1}\big] \times \{0\} \times \big[m49^k, (m+1)49^k\big]\Big) \\ &= (\{(0,0)\} \times \mathbb{R}) \cup \bigcup_{k\in\mathbb{Z}}\bigcup_{m\in\mathbb{Z}} \mathsf{E}_{k,m} \cup \mathsf{E}'_{k,m}.\end{aligned}$$

Therefore the quasiplane $\mathcal{P} = F(\mathsf{P})$ can be expressed as

$$\mathcal{P} = \Gamma \cup \bigcup_{k\in\mathbb{Z}}\bigcup_{m\in\mathbb{Z}} F\big(\mathsf{E}_{k,m}\big) \cup F\big(\mathsf{E}'_{k,m}\big).$$

Denote by $\check{\mathsf{E}}_{k,m}$ (respectively $\check{\mathsf{E}}'_{k,m}$) the union of those rectangles in the above decomposition of P which meet $\mathsf{E}_{k,m}$ (respectively $\mathsf{E}'_{k,m}$). Then $F|\check{\mathsf{E}}_{k,m}$ and $F|\check{\mathsf{E}}'_{k,m}$ are $(7^{-k}, C)$-quasisimilarities for some constant $C > 1$ independent of k and m.

4. Proof of Theorem 1.1

From now on we use (x_1, x_3) for coordinates in $\mathbb{G}$, and recall that the Grushin metric has the dilation property

$$d_{\mathbb{G}}\Big(\big(\lambda x_1, \lambda^2 x_3\big), \big(\lambda y_1, \lambda^2 y_3\big)\Big) = \lambda\, d_{\mathbb{G}}\Big(\big(x_1, x_3\big), \big(y_1, y_3\big)\Big) \quad \text{for } \lambda > 0.$$

Furthermore for any $x = (x_1, x_3)$, $y = (y_1, y_3) \in \mathbb{G}$,

$$d_{\mathbb{G}}(x, y) \simeq |x_1 - y_1| + \min\left\{\sqrt{|x_3 - y_3|}, \frac{|x_3 - y_3|}{\max\{|x_1|, |y_1|\}}\right\}.$$

This estimate is known; it can be deduced from a theorem of Franchi and Lanconelli in [5] (see also Theorem 3 in [4]). In fact, $\max\{|x_1|, |y_1|\}$ may be replaced by $\min\{|x_1|, |y_1|\}$ in the above estimate. Here the expression $a \simeq b$ means that $a, b \geq 0$ and $C^{-1} \leq a/b \leq C$ for a constant $C \geq 1$.

Let $H : \mathbb{G} \to \mathsf{P}$ be the homeomorphism

$$H : (x_1, x_3) \mapsto (x_1|x_1|, 0, x_3);$$

then the composition $F \circ H$ is a homeomorphism from the Grushin plane $\mathbb{G}$ to the quasiplane $\mathcal{P}$.

To check $F \circ H$ is bilipschitz, we let $x = (x_1, x_3)$, $y = (y_1, y_3) \in \mathbb{R}^2$ and assume, as we may, that $|x_1| \geq |y_1|$. We consider four cases based on the relative locations of these points.

Case I. $|x_1| > 0$, $|x_1 - y_1| \leq |x_1|/7$ and $\sqrt{|x_3 - y_3|} \leq |x_1|/7$. In this case, $|x_1| \simeq |y_1|$, the Grushin distance $d_{\mathbb{G}}(x, y) \simeq |x_1 - y_1| + \frac{|x_3 - y_3|}{|x_1|}$ and the Euclidean distance $|H(x) - H(y)| \simeq |x_1||x_1 - y_1| + |x_3 - y_3|$. Suppose $H(x)$ is in a rectangle $\mathsf{E}_{k,m}$, then $7^{k-1} \leq |x_1| \leq 7^k$ and $H(y)$ is in $\check{\mathsf{E}}_{k,m}$. Since $F|\check{\mathsf{E}}_{k,m}$ is a $(7^{-k}, C)$-quasisimilarity for some $C > 1$, $|F \circ H(x) - F \circ H(y)| \simeq 7^{-k}(|x_1||x_1 - y_1| + |x_3 - y_3|) \simeq d_{\mathbb{G}}(x, y)$. The proof is the same if $H(x)$ is in $\mathsf{E}'_{k,m}$.

Case II. $|x_1| > 0$, $|x_1 - y_1| \geq |x_1|/7$ and $\sqrt{|x_3 - y_3|} \leq |x_1|/7$. In this case, $d_{\mathbb{G}}(x, y) \simeq |x_1 - y_1| \simeq |x_1|$ and $|H(x) - H(y)| \simeq |x_1|^2$. Suppose $H(x)$ is in $\mathsf{E}_{k,m}$, as before $\operatorname{diam} \mathsf{E}_{k,m} \simeq 49^k \simeq |x_1|^2$. Since F is quasisymmetric and $F|\mathsf{E}_{k,m}$ is a $(7^{-k}, C)$-quasisimilarity, $|F \circ H(x) - F \circ H(y)| \simeq \operatorname{diam} F(\mathsf{E}_{k,m}) \simeq 7^k \simeq d_{\mathbb{G}}(x, y)$. The proof is the same if $H(x)$ is in $\mathsf{E}'_{k,m}$.

Case III. $|x_1| > 0$ and $\sqrt{|x_3 - y_3|} \geq |x_1|/7$. In this case, $d_{\mathbb{G}}(x, y) \simeq \sqrt{|x_3 - y_3|}$ and $|H(x) - H(y)| \simeq |x_3 - y_3|$. Assume again $H(x)$ is in $\mathsf{E}_{k,m}$. After applying in $\mathbb{G}$ a translation $(z_1, z_3) \mapsto (z_1, z_3 - m49^k)$ followed by a dilation $(z_1, z_3) \to (7^{-k} z_1, 49^{-k} z_3)$, we assume as we may that $H(x) \in \mathsf{P} \cap \mathsf{Q}$. Choose j to be the smallest nonnegative integer j such that $H(y) \in \zeta^{-j}\mathsf{Q}$, then $|H(x) - H(y)| \simeq 49^j$ and $d_{\mathbb{G}}(x, y) \simeq 7^j$. From the definition of F it follows that $|F \circ H(x) - F \circ H(y)| = |F_j \circ H(x) - F_j \circ H(y)| \simeq 7^j$, hence $|F \circ H(x) - F \circ H(y)| \simeq d_{\mathbb{G}}(x, y)$. The case $H(x) \in \mathsf{E}'_{k,m}$ is the same.

Case IV. $x_1 = 0$. $|F \circ H(x) - F \circ H(y)| \simeq d_{\mathbb{G}}(x, y)$ can be obtained by taking limits in Case III.

This shows that $F \circ H : \mathbb{G} \to \mathcal{P}$ is bilipschitz and completes the proof of Theorem 1.1.

5. Appendix

We construct paths J_I and J_L in Section 2.2 following Assouad in [1]. To start, we subdivide the cube $\mathcal{C} = [-\frac{7}{2}, \frac{7}{2}] \times [-\frac{7}{2}, \frac{7}{2}] \times [0, 7]$ into 7^3 unit cubes. The centers of these subcubes are (z_1, z_2, z_3), with $z_1, z_2 \in \{-3, -2, -1, 0, 1, 2, 3\}$ and $z_3 \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2}\}$. Each of the 49 unit segments in J_I (respectively J_L) will pass through the center of and is contained in one of these unit cubes.

We define a polygonal path J' by listing its starting point, its terminal point and all points along the path where it makes a turn, which will be 'half' of the final paths, as follows

$$\begin{aligned} J' : (0,0,0) &\to \left(0,0,\tfrac{3}{2}\right) \to \left(0,1,\tfrac{3}{2}\right) \to \left(-2,1,\tfrac{3}{2}\right) \to \left(-2,-2,\tfrac{3}{2}\right) \\ &\to \left(2,-2,\tfrac{3}{2}\right) \to \left(2,0,\tfrac{3}{2}\right) \to \left(2,0,\tfrac{5}{2}\right) \to \left(2,-2,\tfrac{5}{2}\right) \\ &\to \left(-2,-2,\tfrac{5}{2}\right) \to \left(-2,0,\tfrac{5}{2}\right) \to \left(-1,0,\tfrac{5}{2}\right) \to \left(-1,0,\tfrac{7}{2}\right). \end{aligned}$$

Clearly J' is a simple path; except for its end points, J' is contained in the intersection of the half spaces $x_3 < \frac{7}{2}$ and $x_3 < \frac{7}{2} - x_2$. Denote by J'' the reflection of J' with respect to the plane $x_3 = \frac{7}{2}$, and by J''' the reflection of J' with respect to the plane $x_3 = \frac{7}{2} - x_2$. Both paths $J_I = J' \cup J''$ and $J_L = J' \cup J'''$ are simple.

Since the x_3-coordinate is monotone along path J', J_I is unknotted in Q_I and J_L is unknotted in Q_L. Because $|x_1| + |x_2| \leq 2$ on J', the blocks associated to the 49 unit segments in J_I (respectively J_L) are contained in Q_I (respectively Q_L). Other properties required for pathsJ_I and J_L are obvious.

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