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Generalized stochastic flow associated to the Itô SDE with partially Sobolev coefficients and its application

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Abstract. We consider the Itô SDEs on $\mathbb{R}^n$ with partially Sobolev coefficients. Assuming the exponential integrability of the negative part of the divergence of the drift coefficient and the partial gradient of the diffusion coefficient with respect to the Cauchy measure, we show the existence, uniqueness and stability of generalized stochastic flows associated to such equations. As an application, we prove the weak differentiability in the sense of measure of the stochastic flow generated by the Itô SDE with Sobolev coefficients.

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1. Introduction

We consider the following stochastic differential equation

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t + b(X_t)\,\mathrm{d}t, \quad X_0 = x \in \mathbb{R}^n, \tag{1.1}$$

in which $\sigma = (\sigma^{ik})_{1\le i\le n, 1\le k\le m}$ is a matrix-valued function, $b = (b^1, \dots, b^n)$ is a vector field, and B_t is an m-dimensional standard Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is well known that if σ and b are globally Lipschitz continuous, then equation (1.1) generates a unique stochastic flow X_t of homeomorphisms on $\mathbb{R}^n$. When the coefficients are less regular, for instance, they only have log-Lipschitz continuity, it is still possible to prove the homeomorphic property of the stochastic flow, see [13,24].

On the other hand, recently there are intensive studies on ODEs

$$\frac{\mathrm{d}X_t}{\mathrm{d}t} = b(X_t), \quad X_0 = x \in \mathbb{R}^n, \tag{1.2}$$

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with weakly differentiable coefficients, see for instance [1,8,9]. Here by weakly differentiable coefficients, we mean that they have Sobolev or even BV regularity. The methods adopted in [1,9] are quite indirect, in the sense that the authors first established the well-posedness of the corresponding first order PDEs (transport equation or continuity equation), from which they deduced the existence and uniqueness of generalized flow of measurable maps associated to (1.2) (see also [7] where the standard Gaussian measure γ_n is taken as the reference measure). This strategy can be seen as an extension of the classical characteristics method, and is now widely called the Di Perna–Lions theory. In [18, 19], Le Bris and Lions made use of these ideas to study the Fokker–Planck type equations with Sobolev coefficients; based on Ambrosio's commutator estimate for BV vector fields, we slightly extend their results to the case where the drift coefficient has only BV regularity, see [22]. The generalization of this theory to the infinite dimensional Wiener space has been done in [3, 14], see also [21] in which we studied the Fokker–Planck type equations on the Wiener space. In [10], the authors gave a rather sketchy argument of how to extend the Di Perna–Lions theory to compact Riemannian manifolds; by proving a commutator estimate involving the heat semi-group and Sobolev vector fields on manifolds, this theory was recently generalized in [12] to complete Riemannian manifolds under suitable conditions on the lower bound of the Ricci curvature. Using the pointwise characterization of Sobolev functions, Crippa and De Lellis gave in [8] direct proofs to many of the results in the Di Perna–Lions theory.

It seems that Di Perna and Lions's original method does not work for studying SDE (1.1), as pointed out in the introduction of [26]. X. Zhang successfully implemented in [25] the direct method of Crippa and De Lellis to the Itô SDE and proved the existence and uniqueness of stochastic flow of maps generated by (1.1). A drawback of the main result in [25, Theorem 2.6] is the requirement that $|\nabla\sigma|$ is bounded, a condition which is weakened in [27]. In [15] the authors took the standard Gaussian measure γ_n as the reference measure, and obtained similar results under the exponential integrability of $|\nabla\sigma|^2$, $|\mathrm{div}_{\gamma_n}(\sigma)|^2$ and $|\mathrm{div}_{\gamma_n}(b)|$. Here div_{γ_n} denotes the divergence with respect to the Gaussian measure γ_n. Note that the exponential integrability of $|\nabla\sigma|^2$ is quite weak, but that of $|\mathrm{div}_{\gamma_n}(\sigma)|^2$ prevents us from covering the classical case of globally Lipschitz coefficients, see [15, Theorem 1.2]. This is one of the reasons that we do not take γ_n as the reference measure in this paper. Another reason is that the results in Lemma 6.3 do not hold for the Gaussian measure γ_n. Here we also mention that we choose a finite measure on $\mathbb{R}^n$ as the reference measure and assume the divergences of the coefficients σ and b are exponentially integrable, hence they can be unbounded (both locally and globally, see Theorem 2.3 and [15, 27]), while the papers [1, 8, 9] are set in the framework of the Lebesgue measure, hence the authors naturally assume that the divergence $\mathrm{div}(b)$ (or its negative part $[\mathrm{div}(b)]^-$) is bounded.

The present work is motivated by [4, 8, 18], in which the authors studied the weak differentiability of the generalized flow associated to the ODE (1.2) with Sobolev vector field b. Again the results in [18] are derived from the related transport equation, while the ones in [4, 8] follow from the pointwise inequality of Sobolev functions. Since the generalized stochastic flow of measurable maps has already been established in [15,25,27], we intend to study in this work the differentiability of the stochastic flow. However, we are unable to transfer the methods in [4, 8] to the case of SDE for proving the approximate differentiability of the stochastic flow. The main problem is that the level set G_R (see Lemma 2.4) of the stochastic flow depends on the random element ω, hence one has to take expectation twice in order to estimate an quantity of the form (2.5) in [8]. We do not know how to handle this problem.

Therefore, we follow the idea of [18] to study the differentiability in the sense of measure of the stochastic flow. To this end, we first consider a special form of SDE (1.1) whose coefficients σ and b have the structure below: there is $n_1 \in \{1, \ldots, n-1\}$, such that

$$\sigma_1 := \left(\sigma^{ij}\right)_{1\le i\le n_1, 1\le j\le m} \quad \text{and} \quad b_1 := \left(b^1, \ldots, b^{n_1}\right)$$

only depend on the first n_1-variables $(x^1, \ldots, x^{n_1})$. In the following we also denote by σ_2 (respectively b_2) the last $(n - n_1)$-rows (respectively components) of the diffusion matrix σ (respectively the drift b), and $x_1 = (x^1, \ldots, x^{n_1})$, $x_2 = (x^{n_1+1}, \ldots, x^n)$ (thus $x \in \mathbb{R}^n$ can be written as (x_1, x_2)). Our basic assumptions, among other conditions that will be specified later, are

$$\sigma_1 \in W^{1,2q}_{x_1,\mathrm{loc}}, \quad b_1 \in W^{1,q}_{x_1,\mathrm{loc}}; \tag{1.3}$$

and

$$\sigma_2 \in L^{2q}_{x_1,\mathrm{loc}}\left(W^{1,2q}_{x_2,\mathrm{loc}}\right), \quad b_2 \in L^{q}_{x_1,\mathrm{loc}}\left(W^{1,q}_{x_2,\mathrm{loc}}\right). \tag{1.4}$$

Here $q > 1$ is a fixed number. Note that we don't require σ_2 and b_2 have Sobolev regularity with respect to x_1. Thanks to the key observation (6.4), we are able to deal with this special case.

The paper is organized as follows. In Section 2 we first recall the definition of generalized stochastic flow associated to Itô's SDE (1.1). After that, we extend the known results on the existence and uniqueness of stochastic flows generated by Itô's SDE to allow the coefficients to be locally unbounded. Recall that the main results in [15, 25, 27] require the coefficients σ and b have linear growth. This extension is necessary for proving the differentiability of the stochastic flow, since the linear growth condition for the second equation in (5.2) will basically result in the boundedness of the gradients of σ and b, which is too restrictive.

Then we state and prove an intermediate result in Section 3, where the coefficients $\sigma_2 \in W^{1,2q}_{x_1,x_2,\mathrm{loc}}$ and $b_2 \in W^{1,q}_{x_1,x_2,\mathrm{loc}}$. One reason for establishing such a result is to avoid regularizing the coefficients σ_1 and b_1 in the proof of the existence of stochastic flows generated by Itô's SDE with partially Sobolev coefficients (see Theorem 4.3); otherwise, we cannot apply the a-priori estimate in Lemma 4.1, since the coefficients σ_2 and b_2 have no Sobolev regularity on the variable $x_1 = (x^1, \ldots, x^{n_1})$. We also find a uniform estimate of the Radon–Nikodym density of the form Lemma 3.2, which does not involve the exponential integrability of $|\nabla_{x_1}\sigma_2|^2$.

The main result of this paper is presented in Section 4, where the key step is to prove an a-priori estimate which follows the idea of Crippa and De Lellis [8, Theorem 3.8] and has appeared in [15,25,27] in similar forms. The main difference between this estimate and the previous ones is that we only assume partial Sobolev regularity on the coefficients. As some of the arguments in Sections 3 and 4 are analogous to those of Section 2, we only give relatively detailed proofs in Section 2 and omit them in the subsequent sections to save space.

In Section 5 we apply the results obtained in the previous section to show the weak differentiability in the sense of measure of the generalized stochastic flow of measurable maps, following the ideas in [18, Section 4]. The main part consists in checking that the systems of Itô equations fulfil the assumptions in Section 4.

Finally, we present in the appendix some preliminary results that are used in the paper. Especially, we give a careful analysis of the expression of the Radon–Nikodym density which makes it possible for us to study the SDE with the above-mentioned special structure. We also prove an inequality for the integral of local maximal functions on the whole $\mathbb{R}^n$ with respect to some general finite measure which seems to have independent interest.

2. The Itô SDE with locally unbounded coefficients

First of all we give the precise meaning of the generalized stochastic flow (*cf.* [15, Definition 5.1] and [27, Definition 2.1]). This notion is related to some reference measure on $\mathbb{R}^n$. In this paper, we mainly consider the generalized Cauchy distributions (following the terminology of [5, Section 3]): for some $\alpha > n/2$, set

$$\lambda(x) = -\alpha \log\left(1 + |x|^2\right) \left(x \in \mathbb{R}^n\right) \quad \text{and} \quad \mathrm{d}\mu = e^{\lambda(x)}\,\mathrm{d}x. \tag{2.1}$$

The exact value of α has no importance. It is clear that $\mu(\mathbb{R}^n) < +\infty$. As usual, the space of continuous functions taking values in $\mathbb{R}^n$ is denoted by $C([0,T],\mathbb{R}^n)$. For a measurable map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, we write $\varphi_{\#}\mu = \mu \circ \varphi^{-1}$ for the push-forward of μ by φ (also called the distribution of φ under μ). Denote by $\theta_s B$ the time-shift of the Brownian motion, that is, $(\theta_s B)_t = B_{t+s} - B_s$ for all $t \geq 0$.

Definition 2.1. We say that a measurable map $X\colon \Omega \times \mathbb{R}^d \to C([0,T],\mathbb{R}^n)$ is a generalized stochastic flow associated to the Itô SDE (1.1) if

(i) for each $t \in [0,T]$ and almost all $x \in \mathbb{R}^n$, $\omega \to X_t(\omega, x)$ is measurable with respect to $\mathcal{F}_t$, *i.e.*, the natural filtration generated by the Brownian motion $\{B_s : s \leq t\}$;

(ii) for each $t \in [0,T]$, there exists $K_t \in L^1(\mathbb{P} \times \mu)$ such that $(X_t(\omega,\cdot))_{\#}\mu$ admits K_t as the density with respect to μ;

(iii) for $(\mathbb{P} \times \mu)$-a.e. (ω, x),

$$\int_0^T \left|\sigma\big(X_s(\omega,x)\big)\right|^2 \mathrm{d}s + \int_0^T \left|b\big(X_s(\omega,x)\big)\right| \mathrm{d}s < +\infty;$$

(iv) for μ-a.e. $x \in \mathbb{R}^n$, the integral equation below holds almost surely:

$$X_t(\omega, x) = x + \int_0^t \sigma(X_s(\omega, x))\,\mathrm{d}B_s + \int_0^t b(X_s(\omega, x))\,\mathrm{d}s, \quad \text{for all } t \in [0, T];$$

(v) the flow property holds

$$X_{t+s}(\omega, x) = X_t(\theta_s B, X_s(\omega, x)).$$

In this section we slightly extend the main results of [15,25,27] to allow the coefficients σ and b to be locally unbounded, while the aforementioned papers required that the coefficients have linear growth. To this end, we introduce some notations. Fix some $q > 1$ and take $\alpha > q + n/2$ in the definition (2.1) of the reference measure. We also denote by $\bar\sigma = \frac{\sigma}{1+|x|}$ and $\bar b = \frac{b}{1+|x|}$ to simplify the notations. We assume the following conditions:

(C1) $\sigma \in W^{1,2q}_{\mathrm{loc}}$, $b \in W^{1,q}_{\mathrm{loc}}$;
(C2) there is a $p_0 > 0$ such that $\int_{\mathbb{R}^n} \exp\big[p_0\big([\mathrm{div}(b)]^- + |\bar b| + |\bar\sigma|^2 + |\nabla\sigma|^2\big)\big]\mathrm{d}\mu < +\infty$.

Remark 2.2. We have the following observations.

(i) It is clear that when σ and b are globally Lipschitz continuous, they satisfy the conditions (C1) and (C2).
(ii) The condition (C2) implies $\bar\sigma, \bar b \in L^p(\mu)$ for any $p > 1$. Then by the choice of α, there is p sufficiently big such that $2\alpha - n > 2qp/(p-1)$, hence $\int_{\mathbb{R}^n} (1 + |x|)^{2qp/(p-1)}\mathrm{d}\mu < +\infty$. By Hölder's inequality,

$$\int_{\mathbb{R}^n} |\sigma|^{2q}\,\mathrm{d}\mu \le \bigg[\int_{\mathbb{R}^n} |\bar\sigma|^{2qp}\,\mathrm{d}\mu\bigg]^{1/p}\bigg[\int_{\mathbb{R}^n} (1+|x|)^{2qp/(p-1)}\mathrm{d}\mu\bigg]^{(p-1)/p} < +\infty.$$

Thus $\sigma \in L^{2q}(\mu)$. In the same way we have $b \in L^{2q}(\mu)$.
(iii) Let $\beta \in (0, n/p_0)$ and $Q_1 := \{x \in \mathbb{R}^n : 0 < |x| \le 1 \text{ and } x^i \ge 0,\ 1 \le i \le n\}$. Assume $\mathrm{supp}(b) \subset Q_1$ and $b(x) = \frac{1}{\sqrt{n}}(\log|x|^\beta)(1, \cdots, 1)$, $x \in Q_1$. Then the vector field b satisfies conditions (C1) and (C2). Indeed, $[\mathrm{div}(b)]^- \equiv 0$ and $|\bar b(x)| \le \mathbf{1}_{\{|x|\le 1\}} \log\frac{1}{|x|^\beta}$, thus $\int_{\mathbb{R}^n} e^{p_0|\bar b|}\,\mathrm{d}\mu < +\infty$. This example shows that the coefficient b (and also σ) of the Itô SDE can be locally unbounded. If we strengthen the condition (C2) by requiring that it holds for all $p_0 > 0$, then such a $\beta > 0$ does not exist.

We shall prove:

Theorem 2.3. *Under the conditions* (C1) *and* (C2), *there exists a unique generalized stochastic flow associated to the Itô SDE* (1.1). *Moreover, the Radon–Nikodym density ρ_t of the flow with respect to the reference measure μ satisfies $\rho_t \in L^1 \log L^1$.*

Here by $\rho_t \in L^1 \log L^1$ we mean that $\mathbb{E}\int_{\mathbb{R}^n} \rho_t |\log \rho_t| \,\mathrm{d}\mu < +\infty$. We remark that when t is small enough, the flow X_t is integrable on $\mathbb{R}^n$ with respect to μ, which is an easy consequence of Lemma 2.4 and Proposition 2.8. The integrability of X_t for general $t > 0$ can be proved if we strengthen the condition (C2) by requiring that it holds for any $p_0 > 0$; however, this condition is too restrictive in view of Remark 2.2(iii).

We shall divide the proof of this theorem into several steps, which are presented in the following lemmas and propositions. First we prove an a-priori estimate on the level set of the solution flow X_t. We denote by $\|\cdot\|_{\infty,T}$ the maximum norm in $C([0,T],\mathbb{R}^n)$, the space of continuous curves in $\mathbb{R}^n$. For $R > 0$, define the level set

$$G_R = \big\{(\omega, x) \in \Omega \times \mathbb{R}^n : \|X_\cdot(\omega, x)\|_{\infty,T} \le R\big\}.$$

Lemma 2.4 (Estimate of level sets). *Let X_t be a generalized stochastic flow associated to Itô SDE* (1.1)*, and ρ_t the Radon–Nikodym density with respect to μ. Suppose that*

$$\Lambda_{p,T} := \sup_{0\le t\le T} \|\rho_t\|_{L^p(\mathbb{P}\times\mu)} < +\infty,$$

where p is the conjugate number of q. Then under the condition (C2)*, we have*

$$(\mathbb{P}\times\mu)(G_R^c) \le \frac{C}{R},$$

where C depends on T, $\Lambda_{p,T}$, $\|\sigma\|_{L^{2q}(\mu)}$ and $\|b\|_{L^q(\mu)}$.

Proof. First we deduce from (C2) and Remark 2.2(ii) that $\|\sigma\|_{L^{2q}(\mu)}$ and $\|b\|_{L^q(\mu)}$ are finite. For a.e. $(\omega, x) \in \Omega \times \mathbb{R}^n$, we have

$$X_t(x) = x + \int_0^t \sigma(X_s(x))\,\mathrm{d}B_s + \int_0^t b(X_s(x))\,\mathrm{d}s.$$

Therefore

$$\|X_\cdot(x)\|_{\infty,T} \le |x| + \sup_{0\le t\le T}\left|\int_0^t \sigma(X_s(x))\,\mathrm{d}B_s\right| + \sup_{0\le t\le T}\left|\int_0^t b(X_s(x))\,\mathrm{d}s\right|. \tag{2.2}$$

By Burkholder's inequality,

$$\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t \sigma(X_s(x))\,\mathrm{d}B_s\right| \le 2\left[\mathbb{E}\int_0^T \big|\sigma(X_s(x))\big|^2\,\mathrm{d}s\right]^{\frac12}.$$

Now Cauchy's inequality leads to

$$\begin{aligned}\int_{\mathbb{R}^n}\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t \sigma(X_s(x))\,\mathrm{d}B_s\right|\mathrm{d}\mu &\le 2\mu(\mathbb{R}^n)^{\frac12}\left[\int_0^T \mathbb{E}\int_{\mathbb{R}^n}\big|\sigma(X_s(x))\big|^2\,\mathrm{d}\mu(x)\mathrm{d}s\right]^{\frac12}\\ &= 2\mu(\mathbb{R}^n)^{\frac12}\left[\int_0^T \mathbb{E}\int_{\mathbb{R}^n}\big|\sigma(y)\big|^2\rho_s(y)\,\mathrm{d}\mu(y)\mathrm{d}s\right]^{\frac12}.\end{aligned}$$

We have by Hölder's inequality that

$$\mathbb{E}\int_{\mathbb{R}^n}|\sigma(y)|^2\rho_s(y)\,\mathrm{d}\mu(y)\le\|\sigma\|_{L^{2q}(\mu)}^2\|\rho_s\|_{L^p(\mathbb{P}\times\mu)}\le\Lambda_{p,T}\|\sigma\|_{L^{2q}(\mu)}^2.$$

Therefore

$$\int_{\mathbb{R}^n}\mathbb{E}\sup_{0\le t\le T}\left|\int_0^t\sigma(X_s(x))\,\mathrm{d}B_s\right|\mathrm{d}\mu\le 2(\mu(\mathbb{R}^n)\,T\Lambda_{p,T})^{\frac12}\|\sigma\|_{L^{2q}(\mu)}. \tag{2.3}$$

Next

$$\begin{aligned}\mathbb{E}\int_{\mathbb{R}^n}\sup_{0\le t\le T}\left|\int_0^t b(X_s(x))\,\mathrm{d}s\right|\mathrm{d}\mu&\le\int_0^T\mathbb{E}\int_{\mathbb{R}^n}|b(X_s(x))|\,\mathrm{d}\mu(x)\mathrm{d}s\\&=\int_0^T\mathbb{E}\int_{\mathbb{R}^n}|b(y)|\rho_s(y)\,\mathrm{d}\mu(y)\mathrm{d}s.\end{aligned}$$

Again by Hölder's inequality,

$$\begin{aligned}\mathbb{E}\int_{\mathbb{R}^n}\sup_{0\le t\le T}\left|\int_0^t b(X_s(x))\,\mathrm{d}s\right|\mathrm{d}\mu&\le\int_0^T\|b\|_{L^q(\mu)}\|\rho_s\|_{L^p(\mathbb{P}\times\mu)}\,\mathrm{d}s\\&\le T\Lambda_{p,T}\|b\|_{L^q(\mu)}.\end{aligned}\tag{2.4}$$

Now integrating both sides of (2.2) on $\Omega\times\mathbb{R}^n$ and by (2.3), (2.4), we get

$$\begin{aligned}\mathbb{E}\int_{\mathbb{R}^n}\|X_\cdot(x)\|_{\infty,T}\,\mathrm{d}\mu\le C_1&+2\big(\mu(\mathbb{R}^n)\,T\Lambda_{p,T}\big)^{\frac12}\|\sigma\|_{L^{2q}(\mu)}\\&+T\Lambda_{p,T}\|b\|_{L^q(\mu)},\end{aligned}\tag{2.5}$$

where $C_1:=\int_{\mathbb{R}^n}|x|\,\mathrm{d}\mu(x)<+\infty$. Finally by Chebyshev's inequality,

$$(\mathbb{P}\times\mu)(G_R^c)\le\frac{1}{R}\int_{\Omega\times\mathbb{R}^n}\|X_\cdot(x)\|_{\infty,T}\,\mathrm{d}(\mathbb{P}\times\mu)\le\frac{C}{R},$$

where C is given by the right hand side of (2.5). □

Similar to [25, Lemma 6.1], [15, Theorem 5.2] and [27, Lemma 4.1], we have the following:

Lemma 2.5 (Stability estimate). *Suppose that $\sigma,\tilde\sigma\in W^{1,2q}_{\mathrm{loc}}$ and $b,\tilde b\in W^{1,q}_{\mathrm{loc}}$. Let X_t (respectively $\tilde X_t$) be the stochastic flow associated to the Itô SDE* (1.1) *with coefficients σ and b (respectively $\tilde\sigma$ and $\tilde b$). Denote by ρ_t (respectively $\tilde\rho_t$) the Radon–Nikodym density of X_t (respectively $\tilde X_t$) with respect to μ. Assume that*

$$\Lambda_{p,T}:=\sup_{0\le t\le T}\big(\|\rho_t\|_{L^p(\mathbb{P}\times\mu)}\vee\|\tilde\rho_t\|_{L^p(\mathbb{P}\times\mu)}\big)<+\infty,$$

where p is the conjugate number of q. Then for any $\delta > 0$,

$$\begin{aligned}
&\mathbb{E}\int_{G_R\cap\tilde G_R}\log\bigg(\frac{\|X-\tilde X\|_{\infty,T}^2}{\delta^2}+1\bigg)\mathrm{d}\mu\\
&\le C_T\Lambda_{p,T}\bigg\{C_{n,q}\Big[\|\nabla b\|_{L^q(B(3R))}+\|\nabla\sigma\|_{L^{2q}(B(3R))}+\|\nabla\sigma\|_{L^{2q}(B(3R))}^2\Big]\\
&\qquad+\frac{1}{\delta^2}\|\sigma-\tilde\sigma\|_{L^{2q}(B(R))}^2+\frac{1}{\delta}\Big[\|\sigma-\tilde\sigma\|_{L^{2q}(B(R))}+\|b-\tilde b\|_{L^q(B(R))}\Big]\bigg\},
\end{aligned}$$

where $\tilde G_R := \big\{(\omega,x)\in\Omega\times\mathbb{R}^n : \|\tilde X_\cdot(\omega,x)\|_{\infty,T}\le R\big\}$ is the level set of the flow $\tilde X_t$.

Here the space $L^q(B(R))$ is defined with respect to the Lebesgue measure. The proof of Lemma 2.5 is similar to the above cited references, hence we omit it.

Now we start to prove the existence part of Theorem 2.3. We have to regularize the coefficients σ and b. Let $\chi\in C_c^\infty(\mathbb{R}^n,\mathbb{R}_+)$ be such that $\int_{\mathbb{R}^n}\chi\,\mathrm{d}x = 1$ and its support $\mathrm{supp}(\chi)\subset B(1)$. For $k\ge 1$, define $\chi_k(x)=k^n\chi(kx)$ for all $x\in\mathbb{R}^n$. Next choose $\psi\in C_c^\infty(\mathbb{R}^n,[0,1])$ which satisfies $\psi|_{B(1)}\equiv 1$ and $\mathrm{supp}(\psi)\subset B(2)$. Set $\psi_k(x)=\psi(x/k)$ for all $x\in\mathbb{R}^n$ and $k\ge 1$. Now we define

$$\sigma_k=(\sigma*\chi_k)\,\psi_k \quad\text{and}\quad b_k=(b*\chi_k)\,\psi_k.$$

Then for every $k\ge 1$, the functions σ_k and b_k are smooth with compact supports. Consider the following Itô's SDE:

$$\mathrm{d}X_t^k=\sigma_k(X_t^k)\,\mathrm{d}B_t+b_k(X_t^k)\,\mathrm{d}t,\quad X_0^k=x. \tag{2.6}$$

This equation has a unique strong solution which gives rise to a stochastic flow of diffeomorphisms on $\mathbb{R}^n$. Denote by ρ_t^k the Radon–Nikodym density of $(X_t^k)_\#\mu$ with respect to μ. Applying Lemma 6.1 for $p>1$, we have

$$\|\rho_t^k\|_{L^p(\mathbb{P}\times\mu)}\le\mu(\mathbb{R}^n)^{\frac{1}{p+1}}\bigg(\sup_{t\in[0,T]}\int_{\mathbb{R}^n}\exp\big(p^3t\big|\Lambda_1^{\sigma_k}\big|^2-p^2t\Lambda_2^{\sigma_k,b_k}\big)\mathrm{d}\mu\bigg)^{\frac{1}{p(p+1)}}. \tag{2.7}$$

We shall give a uniform estimate to the density functions.

Lemma 2.6 (Uniform density estimate). *For fixed $p>1$, there are two positive constants $C_{1,p}, C_{2,p}>0$ and sufficiently small $T_0>0$, such that for all $k\ge 1$,*

$$\begin{aligned}
&\sup_{0\le t\le T_0}\|\rho_t^k\|_{L^p(\mathbb{P}\times\mu)}\\
&\le C_{1,p}\bigg(\int_{\mathbb{R}^n}\exp\big[C_{2,p}T_0\big([\mathrm{div}(b)]^-+|\bar b|+|\nabla\sigma|^2+|\bar\sigma|^2\big)\big]\mathrm{d}\mu\bigg)^{\frac{1}{p(p+1)}}<+\infty.
\end{aligned} \tag{2.8}$$

Proof. Using the expressions of $\Lambda_1^{\sigma_k}$, $\Lambda_2^{\sigma_k,b_k}$ and λ, elementary computations lead to

$$|\Lambda_1^{\sigma_k}|^2 \le C_0\big(|\mathrm{div}(\sigma)|^2 + |\bar\sigma|^2\big) * \chi_k$$

and

$$-\Lambda_2^{\sigma_k,b_k} \le C_0\big([\mathrm{div}(b)]^- + |\bar b| + |\nabla\sigma|^2 + |\bar\sigma|^2\big) * \chi_k.$$

Noticing that $|\mathrm{div}(\sigma)| \le |\nabla\sigma|$, we have for any $t > 0$,

$$p^3 t|\Lambda_1^{\sigma_k}|^2 - p^2 t\Lambda_2^{\sigma_k,b_k} \le Cp^3 t\big[\big([\mathrm{div}(b)]^- + |\bar b| + |\nabla\sigma|^2 + |\bar\sigma|^2\big) * \chi_k\big].$$

Substituting this estimate into (2.7), we see that there are two constants $C_{1,p}, C_{2,p} > 0$ such that for any $T > 0$ and all $k \ge 1$,

$$\sup_{0\le t\le T} \|\rho_t^k\|_{L^p(\mathbb{P}\times\mu)}$$
$$\le C_{1,p}\bigg(\int_{\mathbb{R}^n} \exp\big[C_{2,p}T\big([\mathrm{div}(b)]^- + |\bar b| + |\nabla\sigma|^2 + |\bar\sigma|^2\big) * \chi_k\big]\mathrm{d}\mu\bigg)^{\frac{1}{p(p+1)}}.$$

To simplify the notations, we denote by $\Phi = C_{2,p}T\big([\mathrm{div}(b)]^- + |\bar b| + |\nabla\sigma|^2 + |\bar\sigma|^2\big)$; then

$$\sup_{k\ge1}\sup_{0\le t\le T} \|\rho_t^k\|_{L^p(\mathbb{P}\times\mu)} \le C_{1,p}\bigg(\int_{\mathbb{R}^n} \exp\big[(\Phi * \chi_k)(x) + \lambda(x)\big]\mathrm{d}x\bigg)^{\frac{1}{p(p+1)}}. \quad (2.9)$$

We want to show that there is a constant $C > 0$ such that for any $k \ge 1$,

$$\lambda(x) \le (\lambda * \chi_k)(x) + C \quad \text{for all } x \in \mathbb{R}^n. \quad (2.10)$$

Indeed, for any $u \in B(1)$, one has

$$1 + |x-u|^2 \le 1 + 2|x|^2 + 2|u|^2 \le 3(1+|x|^2),$$

hence

$$\lambda(x-u) = -\alpha\log(1+|x-u|^2) \ge -\alpha\log 3 + \lambda(x).$$

As a result, for all $k \ge 1$,

$$(\lambda * \chi_k)(x) = \int_{\mathbb{R}^n} \lambda(x-u)\chi_k(u)\,\mathrm{d}u \ge -\alpha\log 3 + \lambda(x)$$

since $\chi_k \ge 0$ and $\int_{\mathbb{R}^n} \chi_k(u)\,\mathrm{d}u = 1$. Hence (2.10) holds with $C = \alpha\log 3$. Now by (2.10) and Jensen's inequality,

$$\begin{aligned}\int_{\mathbb{R}^n} \exp\big[(\Phi * \chi_k)(x) + \lambda(x)\big]\mathrm{d}x &\le 3^\alpha \int_{\mathbb{R}^n} \exp\big[(\Phi+\lambda) * \chi_k(x)\big]\mathrm{d}x \\ &\le 3^\alpha \int_{\mathbb{R}^n} (e^{\Phi+\lambda} * \chi_k)(x)\,\mathrm{d}x \\ &= 3^\alpha \int_{\mathbb{R}^n} e^{\Phi+\lambda}\,\mathrm{d}x = 3^\alpha \int_{\mathbb{R}^n} e^\Phi\,\mathrm{d}\mu.\end{aligned}$$

Substituting this estimate into (2.9) and by the definition of Φ, we see that if we take $T_0 \le p_0/C_{2,p}$, then the right hand side of (2.8) is finite. □

In the following we fix p as the conjugate number of q and denote by Λ_{p,T_0} the quantity on the right hand side of (2.8). Then we have

$$\sup_{k\ge 1} \sup_{0\le t\le T_0} \|\rho_t^k\|_{L^p(\mathbb{P}\times\mu)} \le \Lambda_{p,T_0}. \tag{2.11}$$

Using Lemma 2.5 and the density estimate (2.11), we can now show that there exists a random field $X : \Omega \times \mathbb{R}^n \to C([0, T_0], \mathbb{R}^n)$, which is the limit of the sequence of stochastic flows generated by (2.6).

Proposition 2.7. *Under the conditions* (C1) *and* (C2)*, there exists a random field* $X : \Omega \times \mathbb{R}^n \to C([0, T_0], \mathbb{R}^n)$ *such that*

$$\lim_{k\to\infty} \mathbb{E} \int_{\mathbb{R}^n} 1 \wedge \|X^k - X\|_{\infty,T_0}\, \mathrm{d}\mu = 0.$$

Proof. The proof is similar to that of [15, Theorem 5.3]. For any $k \ge 1$, we denote by G_R^k the level set of the flow X_t^k on the interval $[0, T_0]$:

$$G_R^k = \Big\{(\omega, x) \in \Omega \times \mathbb{R}^n : \big\|X_\cdot^k(\omega, x)\big\|_{\infty,T_0} \le R\Big\}.$$

By Lemma 2.4,

$$(\mathbb{P}\times\mu)\big[(G_R^k \cap G_R^l)^c\big] \le (\mathbb{P}\times\mu)\big[(G_R^k)^c\big] + (\mathbb{P}\times\mu)\big[(G_R^l)^c\big] \le \frac{C_k + C_l}{R}, \tag{2.12}$$

in which C_k depends on T_0, Λ_{p,T_0}, $\|\sigma_k\|_{L^{2q}(\mu)}$, $\|b_k\|_{L^q(\mu)}$. We have $|\sigma_k| \le |\sigma| * \chi_k$. Jensen's inequality leads to

$$\|\sigma_k\|_{L^{2q}(\mu)}^{2q} \le \int_{\mathbb{R}^n} \big(|\sigma|^{2q} * \chi_k\big)(x)\, \mathrm{d}\mu(x) = \int_{\mathbb{R}^n} |\sigma(y)|^{2q}\, \mathrm{d}y \int_{\mathbb{R}^n} \frac{\chi_k(x-y)}{(1+|x|^2)^\alpha}\, \mathrm{d}x.$$

Notice that for $|x - y| \le 1/k$, one has $|y| \le |x| + 1/k$, hence

$$1 + |y|^2 \le 1 + 2|x|^2 + 2/k^2 \le 3(1 + |x|^2) \quad \text{for all } k \ge 1.$$

Consequently,

$$\int_{\mathbb{R}^n} \frac{\chi_k(x-y)}{(1+|x|^2)^\alpha}\, \mathrm{d}x \le 3^\alpha \int_{\mathbb{R}^n} \frac{\chi_k(x-y)}{(1+|y|^2)^\alpha}\, \mathrm{d}x = \frac{3^\alpha}{(1+|y|^2)^\alpha} \tag{2.13}$$

since $\int_{\mathbb{R}^n} \chi_k\, \mathrm{d}x = 1$. As a result,

$$\|\sigma_k\|_{L^{2q}(\mu)} \le 3^{\alpha/2q} \bigg(\int_{\mathbb{R}^n} |\sigma(y)|^{2q}\, \mathrm{d}\mu(y)\bigg)^{1/2q} = 3^{\alpha/2q} \|\sigma\|_{L^{2q}(\mu)}. \tag{2.14}$$

In the same way, we have $\|b_k\|_{L^q(\mu)} \le 3^{\alpha/q}\|b\|_{L^q(\mu)}$. Therefore the positive constants $(C_k)_{k\ge1}$ are uniformly bounded from above by some $\hat C > 0$. Combining this observation with (2.12), we obtain

$$\sup_{k,l\ge1}(\mathbb{P}\times\mu)\big[(G_R^k\cap G_R^l)^c\big] \le \frac{2\hat C}{R}. \tag{2.15}$$

Now an application of Lemma 2.5 to the flows X_t^k and X_t^l gives us

$$\begin{aligned}
&\mathbb{E}\int_{G_R^k\cap G_R^l}\log\bigg(\frac{\|X^k-X^l\|_{\infty,T_0}^2}{\delta^2}+1\bigg)\mathrm{d}\mu\\
&\quad\le C_{T_0}\Lambda_{p,T_0}\bigg\{C_{n,q}\Big[\|\nabla b_k\|_{L^q(B(3R))}+\|\nabla\sigma_k\|_{L^{2q}(B(3R))}+\|\nabla\sigma_k\|_{L^{2q}(B(3R))}^2\Big]\\
&\qquad\qquad+\frac{1}{\delta^2}\|\sigma_k-\sigma_l\|_{L^{2q}(B(R))}^2\\
&\qquad\qquad+\frac{1}{\delta}\Big[\|\sigma_k-\sigma_l\|_{L^{2q}(B(R))}+\|b_k-b_l\|_{L^q(B(R))}\Big]\bigg\}.
\end{aligned}\tag{2.16}$$

By the definition of b_k, we have

$$|\nabla b_k| \le |\nabla b| * \chi_k + C\frac{|b*\chi_k|}{1+|x|} \le |\nabla b| * \chi_k + 2C|\bar b| * \chi_k.$$

From this we can show that

$$\|\nabla b_k\|_{L^q(B(3R))} \le C_q\big(\|\nabla b\|_{L^q(B(3R+1))}+\|\bar b\|_{L^q(B(3R+1))}\big).$$

In the same way, $\|\nabla\sigma_k\|_{L^{2q}(B(3R))} \le C_q\big(\|\nabla\sigma\|_{L^{2q}(B(3R+1))}+\|\bar\sigma\|_{L^q(B(3R+1))}\big)$. Notice that under the conditions (C1) and (C2), ∇b and $\bar b$ (respectively $\nabla\sigma$ and $\bar\sigma$) are locally integrable. Hence for any $k\ge1$,

$$C_{n,q}\Big[\|\nabla b_k\|_{L^q(B(3R))}+\|\nabla\sigma_k\|_{L^{2q}(B(3R))}+\|\nabla\sigma_k\|_{L^{2q}(B(3R))}^2\Big] \le C'_{n,q,R}.$$

Now we define

$$\delta_{k,l} = \|\sigma_k-\sigma_l\|_{L^{2q}(B(R))}+\|b_k-b_l\|_{L^q(B(R))}$$

which tends to 0 as $k,l\to+\infty$. Taking $\delta=\delta_{k,l}$ in (2.16), we obtain that for any $k,l\ge1$,

$$\mathbb{E}\int_{G_R^k\cap G_R^l}\log\bigg(\frac{\|X^k-X^l\|_{\infty,T_0}^2}{\delta_{k,l}^2}+1\bigg)\mathrm{d}\mu \le C_{T_0,n,q,R} < +\infty. \tag{2.17}$$

We have by (2.15)

$$
\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^n} \big(1 \wedge \|X^k - X^l\|_{\infty,T_0}\big)\,\mathrm{d}\mu \\
&\le (\mathbb{P}\times\mu)\big[(G_R^k \cap G_R^l)^c\big] + \int_{G_R^k\cap G_R^l} \big(1 \wedge \|X^k - X^l\|_{\infty,T_0}\big)\,\mathrm{d}(\mathbb{P}\times\mu) \qquad (2.18)\\
&\le \frac{2\hat{C}}{R} + \int_{G_R^k\cap G_R^l} \big(1 \wedge \|X^k - X^l\|_{\infty,T_0}\big)\,\mathrm{d}(\mathbb{P}\times\mu).
\end{aligned}
$$

Next for $\eta \in (0,1)$, set

$$\Sigma_\eta^{k,l} = \big\{(\omega, x) \in \Omega\times\mathbb{R}^n : \|X^k - X^l\|_{\infty,T_0} \le \eta\big\}.$$

Then

$$
\begin{aligned}
&\int_{G_R^k\cap G_R^l} \big(1 \wedge \|X^k - X^l\|_{\infty,T_0}\big)\,\mathrm{d}(\mathbb{P}\times\mu) \\
&\quad = \bigg(\int_{(G_R^k\cap G_R^l)\cap\Sigma_\eta^{k,l}} + \int_{(G_R^k\cap G_R^l)\setminus\Sigma_\eta^{k,l}}\bigg)\big(1 \wedge \|X^k - X^l\|_{\infty,T_0}\big)\,\mathrm{d}(\mathbb{P}\times\mu) \\
&\quad \le \eta\,\mu(\mathbb{R}^n) + \frac{1}{\log\Big(1+\frac{\eta^2}{\delta_{k,l}^2}\Big)}\int_{G_R^k\cap G_R^l} \log\bigg(1 + \frac{\|X^k - X^l\|_{\infty,T_0}^2}{\delta_{k,l}^2}\bigg)\mathrm{d}(\mathbb{P}\times\mu) \\
&\quad \le \eta\,\mu(\mathbb{R}^n) + \frac{C_{T_0,n,q,R}}{\log\Big(1+\frac{\eta^2}{\delta_{k,l}^2}\Big)},
\end{aligned}
$$

where the last inequality follows from (2.17). Substituting this estimate into (2.18), we get

$$\mathbb{E}\int_{\mathbb{R}^n} \big(1 \wedge \|X^k - X^l\|_{\infty,T_0}\big)\,\mathrm{d}\mu \le \frac{2\hat{C}}{R} + \eta\,\mu(\mathbb{R}^n) + \frac{C_{T_0,n,q,R}}{\log\Big(1+\frac{\eta^2}{\delta_{k,l}^2}\Big)}.$$

First letting $k, l \to +\infty$, and then $R \to +\infty$, $\eta \to 0$, we obtain

$$\lim_{k,l\to+\infty} \mathbb{E}\int_{\mathbb{R}^n} \big(1 \wedge \|X^k - X^l\|_{\infty,T_0}\big)\,\mathrm{d}\mu = 0.$$

Hence there exists a random field $X : \Omega\times\mathbb{R}^n \to C([0,T_0],\mathbb{R}^n)$ such that

$$\lim_{k\to+\infty} \mathbb{E}\int_{\mathbb{R}^n} \big(1 \wedge \|X^k - X\|_{\infty,T_0}\big)\,\mathrm{d}\mu = 0.$$

The proof is complete. □

Proposition 2.8. *For all $t \in [0, T_0]$, there exists $\rho_t : \Omega \times \mathbb{R}^n \to \mathbb{R}_+$ such that $(X_t)_{\#}\mu = \rho_t \mu$. Moreover,* $\sup_{0\le t\le T_0} \|\rho_t\|_{L^p(\mathbb{P}\times\mu)} \le \Lambda_{p,T_0}$.

The proofs are similar to the arguments of [15, Theorem 3.4] and are omitted here. To show that $(X_t)_{0\le t\le T_0}$ solves the Itô SDE (1.1), we need the following preparation.

Lemma 2.9. *We have*

$$\lim_{k\to\infty} \mathbb{E}\int_{\mathbb{R}^n} \left(\sup_{0\le t\le T_0} \left| \int_0^t \big[\sigma_k(X_s^k) - \sigma(X_s)\big] \mathrm{d}B_s \right| \right) \mathrm{d}\mu = 0$$

and

$$\lim_{k\to\infty} \mathbb{E}\int_{\mathbb{R}^n} \left(\sup_{0\le t\le T_0} \left| \int_0^t \big[b_k(X_s^k) - b(X_s)\big] \mathrm{d}s \right| \right) \mathrm{d}\mu = 0.$$

Proof. By elementary calculations, it is easy to show that

$$\lim_{k\to\infty} \|\sigma_k - \sigma\|_{L^{2q}(\mu)} = 0 \quad \text{and} \quad \lim_{k\to\infty} \|b_k - b\|_{L^{2q}(\mu)} = 0.$$

Combining these limits with Propositions 2.7 and 2.8, we can finish the proof as in [15, Proposition 4.1]. □

For any $k \ge 1$, we rewrite the equation (2.6) in the integral form:

$$X_t^k(x) = x + \int_0^t \sigma_k(X_s^k)\,\mathrm{d}B_s + \int_0^t b_k(X_s^k)\,\mathrm{d}s. \tag{2.19}$$

When $k \to +\infty$, by Proposition 2.7 and Lemma 2.9, the two sides of (2.19) converge respectively to X and

$$x + \int_0^{\cdot} \sigma(X_s)\,\mathrm{d}B_s + \int_0^{\cdot} b(X_s)\,\mathrm{d}s.$$

Therefore, for almost all $x \in \mathbb{R}^d$, the following equality holds $\mathbb{P}$-almost surely:

$$X_t(x) = x + \int_0^t \sigma(X_s)\,\mathrm{d}B_s + \int_0^t b(X_s)\,\mathrm{d}s, \quad \text{for all } t \in [0, T_0].$$

That is to say, X_t solves SDE (1.1) over the time interval $[0, T_0]$. Similar to [15, Proposition 5.6], we can prove the uniqueness of the solution flow on $[0, T_0]$.

Now we extend the solution to any time interval $[0, T]$. Let $\theta_{T_0}B$ be the time-shift of the Brownian motion B by T_0 and denote by $X_t^{T_0}$ the corresponding solution to the SDE (1.1) driven by $\theta_{T_0}B$. By the above discussions, $\{X_t^{T_0}(\theta_{T_0}B, x) : 0 \le t \le T_0\}$ is the unique solution to the following SDE over $[0, T_0]$:

$$X_t^{T_0}(x) = x + \int_0^t \sigma\big(X_s^{T_0}(x)\big)\,\mathrm{d}(\theta_{T_0}B)_s + \int_0^t b\big(X_s^{T_0}(x)\big)\,\mathrm{d}s.$$

For $t \in [0, T_0]$, define $X_{t+T_0}(\omega, x) = X_t^{T_0}(\theta_{T_0} B, X_{T_0}(\omega, x))$. Note that X_t is well defined on the interval $[0, 2T_0]$ up to a $(\mathbb{P} \times \mu)$-negligible subset of $\Omega \times \mathbb{R}^n$. Replacing x by $X_{T_0}(x)$ in the above equation, we obtain

$$X_{t+T_0}(x) = x + \int_0^{t+T_0} \sigma\big(X_s(x)\big)\,\mathrm{d}B_s + \int_0^{t+T_0} b\big(X_s(x)\big)\,\mathrm{d}s.$$

Therefore X_t defined as above is a solution to SDE (1.1) on the interval $[0, 2T_0]$. Continuing in this way, we obtain the solution of SDE (1.1) on the interval $[0, T]$. At this stage we can repeat the arguments of [15, Theorem 5.7] to complete the proof of Theorem 2.3.

3. An intermediate result

In this section we prove a technical result which serves as a bridge between Theorem 2.3 and the main result in Section 4. First we introduce some notations. The functions σ_i and b_i $(i = 1, 2)$ are the same as in (1.3) and (1.4). We fix some $q > 1$ and choose $\alpha_1 > q + n_1/2$, $\alpha > \alpha_1 + n_2/2$. Let

$$\mathrm{d}\mu(x) = (1 + |x|^2)^{-\alpha}\,\mathrm{d}x \quad \text{and} \quad \mathrm{d}\mu_1(x_1) = (1 + |x_1|^2)^{-\alpha_1}\,\mathrm{d}x_1.$$

Then μ (respectively μ_1) is a finite measure on $\mathbb{R}^n$ (respectively $\mathbb{R}^{n_1}$). To simplify the notation we write $\bar\sigma_1 = \frac{\sigma_1}{1+|x_1|}$ and $\bar\sigma_2 = \frac{\sigma_2}{1+|x|}$. $\bar b_i$ is defined similarly to $\bar\sigma_i$ $(i = 1, 2)$. Our assumptions in this section are:

(H1) $\sigma_1 \in W^{1,2q}_{x_1,\mathrm{loc}}$, $b_1 \in W^{1,q}_{x_1,\mathrm{loc}}$;

(H2) $\int_{\mathbb{R}^{n_1}} \exp\big[p_0\big([\mathrm{div}_{x_1}(b_1)]^- + |\bar b_1| + |\bar\sigma_1|^2 + |\nabla_{x_1}\sigma_1|^2\big)\big]\mathrm{d}\mu_1 < +\infty$ for some $p_0 > 0$;

(H3) $\sigma_2 \in W^{1,2q}_{x_1,x_2,\mathrm{loc}}$, $b_2 \in W^{1,q}_{x_1,x_2,\mathrm{loc}}$;

(H4) $\int_{\mathbb{R}^n} \exp\big[p_0\big([\mathrm{div}_{x_2}(b_2)]^- + |\bar b_2| + |\bar\sigma_2|^2 + |\nabla_{x_2}\sigma_2|^2\big)\big]\mathrm{d}\mu < +\infty$ for some $p_0 > 0$.

Under the conditions (H1) and (H2), we conclude from Theorem 2.3 that there exists a unique stochastic flow $X_{1,t}$ on $\mathbb{R}^{n_1}$ associated to the Itô SDE (1.1) with coefficients σ_1 and b_1, such that the reference measure μ_1 is absolutely continuous under the action of the flow $X_{1,t}$. In the next result we show that under the additional assumptions (H3)–(H4), the following SDE

$$\begin{cases} \mathrm{d}X_{1,t} = \sigma_1(X_{1,t})\,\mathrm{d}B_t + b_1(X_{1,t})\,\mathrm{d}t, & X_{1,0} = x_1, \\ \mathrm{d}X_{2,t} = \sigma_2(X_{1,t}, X_{2,t})\,\mathrm{d}B_t + b_2(X_{1,t}, X_{2,t})\,\mathrm{d}t, & X_{2,0} = x_2 \end{cases} \tag{3.1}$$

generates a unique flow $X_t = (X_{1,t}, X_{2,t})$ on the whole space $\mathbb{R}^n$, which leaves the measure μ absolutely continuous. Notice that the hypotheses (H1) and (H3) imply $\sigma = (\sigma_1, \sigma_2) \in W^{1,2q}_{x_1,x_2,\mathrm{loc}}$ and $b = (b_1, b_2) \in W^{1,q}_{x_1,x_2,\mathrm{loc}}$, therefore the following theorem can essentially be seen as a special case of Theorem 2.3 (see also [27, Theorem 2.4] and [15, Theorem 1.3]). The main difference between the two results is that we no longer require the exponential integrability of the partial derivatives $\nabla_{x_1}\sigma_2$; the reason for this will become clear in view of (6.4).

Theorem 3.1. *Under the assumptions* (H1)–(H4)*, the Itô SDE* (3.1) *generates a unique stochastic flow X_t of measurable maps on $\mathbb{R}^n$. Moreover, the Radon–Nikodym density ρ_t of the flow with respect to the measure μ satisfies $\rho_t \in L^1 \log L^1$.*

We shall not give a complete proof to the above result, but only mention some arguments that are different from those in Section 2. We focus on the existence part of Theorem 3.1 which needs to regularize the coefficients σ_1, b_1 and σ_2, b_2 separately.

Let $\chi_1 \in C_c^\infty(\mathbb{R}^{n_1}, \mathbb{R}_+)$ be such that $\int_{\mathbb{R}^{n_1}} \chi_1(x_1)\,\mathrm{d}x_1 = 1$ and its support $\mathrm{supp}(\chi_1) \subset B_1(1)$, where $B_1(r)$ is a ball in $\mathbb{R}^{n_1}$ centered at the origin with radius $r > 0$. For $k \geq 1$, define $\chi_{1,k}(x_1) = k^{n_1}\chi_1(kx_1)$. Next choose $\psi_1 \in C_c^\infty(\mathbb{R}^{n_1}, [0,1])$ so that $\psi_1|_{B_1(1)} \equiv 1$ and ψ_1 vanishes outside $B_1(2)$. Denote by $\psi_{1,k}(x_1) = \psi_1(x_1/k)$ for $k \geq 1$. Now we set

$$\sigma_{1,k} = (\sigma_1 * \chi_{1,k})\,\psi_{1,k}, \quad b_{1,k} = (b_1 * \chi_{1,k})\,\psi_{1,k};$$

and

$$\sigma_{2,k} = (\sigma_2 * \chi_k)\,\psi_k, \quad b_{2,k} = (b_2 * \chi_k)\,\psi_k. \tag{3.2}$$

Here χ_k and ψ_k are the same as in Section 2. Then the coefficients $\sigma_{i,k}, b_{i,k} \in C_b^\infty(\mathbb{R}^n)$ $(i = 1, 2)$. Furthermore, simple computations yield

$$\frac{|\sigma_{1,k}|}{1+|x_1|} \leq 2|\bar\sigma_1| * \chi_{1,k}, \quad \frac{|b_{1,k}|}{1+|x_1|} \leq 2|\bar b_1| * \chi_{1,k} \tag{3.3}$$

and

$$\frac{|\sigma_{2,k}|}{1+|x|} \leq 2|\bar\sigma_2| * \chi_k, \quad \frac{|b_{2,k}|}{1+|x|} \leq 2|\bar b_2| * \chi_k. \tag{3.4}$$

We now consider the Itô SDEs

$$\begin{cases} \mathrm{d}X^k_{1,t} = \sigma_{1,k}(X^k_{1,t})\,\mathrm{d}B_t + b_{1,k}(X^k_{1,t})\,\mathrm{d}t, & X^k_{1,0} = x_1, \\ \mathrm{d}X^k_{2,t} = \sigma_{2,k}(X^k_{1,t}, X^k_{2,t})\,\mathrm{d}B_t + b_{2,k}(X^k_{1,t}, X^k_{2,t})\,\mathrm{d}t, & X^k_{2,0} = x_2. \end{cases}$$

For any $k \geq 1$, the above equation determines a unique stochastic flow $X^k_t = (X^k_{1,t}, X^k_{2,t})$ of diffeomorphisms on $\mathbb{R}^n$. Moreover, denoting by $\rho^k_t = \frac{\mathrm{d}[(X^k_t)_\#\mu]}{\mathrm{d}\mu}$, then by Lemma 6.1, we have for any $p > 1$ and $t \in [0, T]$,

$$\|\rho^k_t\|_{L^p(\mathbb{P}\times\mu)} \leq \mu(\mathbb{R}^n)^{\frac{1}{p+1}} \left(\sup_{0\leq t\leq T} \int_{\mathbb{R}^n} \exp\left(tp^3|\Lambda_1^{\sigma_k}|^2 - tp^2\Lambda_2^{\sigma_k,b_k}\right)\mathrm{d}\mu \right)^{\frac{1}{p(p+1)}}, \tag{3.5}$$

where $\sigma_k = (\sigma_{1,k}, \sigma_{2,k})$ and $b_k = (b_{1,k}, b_{2,k})$. We give a uniform estimate for the densities ρ^k_t.

Lemma 3.2 (Uniform density estimate). *For fixed $p > 1$, there are two positive constants $C_{1,p}, C_{2,p} > 0$ and $T_0 > 0$ small enough such that*

$$\sup_{0\le t\le T_0} \|\rho_t^k\|_{L^p(\mathbb{P}\times\mu)}$$
$$\le C_{1,p}\bigg(\int_{\mathbb{R}^{n_1}} \exp\big[C_{2,p}T_0\big([\mathrm{div}_{x_1}(b_1)]^- + |\bar{b}_1| + |\nabla_{x_1}\sigma_1|^2 + |\bar{\sigma}_1|^2\big)\big]\mathrm{d}\mu_1\bigg)^{\frac{1}{p(p+1)}}$$
$$\times\bigg(\int_{\mathbb{R}^{n_1}} \exp\big[C_{2,p}T_0\big([\mathrm{div}_{x_2}(b_2)]^- + |\bar{b}_2| + |\nabla_{x_2}\sigma_2|^2 + |\bar{\sigma}_2|^2\big)\big]\mathrm{d}\mu\bigg)^{\frac{1}{p(p+1)}}.$$

Proof. Direct calculations give us

$$|\Lambda_1^{\sigma_k}|^2 \le C_0\big(|\mathrm{div}_{x_1}(\sigma_1)|^2 + |\bar{\sigma}_1|^2\big) * \chi_{1,k} + C_0\big(|\mathrm{div}_{x_2}(\sigma_2)|^2 + |\bar{\sigma}_2|^2\big) * \chi_k, \quad (3.6)$$

and

$$\begin{aligned}-\Lambda_2^{\sigma_k,b_k} \le\ & C_0\big([\mathrm{div}_{x_1}(b_1)]^- + |\bar{b}_1| + |\bar{\sigma}_1|^2 + |\nabla_{x_1}\sigma_1|^2\big) * \chi_{1,k} \\ &+ C_0\big([\mathrm{div}_{x_2}(b_2)]^- + |\bar{b}_2| + |\bar{\sigma}_2|^2 + |\nabla_{x_2}\sigma_2|^2\big) * \chi_k.\end{aligned} \quad (3.7)$$

Note that $|\mathrm{div}_{x_i}(\sigma_i)| \le |\nabla_{x_i}\sigma_i|$ $(i = 1, 2)$, thus (3.6) becomes

$$|\Lambda_1^{\sigma_k}|^2 \le C_0\big(|\nabla_{x_1}\sigma_1|^2 + |\bar{\sigma}_1|^2\big) * \chi_{1,k} + C_0\big(|\nabla_{x_2}\sigma_2|^2 + |\bar{\sigma}_2|^2\big) * \chi_k.$$

For any $t \in [0, T]$, the above inequality plus (3.7) gives us

$$\begin{aligned}tp^3|\Lambda_1^{\sigma_k}|^2 - tp^2\Lambda_2^{\sigma_k,b_k} \le\ & 2Tp^3C_0\big([\mathrm{div}_{x_1}(b_1)]^- + |\bar{b}_1| + |\bar{\sigma}_1|^2 + |\nabla_{x_1}\sigma_1|^2\big) * \chi_{1,k} \\ &+ 2Tp^3C_0\big([\mathrm{div}_{x_2}(b_2)]^- + |\bar{b}_2| + |\bar{\sigma}_2|^2 + |\nabla_{x_2}\sigma_2|^2\big) * \chi_k.\end{aligned}$$

Denote by

$$\Phi_i = 2Tp^3C_0\big([\mathrm{div}_{x_i}(b_i)]^- + |\bar{b}_i| + |\bar{\sigma}_i|^2 + |\nabla_{x_i}\sigma_i|^2\big), \quad i = 1, 2.$$

Then Φ_1 is a function defined on $\mathbb{R}^{n_1}$, while Φ_2 is a function on the whole $\mathbb{R}^n$. Now we have by Cauchy's inequality,

$$\begin{aligned}\int_{\mathbb{R}^n}\exp\big(tp^3|\Lambda_1^{\sigma_k}|^2 - tp^2\Lambda_2^{\sigma_k,b_k}\big)\mathrm{d}\mu &\le \int_{\mathbb{R}^n} e^{\Phi_1*\chi_{1,k}}\, e^{\Phi_2*\chi_k}\,\mathrm{d}\mu \\ &\le \bigg[\int_{\mathbb{R}^n} e^{2\Phi_1*\chi_{1,k}}\mathrm{d}\mu\bigg]^{\frac{1}{2}}\bigg[\int_{\mathbb{R}^n} e^{2\Phi_2*\chi_k}\mathrm{d}\mu\bigg]^{\frac{1}{2}}.\end{aligned} \quad (3.8)$$

In the following we estimate the two integrals given in (3.8). First we have

$$(1 + |x|^2)^\alpha \ge (1 + |x_1|^2)^{\alpha_1} \times (1 + |x_2|^2)^{\alpha-\alpha_1}.$$

Thus

$$\begin{aligned}\int_{\mathbb{R}^n} e^{2\Phi_1 * \chi_{1,k}}\,\mathrm{d}\mu &\le \int_{\mathbb{R}^n} e^{2(\Phi_1*\chi_{1,k})(x_1)} \frac{\mathrm{d}x_1}{(1+|x_1|^2)^{\alpha_1}} \cdot \frac{\mathrm{d}x_2}{(1+|x_2|^2)^{\alpha-\alpha_1}} \\ &= \mu_2(\mathbb{R}^{n_2}) \int_{\mathbb{R}^{n_1}} e^{2(\Phi_1*\chi_{1,k})(x_1)+\lambda_1(x_1)}\,\mathrm{d}x_1,\end{aligned} \tag{3.9}$$

where $\mathrm{d}\mu_2(x_2) = \frac{\mathrm{d}x_2}{(1+|x_2|^2)^{\alpha-\alpha_1}}$ is a finite measure on $\mathbb{R}^{n_2}$ and $\lambda_1(x_1) = -\alpha_1 \log(1+|x_1|^2)$. Similar to (2.10), we can show that there is a constant $C > 0$ such that for any $k \ge 1$,

$$\lambda_1(x_1) \le (\lambda_1 * \chi_{1,k})(x_1) + C \quad \text{for all } x_1 \in \mathbb{R}^{n_1}. \tag{3.10}$$

Substituting (3.10) into the inequality (3.9) and by Jensen's inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} e^{2\Phi_1 * \chi_{1,k}}\,\mathrm{d}\mu &\le \mu_2(\mathbb{R}^{n_2})\, e^C \int_{\mathbb{R}^{n_1}} e^{[(2\Phi_1+\lambda_1)*\chi_{1,k}](x_1)}\,\mathrm{d}x_1 \\ &\le \mu_2(\mathbb{R}^{n_2})\, e^C \int_{\mathbb{R}^{n_1}} \big[(e^{2\Phi_1+\lambda_1}) * \chi_{1,k}\big](x_1)\,\mathrm{d}x_1 \\ &= \mu_2(\mathbb{R}^{n_2})\, e^C \int_{\mathbb{R}^{n_1}} e^{2\Phi_1}\,\mathrm{d}\mu_1.\end{aligned} \tag{3.11}$$

The second integral on the right hand side of (3.8) can be treated in a similar way, thanks to (2.10). Hence

$$\int_{\mathbb{R}^n} e^{2\Phi_2 * \chi_k}\,\mathrm{d}\mu \le e^{\bar{C}} \int_{\mathbb{R}^n} e^{2\Phi_2}\,\mathrm{d}\mu. \tag{3.12}$$

Now combining the inequalities (3.8), (3.11) and (3.12), we finally obtain from the definition of Φ_1 and Φ_2 that

$$\begin{aligned}&\int_{\mathbb{R}^n} \exp\big(tp^3|\Lambda_1^{\sigma_k}|^2 - tp^2\Lambda_2^{\sigma_k,b_k}\big)\mathrm{d}\mu \\ &\le \big(\mu_2(\mathbb{R}^{n_2})\, e^{C+\bar{C}}\big)^{\frac{1}{2}} \bigg[\int_{\mathbb{R}^{n_1}} \exp\Big\{4Tp^3C_0\big([\mathrm{div}_{x_1}(b_1)]^- + |\bar{b}_1| + |\bar{\sigma}_1|^2 + |\nabla_{x_1}\sigma_1|^2\big)\Big\}\mathrm{d}\mu_1\bigg]^{\frac{1}{2}} \\ &\quad \times \bigg[\int_{\mathbb{R}^n} \exp\Big\{4Tp^3C_0\big([\mathrm{div}_{x_2}(b_2)]^- + |\bar{b}_2| + |\bar{\sigma}_2|^2 + |\nabla_{x_2}\sigma_2|^2\big)\Big\}\,\mathrm{d}\mu\bigg]^{\frac{1}{2}}.\end{aligned}$$

Substituting this inequality into (3.5), we see that for any $k \ge 1$,

$$\begin{aligned}&\sup_{t\le T} \|\rho_t^k\|_{L^p(\mathbb{P}\times\mu)} \\ &\le C_{1,p}\bigg[\int_{\mathbb{R}^{n_1}} \exp\Big\{C_{2,p}T\big([\mathrm{div}_{x_1}(b_1)]^- + |\bar{b}_1| + |\bar{\sigma}_1|^2 + |\nabla_{x_1}\sigma_1|^2\big)\Big\}\,\mathrm{d}\mu_1\bigg]^{\frac{1}{2p(p+1)}} \\ &\quad \times \bigg[\int_{\mathbb{R}^n} \exp\Big\{C_{2,p}T\big([\mathrm{div}_{x_2}(b_2)]^- + |\bar{b}_2| + |\bar{\sigma}_2|^2 + |\nabla_{x_2}\sigma_2|^2\big)\Big\}\,\mathrm{d}\mu\bigg]^{\frac{1}{2p(p+1)}},\end{aligned}$$

where $C_{1,p}, C_{2,p}$ are two positive constants independent on k and T. Under the conditions (H2) and (H4), there exists $T_0 > 0$ small enough such that the quantity on the right hand side is finite. $\square$

Having Lemma 3.2 in hand, we can follow the line of arguments in Section 2 to prove Theorem 3.1. We omit the details.

4. SDE with partially Sobolev coefficients

In this section we aim at generalizing Theorem 3.1 to the case where the coefficients σ_2 and b_2 only have partial Sobolev regularity. More precisely, we replace the condition (H3) by

(H3′) $\sigma_2 \in L^{2q}_{x_1,\mathrm{loc}}(W^{1,2q}_{x_2,\mathrm{loc}}),\ b_2 \in L^{q}_{x_1,\mathrm{loc}}(W^{1,q}_{x_2,\mathrm{loc}})$,

and we shall show that the results of Theorem 3.1 still hold.

To achieve such an extension, we need an a-priori estimate which only involves partial derivatives of σ_2 and b_2. First we introduce some notations. Throughout this section we fix a pair of functions

$$\sigma_1 : \mathbb{R}^{n_1} \to \mathbb{R}^m \otimes \mathbb{R}^{n_1} \quad \text{and} \quad b_1 : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1}$$

which satisfy the assumptions (H1) and (H2) in Section 3. Under these conditions, it is known that the following Itô SDE

$$\mathrm{d}X_{1,t} = \sigma_1(X_{1,t})\,\mathrm{d}B_t + b_1(X_{1,t})\,\mathrm{d}t, \quad X_{1,0} = x_1$$

generates a unique stochastic flow of measurable maps on $\mathbb{R}^{n_1}$, which leaves the reference measure μ_1 absolutely continuous, as shown in Theorem 2.3.

Let

$$\sigma_2, \tilde\sigma_2 : \mathbb{R}^n \to \mathbb{R}^m \otimes \mathbb{R}^{n_2} \quad \text{and} \quad b_2, \tilde b_2 : \mathbb{R}^n \to \mathbb{R}^{n_2}$$

be measurable functions, all verifying the conditions (H3′). Denote by

$$\sigma = (\sigma_1, \sigma_2),\ b = (b_1, b_2) \quad \text{and} \quad \tilde\sigma = (\sigma_1, \tilde\sigma_2),\ \tilde b = (b_1, \tilde b_2).$$

Let $X_t = (X_{1,t}, X_{2,t})$ (respectively $\tilde X_t = (X_{1,t}, \tilde X_{2,t})$) be the stochastic flow generated by the Itô SDE (1.1) with coefficients σ and b (respectively $\tilde\sigma$ and $\tilde b$).

Lemma 4.1 (A-priori estimate). *Suppose that for any $t \in [0,T]$, the push-forwards $(X_t)_\#\mu$ and $(\tilde X_t)_\#\mu$ of the reference measure μ are absolutely continuous with respect to itself, with density functions ρ_t and $\tilde\rho_t$ respectively. Moreover,*

$$\Lambda_{p,T} := \sup_{0\le t\le T} \|\rho_t\|_{L^p(\mathbb{P}\otimes\mu)} \vee \|\tilde\rho_t\|_{L^p(\mathbb{P}\otimes\mu)} < +\infty, \tag{4.1}$$

where p is the conjugate number of q. Then for any $\delta > 0$,

$$\begin{aligned}
&\mathbb{E}\int_{G_R\cap\tilde G_R}\log\left(\frac{\|X_2-\tilde X_2\|_{\infty,T}^2}{\delta^2}+1\right)\mathrm{d}\mu\\
&\le C_T\Lambda_{p,T}\bigg\{C_{n_2,q}\Big[\big\|\nabla_{x_2}b_2\big\|_{L^q(B(4R))}+\big\|\nabla_{x_2}\sigma_2\big\|_{L^{2q}(B(4R))}+\big\|\nabla_{x_2}\sigma_2\big\|_{L^{2q}(B(4R))}^2\Big]\\
&\quad+\frac{1}{\delta^2}\big\|\sigma_2-\tilde\sigma_2\big\|_{L^{2q}(B(R))}^2+\frac{1}{\delta}\Big[\|\sigma_2-\tilde\sigma_2\big\|_{L^{2q}(B(R))}+\big\|b_2-\tilde b_2\big\|_{L^q(B(R))}\Big]\bigg\},
\end{aligned}$$

where G_R and $\tilde G_R$ are the level sets of X_t and $\tilde X_t$ respectively.

Proof. We follow the idea of the proof of [15, Theorem 5.2] (see also [27, Lemma 4.1]). Denote by $\xi_t = X_{2,t} - \tilde X_{2,t}$. Then $\xi_0 = 0$. By the Itô formula,

$$\begin{aligned}
&\mathrm{d}\log(|\xi_t|^2+\delta^2)\\
&=\frac{2\langle\xi_t,[\sigma_2(X_t)-\tilde\sigma_2(\tilde X_t)]\,\mathrm{d}B_t\rangle}{|\xi_t|^2+\delta^2}+\frac{2\langle\xi_t,b_2(X_t)-\tilde b_2(\tilde X_t)\rangle}{|\xi_t|^2+\delta^2}\,\mathrm{d}t\\
&\quad+\frac{\big|\sigma_2(X_t)-\tilde\sigma_2(\tilde X_t)\big|^2}{\big|\xi_t\big|^2+\delta^2}\,\mathrm{d}t-\frac{2\big|[\sigma_2(X_t)-\tilde\sigma_2(\tilde X_t)]^*\xi_t\big|^2}{(|\xi_t|^2+\delta^2)^2}\,\mathrm{d}t \qquad (4.2)\\
&=:\sum_{i=1}^4\mathrm{d}I_i(t).
\end{aligned}$$

Note that the last term is negative, hence we omit it. We shall estimate the other terms in the sequel.

Let $\tau_R(x) = \inf\{t \ge 0 : |X_t(x)| \vee |\tilde X_t(x)| > R\}$ for $x \in \mathbb{R}^n$. Remark that almost surely, $G_R, \tilde G_R \subset \{x : \tau_R(x) > T\}$ and for any $t \ge 0$, $\{\tau_R > t\} \subset B(R)$. Thus by Cauchy's inequality,

$$\begin{aligned}
\mathbb{E}\bigg[\int_{G_R\cap\tilde G_R}\sup_{0\le t\le T}|I_1(t)|\,\mathrm{d}\mu\bigg]&\le\mathbb{E}\bigg[\int_{B(R)}\sup_{0\le t\le T\wedge\tau_R}|I_1(t)|\,\mathrm{d}\mu\bigg]\\
&\le\mu(\mathbb{R}^n)^{\frac12}\bigg[\int_{B(R)}\mathbb{E}\Big(\sup_{0\le t\le T\wedge\tau_R}|I_1(t)|^2\Big)\mathrm{d}\mu\bigg]^{\frac12}.
\end{aligned}$$

Burkholder's inequality gives us

$$\begin{aligned}
\mathbb{E}\Big(\sup_{0\le t\le T\wedge\tau_R}|I_1(t)|^2\Big)&\le16\,\mathbb{E}\bigg(\int_0^{T\wedge\tau_R}\frac{\big|[\sigma_2(X_t)-\tilde\sigma_2(\tilde X_t)]^*\xi_t\big|^2}{(|\xi_t|^2+\delta^2)^2}\,\mathrm{d}t\bigg)\\
&\le16\,\mathbb{E}\bigg(\int_0^{T\wedge\tau_R}\frac{|\sigma_2(X_t)-\tilde\sigma_2(\tilde X_t)|^2}{|\xi_t|^2+\delta^2}\,\mathrm{d}t\bigg).
\end{aligned}$$

As a result, by changing the order of integration, we obtain

$$\begin{aligned}
&\mathbb{E}\bigg[\int_{G_R\cap\tilde G_R}\sup_{0\le t\le T}|I_1(t)|\,\mathrm{d}\mu\bigg]\\
&\le 4C_{\alpha,n}\bigg[\int_{B(R)}\mathbb{E}\bigg(\int_0^{T\wedge\tau_R}\frac{|\sigma_2(X_t)-\tilde\sigma_2(\tilde X_t)|^2}{|\xi_t|^2+\delta^2}\,\mathrm{d}t\bigg)\mathrm{d}\mu\bigg]^{\frac12}\\
&=4C_{\alpha,n}\bigg[\int_0^T\bigg(\mathbb{E}\int_{\{\tau_R>t\}}\frac{|\sigma_2(X_t)-\tilde\sigma_2(\tilde X_t)|^2}{|\xi_t|^2+\delta^2}\,\mathrm{d}\mu\bigg)\mathrm{d}t\bigg]^{\frac12}.
\end{aligned}\tag{4.3}$$

Note that

$$\sigma_2(X_t)-\tilde\sigma_2(\tilde X_t)=\sigma_2(X_t)-\sigma_2(\tilde X_t)+\sigma_2(\tilde X_t)-\tilde\sigma_2(\tilde X_t).$$

We have by (4.1) and Hölder's inequality that

$$\begin{aligned}
\mathbb{E}\int_{\{\tau_R>t\}}\frac{|\sigma_2(\tilde X_t)-\tilde\sigma_2(\tilde X_t)|^2}{|\xi_t|^2+\delta^2}\,\mathrm{d}\mu &\le\frac{1}{\delta^2}\,\mathbb{E}\int_{\{\tau_R>t\}}\big|(\sigma_2-\tilde\sigma_2)\mathbf{1}_{B(R)}\big|^2(\tilde X_t)\,\mathrm{d}\mu\\
&\le\frac{1}{\delta^2}\,\mathbb{E}\int_{B(R)}|\sigma_2-\tilde\sigma_2|^2\tilde\rho_t\,\mathrm{d}\mu\\
&\le\frac{\Lambda_{p,T}}{\delta^2}\|\sigma_2-\tilde\sigma_2\|^2_{L^{2q}(B(R),\mu)}.
\end{aligned}$$

Since $\mu|_{B(R)}\le\mathcal{L}_n|_{B(R)}$ for any $R>0$, where $\mathcal{L}_n$ is the Lebesgue measure on $\mathbb{R}^n$, we obtain

$$\mathbb{E}\int_{\{\tau_R>t\}}\frac{\big|\sigma_2(\tilde X_t)-\tilde\sigma_2(\tilde X_t)\big|^2}{|\xi_t|^2+\delta^2}\,\mathrm{d}\mu\le\frac{\Lambda_{p,T}}{\delta^2}\big\|\sigma_2-\tilde\sigma_2\big\|^2_{L^{2q}(B(R))}.\tag{4.4}$$

Next on the set $\{\tau_R>t\}$, we have $X_t,\tilde X_t\in B(R)$, hence $|X_t-\tilde X_t|_{\mathbb{R}^n}=|X_{2,t}-\tilde X_{2,t}|_{\mathbb{R}^{n_2}}\le 2R$. As $(X_t)_\#\mu\ll\mu$ and $(\tilde X_t)_\#\mu\ll\mu$, we can apply Lemma 6.2(i) to get

$$\big|\sigma_2(X_t)-\sigma_2(\tilde X_t)\big|\le C_{n_2}\big|X_{2,t}-\tilde X_{2,t}\big|\big(M_{2,2R}\big|\nabla_{x_2}\sigma_2\big|(X_t)+M_{2,2R}\big|\nabla_{x_2}\sigma_2\big|(\tilde X_t)\big).$$

Thus

$$\begin{aligned}
&\mathbb{E}\int_{\{\tau_R>t\}}\frac{|\sigma_2(X_t)-\sigma_2(\tilde X_t)|^2}{|\xi_t|^2+\delta^2}\,\mathrm{d}\mu\\
&\le C_{n_2}^2\mathbb{E}\int_{\{\tau_R>t\}}\big(M_{2,2R}|\nabla_{x_2}\sigma_2|(X_t)+M_{2,2R}|\nabla_{x_2}\sigma_2|(\tilde X_t)\big)^2\,\mathrm{d}\mu\\
&\le 2C_{n_2}^2\mathbb{E}\int_{B(R)}\big(M_{2,2R}|\nabla_{x_2}\sigma_2|\big)^2(\rho_t+\tilde\rho_t)\,\mathrm{d}\mu.
\end{aligned}$$

Hölder's inequality gives us

$$\mathbb{E}\int_{\{\tau_R>t\}}\frac{\left|\sigma_2(X_t)-\sigma_2(\tilde{X}_t)\right|^2}{|\xi_t|^2+\delta^2}\,\mathrm{d}\mu\le 4C_{n_2}^2\Lambda_{p,T}\left(\int_{B(R)}\big(M_{2,2R}|\nabla_{x_2}\sigma_2|\big)^{2q}\,\mathrm{d}\mu\right)^{\frac{1}{q}}. \tag{4.5}$$

We have

$$\begin{aligned}\int_{B(R)}\big(M_{2,2R}|\nabla_{x_2}\sigma_2|\big)^{2q}\,\mathrm{d}\mu &\le \int_{B(R)}\big(M_{2,2R}|\nabla_{x_2}\sigma_2|\big)^{2q}\,\mathrm{d}x\\ &\le \int_{B_1(R)}\mathrm{d}x_1\int_{B_2(R)}\big(M_{2,2R}|\nabla_{x_2}\sigma_2|\big)^{2q}\,\mathrm{d}x_2.\end{aligned}$$

Recall that $B_i(R)$ is a ball in $\mathbb{R}^{n_i}$ centered at the origin with radius R, for $i=1,2$. Lemma 6.2(ii) gives us

$$\int_{B_2(R)}\big(M_{2,2R}|\nabla_{x_2}\sigma_2|\big)^{2q}\,\mathrm{d}x_2\le C_{q,n_2}\int_{B_2(3R)}|\nabla_{x_2}\sigma_2|^{2q}\,\mathrm{d}x_2.$$

Therefore

$$\int_{B(R)}\big(M_{2,2R}|\nabla_{x_2}\sigma_2|\big)^{2q}\,\mathrm{d}\mu\le C_{q,n_2}\int_{B(4R)}|\nabla_{x_2}\sigma_2|^{2q}\,\mathrm{d}x.$$

Substituting this estimate into (4.5), we obtain

$$\begin{aligned}\mathbb{E}\int_{\{\tau_R>t\}}\frac{|\sigma_2(X_t)-\sigma_2(\tilde{X}_t)|^2}{|\xi_t|^2+\delta^2}\,\mathrm{d}\mu &\le C'_{q,n_2}\Lambda_{p,T}\left(\int_{B(4R)}|\nabla_{x_2}\sigma_2|^{2q}\,\mathrm{d}x\right)^{\frac{1}{q}}\\ &= C'_{q,n_2}\Lambda_{p,T}\|\nabla_{x_2}\sigma_2\|^2_{L^{2q}(B(4R))}.\end{aligned}$$

Combining this inequality with (4.3) and (4.4), we arrive at

$$\begin{aligned}&\mathbb{E}\left[\int_{G_R\cap\tilde{G}_R}\sup_{0\le t\le T}|I_1(t)|\,\mathrm{d}\mu\right]\\ &\quad\le C_T\Lambda_{p,T}^{\frac{1}{2}}\left[\frac{1}{\delta^2}\|\sigma_2-\tilde{\sigma}_2\|^2_{L^{2q}(B(R))}+C'_{q,n_2}\|\nabla_{x_2}\sigma_2\|^2_{L^{2q}(B(4R))}\right]^{\frac{1}{2}}.\end{aligned} \tag{4.6}$$

Now we begin estimating the term $I_2(t)$. We have

$$\mathbb{E}\left[\int_{G_R\cap\tilde{G}_R}\sup_{0\le t\le T}|I_2(t)|\,\mathrm{d}\mu\right]\le 2\int_0^T\left[\mathbb{E}\int_{G_R\cap\tilde{G}_R}\frac{|b_2(X_t)-\tilde{b}_2(\tilde{X}_t)|}{(|\xi_t|^2+\delta^2)^{\frac{1}{2}}}\,\mathrm{d}\mu\right]\mathrm{d}t.$$

For $x\in G_R\cap\tilde{G}_R$, one has $\tilde{X}_t(x)\in B(R)$ for all $t\in[0,T]$, then

$$\begin{aligned}\mathbb{E}\int_{G_R\cap\tilde{G}_R}\frac{\left|b_2(\tilde{X}_t)-\tilde{b}_2(\tilde{X}_t)\right|}{(|\xi_t|^2+\delta^2)^{\frac{1}{2}}}\,\mathrm{d}\mu &\le \frac{1}{\delta}\,\mathbb{E}\int_{B(R)}|b_2-\tilde{b}_2|\tilde{\rho}_t\,\mathrm{d}\mu\\ &\le \frac{\Lambda_{p,T}}{\delta}\|b_2-\tilde{b}_2\|_{L^q(B(R))}.\end{aligned} \tag{4.7}$$

By Lemma 6.2(i) and Hölder's inequality, analogous arguments as for estimating (4.5) lead to

$$\begin{aligned}
&\mathbb{E}\int_{G_R\cap\tilde{G}_R}\frac{|b_2(X_t)-b_2(\tilde{X}_t)|}{(|\xi_t|^2+\delta^2)^{\frac{1}{2}}}\,\mathrm{d}\mu\\
&\le C_{n_2}\mathbb{E}\int_{G_R\cap\tilde{G}_R}\big(M_{2,2R}|\nabla_{x_2}b_2|(X_t)+M_{2,2R}|\nabla_{x_2}b_2|(\tilde{X}_t)\big)\mathrm{d}\mu\\
&\le C_{n_2}\mathbb{E}\int_{B(R)}\big(M_{2,2R}|\nabla_{x_2}b_2|\big)(\rho_t+\tilde{\rho}_t)\,\mathrm{d}\mu\\
&\le 2C''_{q,n_2}\Lambda_{p,T}\|\nabla_{x_2}b_2\|_{L^q(B(4R))}.
\end{aligned}$$

This together with (4.7) gives us

$$\begin{aligned}
&\mathbb{E}\bigg[\int_{G_R\cap\tilde{G}_R}\sup_{0\le t\le T}|I_2(t)|\,\mathrm{d}\mu\bigg]\\
&\le 2T\Lambda_{p,T}\bigg(\frac{1}{\delta}\|b_2-\tilde{b}_2\|_{L^q(B(R))}+C''_{q,n_2}\|\nabla_{x_2}b_2\|_{L^q(B(4R))}\bigg).
\end{aligned}\tag{4.8}$$

Similarly we can show that

$$\begin{aligned}
&\mathbb{E}\bigg[\int_{G_R\cap\tilde{G}_R}\sup_{0\le t\le T}|I_3(t)|\,\mathrm{d}\mu\bigg]\\
&\le CT\Lambda_{p,T}\bigg(\frac{1}{\delta^2}\|\sigma_2-\tilde{\sigma}_2\|^2_{L^{2q}(B(R))}+C'_{q,n_2}\|\nabla_{x_2}\sigma_2\|^2_{L^{2q}(B(4R))}\bigg).
\end{aligned}\tag{4.9}$$

Combining the estimates (4.6), (4.8) and (4.9), we obtain the result. □

The a-priori estimate in Lemma 4.1 has some direct consequences. The first one is the stability of generalized stochastic flow, whose proof is analogous to that of Proposition 2.7.

Theorem 4.2 (Stability). *Suppose there is a sequence of coefficients $\sigma_{2,k}:\mathbb{R}^n\to\mathbb{R}^m\otimes\mathbb{R}^{n_2}$ and $b_{2,k}:\mathbb{R}^n\to\mathbb{R}^{n_2}$, verifying the conditions* (H3$'$) *and* (H4). *Assume that $\sigma_{2,k}$ (respectively $b_{2,k}$) converge to σ_2 (respectively b_2) in $L^{2q}_{\mathrm{loc}}(\mathbb{R}^n)$ (respectively $L^q_{\mathrm{loc}}(\mathbb{R}^n)$) as $k\to\infty$. We also assume that*

$$C_1:=\sup_{k\ge1}\big[\|\sigma_{2,k}\|_{L^{2q}(\mu)}+\|b_{2,k}\|_{L^q(\mu)}\big]<+\infty,\tag{4.10}$$

and for any $R>0$,

$$C_{2,R}:=\sup_{k\ge1}\big[\|\nabla_{x_2}b_{2,k}\|_{L^q(B(R))}+\|\nabla_{x_2}\sigma_{2,k}\|_{L^{2q}(B(R))}\big]<+\infty.\tag{4.11}$$

Let $X_t^k = (X_{1,t}, X_{2,t}^k)$ *be the stochastic flow generated by the Itô SDE* (1.1) *with the coefficients* $\sigma_k = (\sigma_1, \sigma_{2,k})$ *and* $b_k = (b_1, b_{2,k})$. *Suppose that for all* $k \geq 1$, *the density function* $\rho_t^k := \frac{\mathrm{d}(X_t^k)_\# \mu}{\mathrm{d}\mu}$ *exists and*

$$\Lambda_{p,T} := \sup_{k\geq 1} \sup_{0\leq t\leq T} \|\rho_t^k\|_{L^p(\mathbb{P}\times\mu)} < +\infty. \tag{4.12}$$

Then there exists a random field $X_2 : \Omega \times \mathbb{R}^n \to C([0,T], \mathbb{R}^{n_2})$ *such that*

$$\lim_{k\to\infty} \mathbb{E} \int_{\mathbb{R}^n} 1 \wedge \|X_2^k - X_2\|_{\infty,T} \, \mathrm{d}\mu = 0.$$

Now we are ready to show the existence of generalized stochastic flows to the Itô SDE (1.1) with partially Sobolev coefficients.

Theorem 4.3 (Existence and uniqueness). *Under the assumptions* (H1), (H2), (H3′) *and* (H4), *the Itô SDE* (1.1) *generates a unique stochastic flow* $X_t = (X_{1,t}, X_{2,t})$, *which is well defined on some small interval* $[0, T_1]$. *Moreover, the Radon–Nikodym density* $\rho_t := \frac{\mathrm{d}(X_t)_\# \mu}{\mathrm{d}\mu}$ *exists and satisfies*

$$\sup_{0\leq t\leq T_1} \|\rho_t\|_{L^p(\mathbb{P}\times\mu)} < +\infty.$$

Proof. With the a-priori estimate (Lemma 4.1) in hand, the proof of uniqueness is simple (*cf.* [15, Proposition 5.6]). We split the proof of the existence part into three steps.

Step 1. In this step we shall regularize the coefficients σ_2, b_2, and then apply Theorem 3.1 to get a sequence of stochastic flows.

To this end, we define $\sigma_{2,k}$ and $b_{2,k}$ as in (3.2). We remark that there is no need to regularize the coefficients σ_1 and b_1. Consider the family of Itô's SDE:

$$\begin{cases} \mathrm{d}X_{1,t} = \sigma_1(X_{1,t}) \, \mathrm{d}B_t + b_1(X_{1,t}) \, \mathrm{d}t, & X_{1,0} = x_1, \\ \mathrm{d}X_{2,t}^k = \sigma_{2,k}(X_{1,t}, X_{2,t}^k) \, \mathrm{d}B_t + b_{2,k}(X_{1,t}, X_{2,t}^k) \, \mathrm{d}t, & X_{2,0} = x_2. \end{cases} \tag{4.13}$$

Now we check that the regularized coefficients $\sigma_{2,k}$ and $b_{2,k}$ satisfy the conditions (H3) and (H4) stated at the beginning of Section 3. Under the assumption (H3′), it is clear that $\sigma_{2,k} \in W^{1,2q}_{x_1,x_2,\mathrm{loc}}$, $b_{2,k} \in W^{1,q}_{x_1,x_2,\mathrm{loc}}$, hence (H3) is verified. Now we show that there is $p_1 > 0$ small enough such that

$$\int_{\mathbb{R}^n} \exp\left\{p_1\big([\mathrm{div}_{x_2}(b_{2,k})]^- + |\bar{b}_{2,k}| + |\bar{\sigma}_{2,k}|^2 + |\nabla_{x_2}\sigma_{2,k}|^2\big)\right\} \mathrm{d}\mu < +\infty,$$

where $\bar{b}_{2,k} = \frac{b_{2,k}}{1+|x|}$ and $\bar{\sigma}_{2,k} = \frac{\sigma_{2,k}}{1+|x|}$. In fact, we have

$$[\mathrm{div}_{x_2}(b_{2,k})]^- \leq \big([\mathrm{div}_{x_2}(b_2)]^- + 2C|\bar{b}_2|\big) * \chi_k$$

and

$$|\nabla_{x_2}\sigma_{2,k}|^2 \le C\big(|\nabla_{x_2}\sigma_2|^2 + |\bar\sigma_2|^2\big) * \chi_k.$$

These estimates together with the inequalities (3.4) give us

$$\begin{aligned} &[\mathrm{div}_{x_2}(b_{2,k})]^- + |\bar b_{2,k}| + |\bar\sigma_{2,k}|^2 + |\nabla_{x_2}\sigma_{2,k}|^2 \\ &\quad \le 2C\big([\mathrm{div}_{x_2}(b_2)]^- + |\bar b_2| + |\bar\sigma_2|^2 + |\nabla_{x_2}\sigma_2|^2\big) * \chi_k. \end{aligned} \tag{4.14}$$

Now similar to the proof of Lemma 2.6, we can show that

$$\begin{aligned} &\int_{\mathbb{R}^n} \exp\big\{p\big([\mathrm{div}_{x_2}(b_{2,k})]^- + |\bar b_{2,k}| + |\bar\sigma_{2,k}|^2 + |\nabla_{x_2}\sigma_{2,k}|^2\big)\big\}\,\mathrm{d}\mu \\ &\quad \le \int_{\mathbb{R}^n} \exp\big\{2pC\big([\mathrm{div}_{x_2}(b_2)]^- + |\bar b_2| + |\bar\sigma_2|^2 + |\nabla_{x_2}\sigma_2|^2\big) * \chi_k\big\}\,\mathrm{d}\mu \\ &\quad \le 3^\alpha \int_{\mathbb{R}^n} \exp\big\{2pC\big([\mathrm{div}_{x_2}(b_2)]^- + |\bar b_2| + |\bar\sigma_2|^2 + |\nabla_{x_2}\sigma_2|^2\big)\big\}\,\mathrm{d}\mu, \end{aligned} \tag{4.15}$$

where $C > 0$ is independent on $k \ge 1$. Hence when $p \le p_1 := p_0/2C$, the right hand side is finite; in other words, the condition (H4) is also satisfied.

Next, since σ_1 and b_1 satisfy (H1) and (H2), we can apply Theorem 3.1 to conclude that for every $k \ge 1$, the Itô SDE (4.13) generates a unique stochastic flow $X_t^k = (X_{1,t}, X_{2,t}^k)$ which leaves the reference measure μ absolutely continuous, and by Lemma 3.2, there is T_0 small enough such that the Radon–Nikodym density $\rho_t^k := \frac{\mathrm{d}(X_t^k)_\# \mu}{\mathrm{d}\mu}$ has the following estimate: for all $t \le T_0$,

$$\begin{aligned} &\|\rho_t^k\|_{L^p(\mathbb{P}\times\mu)} \\ &\le C_{1,p}\bigg[\int_{\mathbb{R}^{n_1}} \exp\big\{C_{2,p}T_0\big([\mathrm{div}_{x_1}(b_1)]^- + |\bar b_1| + |\bar\sigma_1|^2 + |\nabla_{x_1}\sigma_1|^2\big)\big\}\mathrm{d}\mu_1\bigg]^{\frac{1}{2p(p+1)}} \\ &\quad \times \bigg[\int_{\mathbb{R}^n} \exp\big\{C_{2,p}T_0\big([\mathrm{div}_{x_2}(b_{2,k})]^- + |\bar b_{2,k}| + |\bar\sigma_{2,k}|^2 + |\nabla_{x_2}\sigma_{2,k}|^2\big)\big\}\mathrm{d}\mu\bigg]^{\frac{1}{2p(p+1)}}. \end{aligned}$$

Since p_1 does not depend on k, T_0 can also be chosen to be independent of $k \ge 1$. Substituting the estimate (4.14) into the above inequality and by an analogous argument of (4.15), we can find two constants $C'_{1,p}, C'_{2,p} > 0$ and $T_1 \le T_0$, still independent on k, such that for all $t \le T_1$,

$$\begin{aligned} &\|\rho_t^k\|_{L^p(\mathbb{P}\times\mu)} \\ &\le C'_{1,p}\bigg[\int_{\mathbb{R}^{n_1}} \exp\big\{C'_{2,p}T_1\big([\mathrm{div}_{x_1}(b_1)]^- + |\bar b_1| + |\bar\sigma_1|^2 + |\nabla_{x_1}\sigma_1|^2\big)\big\}\mathrm{d}\mu_1\bigg]^{\frac{1}{2p(p+1)}} \\ &\quad \times \bigg[\int_{\mathbb{R}^n} \exp\big\{C'_{2,p}T_1\big([\mathrm{div}_{x_2}(b_2)]^- + |\bar b_2| + |\bar\sigma_2|^2 + |\nabla_{x_2}\sigma_2|^2\big)\big\}\mathrm{d}\mu\bigg]^{\frac{1}{2p(p+1)}}. \end{aligned} \tag{4.16}$$

Step 2. We show in this step that the family of flows $(X_t^k)_{k\geq 1}$ are convergent in some sense. For this purpose we check the conditions of Theorem 4.2. First, by Remark 2.2(ii), the inequality (2.14) shows that (4.10) is satisfied. Next by (4.16), we see that under the assumptions (H2) and (H4),

$$\Lambda_{p,T_1} := \sup_{k\geq 1} \sup_{0\leq t\leq T_1} \|\rho_t^k\|_{L^p(\mathbb{P}\times\mu)} < +\infty, \tag{4.17}$$

which is nothing but (4.12). It remains to check (4.11). Direct computations lead to

$$|\nabla_{x_2} b_{2,k}| \leq |\nabla_{x_2} b_2| * \chi_k + 2C|\bar{b}_2| * \chi_k.$$

Thus

$$\int_{B(R)} |\nabla_{x_2} b_{2,k}|^q \,\mathrm{d}x \leq C_q \int_{B(R)} \big[\big(|\nabla_{x_2} b_2| * \chi_k\big)^q + \big(|\bar{b}_2| * \chi_k\big)^q\big]\,\mathrm{d}x. \tag{4.18}$$

By Jensen's inequality,

$$\begin{aligned}\int_{B(R)} |\nabla_{x_2} b_{2,k}|^q \,\mathrm{d}x &\leq C_q \int_{B(R)} \big(|\nabla_{x_2} b_2|^q + |\bar{b}_2|^q\big) * \chi_k \,\mathrm{d}x \\ &\leq C_q \|\nabla_{x_2} b_2\|_{L^q(B(R+1))}^q + C_q \|\bar{b}_2\|_{L^q(B(R+1))}^q.\end{aligned}$$

Therefore

$$\sup_{k\geq 1} \|\nabla_{x_2} b_{2,k}\|_{L^q(B(R))} < +\infty.$$

Analogously, we can show that $\sup_{k\geq 1} \|\nabla_{x_2}\sigma_{2,k}\|_{L^{2q}(B(R))} < +\infty$. Hence (4.11) is also satisfied. By Theorem 4.2, there exists $X_2 : \Omega \times \mathbb{R}^n \to C([0, T_1], \mathbb{R}^{n_2})$ such that

$$\lim_{k\to\infty} \mathbb{E} \int_{\mathbb{R}^n} 1 \wedge \|X_{2,\cdot}^k - X_{2,\cdot}\|_{\infty,T_1} \,\mathrm{d}\mu = 0. \tag{4.19}$$

Step 3. In the last step we prove that the random field $X_t = (X_{1,t}, X_{2,t})$ is the stochastic flow generated by the Itô SDE (1.1). First the same proof as that of Proposition 2.8 shows that there exists a family $\{\rho_t : 0 \leq t \leq T_1\}$ of density functions such that $(X_t)_{\#}\mu = \rho_t\mu$ for any $t \in [0, T_1]$. Moreover $\sup_{0\leq t\leq T_1} \|\rho_t\|_{L^p(\mathbb{P}\times\mu)} \leq \Lambda_{p,T_1}$, where Λ_{p,T_1} is defined in (4.17).

Thanks to (4.19), we have the following analogues of Lemma 2.9:

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \mathbb{E}\bigg(\sup_{0\leq t\leq T} \bigg|\int_0^t \big[\sigma_{2,k}(X_s^k) - \sigma_2(X_s)\big]\mathrm{d}B_s\bigg|\bigg)\mathrm{d}\mu = 0$$

and

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \mathbb{E}\bigg(\sup_{0\leq t\leq T} \bigg|\int_0^t \big[b_{2,k}(X_s^k) - b_2(X_s)\big]\mathrm{d}s\bigg|\bigg)\mathrm{d}\mu = 0.$$

With the above two limit results in hand, we let k goes to $+\infty$ in the following equation

$$X^k_{2,t} = x_2 + \int_0^t \sigma_{2,k}(X^k_s)\,\mathrm{d}B_s + \int_0^t b_{2,k}(X^k_s)\,\mathrm{d}s$$

and conclude that X_t is the flow generated by (1.1). □

Following the arguments of Section 2, we can finally extend the flow X_t to any time interval $[0,T]$; moreover, the push-forward $(X_t)_{\#}\mu = \rho_t\mu$ and the density function $\rho_t \in L^1 \log L^1$.

5. Weak differentiability of generalized stochastic flow

Using the results of the preceding section, we intend to prove in this section that the generalized stochastic flow associated to the Itô SDE with Sobolev coefficients, for which the existence and uniqueness were established in Theorem 2.3 (see also [15,25,27]), is weakly differentiable in the sense of measure, as in [18].

First we introduce some notations and assumptions. Let $d, m \geq 1$ be integers. Suppose we are given a matrix-valued function $\sigma : \mathbb{R}^d \to \mathbb{R}^m \otimes \mathbb{R}^d$ and a vector field $b : \mathbb{R}^d \to \mathbb{R}^d$. B_t is an m-dimensional standard Brownian motion. We consider the following Itô's SDE

$$\mathrm{d}X_t(x) = \sigma(X_t(x))\,\mathrm{d}B_t + b(X_t(x))\,\mathrm{d}t, \quad X_0(x) = x. \tag{5.1}$$

In this section we write $X_t(x)$ to stress the initial condition of the stochastic flow. Fix $q > 1$ and $\alpha_1 > d/2$. We denote by $\mathrm{d}\mu_1(x) = (1+|x|^2)^{-\alpha_1}\,\mathrm{d}x$ which is a finite measure on $\mathbb{R}^d$. We still write $\bar\sigma$ (respectively $\bar b$) for $\frac{\sigma}{1+|x|}$ $\big($respectively $\frac{b}{1+|x|}\big)$. Our assumptions in this section are:

(A1) $\sigma \in W^{1,2q}_{\mathrm{loc}}$ and $b \in W^{1,q}_{\mathrm{loc}}$;
(A2) $\int_{\mathbb{R}^d} \exp\big[p_0\big([\mathrm{div}(b)]^- + |\bar b| + |\bar\sigma|^2 + |\nabla\sigma|^2\big)\big]\,\mathrm{d}\mu_1 < +\infty$ for some $p_0 > 0$.

By Theorem 2.3, we see that under the assumptions (A1) and (A2), the SDE (5.1) generates a unique stochastic flow X_t of measurable maps on $\mathbb{R}^d$, such that the reference measure μ_1 is absolutely continuous under the flow. In order to prove the weak differentiability of the map $X_t : \mathbb{R}^d \to \mathbb{R}^d$, we need one more condition:

(A3) $\int_{\mathbb{R}^d} e^{p_0|\nabla b|}\,\mathrm{d}\mu_1 < +\infty$ for some $p_0 > 0$.

We follow the line of arguments in [18, Section 4]. Consider the Itô SDE on $\mathbb{R}^{2d}$:

$$\begin{cases} \mathrm{d}X_t(x) = \sigma(X_t(x))\,\mathrm{d}B_t + b(X_t(x))\,\mathrm{d}t, & X_0(x) = x, \\ \mathrm{d}Y_t(x,y) = \big[\nabla\sigma(X_t(x))\big]Y_t(x,y)\,\mathrm{d}B_t + \big[\nabla b(X_t(x))\big]Y_t(x,y)\,\mathrm{d}t, & Y_0(x,y) = y. \end{cases} \tag{5.2}$$

As mentioned for the case of ODE in [18, Section 4], the above system of equations should be the limit of a system obtained by perturbing the initial condition of the first equation. That is, for $\varepsilon > 0$, we may consider

$$\mathrm{d}X_t(x+\varepsilon y) = \sigma(X_t(x+\varepsilon y))\,\mathrm{d}B_t + b(X_t(x+\varepsilon y))\,\mathrm{d}t, \quad X_0(x+\varepsilon y) = x+\varepsilon y.$$

Combining this equation together with (5.1), we obtain a system:

$$\begin{cases} \mathrm{d}X_t(x) = \sigma(X_t(x))\,\mathrm{d}B_t + b(X_t(x))\,\mathrm{d}t, & X_0(x) = x, \\ \mathrm{d}\big[\frac{X_t(x+\varepsilon y)-X_t(x)}{\varepsilon}\big] = \frac{\sigma(X_t(x+\varepsilon y))-\sigma(X_t(x))}{\varepsilon}\,\mathrm{d}B_t & \\ \qquad\qquad + \frac{b(X_t(x+\varepsilon y))-b(X_t(x))}{\varepsilon}\,\mathrm{d}t, & \frac{X_0(x+\varepsilon y)-X_0(x)}{\varepsilon} = y. \end{cases} \tag{5.3}$$

Now it is clear that the system of equations (5.2) should be the limit in a certain sense of the above system as $\varepsilon \to 0$.

We now interpret both systems (5.2) and (5.3) as the Itô SDE with partially Sobolev coefficients studied in Section 4:

$$\begin{cases} \mathrm{d}X_{1,t} = \sigma_1(X_{1,t})\,\mathrm{d}B_t + b_1(X_{1,t})\,\mathrm{d}t, & X_{1,0} = x_1, \\ \mathrm{d}X_{2,t} = \sigma_2(X_{1,t}, X_{2,t})\,\mathrm{d}B_t + b_2(X_{1,t}, X_{2,t})\,\mathrm{d}t, & X_{2,0} = x_2. \end{cases}$$

where $x = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and $n_1 + n_2 = n$. In fact,

- for system (5.2), we set $x_1 = x, x_2 = y, n_1 = n_2 = d, X_{1,t} = X_t, X_{2,t} = (\nabla_x X_t)\,y, \sigma_1 = \sigma, b_1 = b$ and $\sigma_2 = (\nabla_x \sigma)\,y, b_2 = (\nabla_x b)\,y$;
- for system (5.3), we introduce the parameter $\varepsilon > 0$ and set $x_1 = x, x_2 = y, n_1 = n_2 = d, X_{1,t} = X_t, X_{2,t}^\varepsilon = \frac{X_t(x+\varepsilon y)-X_t(x)}{\varepsilon}, \sigma_1 = \sigma, b_1 = b$ and $\sigma_2^\varepsilon = \frac{\sigma(x+\varepsilon y)-\sigma(x)}{\varepsilon}, b_2^\varepsilon = \frac{b(x+\varepsilon y)-b(x)}{\varepsilon}$.

In the following we shall show that the two systems (5.2) and (5.3) interpreted as above verify the conditions of Section 4, and that the stochastic flows associated to (5.3) are convergent to that of (5.2) as $\varepsilon \to 0$. To this end, we shall fix $\alpha > 2\alpha_1 + q + d/2$ throughout this section. The reason for this special choice of α will become clear in the following proofs. Denote by

$$\mathrm{d}\mu(x_1, x_2) = \frac{\mathrm{d}x_1 \mathrm{d}x_2}{(1+|x_1|^2+|x_2|^2)^\alpha}.$$

Then μ is obviously a finite measure on $\mathbb{R}^{2d}$. We first prove:

Lemma 5.1. *Under the assumptions* (A1)–(A3), *both systems* (5.2) *and* (5.3) *satisfy the conditions* (H1), (H2), (H3$'$) *and* (H4).

Proof. First, note that for both systems (5.2) and (5.3), the conditions (H1) and (H2) on σ_1 and b_1 are exactly the same assumptions (A1) and (A2) for σ and b. In the following we check the hypotheses (H3$'$) and (H4) for the two systems under the additional assumption (A3) on the drift vector field b.

(1) We first treat the system (5.2). Since $\sigma_2(x_1,x_2) = (\nabla\sigma(x_1))\,x_2$, we have $\nabla_{x_2}\sigma_2(x_1,x_2) = \nabla\sigma(x_1)$, hence for any $R>0$,

$$\begin{aligned}\int_{B_1(R)} \mathrm{d}x_1 \int_{B_2(R)} \big(|\sigma_2(x_1,x_2)|^{2q} + |\nabla_{x_2}\sigma_2(x_1,x_2)|^{2q}\big)\,\mathrm{d}x_2 \\ \leq \int_{B_1(R)} \mathrm{d}x_1 \int_{B_2(R)} \big(|\nabla\sigma(x_1)|^{2q}|x_2|^{2q} + |\nabla\sigma(x_1)|^{2q}\big)\,\mathrm{d}x_2 \\ \leq (1+R^{2q})\,\Sigma_d R^d \int_{B_1(R)} |\nabla\sigma(x_1)|^{2q}\,\mathrm{d}x_1 < +\infty.\end{aligned}$$

Recall that $B_i(R)$ is a ball in $\mathbb{R}^{n_i} = R^d$ centered at the origin with radius R ($i = 1,2$), and Σ_d is the volume of unit ball in $\mathbb{R}^d$. Hence $\sigma_2 \in L^{2q}_{x_1,\mathrm{loc}}\big(W^{1,2q}_{x_2,\mathrm{loc}}\big)$. In the same way we can show that $b_2 \in L^{q}_{x_1,\mathrm{loc}}\big(W^{1,q}_{x_2,\mathrm{loc}}\big)$. As a result, (H3′) is satisfied.

Next note that $\mathrm{div}_{x_2}(b_2)(x_1,x_2) = \mathrm{div}(b)(x_1)$ which is independent on $x_2 \in \mathbb{R}^{n_2} = \mathbb{R}^d$. Since $b_2(x_1,x_2) = (\nabla b(x_1))\,x_2$, we have

$$\big|\bar{b}_2(x_1,x_2)\big| = \frac{\big|\big(\nabla b(x_1)\big)\,x_2\big|}{1+\big|(x_1,x_2)\big|} \leq \big|\nabla b(x_1)\big|;$$

similarly $|\bar{\sigma}_2(x_1,x_2)|^2 \leq |\nabla\sigma(x_1)|^2$. Moreover, $|\nabla_{x_2}\sigma_2(x_1,x_2)|^2 = |\nabla\sigma(x_1)|^2$. Combining these facts, it is clear that the assumptions (A2) and (A3) imply that σ_2 and b_2 satisfy the condition (H4) for some $p_1 \in (0, p_0]$.

(2) Now we deal with the second system (5.3). First we show that $b_2^\varepsilon \in L^{q}_{x_1,\mathrm{loc}}\big(W^{1,q}_{x_2,\mathrm{loc}}\big)$ for any $\varepsilon \leq 1$. By Fubini's theorem,

$$\int_{B_1(R)} \mathrm{d}x_1 \int_{B_2(R)} \big|b_2^\varepsilon(x_1,x_2)\big|^q\,\mathrm{d}x_2 = \int_{B_2(R)} \mathrm{d}x_2 \int_{B_1(R)} \varepsilon^{-q}\big|b(x_1+\varepsilon x_2) - b(x_1)\big|^q\,\mathrm{d}x_1. \tag{5.4}$$

For any fixed $\varepsilon \leq 1$ and $x_2 \in B_2(R)$, by the pointwise characterization of Sobolev functions, we have for a.e. $x_1 \in \mathbb{R}^{n_1}$,

$$\big|b(x_1+\varepsilon x_2) - b(x_1)\big| \leq C_d\,\varepsilon|x_2|\big(M_{|x_2|}\big|\nabla b\big|(x_1+\varepsilon x_2) + M_{|x_2|}|\nabla b|(x_1)\big). \tag{5.5}$$

Therefore

$$\begin{aligned}\int_{B_1(R)} \varepsilon^{-q}|b(x_1+\varepsilon x_2) - b(x_1)|^q\,\mathrm{d}x_1 \\ \leq C_{d,q}|x_2|^q \int_{B_1(R)} \big[\big(M_{|x_2|}|\nabla b|(x_1+\varepsilon x_2)\big)^q + \big(M_{|x_2|}|\nabla b|(x_1)\big)^q\big]\,\mathrm{d}x_1.\end{aligned}$$

For $\varepsilon \le 1$ and $|x_2| \le R$, by the maximal function inequality,

$$\begin{aligned}\int_{B_1(R)} \big(M_{|x_2|}|\nabla b|(x_1+\varepsilon x_2)\big)^q \,\mathrm{d}x_1 &= \int_{\varepsilon x_2 + B_1(R)} \big(M_{|x_2|}|\nabla b|(u)\big)^q \,\mathrm{d}u \\ &\le \int_{B_1(2R)} \big(M_R|\nabla b|(u)\big)^q \,\mathrm{d}u \\ &\le C'_{d,q} \int_{B_1(3R)} |\nabla b(u)|^q \,\mathrm{d}u.\end{aligned}$$

Consequently,

$$\int_{B_1(R)} \varepsilon^{-q}\big|b(x_1+\varepsilon x_2) - b(x_1)\big|^q \,\mathrm{d}x_1 \le \bar{C}_{d,q}\big|x_2\big|^q \int_{B_1(3R)} |\nabla b(u)|^q \,\mathrm{d}u.$$

Substituting this inequality into (5.4), we easily see that

$$\sup_{0<\varepsilon\le 1} \int_{B_1(R)} \mathrm{d}x_1 \int_{B_2(R)} \big|b_2^\varepsilon(x_1,x_2)\big|^q \,\mathrm{d}x_2 \le \bar{C}_{d,q}\,\Sigma_d R^{d+q}\,\|\nabla b\|^q_{L^q(B_1(3R))} < +\infty,$$

where Σ_d is the volume of the unit ball in $\mathbb{R}^d$. Therefore, $b_2^\varepsilon \in L^q_{x_1,\mathrm{loc}}(L^q_{x_2,\mathrm{loc}})$. Next, since $\nabla_{x_2} b_2^\varepsilon(x_1,x_2) = \nabla b(x_1+\varepsilon x_2)$, it is easy to show that $\nabla_{x_2} b_2^\varepsilon \in L^q_{x_1,\mathrm{loc}}(L^q_{x_2,\mathrm{loc}})$. Hence the assertion follows. In the same way we can show that $\sigma_2^\varepsilon \in L^{2q}_{x_1,\mathrm{loc}}(W^{1,2q}_{x_2,\mathrm{loc}})$ for any $\varepsilon \le 1$. Thus we have finished verifying (H3′).

The verifications of (H4) for σ_2^ε and b_2^ε are more complicated. First we have $\mathrm{div}_{x_2}(b_2^\varepsilon)(x_1,x_2) = \mathrm{div}(b)(x_1+\varepsilon x_2)$. Hence for $p>0$,

$$K_{1,\varepsilon} := \int_{\mathbb{R}^{2d}} e^{p[\mathrm{div}_{x_2}(b_2^\varepsilon)]^-} \,\mathrm{d}\mu(x_1,x_2) = \int_{\mathbb{R}^d} \mathrm{d}x_2 \int_{\mathbb{R}^d} \frac{e^{p[\mathrm{div}(b)(x_1+\varepsilon x_2)]^-}}{(1+|x_1|^2+|x_2|^2)^\alpha} \,\mathrm{d}x_1.$$

Making the change of variable $u_1 = x_1 + \varepsilon x_2$ in the inner integral leads to

$$K_{1,\varepsilon} = \int_{\mathbb{R}^d} \mathrm{d}x_2 \int_{\mathbb{R}^d} \frac{e^{p[\mathrm{div}(b)(u_1)]^-}}{(1+|u_1-\varepsilon x_2|^2+|x_2|^2)^\alpha} \,\mathrm{d}u_1.$$

When $\varepsilon \le 1/2$, one has $|u_1|^2 \le 2|u_1-\varepsilon x_2|^2 + 2|\varepsilon x_2|^2 \le 2|u_1-\varepsilon x_2|^2 + |x_2|^2/2$, thus

$$1+|u_1-\varepsilon x_2|^2+|x_2|^2 \ge (1+|u_1|^2+|x_2|^2)/2. \tag{5.6}$$

Therefore

$$\begin{aligned}K_{1,\varepsilon} &\le 2^\alpha \int_{\mathbb{R}^d} \mathrm{d}x_2 \int_{\mathbb{R}^d} \frac{e^{p[\mathrm{div}(b)(u_1)]^-}}{(1+|u_1|^2+|x_2|^2)^\alpha} \,\mathrm{d}u_1 \\ &\le 2^\alpha \mu_2(\mathbb{R}^d) \int_{\mathbb{R}^d} e^{p[\mathrm{div}(b)(u_1)]^-} \,\mathrm{d}\mu_1(u_1),\end{aligned}$$

where $\mathrm{d}\mu_2 = (1+|x_2|^2)^{\alpha_1-\alpha}\,\mathrm{d}x_2$ is a finite measure on $\mathbb{R}^{n_2} = \mathbb{R}^d$. Therefore by (A2), if $p \le p_0$, we have

$$\sup_{\varepsilon\le 1/2}\int_{\mathbb{R}^{2d}} e^{p[\mathrm{div}_{x_2}(b_2^\varepsilon)]^-}\,\mathrm{d}\mu < +\infty. \tag{5.7}$$

We now prove that $\int_{\mathbb{R}^{2d}} e^{p|\bar b_2^\varepsilon|}\,\mathrm{d}\mu < +\infty$ for p sufficiently small. In fact,

$$\int_{\mathbb{R}^{2d}} e^{p|\bar b_2^\varepsilon|}\,\mathrm{d}\mu = \int_{\mathbb{R}^d}\mathrm{d}x_2\int_{\mathbb{R}^d}\frac{\exp\Big\{p\frac{|b(x_1+\varepsilon x_2)-b(x_1)|}{\varepsilon(1+|(x_1,x_2)|)}\Big\}}{(1+|x_1|^2+|x_2|^2)^\alpha}\,\mathrm{d}x_1.$$

Again by the pointwise inequality (5.5), we get

$$\begin{aligned}&\int_{\mathbb{R}^{2d}} e^{p|\bar b_2^\varepsilon|}\,\mathrm{d}\mu\\ &\le \int_{\mathbb{R}^d}\mathrm{d}x_2\int_{\mathbb{R}^d}\frac{\exp\big\{pC_d\big(M_{|x_2|}|\nabla b|(x_1+\varepsilon x_2)+M_{|x_2|}|\nabla b|(x_1)\big)\big\}}{(1+|x_1|^2+|x_2|^2)^\alpha}\,\mathrm{d}x_1.\end{aligned} \tag{5.8}$$

We first estimate the term

$$K_{2,\varepsilon} := \int_{\mathbb{R}^d}\mathrm{d}x_2\int_{\mathbb{R}^d}\frac{\exp\big\{pC_dM_{|x_2|}|\nabla b|(x_1+\varepsilon x_2)\big\}}{(1+|x_1|^2+|x_2|^2)^\alpha}\,\mathrm{d}x_1.$$

Similar to the treatment of $K_{1,\varepsilon}$, changing the variable and by (5.6), we have for all $\varepsilon \le 1/2$,

$$\begin{aligned}K_{2,\varepsilon} &\le 2^\alpha\int_{\mathbb{R}^d}\mathrm{d}x_2\int_{\mathbb{R}^d}\frac{\exp\big\{pC_dM_{|x_2|}|\nabla b|(u_1)\big\}}{(1+|u_1|^2+|x_2|^2)^\alpha}\,\mathrm{d}u_1\\ &\le 2^\alpha\int_{\mathbb{R}^d}\frac{\mathrm{d}x_2}{(1+|x_2|^2)^{\alpha-\alpha_1}}\int_{\mathbb{R}^d}e^{pC_dM_{|x_2|}|\nabla b|(u_1)}\,\mathrm{d}\mu_1(u_1),\end{aligned}$$

where the measure μ_1 is defined at the beginning of this section. We split the right hand side into two parts:

$$\begin{aligned}K_{2,\varepsilon} &\le 2^\alpha\int_{\{|x_2|\le 1\}}\frac{\mathrm{d}x_2}{(1+|x_2|^2)^{\alpha-\alpha_1}}\int_{\mathbb{R}^d}e^{pC_dM_{|x_2|}|\nabla b|(u_1)}\,\mathrm{d}\mu_1(u_1)\\ &\quad+2^\alpha\int_{\{|x_2|>1\}}\frac{\mathrm{d}x_2}{(1+|x_2|^2)^{\alpha-\alpha_1}}\int_{\mathbb{R}^d}e^{pC_dM_{|x_2|}|\nabla b|(u_1)}\,\mathrm{d}\mu_1(u_1).\end{aligned} \tag{5.9}$$

Denote the two terms by $K_{2,\varepsilon}^{(1)}$ and $K_{2,\varepsilon}^{(2)}$ respectively. Now we are going to apply Lemma 6.3. In the present case, $\lambda(z) = (1+|z|^2)^{-\alpha_1}$ ($z \in \mathbb{R}^d$) and $\delta = 1$ or $|x_2|$. It is easy to show that for any $\delta \ge 1$,

$$\Lambda_0 = \sup_{k\ge 1}\left(\frac{1+(k+1)^2\delta^2}{1+(k-1)^2\delta^2}\right)^{\alpha_1} = (1+4\delta^2)^{\alpha_1}.$$

Thus for $|x_2| > 1$, an application of (6.7) gives us

$$\begin{aligned}\int_{\mathbb{R}^d} e^{pC_d M_{|x_2|}|\nabla b|}\,\mathrm{d}\mu_1 \le \int_{\mathbb{R}^d} \big(1 + pC_d M_{|x_2|}|\nabla b|\big)\,\mathrm{d}\mu_1 \\ + 6 \times 5^d\big(1+4|x_2|^2\big)^{\alpha_1}\int_{\mathbb{R}^d} e^{2pC_d|\nabla b|}\,\mathrm{d}\mu_1.\end{aligned} \tag{5.10}$$

By Cauchy's inequality and (6.6), we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} M_{|x_2|}|\nabla b|\,\mathrm{d}\mu_1 &\le \left[\mu_1(\mathbb{R}^d)\int_{\mathbb{R}^d}\big(M_{|x_2|}|\nabla b|\big)^2\,\mathrm{d}\mu_1\right]^{\frac{1}{2}} \\ &\le \left[24 \times 5^d \mu_1(\mathbb{R}^d)\big(1+4|x_2|^2\big)^{\alpha_1}\int_{\mathbb{R}^d}|\nabla b|^2\,\mathrm{d}\mu_1\right]^{\frac{1}{2}} \\ &= C_d'\|\nabla b\|_{L^2(\mu_1)}\big(1+4|x_2|^2\big)^{\alpha_1/2}.\end{aligned} \tag{5.11}$$

Substituting (5.11) into (5.10), we can find some positive constant $C_{p,d} > 0$ such that

$$\int_{\mathbb{R}^d} e^{pC_d M_{|x_2|}|\nabla b|}\,\mathrm{d}\mu_1 \le C_{p,d}\big(1+4|x_2|^2\big)^{\alpha_1}\int_{\mathbb{R}^d} e^{2pC_d|\nabla b|}\,\mathrm{d}\mu_1.$$

Therefore

$$K_{2,\varepsilon}^{(2)} \le 2^\alpha C_{p,d}\left(\int_{\mathbb{R}^d} e^{2pC_d|\nabla b|}\,\mathrm{d}\mu_1\right)\int_{|x_2|>1}\frac{\big(1+4|x_2|^2\big)^{\alpha_1}}{\big(1+|x_2|^2\big)^{\alpha-\alpha_1}}\,\mathrm{d}x_2.$$

Since $\alpha > 2\alpha_1 + d/2$, the second integral is finite. As a result,

$$\sup_{\varepsilon\le 1/2} K_{2,\varepsilon}^{(2)} \le 2^\alpha \tilde{C}_{p,d}\int_{\mathbb{R}^d} e^{2pC_d|\nabla b|}\,\mathrm{d}\mu_1.$$

By (A3), we see that when $p \le p_0/(2C_d)$, the right hand side is finite. Notice that

$$K_{2,\varepsilon}^{(1)} \le 2^\alpha \Sigma_d \int_{\mathbb{R}^d} e^{pC_d M_1|\nabla b|(u_1)}\,\mathrm{d}\mu_1(u_1),$$

where Σ_d is the volume of the d-dimensional unit ball. In the same way we can prove that $\sup_{\varepsilon\le 1/2} K_{2,\varepsilon}^{(1)} < +\infty$ for $p \le p_0/(2C_d)$. Substituting these estimates into (5.9), we conclude that if $p \le p_0/(2C_d)$, $K_{2,\varepsilon}$ is bounded uniformly in $\varepsilon \le 1/2$. In the above discussions, we have indeed proved

$$\int_{\mathbb{R}^d}\mathrm{d}x_2\int_{\mathbb{R}^d}\frac{\exp\big\{pC_d M_{|x_2|}|\nabla b|(x_1)\big\}}{(1+|x_1|^2+|x_2|^2)^\alpha}\,\mathrm{d}x_1 < +\infty.$$

Therefore an application of Cauchy's inequality to (5.8) gives us that for any $p \le p_0/(4C_d)$,

$$\sup_{\varepsilon\le 1/2}\int_{\mathbb{R}^{2d}} e^{p|\bar b_2^\varepsilon|}\,\mathrm{d}\mu < +\infty. \tag{5.12}$$

Analogously, we can show that when p is small enough, it holds

$$\sup_{\varepsilon\le 1/2}\int_{\mathbb{R}^{2d}} e^{p|\bar\sigma_2^\varepsilon|^2}\,\mathrm{d}\mu < +\infty. \tag{5.13}$$

Finally, since $\nabla_{x_2}\sigma_2^\varepsilon(x_1,x_2) = (\nabla\sigma)(x_1+\varepsilon x_2)$, we follow the arguments for estimating $K_{1,\varepsilon}$ and arrive at

$$\sup_{\varepsilon\le 1/2}\int_{\mathbb{R}^{2d}} e^{p|\nabla_{x_2}\sigma_2^\varepsilon|^2}\,\mathrm{d}\mu < +\infty$$

for p sufficiently small. Combining this estimate with (5.7), (5.12) and (5.13), we conclude that σ_2^ε and b_2^ε satisfy the condition (H4), uniformly in $\varepsilon\in(0,1/2]$. □

By Lemma 5.1, we can apply the main result of Section 4 (Theorem 4.3) to both systems (5.2) and (5.3). Therefore, the system (5.2) (respectively (5.3)) generates a unique stochastic flow $Z_t(x,y) = (X_t(x), Y_t(x,y))$ (respectively $Z_t^\varepsilon(x,y) = \big(X_t(x), \varepsilon^{-1}(X_t(x+\varepsilon y)-X_t(x))\big)$); moreover the Radon–Nikodym densities $\rho_t = \frac{\mathrm{d}(Z_t)_\#\mu}{\mathrm{d}\mu}$ and $\rho_t^\varepsilon = \frac{\mathrm{d}(Z_t^\varepsilon)_\#\mu}{\mathrm{d}\mu}$ exist, and there is a $T_0>0$ small enough (note that by the uniform estimate in Lemma 5.1, T_0 does not depend on $\varepsilon\le 1/2$) such that

$$\Lambda_{p,T_0} := \Big(\sup_{0\le t\le T_0}\|\rho_t\|_{L^p(\mathbb{P}\times\mu)}\Big)\bigvee\Big(\sup_{\varepsilon\le 1/2}\sup_{0\le t\le T_0}\|\rho_t^\varepsilon\|_{L^p(\mathbb{P}\times\mu)}\Big) < +\infty, \tag{5.14}$$

where p is the conjugate number of q. Next we want to prove that $Y_t^\varepsilon(x,y) := \varepsilon^{-1}(X_t(x+\varepsilon y)-X_t(x))$ is convergent to $Y_t(x,y)$ in a certain sense, following the idea of Theorem 4.2.

Theorem 5.2. *Under the assumptions* (A1)–(A3)*, we have for any* $T>0$,

$$\lim_{\varepsilon\to 0}\mathbb{E}\int_{\mathbb{R}^{2d}} 1\wedge \big\|Y_\cdot^\varepsilon - Y_\cdot\big\|_{\infty,T}\,\mathrm{d}\mu = 0.$$

Proof. First we show that

$$\lim_{\varepsilon\to 0}\mathbb{E}\int_{\mathbb{R}^{2d}} 1\wedge \big\|Y_\cdot^\varepsilon - Y_\cdot\big\|_{\infty,T_0}\,\mathrm{d}\mu = 0. \tag{5.15}$$

The proof is similar to that of Theorem 4.2, and we shall apply Lemma 4.1 to show the convergence. It is easy to see that for any $R>0$,

$$\big\|\nabla_{x_2}b_2\big\|_{L^q(B(R))} + \big\|\nabla_{x_2}\sigma_2\big\|_{L^{2q}(B(R))} < +\infty.$$

Notice that we already have the uniform density estimate (5.14), hence it only remains to check the following conditions:

$$C_1 := \sup_{\varepsilon\leq 1/2} \big(\|\sigma_2^\varepsilon\|_{L^{2q}(\mu)} + \|b_2^\varepsilon\|_{L^{2q}(\mu)}\big) < +\infty \tag{5.16}$$

and

$$\sigma_2^\varepsilon \to \sigma_2 \text{ in } L^{2q}_{\mathrm{loc}}(\mathbb{R}^{2d}) \quad \text{and} \quad b_2^\varepsilon \to b_2 \text{ in } L^q_{\mathrm{loc}}\big(\mathbb{R}^{2d}\big). \tag{5.17}$$

By Remark 2.2 and (5.12), (5.13), we easily deduce that C_1 defined in (5.16) is finite. Next, since $\sigma_2^\varepsilon(x_1,x_2) = \frac{\sigma(x_1+\varepsilon x_2)-\sigma(x_1)}{\varepsilon}$ and $\sigma_2(x_1,x_2) = (\nabla\sigma(x_1))\,x_2$, the convergence $\sigma_2^\varepsilon \to \sigma_2$ in $L^{2q}_{\mathrm{loc}}(\mathbb{R}^{2d})$ follows from the fact that $\sigma \in W^{1,2q}_{\mathrm{loc}}(\mathbb{R}^d)$. Similarly we conclude that b_2^ε converge to b_2 in $L^q_{\mathrm{loc}}(\mathbb{R}^{2d})$. Hence the convergences in (5.17) are verified. Now we are ready to follow the line of the proof of Theorem 4.2 to obtain the convergence (5.15).

Finally we can use the flow properties of $Z_t = (X_t, Y_t)$ and $Z_t^\varepsilon = (X_t, Y_t^\varepsilon)$ to extend the convergence to the whole interval $[0, T]$. □

This theorem shows that the generalized stochastic flow associated to the Itô SDE (5.1) is weakly differentiable in the sense of measure, provided that its coefficients σ and b satisfy the assumptions (A1)–(A3). Note that if σ and b are globally Lipschitz continuous, then they fulfil (A1)–(A3). In this case, however, our result is weaker than that in [6], where the authors proved that almost surely, the map $X_t : \mathbb{R}^d \to \mathbb{R}^d$ is almost everywhere differentiable with respect to the initial data for any time, by using the theory of Dirichlet form. In [20, Section 5], we considered the Stratonovich SDE with smooth diffusion coefficient σ and Sobolev drift coefficient b, and proved the approximate differentiability of the generalized stochastic flow by using the Ocone-Pardoux decomposition, which essentially reduces the problem to prove the differentiability of the flow generated by some ODE with random Sobolev coefficient.

6. Appendix

In this section we present some results that are used in the paper. We assume the coefficients $\sigma : \mathbb{R}^n \to \mathbb{R}^m \otimes \mathbb{R}^n$ and $b : \mathbb{R}^n \to \mathbb{R}^n$ of the Itô SDE

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t + b(X_t)\,\mathrm{d}t, \qquad X_0 = x \tag{6.1}$$

are smooth and bounded together with their derivatives of all orders. Here B_t is still an m-dimensional standard Brownian motion. Then the above equation generates a stochastic flow X_t of diffeomorphisms on $\mathbb{R}^n$.

First we recall the expression for the Radon–Nikodym density of the stochastic flow with respect to some reference measure. Let $\lambda \in C^2(\mathbb{R}^n)$ and define a measure on $\mathbb{R}^n$ by

$$\mathrm{d}\mu(x) = e^{\lambda(x)}\mathrm{d}x.$$

It is well known that the push-forward $(X_t)_{\#}\mu$ (respectively $(X_t^{-1})_{\#}\mu$) of μ by the flow X_t (respectively the inverse flow X_t^{-1}) is absolutely continuous with respect to μ. Denote by

$$\rho_t(x) = \frac{\mathrm{d}\big[(X_t)_{\#}\mu\big]}{\mathrm{d}\mu}(x) \quad \text{and} \quad \tilde\rho_t(x) = \frac{\mathrm{d}\big[(X_t^{-1})_{\#}\mu\big]}{\mathrm{d}\mu}(x).$$

We have the following simple identity:

$$\rho_t(x) = 1/\tilde\rho_t\big(X_t^{-1}(x)\big). \tag{6.2}$$

Moreover by [17, Lemma 4.3.1], a simple computation gives us (see also [27, (3.6)])

$$\tilde\rho_t(x) = \exp\bigg(\int_0^t \big\langle \Lambda_1^\sigma(X_s(x)), \mathrm{d}B_s\big\rangle + \int_0^t \Lambda_2^{\sigma,b}\big(X_s(x)\big)\,\mathrm{d}s\bigg), \tag{6.3}$$

in which

$$\Lambda_1^\sigma = \mathrm{div}(\sigma) + \sigma^*\nabla\lambda \quad \text{and} \quad \Lambda_2^{\sigma,b} = \mathrm{div}(b) + \mathcal{L}\lambda - \frac{1}{2}\big\langle \nabla\sigma, (\nabla\sigma)^*\big\rangle.$$

Here by $\mathrm{div}(\sigma) = \big(\mathrm{div}(\sigma^{\cdot,1}), \dots, \mathrm{div}(\sigma^{\cdot,m})\big)$ we mean the $\mathbb{R}^m$-valued function whose components are the divergences of the columns of σ; σ^* is the transpose of σ and $\mathcal{L}$ is the second order differential operator associated to (6.1):

$$\mathcal{L}\lambda = \frac{1}{2}\sum_{i,j=1}^n a^{ij}\partial_i\partial_j\lambda + \sum_{i=1}^n b^i\partial_i\lambda$$

with $a^{ij} = \sum_{k=1}^m \sigma^{ik}\sigma^{jk}$ and $\partial_i\lambda = \frac{\partial}{\partial x^i}\lambda$. Finally

$$\big\langle \nabla\sigma, (\nabla\sigma)^*\big\rangle = \sum_{k=1}^m \big\langle \nabla\sigma^{\cdot,k}, \big(\nabla\sigma^{\cdot,k}\big)^*\big\rangle = \sum_{k=1}^m \sum_{i,j=1}^n \big(\partial_i\sigma^{jk}\big)\big(\partial_j\sigma^{ik}\big).$$

From this expression, we see that if the first n_1-rows $\sigma_1 = (\sigma^{ij})_{1\le i\le n_1, 1\le j\le m}$ only depend on the variables $x_1 = (x^1, \dots, x^{n_1})$, then

$$\begin{aligned}\big\langle \nabla\sigma, (\nabla\sigma)^*\big\rangle &= \sum_{k=1}^m \bigg(\sum_{i,j=1}^{n_1} \big(\partial_i\sigma^{jk}\big)\big(\partial_j\sigma^{ik}\big) + \sum_{i,j=n_1+1}^{n} \big(\partial_i\sigma^{jk}\big)\big(\partial_j\sigma^{ik}\big)\bigg)\\ &= \big\langle \nabla_{x_1}\sigma_1, \big(\nabla_{x_1}\sigma_1\big)^*\big\rangle + \big\langle \nabla_{x_2}\sigma_2, \big(\nabla_{x_2}\sigma_2\big)^*\big\rangle,\end{aligned} \tag{6.4}$$

where $x_2 = (x^{n_1+1}, \dots, x^n)$ and σ_2 consists of the last $(n-n_1)$-rows of the matrix σ. Notice that the derivatives $\nabla_{x_1}\sigma_2$ are not involved here. This observation is crucial for the present work.

The following is an L^p-estimate for $\rho_t(x)$ which is proved in [27, Lemma 3.2] (see also [15, Theorem 2.1] for the case where $\mu = \gamma_n$ is the standard Gaussian measure).

Lemma 6.1. *Assume that* $\mu(\mathbb{R}^n) < +\infty$. *Then for any* $t \in [0, T]$ *and* $p > 1$,

$$\|\rho_t\|_{L^p(\mathbb{P}\times\mu)} \le \mu(\mathbb{R}^n)^{1/(p+1)}\bigg(\sup_{t\in[0,T]}\int_{\mathbb{R}^n} \exp\big(tp^3\big|\Lambda_1^\sigma\big|^2 - tp^2\Lambda_2^{\sigma,b}\big)\mathrm{d}\mu\bigg)^{1/p(p+1)}. \tag{6.5}$$

In the following we introduce the pointwise inequality for partially Sobolev functions. To this end, we need the notion of locally maximal function for partial variables. As in the introduction, $n = n_1+n_2$ and for $x \in \mathbb{R}^n$, we write $x = (x_1, x_2)$ where $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$. Let $f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ be locally integrable. For almost every $x_1 \in \mathbb{R}^{n_1}$, define

$$\begin{aligned} M_{2,R} f(x_1, x_2) &= \sup_{0<r\le R} ⨍_{B_2(x_2,r)} \big|f(x_1, y_2)\big|\,\mathrm{d}y_2 \\ &:= \sup_{0<r\le R} \frac{1}{\mathcal{L}_{n_2}\big(B_2(x_2, r)\big)}\int_{B_2(x_2,r)} \big|f(x_1, y_2)\big|\,\mathrm{d}y_2, \quad R > 0. \end{aligned}$$

Here $B_2(x_2, r)$ means the ball in $\mathbb{R}^{n_2}$ centered at x_2 with radius r, and $\mathcal{L}_{n_2}$ is the Lebesgue measure on $\mathbb{R}^{n_2}$. Recall that $B_i(r)$ is the ball in $\mathbb{R}^{n_i}$ of radius r centered at the origin, $i = 1, 2$. The main point of the first result in the next lemma lies in the fact that the exceptional set N is chosen to be a negligible subset of $\mathbb{R}^n$.

Lemma 6.2.

(i) *Suppose that* $f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ *belongs to the space* $L^1_{x_1,\mathrm{loc}}\big(W^{1,1}_{x_2,\mathrm{loc}}\big)$. *Then there is a dimensional constant* $C > 0$ *(independent of* f*) and a negligible set* $N \subset \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, *such that for all* $(x_1, x_2), (x_1, y_2) \notin N$ *with* $|x_2-y_2|_{\mathbb{R}^{n_2}} \le R$, *it holds*

$$\begin{aligned} &|f(x_1, x_2) - f(x_1, y_2)| \\ &\le C|x_2 - y_2|_{\mathbb{R}^{n_2}}\big[M_{2,R}|\nabla_{x_2} f|(x_1, x_2) + M_{2,R}|\nabla_{x_2} f|(x_1, y_2)\big]. \end{aligned}$$

(ii) *If* $f \in L^p_{\mathrm{loc}}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ *for some* $p > 1$, *then there is a constant* $C_{p,n_2} > 0$ *such that*

$$\int_{B_2(r)} \big(M_{2,R} f(x_1, x_2)\big)^p\,\mathrm{d}x_2 \le C_{p,n_2}\int_{B_2(r+R)} \big|f(x_1, x_2)\big|^p\,\mathrm{d}x_2.$$

Proof. (i) Here we present a proof based on the well-known pointwise inequality for Sobolev functions. Let

$$\tilde N = \bigg\{(x_1,x_2)\in\mathbb{R}^n : x_1\in\mathbb{R}^{n_1} \text{ and } \limsup_{\mathcal{L}_{n_2}(B)\to 0,\, x_2\in B}\bigg|⨍_B f(x_1,y_2)\,\mathrm{d}y_2 - f(x_1,x_2)\bigg| > 0\bigg\},$$

where the limit is taken over all balls $B \subset \mathbb{R}^{n_2}$ such that x_2 is contained in B. $\tilde{N}$ is a measurable subset of $\mathbb{R}^n$. We see that for all $x_1 \in \mathbb{R}^{n_1}$, the section

$$\tilde{N}_{x_1} = \left\{ x_2 \in \mathbb{R}^{n_2} : \limsup_{\mathcal{L}_{n_2}(B)\to 0,\, x_2\in B} \left| \fint_B f(x_1, y_2)\,\mathrm{d}y_2 - f(x_1, x_2) \right| > 0 \right\}.$$

Since $f \in L^1_{x_1,\mathrm{loc}}(W^{1,1}_{x_2,\mathrm{loc}})$, there is an $\mathcal{L}_{n_1}$-negligible set $N_1 \subset \mathbb{R}^{n_1}$, such that for every $x_1 \notin N_1$, one has $f(x_1, \cdot) \in W^{1,1}_{x_2,\mathrm{loc}}$. In particular, $f(x_1, \cdot) \in L^1_{x_2,\mathrm{loc}}$. Lebesgue's differentiation theorem gives us $\mathcal{L}_{n_2}(\tilde{N}_{x_1}) = 0$ for all $x_1 \notin N_1$. By Fubini's theorem we have

$$\mathcal{L}_n(\tilde{N}) = \int_{\mathbb{R}^{n_1}} \mathcal{L}_{n_2}(\tilde{N}_{x_1})\,\mathrm{d}x_1 = 0.$$

Define $N = \tilde{N} \cup (N_1 \times \mathbb{R}^{n_2})$. We see that $\mathcal{L}_n(N) = 0$. Now fix any $(x_1, x_2), (x_1, y_2) \notin N$ with $|x_2 - y_2|_{\mathbb{R}^{n_2}} \le R$. Since $x_1 \notin N_1$, we have $f(x_1, \cdot) \in W^{1,1}_{x_2,\mathrm{loc}}$. By the pointwise inequality of Sobolev functions (see *e.g.* [2, page 186] or [15, Theorem A.1]), there exist a constant $C_{n_2} > 0$ such that for all $u_2, v_2 \notin \tilde{N}_{x_1}$ with $|u_2 - v_2|_{\mathbb{R}^{n_2}} \le R$, it holds

$$|f(x_1, u_2) - f(x_1, v_2)| \le C|u_2 - v_2|_{\mathbb{R}^{n_2}} \big[M_{2,R}|\nabla_{x_2} f|(x_1, u_2) + M_{2,R}|\nabla_{x_2} f|(x_1, v_2) \big].$$

Now the result follows by noticing that $x_2, y_2 \notin N_{x_1}$ and $\tilde{N}_{x_1} \subset N_{x_1}$.

(ii) This is obvious from the properties of maximal functions. □

The next result is similar to Lemma 6.2(ii), but the integral is taken with respect to some other reference measure. Perhaps such a result already exists, but we are unaware of its reference. We present its proof for the reader's convenience. Suppose we are given a continuous $\lambda \in C(\mathbb{R}^n, (0, +\infty))$ such that $\mathrm{d}\mu = \lambda\,\mathrm{d}x$ is a finite measure on $\mathbb{R}^n$. Fix $\delta > 0$. For every positive integer k, we denote by $R_k := \{x \in \mathbb{R}^n : (k-1)\delta \le |x| \le k\delta\}$, that is, the ring between the concentric spheres centered at the origin with radii $(k-1)\delta$ and $k\delta$, respectively. Set

$$\overline{\lambda}_k = \sup_{x\in R_k} \lambda(x), \quad \underline{\lambda}_k = \inf_{x\in (R_k)_\delta} \lambda(x),$$

where $(R_k)_\delta$ is the δ-neighborhood of the ring R_k. We shall denote by

$$\Lambda_0 = \sup_{k\ge 1} \frac{\overline{\lambda}_k}{\underline{\lambda}_k}.$$

Obviously $\Lambda_0 \ge 1$. If $\lambda(x) = \phi(|x|)$ and for some $\beta > 1$, $\phi(s) \sim e^{-s^\beta}$ as $s \to \infty$, then $\Lambda_0 = +\infty$. Therefore Lemma 6.3 does not hold for the standard Gaussian measure.

The local maximal function $M_\delta f(x)$ of a locally integrable function $f \in L^1_{\mathrm{loc}}$ is defined as usual:

$$M_\delta f(x) = \sup_{0<r\le\delta} \fint_{B(x,r)} |f(y)|\,\mathrm{d}y := \sup_{0<r\le\delta} \frac{1}{\mathcal{L}_n\big(B(x,r)\big)} \int_{B(x,r)} \big|f(y)\big|\,\mathrm{d}y.$$

Lemma 6.3. *Assume that $\Lambda_0 < +\infty$ and denote by $C_p = 5^n 2^p p/(p-1)$ for $p > 1$. Then*

$$\int_{\mathbb{R}^n} (M_\delta f)^p \,\mathrm{d}\mu \le 3C_p\Lambda_0 \int_{\mathbb{R}^n} |f|^p \,\mathrm{d}\mu. \tag{6.6}$$

As a result, for any $\theta > 0$,

$$\int_{\mathbb{R}^n} e^{\theta M_\delta f}\,\mathrm{d}\mu \le \int_{\mathbb{R}^n} (1+\theta M_\delta f)\,\mathrm{d}\mu + 6\times 5^n\Lambda_0 \int_{\mathbb{R}^n} e^{2\theta|f|}\,\mathrm{d}\mu. \tag{6.7}$$

Proof. Note that

$$\int_{\mathbb{R}^n} (M_\delta f)^p\,\mathrm{d}\mu = \sum_{k=1}^\infty \int_{R_k} (M_\delta f)^p\,\mathrm{d}\mu \le \sum_{k=1}^\infty \overline{\lambda}_k \int_{R_k} (M_\delta f)^p\,\mathrm{d}x. \tag{6.8}$$

Next we follow the idea of [23, Chap. I, Section 1] to show that for any $p > 1$,

$$\int_{R_k} (M_\delta f)^p\,\mathrm{d}x \le C_p \int_{(R_k)_\delta} |f|^p\,\mathrm{d}x, \tag{6.9}$$

where $C_p = 2^p 5^n p/(p-1)$. Indeed, for any $s > 0$, we define $R_k(s) = \{x \in R_k : M_\delta f(x) > s\}$ (note that $s \to \mathcal{L}_n(R_k(s))$ is the distribution function of $M_\delta f$ when restricted on R_k). Then similar to the argument on [23, pp. 6–7], we have

$$\mathcal{L}_n(R_k(s)) \le \frac{2\times 5^n}{s} \int_{(R_k)_\delta \cap \{|f|>s/2\}} |f(y)|\,\mathrm{d}y. \tag{6.10}$$

Next it is easy to show that

$$\int_{R_k} (M_\delta f)^p\,\mathrm{d}x = p\int_0^\infty s^{p-1}\mathcal{L}_n(R_k(s))\,\mathrm{d}s.$$

Substituting (6.10) into the above equality and changing the order of integration, we finally get

$$\int_{R_k} (M_\delta f)^p\,\mathrm{d}x \le \frac{5^n 2^p p}{p-1} \int_{(R_k)_\delta} |f(y)|^p\,\mathrm{d}y.$$

Now by (6.9) and the definition of $\underline{\lambda}_k$, we have

$$\int_{R_k} (M_\delta f)^p\,\mathrm{d}x \le \frac{C_p}{\underline{\lambda}_k} \int_{(R_k)_\delta} |f|^p\,\mathrm{d}\mu.$$

Substituting this inequality into (6.8), we obtain

$$\int_{\mathbb{R}^n} (M_\delta f)^p \,\mathrm{d}\mu \le C_p \sum_{k=1}^{\infty} \frac{\overline{\lambda}_k}{\underline{\lambda}_k} \int_{(R_k)_\delta} |f|^p \,\mathrm{d}\mu \le 3C_p \Lambda_0 \int_{\mathbb{R}^n} |f|^p \,\mathrm{d}\mu.$$

Finally, by expanding the exponential function, we have

$$\int_{\mathbb{R}^n} e^{\theta M_\delta f} \,\mathrm{d}\mu = \int_{\mathbb{R}^n} (1 + \theta M_\delta f) \,\mathrm{d}\mu + \sum_{k=2}^{\infty} \frac{\theta^k}{k!} \int_{\mathbb{R}^n} (M_\delta f)^k \,\mathrm{d}\mu. \tag{6.11}$$

Applying the inequality proved above, we get, for any $k \ge 2$,

$$\int_{\mathbb{R}^n} (M_\delta f)^k \,\mathrm{d}\mu \le 3\Lambda_0 \frac{5^n 2^k k}{k-1} \int_{\mathbb{R}^n} |f|^k \,\mathrm{d}\mu \le 3 \times 5^n \Lambda_0 2^{k+1} \int_{\mathbb{R}^n} |f|^k \,\mathrm{d}\mu.$$

Therefore,

$$\sum_{k=2}^{\infty} \frac{\theta^k}{k!} \int_{\mathbb{R}^n} (M_\delta f)^k \,\mathrm{d}\mu \le 6 \times 5^n \Lambda_0 \sum_{k=2}^{\infty} \frac{(2\theta)^k}{k!} \int_{\mathbb{R}^n} |f|^k \,\mathrm{d}\mu \le 6 \times 5^n \Lambda_0 \int_{\mathbb{R}^n} e^{2\theta |f|} \,\mathrm{d}\mu.$$

The proof is completed by substituting this inequality into (6.11). □

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