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Convergence of flux-limited porous media diffusion equations to their classical counterpart

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Abstract. We prove the convergence of a porous medium type flux-limited diffusion equation to the classical porous medium equation as the parameter c representing the maximum speed of propagation tends to ∞.

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1. Introduction

We are interested in the convergence of entropy solutions of the quasi-linear parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}\left(\dfrac{\Lambda(u)\nabla\Phi(u)}{\sqrt{1+c^{-2}|\nabla\Phi(u)|^2}}\right) & \text{in} \quad Q_T = (0,T)\times\mathbb{R}^N \\ u(0,x) = u_0(x) & \text{in} \quad x\in\mathbb{R}^N, \end{cases} \tag{1.1}$$

where $0 \le u_0 \in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, as $c\to\infty$, to solutions of

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}\left(\Lambda(u)\nabla\Phi(u)\right) & \text{in} \quad Q_T = (0,T)\times\mathbb{R}^N \\ u(0,x) = u_0(x) & \text{in} \quad x\in\mathbb{R}^N. \end{cases} \tag{1.2}$$

We assume that $\Lambda, \Phi : [0,\infty) \to [0,\infty)$ are continuous and strictly increasing functions such that $\Phi(0) = 0$, with some additional regularity that will be made

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precise later on. All these assumptions are satisfied by

$$u_t = \nu \operatorname{div} \left(\frac{u \nabla u^m}{\sqrt{1 + \frac{\nu^2}{c^2} |\nabla u^m|^2}} \right), \tag{1.3}$$

where $\nu, c > 0, m \geq 1$. In this case, as we shall prove in this paper, solutions of (1.3) converge as $c \to \infty$ to solutions of the classical porous medium equation

$$u_t = \nu \operatorname{div} \left(u \nabla u^m \right). \tag{1.4}$$

Equation (1.1) is an example of flux-limited or tempered diffusion equation

$$\frac{\partial u}{\partial t} = \operatorname{div} \mathbf{a}(u, D\Phi(u)) \tag{1.5}$$

characterized by a bounded flux $\mathbf{a}(z, \xi) = \nabla_\xi f(z, \xi)$, where $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^+$ is a continuous function, convex in ξ, with linear growth as $\|\xi\| \to \infty$ such that $\mathbf{a}(z, \xi) \in C(\mathbb{R} \times \mathbb{R}^N)$ and $\Phi : [0, \infty) \to [0, \infty)$ is as above. One of the first such models was proposed by J. R. Wilson in the theory of radiation hydrodynamics [34] and corresponds to the flux $\mathbf{a}(u, Du) = \nu \frac{u Du}{u + \frac{\nu}{c}|Du|}$. In this way, one can enforce the physical restriction that the flux cannot exceed the energy density times the speed of light c, that is, the flux cannot violate causality. Another example contained in the general class of models (1.5) is given by the so-called relativistic heat equation [37] (see also [18])

$$u_t = \nu \operatorname{div} \left(\frac{u \nabla u}{\sqrt{u^2 + \frac{\nu^2}{c^2} |\nabla u|^2}} \right), \tag{1.6}$$

where $\nu > 0$ is a constant representing the kinematic viscosity and $c > 0$ is the maximum speed of propagation. In this case, $\Phi(u) = u$ and the Lagrangian is $f(z, \xi) = \frac{c^2}{\nu} |z| \sqrt{z^2 + \frac{\nu^2}{c^2} |\xi|^2}$.

Rosenau [37] derived (1.6) starting from the observation that the speed of sound is the highest admissible free velocity in a medium. This property is lost in the classical transport theory that predicts the nonphysical divergence of the flux with the gradient, as it happens also with the classical theory of heat conduction (based on Fourier's law) and with the linear diffusion theory (based on Fick's law). To overcome this problem, Rosenau [37] proposed to change the classical flux $\mathcal{F} = -\nu \nabla u$, $\nu > 0$ associated with the heat equation (or the Fokker-Plank equation) $u_t = \nu \Delta u$ by a flux that saturates as the gradient becomes unbounded. The flux can be derived from $\mathcal{F} = u\mathbf{v}$, by replacing the classical relation $\mathbf{v} = -\nu \frac{\nabla u}{u}$ by $\frac{\mathbf{v}}{\sqrt{1 - \frac{|\mathbf{v}|^2}{c^2}}} = -\nu \frac{\nabla u}{u}$ (when c is the acoustic speed, this forces $\mathbf{v}$ to stay in the subsonic regime, the sonic limit being approached only if $|\frac{\nabla u}{u}| \uparrow \infty$), obtaining

$$\mathcal{F} = u\mathbf{v} = \frac{-u\nabla u}{\sqrt{1 + \left(\frac{\nu|\nabla u|}{cu}\right)^2}}. \tag{1.7}$$

Using this new flux (1.7) in the conservation energy equation, we obtain (1.6). Besides Rosenau's derivation [37], (1.6) was also formally derived by Brenier by means of Monge-Kantorovich's mass transport theory in [18], where he named it as the *relativistic heat equation*.

The same argument can be applied if we use Darcy's law

$$\mathbf{v} = -\nu\nabla u^m \qquad m > 0. \tag{1.8}$$

In that case we derive the flux-limited porous medium equation (1.3), see [37]. As we shall see in Section 2, equation (1.3) can be also derived following the transport approach argument proposed by Brenier in [18].

Many other models of nonlinear degenerate parabolic equations with flux saturation as the gradient becomes unbounded have been proposed by Rosenau and his coworkers [26,36,37], Bertsch and Dal Passo [13,29], and Blanc [15,16].

Existence and uniqueness of entropy solutions for problem (1.5) have been proved in [4,5] and an extension to the general class of equations (1.1) has been given in [24]. Our main purpose here is to prove the convergence as $c \to \infty$ of entropy solutions of (1.1) to solutions of (1.2). The convergence of entropy solutions of (1.6) to solutions of the heat equation was proved in [21]. While the proof in [21] was based on a uniform estimate of the gradient, the proof given here for (1.2) is based on a different approach and applies also to (1.6). Indeed, we prove the convergence of the resolvents associated to the problem (1.1) to the resolvent equation for (1.2). And to do that we use weaker estimates and the representation of the energies associated to both problems in terms of linear functionals. As a technical tool in our approach we use some lower semi-continuity results for energy functionals whose density is a function $g(x, u, Du)$ convex in Du with a linear growth rate as $|Du| \to \infty$, which were proved in [28] and [30]. We refer to Section 3 for details.

Let us explain the plan of the paper. In Section 2 we review the derivation of the model (1.3) following the transport approach argument proposed by Brenier in [18]. In Section 3 we prove the convergence of solutions of (1.1) to solutions of (1.2) as $c \to \infty$. Since the Lagrangian function $f(z, \xi)$ associated to a flux-limited diffusion equation has a linear growth as $|\xi| \to \infty$, the notion of (entropy) solution and the corresponding existence and uniqueness results are formulated using functions of bounded variation. To avoid a long set of technical preliminaries and at the same time facilitate reading the paper, we include in Appendix A some basic material to describe the notion of entropy solutions for (1.1) and recall its basic existence and uniqueness results for any initial datum $u_0 \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+$.

2. A transport derivation of flux-limited porous media type diffusion equations

Let us follow the transport approach to generate the flux-limited porous medium type diffusion equation (1.3). Let $k : \mathbb{R}^N \to [0, \infty]$ be a convex cost function and let us define the associated Wasserstein distance between two probability distributions ρ_0 and ρ_1 by

$$W_k^h(\rho_0, \rho_1) := \inf\left\{ \int_{\mathbb{R}^N \times \mathbb{R}^N} k\left(\frac{x-y}{h}\right) d\gamma(x, y) : \gamma \in \Gamma(\rho_0, \rho_1) \right\},$$

where $h > 0$ and $\Gamma(\rho_0, \rho_1)$ is the set of probability measures in $\mathbb{R}^N \times \mathbb{R}^N$ whose marginals are ρ_0 and ρ_1.

Let $F : [0, \infty) \to [0, \infty)$ be a convex function and let $\mathcal{P}(\mathbb{R}^N)$ be the set of probability density functions $\rho : \mathbb{R}^N \to [0, \infty)$. Let $h > 0$. Starting from $\rho_0^h = \rho_0 \in \mathcal{P}(\mathbb{R}^N)$, we solve iteratively

$$\inf_{\rho \in \mathcal{P}(\mathbb{R}^N)} h W_k^h(\rho_{n-1}^h, \rho) + \int_{\mathbb{R}^N} F(\rho(x))\, dx.$$

This is a gradient descent with respect to the Wasserstein distance. Formally, if we define $\rho^h(t) := \rho_n^h$ for $t \in [nh, (n+1)h)$, then as $h \to 0+$ the solution converges to the solution of the diffusion equation

$$u_t = \operatorname{div}\left(u \nabla k^*\left(\nabla F'(u)\right)\right). \tag{2.1}$$

This has been the object of intensive research. For more information we refer to the monographs [3,39] or to the PhD thesis of M. Agueh [1]. For a transport based approach to the so-called relativistic heat equation given by (1.6) we refer to [33].

Let us consider $F(r) = \nu \frac{r^{m+1}}{m+1}$, $m > 0$. The case $m = 0$ is identified with $F(r) = \nu(r \log r - r)$. If we take k defined by

$$k(z) := \begin{cases} c^2\left(1 - \sqrt{1 - \frac{|z|^2}{c^2}}\right) & \text{if } |z| \le c \\ +\infty & \text{if } |z| > c, \end{cases} \tag{2.2}$$

then

$$k^*(\xi) = c^2\left(\sqrt{1 + \frac{|\xi|^2}{c^2}} - 1\right)$$

and $\nabla k^*(\xi) = \frac{\xi}{\sqrt{1 + \frac{|\xi|^2}{c^2}}}$, $\xi \in \mathbb{R}^N$. If $m > 0$, we have

$$\nabla k^*(\nabla F'(u)) = \nu \frac{\nabla u^m}{\sqrt{1 + \frac{\nu^2}{c^2}|\nabla u^m|^2}}.$$

If $m = 0$, we have

$$\nabla k^*(\nabla F'(u)) = \nu \frac{\nabla u}{\sqrt{u^2 + \frac{\nu^2}{c^2}|\nabla u|^2}}.$$

The corresponding diffusion equations are (1.3) and (1.6), respectively. Existence, uniqueness, and qualitative properties of model (1.6) have been studied in [5,6,22]. The study of (1.3) (and, more generally, of (1.1)) was the object of [24]. We call (1.3) the flux-limited porous medium equation.

3. Convergence of entropy solutions of generalized flux-limited porous medium equations to their classical counterpart

Let us consider the generalized flux-limited porous medium model

$$u_t = \operatorname{div}\left(\frac{\Lambda(u)\nabla\Phi(u)}{\sqrt{1 + c^{-2}|\nabla\Phi(u)|^2}}\right), \tag{3.1}$$

where $c > 0$ and we assume

$(H)_{\Phi,\Lambda}$ $\Phi : [0, \infty) \to [0, \infty)$ is a continuous strictly increasing function such that $\Phi(0) = 0$ and $\Phi, \Phi^{-1} \in W^{1,\infty}([a, b])$ for any $0 < a < b$. Let $\Lambda : [0, \infty) \to [0, \infty)$ be a continuous function such that $\Lambda(0) = 0$ and $\Lambda(z) > 0$ for all $z > 0$. We assume that $\Lambda(z) = \widetilde{\Lambda}(z^{\bar{m}})$, where $\widetilde{\Lambda}(z) \geq c_0 z$ for some $c_0 > 0$ and all $z \geq 0$, $\widetilde{\Lambda} \in W^{1,\infty}_{\text{loc}}([0, \infty))$, and $\bar{m} \geq 1$.

Let $f_c(z, \xi) = c^2\Lambda(z)\sqrt{1 + c^{-2}|\xi|^2} - c^2\Lambda(z)$ be the Lagrangian associated to (3.1). Since f_c satisfies the assumptions of Appendix A, by Theorem A.4 there is a unique entropy solution $u(t)$ of (3.1) for any initial condition $u_0 \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+$.

Theorem 3.1. *Assume that Λ, Φ satisfy $(H)_{\Phi,\Lambda}$, $\Phi \in C^1(0, \infty)$ satisfies* (3.3) *below, and Λ is increasing. Let u_c be the entropy solution of* (3.1) *with $u(0, x) = u_0(x) \in (L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))^+$. As $c \to \infty$, u_c converges in $C([0, T], L^1(\mathbb{R}^N))$ to the solution of u the generalized porous medium equation*

$$u_t = \operatorname{div}(\Lambda(u)\nabla\Phi(u)) \tag{3.2}$$

with $u(0, x) = u_0(x)$.

When $\Lambda(u) = u$ and $\Phi(u) = \nu u^m$, $\nu, m > 0$, and $c \to \infty$ we obtain the convergence of solutions of (1.3) to solutions of the classical porous medium equation (1.4). The corresponding result for (1.6) was proved in [21] using a Lipschitz estimate on solutions. Let us mention that partial convergence results have been given in [19] using a similar proof based on a Lipschitz estimate.

Lemma 3.2. *Assume that Φ, Λ satisfy the assumptions $(H)_{\Phi,\Lambda}$, Λ is increasing, and Φ satisfies: for any $\alpha, \kappa > 0$ there exist $\beta, \bar{\kappa} > 0$ such that*

$$\Phi\left(\kappa e^{-\frac{\alpha}{2}|x|^2}\right) \geq \bar{\kappa} e^{-\frac{\beta}{2}|x|^2}. \tag{3.3}$$

Let us consider the equation

$$u - \lambda \mathrm{div}\left(\frac{\Lambda(u)\nabla\Phi(u)}{\sqrt{1+\frac{1}{c^2}|\nabla\Phi(u)|^2}}\right) = v. \tag{3.4}$$

Let $u_{\gamma,\beta} = \Phi^{-1}\left(\gamma e^{-\beta|x|^2}\right)$, $\gamma > 0$. If $v \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+$ and $v(x) \geq \kappa e^{-\alpha|x|^2}$ for some $\alpha, \kappa > 0$, then for some $\gamma, \beta > 0$ $u_{\gamma,\beta}$ is a subsolution of (3.4) *for all $c > 0$.*

Proof. Let $w = \Phi(u)$, $B(w) = \Lambda(\Phi^{-1}(w))$. If w is smooth, it satisfies the PDE

$$\Phi^{-1}(w) - \lambda B(w)\mathrm{div}\left(\frac{\nabla w}{\sqrt{1+\frac{1}{c^2}|\nabla w|^2}}\right) - \lambda B'(w)\frac{|\nabla w|^2}{\sqrt{1+\frac{1}{c^2}|\nabla w|^2}} = v. \tag{3.5}$$

Let $\gamma, \beta > 0$, $\bar{w} = \gamma e^{-\beta|x|^2}$. Let $D = 1 + 4c^{-2}\beta^2\gamma^2|x|^2 e^{-\beta|x|^2}$. An elementary computation shows that $\bar{w}$ is a subsolution of (3.5) if and only if

$$\Phi^{-1}(\bar{w}) - \lambda B(\bar{w})\left(\frac{-2\beta\gamma N + 4\beta^2\gamma|x|^2}{D^{1/2}} e^{-\beta|x|^2} + \frac{8\beta^3\gamma^3}{c^2 D^{3/2}}|x|^2 e^{-2\beta|x|^2}\left(1 - \beta|x|^2\right)\right) \tag{3.6}$$

$$- \lambda B'(\bar{w})\frac{4\beta^2\gamma^2}{D^{1/2}}|x|^2 e^{-2\beta|x|^2} \leq v.$$

Since

$$\lambda B(\bar{w})\frac{8\beta^3\gamma^3}{c^2 D^{3/2}}|x|^2 e^{-2\beta|x|^2}\beta|x|^2 \leq \lambda B(\bar{w})\frac{2\beta^2\gamma}{D^{1/2}}|x|^2 e^{-\beta|x|^2},$$

then (3.6) is implied by

$$\begin{aligned}\Phi^{-1}(\bar{w}) + \lambda B(\bar{w})\frac{2\beta\gamma N}{D^{1/2}} e^{-\beta|x|^2} - \lambda B(\bar{w})\frac{2\beta^2\gamma}{D^{1/2}}|x|^2 e^{-\beta|x|^2} \\ -\lambda B(\bar{w})\tfrac{8\beta^3\gamma^3}{c^2 D^{3/2}}|x|^2 e^{-2\beta|x|^2} - \lambda B'(\bar{w})\tfrac{4\beta^2\gamma^2}{D^{1/2}}|x|^2 e^{-2\beta|x|^2} \leq \kappa e^{-\alpha|x|^2}.\end{aligned} \tag{3.7}$$

Since the last two terms in the left-hand side are negative, then it suffices to prove that

$$\Phi^{-1}(\bar{w}) + \lambda B(\bar{w})\frac{2\beta\gamma}{D^{1/2}}\left(N - \beta|x|^2\right) e^{-\beta|x|^2} \leq \kappa e^{-\alpha|x|^2}. \tag{3.8}$$

Since $\bar{w} \leq \gamma$, by taking $\beta \geq N$, it suffices to prove that

$$\Phi^{-1}(\bar{w}) + \lambda B(\gamma) \frac{2\beta\gamma N}{D^{1/2}} \chi_{|x|\leq 1} e^{-\beta|x|^2} \leq \kappa e^{-\alpha|x|^2}. \tag{3.9}$$

By our assumption (3.3) we may find β_0, γ_0 such that for any $\beta \geq \beta_0$ and $\gamma \leq \gamma_0$,

$$\Phi^{-1}(\bar{w}) \leq \frac{\kappa}{2} e^{-\alpha|x|^2}. \tag{3.10}$$

Thus, (3.9) holds if we choose β, γ so that

$$\lambda B(\gamma) \frac{2\beta\gamma N}{D^{1/2}} \chi_{|x|\leq 1} e^{-\beta|x|^2} \leq \frac{\kappa}{2} e^{-\alpha|x|^2}.$$

Since $D \geq 1$ it suffices to choose β, γ so that

$$\lambda B(\gamma) 2\beta\gamma N \chi_{|x|\leq 1} e^{-\beta|x|^2} \leq \frac{\kappa}{2} e^{-\alpha|x|^2}.$$

This clearly holds by choosing either γ small enough or β large enough.

Notice that $\bar{w} \in L^1(\mathbb{R}^N)$. Thus we have that $\bar{w}$ is a subsolution of

$$\Phi^{-1}(w) - \lambda \mathrm{div} \left(\frac{B(w)\nabla w}{\sqrt{1 + \frac{1}{c^2}|\nabla w|^2}} \right) = v. \tag{3.11}$$

Since $\bar{w}(x) > 0$ for al $x \in \mathbb{R}^N$, then $\bar{u} = \Phi^{-1}(\bar{w})$ is locally Lipschitz, and u is entropy sub-solution of (3.4) for all $c > 0$. □

The following result is due to G. Bouchitté and A. Chambolle [17] (see also [35, Theorem 3.2], or [25, Theorem 4.1]):

Theorem 3.3. *Let $f(x, z, \xi)$ be a continuous function of $(x, z) \in \mathbb{R}^N \times [0, \infty)$ and convex in $\xi \in \mathbb{R}^N$. Let $F : L^1(\mathbb{R}^N) \to [0, \infty]$ be given by*

$$F(u) := \int_{\mathbb{R}^N} f(x, u(x), \nabla u(x))\, dx.$$

For any function in $W^{1,1}(\mathbb{R}^N)$ we have

$$F(u) = \sup_{\phi \in \mathcal{K}} \int_{\Omega\times\mathbb{R}} \phi \cdot D\mathbf{1}_u,$$

where the convex set $\mathcal{K}$ is given by

$$\mathcal{K} = \{\phi = (\phi^x, \phi^z) \in C_c^1(\mathbb{R}^N \times [0, \infty); \mathbb{R}^N \times \mathbb{R}) : \phi^z(x, z) \geq f^*(x, z, \phi^x(x, z)) \forall (x, z) \in \mathbb{R}^N \times [0, \infty)\}.$$

Moreover, for any $u \in BV(\Omega)$ we have

$$\mathcal{R}_f(u) = \sup_{\phi \in \mathcal{K}} \int_{\Omega \times \mathbb{R}} \phi \cdot D\mathbf{1}_u \,,$$

where $\mathcal{R}_f(u)$ has been defined in (A.7) *in Appendix* A.3 *and coincides with the lower semi-continuous relaxation of F.*

Example 3.4. Let $h(x)$, $x \in \mathbb{R}^N$, be a positive and continuous function, and Ψ be a strictly increasing function which is Lipschitz in $[a, b]$ for all $0 < a < b$.
a) Let $f_\Psi(x, z, \xi) = \frac{1}{2\alpha}\Psi(\max(z, h(x)))|\xi|^2, \alpha > 0$.
Then $f^*_\Psi(x, z, p) = \alpha \frac{|p|^2}{2\Psi(\max(z,h(x)))}$.
b) Let $f_{\Psi,\varepsilon}(x, z, \xi) = \frac{1}{\varepsilon}\Psi(\max(z, h(x)))\sqrt{1 + \varepsilon|\xi|^2} - \frac{1}{\varepsilon}\Psi(\max(z, h(x))), \varepsilon > 0$.
Then $f^*_{\Psi,\varepsilon}(x, z, p) = -\frac{1}{\varepsilon}\sqrt{\Psi(\max(z, h(x)))^2 - \varepsilon|p|^2} + \frac{\Psi(\max(z,h(x)))}{\varepsilon}$ if $\varepsilon|p|^2 \leq \Psi(\max(z, h(x)))^2$, $+\infty$ otherwise.

Proof of Theorem 3.1. For notation and basic facts on entropy solutions, we refer to Appendix A.

Step 1. *Reduction to the convergence of resolvents*. Entropy solutions of (3.1) coincide with the semigroup solutions (see Theorem A.4 in Appendix A.5). Let $T_c(t)$ be the semigroup generated by the flux-limited generalized porous medium equation (3.1) in $(L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+$. Let us write the associated accretive operator by $\mathcal{B}_c$, reflecting also its dependence on c. We have that

$$T_c(t)u_0 = \lim_{n\to\infty} \left(I + \frac{t}{n}\mathcal{B}_c\right)^{-n} u_0$$

for any $u_0 \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+$, where for any $v \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+$, $\lambda > 0$, $(I + \lambda\mathcal{B}_c)^{-1} v$ denotes the entropy solution of

$$u - \lambda \mathrm{div} \left(\frac{\Lambda(u)\nabla\Phi(u)}{\sqrt{1 + \frac{1}{c^2}|\nabla\Phi(u)|^2}} \right) = v. \tag{3.12}$$

Similarly, it is well known [38] that if $S(t)$ denotes the semigroup generated by the generalized porous medium equation (3.2) in $L^1(\mathbb{R}^N)^+$, then $S(t)u_0 = \lim_{n\to\infty}\left(I + \frac{t}{n}\mathcal{A}\right)^{-n} u_0$ for any $u_0 \in L^1(\mathbb{R}^N)^+$, where for any $v \in L^1(\mathbb{R}^N)^+$, $\lambda > 0$, $(I + \lambda\mathcal{A})^{-1} v$ denotes the weak solution [10] of $u - \lambda\mathrm{div}\,(\Lambda(u)\nabla\Phi(u)) = v$.

Since the convergence of resolvents implies the convergence of semigroups generated by accretive operators [12], to prove the theorem it suffices to prove that

$$(I + \lambda\mathcal{B}_c)^{-1} v \to (I + \lambda\mathcal{A})^{-1} v \qquad \text{as } c \to \infty \tag{3.13}$$

for any $v \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+$. By the contractivity estimate it suffices to prove that this convergence holds for a dense set in $(L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+$.

Step 2. *Basic estimates and convergence of solutions of* (3.12). Let $v \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+ \cap BV(\mathbb{R}^N)$, $\lambda > 0$. Let u_c be the entropy solution of (3.12). Let us collect the basic estimates for the solution which are independent of c. By (A.13) and (A.14) (see Remark A.5) we have

$$\|u_c\|_p \leq \|v\|_p \qquad \text{for any } p \in [1, \infty], \tag{3.14}$$

$$\|u_c\|_{BV} \leq \|v\|_{BV}. \tag{3.15}$$

Let us write

$$\mathbf{a}(u_c, \nabla\Phi(u_c)) = \sqrt{\Lambda(u_c)}\widetilde{\mathbf{a}}(u_c, \nabla\Phi(u_c)), \tag{3.16}$$

where

$$\widetilde{\mathbf{a}}(u_c, \nabla\Phi(u_c)) = \frac{\sqrt{\Lambda(u_c)}\nabla\Phi(u_c)}{\sqrt{1 + \frac{1}{c^2}|\nabla\Phi(u_c)|^2}},$$

and prove that $\widetilde{\mathbf{a}}(u_c, \nabla\Phi(u_c))$ is bounded in $L^2(\mathbb{R}^N)^N$ with a bound independent of $c > 0$. Let us first proceed formally and then sketch the correct proof. Multiplying (3.12) by $\Phi(u_c)$ and integrating by parts, we have

$$\int_{\mathbb{R}^N} u_c\Phi(u_c) + \lambda \int_{\mathbb{R}^N} \frac{\Lambda(u_c)\left|\nabla\Phi(u_c)\right|^2}{\sqrt{1 + c^{-2}\left|\nabla\Phi(u_c)\right|^2}} = \int_{\mathbb{R}^N} v\Phi(u_c). \tag{3.17}$$

Then we have

$$\begin{aligned}\lambda \int_{\mathbb{R}^N} \left|\widetilde{\mathbf{a}}\,(u_c, \nabla\Phi(u_c))\,\right|^2 dx &\leq \lambda \int_{\mathbb{R}^N} \frac{\Lambda(u_c)\left|\nabla\Phi(u_c)\right|^2}{\sqrt{1 + c^{-2}\left|\nabla\Phi(u_c)\right|^2}} \\ &\leq \Phi\left(\left\|u_c\right\|_\infty\right) \int_{\mathbb{R}^N} v\,dx \\ &\leq \Phi\left(\left\|v\right\|_\infty\right) \int_{\mathbb{R}^N} v\,dx,\end{aligned} \tag{3.18}$$

where the last inequality follows from (3.14). Since we are working with BV functions these computations require an explanation. Thus, to prove the boundedness of $\widetilde{\mathbf{a}}(u_c, \nabla\Phi(u_c))$ in $L^2(\mathbb{R}^N)^N$ we use the approximation (A.15) and we get solutions u_{cn}. Now it is easy to prove that the estimate (3.18) holds for u_{cn}. By passing to the limit we get that (3.18) holds for $\widetilde{\mathbf{a}}(u_c, \nabla\Phi(u_c))$.

Since $\{u_c\}_c$ is bounded in $L^\infty(\mathbb{R}^N)$, we also have that $\mathbf{a}(u_c, \nabla\Phi(u_c))$ is bounded in $L^2(\mathbb{R}^N)^N$.

Thus, by extracting a subsequence if necessary, we may pass to the limit as $c \to \infty$ and get a function $u \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+ \cap BV(\mathbb{R}^N)$ and vector fields $\mathbf{z}, \mathbf{z}_0 \in L^2(\mathbb{R}^N)^N$ such that

$$u_c \to u \qquad \text{as } c \to \infty \text{ a.e. and in } L^p(\mathbb{R}^N) \text{ for all } p \in [1, \infty),$$

weakly* in $L^\infty(\mathbb{R}^N)$,

$$\widetilde{\mathbf{a}}(u_c, \nabla\Phi(u_c)) \rightharpoonup \mathbf{z}_0 \quad \text{and} \quad \mathbf{a}\,(u_c, \nabla\Phi(u_c)) \rightharpoonup \mathbf{z} \tag{3.19}$$

weakly in $L^2(\mathbb{R}^N)$,

$$\mathbf{z} = \sqrt{\Lambda(u)}\mathbf{z}_0, \tag{3.20}$$

and $u - \lambda \operatorname{div} \mathbf{z} = v$ in $\mathcal{D}'(\mathbb{R}^N)$. Only the equality (3.20) requires some explanation. Indeed, (3.20) holds since $\sqrt{\Lambda(u_c)}$ is uniformly bounded in $L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\sqrt{\Lambda(u)} \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $u_c \to u$ a.e. and this implies that $\sqrt{\Lambda(u_c)} \to \sqrt{\Lambda(u)}$ in $L^2(\mathbb{R}^N)$ as $c \to \infty$.

Step 3. *Identification of* $\mathbf{z}(x)$. Let us prove that

$$\mathbf{z}(x) = \mathbf{A}\,(u(x), \nabla\Phi(u(x))) \qquad \text{a.e. in } \mathbb{R}^N, \tag{3.21}$$

where $\mathbf{A}(u, \nabla\Phi(u)) = \Lambda(u)\nabla\Phi(u)$. For that we proceed as in [4,24] and we use Minty-Browder's technique. Although the proof is similar, there are some differences, and we include it for the sake of completeness.

Let $0 < a < b$, let $0 \le \phi \in C_c^1(\mathbb{R}^N)$ and $g \in C^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$. For simplicity, let us write $w_c = \Phi(u_c)$, $w = \Phi(u)$. By the monotonicity of the flux $\mathbf{a}(z, \xi)$ in ξ, we have

$$\int_{\mathbb{R}^N} \phi[\mathbf{a}(u_c, \nabla w_c) - \mathbf{a}(u_c, \nabla g)) \cdot \nabla(w_c - g)]T'_{a,b}(w_c)\, dx \ge 0.$$

We use the notation $T'_{a,b}$ to refer to $\chi_{(a,b)}$ and $\overline{T}'_{a,b}$ to refer to $\chi_{[a,b]}$. Now, since

$$\begin{aligned}
&\int_{\mathbb{R}^N} \phi\mathbf{a}(u_c, \nabla w_c) \cdot \nabla(w_c - g)T'_{a,b}(w_c)\, dx \\
&= \int_{\mathbb{R}^N} \phi\mathbf{a}(u_c, \nabla w_c) \cdot \nabla(T_{a,b}(w_c) - g) dx + \int_{\mathbb{R}^N} \phi\mathbf{a}(u_c, \nabla w_c) \cdot \nabla g(1 - T'_{a,b}(w_c)) dx \\
&= \int_{\mathbb{R}^N} \phi\mathbf{a}(u_c, \nabla w_c) \cdot D(T_{a,b}(w_c) - g) + \int_{\mathbb{R}^N} \phi\mathbf{a}(u_c, \nabla w_c) \cdot \nabla g\,(1 - T'_{a,b}(w_c))\, dx \\
&\quad - \int_{\mathbb{R}^N} \phi\,(\mathbf{a}(u_c, \nabla w_c) \cdot DT_{a,b}(w_c))^s
\end{aligned}$$

(and, the last term being ≤ 0)

$$\begin{aligned}\leq &- \int_{\mathbb{R}^N} \operatorname{div}(\mathbf{a}(u_c, \nabla w_c))\phi(T_{a,b}(w_c) - g)\, dx \\ &- \int_{\mathbb{R}^N} (T_{a,b}(w_c) - g)\mathbf{a}(u_c, \nabla w_c) \cdot \nabla\phi\, dx, \\ &+ \int_{\mathbb{R}^N} \phi\mathbf{a}(u_c, \nabla w_c) \cdot \nabla g\, (1 - T'_{a,b}(w_c))\, dx.\end{aligned}$$

If we denote the last term by $X_c := \int_{\mathbb{R}^N} \phi\mathbf{a}(u_c, \nabla\Phi(u_c)) \cdot \nabla g\, (1 - T'_{a,b}(\Phi(u_c)))\, dx$, we have

$$|X_c| \leq \int_{\mathbb{R}^N} \phi\big|\mathbf{a}(u_c, \nabla\Phi(u_c))\big|\, \big|\nabla g\big| \left(1 - T'_{a,b}(\Phi(u_c))\right) dx.$$

Notice that $\mathbf{a}(u_c, \nabla\Phi(u_c))$ is bounded in $L^2(\mathbb{R}^N)$. By passing to a subsequence if necessary, we may assume that $|\mathbf{a}(u_c, \nabla\Phi(u_c))| \rightharpoonup A$ weakly in $L^2(\mathbb{R}^N)$ as $c \to \infty$ for some function $A \in L^2(\mathbb{R}^N)^+$. Thus passing to the limit as $c \to \infty$ we have

$$\begin{aligned}&\lim_{c\to\infty} \int_{\mathbb{R}^N} \phi\mathbf{a}(u_c, \nabla w_c) \cdot \nabla(w_c - g)\, T'_{a,b}(w_c)\, dx \\ &\leq - \int_{\mathbb{R}^N} \operatorname{div}(\mathbf{z})\phi(T_{a,b}(w) - g)\, dx \\ &\quad - \int_{\mathbb{R}^N} (T_{a,b}(w) - g)\mathbf{z} \cdot \nabla\phi\, dx + \int_{\mathbb{R}^N} \phi\, A(x)|\nabla g|(1 - T'_{a,b}(\Phi(u)))\, dx \\ &= \int_{\mathbb{R}^N} \phi(\mathbf{z} \cdot D(T_{a,b}(w) - g)) + \int_{\mathbb{R}^N} \phi\, A(x)|\nabla g|(1 - T'_{a,b}(\Phi(u)))\, dx.\end{aligned}$$

On the other hand, let us denote by $\mathbf{a}_i$ the coordinates of $\mathbf{a}$, $\mathbf{a}_i^\Phi(\bar{z}, \xi) = \mathbf{a}_i(\Phi^{-1}(\bar{z}), \xi)$, and

$$J_{\mathbf{a}_i^\Phi}(x, r) := \int_0^r \mathbf{a}_i^\Phi(s, \nabla g(x))\, ds, \quad \text{and} \quad J_{\frac{\partial \mathbf{a}_i^\Phi}{\partial x_j}}(x, r) := \int_0^r \frac{\partial}{\partial x_j}\mathbf{a}_i^\Phi(s, \nabla g(x))\, ds,$$

$i, j \in \{1, \dots, N\}$, and observe that

$$\frac{\partial}{\partial x_j} J_{\mathbf{a}_i^\Phi}(x, T_{a,b}(w_c(x))) = \mathbf{a}_i^\Phi(w_c(x), \nabla g(x)) \frac{\partial T_{a,b}(w_c)}{\partial x_j}(x) + J_{\frac{\partial \mathbf{a}_i^\Phi}{\partial x_j}}(x, T_{a,b}(w_c(x))).$$

Now, since the right-hand side of the above equality is bounded in $L^1(\mathbb{R}^N)$ and

$$\frac{\partial}{\partial x_j} J_{\mathbf{a}_i^\Phi}(x, T_{a,b}(w_c)) \rightharpoonup \frac{\partial}{\partial x_j} J_{\mathbf{a}_i^\Phi}(x, T_{a,b}(w))$$

weakly as measures, and $J_{\frac{\partial \mathbf{a}_i^\Phi}{\partial x_j}}\left(x, T_{a,b}(w_c(x))\right) \to J_{\frac{\partial \mathbf{a}_i^\Phi}{\partial x_j}}(x, T_{a,b}(w(x)))$ a.e., we have

$$
\begin{aligned}
&\lim_{c\to\infty} \int_{\mathbb{R}^N} \phi\, \mathbf{a}(u_c, \nabla g) \cdot \nabla(w_c - g)\, T'_{a,b}(w_c)\, dx \\
&= \lim_{c\to\infty} \int_{\mathbb{R}^N} \phi \sum_{i=1}^N \left[\frac{\partial}{\partial x_i} J_{\mathbf{a}_i^\Phi}(x, T_{a,b}(w_c(x))) - J_{\frac{\partial \mathbf{a}_i^\Phi}{\partial x_i}}(x, T_{a,b}(w_c(x))) \right] \\
&\quad - \lim_{c\to\infty} \int_{\mathbb{R}^N} \phi\, \mathbf{a}(u_c, \nabla g) \cdot \nabla g\, T'_{a,b}(w_c)\, dx \\
&\geq \int_{\mathbb{R}^N} \phi \sum_{i=1}^N \left[\frac{\partial}{\partial x_i} J_{\mathbf{a}_i^\Phi}(x, T_{a,b}(w)) - J_{\frac{\partial \mathbf{a}_i^\Phi}{\partial x_i}}(x, T_{a,b}(w(x))) \right] \\
&\quad - \int_{\mathbb{R}^N} \phi\, \mathbf{a}(u, \nabla g) \cdot \nabla g\, \overline{T}'_{a,b}(w)\, dx.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \phi(\mathbf{z}, D(T_{a,b}(\Phi(u)) - g)) + \int_{\mathbb{R}^N} \phi\, A(x)|\nabla g|(1 - T'_{a,b}(\Phi(u)))\, dx \\
&\quad + \int_{\mathbb{R}^N} \phi \mathbf{A}(u, \nabla g) \cdot \nabla g\, \overline{T}'_{a,b}(\Phi(u)) \\
&\quad - \int_{\mathbb{R}^N} \phi \left(\sum_{i=1}^N \left[\frac{\partial}{\partial x_i} J_{\mathbf{A}_i^\Phi}(x, T_{a,b}(\Phi(u(x)))) - J_{\frac{\partial \mathbf{A}_i^\Phi}{\partial x_i}}(x, T_{a,b}(\Phi(u(x)))) \right] \right) \\
&\geq 0,
\end{aligned}
\tag{3.22}
$$

for some $A \in L^2_{\text{loc}}(\mathbb{R}^N)^+$, for all $0 \leq \phi \in C^1_c(\mathbb{R}^N)$, and all $T_{a,b}$, $0 < a < b$. Thus the measure

$$
\begin{aligned}
&(\mathbf{z} \cdot D(T_{a,b}(w) - g)) - \sum_{i=1}^N \left[\frac{\partial}{\partial x_i} J_{\mathbf{a}_i^\Phi}(x, T_{a,b}(w(x))) - J_{\frac{\partial \mathbf{a}_i^\Phi}{\partial x_i}}(x, T_{a,b}(w(x))) \right] \\
&\quad + \mathbf{a}(u, \nabla g) \cdot \nabla g\, \overline{T}'_{a,b}(w)\, \mathcal{L}^N + A(x)|\nabla g|(1 - T'_{a,b}(\Phi(u)))\mathcal{L}^N \geq 0.
\end{aligned}
\tag{3.23}
$$

Using chain's rule for BV functions ([2, Theorem 3.96]) applied to $J_{\mathbf{a}_i}(u_1, u_2)$ with $u_1(x) = x$, $u_2(x) = T_{a,b}(w(x))$, $x \in \mathbb{R}^N$, we deduce that the absolutely continuous part of

$$
\sum_{i=1}^N \left[\frac{\partial}{\partial x_i} J_{\mathbf{a}_i^\Phi}(x, T_{a,b}(w(x))) - J_{\frac{\partial \mathbf{a}_i^\Phi}{\partial x_i}}(x, T_{a,b}(w(x))) \right]
$$

is $\mathbf{a}(u, \nabla g) \cdot \nabla T_{a,b}(w)\, \mathcal{L}^N$. Taking absolutely continuous parts in (3.23) we obtain

$$\mathbf{z} \cdot \nabla(T_{a,b}(w)) - g) - \mathbf{A}(u, \nabla g) \cdot \nabla T_{a,b}(w) + \mathbf{A}(u, \nabla g) \cdot \nabla g \overline{T}'_{a,b}(w)$$

$$+\phi\, A(x)|\nabla g|(1 - T'_{a,b}(w)) \geq 0.$$

In particular, for $x \in \{a < w < b\}$ we have

$$(\mathbf{z} - \mathbf{A}(u, \nabla g)) \cdot \nabla(w - g) \geq 0 \quad \text{a.e..}$$

Since we may take a countable set of functions $g \in C^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ dense in $C^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ we have that the above inequality holds for all $x \in \Omega \cap \{a < \Phi(u) < b\}$, where $\Omega \subset \mathbb{R}^N$ is such that $\mathcal{L}^N(\mathbb{R}^N \setminus \Omega) = 0$, and all $g \in C^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$. Now, fixed $x \in \Omega \cap \{a < \Phi(u) < b\}$ and given $\xi \in \mathbb{R}^N$, there is $g \in C^1(\mathbb{R}^N)$ such that $\nabla g(x) = \xi$. Then

$$(\mathbf{z}(x) - \mathbf{A}(u(x), \xi)) \cdot (\nabla\Phi(u(x)) - \xi) \geq 0 \quad \forall\, \xi \in \mathbb{R}^N.$$

By an application of Minty-Browder's method in $\mathbb{R}^N$, these inequalities imply that

$$\mathbf{z}(x) = \mathbf{A}(u(x), \nabla\Phi(u(x))) \quad \text{a.e. on } \{a < \Phi(u) < b\}.$$

Since this holds for any $0 < a < b$, we obtain (3.21) a.e. on the points x of $\mathbb{R}^N$ such that $\{\Phi(u(x)) \neq 0\} = \{u(x) \neq 0\}$. Now, by our assumptions on $\mathbf{A}$ and (3.20) we deduce that $\mathbf{z}(x) = \mathbf{A}(u(x), \nabla\Phi(u(x))) = 0$ a.e. on $\{u = 0\}$. We have proved (3.21).

Note that we only know that $u \in BV(\mathbb{R}^N)$ and we do not know yet if $Du = \nabla u$. To show this is our next purpose.

Step 4. We need the following result. Since $f_c(z, 0) = 0$, we have

$$f_c^{\Phi}(\Phi(u_c), DT_{a,b}(\Phi(u_c))) \leq (\mathbf{a}(u_c, \nabla\Phi(u_c)) \cdot DT_{a,b}(\Phi(u_c)). \tag{3.24}$$

For a proof see [4, Lemma 4.4], or [22, Lemma 3.11].

Step 5. Let us prove that

$$\frac{\lambda}{2} \int_{\mathbb{R}^N} \Lambda(u)|D\Phi(u)|^2\, dx \leq \int_{\mathbb{R}^N} (v - u)\Phi(u)\, dx. \tag{3.25}$$

This implies that $Du = \nabla u$.

Let $0 < a < b$. Let us observe that since $f_c(z, 0) = 0$, using (A.10) and (3.24), we have

$$\begin{aligned}\int_{\mathbb{R}^N} f_c^{\Phi}(T_{a,b}(\Phi(u_c)), DT_{a,b}(\Phi(u_c))) &= \int_{\mathbb{R}^N} f_c^{\Phi}(\Phi(u_c), DT_{a,b}(\Phi(u_c))) \\ &= \int_{\mathbb{R}^N} f_c(u_c, DT_{a,b}(\Phi(u_c))) \\ &\leq \int_{\mathbb{R}^N} (\mathbf{a}(u_c, D\Phi(u_c)) \cdot DT_{a,b}(\Phi(u_c))\end{aligned}$$

for any $0 < a < b$. Integrating by parts on the right-hand side we have

$$\int_{\mathbb{R}^N} f_c^{\Phi}(T_{a,b}(\Phi(u_c)), DT_{a,b}(\Phi(u_c))) \leq \int_{\mathbb{R}^N} (v - u_c) T_{a,b}(\Phi(u_c)). \qquad (3.26)$$

Let $\alpha > 1$, $\Psi = \Lambda^{\Phi} = \Lambda \circ \Phi^{-1}$, and $h(x) = \gamma e^{-\beta |x|^2}$ with $\gamma, \beta > 0$ such that $\Phi^{-1}(h(x))$ is an exponential subsolution of all u_c as given in Lemma 3.2. Let us consider the functions $f_{\Psi}(x, z, p)$ with the previous value of α and $f_{\Psi,\varepsilon}(x, z, p)$ with $\varepsilon = c^{-1}$, as given in the Examples 3.4. To shorten the notation, let us denote $\mathcal{X} = C_c^1(\mathbb{R}^N \times [0, \infty); \mathbb{R}^N \times \mathbb{R})$. Let

$$\mathcal{K}(f_{\Psi}) = \Big\{\phi = (\phi^x, \phi^z) \in \mathcal{X} : \phi^z(x, z) \geq f_{\Psi}^*(x, z, \phi^x(x, z)) \; \forall (x, z) \in \mathbb{R}^N \times [0, \infty)\Big\},$$
$$\mathcal{K}(f_{\Psi,\varepsilon}) = \Big\{\phi = (\phi^x, \phi^z) \in \mathcal{X} : \phi^z(x, z) \geq f_{\Psi,\varepsilon}^*(x, z, \phi^x(x, z)) \; \forall (x, z) \in \mathbb{R}^N \times [0, \infty)\Big\},$$

where

$$f_{\Psi}^*(x, z, \phi^x(x, z)) = \alpha \frac{|\phi^x(x, z)|^2}{2\Psi(\max(z, h(x)))}, \qquad (3.27)$$
$$f_{\Psi,\varepsilon}^*(x, z, \phi^x(x, z)) = -\frac{1}{\varepsilon}\sqrt{\Psi(\max(z, h(x)))^2 - \varepsilon |\phi^x|^2} + \frac{\Psi(\max(z, h(x)))}{\varepsilon}. \qquad (3.28)$$

Let $\phi \in \mathcal{K}(f_{\Psi})$, $w = T_{a,b}(\Phi(u))$, $w_c = T_{a,b}(\Phi(u_c))$. Then

$$\int_{\mathbb{R}^N} \phi \cdot D\mathbf{1}_w = -\int_{\mathbb{R}^N} \operatorname{div}\phi \, \mathbf{1}_w \, dx = -\lim_c \int_{\mathbb{R}^N} \operatorname{div}\phi \, \mathbf{1}_{w_c} \, dx.$$

Let us prove that for ε small enough we have that $\phi \in \mathcal{K}(f_{\Psi,\varepsilon})$. For that we have to check that

$$-\frac{1}{\varepsilon}\sqrt{\Psi(\max(z, h(x)))^2 - \varepsilon \left|\phi^x\right|^2} + \frac{\Psi(\max(z, h(x)))}{\varepsilon} \leq \phi^z. \qquad (3.29)$$

Indeed, since ϕ^x has compact support and $\max(z, h(x))$ is bounded away from 0 in the support of ϕ^x, for $\varepsilon > 0$ small enough we have

$$\begin{aligned}
&\frac{1}{\varepsilon}\sqrt{\Psi(\max(z, h(x)))^2 - \varepsilon \left|\phi^x\right|^2} - \frac{\Psi(\max(z, h(x)))}{\varepsilon} \\
&= \frac{\Psi(\max(z, h(x)))}{\varepsilon}\sqrt{1 - \frac{\varepsilon}{\Psi(\max(z, h(x)))^2}\left|\phi^x\right|^2} - \frac{\Psi(\max(z, h(x)))}{\varepsilon} \\
&\geq \frac{\Psi(\max(z, h(x)))}{\varepsilon}\left(1 - \frac{\alpha\varepsilon}{\Psi(\max(z, h(x)))^2}\left|\phi^x\right|^2\right) - \frac{\Psi(\max(z, h(x)))}{\varepsilon} \\
&\geq -\alpha \frac{|\phi^x|^2}{2\Psi(\max(z, h(x)))} \geq -\phi^z,
\end{aligned}$$

the last inequality being true since $\phi \in \mathcal{K}(f_\Psi)$. Thus, for $\varepsilon > 0$ small enough, $\phi \in \mathcal{K}(f_{\Psi,\varepsilon})$. Then we may write

$$\begin{aligned}\int_{\mathbb{R}^N} \phi \cdot D\mathbf{1}_w &\leq \lim_c \sup_{\phi\in\mathcal{K}(f_{\Psi,\varepsilon})} \int_{\mathbb{R}^N} \phi \cdot D\mathbf{1}_{w_c}\, dx \\ &= \liminf_c \int_{\mathbb{R}^N} f_{\Psi,\varepsilon}(x, w_c, Dw_c) \\ &= \liminf_c \int_{\mathbb{R}^N} f_c^\Phi(T_{a,b}(\Phi(u_c)), DT_{a,b}(\Phi(u_c))) \\ &\leq \frac{1}{\lambda} \liminf_c \int_{\mathbb{R}^N} (v - u_c)T_{a,b}(\Phi(u_c)) = \frac{1}{\lambda}\int_{\mathbb{R}^N} (v-u)T_{a,b}(\Phi(u)),\end{aligned}$$

where (3.26) was used in the last of the inequalities. Taking sup values on $\phi \in \mathcal{K}(f_\Psi)$ we obtain

$$\begin{aligned}\frac{1}{2\alpha}\int_{\mathbb{R}^N} \Lambda^\Phi(T_{a,b}(\Phi(u)))|DT_{a,b}(\Phi(u))|^2 &= \sup_{\phi\in\mathcal{K}_2}\int_{\mathbb{R}^N} \phi \cdot D\mathbf{1}_u \\ &\leq \frac{1}{\lambda}\int_{\mathbb{R}^N}(v-u)T_{a,b}(\Phi(u)).\end{aligned}$$

Letting $\alpha \to 1$ and $a \to 0+, b > \|\Phi(u)\|_\infty$, we obtain (3.25).

Let $J(r)$ be the primitive of $\sqrt{\Lambda(r)}\Phi'(r)$. The finiteness of the integral

$$\int_{\mathbb{R}^N} f_\Psi(x, u, Du)\, dx = \int_{\mathbb{R}^N} \Lambda(u)|D\Phi(u)|^2\, dx < \infty \tag{3.30}$$

implies that $J(u) \in W^{1,2}(\mathbb{R}^N)$, hence also $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N)$. Indeed, since $u \in BV(\mathbb{R}^N)$, by the definition of f_Ψ we deduce that the singular part of Du is zero, *i.e.* $(Du)^s = 0$. Thus $u \in W^{1,1}(\mathbb{R}^N)$. Hence u is approximately differentiable a.e. and the approximate differential coincides with $\nabla u = Du$. Since $u \in L^\infty(\mathbb{R}^N)$, $u(x) \geq \Phi^{-1}(h(x))$, and J is differentiable in $(0,\infty)$, then $J(u) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$, $J(u(x))$ is approximately differentiable at all points $x \in \mathbb{R}^N$ where u is ([2], Proposition 3.71), and

$$\nabla J(u) = \sqrt{\Lambda(u)}\nabla\Phi(u) = \sqrt{\Lambda(u)}\Phi'(u)\nabla u.$$

Then (3.30) implies that $J(u) \in W^{1,2}(\mathbb{R}^N)$.

Similarly, we prove that $\tilde{J}(u) \in W^{1,2}(\mathbb{R}^N)$, where $\tilde{J}$ is the primitive of $\Lambda(r)\Phi'(r)$.

Step 6. *Conclusion*. Since $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\tilde{J}(u) \in W^{1,2}(\mathbb{R}^N)$, and

$$u - \lambda\Delta\tilde{J}(u) = v \qquad \text{in } \mathcal{D}'(\mathbb{R}^N), \tag{3.31}$$

then u is the weak solution of (3.31) [10]. This concludes the proof of (3.13), hence of the theorem. □

Remark 3.5. The assumption that $\Phi \in C^1(0,\infty)$ has only been used in the last paragraph of the proof to ensure that $J(u) \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)$ and $J(u(x))$ is approximately differentiable at all points $x \in \mathbb{R}^N$ where u is.

Remark 3.6. In a similar way, for $r \geq 1$, if u_c is the entropy solution of

$$u_t = \operatorname{div}\left(\frac{u^r \nabla u}{\sqrt{u^2 + \frac{1}{c^2}|\nabla u|^2}}\right) \tag{3.32}$$

with $u(0,x) = u_0(x) \in (L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))^+$, then u_c converges in $C([0,T], L^1(\mathbb{R}^N))$ as $c \to \infty$ to the solution of u the porous medium equation

$$u_t = \operatorname{div}\left(u^{r-1}\nabla u\right) \tag{3.33}$$

with $u(0,x) = u_0(x)$. In this case, we need the following lemma, whose proof is elementary by a direct computation as in Lemma 3.2.

Lemma 3.7. *Let $\lambda > 0$. If $v \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+$ and $v(x) \geq \kappa e^{-\alpha|x|^2}$ for some $\alpha, \kappa > 0$, then for $\gamma > 0$ small enough and $\beta > 0$ large enough, $u_{\gamma,\beta} = \gamma e^{-\beta|x|^2}$ is a subsolution of*

$$u - \lambda \operatorname{div}\left(\frac{u^r \nabla u}{\sqrt{u^2 + \frac{1}{c^2}|\nabla u|^2}}\right) = v$$

for all $c > 0$.

As in the proof of Step 2 of Theorem 3.1, we first prove that

$$\int_{\mathbb{R}^N} \frac{u_c^r |\nabla u_c|^2}{\sqrt{u_c^2 + \frac{1}{c^2}|\nabla u_c|^2}} dx \leq C,$$

where C does not depend on c. Then

$$\begin{aligned}\int_{\mathbb{R}^N} \left|\frac{u_c^r \nabla u_c}{\sqrt{u_c^2 + \frac{1}{c^2}|\nabla u_c|^2}}\right|^2 dx &\leq \|u_c^{r-1}\|_\infty \int_{\mathbb{R}^N} \frac{u_c^r |\nabla u_c|^2}{\sqrt{u_c^2 + \frac{1}{c^2}|\nabla u_c|^2}} dx \\ &\leq \|v^{r-1}\|_\infty C.\end{aligned} \tag{3.34}$$

This estimate is first obtained for the corresponding approximations (A.15) and then for $\frac{u_c^r \nabla u_c}{\sqrt{u_c^2 + \frac{1}{c^2}|\nabla u_c|^2}}$ by passing to the limit as $n \to \infty$. With this estimate we may repeat the proof of Step 3 in Theorem 3.1 and prove that

$$\mathbf{z}(x) = \mathbf{A}(u(x), \nabla\Phi(u(x))) \qquad \text{a.e. in } \mathbb{R}^N, \tag{3.35}$$

where $\mathbf{A}(u, \nabla\Phi(u)) = u^r \nabla u$. This time we may use that $u_c \in BV(\mathbb{R}^N)$ and we do not need to use truncatures of u_c. We obtain that for almost any $x \in \mathbb{R}^N$ we have

$$(\mathbf{z}(x) - \mathbf{A}(u(x), \xi)) \cdot (\nabla\Phi(u(x)) - \xi) \geq 0 \quad \forall\, \xi \in \mathbb{R}^N.$$

Again, by an application of Minty-Browder's method in $\mathbb{R}^N$, these inequalities imply (3.35).

Let $h(x)$ be an exponential subsolution of (3.32) independent of $c>0$ (Lemma 3.7). Let $\alpha>1$. This time we define $f_{r,\varepsilon}(x,z,\xi)=\frac{1}{\varepsilon}\max(z,h(x))^r\sqrt{\max(z,h(x))^2+\varepsilon|\xi|^2}-\frac{1}{\varepsilon}\max(z,h(x))^{r+1}, \varepsilon>0$. Then $f^*_{r,\varepsilon}(x,z,p)=-\frac{\max(z,h(x))}{\varepsilon}\sqrt{\max(z,h(x))^{2r}-\varepsilon|p|^2}+\frac{\max(z,h(x))^{r+1}}{\varepsilon}$ if $\varepsilon|p|^2 \leq \max(z, h(x))^{2r}$, $+\infty$ otherwise.

We also define $f_r(x, z, \xi) = \frac{1}{2\alpha}\max(z, h(x))^r|\xi|^2, \alpha > 0$. Then $f^*_r(x, z, p)=\alpha\frac{|p|^2}{2\max(z,h(x))^r}$.

Consider the functionals associated to $f_r(x, z, p)$ with the previous value of α and $f_{r,\varepsilon}(x, z, p)$, with $\varepsilon = c^{-1}$. Let

$$\mathcal{K}(f_r)=\{\phi=(\phi^x, \phi^z)\in\mathcal{X} : \phi^z(x, z)\geq f^*_r(x, z, \phi^x(x, z))\ \forall(x, z)\in\mathbb{R}^N \times [0, \infty)\},$$

$$\mathcal{K}(f_{r,\varepsilon})=\{\phi=(\phi^x,\phi^z)\in\mathcal{X} : \phi^z(x, z)\geq f^*_{r,\varepsilon}(x,z,\phi^x(x, z))\ \forall(x, z)\in\mathbb{R}^N \times [0, \infty)\}.$$

We proceed as in Step 5 observing that if $\phi \in \mathcal{K}(f_r)$, then for $\varepsilon > 0$ small enough we also have $\phi \in \mathcal{K}(f_{r,\varepsilon})$.

Remark 3.8. Notice that in the last remark we used that $u_c \in BV(\mathbb{R}^N)$ and we did not need to prove an identity like (3.20) implying that that $\mathbf{z}(x) = 0$ a.e. on $\{u = 0\}$. Thanks to this, we have been able to prove that entropy solutions of (3.32) converge to solutions of (3.33) for any $r \geq 1$. The same approach can be applied when $\Phi \in W^{1,\infty}_{\text{loc}}([0, \infty))$.

Remark 3.9. The case $r = 1$ which shows the convergence of the relativistic heat equation to the heat equation as $c \to \infty$ was treated in [21]. The methods used were based on some a priori Lipschitz estimates.

A. Appendix: A primer on entropy solutions

We collect in this appendix some definitions that are needed to work with entropy solutions of flux-limited diffusion equations.

Note that the equation (3.1) can be written as

$$u_t = \operatorname{div} \mathbf{a}(u, D\Phi(u)), \qquad \text{in} \quad Q_T = (0, T) \times \mathbb{R}^N \tag{A.1}$$

where $\mathbf{a}(z, \xi) = \nabla_\xi f_c(z, \xi)$ and

$$f_c(z, \xi) = c^2\Lambda(z)\sqrt{1 + c^{-2}|\xi|^2} - c^2\Lambda(z). \tag{A.2}$$

As usual, we define

$$h_c(z,\xi) = \mathbf{a}(z,\xi)\cdot\xi = \frac{\Lambda(z)|\xi|^2}{\sqrt{1+c^{-2}|\xi|^2}}. \tag{A.3}$$

Note that f_c is convex in ξ and both f_c, h_c have linear growth as $|\xi| \to \infty$. For the sake of simplicity, in this Appendix we write f, h instead of f_c, h_c. We assume that $\Phi : [0,\infty) \to [0,\infty)$ is a continuous strictly increasing function such that $\Phi(0) = 0$ and $\Phi, \Phi^{-1} \in W^{1,\infty}([a,b])$ for any $0 < a < b$.

A.1. Functions of bounded variation and some generalizations

Denote by $\mathcal{L}^N$ and $\mathcal{H}^{N-1}$ the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$, respectively. Given an open set Ω in $\mathbb{R}^N$ we denote by $\mathcal{D}(\Omega)$ the space of infinitely differentiable functions with compact support in Ω. The space of continuous functions with compact support in $\mathbb{R}^N$ will be denoted by $C_c(\mathbb{R}^N)$.

Recall that if Ω is an open subset of $\mathbb{R}^N$, a function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a vector valued Radon measure with finite total variation in Ω is called a *function of bounded variation*. The class of such functions will be denoted by $BV(\Omega)$. For $u \in BV(\Omega)$, the vector measure Du decomposes into its absolutely continuous and singular parts $Du = D^{ac}u + D^s u$. Then $D^{ac}u = \nabla u\, \mathcal{L}^N$, where ∇u is the Radon–Nikodym derivative of the measure Du with respect to the Lebesgue measure $\mathcal{L}^N$. We also split $D^s u$ in two parts: the *jump* part $D^j u$ and the *Cantor* part $D^c u$. It is well known (see for instance [2]) that

$$D^j u = (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner J_u,$$

where $u^+(x), u^-(x)$ denote the upper and lower approximate limits of u at x, J_u denotes the set of approximate jump points of u (*i.e.* points $x \in \Omega$ for which $u^+(x) > u^-(x)$), and $\nu_u(x) = \frac{Du}{|Du|}(x)$, being $\frac{Du}{|Du|}$ the Radon–Nikodym derivative of Du with respect to its total variation $|Du|$. For further information concerning functions of bounded variation we refer to [2].

We need to consider the following truncation functions. For $a < b$, let $T_{a,b}(r) := \max(\min(b,r),a)$, $T^l_{a,b} = T_{a,b} - l$. We denote

$$\mathcal{T}_r := \left\{ T_{a,b} \ : \ 0 < a < b \right\},$$

$$\mathcal{T}^+ := \left\{ T^l_{a,b} \ : \ 0 < a < b,\ l \in \mathbb{R},\ T^l_{a,b} \geq 0 \right\}.$$

Given any function w and $a, b \in \mathbb{R}$ we shall use the notation $\{w \geq a\} = \{x \in \mathbb{R}^N : w(x) \geq a\}$, $\{a \leq w \leq b\} = \{x \in \mathbb{R}^N : a \leq w(x) \leq b\}$, and similarly for the sets $\{w > a\}$, $\{w \leq a\}$, $\{w < a\}$, etc. We need to consider the following function space

$$TBV_{\rm r}^+(\mathbb{R}^N) := \left\{ w \in L^1(\mathbb{R}^N)^+ \ : \ T_{a,b}(w) - a \in BV(\mathbb{R}^N), \ \ \forall\, T_{a,b} \in \mathcal{T}_r \right\}.$$

Notice that $TBV_{\rm r}^{+}(\mathbb{R}^N)$ is closely related to the space $GBV(\mathbb{R}^N)$ of generalized functions of bounded variation introduced by E. De Giorgi and L. Ambrosio in [2]. Using the chain rule for BV-functions (see for instance [2]), one can give a sense to ∇u for a function $u \in TBV^{+}(\mathbb{R}^N)$ as the unique function v which satisfies

$$\nabla T_{a,b}(u) = v\chi_{\{a<u<b\}} \quad \mathcal{L}^N - \text{a.e.}, \;\; \forall\, T_{a,b} \in \mathcal{T}_r.$$

We refer to [2] for details.

A.2. A generalized Green's formula

Let us denote

$$X_{1,\infty}(\mathbb{R}^N) = \left\{ \mathbf{z} \in L^\infty\left(\mathbb{R}^N, \mathbb{R}^N\right) : \operatorname{div}(\mathbf{z}) \in L^1\left(\mathbb{R}^N\right)\right\}. \tag{A.4}$$

If $\mathbf{z} \in X_{1,\infty}(\mathbb{R}^N)$ and $w \in L^1_{\rm loc}(\mathbb{R}^N) \cap L^\infty_{\rm loc}(\mathbb{R}^N)$, we define the functional $(\mathbf{z}\cdot Dw) : C_c^\infty(\mathbb{R}^N) \to \mathbb{R}$ by the formula

$$\big\langle (\mathbf{z}\cdot Dw), \varphi \big\rangle := -\int_{\mathbb{R}^N} w\,\varphi \operatorname{div}(\mathbf{z})\, dx - \int_{\mathbb{R}^N} w\,\mathbf{z}\cdot\nabla\varphi\, dx, \quad \varphi \in C_c^\infty\left(\mathbb{R}^N\right). \tag{A.5}$$

If $\mathbf{z} \in X_{1,\infty}(\mathbb{R}^N)$ and $w \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then $(\mathbf{z}\cdot Dw)$ is a Radon measure in $\mathbb{R}^N$ [9], and

$$\int_{\mathbb{R}^N} (\mathbf{z}\cdot Dw) = \int_{\mathbb{R}^N} \mathbf{z}\cdot\nabla w\, dx, \quad \forall\, w \in W^{1,1}(\mathbb{R}^N) \cap L^\infty\left(\mathbb{R}^N\right). \tag{A.6}$$

Moreover, $(\mathbf{z}\cdot Dw)$ is absolutely continuous with respect to $|Dw|$ [9].

In the case where the distribution $(\mathbf{z}\cdot Dw)$ is a Radon measure we denote by $(\mathbf{z}\cdot Dw)^{ac}$, $(\mathbf{z}\cdot Dw)^{s}$ its absolutely continuous and singular parts with respect to $\mathcal{L}^N$. One has that $(\mathbf{z}\cdot Dw)^s$ is absolutely continuous with respect to $D^s w$ and $(\mathbf{z}\cdot Dw)^{ac} = \mathbf{z}\cdot\nabla w$.

A.3. Functionals defined on BV

In order to define the notion of entropy solutions of (A.1) and give a characterization of them, we need a functional calculus defined on functions whose truncations are in BV. Let Ω be an open subset of $\mathbb{R}^N$. Let $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0,\infty)$ be a Borel function such that

$$C(x)\,|\xi| - D(x) \le g(x,z,\xi) \le M'(x) + M\,|\xi|$$

for any $(x,z,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, $|z| \le R$, and any $R > 0$, where M is a positive constant and $C, D, M' \ge 0$ are bounded Borel functions which may depend on R. Assume that $C, D, M' \in L^1(\Omega)$.

Following Dal Maso [28] we consider the functional

$$\mathcal{R}_g(u) := \int_\Omega g(x, u(x), \nabla u(x))\, dx + \int_\Omega g^0 \left(x, \tilde{u}(x), \frac{Du}{|Du|}(x) \right) d \left| D^c u \right| \\ + \int_{J_u} \left(\int_{u_-(x)}^{u_+(x)} g^0(x, s, \nu_u(x))\, ds \right) d\mathcal{H}^{N-1}(x), \tag{A.7}$$

for $u \in BV(\Omega) \cap L^\infty(\Omega)$, being $\tilde{u}$ is the approximated limit of u [2]. The recession function g^0 of g is defined by

$$g^0(x, z, \xi) = \lim_{t \to 0^+} tg \left(x, z, \frac{\xi}{t} \right).$$

It is convex and homogeneous of degree 1 in ξ.

In case that Ω is a bounded set, and under standard continuity and coercivity assumptions, Dal Maso proved in [28] that $\mathcal{R}_g(u)$ is L^1-lower semi-continuous for $u \in BV(\Omega)$. More recently, De Cicco, Fusco, and Verde [30] have obtained a very general result about the L^1-lower semi-continuity of $\mathcal{R}_g$ in $BV(\mathbb{R}^N)$.

Assume that $g : \mathbb{R} \times \mathbb{R}^N \to [0, \infty)$ is a Borel function such that

$$C|\xi| - D \le g(z, \xi) \le M(1 + |\xi|) \qquad \forall (z, \xi) \in \mathbb{R}^N, \ |z| \le R, \tag{A.8}$$

for any $R > 0$ and for some constants $C, D, M \ge 0$ which may depend on R. Observe that both functions f, h defined in (A.2), (A.3) satisfy (A.8).

Assume that

$$\chi_{\{u \le a\}} \big(g(u(x), 0) - g(a, 0) \big), \chi_{\{u \ge b\}} \big(g(u(x), 0) - g(b, 0) \big) \in L^1 \left(\mathbb{R}^N \right), \tag{A.9}$$

for any $u \in L^1(\mathbb{R}^N)^+$. Let $u \in TBV_{\rm r}^+(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $T = T_{a,b} - l \in \mathcal{T}^+$. For each $\phi \in C_c(\mathbb{R}^N)$, $\phi \ge 0$, we define the Radon measure $g(u, DT(u))$ by

$$\big\langle g(u, DT(u)), \phi \big\rangle := \mathcal{R}_{\phi g} \left(T_{a,b}(u) \right) + \int_{\{u \le a\}} \phi(x) \big(g(u(x), 0) - g(a, 0) \big)\, dx \\ + \int_{\{u \ge b\}} \phi(x) \big(g(u(x), 0) - g(b, 0) \big)\, dx. \tag{A.10}$$

If $\phi \in C_c(\mathbb{R}^N)$, we write $\phi = \phi^+ - \phi^-$ with $\phi^+ = \max(\phi, 0)$, $\phi^- = -\min(\phi, 0)$, and we define $\langle g(u, DT(u)), \phi \rangle := \langle g(u, DT(u)), \phi^+ \rangle - \langle g(u, DT(u)), \phi^- \rangle$.

Recall that, if $g(z, \xi)$ is continuous in (z, ξ), convex in ξ for any $z \in \mathbb{R}$, and $\phi \in C^1(\mathbb{R}^N)^+$ has compact support, then $\langle g(u, DT(u)), \phi \rangle$ is lower semi-continuous in $TBV^+(\mathbb{R}^N)$ with respect to $L^1(\mathbb{R}^N)$-convergence [30]. This property is used to prove existence of solutions of (A.1).

We can now define the required functional calculus (see [4, 5, 23]). Let us denote by $\mathcal{P}$ the set of Lipschitz continuous functions $p : [0, +\infty) \to \mathbb{R}$ satisfying

$p'(s) = 0$ for s large enough. Let $S, T \in \mathcal{P}$. Assume that, if p represents either S or T, then we may write $p(\bar{z}) = \tilde{p}(T_{a,b}(\bar{z}))$ for some $0 < a < b$, for some $\tilde{p}$ which is differentiable in a neighborhood of $[a, b]$. We assume that $u \in TBV_{\rm r}^{+}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, so that $w = \Phi(u) \in TBV_{\rm r}^{+}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We denote

$$f_{S:T}^{\Phi}(\bar{z}, \xi) = S(\bar{z})T'(\bar{z})f\big(\Phi^{-1}(\bar{z}), \xi\big), \qquad h_{S:T}^{\Phi}(\bar{z}, \xi) = S(\bar{z})T'(\bar{z})h\big(\Phi^{-1}(\bar{z}), \xi\big).$$

We denote by

$$f_{S:T}^{\Phi}\big(\Phi(u), DT_{a,b}(\Phi(u))\big), \qquad h_{S:T}^{\Phi}\big(\Phi(u), DT_{a,b}(\Phi(u))\big),$$

the Radon measures defined by (A.10) with $g(\bar{z}, \xi) = f_{S:T}^{\Phi}(\bar{z}, \xi)$, and $g(\bar{z}, \xi) = h_{S:T}^{\Phi}(\bar{z}, \xi)$ applied to $w = \Phi(u)$, respectively. Note that (A.9) holds since $f_c(z, 0) = h_c(z, 0) = 0$.

A.4. Entropy solutions of the elliptic problem

We define the notion of entropy solution for the elliptic problem

$$u - \operatorname{div} \mathbf{a}(u, \nabla\Phi(u)) = v \quad \text{ in } \mathcal{D}'\big(\mathbb{R}^N\big). \tag{A.11}$$

We define $\mathcal{TSUB}$ as the class of functions $S, T \in \mathcal{P}$ such that

$$S \geq 0, S' \geq 0 \text{ and } T \geq 0, T' \geq 0,$$

where $p(\bar{z}) = \tilde{p}(T_{a,b}(\bar{z}))$ for some $0 < a < b$, and some $\tilde{p}$ which is differentiable in a neighborhood of $[a, b]$, and p represents either S or T. For any function q, $J_q(r)$ denotes the primitive of q, *i.e.*, $J_q(r) = \int_0^r q(s)\, ds$.

Definition A.1. Given $v \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $v \geq 0$, we say that $u \geq 0$ is an *entropy solution* of (A.11) if $u \in TBV_r^{+}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\mathbf{a}(u, \nabla\Phi(u))$ is a vector field in $X_{1,\infty}(\mathbb{R}^N)$ satisfying (A.11) in $\mathcal{D}'(\mathbb{R}^N)$, and

$$\begin{aligned} h_{S:T}^{\Phi}\big(\Phi(u), DT_{a,b}(\Phi(u))\big) \leq \big(\mathbf{a}(u, \nabla\Phi(u)) \cdot DJ_{T'S}(\Phi(u))\big) \\ \text{as measures } \forall\, (S, T) \in \mathcal{TSUB}. \end{aligned} \tag{A.12}$$

Inequality (A.12) holds in the sense of distributions. Since $h_{S:T}^{\Phi}(\Phi(u), DT_{a,b}(\Phi(u)))$ is a Radon measure, then $(\mathbf{a}(u, \nabla\Phi(u)) \cdot DJ_{T'S}(\Phi(u)))$ is also a Radon measure and (A.12) holds in the sense of measures.

Theorem A.2. *For any* $0 \leq v \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *there exists a unique entropy solution* $u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *of the problem* (A.11). *We have*

$$\|u\|_p \leq \|v\|_p \qquad \text{for any } p \in [1, \infty]. \tag{A.13}$$

If u_1, u_2 *are entropy solutions of the problem* (A.11) *corresponding to right-hand sides* $v_1, v_2 \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+$, *then*

$$\int_{\mathbb{R}^N} (u_1 - u_2)^+ \leq \int_{\mathbb{R}^N} (v_1 - v_2)^+. \tag{A.14}$$

Recall that existence is proved by approximating (A.11) by the sequence of problems

$$u_n - \operatorname{div} \mathbf{a}(u_n, \nabla\Phi(u_n)) - \frac{1}{n}\Delta u_n = v \qquad \text{in } \mathcal{D}'(\mathbb{R}^N). \tag{A.15}$$

We define the operator $(u, w) \in B$ if and only if $0 \le u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $u \in TBV_r^+(\mathbb{R}^N)$, $0 \le w \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and u is an entropy solution of (A.11) with $v = u + w$. The operator B is accretive in $L^1(\mathbb{R}^N)$, $\left(L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\right)^+ \subset$ Range$(I + B)$ and $D(B)$ is dense in $L^1(\mathbb{R}^N)^+$ [4, 24]. If we denote by $\mathcal{B}$ the closure in $L^1(\mathbb{R}^N)$ of the operator B, it follows that $\mathcal{B}$ is accretive in $L^1(\mathbb{R}^N)$, it satisfies the comparison principle, and verifies the range condition $\overline{D(\mathcal{B})}^{L^1(\mathbb{R}^N)} = L^1(\mathbb{R}^N)^+ \subset$ Range$(I + \lambda\mathcal{B})$ for all $\lambda > 0$. Therefore, according to the Crandall-Liggett Theorem [12, 27], for any $0 \le u_0 \in L^1(\mathbb{R}^N)$ there exists a unique mild solution $u \in C([0, T]; L^1(\mathbb{R}^N))$ of the abstract Cauchy problem

$$u'(t) + \mathcal{B}u(t) \ni 0, \quad u(0) = u_0. \tag{A.16}$$

Moreover, $u(t) = T(t)u_0$ for all $t \ge 0$, where $(T(t))_{t\ge 0}$ is the semigroup in $L^1(\mathbb{R}^N)^+$ generated by the Crandall-Liggett's exponential formula, *i.e.*,

$$T(t)u_0 = \lim_{n\to\infty} \left(I + \frac{t}{n}\mathcal{B}\right)^{-n} u_0.$$

Finally, the comparison principle also holds for $T(t)$, *i.e.*, if $u_0, \overline{u}_0 \in L^1(\mathbb{R}^N)^+$, we have the estimate

$$\left\|(T(t)u_0 - T(t)\overline{u}_0)^+\right\|_1 \le \left\|(u_0 - \overline{u}_0)^+\right\|_1. \tag{A.17}$$

A.5. Entropy solutions of the evolution problem

Let $L^1_w(0,T,BV(\mathbb{R}^N))$ be the space of weakly* measurable functions $w : [0, T] \to BV(\mathbb{R}^N)$ (*i.e.*, $t \in [0,T] \to \langle w(t), \phi\rangle$ is measurable for every ϕ in the predual of $BV(\mathbb{R}^N)$) such that $\int_0^T \|w(t)\|_{BV}\, dt < \infty$. Observe that, since $BV(\mathbb{R}^N)$ has a separable predual (see [2]), it follows easily that the map $t \in [0, T] \to \|w(t)\|_{BV}$ is measurable. By $L^1_{\mathrm{loc},w}(0, T, BV(\mathbb{R}^N))$ we denote the space of weakly* measurable functions $w : [0, T] \to BV(\mathbb{R}^N)$ such that the map $t \in [0, T] \to \|w(t)\|_{BV}$ is in $L^1_{\mathrm{loc}}(]0, T[)$.

Definition A.3. Let $u_0 \in (L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))^+$. A measurable function $u : (0, T) \times \mathbb{R}^N \to \mathbb{R}$ is an *entropy solution* of (1.1) in $Q_T = (0, T) \times \mathbb{R}^N$ if $u \in C([0, T], L^1(\mathbb{R}^N)^+)$, $T_{a,b}(\Phi(u(\cdot))) - a \in L^1_{\mathrm{loc},w}(0, T; BV(\mathbb{R}^N))$ for all $0 < a < b \le \infty$, and

(i) $\mathbf{a}(u(t), \nabla\Phi(u(t))) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ for a.e. $t \in (0, T)$,

(ii) $u_t = \operatorname{div} \mathbf{a}(u(t), \nabla\Phi(u(t)))$ in $\mathcal{D}'(Q_T)$,

(iii) $u(0) = u_0$, and
(iv) the following inequality is satisfied

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} \phi h^{\Phi}_{S:T}\big(\Phi(u), DT_{a,b}(\Phi(u))\big)\, dt \\ &\quad + \int_0^T \int_{\mathbb{R}^N} \phi h^{\Phi}_{T:S}\big(\Phi(u), DS_{c,d}(\Phi(u))\big)\, dt \\ &\leq \int_0^T \int_{\mathbb{R}^N} J_{T\circ\Phi S\circ\Phi}(u(t))\phi'(t)\, dxdt \\ &\quad - \int_0^T \int_{\mathbb{R}^N} \mathbf{a}\big(u(t), \nabla\Phi(u(t))\big)\cdot\nabla\phi\, T\big(\Phi(u(t))\big)S\big(\Phi(u(t))\big)\, dxdt \end{aligned} \tag{A.18}$$

for truncatures $(S, T) \in \mathcal{TSUB}$ with $T = \tilde{T} \circ T_{a,b}$, $S = \tilde{S} \circ S_{c,d}$ and any smooth function ϕ of compact support, in particular of the form $\phi(t, x) = \phi_1(t)\rho(x)$, $\phi_1 \in \mathcal{D}((0, T))$, $\rho \in \mathcal{D}(\mathbb{R}^N)$.

We observe that the functions that appear in (A.18) are measurable. For a proof we refer to Proposition 6.1 in [7]. We have the following existence and uniqueness result:

Theorem A.4. *Assume we are under assumptions* (H). *Then, for any initial datum* $0 \leq u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *there exists a unique entropy solution* u *of (1.1) in* $Q_T = (0, T)\times\mathbb{R}^N$ *for every* $T > 0$ *such that* $u(0) = u_0$. *Moreover, if* $u(t), \overline{u}(t)$ *are the entropy solutions corresponding to initial data* $u_0, \overline{u}_0 \in L^1(\mathbb{R}^N)^+$, *respectively, then*

$$\big\|(u(t) - \overline{u}(t))^+\big\|_1 \leq \big\|(u_0 - \overline{u}_0)^+\big\|_1 \qquad \text{for all } t \geq 0. \tag{A.19}$$

Moreover, entropy and semigroup solutions with initial condition $0\leq u_0\in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *coincide.*

The proof of this theorem has been given in [24] and follows the Steps of Theorem 4.5 in [5,22]. Existence of entropy solutions is proved by using Crandall-Liggett's scheme [27] and uniqueness is proved using Kruzhkov's doubling variables technique [20,32].

Remark A.5. We observe that $u(t) \in BV(\mathbb{R}^N)$ for any $t > 0$ if $u_0 \in BV(\mathbb{R}^N)$. Indeed, let $\tau_h u_0(x) = u_0(x + h)$, $h \in \mathbb{R}^N$. Let $u_h(t)$ be the entropy solution corresponding to the initial datum $\tau_h u_0$. Then by the uniqueness result of Theorem A.4 we have that $u_h(t) = \tau_h u(t)$ for any $t \geq 0$. By applying estimate (A.19) we have

$$\big\|u(t) - \tau_h u(t)\big\|_1 \leq \big\|u_0 - \tau_h u_0\big\|_1 \qquad \forall t > 0.$$

Since $u_0 \in BV(\mathbb{R}^N)$ we deduce that $u(t) \in BV(\mathbb{R}^N)$ for all $t > 0$ and $\|u(t)\|_{BV} \leq \|u_0\|_{BV}$. Clearly $u \in L^1_w(0, T; BV(\mathbb{R}^N))$.

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