Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XIV (2015), 263-292

The geometry of planar p-harmonic mappings: convexity, level curves and the isoperimetric inequality

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Abstract. We discuss various representations of planar p-harmonic systems of equations and their solutions. For coordinate functions of p-harmonic maps we analyze signs of their Hessians, the Gauss curvature of p-harmonic surfaces, the length of level curves as well as we discuss curves of steepest descent. The isoperimetric inequality for the level curves of coordinate functions of planar p-harmonic maps is proven. Our main techniques involve relations between quasiregular maps and planar PDEs. We generalize some results due to P. Lindqvist, G. Alessandrini, G. Talenti and P. Laurence.

Mathematics Subject Classification (2010): 35J47 (primary); 35J70, 35J92, 30C65, 35J45 (secondary).

1. Introduction

In this note we discuss the geometry of solutions to a p-harmonic system of equations in the plane. That is, for a map $u = (u^1, u^2) : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ and $1 < p < \infty$ we will investigate the following nonlinear system of equations:

$$\operatorname{div}(|Du|^{p-2} Du) = 0,$$

where Du stands for the Jacobi matrix of u and $|Du|^2 = |\nabla u^1|^2 + |\nabla u^2|^2$. If a solution exists it is called a p-harmonic map. The system originates from the Euler-Lagrange system for the energy $\int_\Omega |Du|^p$ and therefore, the natural domain of definition for solutions is the Sobolev space $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. However, in the discussion below we will deal mainly with C^2-regular maps. Equivalently, this system can be written as follows.

$$\begin{cases} \operatorname{div}(|Du|^{p-2}\nabla u^1) = 0 \\ \\ \operatorname{div}(|Du|^{p-2}\nabla u^2) = 0. \end{cases} \tag{1.1}$$

Furthermore, for $p = 2$ the system reduces to the harmonic one and so from that point of view p-harmonic maps are the nonlinear counterparts of harmonic transfor-

Received January 23, 2012; accepted in revised version January 28, 2013.

mations. On the other hand, if map u degenerates to a single function $u = (u^1, 0)$, we retrieve from (1.1) the classical p-harmonic equation $\operatorname{div}(|\nabla u^1|^{p-2}\nabla u^1) = 0$. In spite of similarity to the definition of the p-harmonic equation, the p-harmonic system is far more complicated, as the component functions are tangled together by the appearance of Du in both equations. This property, together with degeneracy of the system at points where $Du = 0$ makes the analysis of p-harmonics difficult and challenging.

The p-harmonic operators and systems arise naturally in a variety of applications *e.g.* in nonlinear elasticity theory [16,17], nonlinear fluid dynamics [1,12], as well as in cosmology or climate sciences and several other areas (see *e.g.* [3] and references therein). In pure mathematics the p-harmonic maps appear for instance in differential geometry [14,29,30] or in relation to differential forms and quasiregular maps [11]. In what follows we will confine our discussion to maps between planar domains. The reason for this is twofold. First, already in the two dimensional setting computations for nonlinear Laplace operators are complicated and in higher dimensions the complexity increases even further. The second reason is related to the fact that we will often appeal to relations between the complex gradients of coordinate functions of a p-harmonic map and quasiregular maps. Such relations known for planar p-harmonic equation [10,26] has been recently established also in the setting of p-harmonic maps (see [3] or discussion in Section 2 and Appendix A.1). The corresponding relations between nonlinear PDEs and quasiregular maps beyond the plane remains an open problem.

We recall that in the planar case quasiregular map can be defined in terms of the Beltrami coefficient μ. Namely, a map F is quasiregular if there exists a constant k such that

$$|\mu| = \frac{|F_{\bar{z}}|}{|F_z|} \leq k < 1 \quad \text{a.e. in } \Omega. \tag{1.2}$$

For the equivalent definitions of quasiregular maps and further information on this topic we refer to *e.g.* [20], [15, Chapter 14], [9, Chapter 3]. Other properties of quasiregular maps needed in our presentation will be recalled throughout the discussion.

The subject of our interest will be the geometry of p-harmonic surfaces, that is the geometry of the graphs of coordinate functions of planar p-harmonic maps. The main difficulty lies in the fact that functions u^1 and u^2 are coupled by Du, and so many of our estimates involve both coordinates and depend on the Jacobi matrix norm $|Du|$.

In Section 2 we recall and introduce various representations of the p-harmonic operator and p-harmonic system needed in further sections, as depending on the discussed problem we will adopt different points of view on p-harmonicity.

In Section 3 we show that for some range of parameter p the positivity of Hessian determinant for one coordinate function of u implies that the second Hessian determinant is negative. Such a phenomenon has not been noticed before for p-harmonic maps. From this observation we infer number of conclusions regarding convexity of coordinate functions and their level sets and the Gauss curvature of the corresponding surfaces. In the latter case, we generalize work of Lindqvist [23] on p-harmonic surfaces. Using the class of radial maps we illustrate Section 3 by example postponed to Appendix A.3 due to complexity and technical nature.

Section 4 is devoted to studying the curvature of level curves. Following the ideas of Alessandrini [4] and Lindqvist [23] we prove Theorem 4.3 providing the local estimates of lengths of a level curves of u^1 and u^2. To our best knowledge such estimates in the nonlinear vectorial setting are not present in the literature so far.

We continue investigation of level curves in Section 5, where basing on the work of Talenti [27] we discuss level curves of steepest descent and provide some estimates for the curvature functions involving both the level curves and their orthogonal trajectories. Results in Sections 3-5 are based on techniques developed in earlier work by the author [3] and therefore, for the sake of completeness and for the readers convenience we recall the necessary results from [3] in Appendix A.1. There we also extend some of the estimates from [3] due to C^2 assumption on p-harmonic maps.

Section 6 contains discussion of an isoperimetric inequality for p-harmonic maps and generalizes works of Laurence [19] and Alessandrini [5] to the setting of vector transformations. Again, in the setting of systems of coupled differential equations such a result is new.

We believe that our approach based on mixture of complex analysis, theory of quasiregular maps and PDEs techniques can be extended to some other nonlinear systems of equations in the plane.

ACKNOWLEDGEMENTS. The paper was written during the author employment at the University of Cincinnati and then at Linköping University. Tomasz Adamowicz would like to express his gratitude to these institutions for hospitality and support.

2. Representations of p-harmonic equations and systems

In this section we recall and develop various representation formulas for p-harmonic operator and system in the plane. The presentation is of mainly technical nature and the results here will serve as auxiliary tools for the discussion in the following sections. Also, our goal is to compare p-harmonic transformations with their scalar counterparts (p-harmonic functions) and, therefore, illustrate the differences between these cases. We would like to emphasize that despite formal similarity to the scalar case one should expect new phenomena in the setting of p-harmonic mappings.

Let $u = (u^1, u^2) : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a planar p-harmonic mapping. Assume that $u \in C^2(\Omega)$. Following notation in [3] we denote by f, g the complex gradients of the first and the second coordinate function of u, respectively.

$$f = \frac{1}{2}(u^1_x - i\, u^1_y), \quad g = \frac{1}{2}(u^2_x - i\, u^2_y). \tag{2.1}$$

In what follows we will frequently appeal to the following equations for $|f|$, $|g|$, $f_z, g_z, f_{\bar z}, g_{\bar z}$.

$$
\begin{aligned}
&|f| = \tfrac{1}{2}|\nabla u^1|, \qquad |g| = \tfrac{1}{2}|\nabla u^2|,\\
&f_z = \tfrac{1}{4}\left(u^1_{xx} - u^1_{yy} - 2iu^1_{xy}\right), \qquad f_{\bar z} = \overline{f_{\bar z}} = \tfrac{1}{4}\left(u^1_{xx} + u^1_{yy}\right) = \tfrac{1}{4}\Delta u^1,\\
&g_z = \tfrac{1}{4}\left(u^2_{xx} - u^2_{yy} - 2iu^2_{xy}\right), \qquad g_{\bar z} = \overline{g_{\bar z}} = \tfrac{1}{4}\left(u^2_{xx} + u^2_{yy}\right) = \tfrac{1}{4}\Delta u^2.
\end{aligned}
\tag{2.2}
$$

Next, we define the p-harmonic operator and express it by using the complex notation. Let $\Omega \subset \mathbb{R}^2$ and $v \in C^2(\Omega, \mathbb{R})$ be a Sobolev function for a given $1 < p < \infty$. The following differential operator is called the scalar p-harmonic operator:

$$
\begin{aligned}
\Delta_p v &= \operatorname{div}(|\nabla v|^{p-2}\nabla v)\\
&= |\nabla v|^{p-4}\big(|\nabla v|^2\Delta v + (p-2)((v_x)^2 v_{xx} + 2v_x v_y v_{xy} + (v_y)^2 v_{yy})\big)\\
&= |\nabla v|^{p-4}\big(|\nabla v|^2\Delta v + \tfrac{p-2}{2}\langle\nabla v, \nabla|\nabla v|^2\rangle\big).
\end{aligned}
\tag{2.3}
$$

We will also need the complex formulation of a scalar p-harmonic operator in the plane. Using (2.1) with (2.3) (with the abuse of notation that here $f = \frac{1}{2}(v_x - i\, v_y)$) we obtain that

$$
\begin{aligned}
\operatorname{div}(|\nabla v|^{p-2}\nabla v) &= \tfrac{\partial}{\partial x}\left(|\nabla v|^{p-2}v_x\right) + \tfrac{\partial}{\partial y}\left(|\nabla v|^{p-2}v_y\right)\\
&= \tfrac{\partial}{\partial x}\left(2^{p-2}|f|^{p-2}(f + \overline{f})\right) + i\tfrac{\partial}{\partial y}\left(2^{p-2}|f|^{p-2}(f - \overline{f})\right)\\
&= 2^p\mathfrak{Re}\left(|f|^{p-2}f\right)_{\bar z} = 2^{p-1}\left(\left(|f|^{p-2}f\right)_{\bar z} + \overline{\left(|f|^{p-2}f\right)_{\bar z}}\right)\\
&= 2^{p-2}|f|^{p-2}\left(2pf_{\bar z} + (p-2)\left(\tfrac{f}{\overline{f}}\overline{f_z} + \tfrac{\overline{f}}{f}f_z\right)\right).
\end{aligned}
\tag{2.4}
$$

Let us now focus our attention on p-harmonic maps. Using the definition of the p-harmonic operator (2.3) the following form of the p-harmonic system (1.1) in the plane can be established at the points where $\nabla u^1 \neq 0$ and $\nabla u^2 \neq 0$:

$$
\begin{cases}
|\nabla u^2|^{p-4}\Delta_p u^1 + |\nabla u^1|^{p-4}\operatorname{div}(|\nabla u^2|^{p-2}\nabla u^1) = 0\\[1ex]
|\nabla u^1|^{p-4}\Delta_p u^2 + |\nabla u^2|^{p-4}\operatorname{div}(|\nabla u^1|^{p-2}\nabla u^2) = 0.
\end{cases}
\tag{2.5}
$$

Remark 2.1. The analogous representation can be stated in any dimension $n \geq 2$. However, here we confine our discussion to the case $n = 2$ only.

Equivalently, system (2.5) can also be written as follows:

$$
\begin{cases}
|\nabla u^1|^{4-p}\Delta_p u^1 + |\nabla u^2|^2\Delta u^1 + \tfrac{p-2}{2}\langle\nabla u^1, \nabla|\nabla u^2|^2\rangle = 0\\[1ex]
|\nabla u^2|^{4-p}\Delta_p u^2 + |\nabla u^1|^2\Delta u^2 + \tfrac{p-2}{2}\langle\nabla u^2, \nabla|\nabla u^1|^2\rangle = 0.
\end{cases}
\tag{2.6}
$$

We will sketch the proof only for the first equation of system (2.5) as the derivation of the second equation goes along the same lines:

$$
\begin{aligned}
\operatorname{div}(|Du|^{p-2}\nabla u^1) &= \tfrac{\partial}{\partial x}\left((|\nabla u^1|^2+|\nabla u^2|^2)^{\frac{p-2}{2}}u^1_x\right)\\
&\quad+\tfrac{\partial}{\partial y}\left((|\nabla u^1|^2+|\nabla u^2|^2)^{\frac{p-2}{2}}u^1_y\right)=0\\
|\nabla u^1|^2\Delta u^1 &+|\nabla u^2|^2\Delta u^1+\tfrac{p-2}{2}\Big(\left(|\nabla u^1|^2\right)_x u^1_x+\left(|\nabla u^2|^2\right)_x u^1_x\\
&\qquad+\left(|\nabla u^1|^2\right)_y u^1_y+\left(|\nabla u^2|^2\right)_y u^1_y\Big)=0\\
|\nabla u^1|^2\Delta u^1 &+\tfrac{p-2}{2}\langle\nabla u^1,\nabla|\nabla u^1|^2\rangle+|\nabla u^2|^2\Delta u^1+\tfrac{p-2}{2}\langle\nabla u^1,\nabla|\nabla u^2|^2\rangle=0\\
|\nabla u^2|^{p-4}\Delta_p u^1 &+|\nabla u^1|^{p-4}(|\nabla u^2|^{p-2}\Delta u^1+(p-2)|\nabla u^2|^{p-4})=0\\
|\nabla u^2|^{p-4}\Delta_p u^1 &+|\nabla u^1|^{p-4}\left(\tfrac{\partial}{\partial x}(|\nabla u^2|^{p-2}u^1_x)+\tfrac{\partial}{\partial y}(|\nabla u^2|^{p-2}u^1_y)\right)=0. \qquad (2.7)
\end{aligned}
$$

Using the definition of the divergence operator we arrive at the first equation of (2.5). Multiplying (2.7) by $|\nabla u^1|^{4-p}|\nabla u^2|^{4-p}$ we produce the first equation of (2.6). Similar computations allow us to obtain second equation of (2.5) and (2.6), respectively.

The complex notation and the connection between PDEs in the plane and functions of complex variable is nowadays classical and has become very fruitful and brought lots of insight into both complex analysis and theory of differential equations, to mention for instance the theory of Beltrami equation or the theory of generalized analytic functions (see *e.g.* [9,31]). It turns out that also in the setting of p-harmonic systems such relations can be discovered. Indeed, in [3] we proved that f and g satisfy the following system equivalent to (1.1). We will frequently appeal to this result and its consequences in further sections.

Theorem 2.2 (Theorem 1, [3]). *For $1<p<\infty$ let $u=(u^1,u^2)$ be a $C^2(\Omega,\mathbb{R}^2)$ p-harmonic mapping. Consider complex gradients f,g of coordinate functions u^1,u^2, respectively* (eqs. (2.1)). *We have the following system of quasilinear equations*

$$
\begin{cases}
\left(2p+\frac{4|g|^2}{|f|^2}\right)f_{\bar z} = (2-p)\left(\frac{\bar f}{f}f_z+\frac{f}{\bar f}\overline{f_z}\right)\\
\qquad\qquad +(2-p)\left[\frac{\bar g}{f}g_z+\frac{g}{\bar f}\overline{g_z}+\left(\frac{\bar g}{\bar f}+\frac{g}{f}\right)g_{\bar z}\right]\\
\left(2p+\frac{4|f|^2}{|g|^2}\right)g_{\bar z} = (2-p)\left(\frac{\bar g}{g}g_z+\frac{g}{\bar g}\overline{g_z}\right)\\
\qquad\qquad +(2-p)\left[\frac{\bar f}{g}f_z+\frac{f}{\bar g}\overline{f_z}+\left(\frac{\bar f}{\bar g}+\frac{f}{g}\right)f_{\bar z}\right]
\end{cases}
\qquad (2.8)
$$

at the points where $f\neq 0$ and $g\neq 0$.

Remark 2.3. In [3] the above result is proven for $p \geq 2$. Here, the C^2 assumption on u allows us to extend theorem to the whole range of $1 < p < \infty$ (see Appendix A.1 for further discussion).

Let us compare the above system to its scalar counterpart. For that purpose, note that if $u^2 \equiv 0$, then the mapping u reduces to one coordinate function u^1. In such a case system (2.8) reduces to well known equation, see *e.g.* [10, equation (5)]:

$$f_{\bar{z}} = \left(\frac{1}{p} - \frac{1}{2}\right)\left(\frac{\overline{f}}{f} f_z + \frac{f}{\overline{f}}\overline{f_z}\right).$$

From this we immediately infer that f is a quasiregular mapping (more on this topic can be found in [9]).

Let us also mention that system (2.8) can be solved for $f_{\bar{z}}$ and $g_{\bar{z}}$. As a result we arrive at the following representation for f and g. (We refer to Appendix A.1 and discussion in [3] for the definition of matrix $A(f, g)$ and further estimates).

$$\begin{bmatrix} f \\ g \end{bmatrix}_{\bar{z}} = A(f, g)\begin{bmatrix} f \\ g \end{bmatrix}_z + \overline{A(f, g)\begin{bmatrix} f \\ g \end{bmatrix}_z}. \tag{2.9}$$

The ellipticity of such quasilinear system has been proven in [3, Theorem 2]. There, we also showed that, perhaps surprisingly, the coefficients of $A(f, g)$ can be estimated in terms of parameter p only.

Using the above system one can investigate when f and g are quasiregular maps, extending the results known for p-harmonic functions in the plane [3, Section 3].

We would like to add that in the planar case the relation between quasiregular mappings and p-harmonic functions is known in much deeper details than we just sketched it above. For instance one can prove that the coordinates of a planar quasiregular map, as well as the logarithm of the modulus of such map satisfy certain elliptic equations and the same holds for the logarithm of the modulus of the gradient of p-harmonic function (see [26]). The higher-dimensional counterparts of such properties remain unknown neither for p-harmonic functions nor p-harmonic mappings, due to lack of Stoïlow factorization beyond the complex plane.

3. Convexity of coordinate functions, the Gaussian curvature of p-harmonic surfaces

Below we use system (2.9) and the estimates for the entries of matrix $A(f, g)$ (see (A.4) in Appendix A.1) to determine mutual relations between convexity of coordinate functions of p-harmonic map. We discover an interesting phenomenon that for a certain range of p convexity of one coordinate function implies the concavity of the other. Theorem 3.1 has not been noticed before in the literature, mainly due to the lack of enough wide classes of examples of p-harmonic maps. Among equivalent formulations of Theorem 3.1 and its corollaries we discuss the sign of Gauss curvature for p-harmonic surfaces and convexity of their level sets.

Since the convexity properties of a function are govern by the second derivatives matrix of such function, the Hessian matrices for u^1 and u^2 and the analysis of their signs will be of our main interest. Using equations (2.2) we express the determinant of Hessian $H(u^1)$ as follows:

$$\det H(u^1) = u^1_{xx}u^1_{yy} - (u^1_{xy})^2 = 4(|f_{\bar{z}}|^2 - |f_z|^2),$$

where f is a complex gradient of u^1 (see (2.1)). Related is the Gauss curvature of a surface $z = u^1(x, y)$.

$$K_{u^1} = \frac{u^1_{xx}u^1_{yy} - (u^1_{xy})^2}{(1 + (u^1_x)^2 + (u^1_y)^2)^2} = \frac{\det H(u^1)}{(1 + (u^1_x)^2 + (u^1_y)^2)^2}. \tag{3.1}$$

Similar formulas hold for $H(u^2)$ and K_{u^2}.

Theorem 3.1. *Suppose $u = (u^1, u^2)$ is a p-harmonic mapping and let $p \in [\frac{4}{3}, 2 + \sqrt{2}]$. If* $\det H(u^2) \geq 0$, *then* $\det H(u^1) \leq 0$.

Moreover, we have that if $\det H(u^2) \geq 0$ ($\det H(u^1) \geq 0$, *respectively*) *holds in the whole domain Ω, then the Gauss curvature $K_{u^1} \leq 0$ ($K_{u^2} \leq 0$, respectively) in Ω.*

Before presenting the proof we will compare this observation to the case of a single p-harmonic equation and discuss some consequences of the above result.

Remark 3.2. If $u^2 \equiv 0$ ($u^1 \equiv 0$) then p-harmonic system (1.1) reduces to a single p-harmonic equation for $u := u^1$ ($u := u^2$, respectively). In such a case from Theorem 3.1 we retrieve first part of the assertion of [23, Theorem 5.3] which stays that for p-harmonic surfaces $K_u \leq 0$. Furthermore, the quasiregularity of the complex gradient of u implies that $K_u = 0$ at most at isolated points or u is an affine function (*cf.* [23]).

Recall that the Jacobian of f satisfies $4J(z, f) = -\det H(u^1)$ (similarly, $4J(z, g) = -\det H(u^2)$). This, together with the characterization of quasiregularity via the Beltrami coefficient (1.2) leads us to the equivalent formulation of Theorem 3.1.

Corollary 3.3. *Under the assumptions of Theorem* 3.1 *it holds that if $J(z, g) \leq 0$, then $J(z, f) \geq 0$.*

Remark 3.4. Taking into account that if $J(z, g) < 0$ in Ω, then $\overline{g}$ is quasiregular, Theorem 3.1 can be equivalently rephrased as follows: *if $\overline{g}$ is quasiregular, then so is f.*

Theorem 3.1 allows us also to explore the convexity properties of p-harmonic surfaces. Recall, that if a function v is convex, then level sets $\{v \leq c\}$ are convex as well. Similarly, if v is concave, then level sets $\{v \geq c\}$ are concave. The analysis of convexity of level sets has been the subject of several interesting papers, for instance due to Kawohl [18, Chapter 3], Lewis [22] or recently Ma *et al.* [25], to mention only some.

Corollary 3.5. *Suppose that the assumptions of Theorem* 3.1 *hold. If* u^2 *is a convex component function of* u*, then* u^1 *is concave and so are level sets* $\{u^1 \geq c\}$*, provided that* $u^1_{xx} \leq 0$.

The assertion of Theorem 3.1 can also be related to work of Laurence [19] on the derivatives of geometric functionals emerging in analysis, such as the length of the level curves (Laurence's work will be discussed in greater details in Section 6 below). The results of [19] specialized to our setting read as follows.

Corollary 3.6. *Let* u *satisfies the assumptions in* [19, *Theorems* 1, 2 *and* 5] *and Theorem* 3.1 *above. Denote by* $L(s) = \int_{\{u^2=s\}} \mathrm{d}\mathcal{H}^1$ *the length of level curve of* u^2 *corresponding to* s*. It holds, that if* u^1 *is convex, then* $L''(s) > 0$.

Proof of Theorem 3.1. From formula (2.9) we get the following important estimate (see also Remark A.1 and inequality (15) in [3]):

$$\begin{aligned}|f_{\bar z}| &\leq |A_{11}(f,g)||f_z| + |A_{12}(f,g)||g_z| + |\overline{A_{11}(f,g)}||\overline{f_z}| + |\overline{A_{12}(f,g)}||\overline{g_z}| \\ &\leq 2A_p\left(|f_z| + |g_z|\right).\end{aligned}$$

Here A_p is the upper bound for the entries of matrix $A(f,g)$ (see Appendix A.1 for computations):

$$A_p = \begin{cases} \frac{2-p}{2p} & \text{for} \quad 1 < p < 2, \\ \frac{p-2}{2p} & \text{for} \quad 2 \leq p \leq 3, \\ \frac{(p-2)(p-1)}{4p} & \text{for} \quad 3 < p. \end{cases} \tag{3.2}$$

This, together with the analogous inequality for $|g_{\bar z}|$ and arithmetic-geometric mean inequality results in the estimate:

$$|f_{\bar z}|^2 + |g_{\bar z}|^2 \leq 4(2A_p)^2(|f_z|^2 + |g_z|^2). \tag{3.3}$$

With the above notation we infer from (3.3) the following chain of estimates.

$$\begin{aligned}&\tfrac14\left(\det H(u^1) - \det H(u^2)\right) = (|f_{\bar z}|^2 - |f_z|^2) - (|g_{\bar z}|^2 - |g_z|^2) \\ &\leq 16A_p^2(|f_z|^2 + |g_z|^2) - 2|g_{\bar z}|^2 + |g_z|^2 - |f_z|^2 \\ &= \left(16A_p^2 - 1\right)|f_z|^2 + \left(16A_p^2 + 1\right)|g_z|^2 - 2|g_{\bar z}|^2\end{aligned}$$

(next, we use explicit formulas (2.2) for g_z and $g_{\bar z}$)

$$\begin{aligned}&= \left(16A_p^2 - 1\right)|f_z|^2 + \frac{1}{16}\left(16A_p^2 + 1\right) \\ &\quad \times \left((u^2_{xx})^2 + (u^2_{yy})^2 - 2u^2_{xx}u^2_{yy} + 4(u^2_{xy})^2\right) - \frac18(\Delta u^2)^2 \\ &= \left(16A_p^2 - 1\right)|f_z|^2 + \frac{1}{16}\left(16A_p^2 - 1\right)(\Delta u^2)^2 \\ &\quad - \frac14\left(16A_p^2 + 1\right)u^2_{xx}u^2_{yy} + \frac14\left(16A_p^2 + 1\right)(u^2_{xy})^2 \\ &= \left(16A_p^2 - 1\right)|f_z|^2 + \frac{1}{16}\left(16A_p^2 - 1\right)(\Delta u^2)^2 - \frac14\left(16A_p^2 + 1\right)\det H(u^2).\end{aligned}$$

It follows that

$$\det H(u^1) \leq 4(16A_p^2 - 1)\left(|f_z|^2 + \frac{1}{16}(\Delta u^2)^2\right) - 16A_p^2 \det H(u^2). \tag{3.4}$$

Computations involving the appropriate values of A_p (see (3.2) and (A.4) in Appendix A.1) give us that

$$16A_p^2 \leq 1 \quad \text{if} \quad \begin{cases} (4-3p)(4-p) \leq 0 & \text{for} \quad 1 < p < 2, \\ (p-4)(3p-4) \leq 0 & \text{for} \quad 2 \leq p \leq 3, \\ (p^2-4p+2)(p^2-2p+2) \leq 0 & \text{for} \quad 3 < p. \end{cases}$$

From these conditions we derive that $16A_p^2 \leq 1$ holds provided $p \in [\frac{4}{3}, 2+\sqrt{2}]$. From this and (3.4) the first assertion of theorem follows immediately.

The second assertion of the theorem is the straightforward consequence of the first part and equation (3.1). □

Remark 3.7. The range of parameter p in the assertion of Theorem 3.1 is the consequence of estimates for entries of the matrix $A(f, g)$. In Appendix A.3 we discuss the counterexample to Theorem 3.1 for some p outside the interval $[\frac{4}{3}, 2+\sqrt{2}]$. The problem of finding such examples is the general feature of p-harmonic world, as we know only few classes of p-harmonic maps and few explicit solutions of the p-harmonic system of equations, namely affine, radial and quasiradial (see [2, Chapter 2] for the definition of the latter one class of mappings).

Open problem 1. Let $u = (u^1, \dots, u^n)$ be a non-trivial p-harmonic map between domains in $\mathbb{R}^n$ for $n \geq 3$ (that is u is not an affine or constant map). Suppose that u^i is convex for some $i = 1, \dots, n$ (and so $\det H(u^i) > 0$). Does it then hold that $\det H(u^j) \leq 0$ for $j \neq i$? Describe the conditions for concavity of u^j for $j \neq i$.

4. The curvature of level curves

Below we discuss various curvature functions of level curves for the component functions of a map $u = (u^1, u^2)$ and employ such curvatures to estimate the length of the level curves. It appears that such estimates require integrability of Hessians or quasiregularity of complex gradients of u^1 and u^2. Therefore, the complex linearization of p-harmonic system (2.9) comes in handy. The results below extend the work of Lindqvist [23] for a p-harmonic equation.

Let $\{u^1 = c\}$ be a nonempty level curve with the property that none of the critical points of u^1 lies on this level curve. The curvature function k_{u^1} of $\{u^1 = c\}$ can be computed by the following formula:

$$k_{u^1} = -\frac{(u_y^1)^2 u_{xx}^1 - 2u_x^1 u_y^1 u_{xy}^1 + (u_x^1)^2 u_{yy}^1}{|\nabla u^1|^3} = -\operatorname{div}\left(\frac{\nabla u^1}{|\nabla u^1|}\right) = -\Delta_1(u^1). \tag{4.1}$$

Consider the above formula for a harmonic function v. Theorem 3 in Talenti [27] shows that if v has no critical points, then $\frac{k_v}{|\nabla v|}$ is harmonic and $-\ln|k_v|$ is subharmonic. As far as we know the similar results for a single p-harmonic equation with $p \neq 2$ are not known (see also presentation in Section 5 below). In the next observation we further illustrate differences between scalar and vector cases by computing curvatures k for p-harmonic functions and coordinate functions of u. Moreover, the second part of the observation can be considered as a starting point for obtaining the counterparts of aforementioned Talenti's results in the nonlinear setting (see also Remark 1.5 and the discussion in [25] for some recent developments in this topic).

Observation 1. Let $p \neq 2$ and suppose that the component function u^1 of a p-harmonic map u has no critical points on the level curve $\{u^1 = c\}$. Then

$$k_{u^1} = -\frac{\Delta u^1}{|\nabla u^1|} + \frac{1}{|\nabla u^1|}\left\langle \nabla|\nabla u^1|, \frac{\nabla u^1}{|\nabla u^1|}\right\rangle \tag{4.2}$$

$$= -\frac{p-1}{p-2}\frac{\Delta u^1}{|\nabla u^1|} + \frac{\Delta_p u^1}{(p-2)|\nabla u^1|^{p-1}}. \tag{4.3}$$

Equivalently, in the complex notation k_{u^1} becomes

$$2|f|k_{u^1} = -2f_{\bar{z}} + \frac{f}{\overline{f}}\overline{f_z} + \frac{\overline{f}}{f}f_z \tag{4.4}$$

$$= -2\left(\ln|f|^2\right)_z + \frac{f}{\overline{f}}\overline{f_z} + 3\frac{\overline{f}}{f}f_z, \tag{4.5}$$

where f is a complex gradient of u^1 as defined in (2.1). Similar formulas hold for the second component function u^2 as well.

Furthermore, if $u^2 \equiv 0$, then (4.3) reduces to the following:

$$k_{u_1} = -\frac{p-1}{p-2}\frac{\Delta u_1}{|\nabla u_1|}. \tag{4.6}$$

Before proving the observation, we would like to make some remarks in order to motivate above computations and present our discussion in the wider perspective.

Remark 4.1. The formula (4.5) is convenient, for instance, if one knows additionally that f is a quasiregular map. In such a case function $-\ln|f|$ solves the A-harmonic type equation at points where $f \neq 0$, see *e.g.* [13]. The integral estimates which follow from this fact will be of use for us when discussing the integrability of k_{u_1}.

Remark 4.2. For $p = 2$ the nonlinear p-harmonic system reduces to the uncoupled system of two harmonic equations, for which the curvature functions are already present in the literature, see *e.g.* [23,27].

Proof of Observation 1. Equation (4.2) follows immediately from the divergence formulation of curvature (4.1). The same formulation used again leads us to the following identity:

$$
\begin{aligned}
\Delta u^1 &= \operatorname{div}\left(\frac{\nabla u^1}{|\nabla u^1|}\,|\nabla u^1|\right)\\
&= -|\nabla u^1| k_{u^1} + \frac{u^1_x}{|\nabla u^1|^2}(u^1_x u^1_{xx} + u^1_y u^1_{xy}) + \frac{u^1_y}{|\nabla u^1|^2}(u^1_x u^1_{xy} + u^1_y u^1_{yy}) \qquad (4.7)\\
&= -|\nabla u^1| k_{u^1} + \frac{(u^1_x)^2 u^1_{xx} + 2u^1_x u^1_y u^1_{xy} + (u^1_y)^2 u^1_{yy}}{|\nabla u^1|^2}.
\end{aligned}
$$

Applying the definition of p-harmonic operator (2.3) to the last term, we express (4.7) in the following form:

$$|\nabla u^1|^2 \Delta u^1 = -|\nabla u^1|^3 k_{u^1} + \tfrac{1}{p-2}|\nabla u^1|^{4-p}\Delta_p u^1 - \tfrac{1}{p-2}|\nabla u^1|^2 \Delta u^1.$$

From this, formula (4.3) follows immediately. In order to show the complex representation of k_{u^1} we use (2.2) and (2.4) together with (4.2) to obtain equation from which (4.4) follows straightforwardly:

$$k_{u^1} = -\frac{2(p-1)}{p-2}\frac{f_{\bar z}}{|f|} + \frac{2^{p-2}|f|^{p-2}\Big(2p f_{\bar z} + (p-2)\Big(\frac{f}{\bar f}\overline{f_z} + \frac{\bar f}{f} f_z\Big)\Big)}{2^{p-1}(p-2)|f|^{p-1}}.$$

We show (4.5) by first observing that $\big(\ln(f\bar f)\big)_z = \frac{f_z}{f} + \frac{f_{\bar z}}{\bar f}$ since $(\bar f)_z = f_{\bar z}$. Then, by (4.4) we obtain that

$$2|f|k_{u^1} = -2\left(\frac{f_{\bar z}}{\bar f} + \frac{f_z}{f}\right)\bar f + \frac{f}{\bar f}\overline{f_z} + 3\frac{\bar f}{f} f_z = -2\left(\ln|f|^2\right)_z + \frac{f}{\bar f}\overline{f_z} + 3\frac{\bar f}{f} f_z.$$

Finally, (4.6) follows from the observation that if u^1 is a p-harmonic function, then $\Delta_p u^1 = 0$ and so (4.6) is a special case of (4.3). $\square$

We would like now to show one of the main results of the paper, namely the length estimates for the level curves. We follow the approach of Talenti [27] for planar linear elliptic equations and of Lindqvist [23] for planar p-harmonic functions.

Theorem 4.3. *Let $u = (u^1, u^2)$ be a C^2 p-harmonic map in the planar domain Ω for $p \neq 2$. Let also $C > 0$ be a constant. Denote by $B = B(z_0, R)$ a ball in Ω and consider a nonempty level curve $\{u^1 = c\} \cap B \neq \emptyset$. Suppose that, either*

(1) *Euclidean norms of Hessians $\|H(u^1)\|, \|H(u^2)\|$ are in $L^2(B)$ and $|f| > C$ in B or*
(2) *$f_z, g_z \in L^2(B)$ and $|f| > C$ in B or*

(3) *f and g are quasiregular in B and $|f| > C$ in B or*
(4) *f and g are quasiregular in B and $|f(z)| > C|z - z_0|^\alpha$ in B with $\alpha < 1$.*

Then $k_{u^1} \in L^1(B)$ and the same result holds for u^2 with f replaced by g in the above assumptions.

Moreover, suppose that the singular set of u^1 consists of isolated critical points only and that there are no such points in $\{u^1 = c\} \cap B$. Then

$$L(s) \leq \int_{\Omega \cap B} |k_{u^1}| + 2\pi R. \tag{4.8}$$

The analogous estimate holds for u^2.

Remark 4.4. Note that the second parts of Assumptions (1), (2) and (3) above can be weaken, as we need $|f| > C$ to hold only on the level curve. Then, by the continuity of u there exists an open neighborhood, where the lower bound for $|f|$ holds as well. Therefore, in Theorem 4.3 it is enough to assume that $|f| > C$ on some open neighborhood of $\{u^1 = c\}$, only.

Proof of Theorem 4.3. Formula (4.4) implies that

$$|f||k_{u^1}| \leq |f_{\bar{z}}| + |f_z|.$$

From the linearization of p-harmonic system in (2.9) we infer that $f_{\bar{z}} = A_{11}(f,g) f_z + A_{12}(f, g) g_z + \overline{A_{11}(f, g) f_z} + \overline{A_{12}(f, g) g_z}$. Therefore,

$$|f||k_{u^1}| \leq (2|A_{11}(f, g)| + 1)|f_z| + 2|A_{12}(f, g)||g_z|.$$

Recall from (A.4) in Appendix A.1 that entries of matrix $A(f, g)$ can be estimated in terms of p only and hence the following inequalities hold:

$$\begin{aligned}
&|f||k_{u^1}| \leq \tfrac{2}{p}|f_z| + \tfrac{2-p}{p}|g_z| \leq \tfrac{2}{p}(|f_z| + |g_z|) && \text{if } 1 < p < 2, \\
&|f||k_{u^1}| \leq \tfrac{2(p-1)}{p}|f_z| + \tfrac{p-2}{p}|g_z| \leq \tfrac{2(p-1)}{p}(|f_z| + |g_z|) && \text{if } 2 \leq p \leq 3, \\
&|f||k_{u^1}| \leq \tfrac{p^2-p+2}{2p}|f_z| + \tfrac{(p-2)(p-1)}{p}|g_z| \leq \tfrac{p^2-p+2}{2p}(|f_z| + |g_z|) && \text{if } 3 < p.
\end{aligned} \tag{4.9}$$

Denote, by $A(p)$ the maximum of constants on the right hand sides of inequalities (4.9). Then, by the Hölder inequality we have that

$$\int_B |k_{u_1}| \leq 2A(p) \left(\int_B |f_z|^2 + |g_z|^2 \right)^{\frac{1}{2}} \left(\int_B \frac{1}{|f|^2} \right)^{\frac{1}{2}}. \tag{4.10}$$

It is then clear that Assumption (1) or Assumption (2) imply the assertion. So is Assumption (3), as if f and g are quasiregular in B, then $f_z, g_z \in L^2(B)$ (see *e.g.* [26]). If Assumption (4) holds, then integration in polar coordinates gives us the following estimate:

$$\int_{B(z_0,R)} \frac{1}{|f(z)|^2} dz \leq \tfrac{2\pi}{C} \int_0^R r^{1-2\alpha} dr = \tfrac{2\pi}{C} \tfrac{1}{2(1-\alpha)} R^{2(1-\alpha)} < \infty.$$

Inequality (4.10) then implies that $\|k_{u^1}\|_{L^1(B)} < \infty$.

By the discussion in [4] (see also in [23, Theorem 4.11]) we know, that if function u^1 defined on Ω has isolated critical points and none of them lies on the level curve $\{x \in \Omega : u^1(x) = c\} \cap G$ for $G \subset \Omega$, then the following "integration by parts" can be performed in a set G (here the definition of k_{u^1} in (4.1) is used as well):

$$\begin{aligned} - \int_{\{x \in G : u^1(x) < c\}} k_{u^1} dz &= \int_{\{x \in G : u^1(x) < c\}} \operatorname{div} \left(\frac{\nabla u^1}{|\nabla u^1|} \right) dz \\ &= \int_{\{x \in G : u^1(x) = c\}} ds + \int_{\{x \in \partial G : u^1(x) < c\}} \left\langle \frac{\nabla u^1}{|\nabla u^1|}, n \right\rangle ds, \end{aligned}$$

where n denotes the outer normal vector to ∂G. Using the notation of Corollary 3.6 we get (*cf.* in [4, Formula (v)]) that

$$L(c) := \text{length}(\{x \in G : u^1(x) = c\}) \leq \int_G |k_{u^1}| + \text{perimeter of } G. \tag{4.11}$$

Combining this inequality for $G = \Omega \cap B(z_0, R)$ with the above integrability result of k_{u_1} we obtain (4.8). □

Remark 4.5. In Section 3 in [3] (see also Appendix A.1 below) we discuss an inequality relating f_z and g_z which implies that f is quasiregular: $|g_z| \leq C(p)|f_z|$. This condition allows us to weaken Assumptions (3) and (4) and require only f to be quasiregular. Indeed, in the proof of Theorem 4.3 quasiregularity of g is used only to obtain L^2-integrability of g_z.

A similar simplification occurs when using the logarithmic representation (4.5) of k_{u^1}. Namely, if f is quasiregular, then components of f and $-\ln|f|$ satisfy certain elliptic equation, from which the integrability of $|\nabla \ln|f||$ can be inferred, see discussion in [23, Section 2]. The L^1-integrability of k_{u^1} then follows immediately from L^1-integrability of f_z (*cf.* Remark 4.1).

Remark 4.6. Observe that under the quasiregularity assumptions (3) or (4) of Theorem 4.3 the requirement for u^1 to have only isolated critical points is satisfied automatically, since quasiregular maps are discrete and open (see *e.g.* discussion in [23]).

5. Level curves and curves of steepest descent

In the previous section we defined k_v, the curvature of the level curves of function v. Similarly, one may introduce function h_v, the curvature of the orthogonal trajectories of the level curves (also called lines of steepest descent):

$$h_v = \frac{(v_{xx} - v_{yy})v_x v_y - v_{xy}((v_x)^2 - (v_y)^2)}{|\nabla v|^3}.$$

By considering a function

$$\phi_v = k_v + i h_v = -2\frac{\partial}{\partial z}\left(\frac{\overline{f}}{|f|}\right) \qquad \text{for } f = v_x - i v_y \tag{5.1}$$

we obtain yet another tool to analyze the geometry of solutions of partial differential equations, see *e.g.* [23,27]. Theorem 3 in [27] shows, that if v is a harmonic function without critical points, then ϕ_v satisfies certain nonlinear PDE. Moreover, properties of ϕ_v can be used to show that $\frac{k_v}{|\nabla v|}$ and $\frac{h_v}{|\nabla v|}$ are conjugate harmonic and that $-\ln|k_v|$ and $-\ln|h_v|$ are subharmonic. Theorem 3 in [27] has been partially extended by Lindqvist to the nonlinear setting of p-harmonic functions. Namely, Theorem 4.3 in [23] asserts that for a planar p-harmonic function u it holds that

$$\begin{aligned}\phi_u|f| &= |f|^2\frac{\partial}{\partial z}\left(-\frac{1}{f}\right) + \frac{p-2}{p}|f|^2\,\mathfrak{Re}\frac{\partial}{\partial z}\left(-\frac{1}{f}\right) \\ &= \tfrac{\overline{f}}{f}f_z + \tfrac{p-2}{2p}\left(\tfrac{\overline{f}}{f}f_z + \tfrac{f}{\overline{f}}\overline{f_z}\right), \qquad \text{when } f \neq 0.\end{aligned} \tag{5.2}$$

Since $|k_v| \leq |\phi_v|$, Lindqvist employed properties of ϕ_v together with the theory of quasiregular mappings and stream functions to prove the estimates for the length of level curves, similar to (4.8) above, in terms of integral of ϕ_v, see [23, (4.12)].

Theorem 5.1. *If $u = (u^1, u^2)$ is p-harmonic, then at the points where $f, g \neq 0$ it holds that*

$$|\phi_{u^1}||f| \leq C(p)(|f_z| + |g_z|), \qquad |\phi_{u^2}||g| \leq C(p)(|f_z| + |g_z|). \tag{5.3}$$

Proof. Computing the right-hand side of (5.1) we get that

$$\phi_{u^1}|f| = |f|^2\left(\frac{f_z}{f^2} - \frac{f_{\overline{z}}}{|f|^2}\right) = -f_{\overline{z}} + \frac{\overline{f}}{f}f_z. \tag{5.4}$$

From (2.9) we know that $f_{\overline{z}} = A_{11}(f,g)f_z + A_{12}(f,g)g_z + \overline{A_{11}(f,g)f_z} + \overline{A_{12}(f,g)g_z}$. Substituting this in (5.4) we obtain

$$\phi_{u^1}|f| = \left(\tfrac{\overline{f}}{f} - A_{11}(f,g)\right)f_z - A_{12}(f,g)g_z - \overline{A_{11}(f,g)f_z} - \overline{A_{12}(f,g)g_z}, \tag{5.5}$$

$$\phi_{u^2}|g| = A_{21}(f,g)f_z + \left(\tfrac{\overline{g}}{g} - A_{22}(f,g)\right)g_z - \overline{A_{21}(f,g)f_z} - \overline{A_{22}(f,g)g_z}.$$

Computing ϕ_{u^2} and then $g_{\overline{z}}$ from (2.9) results in the analogous formula for $\phi_{u^2}|g|$. The estimates for the entries of matrix $A(f, g)$ in (A.4) applied to equations (5.5) immediately give us the estimates (5.3). □

The case of p-harmonic functions can now be identified as a special case of Theorem 5.1. By using matrix (A.1) and (A.2) in Appendix A.1 we may easily check that if $u^2 \equiv 0$, and hence also $g \equiv 0$, then $A_{11}(f, 0) = \frac{2-p}{2p}\frac{\overline{f}}{f}$, whereas $A_{12}(f, 0) \equiv 0$. Thus, equation (5.5) reduces to (5.2).

In Theorem 5.1 we use equation (5.1) to extend the aforementioned [23, Theorem 4.5] to vectorial setting. Furthermore, under Assumptions (2) or (3) or (4) of Theorem 4.3 we may prove the similar integrability result for ϕ_{u^1} (ϕ_{u^2}) as obtained for k_{u^1} (k_{u^2}) and, in a consequence, obtain a counterpart of level curves length estimate (4.8) expressed in terms of functions ϕ_{u^1} (ϕ_{u^2}), respectively. This result generalizes [23, Theorem 4.11] on the integrability of ϕ_v for a p-harmonic function v in the plane.

We would like to emphasize that in the setting of p-harmonic maps the concept of stream functions (*cf.* [7]), used by Lindqvist [23] to extend the harmonic result to the nonlinear setting, is not available and it is only due to estimates for $A(f, g)$ in (A.4) for the operator form of p-harmonic map (2.9) that we are able to prove the above result and the mentioned counterpart of [23, Theorem 4.11].

6. The isoperimetric inequality for the level curves

Below we derive a variant of an isoperimetric inequality for p-harmonic mappings in the plane. In the setting of p-harmonic functions on annuli this type results are due to Alessandrini [5] and Longinetti [24] (*cf.* Remark 6.2). Our approach is based on the work by Laurence [19] and extends [5, 24]. To our best knowledge our isoperimetric inequality is new in the setting of coupled nonlinear systems. We hope, the techniques used here can be applied in the framework of more general systems of PDEs. In some parts of the proof we use the complex notation, but we do not appeal to the complex representation of p-harmonic system (2.8). This section is, therefore, selfcontained and independent of earlier results of the paper. Nevertheless, similarly to previous sections, the properties of quasiregular maps will appear to be vital in discussion.

Let v be a function from $\Omega \subset \mathbb{R}^2$ to $\mathbb{R}$ and define

$$\Omega_{a,b} = \{x \in \Omega : a < v(x) < b\}, \quad -\infty \leq a < b \leq \infty.$$

Recall the function of length of a level curve:

$$L(s) = \int_{\{x\in\Omega\,:\,v(x)=s\}} \mathrm{d}\mathcal{H}^1, \tag{6.1}$$

where $\mathrm{d}\mathcal{H}^1$ stands for the 1-Hausdorff measure. However, in what follows for the sake of simplicity we will often omit the measure in notation. Theorem 1 in [19] asserts, that for a function $v \in C^3(\overline{\Omega})$ such that v is constant on the boundary of Ω, it holds that if $|\nabla v| \geq c$ in $\Omega_{a,b}$ for some $c > 0$ and given a, b, then

$$L'(s) = \int_{\{x\in\Omega\,:\,v(x)=s\}} \operatorname{div}\left(\frac{\nabla v}{|\nabla v|}\right) \frac{d\mathcal{H}^1}{|\nabla v|}, \tag{6.2}$$

$$L''(s) = \int_{\{x\in\Omega\,:\,v(x)=s\}} \left[\operatorname{div}\left(\frac{\nabla v \Delta v}{|\nabla v|^3}\right) + \Delta\left(\frac{1}{|\nabla v|}\right)\right] \frac{d\mathcal{H}^1}{|\nabla v|}. \tag{6.3}$$

We would like to point out that the lower bound assumption on $|\nabla v|$ can be weaken (see part 2 of Remark 6.3 below). However, in what follows we will not explore this observation any further. In addition to (6.2) and (6.3) we will use two other interesting formulas, holding for C^3 functions (*cf.* in [4, (1.4) and (2.6)]):

$$\Delta \ln|\nabla v| = \operatorname{div}\left(\frac{\Delta v}{|\nabla v|^2}\nabla v\right),$$

$$\operatorname{div}\left(\frac{\Delta v}{|\nabla v|^3}\nabla v\right) + \Delta\left(\frac{1}{|\nabla v|}\right) = \left\langle \nabla\left(\frac{1}{|\nabla v|}\right), \frac{\Delta v}{|\nabla v|^2}\nabla v - \frac{1}{|\nabla v|}\nabla|\nabla v|\right\rangle. \tag{6.4}$$

To this end, we will focus our discussion on the case $v = u^1$, but the similar results hold for u^2 as well. Recall that, by f and g we denote the complex gradient of u^1 and u^2, respectively.

Theorem 6.1. *Let $\Omega' \subset B(z_0, R) \subset B(z_0, 4R) \Subset \Omega$ and let $u = (u^1, u^2)$ be a $C^3(\overline{\Omega})$-regular p-harmonic map. Suppose that the coordinate function u^1 is constant on $\partial\Omega'$ and that there exists a positive constant c such that $|\nabla u^1| > c$ in Ω'. Furthermore, let us assume that f and g are quasiregular in Ω' and consider $L(s)$ in* (6.1) *for $v = u_1$. Then the following formulas hold at the points, where $\nabla u^1 \neq 0$ and $\nabla u^2 \neq 0$.*
If $p = 2$, then

$$(\ln L(s))'' \geq 0. \tag{6.5}$$

Otherwise, if $p \neq 2$, then

$$L^{\frac{1}{p}}(s)\left(\tfrac{p}{p-1}\, L^{\frac{p-1}{p}}(s)\right)'' \geq -\frac{C}{R^{\frac{4}{p}}}, \tag{6.6}$$

where $C = C(c, p, \|Du\|_{L^p(B_{2R})}, \operatorname{dist}(\Omega', \partial\Omega))$ is positive.

Moreover if $u^2 \equiv 0$ or $u^1 \equiv u^2$, and Ω' is a circular annulus, then the inequalities in (6.5) *and* (6.6) *become equalities, with $C = 0$, is the latter; and, in this case, u^1 is a radial p-harmonic function on Ω'.*

This theorem generalizes the linear case of harmonic functions, as well as the case of p-harmonic functions for $p \neq 2$.

Remark 6.2.

1. If $p = 2$ and Ω is a planar annulus, then we retrieve the well known harmonic case discussed for instance by Laurence [19, Theorem 6] and Alessandrini [5, formula (1.3a), Theorem 1.1].
2. If $p \neq 2$ and $u^2 \equiv 0$ or $u^1 \equiv u^2$, then u degenerates to a p-harmonic function u^1. In such a case, the analysis of steps of the proof below allows us to retrieve the nonlinear part of assertion in [5, formula (1.3b), Theorem 1.1] with $\Lambda = p - 1$. In [5], it is assumed that Ω is an annuli. This is because for such Ω it can be showed that the gradient norm is strictly positive (the argument goes back to Lewis [22]), and therefore (6.2) and (6.3) can be applied. Instead, in Theorem 6.1 we localize the discussion on the subset $\Omega' \Subset \Omega$ and assume that $|\nabla u^1| > c > 0$ (see also the discussion of equality in the proof of Theorem 6.1).
3. In Theorem 6.1 we may assume that $s \leq \max_{\partial\Omega'} u^1$ due to the maximum principle for coordinates of p-harmonic maps (see Observation 2 in Appendix A.2).

Let us comment the assumptions and hypothesis of the theorem.

Remark 6.3.

1. The C^3-regularity of u is assumed in order to be able to apply formulas (6.2) and (6.3) for L' and L'', respectively.
2. According to [19, Remark on page 266], the assumption that $|\nabla u^1| > c$ can be weaken due to the Sard theorem. Namely, one can require formulas (6.2) and (6.3) to hold only for the regular values of u^1. Since, let t be a regular value of u^1. Then there exist $\epsilon > 0$ and $c > 0$ such that $|\nabla u^1| > c$ on the set $\{x \in \Omega : t - \epsilon < u^1(x) < t + \epsilon\}$. Furthermore, by [19, Remark 2 on page 267], the assumption on lower bounds for $|\nabla u^1|$ can be replaced by the integrability condition of a suitable power of the gradient of u^1.
3. We require Ω' to be enough far away from the boundary of Ω, since in the proof we use the following important estimate for p-harmonic maps due to Uhlenbeck [28, Theorem 3.2]:
$$\sup_{B_R} |Du| \leq \frac{C}{R^{\frac{2}{p}}} \|Du\|_{L^p(B_{2R})}, \tag{6.7}$$
where $C(p)$ is a constant in the Sobolev imbedding theorem.
4. One of the assumptions of Theorem 6.1 is that f and g are quasiregular maps. Using computations similar to [3, formula (17), Section 3] we may determine conditions under which complex gradients f and g of coordinate functions of a p-harmonic map are quasiregular (*cf.* Remark 4.5).

Proof of Theorem 6.1. The definition of the p-harmonic operator (2.3) and the representation of the p-harmonic system (2.6) allow us to write the first equation of such system as follows:

$$
\begin{aligned}
0 = |\nabla u^1|^2 &\left(\Delta u^1 + (p-2)\left\langle \nabla|\nabla u^1|, \frac{\nabla u^1}{|\nabla u^1|}\right\rangle\right) \\
&+ |\nabla u^2|^2 \left(\Delta u^1 + (p-2)\left\langle \nabla u^1, \frac{\nabla|\nabla u^2|}{|\nabla u^2|}\right\rangle\right).
\end{aligned}
\tag{6.8}
$$

From this equation we compute the Laplacian of u^1 and use (4.1), (4.2) and (6.2) to obtain the formula for $L'(s)$:

$$
\begin{aligned}
L'(s) &= \int_{\{u^1=s\}} \operatorname{div}\left(\frac{\nabla u^1}{|\nabla u^1|}\right)\frac{\mathrm{d}s}{|\nabla u^1|} \\
&= \int_{\{u^1=s\}} \frac{1}{|\nabla u^1|}\left(\frac{\Delta u^1}{|\nabla u^1|} - \frac{1}{|\nabla u^1|}\left\langle \nabla|\nabla u^1|, \frac{\nabla u^1}{|\nabla u^1|}\right\rangle\right) \\
&= -\int_{\{u^1=s\}} \frac{1}{|\nabla u^1|^2}\Bigg[(p-2)\frac{|\nabla u^1|^2}{|Du|^2}\left\langle \nabla|\nabla u^1|, \frac{\nabla u^1}{|\nabla u^1|}\right\rangle \\
&\quad + (p-2)\frac{|\nabla u^2|^2}{|Du|^2}\left\langle \nabla u^1, \frac{\nabla|\nabla u^2|}{|\nabla u^2|}\right\rangle + \left\langle \nabla|\nabla u^1|, \frac{\nabla u^1}{|\nabla u^1|}\right\rangle\Bigg].
\end{aligned}
\tag{6.9}
$$

Next we compute $L''(s)$. Combining (6.3) with (6.4) and (6.8), together with the fact that $\nabla(|\nabla u^1|^{-1}) = -\frac{\nabla|\nabla u^1|}{|\nabla u^1|^2}$ we obtain equation:

$$
\begin{aligned}
L''(s) &= \int_{\{u^1=s\}} \left\langle \nabla\left(\frac{1}{|\nabla u^1|}\right), \frac{\Delta u^1}{|\nabla u^1|^2}\nabla u^1 - \frac{\nabla|\nabla u^1|}{|\nabla u^1|}\right\rangle \frac{\mathrm{d}s}{|\nabla u^1|} \\
&= \int_{\{u^1=s\}} \frac{1}{|\nabla u^1|}\Bigg\langle \nabla\left(\frac{1}{|\nabla u^1|}\right), -(p-2)\frac{|\nabla u^1|^2}{|Du|^2}\left\langle \nabla|\nabla u^1|, \frac{\nabla u^1}{|\nabla u^1|}\right\rangle\frac{\nabla u^1}{|\nabla u^1|^2} \\
&\qquad - (p-2)\frac{|\nabla u^2|^2}{|Du|^2}\left\langle \nabla u^1, \frac{\nabla|\nabla u^2|}{|\nabla u^2|}\right\rangle\frac{\nabla u^1}{|\nabla u^1|^2} - \frac{\nabla|\nabla u^1|}{|\nabla u^1|}\Bigg\rangle. \\
&= \int_{\{u^1=s\}} \frac{1}{|\nabla u^1|^4}\Bigg[(p-2)\frac{|\nabla u^1|^2}{|Du|^2}\left\langle \nabla|\nabla u^1|, \frac{\nabla u^1}{|\nabla u^1|}\right\rangle^2 \\
&\qquad + (p-2)\frac{|\nabla u^2|^2}{|Du|^2}\left\langle \nabla u^1, \frac{\nabla|\nabla u^2|}{|\nabla u^2|}\right\rangle\left\langle \nabla|\nabla u^1|, \frac{\nabla u^1}{|\nabla u^1|}\right\rangle \\
&\qquad + \big|\nabla|\nabla u^1|\big|^2\Bigg].
\end{aligned}
$$

In order to simplify the discussion, we introduce the following notation for the terms of $L''(s)$:

$$
\begin{aligned}
A_u &:= (p-2)\frac{|\nabla u^1|^2}{|Du|^2}\left\langle \nabla|\nabla u^1|, \frac{\nabla u^1}{|\nabla u^1|}\right\rangle^2, \\
B_u &:= (p-2)\frac{|\nabla u^2|^2}{|Du|^2}\left\langle \nabla u^1, \frac{\nabla|\nabla u^2|}{|\nabla u^2|}\right\rangle\left\langle \nabla|\nabla u^1|, \frac{\nabla u^1}{|\nabla u^1|}\right\rangle, \\
C_u &:= \big|\nabla|\nabla u^1|\big|^2.
\end{aligned}
\tag{6.10}
$$

With this notation $L''(s)$ reads:

$$
L''(s) = \int_{\{u^1=s\}} \frac{1}{|\nabla u^1|^4}\,(A_u + B_u + C_u)\,. \tag{6.11}
$$

Using the Hölder inequality at (6.9) we obtain the following estimate:

$$
\begin{aligned}
\frac{(L'(s))^2}{L(s)} \le \int_{\{u^1=s\}} \frac{1}{|\nabla u^1|^4}\Bigg[&(p-2)^2\frac{|\nabla u^1|^4}{|Du|^4}\left\langle \nabla|\nabla u^1|, \frac{\nabla u^1}{|\nabla u^1|}\right\rangle^2 \\
&+ (p-2)^2\frac{|\nabla u^2|^4}{|Du|^4}\left\langle \nabla u^1, \frac{\nabla|\nabla u^2|}{|\nabla u^2|}\right\rangle^2 + \left\langle \nabla|\nabla u^1|, \frac{\nabla u^1}{|\nabla u^1|}\right\rangle^2 \\
&+ 2(p-2)^2\frac{|\nabla u^1|^2|\nabla u^2|^2}{|Du|^4}\left\langle \nabla|\nabla u^1|, \frac{\nabla u^1}{|\nabla u^1|}\right\rangle\left\langle \nabla u^1, \frac{\nabla|\nabla u^2|}{|\nabla u^2|}\right\rangle \\
&+ 2(p-2)\frac{|\nabla u^2|^2}{|Du|^2}\left\langle \nabla|\nabla u^1|, \frac{\nabla u^1}{|\nabla u^1|}\right\rangle\left\langle \nabla u^1, \frac{\nabla|\nabla u^2|}{|\nabla u^2|}\right\rangle \\
&+ 2(p-2)\frac{|\nabla u^1|^2}{|Du|^2}\left\langle \nabla|\nabla u^1|, \frac{\nabla u^1}{|\nabla u^1|}\right\rangle^2\Bigg].
\end{aligned}
\tag{6.12}
$$

Using notation (6.10) we express the above inequality in a more suitable and compact form:

$$
\frac{(L'(s))^2}{L(s)} \le \int_{\{u^1=s\}} \frac{1}{|\nabla u^1|^4}\left((p-2)\frac{|\nabla u^1|^2}{|Du|^2}A_u + C_u + 2B_u + 2A_u + E_u\right), \tag{6.13}
$$

where E_u stands for the sum of the remaining terms in formula (6.12):

$$E_u = 2(p-2)^2 \frac{|\nabla u^1|^2 |\nabla u^2|^2}{|Du|^4} \left\langle \nabla |\nabla u^1|, \frac{\nabla u^1}{|\nabla u^1|} \right\rangle \left\langle \nabla u^1, \frac{\nabla |\nabla u^2|}{|\nabla u^2|} \right\rangle$$
$$+ (p-2)^2 \frac{|\nabla u^2|^4}{|Du|^4} \left\langle \nabla u^1, \frac{\nabla |\nabla u^2|}{|\nabla u^2|} \right\rangle^2$$
$$= (p-2)^2 \left(\frac{|\nabla u^2|^2}{|Du|^2} \left\langle \nabla u^1, \frac{\nabla |\nabla u^2|}{|\nabla u^2|} \right\rangle + \frac{|\nabla u^1|^2}{|Du|^2} \left\langle \nabla |\nabla u^1|, \frac{\nabla u^1}{|\nabla u^1|} \right\rangle \right)^2$$
$$- (p-2)^2 \frac{|\nabla u^1|^4}{|Du|^4} \left\langle \nabla |\nabla u^1|, \frac{\nabla u^1}{|\nabla u^1|} \right\rangle^2 .$$

By the Schwarz inequality we have that

$$\left| \frac{|\nabla u^2|^2}{|Du|^2} \left\langle \nabla u^1, \frac{\nabla |\nabla u^2|}{|\nabla u^2|} \right\rangle \right| \leq \frac{|\nabla u^1|}{|Du|} \frac{|\nabla |\nabla u^2|^2|}{2|Du|} \leq \frac{|\nabla |\nabla u^2|^2|}{2|Du|}.$$

Therefore, E_u can be estimated as follows:

$$E_u \leq (p-2)^2 \left(\frac{\left|\nabla |\nabla u^2|^2\right|}{2|Du|} + \left|\nabla |\nabla u^1|\right| \right)^2 \leq 2(p-2)^2 \frac{\left|\nabla |\nabla u^2|^2\right|^2}{|\nabla u^1|^2} + 2(p-2)^2 C_u.$$

As a consequence, inequality (6.13) becomes:

$$\frac{(L'(s))^2}{L(s)} \leq \int_{\{u^1=s\}} \frac{1}{|\nabla u^1|^4} \Bigg((p-2) \frac{|\nabla u^1|^2}{|Du|^2} A_u + C_u + 2B_u + 2A_u$$
$$+ 2(p-2)^2 C_u + 2(p-2)^2 \frac{\left|\nabla |\nabla u^2|^2\right|^2}{|\nabla u^1|^2} \Bigg). \tag{6.14}$$

We may now proceed to the crucial inequality combining (6.11) and (6.14):

$$\frac{(L'(s))^2}{L(s)}$$
$$\leq \int_{\{u^1=s\}} \frac{1}{|\nabla u^1|^4} \left(pA_u + 2B_u + (1+2(p-2)^2) C_u + 2(p-2)^2 \frac{\left|\nabla |\nabla u^2|^2\right|^2}{|\nabla u^1|^2} \right)$$
$$= \int_{\{u^1=s\}} \frac{1}{|\nabla u^1|^4} \, p(A_u + B_u + C_u)$$
$$+ \int_{\{u^1=s\}} \frac{1}{|\nabla u^1|^4} \left((2-p)B_u + (1-p+2(p-2)^2) C_u + 2(p-2)^2 \frac{\left|\nabla |\nabla u^2|^2\right|^2}{|\nabla u^1|^2} \right). \tag{6.15}$$

In order to complete the above estimate we need the following upper bound on B_u:

$$\begin{aligned}|B_u| &\le |p-2|\frac{|\nabla u^1||\nabla u^2|}{|Du|^2}\,\big|\nabla|\nabla u^1|\big|\,\big|\nabla|\nabla u^2|\big| \\ &\le |p-2|\big|\nabla|\nabla u^1|\big|\frac{|\nabla|\nabla u^2|^2|}{2|\nabla u^1|} \\ &\le \frac{|p-2|}{4}\big|\nabla|\nabla u^1|\big|^2+|p-2|\frac{\big|\nabla|\nabla u^2|^2\big|^2}{4|\nabla u^1|^2}.\end{aligned}$$

Using this inequality in (6.15) we obtain:

$$\begin{aligned}\frac{(L'(s))^2}{L(s)} \le &\int_{\{u^1=s\}}\frac{1}{|\nabla u^1|^4}\,p(A_u+B_u+C_u) \\ &+\Big(\tfrac{9}{4}(p-2)^2+1-p\Big)\int_{\{u^1=s\}}\frac{C_u}{|\nabla u^1|^4} \\ &+\tfrac{9}{4}(p-2)^2\int_{\{u^1=s\}}\frac{\big|\nabla|\nabla u^2|^2\big|^2}{|\nabla u^1|^6}.\end{aligned}\tag{6.16}$$

Upon defining

$$\alpha(p):=\tfrac{9}{4}(p-2)^2+1-p \quad\text{and}\quad \beta(p):=\tfrac{9}{4}(p-2)^2,$$

we arrive at the inequality

$$\frac{(L'(s))^2}{L(s)}\le pL''(s)+\alpha(p)\int_{\{u^1=s\}}\frac{\big|\nabla|\nabla u^1|\big|^2}{|\nabla u^1|^4}+\beta(p)\int_{\{u^1=s\}}\frac{\big|\nabla|\nabla u^2|^2\big|^2}{|\nabla u^1|^6}.$$

With such $\alpha(p)$ and $\beta(p)$, if $p=2$ (*i.e.* $\alpha(p)=-1$, $\beta(p)=0$) we retrieve (6.5), the harmonic case of the hypothesis (see also Remark 6.2 above). Indeed, for $p=2$ inequality (6.16) reads (*cf.* equation (6.11))

$$\frac{(L'(s))^2}{L(s)}\le\int_{\{u^1=s\}}\frac{\big|\nabla|\nabla u^1|\big|^2}{|\nabla u^1|^4}=L''(s),$$

and thus

$$(L'(s))^2-L(s)L''(s)\le 0 \Leftrightarrow (\,\ln L(s)\,)''\ge 0.$$

Whereas, if $p \neq 2$, we have:

$$\begin{aligned}\left(\tfrac{p}{p-1}L^{\frac{p-1}{p}}(s)\right)'' &= -\tfrac{1}{p}L^{-1-\frac{1}{p}}(s)\left((L'(s))^2 - pL(s)L''(s)\right)\\ &\geq -L^{-\frac{1}{p}}(s)\left(\frac{\alpha(p)}{p}\int_{\{u^1=s\}}\frac{\left|\nabla|\nabla u^1|\right|^2}{|\nabla u^1|^4}\right.\\ &\qquad\left.+\frac{\beta(p)}{p}\int_{\{u^1=s\}}\frac{\left|\nabla|\nabla u^2|^2\right|^2}{|\nabla u^1|^6}\right).\end{aligned} \tag{6.17}$$

Let us now turn to estimates for the right-hand side of (6.17) and first analyze the last term of this inequality. Using the complex notation and the assumption that g, the complex gradient of u^2, is quasiregular we have that

$$|\nabla|\nabla u^2|^2| = 8|(|g|^2)_z| \leq 8|g|(|g_z| + |g_{\bar{z}}|) = 8|g||g_z|\left(1 + \tfrac{|g_{\bar{z}}|}{|g_z|}\right) < 16|g||g_z|.$$

Then

$$\int_{\{u^1=s\}}\frac{|\nabla|\nabla u^2|^2|^2}{|\nabla u^1|^6} \leq \int_{\{u^1=s\}}\frac{4|g|^2|g_z|^2}{|f|^6}.$$

By the assumptions, it holds that $2|f| = |\nabla u^1| > c$. From this and from the Hölder inequality we immediately obtain the following estimate:

$$\int_{\{u^1=s\}}\frac{4|g|^2|g_z|^2}{|f|^6} \leq \frac{256}{c^6}\,(\sup_{\{u^1=s\}}|g|^2)\int_{\{u^1=s\}}|g_z|^2. \tag{6.18}$$

The first factor on the right-hand side can be estimated by the Uhlenbeck inequality (6.7). Moreover, the same inequality allows us to estimate also the second integral in (6.18), as if a quasiregular transformation g is bounded (and here $|g| < 2|Du|$ in Ω), then $\|g\|_{W^{1,2}(\Omega')} < C(\|g\|_{L^\infty(\Omega')}, \operatorname{dist}(\Omega', \partial\Omega)) < \infty$ (see *e.g.* [20, 26]). Hence,

$$\int_{\{u^1=s\}}\frac{|\nabla|\nabla u^2|^2|^2}{|\nabla u^1|^6} \leq \frac{256}{c^6}\,(\sup_{\{u^1=s\}}|g|^2)\int_{\{u^1=s\}}|g_z|^2 \leq \frac{C}{R^{\frac{4}{p}}}\|Du\|^2_{L^p(B_{2R})}, \tag{6.19}$$

where $C = C(c, \|Du\|_{L^p(B_{2R})}, \operatorname{dist}(\Omega', \partial\Omega))$.

In order to estimate the first term on the right-hand side of (6.17), we again use the complex notation. Then, quasiregularity of f and the assumption that $2|f| = |\nabla u^1| > c$ imply that

$$\frac{\left|\nabla|\nabla u^1|\right|^2}{|\nabla u^1|^4} = \frac{|f_z|^2}{4|f|^4}\left(1 + \frac{|f_{\bar{z}}|}{|f_z|}\right)^2 \leq \frac{1}{c^4}|f_z|^2.$$

Discussion similar to that in the paragraph following (6.18) leads us to inequality

$$\int_{\{u^1=s\}} \frac{\big|\nabla|\nabla u^1|\big|^2}{|\nabla u^1|^4} \le \frac{C}{R^{\frac{4}{p}}} \|Du\|^2_{L^p(B_{2R})}, \tag{6.20}$$

with $C = C(c, \|Du\|_{L^p(B_{2R})}, \mathrm{dist}(\Omega', \partial\Omega))$. Applying inequalities (6.20) and (6.19) in (6.17), we obtain the first part of assertion:

$$\left(\frac{p}{p-1} L^{\frac{p-1}{p}}(s)\right)'' \ge -\frac{C}{R^{\frac{4}{p}}} L^{-\frac{1}{p}}(s).$$

Here the constant $C = C(c, p, |\alpha(p)| + |\beta(p)|, \|Du\|_{L^p(B_{2R})}, \mathrm{dist}(\Omega', \partial\Omega))$. Let us now discuss the case of equality in (6.6). Let $u^2 \equiv 0$ or $u^1 \equiv u^2$. Then the assertion of theorem reduces to the case of p-harmonic functions previously discussed in [5] and $L'(s)$ and $L''(s)$ take the following form (*cf.* formulas in the proof of [5, Theorem 1.1]):

$$L'(s) = -\int_{\{u^1=s\}} \frac{p-1}{|\nabla u^1|^2} \left\langle \nabla|\nabla u^1|, \frac{\nabla u^1}{|\nabla u^1|} \right\rangle,$$

$$L''(s) = \int_{\{u^1=s\}} \frac{1}{|\nabla u^1|^4} \left[(p-2) \left\langle \nabla|\nabla u^1|, \frac{\nabla u^1}{|\nabla u^1|} \right\rangle^2 + \big|\nabla|\nabla u^1|\big|^2 \right].$$

By discussion in the proof of [19, Theorem 6] and [5, Theorem 1.1], we know that equalities in [5, formulas (1.3a) and (1.3b)] hold provided that the level curves are circles (see also Remark 6.2). This observation leads us to two cases: either Ω' is a ball or an annulus. In the first case, the assumption that $u^1 = k$ on $\partial\Omega'$ together with the maximum principle for p-harmonic functions (see *e.g.* [15, Chapter 6]) imply that $u^1 \equiv k$ in Ω'. If Ω' is an annulus, then since u^1 is constant on two components of the boundary of Ω', the boundary data is rotationally invariant. This, together with the uniqueness of Dirichlet problem for the strictly convex p-harmonic energy $\int_{\Omega'} |\nabla u^1|^p$ implies that the solution inside Ω' is a radial p-harmonic function

$$u^1(r) = c_1 H(r) + c_2, \quad \text{where} \quad H(r) = r^{\frac{1}{1-p}} \quad \text{and} \quad r = \sqrt{x^2+y^2},$$

whereas constants c_1 and c_2 depend on the values of u^1 on $\partial\Omega'$. Easy computations reveal that for such radial u^1 it holds that $L'(s) = -4\pi(p-1)s \ln H'(s)$, $L''(s) = 8\pi(p-1)s\left(\ln H'(s)\right)^2$ and so

$$\frac{(L'(s))^2}{L(s)} = (p-1)L''(s). \tag{6.21}$$

Now, if $p = 2$, then the last equation reads: $\frac{(L'(s))^2}{L(s)} - L''(s) = 0 \Leftrightarrow (\ln L(s))'' = 0$, resulting in the equality in (6.5). If $p \ne 2$, then (6.21) can be equivalently written as $\left(\frac{p-1}{p-2}L(s)^{\frac{p-2}{p-1}}\right)'' = 0$. We, therefore, retrieve [5, formula (1.3b)] and the claim follows. □

A. Appendix

A.1. The matrix $A(f, g)$

One of the main observations used throughout the paper is that one can associate with the p-harmonic system in the plane a quasilinear system (2.8) and a matrix $A(f, g)$ (see for instance the work of Vekua [31] for more applications of such an approach as well as Section 2 in Alessandrini-Magnanini [6]). For the readers convenience we now recall the estimates for the entries of matrix $A(f, g)$ in formula (2.9) and necessary definitions and notation (*cf.* [3] for the complete discussion). Additionally, we improve the norm estimates for $A(f, g)$ comparing to [3, Theorem 2], as now we allow $1 < p < \infty$. This extension is due to C^2 assumption on mapping u. Indeed, in [3] we need Lemma 1 in order to infer the higher regularity of auxiliary expression depending on Du, available only for $p \geq 2$, due to techniques we use. However, if u is C^2, then [3, Lemma 1] is no longer needed to formulate system (2.8). Let us also comment, that in the setting of planar p-harmonic functions the similar analysis for a Sobolev solutions in the full range of parameter p is possible, if one uses the stream functions [8] or a variational approach [26], both unknown in the vectorial setting.

The following matrix A is introduced for the purpose of solving system (2.8) in the operator form (2.9):

$$A(f,g) = \frac{2-p}{\Phi}\begin{bmatrix} B\frac{\overline{f}}{f} + (2-p)D & \frac{\overline{g}}{f}\left(B\frac{\overline{f}}{f} + (2-p)D\right) \\ \frac{\overline{f}}{Bg}\left(\Phi + (2-p)^2C\right) + (2-p)\frac{\overline{f}}{\overline{g}}D & \frac{\overline{g}}{Bg}\left(\Phi + (2-p)^2C\right) + (2-p)D \end{bmatrix} \tag{A.1}$$

for

$$\begin{aligned} \Phi &:= \Phi(f,g) = \left(2p + \frac{4|g|^2}{|f|^2}\right)\left(2p + \frac{4|f|^2}{|g|^2}\right) - (2-p)^2\left(2 + \frac{\overline{g}}{g}\frac{f}{\overline{f}} + \frac{g}{\overline{g}}\frac{\overline{f}}{f}\right), \\ B &:= B(f,g) = 2p + 4\frac{|f|^2}{|g|^2}, \\ C &:= C(f,g) = 2 + \frac{\overline{g}}{g}\frac{f}{\overline{f}} + \frac{g}{\overline{g}}\frac{\overline{f}}{f}, \\ D &:= D(f,g) = \frac{\overline{g}}{g} + \frac{\overline{f}}{f}. \end{aligned} \tag{A.2}$$

As in [3] we show that

$$|\Phi| \geq 16p + 8p\left(\frac{|f|^2}{|g|^2} + \frac{|g|^2}{|f|^2}\right).$$

Then

$$\begin{aligned}
|A_{11}(f,g)| &= \left|\frac{(2-p)\big[\big(2p+\frac{4|f|^2}{|g|^2}\big)\frac{\overline{f}}{f} + (2-p)\big(\frac{\overline{g}}{g}+\frac{\overline{f}}{f}\big)\big]}{\big(2p+\frac{4|g|^2}{|f|^2}\big)\big(2p+\frac{4|f|^2}{|g|^2}\big) - (2-p)^2\big(2+\frac{\overline{g}}{g}\frac{f}{\overline{f}}+\frac{g}{\overline{g}}\frac{\overline{f}}{f}\big)}\right| \\
&\leq \frac{|2-p|\big[2p+\frac{4|f|^2}{|g|^2}+2|2-p|\big]}{16p+8p\big(\frac{|f|^2}{|g|^2}+\frac{|g|^2}{|f|^2}\big)} \\
&\leq \frac{2|2-p|(p+|2-p|)|g|^2|f|^2+4|2-p||f|^4}{8p(|f|^2+|g|^2)^2} \\
&\leq (|f|^2+|g|^2)^2\frac{\max\{4|2-p|,(p+|2-p|)|2-p|\}}{8p(|f|^2+|g|^2)^2}.
\end{aligned} \tag{A.3}$$

Now, we find that

$$A_p := |A_{11}(f,g)| \leq \begin{cases} \frac{2-p}{2p} & \text{for} \quad 1 < p < 2, \\ \frac{p-2}{2p} & \text{for} \quad 2 \leq p \leq 3, \\ \frac{(p-2)(p-1)}{4p} & \text{for} \quad 3 < p. \end{cases} \tag{A.4}$$

Similarly we find that the remaining entries $A_{12}(f,g)$, $A_{21}(f,g)$, $A_{22}(f,g)$ satisfy the same estimates in the corresponding ranges of p.

Remark A.1. Formulas (10) and (11) in [3] are slightly different then the above estimates for $A(f,g)$, but one can show that in fact we have now improved estimates used in the proof in [3, Theorem 2].

The definition of quasiregular maps in terms of the Beltrami coefficient (1.2) together with the above estimates allow us to describe when the complex gradients f and g are quasiregular (see [3, Section 3] for more details). Indeed, as mentioned in Section 2, system of equations (2.8) can be solved with the help of matrix $A(f,g)$ resulting in equation (2.9):

$$\begin{bmatrix} f \\ g \end{bmatrix}_{\overline{z}} = A(f,g)\begin{bmatrix} f \\ g \end{bmatrix}_z + \overline{A(f,g)\begin{bmatrix} f \\ g \end{bmatrix}_z}.$$

From this, we have that

$$\begin{aligned}
|f_{\overline{z}}| &\leq |A_{11}(f,g)||f_z| + |A_{12}(f,g)||g_z| + |\overline{A_{11}(f,g)}||\overline{f_z}| + |\overline{A_{12}(f,g)}||\overline{g_z}| \\
&\leq 2A_p\left(|f_z|+|g_z|\right),
\end{aligned} \tag{A.5}$$

where A_p is as in (A.4). From inequality (A.5) we immediately obtain that $\frac{|f_{\bar{z}}|}{|f_z|} \leq 2A_p\left(1+\frac{|g_z|}{|f_z|}\right) < 1$ provided that $\frac{|g_z|}{|f_z|} < \frac{1-2A_p}{A_p}$. The similar condition can be derived for g.

A.2. Maximum principle for coordinate functions of p-harmonic maps

The purpose of this short section is to show the maximum principle for the coordinate functions of the p-harmonic mapping. We use this principle in part (3) of Remark 6.2. To our best knowledge this result has not appeared in the literature so far.[1] In the proof below we will use the approach by Leonetti and Siepe, see the proofs of [21, Theorems 2.1 and 2.2].

Observation 2. Let $u \in W^{1,p}(\Omega, \mathbb{R}^2)$ be a p-harmonic mapping in the domain $\Omega \subset \mathbb{R}^2$. If for some u^i, $i = 1, 2$ there exists $k \in \mathbb{R}$ such that $u^i \leq k$ on $\partial\Omega$, then $u^i \leq k$ in Ω.

Before giving the proof, let us state the following remark.

Remark A.2. Let $v \in W^{1,p}(\Omega, \mathbb{R})$. Then the assumption $v \leq l$ on $\partial\Omega$ means that there exists a sequence $\{v_k\}$ of a Lipschitz functions on the closure of Ω such that $v_k(x) \leq l$ for every $x \in \partial\Omega$, for each $k \in \mathbb{N}$ and $\|v - v_k\|_{W^{1,p}(\Omega,\mathbb{R})} \to 0$, as $k \to \infty$.

Proof of Observation 2. Without loss of generality, let us assume that $i = 1$. Consider the following perturbation of mapping u:

$$\tilde{u} = (u^1 + \phi, u^2),$$

where $\phi = -\max\{u^1 - k, 0\}$. As $\max\{u^1 - k, 0\} \in W_0^{1,p}(\Omega, \mathbb{R})$ we have that u and $\tilde{u}$ have the same trace. Define sets

$$\Omega_1 = \{u^1 \leq k\} \cup \{u^1 > k,\ \nabla u^1 = 0\}, \qquad \Omega_2 = \Omega \setminus \Omega_1.$$

Then

$$|D\tilde{u}|^p = |Du|^p \quad \text{on } \Omega_1 \text{ a.e.} \quad \text{and} \quad |D\tilde{u}|^p < |Du|^p \quad \text{on } \Omega_2 \text{ a.e.}$$

Uniqueness of the p-harmonic minimizer implies that $|\Omega_2| = 0$. From this we obtain that $\nabla\phi = 0$ a.e. in Ω. Since $\phi \in W_0^{1,p}(\Omega, \mathbb{R})$ we have by the Poincaré inequality

$$\|\phi\|_{L^p(\Omega)} \leq c(p)|\Omega|^{\frac{1}{2}}\|\nabla\phi\|_{L^p(\Omega)} = 0.$$

Thus $\phi = 0$ a.e. in Ω and the definition of ϕ immediately implies that $|\{x \in \Omega |\, u^1 > k\}| = 0$, completing the proof. □

[1] This section is adapted from [2, Section 4.4]. The result holds for all dimensions $n \geq 2$.

A.3. Radial p-harmonic surfaces and Theorem 3.1

In Remark 3.7 we mention the difficulty with finding wide classes of nontrivial examples when dealing with the p-harmonic world. The class of p-harmonic solutions that comes most in handy is the one of radial transformations. In Observation 3 we use radial p-harmonic surfaces to show that Theorem 3.1 may fail beyond the range of parameter $p \in \left\langle \frac{4}{3}, 2+\sqrt{2} \right\rangle$. Let

$$u(x, y) = (u^1, u^2) = (H(r)x, H(r)y), \quad \text{for} \quad r = \sqrt{x^2 + y^2}$$

be a radial map in a planar domain. For such u the p-harmonic system (1.1) reduces to a single ODE:

$$\begin{aligned}(p-1)H''(H')^2r^3 + (2p-1)(H')^3r^2 + 2(p-1)HH'H''r^2 \\ + (5p-4)H(H')^2r + pH^2H''r + 3pH^2H' = 0.\end{aligned} \tag{A.6}$$

The following formulas hold for u^1:

$$\begin{aligned}
\nabla u^1 &= \big(H'(r)\frac{x^2}{r} + H(r), H'(r)\frac{xy}{r}\big), \\
u^1_{xx} &= H''(r)\frac{x^3}{r^2} + H'(r)\frac{2y^3 + 3x^2y}{r^3}, \\
u^1_{xy} &= H''(r)\frac{x^2y}{r^2} + H'(r)\frac{y^3}{r^3}, \\
u^1_{yy} &= H''(r)\frac{xy^2}{r^2} + H'(r)\frac{x^3}{r^3}.
\end{aligned}$$

Similarly we find ∇u^2 and $u^2_{xx}, u^2_{xy}, u^2_{yy}$. After lengthy computations, we arrive at equations for Hessian determinants of u^1 and u^2:

$$\begin{aligned}
\det H(u^1) &= H'(r)H''(r)\frac{x^2}{r} + (H'(r))^2\frac{2x^2 - y^2}{r^2}, \\
\det H(u^2) &= H'(r)H''(r)\frac{y^2}{r} + (H'(r))^2\frac{2y^2 - x^2}{r^2}.
\end{aligned}$$

Observation 3. If $p > 6 + 4\sqrt{2}$, then there exist a constant $c > 1$ and a radial p-harmonic map $u = (u^1, u^2) = H(r)(x, y)$ with $H' \leq 0$ defined in the domain $\Omega \subset \{(x, y) \in \mathbb{R}^2 : cy^2 > x^2 > y^2\}$ such that $\det H(u^2) \geq 0$ and $\det H(u^1) \geq 0$. Thus, Theorem 3.1 does not hold in general.

Proof. Using the above computations for Hessians of u^1 and u^2, the proof reduces to finding u with the following properties:

$$\begin{aligned}
\det H(u^2) \geq 0 \Leftrightarrow & \left(H' \geq 0 \text{ and } H''r + H'\left(2 - \frac{x^2}{y^2}\right) \geq 0\right) \text{ or} \\
& \left(H' \leq 0 \text{ and } H''r + H'\left(2 - \frac{x^2}{y^2}\right) \leq 0\right), \\
\det H(u^1) \geq 0 \Leftrightarrow & \left(H' \leq 0 \text{ and } H''r + H'\left(2 - \frac{y^2}{x^2}\right) \leq 0\right) \text{ or} \\
& \left(H' \geq 0 \text{ and } H''r + H'\left(2 - \frac{y^2}{x^2}\right) \geq 0\right).
\end{aligned} \tag{A.7}$$

For the simplicity of discussion, from now on we will assume that $H > 0$. Such an assumption is justified by the fact that if u is p-harmonic, then so is $\tilde{u} = u + (Cx, Cy)$ for a constant C. Thus, by shifting H by a constant we may ensure the positivity of H. Therefore, condition (A.7) together with requirement that $H' \leq 0$ reduce the hypothesis of observation to showing the following inequalities:

$$\begin{cases} -2 - \frac{H''r}{H'} & \leq -\frac{x^2}{y^2} \\ -2 - \frac{H''r}{H'} & \leq -\frac{y^2}{x^2}. \end{cases} \tag{A.8}$$

Furthermore, assumption that $c > 1$ and definition of Ω allow us to check only that

$$-2 - \frac{H''r}{H'} \leq -c < -\frac{x^2}{y^2} \quad \left(< -1 \leq -\frac{y^2}{x^2}\right).$$

From (A.6) we find that

$$-2 - \frac{H''r}{H'} = \frac{(H'r + H)^2 + (p-2)H(H'r + H) + H^2}{(p-1)(H'r + H)^2 + H^2}.$$

Upon defining $t = \frac{H'r+H}{H}$, condition $-2 - \frac{H''r}{H'} \leq -c$ reads:

$$(1 + c(p-1))t^2 + (p-2)t + 1 + c \leq 0. \tag{A.9}$$

Solutions exist provided that $4(1-p)c^2 - 4pc + p(p-4) \geq 0$. In such a case c must satisfy $-\frac{p}{2(p-1)} - \frac{|p-2|}{2(p-1)}\sqrt{p} \leq c \leq -\frac{p}{2(p-1)} + \frac{|p-2|}{2(p-1)}\sqrt{p}$. Requiring that $c > 0$ gives us condition $p > 4$, while $c > 1$ holds if $p > 6 + 4\sqrt{2}$. By solving inequality (A.9) for t we may determine conditions for H and H' under which mapping u satisfies (A.8). Thus, the proof of the observation is complete. □

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