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An existence theorem for steady Navier-Stokes equations in the axially symmetric case

MIKHAIL KOROBKOV, KONSTANTIN PILECKAS AND REMIGIO RUSSO

Abstract. We study the nonhomogeneous boundary value problem for the Navier-Stokes equations of steady motion of a viscous incompressible fluid in a bounded three-dimensional domain with multiply connected boundary. We prove that this problem has a solution in some axially symmetric cases, in particular, when all components of the boundary intersect the axis of symmetry.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary $\partial\Omega = \Gamma_0 \cup \ldots \cup \Gamma_N$, consisting of $N+1$ disjoint connected components Γ_j. Consider the stationary Navier–Stokes system with nonhomogeneous boundary conditions

$$\begin{cases} -\nu\Delta\mathbf{u} + \big(\mathbf{u}\cdot\nabla\big)\mathbf{u} + \nabla p = 0 & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{a} & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The continuity equation (1.1_2) implies the compatibility condition

$$\int_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\,dS = \sum_{j=0}^{N} \int_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,dS = \sum_{j=0}^{N} \mathcal{F}_i = 0 \tag{1.2}$$

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necessary for the solvability of (1.1), where $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$ and $\mathcal{F}_j = \int\limits_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,dS$.

Starting from the famous paper of J. Leray [23] published in 1933, problem (1.1) was studied in many articles (see, *e.g.*, [1], [2], [8]– [13], [18]– [21], [26]– [36], etc.). However, for a long time the existence of a weak solution $\mathbf{u} \in W^{1,2}(\Omega)$ to problem (1.1) was established only under the assumption that

$$\mathcal{F}_j = \int\limits_{\Gamma_j} \mathbf{a}\cdot\mathbf{n}\,dS = 0, \qquad j = 1, 2, \dots, N, \tag{1.3}$$

or for sufficiently small fluxes (see [23], [20]– [21], [9], [36], [18], etc.). Condition (1.3) requires the flux of the boundary value $\mathbf{a}$ to vanish separately through each component Γ_j of the boundary $\partial\Omega$, while the compatibility condition (1.2) means only that the total flux vanishes. Thus, (1.3) is stronger than (1.2) (condition (1.3) excludes the presence of sinks and sources).

A detailed survey of available results appeared in the recent papers [15] and [28]– [29]. In particular, in the latter papers Pukhnachev established the existence of a solution to (1.1) in the three-dimensional case when the domain Ω and the boundary value $\mathbf{a}$ have a symmetry axis and a symmetry plane perpendicular to this axis, moreover, this plane intersects each boundary component (for a more precise formulation, see below).

In this paper we study the problem in the axially symmetric case. Take coordinate axes $O_{x_1}, O_{x_2}, O_{x_3}$ in $\mathbb{R}^3$ and consider cylindrical coordinates $\theta = \mathrm{arctg}(x_2/x_1)$, $r = (x_1^2 + x_2^2)^{1/2}$, $z = x_3$. Denote by v_θ, v_r, v_z the projections of a vector $\mathbf{v}$ on the axis θ, r, z.

A function f is said to be *axially symmetric* if it is independent of θ. A vector-valued function $\mathbf{h} = (h_\theta, h_r, h_z)$ is called *axially symmetric* if h_θ, h_r and h_z are independent of θ. A vector-valued function $\mathbf{h} = (h_\theta, h_r, h_z)$ is called *axially symmetric without rotation* if $h_\theta = 0$ while h_r and h_z are independent of θ.

We will use the following symmetry assumptions.

(SO) $\Omega \subset \mathbb{R}^3$ is a bounded domain with Lipschitz boundary and O_{x_3} is a symmetry axis of Ω.

(AS) The assumptions (SO) are fulfilled and the boundary value $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ is axially symmetric.

(ASwR) The assumptions (SO) are fulfilled and the boundary value $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ is axially symmetric without rotation.

Denote by Ω_j the bounded simply connected domain with $\partial\Omega_j = \Gamma_j$, $j = 0, \dots, N$. Let Ω_0 be the largest domain, *i.e.*,

$$\Omega = \Omega_0 \setminus \big(\cup_{j=1}^N \bar{\Omega}_j\big).$$

Here and henceforth we denote by $\bar{A}$ the closure of a set A.

Let

$$\Gamma_j \cap O_{x_3} \neq \emptyset, \quad j = 0, \dots, M,$$
$$\Gamma_j \cap O_{x_3} = \emptyset, \quad j = M+1, \dots, N.$$

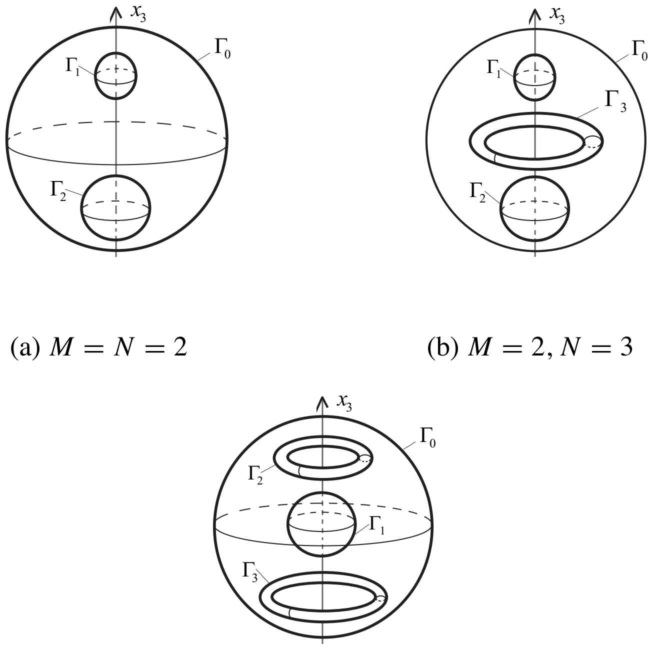
We shall prove the existence theorem provided that one of the following two additional conditions is fulfilled:

$$M = N - 1, \qquad \mathcal{F}_N \geq 0, \tag{1.4}$$

or

$$|\mathcal{F}_j| < \delta, \quad j = M+1, \dots, N, \tag{1.5}$$

where $\delta = \delta(\nu, \Omega)$ is sufficiently small (we specify $\delta(\nu, \Omega)$ in Section 4). In particular, (1.5) includes the case $N = M$ when each component of the boundary intersects the axis of symmetry. Notice that in (1.4), (1.5) the fluxes $\mathcal{F}_j$, $j = 1, \dots, M$, could be arbitrarily large.



(a) $M = N = 2$ (b) $M = 2, N = 3$

(c) $M = 1, N = 3$

Figure 1.1. Domain Ω.

Figure 1.1 depicts several possible domains Ω. In case (a) all fluxes $\mathcal{F}_0, \mathcal{F}_1$ and $\mathcal{F}_2$ are arbitrary; in case (b) the fluxes $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$ are arbitrary, while $\mathcal{F}_3$ has to be nonnegative, but there is no restriction on its size; in the case (c) the fluxes $\mathcal{F}_0, \mathcal{F}_1$ are arbitrary, while $\mathcal{F}_2$ and $\mathcal{F}_3$ have to be "sufficiently small".

The main result of this paper reads as follows.

Theorem 1.1. *Let conditions* (AS) *and* (1.2) *be fulfilled. Suppose that either* (1.4) *or* (1.5) *holds. Then problem* (1.1) *admits at least one weak axially symmetric solution* $\mathbf{u} \in W^{1,2}(\Omega)$.

If, in addition, conditions (ASwR) *are fulfilled, then problem* (1.1) *admits at least one weak axially symmetric solution without rotation.*

For the definition of a weak solution, see Section 2.1. Analogous results in the plane case were established in [15].

Note that in [29] the existence theorem was obtained under the following assumptions on the axially symmetric boundary data $\mathbf{a} = (a_\theta, a_r, a_z)$: $a_\theta \equiv 0$ (*i.e.*, the axially symmetric case without rotation), $\{z = 0\}$ is a symmetry plane of Ω, and each boundary component Γ_j intersects this plane; furthermore, a_r is an even function of z, while a_z is an odd function of z (no restrictions on the size of the fluxes). Under these assumptions, the number N of boundary components can be arbitrarily large, but only at most two of them can intersect the symmetry axis, in our notation that means $M \leq 1$. Therefore, neither our Theorem 1.1 implies Pukhnachev's result, nor the latter implies the former, and so these results are in a sense independent. Moreover, the proof in [29] is based on different ideas; in particular, in [29] a priori estimates for the velocity field were obtained without using Leray's contradiction argument.

Let us also remark that Alekseev and Pukhnachev recently obtained [30] an existence theorem for the steady Navier–Stokes equations in the axially symmetric case with boundary conditions formulated in terms of stream function and vorticity. Of course, this result is even farther from our Theorem 1.1 than the previous one.

Our proof of Theorem 1.1 uses Bernoulli's law for a weak solution of the Euler equations and the weak one-sided maximum principle for the total head pressure corresponding to this solution (see Section 3). These results were obtained in [14] for the plane case (see [15] for more detailed proofs). The proof of Bernoulli's law for solutions in Sobolev spaces is based on the recent results of [3] (see also Section 2.2).

A short version of this paper appeared as [16].

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2. Notation and preliminary results

By *a domain* we mean a connected open set. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary $\partial\Omega$. We use standard notation for function spaces: $C^k(\bar{\Omega})$, $C^k(\partial\Omega)$, $W^{k,q}(\Omega)$, $\mathring{W}^{k,q}(\Omega)$, $W^{\alpha,q}(\partial\Omega)$, where $\alpha \in (0,1)$, $k \in \mathbb{N}_0$, and $q \in [1, +\infty]$. In our notation we do not distinguish function spaces of scalar- and vector-valued functions; it is clear from the context whether we use scalar, vector, or tensor-valued function spaces. Denote by $H(\Omega)$ the subspace of all solenoidal vector fields ($\operatorname{div}\mathbf{u}=0$) from $\mathring{W}^{1,2}(\Omega)$ equipped with the norm $\|\mathbf{u}\|_{H(\Omega)} = \|\nabla\mathbf{u}\|_{L^2(\Omega)}$. Observe that for functions $\mathbf{u} \in H(\Omega)$ the norm $\|\cdot\|_{H(\Omega)}$ is equivalent to $\|\cdot\|_{W^{1,2}(\Omega)}$.

Working with Sobolev functions, we always assume that the "best representatives" are chosen. For $w \in L^1_{\text{loc}}(\Omega)$ the best representative w^* is defined as

$$w^*(x) = \begin{cases} \lim\limits_{r\to 0} ⨍_{B_r(x)} w(z)dz, & \text{if the finite limit exists;} \\ 0 & \text{otherwise,} \end{cases}$$

where

$$⨍_{B_r(x)} w(z)dz = \frac{1}{\operatorname{meas}(B_r(x))} \int_{B_r(x)} w(z)\,dz,$$

and $B_r(x) = \{y : |y-x| < r\}$ is the ball of radius r centered at x.

Below (see Theorem 3.7) we discuss some properties of the best representatives for Sobolev functions.

2.1. Some facts about solenoidal functions

The following lemmas concern the existence of solenoidal extensions of boundary values and an integral representation for bounded linear functionals vanishing on the solenoidal functions.

Lemma 2.1 (see Corollary 2.3 in [22]). *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary. If $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ and* (1.2) *holds, then there exists a solenoidal extension $\mathbf{A} \in W^{1,2}(\Omega)$ of $\mathbf{a}$ with*

$$\|\mathbf{A}\|_{W^{1,2}(\Omega)} \le c\|\mathbf{a}\|_{W^{1/2,2}(\partial\Omega)}. \tag{2.1}$$

From this lemma we can deduce some assertions for the symmetric case.

Lemma 2.2. *If conditions* (AS) *and* (1.2) *are fulfilled, then there exists an axially symmetric solenoidal extension $\mathbf{A} \in W^{1,2}(\Omega)$ of $\mathbf{a}$ such that estimate* (2.1) *holds.*

Proof. Take a solenoidal extension $\mathbf{A_0} \in W^{1,2}(\Omega)$ of $\mathbf{a}$ from Lemma 2.1. Put

$$\mathbf{A}_i(\theta, r, z) = \frac{1}{i!}\sum_{j=0}^{i!} \mathbf{A_0}\left(\theta + \frac{2\pi j}{i!}, r, z\right).$$

Clearly, each $\mathbf{A}_i$ is also a solenoidal extension of $\mathbf{a}$ and the estimate (2.1) holds for $\mathbf{A}_i$ with the same c (independent of i). By construction

$$\mathbf{A}_i(\theta + \frac{2\pi j}{m}, r, z) = \mathbf{A}_i(\theta, r, z) \quad \text{for all } m = 1, \dots, i. \tag{2.2}$$

Take a weakly convergent sequence $\mathbf{A}_{i_k} \rightharpoonup \mathbf{A}$ in $W^{1,2}(\Omega)$. Then by construction $\operatorname{div}\mathbf{A} = 0$, $\mathbf{A}|_{\partial\Omega} = \mathbf{a}$, and (2.1) holds. Now (2.2) implies that $\mathbf{A}(\theta + \frac{2\pi j}{m}, r, z) = \mathbf{A}(\theta, r, z)$ for all m, j. Hence $\mathbf{A}$ is axially symmetric. □

Lemma 2.3. *Let conditions* (ASwR) *and* (1.2) *be fulfilled. Then there exists a solenoidal extension* $\mathbf{A} \in W^{1,2}(\Omega)$ *of* $\mathbf{a}$ *such that* $\mathbf{A}$ *is axially symmetric without rotation and estimate* (2.1) *holds.*

Proof. Take a solenoidal extension $\tilde{\mathbf{A}} = (\tilde{A}_\theta, \tilde{A}_r, \tilde{A}_z) \in W^{1,2}(\Omega)$ of $\mathbf{a}$ from Lemma 2.2. Then a classical formula yields

$$\begin{aligned}\operatorname{div} \tilde{\mathbf{A}}(\theta, r, z) &= \frac{1}{r}\frac{\partial}{\partial \theta}(\tilde{A}_\theta) + \frac{1}{r}\frac{\partial}{\partial r}(\tilde{A}_r r) + \frac{\partial}{\partial z}(\tilde{A}_z) \\ &= \frac{1}{r}\frac{\partial}{\partial r}(\tilde{A}_r r) + \frac{\partial}{\partial z}(\tilde{A}_z) = 0.\end{aligned} \tag{2.3}$$

Here $\frac{\partial \tilde{A}_\theta}{\partial \theta} = 0$ because of axial symmetry. Define the vector field $\mathbf{A} = (A_\theta, A_r, A_z)$ by putting

$$A_\theta = 0, \quad A_r = \tilde{A}_r, \quad A_z = \tilde{A}_z.$$

Then by construction $\mathbf{A}$ is axially symmetric without rotation, $\mathbf{A}|_{\partial\Omega} = \mathbf{a}$, and estimate (2.1) holds. Now (2.3) implies that $\operatorname{div} \mathbf{A} = 0$. □

Lemma 2.4 (see [33]). *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with Lipschitz boundary and* $R(\boldsymbol{\eta})$ *a continuous linear functional defined on* $\mathring{W}^{1,2}(\Omega)$. *If*

$$R(\boldsymbol{\eta}) = 0 \qquad \forall\ \boldsymbol{\eta} \in H(\Omega),$$

then there exists a unique function $p \in L^2(\Omega)$ *with* $\int\limits_\Omega p(x)\,dx = 0$ *such that*

$$R(\boldsymbol{\eta}) = \int\limits_\Omega p \operatorname{div} \boldsymbol{\eta}\, dx \qquad \forall\ \boldsymbol{\eta} \in \mathring{W}^{1,2}(\Omega).$$

Moreover, $\|p\|_{L^2(\Omega)}$ *is equivalent to* $\|R\|_{(\mathring{W}^{1,2}(\Omega))^*}$.

Lemma 2.5. *If, in addition to the hypotheses of Lemma* 2.4, *the domain* Ω *satisfies assumption* (SO) *and* $R(\boldsymbol{\eta}) \equiv R(\boldsymbol{\eta}_{\theta_0})$ *for all* $\boldsymbol{\eta} \in H(\Omega)$ *and* $\theta_0 \in [0, 2\pi]$, *where* $\boldsymbol{\eta}_{\theta_0}(\theta, r, z) := \boldsymbol{\eta}(\theta + \theta_0, r, z)$, *then the function* p *is axially symmetric.*

Proof. Take the function p of Lemma 2.4. For $\theta_0 \in [0, 2\pi]$ define a function p_{θ_0} by $p_{\theta_0}(\theta, r, z) = p(\theta - \theta_0, r, z)$. By construction,

$$\begin{aligned}\int\limits_\Omega p \operatorname{div} \boldsymbol{\eta}\, dx = R(\boldsymbol{\eta}) = R(\boldsymbol{\eta}_{\theta_0}) &= \int\limits_\Omega p \operatorname{div} \boldsymbol{\eta}_{\theta_0}\, dx \\ &= \int\limits_\Omega p_{\theta_0} \operatorname{div} \boldsymbol{\eta}\, dx \qquad \forall\ \boldsymbol{\eta} \in \mathring{W}^{1,2}(\Omega).\end{aligned}$$

Since p is unique, the identity $p(x) \equiv p_{\theta_0}(x)$ follows. □

Lemma 2.6 (see [21]). *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary and let $\mathbf{A} \in W^{1,2}(\Omega)$ be divergence-free. Then there exists a unique weak solution $\mathbf{U} \in W^{1,2}(\Omega)$ to the Stokes problem satisfying the boundary condition $\mathbf{U}|_{\partial\Omega} = \mathbf{A}|_{\partial\Omega}$,* i.e., *$\mathbf{U} - \mathbf{A} \in H(\Omega)$ and*

$$\int_\Omega \nabla\mathbf{U} \cdot \nabla\boldsymbol{\eta}\, dx = 0 \quad \forall\, \boldsymbol{\eta} \in H(\Omega). \tag{2.4}$$

Moreover,

$$\|\mathbf{U}\|_{W^{1,2}(\Omega)} \le c\|\mathbf{A}\|_{W^{1,2}(\Omega)}. \tag{2.5}$$

Lemma 2.7. *If, in addition to the hypotheses of Lemma* 2.6*, the domain Ω satisfies assumptions* (SO) *and also* $\mathbf{A}$ *is axially symmetric, then* $\mathbf{U}$ *is axially symmetric too.*

Proof. Take a solution $\mathbf{U}$ to the Stokes problem of Lemma 2.6. For $\theta_0 \in [0, 2\pi]$ define the function $\mathbf{U}_{\theta_0}$ by the formula $\mathbf{U}_{\theta_0}(\theta, r, z) := \mathbf{U}(\theta - \theta_0, r, z)$. By construction, $\mathbf{U}_{\theta_0} - \mathbf{A} \in H(\Omega)$. Moreover,

$$\int_\Omega \nabla\mathbf{U}_{\theta_0} \cdot \nabla\boldsymbol{\eta}\, dx = \int_\Omega \nabla\mathbf{U} \cdot \nabla\boldsymbol{\eta}_{\theta_0}\, dx = 0 \quad \forall\, \boldsymbol{\eta} \in H(\Omega),$$

where $\boldsymbol{\eta}_{\theta_0}(\theta, r, z) := \boldsymbol{\eta}(\theta + \theta_0, r, z)$. By uniqueness, the identity $\mathbf{U}(x) \equiv \mathbf{U}_{\theta_0}(x)$ follows. $\square$

Lemma 2.8. *If, in addition to the hypotheses of Lemma* 2.6*, the vector field* $\mathbf{A}$ *is axially symmetric without rotation, then* $\mathbf{U}$ *is also axially symmetric without rotation.*

Proof. Take the axially symmetric function $\mathbf{U} = (U_\theta, U_r, U_z)$ of Lemmas 2.6–2.7 and define $\boldsymbol{\eta} = (\eta_\theta, \eta_r, \eta_z)$ by putting

$$\eta_\theta \equiv U_\theta, \quad \eta_r = \eta_z \equiv 0.$$

Then Lemma 2.7 implies that $\boldsymbol{\eta} \in H(\Omega)$ (see also (2.3)). Consequently, (2.4) yields

$$\int_\Omega \nabla\mathbf{U} \cdot \nabla\boldsymbol{\eta}\, dx = 0. \tag{2.6}$$

However,

$$\nabla\mathbf{U} \cdot \nabla\boldsymbol{\eta} \equiv \left(\frac{U_\theta}{r}\right)^2 + \left(\frac{\partial U_\theta}{\partial r}\right)^2 + \left(\frac{\partial U_\theta}{\partial z}\right)^2 \tag{2.7}$$

by a straightforward calculation, and the required equality $U_\theta \equiv 0$ follows from (2.6), (2.7). $\square$

Given a function $\mathbf{f} \in L^q(\Omega)$ with $1 \le q \le 6/5$ consider the continuous linear functional $H(\Omega) \ni \boldsymbol{\eta} \mapsto \int\limits_\Omega \mathbf{f} \cdot \boldsymbol{\eta}\, dx$. By the Riesz representation theorem, there exists a unique function $\mathbf{g} \in H(\Omega)$ with

$$\int\limits_\Omega \mathbf{f} \cdot \boldsymbol{\eta}\, dx = \int\limits_\Omega \nabla \boldsymbol{\eta} \cdot \nabla \mathbf{g}\, dx = \langle \mathbf{g}, \boldsymbol{\eta} \rangle_{H(\Omega)} \quad \forall \boldsymbol{\eta} \in H(\Omega).$$

Put $\mathbf{g} = T_0 \mathbf{f}$. Evidently, T_0 is a continuous linear operator from $L^q(\Omega)$ to $H(\Omega)$.

Denote by $L^q_{AS}(\Omega)$ the space of all axially symmetric vector-valued functions in $L^q(\Omega)$. Similarly define the spaces $L^q_{ASwR}(\Omega)$, $H_{AS}(\Omega)$, $H_{ASwR}(\Omega)$, $W^{1,2}_{AS}(\Omega)$, $W^{1,2}_{ASwR}(\Omega)$, etc.

Lemma 2.9. *The operator $T_0 : L^{3/2}(\Omega) \to H(\Omega)$ has the following symmetry properties:*

$$\forall \mathbf{f} \in L^{3/2}_{AS}(\Omega) \quad T_0 \mathbf{f} \in H_{AS}(\Omega), \tag{2.8}$$

$$\forall \mathbf{f} \in L^{3/2}_{ASwR}(\Omega) \quad T_0 \mathbf{f} \in H_{ASwR}(\Omega). \tag{2.9}$$

Proof. We can prove (2.8) in the same way as Lemma 2.7 and (2.9) as Lemma 2.8. □

Lemma 2.10. *The following inclusions are valid:*

$$\forall \mathbf{u}, \mathbf{v} \in H_{AS}(\Omega) \qquad (\mathbf{u} \cdot \nabla)\mathbf{v} \in L^{3/2}_{AS}(\Omega), \tag{2.10}$$

$$\forall \mathbf{u}, \mathbf{v} \in H_{ASwR}(\Omega) \qquad (\mathbf{u} \cdot \nabla)\mathbf{v} \in L^{3/2}_{ASwR}(\Omega). \tag{2.11}$$

Proof. Direct calculation. □

Take $\mathbf{a} \in W^{1/2,2}(\partial\Omega)$ and assume that conditions (1.2) and (AS) (or (ASwR)) are fulfilled. Take the corresponding axially symmetric functions $\mathbf{A}$ and $\mathbf{U}$ of Lemmas 2.2–2.3, 2.7–2.8. Put $\mathbf{w} = \mathbf{u} - \mathbf{U}$. Then problem (1.1) is equivalent to

$$\begin{cases} -\nu \Delta \mathbf{w} + \big(\mathbf{U} \cdot \nabla\big)\mathbf{w} + \big(\mathbf{w} \cdot \nabla\big)\mathbf{w} + \big(\mathbf{w} \cdot \nabla\big)\mathbf{U} & \\ \qquad = -\nabla p - \big(\mathbf{U} \cdot \nabla\big)\mathbf{U} & \text{in } \ \Omega, \\ \operatorname{div} \mathbf{w} = 0 & \text{in } \ \Omega, \\ \mathbf{w} = 0 & \text{on } \ \partial\Omega. \end{cases} \tag{2.12}$$

By a *weak solution* to problem (1.1) we understand a function $\mathbf{u}$ such that $\mathbf{w} = \mathbf{u} - \mathbf{U} \in H(\Omega)$ and

$$\begin{aligned} \nu \langle \mathbf{w}, \boldsymbol{\eta} \rangle_{H(\Omega)} = &- \int\limits_\Omega \big(\mathbf{U} \cdot \nabla\big)\mathbf{U} \cdot \boldsymbol{\eta}\, dx - \int\limits_\Omega \big(\mathbf{U} \cdot \nabla\big)\mathbf{w} \cdot \boldsymbol{\eta}\, dx \\ &- \int\limits_\Omega \big(\mathbf{w} \cdot \nabla\big)\mathbf{w} \cdot \boldsymbol{\eta}\, dx - \int\limits_\Omega \big(\mathbf{w} \cdot \nabla\big)\mathbf{U} \cdot \boldsymbol{\eta}\, dx \qquad \forall \boldsymbol{\eta} \in H(\Omega). \end{aligned} \tag{2.13}$$

By the Riesz representation theorem, for any $\mathbf{w} \in H(\Omega)$ there exists a unique function $T\mathbf{w} \in H(\Omega)$ such that the right-hand side of (2.13) is equivalent to $\langle T\mathbf{w}, \boldsymbol{\eta}\rangle_{H(\Omega)}$ for all $\boldsymbol{\eta} \in H(\Omega)$. Obviously, T is a nonlinear operator from $H(\Omega)$ to $H(\Omega)$.

Lemma 2.11. *The operator $T : H(\Omega) \to H(\Omega)$ is compact. Moreover, T has the following symmetry properties:*

$$\forall \mathbf{w} \in H_{AS}(\Omega) \quad T\mathbf{w} \in H_{AS}(\Omega), \tag{2.14}$$

$$\forall \mathbf{w} \in H_{ASwR}(\Omega) \quad T\mathbf{w} \in H_{ASwR}(\Omega). \tag{2.15}$$

Proof. The first claim is well-known (see [21]). The symmetry claims follow from Lemmas 2.9–2.10. □

Obviously, (2.13) is equivalent to the operator equation

$$\nu \mathbf{w} = T\mathbf{w} \tag{2.16}$$

in the space $H(\Omega)$. Thus, we can apply the Leray–Schauder fixed point theorem to the compact operators $T|_{H_{AS}(\Omega)}$ and $T|_{H_{ASwR}(\Omega)}$. The following statements hold.

Lemma 2.12. *Let conditions* (AS), (1.2) *be fulfilled. Suppose that all possible solutions to the equation $\nu\mathbf{w} = \lambda T\mathbf{w}$ with $\lambda \in [0, 1]$ and $\mathbf{w} \in H_{AS}(\Omega)$ are uniformly bounded in $H_{AS}(\Omega)$. Then problem* (1.1) *admits at least one weak axially symmetric solution.*

Lemma 2.13. *Let conditions* (ASwR), (1.2) *be fulfilled. Suppose that all possible solutions to the equation $\nu\mathbf{w} = \lambda T\mathbf{w}$ with $\lambda \in [0, 1]$ and $\mathbf{w} \in H_{ASwR}(\Omega)$ are uniformly bounded in $H_{ASwR}(\Omega)$. Then problem* (1.1) *admits at least one weak axially symmetric solution without rotation.*

2.2. On the Morse–Sard and Luzin N-properties of Sobolev functions in $W^{2,1}$

First we recall some classical differentiability properties of Sobolev functions.

Lemma 2.14 (see Proposition 1 in [6]). *If $\psi \in W^{2,1}(\mathbb{R}^2)$, then ψ is continuous and there exists a set A_ψ such that $\mathfrak{H}^1(A_\psi) = 0$ and ψ is differentiable (in the classical sense) at each $x \in \mathbb{R}^2 \setminus A_\psi$. Furthermore, the classical derivative at these points x coincides with $\nabla\psi(x) = \lim\limits_{r\to 0} \fint_{B_r(x)} \nabla\psi(z)dz$, where $\lim\limits_{r\to 0} \fint_{B_r(x)} |\nabla\psi(z) - \nabla\psi(x)|^2 dz = 0$.*

Here and henceforth we denote by $\mathfrak{H}^1$ the one-dimensional Hausdorff measure, *i.e.*, $\mathfrak{H}^1(F) = \lim\limits_{t\to 0+} \mathfrak{H}^1_t(F)$, where

$$\mathfrak{H}^1_t(F) = \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam} F_i \,:\, \operatorname{diam} F_i \le t,\ F \subset \bigcup_{i=1}^{\infty} F_i\right\}.$$

The following theorems have been proved recently by J. Bourgain, M. Korobkov and J. Kristensen [3].

Theorem 2.15. *Let $\mathcal{D} \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary and $\psi \in W^{2,1}(\mathcal{D})$. Then:*

(i) $\mathfrak{H}^1(\{\psi(x) : x \in \bar{\mathcal{D}} \setminus A_\psi \ \& \ \nabla\psi(x) = 0\}) = 0$;

(ii) *for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every set $U \subset \bar{\mathcal{D}}$ with $\mathfrak{H}^1_\infty(U) < \delta$ the inequality $\mathfrak{H}^1(\psi(U)) < \varepsilon$ holds;*

(iii) *for $\mathfrak{H}^1$-almost all $y \in \psi(\bar{\mathcal{D}}) \subset \mathbb{R}$ the preimage $\psi^{-1}(y)$ is a finite disjoint family of C^1-curves S_j, $j = 1, 2, \ldots, N(y)$. Each S_j is either a cycle in $\mathcal{D}$* (i.e., *$S_j \subset \mathcal{D}$ is homeomorphic to the unit circle $\mathbb{S}^1$*) *or a simple arc with endpoints on $\partial\mathcal{D}$ (in this case S_j is transversal to $\partial\mathcal{D}$).*

Theorem 2.16. *Let $\mathcal{D} \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary and $\psi \in W^{2,1}(\mathcal{D})$. Then for every $\varepsilon > 0$ there exist an open set $V \subset \mathbb{R}$ and a function $g \in C^1(\mathbb{R}^2)$ such that $\psi(A_\psi) \subset V$, $\mathfrak{H}^1(V) < \varepsilon$, and the identities $\psi(x) \equiv g(x)$, $\nabla\psi(x) = \nabla g(x) \neq 0$ hold for all $x \in \bar{\mathcal{D}}$ provided that $\psi(x) \notin V$.*

We say that a value $y \in \psi(\bar{\mathcal{D}})$ is *regular* if it satisfies condition (iii) of Theorem 2.15 and $\psi(x) \notin V$ for some g and V of Theorem 2.16. Observe that by Theorems 2.15 and 2.16 almost all values $y \in \psi(\bar{\mathcal{D}})$ are regular.

3. Euler equation

We study the Euler equation under the following assumptions.

(E) Let conditions (SO) be fulfilled. Suppose that some axially symmetric functions $\mathbf{v} \in W^{1,2}(\Omega)$ and $p \in W^{1,3/2}(\Omega)$ satisfy the Euler system

$$\begin{cases} \lambda_0\big(\mathbf{v} \cdot \nabla\big)\mathbf{v} + \nabla p = 0, \\ \operatorname{div} \mathbf{v} = 0 \end{cases} \tag{3.1}$$

for almost all $x \in \Omega$. Moreover, suppose that

$$\mathbf{v}|_{\partial\Omega} = 0. \tag{3.2}$$

Put $P_+ = \{(0, x_2, x_3) : x_2 > 0,\ x_3 \in \mathbb{R}\}$, $\mathcal{D} = \Omega \cap P_+$, $\mathcal{D}_j = \Omega_j \cap P_+$. Of course, on P_+ the coordinates x_2 and x_3 coincides with r and z. From (SO) we can easily infer that

(S_1) $\mathcal{D}$ is a bounded plane domain with Lipschitz boundary. Moreover, $C_j := P_+ \cap \Gamma_j$ is a connected set for each $j = 0, \ldots, N$. In other words, $\{C_j : j = 0, \ldots, N\}$ coincides with the set of all connected components of $P_+ \cap \partial\mathcal{D}$.

Then $\mathbf{v}$ and p satisfy the following system of equations in the plane domain $\mathcal{D}$:

$$\begin{cases} \frac{\partial p}{\partial z} + \lambda_0 v_r \frac{\partial v_z}{\partial r} + \lambda_0 v_z \frac{\partial v_z}{\partial z} = 0, \\ \frac{\partial p}{\partial r} - \lambda_0 \frac{(v_\theta)^2}{r} + \lambda_0 v_r \frac{\partial v_r}{\partial r} + \lambda_0 v_z \frac{\partial v_r}{\partial z} = 0, \\ \frac{v_\theta v_r}{r} + v_r \frac{\partial v_\theta}{\partial r} + v_z \frac{\partial v_\theta}{\partial z} = 0, \\ \frac{\partial (r v_r)}{\partial r} + \frac{\partial (r v_z)}{\partial z} = 0 \end{cases} \tag{3.3}$$

(the equations are satisfied for almost all $x \in \mathcal{D}$).

The next statement was proved in [13, Lemma 4] and [1, Theorem 2.2].

Theorem 3.1. If conditions (E) are fulfilled, then

$$\forall j \in \{0, \dots, N\}\ \exists\, p_j \in \mathbb{R}: \quad p(x) \equiv p_j \quad \text{for } \mathfrak{H}^2\text{-almost all } x \in \Gamma_j. \tag{3.4}$$

In particular, by axial symmetry,

$$p(x) \equiv p_j \quad \text{for } \mathfrak{H}^1\text{-almost all } x \in C_j. \tag{3.5}$$

Lemma 3.2 (*e.g.*, [18], [27]). Under the assumptions of Theorem 3.1, the estimate

$$\max_{i,j=0,\dots N} |p_i - p_j| \le \delta_1 \lambda_0 \|\mathbf{v}\|^2_{H(\Omega)} \tag{3.6}$$

holds, where the constant δ_1 depends only on Ω.

One of the main purposes of this section is to prove the following fact.

Theorem 3.3. Under the assumptions of Theorem 3.1, the equalities

$$p_0 = p_1 = \dots = p_M \tag{3.7}$$

are fulfilled.

To prove the last theorem, we need some preparation, in particular, a version of Bernoulli's Law in the Sobolev case (see Theorem 3.4 below).

The last equality in (3.3) and (3.2) imply that there exists a stream function $\psi \in W^{2,2}_{\mathrm{loc}}(\mathcal{D})$ with

$$\frac{\partial \psi}{\partial r} = -r v_z, \qquad \frac{\partial \psi}{\partial z} = r v_r. \tag{3.8}$$

We have the following integral estimates: $\mathbf{v} \in W^{1,2}_{\mathrm{loc}}(\mathcal{D})$,

$$\int_{\mathcal{D}} r |\mathbf{v}(r,z)|^2 \, dr dz < \infty. \tag{3.9}$$

Using (3.8), we can rewrite this as

$$\int_{\mathcal{D}} \frac{|\nabla\psi(r,z)|^2}{r}\,drdz < \infty. \tag{3.10}$$

Fix a point $x_* \in \mathcal{D}$. For $\varepsilon > 0$ denote by $\mathcal{D}_\varepsilon$ the connected component of $\mathcal{D} \cap \{(r,z) : r > \varepsilon\}$ containing x_*. Since

$$\psi \in W^{2,2}(\mathcal{D}_\varepsilon) \quad \forall \varepsilon > 0, \tag{3.11}$$

the Sobolev Embedding Theorem yields $\psi \in C(\bar{\mathcal{D}}_\varepsilon)$. Hence, ψ is continuous at the points of $\bar{\mathcal{D}} \setminus O_z = \bar{\mathcal{D}} \setminus \{(0,z) : z \in \mathbb{R}\}$.

Denote by $\Phi = p + \lambda_0 \frac{|\mathbf{v}|^2}{2}$ the total head pressure corresponding to the solution $(\mathbf{v}, p)$. Obviously,

$$\Phi \in W^{1,3/2}(\mathcal{D}_\varepsilon) \quad \forall \varepsilon > 0. \tag{3.12}$$

Straightforward calculations yield the identity

$$v_r \frac{\partial \Phi}{\partial r} + v_z \frac{\partial \Phi}{\partial z} = 0 \tag{3.13}$$

for almost all $x \in \mathcal{D}$.

Theorem 3.4. *Assume that conditions* (E) *are fulfilled (see the beginning of this section). Then there exists a set* $A_{\mathbf{v}} \subset P_+$ *with* $\mathfrak{H}^1(A_{\mathbf{v}}) = 0$ *such that if for a compact connected*[1] *set* $K \subset \bar{\mathcal{D}} \setminus O_z$

$$\psi\big|_K = \text{const}, \tag{3.14}$$

then

$$\Phi(x_1) = \Phi(x_2) \quad \textit{for all } x_1, x_2 \in K \setminus A_{\mathbf{v}}. \tag{3.15}$$

Theorem 3.4 was obtained in the plane case in [14, Theorem 1] (see also [15] for a detailed proof).

To prove Theorem 3.4, we need some preliminaries.

Lemma 3.5. If conditions (E) are fulfilled, then

$$p \in W^{2,1}_{\text{loc}}(\mathcal{D}). \tag{3.16}$$

Proof. Clearly, p is the (unique) weak solution to the Poisson equation

$$\begin{cases} \Delta p + \lambda_0 \nabla \mathbf{v} \cdot \nabla \mathbf{v}^\top = 0 & \text{in } \Omega \\ p = \tilde{p}, & \text{in } \partial\Omega, \end{cases} \tag{3.17}$$

[1] We understand connectedness in the sense of general topology.

with $\tilde{p} = \mathrm{tr}_{|\partial\Omega} p \in W^{1/3,3/2}(\partial\Omega)$. Put

$$G(x) = \frac{\lambda_0}{4\pi} \int\limits_{\Omega} \frac{(\nabla \mathbf{v} \cdot \nabla \mathbf{v}^\top)(y)}{|x-y|} dv_y.$$

By the results of [5], $\nabla \mathbf{v} \cdot \nabla \mathbf{v}^\top$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^3)$. Therefore, the Calderón–Zygmund theorem for Hardy spaces [34] yields $G \in W^{2,1}(\Omega)$. Take the trace $\bar{G} \in W^{1/3,3/2}(\partial\Omega)$ of G on $\partial\Omega$ and the solution $p_* \in C^\infty(\Omega)$ to the problem

$$\begin{cases} \Delta p_* = 0 & \text{in } \Omega, \\ p_* = \tilde{p} - \bar{G} & \text{in } \partial\Omega. \end{cases} \tag{3.18}$$

The uniqueness theorem yields

$$p = p_* + G(x) \in W^{2,1}_{\mathrm{loc}}(\Omega). \qquad \square$$

From (3.16) we infer that $p_{rz} \equiv p_{zr}$ for almost all $x \in \mathcal{D}$. Denote $Z = \{x \in \mathcal{D} : v_r(x) = v_z(x) = 0\}$. By (3.3), we have

$$\frac{\partial p}{\partial z}(x) = 0, \quad \frac{\partial p}{\partial r}(x) = \lambda_0 \frac{(v_\theta)^2}{r} \quad \text{for almost all } x \in Z,$$

and it is easy to deduce that

$$\frac{\partial \Phi}{\partial z}(x) = 0 \quad \text{for almost all } x \in \mathcal{D} \text{ such that } v_r(x) = v_z(x) = 0. \tag{3.19}$$

Consider the stream function ψ. By (3.2) and (3.8) we have $\nabla\psi(x) = 0$ for $\mathfrak{H}^1$-almost all $x \in \partial\mathcal{D} \setminus O_z$. Then the Morse–Sard property (see Theorem 2.15) implies that

for every connected set $C \subset \partial\mathcal{D} \setminus O_z$ $\exists \alpha = \alpha(C) \in \mathbb{R}: \quad \psi(x) \equiv \alpha \ \forall x \in C.$

Then by (S_1) (see the beginning of Section 3)

$$\forall j \in \{0, \dots, N\} \ \exists \xi_j \in \mathbb{R}: \quad \psi(x) \equiv \xi_j \ \forall x \in C_j. \tag{3.20}$$

Remark 3.6. Since $\nabla\psi = 0$ on $\partial D \setminus O_z$ (in the sense of traces), the function ψ extends to the whole half-plane P_+:

$$\psi(x) := \xi_0, \ x \in P_+ \setminus \mathcal{D}_0, \quad \psi(x) := \xi_j, \ x \in P_+ \cap \bar{\mathcal{D}}_j, \ j = 1, \dots, N. \tag{3.21}$$

The functions $\mathbf{v}$, p and Φ extend to P_+ as

$$\mathbf{v}(x) = 0, \quad x \in P_+ \setminus \mathcal{D}, \tag{3.22}$$

$$p(x) = \Phi(x) = \begin{cases} p_0, & x \in P_+ \setminus \mathcal{D}_0, \\ p_j, & x \in P_+ \cap \bar{\mathcal{D}}_j, \ j = 1, \dots, N. \end{cases} \tag{3.23}$$

The extended functions inherit the properties of the original ones. Namely, (3.3), (3.8)–(3.13), (3.19) hold with $\mathcal{D}$ and $\mathcal{D}_\varepsilon$ replaced by P_+ and

$$P_\varepsilon := \left\{(r,z) : r \in \left[\varepsilon, \frac{1}{\varepsilon}\right],\ z \in \left[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}\right]\right\}, \tag{3.24}$$

respectively.

For $r_0 > 0$ denote by L_{r_0} the straight line parallel to the z-axis: $L_{r_0} = \{(r_0, z) : z \in \mathbb{R}\}$.

Working with Sobolev functions, we always assume that the "best representatives" are chosen. We collect the basic properties of these "best representatives" in the next theorem.

Theorem 3.7. *There exists a set $A_{\mathbf{v}} \subset P_+$ with the following properties.*

(i) $\mathfrak{H}^1(A_{\mathbf{v}}) = 0$.

(ii) *For all $x \in P_+ \setminus A_{\mathbf{v}}$*

$$\lim_{r\to 0} \fint_{B_r(x)} |\mathbf{v}(y) - \mathbf{v}(x)|^2 dy = \lim_{r\to 0} \fint_{B_r(x)} |\Phi(y) - \Phi(x)|^{3/2} dy = 0,$$
$$\lim_{r\to 0} \frac{1}{r} \int_{B_r(x)} |\nabla\Phi(y)|^{3/2} dy = 0, \tag{3.25}$$

and moreover, the function ψ is differentiable at x and $\nabla\psi(x) = (-r v_z(x), r v_r(x))$.

(iii) *For all $\varepsilon > 0$ there exists an open set $U \subset \mathbb{R}^2$ with $\mathfrak{H}^1_\infty(U) < \varepsilon$ and $A_{\mathbf{v}} \subset U$ such that the functions $\mathbf{v}$ and Φ are continuous on $P_+ \setminus U$.*

(iv) *For each $x_0 = (r_0, z_0) \in P_+ \setminus A_{\mathbf{v}}$ and for every $\varepsilon > 0$ we have the convergence*

$$\lim_{\rho\to 0+} \frac{1}{2\rho} \mathfrak{H}^1(E(x_0, \varepsilon, \rho)) \to 1, \tag{3.26}$$

where

$$\begin{aligned} E(x_0,\varepsilon,\rho) := \Bigg\{ t \in (-\rho,\rho) : \int\limits_{r_0-\rho}^{r_0+\rho} \left|\frac{\partial\Phi}{\partial r}(r, z_0+t)\right| dr + \int\limits_{z_0-\rho}^{z_0+\rho} \left|\frac{\partial\Phi}{\partial z}(r_0+t, z)\right| dz \\ + \sup_{r\in[r_0-\rho, r_0+\rho]} |\Phi(r, z_0+t) - \Phi(x_0)| \\ + \sup_{z\in[z_0-\rho, z_0+\rho]} |\Phi(r_0+t, z) - \Phi(x_0)| < \varepsilon \Bigg\}. \end{aligned}$$

(v) *Take a function $g \in C^1(\mathbb{R}^2)$ and a closed set $F \subset P_+$ such that $\nabla g \neq 0$ on F. Then for almost all $y \in g(F)$ and for all connected components K of the set $F \cap g^{-1}(y)$ we have $K \cap A_{\mathbf{v}} = \emptyset$, the restriction $\Phi|_K$ is an absolutely continuous function, while* (3.3) *and* (3.13) *hold $\mathfrak{H}^1$-almost everywhere on K.*

Most of these properties are from [7]. For a detailed proof of Theorem 3.7 see, for example, [15]. The property (iv) follows directly from (3.25). The last property (v) follows (by coordinate transformation, *cf.* [24, Section 1.1.7]) from the well-known fact that every function $f \in W^{1,1}$ is absolutely continuous along almost all coordinate lines. The same fact together with (3.19) and (3.13) implies

Lemma 3.8. *For almost all $r_0 > 0$ we have $L_{r_0} \cap A_{\mathbf{v}} = \emptyset$; moreover, $p(r_0, \cdot)$ and $\mathbf{v}(r_0, \cdot)$ are absolutely continuous functions (locally) and*

$$\frac{\partial \Phi}{\partial z}(r_0, z) = 0 \quad \text{for almost all } z \in \mathbb{R} \text{ with } v_r(r_0, z) = 0. \tag{3.27}$$

Below we prove that Bernoulli's Law (Theorem 3.4) holds with the set $A_{\mathbf{v}}$ from Theorem 3.7. This requires several lemmas.

Lemma 3.9. *For almost all $y \in \psi(P_+)$ we have*

$$\psi^{-1}(y) \cap A_{\mathbf{v}} = \emptyset, \tag{3.28}$$

and for each continuum[2] *$K \subset \psi^{-1}(y)$ the identities*

$$\Phi(x_1) = \Phi(x_2) \quad \text{for all } x_1, x_2 \in K \tag{3.29}$$

hold.

Proof. Fix some $\varepsilon > 0$ and consider a function $g \in C^1(\mathbb{R}^2)$ and an open set V with $\mathfrak{H}^1(V) < \varepsilon$ from Theorem 2.16 applied to the function $\psi|_{P_\varepsilon}$, where the rectangle P_ε is defined by (3.24). Put $F = P_\varepsilon \setminus \psi^{-1}(V)$. Then $\psi(x) = g(x)$ and $\nabla\psi(x) = \nabla g(x) \neq 0$ for all $x \in F$. Thus, by Theorem 3.7 (v) for almost all $y \in \psi(P_\varepsilon) \setminus V = g(F)$ and for every connected component K of the set $\{x \in P_\varepsilon : \psi(x) = y\}$ the equality $K \cap A_{\mathbf{v}} = \emptyset$ holds and the restriction $\Phi|_K$ is absolutely continuous; moreover, for every C^1-smooth parametrization $\gamma : [0, 1] \to K$ the identity (3.13) gives

$$[\Phi(\gamma(t))]' = \nabla\Phi(\gamma(t)) \cdot \gamma'(t) = 0 \quad \text{for } \mathfrak{H}^1\text{-almost all } t \in [0, 1]$$

(the last equality is valid because $\psi(x) = \text{const}$ on K, and hence $\nabla\psi(\gamma(t))\cdot\gamma'(t) = r(-v_z(\gamma(t)), v_r(\gamma(t))) \cdot \gamma'(t) = 0$). Thus, $\Phi(x) = \text{const}$ on K. Since $\varepsilon > 0$ is arbitrary, the assertion of the lemma follows. □

We also need certain technical facts about the continuity properties of Φ at "good" points $x \in P_+ \setminus A_{\mathbf{v}}$.

Lemma 3.10. *Take $x_0 \in P_+ \setminus A_{\mathbf{v}}$. Suppose that there exist a constant $\sigma > 0$ and a sequence of continua $K_j \subset P_+ \setminus A_{\mathbf{v}}$ with $\Phi|_{K_j} \equiv \beta_j$ and $K_j \subset B_{x_0}(\rho_j)$, where $\rho_j \to 0$ as $j \to \infty$, and $\operatorname{diam}(K_j) \geq \sigma\rho_j$. Then $\beta_j \to \Phi(x_0)$ as $j \to \infty$.*

[2] By *a continuum* we mean a compact connected set.

Proof. Without loss of generality we may assume that the projection of each K_j on the O_r-axis is a segment $I_j \subset [r_0 - \rho_j, r_0 + \rho_j]$ of length $\geq \frac{1}{2}\sigma\rho_j$ (otherwise the same fact holds for the projection of K_j on the O_z-axis). By Theorem 3.7 (iv), for every $\varepsilon > 0$ we have $I_j - \{r_0\} \cap E(x_0, \varepsilon, \rho_j) \neq \emptyset$ for sufficiently large j. Thus, $|\beta_j - \Phi(x_0)| < \varepsilon$ for sufficiently large j. □

Lemma 3.11. *Suppose that for $r_0 > 0$ the assertion of Lemma 3.8 is fulfilled,* i.e., *$L_{r_0} \cap A_{\mathbf{v}} = \emptyset$, $p(r_0, \cdot)$ and $\mathbf{v}(r_0, \cdot)$ are absolutely continuous functions, and formula (3.27) is valid. Assume that $F \subset \mathbb{R}$ is a compact set such that*

$$\psi(r_0, z) \equiv \text{const} \quad \textit{for all } z \in F \tag{3.30}$$

and

$$\Phi(r_0, \alpha) = \Phi(r_0, \beta) \quad \textit{for every interval } (\alpha, \beta) \textit{ adjoining } F \tag{3.31}$$

(recall that (α, β) is called an interval adjoining F if $\alpha, \beta \in F$ and $(\alpha, \beta) \cap F = \emptyset$). Then

$$\Phi(r_0, z) \equiv \text{const} \quad \textit{for all } z \in F. \tag{3.32}$$

Proof. Take a pair $z', z'' \in F$ with $z' < z''$. Define a function $g(z)$ on the interval $[z', z'']$ by the rule $g(z) = \Phi(r_0, z)$. By construction, $g(\cdot)$ is an absolutely continuous function, and (3.31) implies that $g(\alpha) = g(\beta)$ for every interval $(\alpha, \beta) \subset [z', z'']$ adjoining F. Since by definition the absolutely continuous function $g(z)$ is differentiable almost everywhere and coincides with the Lebesgue integral of its derivative, we obtain

$$\int_\alpha^\beta g'(z)\, dz = 0.$$

Hence,

$$\int_\mu^\nu g'(z)\, dz = 0 \tag{3.33}$$

if $\mu, \nu \in F \cap [z', z'']$ and the interval (μ, ν) contains only finitely many points of F.

Consider now the closed set

$$F_\infty = \{z \in [z', z''] : \textit{every neighborhood of } z \textit{ contains infinitely many points of } F\}.$$

By (3.33),

$$\int_{[z', z''] \setminus F_\infty} g'(z)\, dz = 0. \tag{3.34}$$

According to properties (ii) of Theorem 3.7, the function ψ is differentiable at every point (r_0, z) with $z \in (z', z'')$. Hence, (3.30) yields $\psi_z(r_0, z) = 0$ for all $z \in F_\infty$. Using (3.8), we can rewrite the last fact as $v_r(r_0, z) = 0$ for all $z \in F_\infty$. Then (3.27) immediately implies that

$$\int\limits_{F_\infty} g'(z)\,dz = 0. \tag{3.35}$$

Adding up (3.34) and (3.35), we obtain

$$g(z') - g(z'') = \int\limits_{z'}^{z''} g'(z)\,dz = 0.$$

This is equivalent to the required equality $\Phi(r_0, z') = \Phi(r_0, z'')$. The proof of the lemma is complete. □

Proof of Theorem 3.4.

STEP 1. By Remark 3.6, we may assume without loss of generality that the continuum K is a connected component of the set $\{x \in P : \psi(x) = y_0\}$, where $y_0 \in \mathbb{R}$ and $P \subset P_+$ is a rectangle $P := \{(r, z) : r \in [r_1, r_2],\ z \in [z_1, z_2]\}$ with $r_1 > 0$, while $\psi(x) \equiv \xi_0$ and $\Phi(x) \equiv p_0$ for each $x \in \partial^* P$, where we denote

$$\partial^* P = \partial P \setminus \{(r_1, z) : z \in (z_1, z_2)\}.$$

Put $P^\circ = \operatorname{Int} P = (r_1, r_2) \times (z_1, z_2)$. Given $\varepsilon > 0$, denote by K_ε the connected component of the compact set $\{x \in P : \psi(x) \in [y_0 - \varepsilon, y_0 + \varepsilon]\}$ which includes K. Clearly, $K_\varepsilon \to K$ as $\varepsilon \to 0$ in the Hausdorff metric[3]. By Theorem 2.15 and Lemma 3.9 for almost all $\varepsilon > 0$ the set $P^\circ \cap \partial K_\varepsilon$ is a finite disjoint union of C^1-curves on which the functions ψ and Φ are constant. This implies that for each component U_i of the open set $P^\circ \setminus K$ there exists a sequence of continua $K^i_j \subset \bar{U}_i \setminus A_{\mathbf{v}}$ such that each K^i_j is a C^1-curve homeomorphic to the segment $[0, 1]$ or to the circle $\mathbb{S}^1$, furthermore, K^i_j is a connected component of the set $\{x \in P : \psi(x) = \alpha^i_j \neq y_0\}$, $\Phi|_{K^i_j} \equiv \beta^i_j$, $K^i_j \to K \cap \partial U_i$ as $j \to \infty$ in the Hausdorff metric, and for every $x \in U_i$ there exists an index j_x such that x and K lie in the different connected components of the set $P \setminus K^i_j$ for $j \geq j_x$. Using Lemma 3.10, it is easy to deduce from these facts that for every U_i the limit $\beta_i = \lim_{j\to\infty} \beta^i_j$ exists and

$$\Phi(x) = \beta_i \quad \text{for all } x \in K \cap \partial U_i \setminus A_{\mathbf{v}}. \tag{3.36}$$

[3] Recall that the Hausdorff metric d_H between two compact sets $A, B \subset \mathbb{R}^n$ is defined as follows: $d_H(A, B) = \max\big(\sup_{a\in A} \operatorname{dist}(a, B), \sup_{b\in B} \operatorname{dist}(b, A)\big)$ (*e.g.*, §7.3.1 in [4]).

STEP 2. We claim that for almost all $r_0 \in (r_1, r_2)$ the identities

$$\Phi(r_0, z') = \Phi(r_0, z'') \quad \forall\, (r_0, z'), (r_0, z'') \in K \tag{3.37}$$

hold. Indeed, take $r_0 \in (r_1, r_2)$ satisfying the assertion of Lemma 3.8 and points $(r_0, z'), (r_0, z'') \in K$. Put $F = \{z \in [z', z''] : (r_0, z) \in K\}$. By (3.36), we infer that $\Phi(r_0, \alpha) = \Phi(r_0, \beta)$ for every interval $(\alpha, \beta) \subset [z', z'']$ adjoining F. Thus, the required identity (3.37) follows directly from Lemma 3.11.

STEP 3. We claim that there exists $\beta_0 \in \mathbb{R}$ with

$$\beta_i \equiv \beta_0 \tag{3.38}$$

for each component U_i (see formula (3.36)). We split the proof of this claim into two cases.

(3a) Suppose that $K \cap \partial^* P \neq \emptyset$. Then by construction (see the beginning of Step 1) $y_0 = \xi_0$, $K \supset \partial^* P$, $\Phi|_{\partial^* P} \equiv p_0$, and we easily deduce from (3.36)–(3.37) that $\beta_i \equiv p_0$.

(3b) Suppose now that $K \cap \partial^* P = \emptyset$. Let U_1 be the component with $\partial^* P \subset \partial U_1$. Then for each horizontal line L_{r_0} if $L_{r_0} \cap K \neq \emptyset$, then $L_{r_0} \cap K \cap \partial U_1 \neq \emptyset$. Hence, (3.36)–(3.37) imply that $\beta_i \equiv \beta_1$. This justifies (3.38).

Now we can rewrite (3.36)–(3.37) as

$$\Phi(x) = \beta_0 \qquad \text{for all } x \in K \cap \partial U_i \setminus A_{\mathbf{v}} \text{ and each } i, \tag{3.39}$$

$$\Phi(r, z) = \beta_0 \quad \text{for almost all } r \in (r_1, r_2) \text{ and for every } (r, z) \in K \tag{3.40}$$

(here β_0 equals either p_0 or β_1).

STEP 4. We claim that

$$\Phi(x_0) = \beta_0 \tag{3.41}$$

for every $x_0 \in K \setminus A_{\mathbf{v}}$. Indeed, fix $x_0 = (r_0, z_0) \in K \setminus A_{\mathbf{v}}$ (for simplicity we suppose that $x_0 \in P^\circ$). We divide the proof of the claim into two cases.

(4a) Suppose that there exists $\delta > 0$ such that $K \cap \{(r_0 + t, z) : |z - z_0| \le |t|\} \neq \emptyset$ for every $t \in (-\delta, \delta)$. Then (3.41) follows from (3.40) and assertion (iv) of Theorem 3.7. Namely, fix $\varepsilon > 0$ and take $t \in (-\delta, \delta) \cap E(x_0, \varepsilon, \rho)$ (this intersection is nonempty for sufficiently small ρ) such that $L_{r_0+t} \cap A_{\mathbf{v}} = \emptyset$ and (3.40) holds for $r = r_0 + t$, *i.e.*,

$$\Phi(r_0 + t, z) = \beta_0 \qquad \text{for every } z \text{ with } (r_0 + t, z) \in K. \tag{3.42}$$

By construction, $|t| < \rho$. By our assumption (4a) there exists a point $(r_0 + t, z_t) \in K$ with $|z_t - z_0| \le |t| < \rho$. Theorem 3.7 (iv) implies that $|\Phi(r_0 + t, z_t) - \Phi(x_0)| < \varepsilon$. Using (3.42), we finally obtain $|\beta_0 - \Phi(x_0)| < \varepsilon$.

(4b) Suppose now that assumption (4a) is false. Then there exists a sequence $0 \neq t_k \to 0$ with

$$K \cap \{(r_0 + t_k, z) : |z - z_0| \leq |t_k|\} = \emptyset. \tag{3.43}$$

This implies that each segment $\{(r_0 + t_k, z) : |z - z_0| \leq |t_k|\}$ is included into some U_{i_k}. Denote by Q_k the open squares $Q_k = (r_0 - |t_k|, r_0 + |t_k|) \times (z_0 - |t_k|, z_0 + |t_k|)$. It is easy to deduce that for sufficiently large k each set $\overline{Q_k} \cap K \cap \partial U_{i_k}$ includes a continuum K_k with $\operatorname{diam}(K_k) \geq |t_k|$. Indeed, by construction there exists $r_k \in [r_0, r_0 + t_k)$ with $(r_k, z_0) \in \partial U_{i_k}$ (the existence of r_k follows from the inclusions $(r_0, z_0) \in K \subset \mathbb{R}^2 \setminus U_{i_k}$ and $(r_0 + t_k, z_0) \in U_{i_k}$). Take as K_k the closure of the connected component of $Q_k \cap \partial U_{i_k}$ containing the point (r_k, z_0). Then $K_k \cap \partial Q_k \neq \emptyset$, as otherwise there would be a contradiction with the connectedness of K. However, K_k does not intersect the segment $\{(r_0 + t_k, z) : |z - z_0| \leq |t_k|\}$ by assumption (3.43). Hence, K_k intersects at least one of the other three sides of ∂Q_k. In each case $\operatorname{diam}(K_k) \geq |t_k|$. Therefore, (3.41) follows from (3.39) and Lemma 3.10.

This justifies (3.41) for all $x_0 \in K \setminus A_{\mathbf{v}}$. Thus, the proof of Theorem 3.4 is complete. □

Proof of Theorem 3.3. To prove (3.7), we use Bernoulli's law and the fact that the axis O_z is "almost" a stream line. More precisely, O_z is a singularity line for $\mathbf{v}$, ψ and p, but it can be accurately approximated by regular stream lines (on which $\Phi = \text{const}$).

First of all, let us simplify the geometrical setting. Put

$$\tilde{\mathcal{D}} = \mathcal{D} \cup \bar{\mathcal{D}}_{M+1} \cup \cdots \cup \bar{\mathcal{D}}_N \tag{3.44}$$

and consider extensions of ψ and Φ to $\tilde{\mathcal{D}}$ by the formulas of Remark 3.6. Then the extended functions ψ and Φ inherit the properties of the original ones. Namely, Bernoulli's Law (see the assertion of Theorem 3.4) and (3.10)–(3.12) hold with $\mathcal{D}$ and $\mathcal{D}_\varepsilon$ replaced by $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{D}}_\varepsilon$. Below these facts suffice. Thus, we may assume without loss of generality that $N = M$, *i.e.*, that $\tilde{\mathcal{D}} = \mathcal{D}$ is a *simply connected* plane domain.

By (3.10), there exists a sequence $r_i \to 0+$ such that

$$\int_{L_i} |\nabla \psi| \, dz \to 0 \quad \text{as } i \to +\infty \tag{3.45}$$

for the lines $L_i = \{(r, z) \in \bar{\mathcal{D}} : r = r_i\}$. Fix a point $x_0 \in \mathcal{D}$ and denote by $\mathcal{D}^i$ the connected component of the open set $\{(r, z) \in \mathcal{D} : r > r_i\}$ containing x_0. Obviously, for sufficiently large i the open set $\mathcal{D}^i$ is a simply connected plane domain with a Lipschitz boundary, $\psi \in W^{2,1}(\mathcal{D}^i) \subset C(\bar{\mathcal{D}}^i)$. In addition,

$$\partial\mathcal{D}^i \setminus L_i = C_0^i \cup \cdots \cup C_M^i, \tag{3.46}$$

$$C_j^i \cap L_i \neq \emptyset, \quad j = 0, \dots, M, \tag{3.47}$$

where $C_j^i = C_j \cap \{(r,z) \in \mathcal{D} : r \geq r_i\}$, $j = 0, \dots, M$. Using (3.20) and (3.45), we conclude that

$$\operatorname{diam}(\psi(\partial\mathcal{D}^i)) = \sup_{x,y\in\partial\mathcal{D}^i} |\psi(x) - \psi(y)| \to 0. \tag{3.48}$$

In particular, $\xi_0 = \cdots = \xi_M$, *i.e.*,

$$\psi|_{P_+\cap\partial\mathcal{D}} \equiv \xi_0 \equiv \psi|_{\partial\mathcal{D}^i\setminus L_i}, \qquad \sup_{x\in\partial\mathcal{D}^i} |\psi(x) - \xi_0| \to 0. \tag{3.49}$$

Our plan for the remainder of the proof is as follows. First, we prove that for every $x \in P_+ \cap \bar{\mathcal{D}}$ there exists a set $U(x)$ such that

$$x \in U(x) \subset P_+ \cap \bar{\mathcal{D}}, \quad O_z \cap \partial U(x) \neq \emptyset,$$

$$\psi|_{P_+\cap\partial U(x)} \equiv \xi_0, \tag{3.50}$$

$$\exists\, \beta(x) \in \mathbb{R} : \quad \Phi(y) = \beta(x) \;\; \forall y \in P_+ \cap (\partial U(x)) \setminus A_{\mathbf{v}}. \tag{3.51}$$

Observe that $\psi|_{P_+\cap\partial U(x)} = \xi_0$ is independent of x, while $\Phi|_{P_+\cap\partial U(x)} = \beta(x)$ can a priori depend on x. However, we prove eventually that $\beta(x) \equiv p_0$ for all $x \in P_+ \cap \bar{\mathcal{D}}$. This fact will easily imply the required equalities (3.7).

Define an equivalence relation on $\bar{\mathcal{D}}^i$ by the rule $x \sim_i y \Leftrightarrow \exists$ a continuum $K \subset \bar{\mathcal{D}}^i$ such that $\psi|_K \equiv \text{const}$ and both x, y lie outside the unbounded connected component of the open set $\mathbb{R}^2 \setminus K$. Denote by $U_i(x)$ the corresponding equivalence class. Let us illustrate this definition by some examples.

- (I$\sim$) If $K \subset \bar{\mathcal{D}}^i$ is a continuum and $\psi|_K = \text{const}$, then $x \sim_i y$ for all $x, y \in K$.
- (II$\sim$) If $K \subset \bar{\mathcal{D}}^i$ is homeomorphic to the circle and $\psi|_K \equiv \text{const}$, then $x \sim_i y$ for all $x, y \in U$, where U is a bounded domain with $\partial U = K$.

The following properties of the relation $\sim_i$ hold for every $x \in \bar{\mathcal{D}}^i$ (for a proof see Appendix).

- (III$\sim$) $U_i(x) \subset U_{i+1}(x)$ and every $U_i(x)$ is a compact set.
- (IV$\sim$) The set $U_i(x)$ is connected.
- (V$\sim$) $\psi|_{\partial U_i(x)} \equiv \text{const}$.
- (VI$\sim$) The set $\mathbb{R}^2 \setminus U_i(x)$ is connected.
- (VII$\sim$) The set $\partial U_i(x)$ is connected.

(VIII$_\sim$) We have

$$L_i \cap \partial U_i(x) \neq \emptyset. \tag{3.52}$$

For $x \in \bar{\mathcal{D}} \setminus O_z$ put $U(x) = \bigcup_i U_i(x)$. It is topologically obvious that

$$\forall y \in P_+ \cap \partial U(x) \ \exists \ \text{a sequence} \ \ \partial U_i(x) \ni y_i \to y. \tag{3.53}$$

Then (V$_\sim$), (3.48)–(3.49), and (3.52) imply (3.50).

Bernoulli's Law (see Theorem 3.4) implies that

$$\forall x \in P_+ \cap \bar{\mathcal{D}} \ \ \exists \beta_i(x): \quad \Phi(y) = \beta_i(x) \ \text{ for all } \ y \in \partial U_i(x) \setminus A_{\mathbf{v}}. \tag{3.54}$$

Fix a point $y_* \in P_+ \cap \partial U(x) \setminus A_{\mathbf{v}}$ and j such that $y_* \in \bar{\mathcal{D}}^j \setminus L_j$. By construction (see properties (VII$_\sim$)–(VIII$_\sim$) and (3.53)) there exist sequences of continua $K_i \subset \bar{\mathcal{D}}^j \cap \partial U_i(x)$ and points $y_i \in K_i$ such that $K_i \cap L_j \neq \emptyset$ for all sufficiently large i, $y_i \to y_*$, and K_i converges as $i \to \infty$ to some set K with respect to the Hausdorff metric. Hence, $y_* \in K$, K is a compact connected set, $\psi|_K \equiv \xi_0 = \text{const}$, and $K \cap L_j \neq \emptyset$. Consequently,

$$\operatorname{diam} K > 0. \tag{3.55}$$

Again Bernoulli's Law implies that

$$\exists \beta \in \mathbb{R}: \qquad \Phi(y) = \beta \ \text{ for all } y \in K \setminus A_{\mathbf{v}}. \tag{3.56}$$

Using (3.54)–(3.55), the connectedness of K and K_i, and the continuity properties of Φ (see Theorem 3.7 (iii)), we obtain

$$\lim_{i\to\infty} \beta_i(x) = \beta.$$

In particular,

$$\Phi(y_*) = \lim_{i\to\infty} \beta_i(x).$$

Because the right-hand side here is independent of the choice of $y_* \in P_+ \cap \partial U(x) \setminus A_{\mathbf{v}}$, we have justified (3.51) with $\beta(x) = \lim_{i\to\infty} \beta_i(x)$.

Now take $r_0 > 0$ satisfying the assertion of Lemma 3.8 and the points (r_0, z'), $(r_0, z'') \in P_+ \cap \partial\mathcal{D}$ such that $\{(r_0, z) : z \in (z', z'')\} \subset \mathcal{D}$. To complete the proof of the theorem, we must show that

$$\Phi(r_0, z') = \Phi(r_0, z''). \tag{3.57}$$

Put

$$F = \{z \in [z', z''] : \ (r_0, z) \in \partial U((r_0, z))\}.$$

Then by construction $z', z'' \in F$ and the set F is compact. Indeed, put $x' = (r_0, z')$ and $x'' = (r_0, z'')$. Since $U(x') \subset P_+ \cap \bar{\mathcal{D}}$, we have $\partial\mathcal{D} \ni x' \notin \operatorname{Int} U(x')$, and consequently, $x' \in \partial U(x')$. Similarly, $x'' \in \partial U(x'')$, *i.e.*, $z', z'' \in F$. Furthermore, take $z_k \to z_0$, $z_k \in F$. Put $x_k = (r_0, z_k)$. Then $x_k \in \partial U(x_k)$ and $x_k \to x_0 = (r_0, z_0)$. Of course, $x_0 \notin \operatorname{Int} U(x_0)$, for otherwise $x_k \in \operatorname{Int} U(x_0) = \operatorname{Int} U(x_k)$ for large k. Therefore, $x_0 \in \partial U(x_0)$, *i.e.*, $z_0 \in F$. Hence, $z', z'' \in F$ and F is a compact set.

Now (3.50)–(3.51) yield (3.30)–(3.31). Thus, Lemma 3.11 implies the required equality (3.57). □

In the course of the last proof we established in particular the following assertion.

Lemma 3.12. *Assume that conditions* (E) *are fulfilled. Let K_i be a sequence of compact sets such that $K_i \subset \bar{\mathcal{D}} \cap P_+$, $\psi|_{K_i} = \text{const}$, and there exist $x_i, y_i \in K_i$ with* $\operatorname{dist}(x_i, O_z) \to 0$ *and* $\operatorname{dist}(y_i, O_z) \not\to 0$. *Then there exist $\beta_i \in \mathbb{R}$ such that $\Phi(x) \equiv \beta_i$ for all $x \in K_i \setminus A_{\mathbf{v}}$ and $\beta_i \to p_0$ as $i \to \infty$.*

Let $U \subset \mathbb{R}^2$ be a domain with Lipschitz boundary. We say that a function $f \in W^{1,s}(U)$ *satisfies the weak one-sided maximum principle locally* in U if

$$\operatorname*{ess\,sup}_{x \in U'} f(x) \le \operatorname*{ess\,sup}_{x \in \partial U'} f(x) \tag{3.58}$$

for every strictly interior subdomain U' (*i.e.*, $\bar{U}' \subset U$) whose boundary $\partial U'$ avoids the singleton connected components. (In (3.58) the negligible sets are those of the two-dimensional Lebesgue measure zero in the left esssup, and those of the one-dimensional Hausdorff measure zero in the right esssup.)

If (3.58) holds for every $U' \subset U$ (not necessarily strictly interior) whose boundary $\partial U'$ avoids the singleton connected components, then we say that $f \in W^{1,s}(U)$ *satisfies the weak one-sided maximum principle globally* in U (in particular, we can take $U' = U$ in (3.58)).

Theorem 3.13. *Let conditions* (E) *be fulfilled. Assume that there exists a sequence of functions $\{\Phi_\mu\}$ such that $\Phi_\mu \in W^{1,s}_{\text{loc}}(\mathcal{D})$ and $\Phi_\mu \rightharpoonup \Phi$ weakly in $W^{1,s}_{\text{loc}}(\mathcal{D})$ for some $s \in [4/3, 2)$. If all Φ_μ satisfy the weak one–side maximum principle locally in $\mathcal{D}$, then*

$$\operatorname*{ess\,sup}_{x \in \mathcal{D}} \Phi(x) \le \max_{j=0,\dots,N} p_j. \tag{3.59}$$

Proof. If the hypotheses of Theorem 3.13 hold, then Theorem 2 of [14] (see also [15] for a more detailed proof) implies

(∗) for every subdomain $U \subset \mathcal{D}$ with $\bar{U} \cap O_z = \emptyset$ the function $\Phi|_{\bar{U}}$ satisfies the weak one-sided maximum principle globally.

In order to prove the estimate (3.59) on the whole domain $\mathcal{D}$, we use the same methods as in the proof of Theorem 3.3. First of all, we simplify the situation: as above, define the domain $\tilde{\mathcal{D}}$ by (3.44) and extend the functions ψ and Φ into $\tilde{\mathcal{D}}$ using (3.21)–(3.23). The extended functions ψ and Φ inherit the properties of the original ones. Namely, (3.10)–(3.12) and Bernoulli's Law (see Theorem 3.4) hold with $\mathcal{D}$ and $\mathcal{D}_\varepsilon$ replaced by $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{D}}_\varepsilon$. Moreover, the maximum property $(*)$ holds with $\mathcal{D}$ replaced by $\tilde{\mathcal{D}}$. Since these facts suffice for the proof below, we may assume without loss of generality that $N = M$, *i.e.*, $\tilde{\mathcal{D}} = \mathcal{D}$ is a simply connected plane domain.

Suppose that the assertion of the theorem is false. Then there exists a point $x_* \in \mathcal{D} \setminus A_{\mathbf{v}}$ with

$$\Phi(x_*) = p_* > \max_{j=0,\dots,N} p_j. \tag{3.60}$$

Take a sequence of numbers $r_i \to +0$, the corresponding lines L_i and domains $\mathcal{D}^i$ as in the proof of Theorem 3.3 (in particular, (3.45) holds). Denote by K_i^* the connected component of the level set $\{x \in \bar{\mathcal{D}}^i \ : \ \psi(x) = \psi(x_*)\}$ containing x_*. By Bernoulli's Law,

$$\Phi(x) = p_* \quad \text{for all } i \text{ and for all } x \in K_i^* \setminus A_{\mathbf{v}}. \tag{3.61}$$

There are two possibilities:

(I) $K_i^* \cap L_i \neq \emptyset$ for all i. Then Lemma 3.12 yields $p_* = p_0$ and we arrive at a contradiction with assumption (3.60).

(II) There exists i_0 with $K_{i_0}^* \cap L_{i_0} = \emptyset$. Then the sequence K_i^* stabilizes after $i = i_0$:

$$K_i^* = K_{i_0}^*, \quad K_i^* \cap L_i = \emptyset \quad \text{for all } i \geq i_0. \tag{3.62}$$

Put $K^* = K_{i_0}^*$. Then by construction

$$K^* \cap \partial\mathcal{D}^i = \emptyset \quad \text{for all } i \geq i_0. \tag{3.63}$$

Now consider the family of sets $U_i(x_*)$ introduced in the proof of Theorem 3.3. By (3.54),

$$\Phi(y) = \beta_i(x_*) \quad \forall y \in \partial U_i(x_*) \setminus A_{\mathbf{v}}, \tag{3.64}$$

where

$$\lim_{i\to\infty} \beta_i(x_*) = p_0 \tag{3.65}$$

(the last convergence follows from Lemma 3.12). Take sufficiently large $i_1 \geq i_0$ with

$$\beta_i(x_*) < p_* \quad \text{for all } i \geq i_1. \tag{3.66}$$

Put $U = \operatorname{Int} U_{i_1}(x_*)$. By construction,

$$\operatorname*{ess\,sup}_{x\in U} \Phi(x) \geq p_* > \beta_{i_1}(x_*) = \operatorname*{ess\,sup}_{x\in\partial U} \Phi(x). \tag{3.67}$$

However, this contradicts $(*)$. The proof of Theorem 3.13 is complete. □

4. Proof of the existence theorem

Consider firstly the axially symmetric case with possible rotation. According to Lemma 2.12, in order to prove the existence of a solution to problem (1.1), it suffices to show that all possible solutions to the operator equation

$$\nu \mathbf{w} = \lambda T \mathbf{w}, \quad \lambda \in [0,1], \quad \mathbf{w} \in H_{AS}(\Omega) \tag{4.1}$$

are uniformly bounded in $H_{AS}(\Omega)$. We prove this estimate by contradiction, following the famous argument of J. Leray [23] (many other authors used it, *e.g.*, [20], [21], [13], [1], see also [15]).

Suppose that the solutions to (4.1) are not uniformly bounded in $H_{AS}(\Omega)$. Then there exists a sequence of functions $\mathbf{w}_k \in H_{AS}(\Omega)$ such that $\nu \mathbf{w_k} = \lambda_k T \mathbf{w_k}$ with $\lambda_k \in [0,1]$ and $J_k = \|\mathbf{w}_k\|_{H(\Omega)} \to \infty$. Observe that $\mathbf{w}_k$ and the corresponding axially symmetric pressures $p_k \in L^2_{AS}(\Omega)$ satisfy the integral identity

$$\begin{aligned} \nu \int_\Omega \nabla \mathbf{w}_k \cdot \nabla \boldsymbol{\eta}\, dx = &-\lambda_k \int_\Omega \big(\mathbf{U}\cdot\nabla\big)\mathbf{U}\cdot\boldsymbol{\eta}\, dx - \lambda_k \int_\Omega \big(\mathbf{U}\cdot\nabla\big)\mathbf{w}_k\cdot\boldsymbol{\eta}\, dx \\ &- \lambda_k \int_\Omega \big(\mathbf{w}_k\cdot\nabla\big)\mathbf{w}_k\cdot\boldsymbol{\eta}\, dx - \lambda_k \int_\Omega \big(\mathbf{w}_k\cdot\nabla\big)\mathbf{U}\cdot\boldsymbol{\eta}\, dx \\ &+ \int_\Omega p_k \operatorname{div} \boldsymbol{\eta}\, dx \end{aligned} \tag{4.2}$$

for every $\boldsymbol{\eta} \in \mathring{W}^{1,2}(\Omega)$. Here $\mathbf{U}$ is an axially symmetric solution to the Stokes problem (see Lemmas 2.6–2.7).

Put $\mathbf{u}_k = \mathbf{w}_k + \mathbf{U}$, $\widehat{\mathbf{u}}_k = \frac{1}{J_k}\mathbf{u}_k$, $\widehat{\mathbf{w}}_k = \frac{1}{J_k}\mathbf{w}_k$, and $\widehat{p}_k = \frac{1}{J_k^2} p_k$. Then $\|\widehat{\mathbf{w}}_k\|_{H(\Omega)} = 1$ and we have

$$\|\widehat{p}_k\|_{L^2(\Omega)} \le \text{const}, \qquad \|\widehat{p}_k\|_{W^{1,3/2}(\Omega')} \le \text{const}$$

for every $\bar{\Omega}' \subset \Omega$ (for a detailed proof of the above estimates see [15] for instance). Extracting subsequences, we may assume without loss of generality that

$$\lambda_k \to \lambda_0 \in [0,1], \tag{4.3}$$

$$\widehat{\mathbf{u}}_k \rightharpoonup \mathbf{v} \in H(\Omega) \quad \text{weakly in } W^{1,2}(\Omega), \tag{4.4}$$

$$\widehat{p}_k \rightharpoonup p \in W^{1,3/2}_{\mathrm{loc}}(\Omega) \cap L^2(\Omega) \quad \text{weakly in } L^2(\Omega) \text{ and in } W^{1,3/2}_{\mathrm{loc}}(\Omega). \tag{4.5}$$

Multiplying (4.2) for an arbitrary fixed $\boldsymbol{\eta} \in \mathring{W}^{1,2}(\Omega)$ by J_k^{-2} and passing to the limit as $k \to \infty$, we find that the limit functions $\mathbf{v}$ and p satisfy the Euler equations

$$\begin{cases} \lambda_0 \big(\mathbf{v}\cdot\nabla\big)\mathbf{v} + \nabla p = 0, \\ \operatorname{div} \mathbf{v} = 0 \\ \mathbf{v}|_{\partial\Omega} = 0 \end{cases} \tag{4.6}$$

(for details of the proof see [15] for example). By (4.4) and (4.5), this implies that $p \in W^{1,3/2}(\Omega)$. Thus, assumptions (E) in the beginning of Section 3 are fulfilled. Moreover, $\|\mathbf{v}\|_{H(\Omega)} \le 1$.

Now, taking $\boldsymbol{\eta} = J_k^{-2}\mathbf{w}_k$ in (4.2), we obtain

$$\nu \int_\Omega |\nabla \widehat{\mathbf{w}}_k|^2 \, dx = \lambda_k \int_\Omega \big(\widehat{\mathbf{w}}_k \cdot \nabla\big)\widehat{\mathbf{w}}_k \cdot \mathbf{U}\, dx + J_k^{-1}\lambda_k \int_\Omega \big(\mathbf{U} \cdot \nabla\big)\widehat{\mathbf{w}}_k \cdot \mathbf{U}\, dx \qquad (4.7)$$

Using the compact embedding $H(\Omega) \hookrightarrow L^r(\Omega)$ with $r < 6$, we can pass to the limit as $k \to \infty$ in equality (4.7). This yields

$$\nu = \lambda_0 \int_\Omega \big(\widehat{\mathbf{v}} \cdot \nabla\big)\widehat{\mathbf{v}} \cdot \mathbf{U}\, dx. \qquad (4.8)$$

From the last formula and the Euler equation (4.6), we derive

$$\nu = -\int_\Omega \nabla p \cdot \mathbf{U}\, dx = -\int_\Omega \operatorname{div}(p\,\mathbf{U})\, dx = -\int_{\partial\Omega} p\,\mathbf{a} \cdot \mathbf{n}\, dS. \qquad (4.9)$$

Because of (3.4), we can rearrange this as

$$\sum_{j=0}^{N} p_j \mathcal{F}_j = -\nu. \qquad (4.10)$$

Now, using (1.2) and (3.7), we deduce from (4.10) that

$$p_0 \sum_{j=0}^{M} \mathcal{F}_j + \sum_{j=M+1}^{N} p_j \mathcal{F}_j = \sum_{j=M+1}^{N} \mathcal{F}_j (p_j - p_0) = -\nu. \qquad (4.11)$$

To begin with, consider case (1.5). If condition (1.5) is fulfilled with $\delta = \frac{1}{\delta_1 (N-M)}\nu$, where δ_1 is a constant of Lemma 3.2, then (4.11) and (3.6) lead to a contradiction (recall that $\|\mathbf{v}\|_{H(\Omega)} \le 1$ and $\lambda_0 \in [0,\ 1]$). Thus, the proof of case (1.5) is complete.

Consider now the case that condition (1.4) is fulfilled. Then (4.11) becomes

$$\mathcal{F}_N (p_0 - p_N) = \nu. \qquad (4.12)$$

By (1.4) and (4.12),

$$p_0 > p_N. \qquad (4.13)$$

Consider the identity

$$\begin{aligned} \operatorname{div}\big(xp + \lambda_0(\mathbf{v} \cdot x)\mathbf{v}\big) &= \big(x \cdot \nabla p + x \cdot \lambda_0 (\mathbf{v} \cdot \nabla)\mathbf{v}\big) + 3p + \lambda_0 |\mathbf{v}|^2 \\ &= 3\left(p + \frac{\lambda_0}{2}|\mathbf{v}|^2\right) - \frac{\lambda_0}{2}|\mathbf{v}|^2 = 3\Phi - \frac{\lambda_0}{2}|\mathbf{v}|^2. \end{aligned}$$

Integrating by parts in Ω, we obtain

$$
\begin{aligned}
3\int_{\Omega} \Phi dx - \frac{\lambda_0}{2}\int_{\Omega} |\mathbf{v}|^2 dx &= \int_{\partial\Omega} p\,(x\cdot\mathbf{n}) dS \\
&= p_0 \int_{\Gamma_0} (x\cdot\mathbf{n}) dS + p_0 \sum_{j=1}^{N-1} \int_{\Gamma_j} (x\cdot\mathbf{n}) dS + p_N \int_{\Gamma_N} (x\cdot\mathbf{n}) dS \\
&= p_0 \int_{\Omega_0} \operatorname{div} x dx - p_0 \sum_{j=1}^{N-1} \int_{\Omega_j} \operatorname{div} x dx - p_N \int_{\Omega_N} \operatorname{div} x dx \\
&= 3p_0\big(|\Omega_0| - \sum_{j=1}^{N-1} |\Omega_j|\big) - 3p_N|\Omega_N| \\
&= 3p_0|\Omega| + 3(p_0 - p_N)|\Omega_N|.
\end{aligned}
$$

Hence,

$$
\int_{\Omega} \Phi dx \ge \int_{\Omega} \Phi dx - \frac{\lambda_0}{6}\int_{\Omega} |\mathbf{v}|^2 dx = p_0|\Omega| + (p_0 - p_N)|\Omega_N|. \tag{4.14}
$$

The total head pressures $\Phi_k = p_k + \frac{\lambda_k}{2}|\mathbf{u}_k|^2$ for the Navier–Stokes system (1.1) satisfy the equations

$$
\nu\Delta\Phi_k - \lambda_k \mathbf{u}_k \cdot \nabla\Phi_k = \nu|\operatorname{curl}\mathbf{u}_k|^2 \ge 0.
$$

Hence (*e.g.*, [25]), Φ_k satisfy the one-sided maximum principle locally in Ω. Put $\widehat{\Phi}_k = \frac{1}{J_k^2}\Phi_k$. By (4.4)–(4.5) and the symmetry assumptions, the sequence $\{\widehat{\Phi}_k\}$ weakly converges to $\Phi = p + \dfrac{\lambda_0}{2}|\mathbf{v}|^2$ in the space $W^{1,3/2}_{\mathrm{loc}}(\mathcal{D})$. Therefore, by Theorem 3.13,

$$
\operatorname*{ess\,sup}_{x\in\Omega} \Phi(x) = \operatorname*{ess\,sup}_{x\in\mathcal{D}} \Phi(x) \le \max_{j=0,\dots,N} p_j = p_0 \tag{4.15}
$$

(the last equality follows from the conditions $N = M + 1$ and (4.13)). Then (4.14) yields

$$
p_0|\Omega| + (p_0 - p_N)|\Omega_N| \le p_0|\Omega| \quad \Leftrightarrow \quad p_0 \le p_N,
$$

and we arrived at a contradiction with (4.13), which proves the theorem in case (1.4).

If the boundary value $\mathbf{a}$ is axially symmetric without rotation, then the proof of Theorem 1.1 is the same as in the first part; we only have to use Lemma 2.13 instead of Lemma 2.12.

5. Appendix

Let us establish the topological properties (III$_\sim$)–(VIII$_\sim$) of the equivalence class $U_i(x)$ for $x \in \bar{\mathcal{D}}^i$ used in the proof of Theorem 3.3.

(III$_\sim$) Indeed, if $U_i(x) \ni y_j \to y$, then by definition there exists a sequence of continua K_j with $\psi|_{K_j} = \text{const}$ such that x and y_j lie outside the unbounded connected component of the set $\mathbb{R}^2 \setminus K_j$. Without loss of generality we may assume that K_j converge in the Hausdorff metric to a set K. Then K is a continuum, $\psi|_K = \text{const}$, and it is easy to see that neither x nor y belongs to the unbounded connected component of the open set $\mathbb{R}^2 \setminus K$.

(IV$_\sim$) Fix some $y \in U_i(x)$. Take the corresponding set K in the definition of $x \sim_i y$. Then $K \subset \bar{\mathcal{D}}^i$ is a compact connected set with $\psi|_K \equiv \text{const}$ such that both x and y lie outside the unbounded connected component of the open set $\mathbb{R}^2 \setminus K$. Denote the family of connected components of $\mathbb{R}^2 \setminus K$ by V_j and take an unbounded component V_0. Since $\mathcal{D}^i$ is a simply connected domain, we have $\bar{V}_j \subset \bar{\mathcal{D}}^i$ for all $j \neq 0$. Hence, the definition of $\sim_i$ yields $\bar{V}_j \subset U_i(x)$ for all $j \neq 0$. By construction, the sets K and $\bar{V}_j$ are connected and $K \cap \bar{V}_j \neq \emptyset$. Therefore, the set $S_y = K \cup \left(\bigcup_{j \neq 0} \bar{V}_j \right)$ is connected and $\{x, y\} \subset S_y \subset U_i(x)$. Since $y \in U_i(x)$ is arbitrary, the connectedness of $U_i(x)$ follows.

(V$_\sim$) To prove that $\psi|_{\partial U_i(x)} = \text{const}$, we may assume without loss of generality that $x \in \partial U_i(x)$. Fix some $y \in \partial U_i(x)$. Take the corresponding set K in the definition of $x \sim_i y$ and the sets V_j in the proof of property (IV$_\sim$). Then it is easy to see that

$$x, y \in K. \tag{5.1}$$

Indeed, if $y \notin K$ for example, then $y \in V_j$ for some $j \neq 0$. But by construction V_j is an open set and $V_j \subset U_i(x)$, in contradiction with the assumption that $y \in \partial U_i(x)$. This proves (5.1), using which and the assumption $\psi|_K \equiv \text{const}$, we obtain the required equality $\psi(y) = \psi(x)$.

Using similar elementary arguments, we can easily prove the next two properties (VI$_\sim$)–(VII$_\sim$). Therefore, we prove in detail only the last property (VIII$_\sim$).

(VIII$_\sim$) Suppose that (3.52) fails, *i.e.*,

$$L_i \cap \partial U_i(x) = \emptyset. \tag{5.2}$$

Hence,

$$L_i \cap U_i(x) = \emptyset. \tag{5.3}$$

By (V$_\sim$), $\psi|_{\partial U_i(x)}$ equals some constant, denote it by c_0. Fix $y_0 \in \partial U_i(x)$. Properties (I$_\sim$), (V$_\sim$) and (VII$_\sim$) yield

$$\partial U_i(x) \subset K_0 \subset U_i(x), \tag{5.4}$$

where we denote by K_0 the connected component of the level set $\{y \in \bar{\mathcal{D}}^i : \psi(y) = c_0\}$ containing the point y_0.

By construction, the closure of each connected component $\tilde{C}$ of $(\partial\mathcal{D}^i) \setminus L_i$ intersects the line L_i and $\psi|_{\tilde{C}} \equiv \text{const}$ (see (3.46)–(3.47), (3.49)). Hence, conditions (5.3)–(5.4) imply that

$$K_0 \cap \partial\mathcal{D}^i = U_i(x) \cap \partial\mathcal{D}^i = \emptyset. \tag{5.5}$$

Take a sequence $0 < \delta_j \to 0$ such that all values $c_0 + \delta_j$ and $c_0 - \delta_j$ are regular from the viewpoint of the Morse–Sard theorem (see Theorem 2.15 (iii)). Denote by B_j the connected component of the level set $\{y \in \bar{\mathcal{D}}^i : \psi(y) \in [c_0 - \delta_j, c_0 + \delta_j]\}$ containing K_0. Then for sufficiently large j the boundary ∂B_j amounts to a finite disjoint family of C^1–cycles in $\mathcal{D}^i$ (this follows from (5.5) and from the evident convergence $\sup\limits_{y\in B_j} \operatorname{dist}(y, K_0) \to 0$).

Denote by $K_j \subset \partial B_j$ the cycle separating B_j from infinity, and by U_j the bounded domain with $\partial U_j = K_j$. Then by construction $\psi|_{K_j} \equiv \text{const}$, $K_0 \cap K_j = \emptyset$, and $K_0 \subset U_j$. Consequently,

$$U_i(x) \subsetneqq U_j. \tag{5.6}$$

On the other hand, by property (II$_\sim$) all points of U_j are $\sim_i$ equivalent, which contradicts (5.6) and the definition of $U_i(x)$. This justifies (3.52).

ADDED IN PROOF. After the paper was submitted, these results and techniques allowed us to prove the existence theorem for plane and 3D axially symmetric spatial stationary flows in the general situation: under the necessary and sufficient condition of the zero total flux (see [17]).

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Sobolev Institute of Mathematics
Koptyuga pr. 4
and Novosibirsk State University
Pirogova Str. 2
630090 Novosibirsk, Russia
korob@math.nsc.ru

Faculty of Mathematics and Informatics
Vilnius University
Naugarduko Str., 24
Vilnius, 03225 Lithuania
pileckas@ktl.mii.lt

Dipartimento di Matematica e Fisica
Seconda Università di Napoli
Via Vivaldi, 43
81100 Caserta, Italia
remigio.russo@unina2.it