

Partial Gaussian bounds for degenerate differential operators II

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Abstract. Let $A = -\sum \partial_k c_{kl} \partial_l$ be a degenerate sectorial differential operator with complex bounded measurable coefficients. Let $\Omega \subset \mathbb{R}^d$ be open and suppose that A is strongly elliptic on Ω . Further, let $\chi \in C_b^\infty(\mathbb{R}^d)$ be such that an ε -neighbourhood of $\text{supp } \chi$ is contained in Ω . Let $\nu \in (0, 1]$ and suppose that the $c_{kl}|_\Omega \in C^{0,\nu}(\Omega)$. Then we prove (Hölder) Gaussian kernel bounds for the kernel of the operator $u \mapsto \chi S_t(\chi u)$, where S is the semigroup generated by $-A$. Moreover, if $\nu = 1$ and the coefficients are real, then we prove Gaussian bounds for the kernel of the operator $u \mapsto \chi S_t u$ and for the derivatives in the first variable. Finally we show boundedness on $L_p(\mathbb{R}^d)$ of restricted Riesz transforms.

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1. Introduction

If A is a strongly elliptic second-order operator on \mathbb{R}^d in divergence form with complex bounded Hölder continuous coefficients, then it is well known that it generates a holomorphic semigroup S which satisfies Gaussian kernel bounds and Gaussian bounds for first order derivatives in each of the variables. If A is merely partially strongly elliptic on an open set $\Omega \subset \mathbb{R}^d$ then in general Gaussian bounds on \mathbb{R}^d fail, but in a previous paper [5] we showed Gaussian kernel bounds on good parts of Ω if the coefficients of A are real and measurable. Precisely, if $\chi \in C_b^\infty(\Omega, \mathbb{R})$ and if A is strongly elliptic on $\text{supp } \chi$, then for all $t > 0$ the operator $M_\chi S_t M_\chi$ has a Hölder continuous kernel satisfying (Hölder) Gaussian bounds, where M_χ is the multiplication operator with the function χ . In this paper we extend this to (Hölder) derivatives of the kernel if the coefficients of the operator A are complex Hölder continuous on Ω and the distance $d(\text{supp } \chi, \Omega^c) > 0$, that is an ε -neighbourhood of $\text{supp } \chi$ is still in Ω . If in addition the coefficients are in $W^{1,\infty}(\Omega)$ and real on

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\mathbb{R}^d , then we also show that for all $t > 0$ the operator $M_\chi S_t$ has a kernel K_t satisfying Gaussian bounds. This is remarkable, since there is no cut-off for the operator $M_\chi S_t$ on the right. Moreover, we show that there exists a representative of the kernel K_t such that $(t, x, y) \mapsto K_t(x, y)$ is measurable on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ and $x \mapsto K_t(x, y)$ is once differentiable for all $y \in \mathbb{R}^d$ and $t > 0$, and the derivatives satisfy (Hölder) Gaussian bounds. This allows to prove boundedness of the restricted Riesz transform $\nabla M_\chi (I + A)^{-1/2}$ on $L_p(\mathbb{R}^d)$ for all $p \in (1, \infty)$.

Throughout this paper the field is \mathbb{C} . Fix $d \in \mathbb{N}$ and for all $k, l \in \{1, \dots, d\}$ let $c_{kl}: \mathbb{R}^d \rightarrow \mathbb{C}$ be a measurable bounded function. Suppose that the matrix $C(x) := (c_{kl}(x))$ is uniformly sectorial for all $x \in \mathbb{R}^d$, i.e., there exists a $\theta \in [0, \frac{\pi}{2})$ such that

$$\sum_{k,l=1}^d c_{kl}(x) \xi_k \bar{\xi}_l \in \Sigma_\theta$$

for all $(\xi_1, \dots, \xi_d) \in \mathbb{C}^d$ and $x \in \mathbb{R}^d$, where

$$\Sigma_\theta = \{r e^{i\alpha} : r \geq 0 \text{ and } \alpha \in [-\theta, \theta]\}.$$

Define the form $\mathfrak{a}: W^{1,2}(\mathbb{R}^d) \times W^{1,2}(\mathbb{R}^d) \rightarrow \mathbb{C}$ by

$$\mathfrak{a}(u, v) = \sum_{k,l=1}^d \int_{\mathbb{R}^d} c_{kl} (\partial_k u) \overline{\partial_l v}.$$

Then \mathfrak{a} is a densely defined sectorial form. In general \mathfrak{a} is not closable, but nevertheless one can assign a semigroup generator A with \mathfrak{a} as follows. If $u, f \in L_2(\mathbb{R}^d)$ then $u \in D(A)$ and $Au = f$ if and only if there exist $u_1, u_2, \dots \in W^{1,2}(\mathbb{R}^d)$ such that $\lim u_n = u$ in $L_2(\mathbb{R}^d)$, $\sup \operatorname{Re} \mathfrak{a}(u_n) < \infty$ and $\lim \mathfrak{a}(u_n, v) = (f, v)$ for all $v \in W^{1,2}(\mathbb{R}^d)$. The operator A is well defined and is m -sectorial by Theorem 1.1 in [3]. If \mathfrak{a} is closable then A is the operator associated with the closure $\bar{\mathfrak{a}}$ of the form \mathfrak{a} in the sense of Kato [15]. We call A the sectorial degenerate differential operator with coefficients (c_{kl}) . Formally, $A = -\sum_{k,l} \partial_l c_{kl} \partial_k$. We denote by $S = (S_t)_{t>0}$ the contraction semigroup generated by $-A$ on $L_2(\mathbb{R}^d)$. Then S is holomorphic on the open sector $\Sigma_{\theta_\alpha}^\circ$, where throughout this paper we define $\theta_\alpha = \frac{\pi}{2} - \theta$. Let $\Omega \subset \mathbb{R}^d$ open. We suppose that the coefficients of A are strongly elliptic on Ω , that is, there exists a $\mu > 0$ such that

$$\operatorname{Re} \sum_{k,l=1}^d c_{kl}(x) \xi_k \bar{\xi}_l \geq \mu |\xi|^2$$

for all $\xi \in \mathbb{C}^d$ and a.e. $x \in \Omega$.

The main results of this paper are the following. The first theorem is for complex Hölder continuous coefficients on Ω , but with a multiplication operator on both sides of the semigroup.

Theorem 1.1. *Let A be a sectorial degenerate differential operator with coefficients (c_{kl}) , where $c_{kl}: \mathbb{R}^d \rightarrow \mathbb{C}$ is a bounded measurable function for all $k, l \in \{1, \dots, d\}$. Let $\Omega \subsetneq \mathbb{R}^d$ be open and suppose that (c_{kl}) is strongly elliptic on Ω . Let $\nu \in (0, 1)$ and suppose that $c_{kl}|_{\Omega} \in C^{0,\nu}(\Omega)$ for all $k, l \in \{1, \dots, d\}$. Let $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\chi \neq 0$ and suppose $d(\text{supp } \chi, \Omega^c) > 0$. Then there exists a continuous function $(z, x, y) \mapsto K_z(x, y)$ from $\Sigma_{\theta_a}^\circ \times \mathbb{R}^d \times \mathbb{R}^d$ into \mathbb{C} such that the following is valid.*

- *The function K_z is the kernel of the operator $M_\chi S_z M_\chi$ for all $z \in \Sigma_{\theta_a}^\circ$, where S is the semigroup generated by $-A$.*
- *The function K_z is once differentiable in each variable and the derivative with respect to one variable is differentiable in the other variable.*
- *For every multi-index α, β with $|\alpha|, |\beta| \leq 1, \kappa > 0$ and $\tau \in [0, 1)$ there exist $a, b > 0$ such that*

$$|(\partial_x^\alpha \partial_y^\beta K_z)(x, y)| \leq a |z|^{-d/2} |z|^{-(|\alpha|+|\beta|)/2} (1 + |z|)^{\frac{d+|\alpha|+|\beta|}{2}} e^{-b \frac{|x-y|^2}{|z|}}$$

and

$$\begin{aligned} & |(\partial_x^\alpha \partial_y^\beta K_z)(x+h, y+k) - (\partial_x^\alpha \partial_y^\beta K_z)(x, y)| \\ & \leq a |z|^{-d/2} |z|^{-(|\alpha|+|\beta|)/2} \left(\frac{|h| + |k|}{|x-y| + \sqrt{|z|}} \right)^\nu (1 + |z|)^{\frac{d+|\alpha|+|\beta|+\nu}{2}} e^{-b \frac{|x-y|^2}{|z|}} \end{aligned}$$

for all $z \in \Sigma_{\theta_a}^\circ$ and $x, y, h, k \in \mathbb{R}^d$ with $|h| + |k| \leq \tau |x-y| + \kappa \sqrt{|z|}$.

The second result is for merely one multiplication operator on the left of the semigroup, but it requires that the coefficients of the operator are real on \mathbb{R}^d and uniformly Lipschitz on Ω .

Theorem 1.2. *Let A be a sectorial degenerate differential operator with real coefficients (c_{kl}) , where $c_{kl}: \mathbb{R}^d \rightarrow \mathbb{R}$ is a bounded measurable function for all $k, l \in \{1, \dots, d\}$. Let S be the semigroup generated by $-A$. Let $\Omega \subsetneq \mathbb{R}^d$ be open and suppose that (c_{kl}) is strongly elliptic on Ω . Suppose that $c_{kl}|_{\Omega} \in W^{1,\infty}(\Omega)$ for all $k, l \in \{1, \dots, d\}$. Let $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\chi \neq 0$ and suppose $d(\text{supp } \chi, \Omega^c) > 0$. Then there exists a measurable function $(t, x, y) \mapsto K_t(x, y)$ from $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ into \mathbb{R} such that the following is valid.*

- *The function K_t is a kernel of $M_\chi S_t$ for all $t > 0$.*
- *The function $x \mapsto K_t(x, y)$ is continuously differentiable on \mathbb{R}^d for all $t > 0$ and $y \in \mathbb{R}^d$.*
- *The function $t \mapsto K_t(x, y)$ is continuous for all $x, y \in \mathbb{R}^d$.*
- *For every multi-index α with $|\alpha| \leq 1, \nu \in (0, 1), \varepsilon > 0, \kappa > 0$ and $\tau \in [0, 1)$ there exist $a, b > 0$ such that*

$$|(\partial_x^\alpha K_t)(x, y)| \leq a t^{-d/2} t^{-|\alpha|/2} e^{\varepsilon t} e^{-b \frac{|x-y|^2}{t}}$$

and

$$|(\partial_x^\alpha K_t)(x+h, y) - (\partial_x^\alpha K_t)(x, y)| \leq at^{-d/2}t^{-|\alpha|/2} \left(\frac{|h|}{|x-y| + \sqrt{t}} \right)^v e^{\varepsilon t} e^{-b \frac{|x-y|^2}{t}}$$

for all $t > 0$ and $x, y, h \in \mathbb{R}^d$ with $|h| \leq \tau |x - y| + \kappa \sqrt{t}$.

In Theorem 1.1 the function $(t, x, y) \mapsto K_t(x, y)$ is continuous, whilst it is not clear whether the function $(t, x, y) \mapsto K_t(x, y)$ is continuous in the setting of Theorem 1.2. Likely, there even does not exist a continuous function which is equal to this function almost everywhere on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. On the other hand, we prove measurability jointly in the three variables and do not work with an equivalent class of functions, for which the representative changes all the time. Since there are uncountable many $y \in \mathbb{R}^d$ this complicates the proof.

We also investigate boundedness on L_p of Riesz transform type operators. We obtain the following result.

Theorem 1.3. *Let A be a sectorial degenerate differential operator with complex coefficients (c_{kl}) . Let $\Omega \subsetneq \mathbb{R}^d$ be open and suppose that (c_{kl}) is strongly elliptic on Ω . Let $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\chi \neq 0$ and $d(\text{supp } \chi, \Omega^c) > 0$. Then one has the following.*

- (a) *Let $v \in (0, 1)$ and suppose that $c_{kl}|_\Omega \in C^{0,v}(\Omega)$ for all $k, l \in \{1, \dots, d\}$. Then the Riesz transforms $\nabla M_\chi (I + A)^{-1/2} M_\chi$ are bounded on $L_p(\mathbb{R}^d)$ for all $p \in (1, \infty)$.*
- (b) *Suppose that $c_{kl}|_\Omega \in W^{1,\infty}(\Omega)$ and c_{kl} is real valued for all $k, l \in \{1, \dots, d\}$. Then the Riesz transforms $\nabla M_\chi (I + A)^{-1/2}$ are bounded on $L_p(\mathbb{R}^d)$ for all $p \in (1, \infty)$.*
- (c) *Let $v \in (0, 1)$ and suppose that $c_{kl}|_\Omega \in C^{0,v}(\Omega)$ for all $k, l \in \{1, \dots, d\}$. Then the Riesz transforms $\nabla M_\chi (I + A)^{-1/2}$ are bounded on $L_2(\mathbb{R}^d)$.*

Using Morrey and Campanato spaces we prove Theorem 1.1 as in [5], if the operator is strongly elliptic on \mathbb{R}^d and the coefficients are Hölder continuous on \mathbb{R}^d . We carefully control all the constants and show that they depend only on the ellipticity constant on Ω and on the Hölder continuity of the coefficients on Ω . Then the theorem follows by approximation arguments. We prove a quantitative version of Theorem 1.1 in Section 2.

Since

$$M_\chi^2 S_t = M_\chi S_t M_\chi + M_\chi [M_\chi, S_t]$$

and one can use Theorem 1.1 to handle the first term, it suffices to obtain good estimates on the commutator to derive the bounds of Theorem 1.2. This is done in Section 3.

Finally, in Section 4 we prove the boundedness of the Riesz transforms of Theorem 1.3 and the boundedness of several other Riesz transforms. For strongly elliptic operators in divergence form with complex bounded measurable coefficients the boundedness of the Riesz transforms on $L_2(\mathbb{R}^d)$ was the longstanding

open Kato problem until it was solved by Auscher–Hofmann–Lacey–McIntosh–Tchamitchian [4]. For Hölder continuous coefficients the Kato problem was solved earlier by McIntosh [16] and a simplified proof was given in [7]. In the proof of Theorem 1.3(a) we adapt this simplified proof. In [6] Theorem 1.2 we proved Theorem 1.3(a) for merely measurable coefficients, but with the restriction that the coefficients are real symmetric and $p \in (1, 2]$.

In the proofs we need various times to transfer semigroup estimates into Gaussian bounds, with control of large time behaviour, using the Davies perturbation method. Note that we deduce polynomial growth for large time for the kernel bounds in Theorem 1.1. The techniques are more or less folklore, however scattered over the literature. In the appendix we collect them together for the convenience of the reader. Finally, by decomposing χ into its real and imaginary part, for simplicity we may and do assume throughout the rest of this paper that the various cut-off functions $\chi, \tilde{\chi}, \dots$ are always real valued.

2. Complex Hölder continuous coefficients

We start this section with the definition of a number of classes of coefficients and operators. The main aim is to obtain results for elements of these classes and that the constants involved are uniformly for a given class.

Let $\theta \in [0, \frac{\pi}{2})$ and $M > 0$. Define $\mathcal{S}(\theta, M)$ to be the set of all measurable $C : \mathbb{R}^d \rightarrow \mathbb{C}^{d \times d}$ such that

$$\begin{aligned} \langle C(x) \xi, \xi \rangle &\in \Sigma_\theta \text{ for all } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{C}^d, \text{ and,} \\ \|C(x)\| &\leq M \quad \text{for all } x \in \mathbb{R}^d, \end{aligned}$$

where $\|C(x)\|$ is the ℓ_2 -norm of $C(x)$ in \mathbb{C}^d and $\langle \cdot, \cdot \rangle$ is the inner product on \mathbb{C}^d . For all $C \in \mathcal{S}(\theta, M)$ define the sectorial form $\mathfrak{a}_C : W^{1,2}(\mathbb{R}^d) \times W^{1,2}(\mathbb{R}^d) \rightarrow \mathbb{C}$ by

$$\mathfrak{a}_C(u, v) = \int_{\mathbb{R}^d} \sum_{k,l=1}^d c_{kl} (\partial_k u) \overline{(\partial_l v)}$$

and let A_C and S^C be the associated operator and semigroup. Here and in the sequel $c_{kl}(x)$ is the appropriate matrix coefficient of $C(x)$. If no confusion is possible then we drop the C and write $\mathfrak{a} = \mathfrak{a}_C$, $A = A_C$ and $S = S^C$. For all $C \in \mathcal{S}(\theta, M)$ define $\Re C : \mathbb{R}^d \rightarrow \mathbb{C}$ by

$$(\Re C)(x) = \frac{1}{2} (C(x) + C(x)^*).$$

Then $\Re C \in \mathcal{S}(0, M)$ and $A_{\Re C}$ is self-adjoint. Moreover, $\mathfrak{a}_{\Re C}(u) = \operatorname{Re} \mathfrak{a}(u)$ for all $u \in W^{1,2}(\mathbb{R}^d)$. Next, let $\mathcal{E}(\theta, M)$ be the set of all $C \in \mathcal{S}(\theta, M)$ such that there exists a $\mu_0 > 0$ such that $\operatorname{Re} \langle C(x) \xi, \xi \rangle \geq \mu_0 |\xi|^2$ for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$. We emphasise that the constant μ_0 depends on C .

Let $Y \subset \mathbb{R}^d$ be a set, $\theta \in [0, \frac{\pi}{2})$ and $\mu, M > 0$. Let $\mathcal{S}(Y, \theta, \mu, M)$ be the set of all $C \in \mathcal{S}(\theta, M)$ such that

$$\operatorname{Re}\langle C(x)\xi, \xi \rangle \geq \mu |\xi|^2 \text{ for all } x \in Y \text{ and } \xi \in \mathbb{C}^d$$

and define

$$\mathcal{E}(Y, \theta, \mu, M) = \mathcal{S}(Y, \theta, \mu, M) \cap \mathcal{E}(\theta, M).$$

Next, let $\nu \in (0, 1]$ and suppose that Y contains at least two elements x, y with $0 < |x - y| \leq 1$. The space $C^{0,\nu}(Y)$ is the space of all Hölder continuous functions on Y with seminorm

$$\|u\|_{C^{0,\nu}(Y)} = \sup \left\{ \frac{|u(x) - u(y)|}{|x - y|^\nu} : x, y \in Y, 0 < |x - y| \leq 1 \right\}.$$

Let $\mathcal{S}^\nu(Y, \theta, \mu, M)$ be the set of all $C \in \mathcal{S}(Y, \theta, \mu, M)$ such that

$$\|c_{kl}\|_{C^{0,\nu}(Y)} \leq M \text{ for all } k, l \in \{1, \dots, d\},$$

and define

$$\mathcal{E}^\nu(Y, \theta, \mu, M) = \mathcal{S}^\nu(Y, \theta, \mu, M) \cap \mathcal{E}(\theta, M).$$

Finally, let $\mathcal{EH}^\nu(Y, \theta, \mu, M)$ be the set of all $C \in \mathcal{E}^\nu(Y, \theta, \mu, M)$ such that $\|c_{kl}\|_{C^{0,\nu}(\mathbb{R}^d)} < \infty$ for all $k, l \in \{1, \dots, d\}$. If $C \in \mathcal{S}^\nu(Y, \theta, \mu, M)$ then A_C is sectorial on $L_2(\mathbb{R}^d)$, whilst A_C is strongly elliptic on \mathbb{R}^d if $C \in \mathcal{E}^\nu(Y, \theta, \mu, M)$. Finally, A_C is strongly elliptic with Hölder continuous coefficients on \mathbb{R}^d if $C \in \mathcal{EH}^\nu(Y, \theta, \mu, M)$. In any case, A_C is strongly elliptic on the set Y with ellipticity constant at least μ .

In the proof of the theorems we frequently need the Davies perturbation. For all $\rho \in \mathbb{R}$ and $\psi \in W^{1,\infty}(\mathbb{R}^d)$ define the multiplication operator U_ρ by $U_\rho u = e^{-\rho\psi} u$. For all $n \in \mathbb{N}$ let

$$\mathcal{D}_n = \left\{ \psi \in W^{n,\infty}(\mathbb{R}^d, \mathbb{R}) : \left\| \sum_{1 \leq |\alpha| \leq n} |\partial^\alpha \psi|^2 \right\|_\infty \leq 1 \right\}.$$

Thus

$$\mathcal{D}_1 = \{\psi \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}) : \|\nabla \psi\|_\infty \leq 1\}.$$

Let $C \in \mathcal{S}(\theta, M)$. Define $S_t^{(C,\rho)} = S_t^{(\rho)} = U_\rho S_t U_{-\rho}$ to be the Davies perturbation of S_t for all $t > 0$. Let $-A_{C,\rho} = -A_\rho$ be the generator of $S^{(C,\rho)}$. Moreover, define the form $\mathfrak{a}_{C,\rho}$ by

$$\mathfrak{a}_{C,\rho}(u, v) = \mathfrak{a}_C(U_{-\rho}u, U_\rho v)$$

with form domain $D(\mathfrak{a}_{C,\rho}) = W^{1,2}(\mathbb{R}^d)$. Then $A_{C,\rho}$ is the operator associated with $\mathfrak{a}_{C,\rho}$.

We frequently need the following lemma for estimates on L_2 .

Lemma 2.1. *Let $\theta \in [0, \frac{\pi}{2})$ and $\mu, M > 0$. Then one has the following.*

- (a) *If $C \in \mathcal{S}(\theta, M)$ then $\|S_t^{(\rho)}\|_{2 \rightarrow 2} \leq e^{\omega \rho^2 t}$ for all $t > 0$, $\psi \in \mathcal{D}_1$ and $\rho \in \mathbb{R}$, where $\omega = 3(1 + \tan \theta)^2(1 + d^2 M)$.*
 (b) *Let $Y \subset \mathbb{R}^d$ and $\alpha \in (-\theta_\alpha, \theta_\alpha)$. Then*

$$e^{i\alpha} C \in \mathcal{S}(Y, \theta + |\alpha|, \mu \frac{\cos(\theta + |\alpha|)}{\cos \theta}, M)$$

for all $C \in \mathcal{S}(Y, \theta, \mu, M)$.

- (c) *If $\chi \in W^{1,\infty}(\mathbb{R}^d)$ and $C \in \mathcal{S}(\text{supp } \chi, \theta, \mu, M)$, then $M_\chi D(A_{\mathfrak{R}C}^{1/2}) \subset W^{1,2}(\mathbb{R}^d)$ and*

$$\sum_{m=1}^d \|(\partial_m M_\chi - M_{\partial_m \chi})u\|_2^2 \leq \frac{\|\chi\|_\infty^2}{\mu} \|A_{\mathfrak{R}C}^{1/2} u\|_2^2 \quad (2.1)$$

for all $u \in D(A_{\mathfrak{R}C}^{1/2})$.

- (d) *If $\chi \in W^{1,\infty}(\mathbb{R}^d)$ and $C \in \mathcal{S}(\text{supp } \chi, \theta, \mu, M)$, then $M_\chi S_t L_2(\mathbb{R}^d) \subset W^{1,2}(\mathbb{R}^d)$ and*

$$\|(\partial_m M_\chi - M_{\partial_m \chi})S_t u\|_2 \leq \frac{\|\chi\|_\infty}{\sqrt{\mu \sin \theta_\alpha}} t^{-1/2} \|u\|_2$$

for all $t > 0$, $u \in L_2(\mathbb{R}^d)$ and $m \in \{1, \dots, d\}$.

Remark 2.2. Note that $\partial_m M_\chi - M_{\partial_m \chi} \supset M_\chi \partial_m$ in Statements (c) and (d).

The rotational invariance of Statement (b) allows to consider various bounds on kernels K_z merely for $z \in (0, \infty)$. Then the uniform bounds for z in a sector $\Sigma_{\theta'}^\circ$ with $\theta' \in (0, \theta_\alpha)$ follow since all bounds depend only on $\chi, \Omega, \mu, M, \nu$ and θ .

Proof. (a). This follows from (14) in [3].

(b). Let $x \in Y$ and $\xi \in \mathbb{C}^d$. Then

$$\begin{aligned} \operatorname{Re}(e^{i\alpha} C(x) \xi, \xi) &= \cos \alpha \operatorname{Re}\langle C(x) \xi, \xi \rangle - \sin \alpha \operatorname{Im}\langle C(x) \xi, \xi \rangle \\ &\geq \cos \alpha \operatorname{Re}\langle C(x) \xi, \xi \rangle - \sin |\alpha| \tan \theta \operatorname{Re}\langle C(x) \xi, \xi \rangle \\ &= \frac{\cos(\theta + |\alpha|)}{\cos \theta} \operatorname{Re}\langle C(x) \xi, \xi \rangle \\ &\geq \mu |\xi|^2 \frac{\cos(\theta + |\alpha|)}{\cos \theta}. \end{aligned}$$

This proves Statement (b).

(c). Let $\varepsilon > 0$ and $n \in \mathbb{N}$. Set $C_n = C + \frac{1}{n} I$. Then $C_n \in \mathcal{E}(\text{supp } \chi, \theta, \mu, M + 1)$.

Let $u \in W^{1,2}(\mathbb{R}^d)$. Then

$$\begin{aligned} \sum_{m=1}^d \|(\partial_m M_\chi - M_{\partial_m \chi})u\|_2^2 &= \sum_{m=1}^d \int |\chi \partial_m u|^2 \\ &\leq \frac{1}{\mu} \int \operatorname{Re} \sum_{k,l=1}^d (c_{kl} + \frac{1}{n} \delta_{kl}) \chi^2 (\partial_k u) \overline{\partial_l u} \\ &\leq \frac{\|\chi\|_\infty^2}{\mu} \|A_{\mathfrak{R}C_n}^{1/2} u\|_2^2 \\ &\leq \frac{\|\chi\|_\infty^2}{\mu} \|(\varepsilon I + A_{\mathfrak{R}C_n})^{1/2} u\|_2^2. \end{aligned}$$

Now let $u \in L_2(\mathbb{R}^d)$, $v \in W^{1,2}(\mathbb{R}^d)$ and $m \in \{1, \dots, d\}$. Then

$$\begin{aligned} |((\varepsilon I + A_{\mathfrak{R}C_n})^{-1/2} u, \partial_m(\chi v))| &= |((\partial_m M_\chi - M_{\partial_m \chi})(\varepsilon I + A_{\mathfrak{R}C_n})^{-1/2} u, v)| \\ &\leq \frac{\|\chi\|_\infty^2}{\mu} \|u\|_2 \|v\|_2. \end{aligned}$$

Taking the limit $n \rightarrow \infty$ and using [3, Theorem 3.7] (cf. [19, Theorem 3.2]), one deduces that

$$|((\varepsilon I + A_{\mathfrak{R}C})^{-1/2} u, \partial_m(\chi v))| \leq \frac{\|\chi\|_\infty^2}{\mu} \|u\|_2 \|v\|_2.$$

So

$$|(u, \partial_m(\chi v))| \leq \frac{\|\chi\|_\infty^2}{\mu} \|(\varepsilon I + A_{\mathfrak{R}C})^{1/2} u\|_2 \|v\|_2$$

for all $u \in D(A_{\mathfrak{R}C}^{1/2})$, $v \in W^{1,2}(\mathbb{R}^d)$ and $m \in \{1, \dots, d\}$. Taking the limit $\varepsilon \downarrow 0$ one obtains

$$|(u, (M_\chi \partial_m + M_{\partial_m \chi})v)| \leq \frac{\|\chi\|_\infty^2}{\mu} \|A_{\mathfrak{R}C}^{1/2} u\|_2 \|v\|_2. \quad (2.2)$$

Therefore

$$|(M_\chi u, \partial_m v)| \leq \left(\frac{\|\chi\|_\infty^2}{\mu} \|A_{\mathfrak{R}C}^{1/2} u\|_2 + \|\partial_m \chi\|_\infty \|u\|_2 \right) \|v\|_2$$

for all $v \in W^{1,2}(\mathbb{R}^d)$ and $m \in \{1, \dots, d\}$. So $M_\chi u \in W^{1,2}(\mathbb{R}^d)$ and then the estimate (2.1) follows from (2.2).

(d). Let $\alpha \in (0, \theta_a)$. Then it follows from Statements (a) and (b) that $\|S_z^{(\rho)}\|_{2 \rightarrow 2} \leq e^{\omega \rho^2 |z|}$ for all $z \in \Sigma_\alpha^\circ$, $\rho \in \mathbb{R}$ and $\psi \in \mathcal{D}_1$, where $\omega = 3(1 + \tan(\theta + \alpha))^2(1 + d^2 M)$. Hence the Cauchy representation formula gives

$$\|A_\rho S_t^{(\rho)}\|_{2 \rightarrow 2} \leq \frac{1}{t \sin \alpha} e^{2\omega \rho^2 t} \quad (2.3)$$

for all $t > 0, \rho \in \mathbb{R}$ and $\psi \in \mathcal{D}_1$. If $C \in \mathcal{E}(\text{supp } \chi, \theta, \mu, M)$ then

$$\begin{aligned} \|(\partial_m M_\chi - M_{\partial_m \chi}) S_t^C u\|_2^2 &\leq \frac{\|\chi\|_\infty^2}{\mu} \|A_{\Re C}^{1/2} S_t^C u\|_2^2 \\ &= \frac{\|\chi\|_\infty^2}{\mu} \text{Re}(A_C S_t^C u, S_t^C u) \\ &\leq \frac{\|\chi\|_\infty^2}{\mu} \|A_C S_t^C u\|_2 \|S_t^C u\|_2 \\ &\leq \frac{\|\chi\|_\infty^2}{\mu} \frac{1}{t \sin \alpha} \|u\|_2^2. \end{aligned}$$

Now take the limit $\alpha \uparrow \theta_a$. Finally, let $C \in \mathcal{S}(\text{supp } \chi, \theta, \mu, M)$. For all $n \in \mathbb{N}$ define $C_n = C + \frac{1}{n} I$. Since $\lim_{n \rightarrow \infty} S_t^{C_n} = S_t^C$ strongly in $\mathcal{L}(L_2(\mathbb{R}^d))$ by [3] Corollary 3.9, now Statement (d) follows as in the proof of Statement (c). \square

The next theorem is a uniform version of Theorem 1.1.

Theorem 2.3. *Let $\Omega \subsetneq \mathbb{R}^d$ be open, $\theta \in [0, \frac{\pi}{2})$, $\mu, M > 0$, $\nu \in (0, 1)$, $\kappa > 0$, $\tau \in [0, 1)$, $\theta' \in (0, \theta_a)$ and $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\chi \neq 0$ and $d(\text{supp } \chi, \Omega^c) > 0$. Then there exist $a, b > 0$ such that for every $C \in \mathcal{S}^\nu(\Omega, \theta, \mu, M)$ there exists a function $(z, x, y) \mapsto K_z(x, y)$ from $\Sigma_{\theta_a}^\circ \times \mathbb{R}^d \times \mathbb{R}^d$ into \mathbb{C} such that the following is valid.*

- (a) *The function $(z, x, y) \mapsto K_z(x, y)$ is continuous from $\Sigma_{\theta_a}^\circ \times \mathbb{R}^d \times \mathbb{R}^d$ into \mathbb{C} .*
- (b) *For all $z \in \Sigma_{\theta'}^\circ$, the function K_z is the kernel of the operator $M_\chi S_z M_\chi$.*
- (c) *For all $x, y \in \mathbb{R}^d$ the function $z \mapsto K_z(x, y)$ is holomorphic from $\Sigma_{\theta'}^\circ$ into \mathbb{C} .*
- (d) *For all $z \in \Sigma_{\theta'}^\circ$ the function K_z is once differentiable in each variable and the derivative with respect to one variable is differentiable in the other variable. Moreover, for every multi-index α, β with $0 \leq |\alpha|, |\beta| \leq 1$ one has*

$$|(\partial_x^\alpha \partial_y^\beta K_z)(x, y)| \leq a |z|^{-d/2} |z|^{-(|\alpha|+|\beta|)/2} (1 + |z|)^{\frac{d+|\alpha|+|\beta|}{2}} e^{-b \frac{|x-y|^2}{|z|}}$$

and

$$\begin{aligned} &|(\partial_x^\alpha \partial_y^\beta K_z)(x+h, y+k) - (\partial_x^\alpha \partial_y^\beta K_z)(x, y)| \\ &\leq a |z|^{-d/2} |z|^{-(|\alpha|+|\beta|)/2} \left(\frac{|h| + |k|}{|x-y| + \sqrt{|z|}} \right)^\nu (1 + |z|)^{\frac{d+|\alpha|+|\beta|+\nu}{2}} e^{-b \frac{|x-y|^2}{|z|}} \end{aligned}$$

for all $x, y, h, k \in \mathbb{R}^d$ with $|h| + |k| \leq \tau |x-y| + \kappa \sqrt{|z|}$.

Proof. We first prove the theorem with $\mathcal{S}^\nu(\Omega, \theta, \mu, M)$ replaced by $\mathcal{E}\mathcal{H}^\nu(\Omega, \theta, \mu, M)$. For strongly elliptic operators on \mathbb{R}^d in divergence form and Hölder continuous coefficients all the kernels with stated holomorphy and continuity properties are well known. The main point is to derive the uniform bounds. We emphasise that the

constants in the proof do not depend on the ellipticity constant μ_0 for elements in $\mathcal{E}\mathcal{H}^\nu(\Omega, \theta, \mu, M)$, nor on the Hölder continuity of the coefficients on Ω^c . Then we will approximate elements of $\mathcal{E}^\nu(\Omega, \theta, \mu, M)$ by elements of $\mathcal{E}\mathcal{H}^\nu(\Omega, \theta, \mu, M)$ and finally approximate elements of $\mathcal{S}^\nu(\Omega, \theta, \mu, M)$ by elements of $\mathcal{E}^\nu(\Omega, \theta, \mu, M)$.

Without loss of generality we may assume that $d \geq 3$. Let $F \subset \Omega$ be a closed set with $F \neq \emptyset$ and $d(F, \Omega^c) > 0$. Let $r_0 = \frac{1}{3}d(F, \Omega^c)$. There exist $\chi_1, \chi_2 \in C_b^\infty(\mathbb{R}^d)$ such that $0 \leq \chi_1, \chi_2 \leq 1$ and

$$\chi_1(x) = \begin{cases} 0 & \text{if } d(x, F) < r_0, \\ 1 & \text{if } d(x, \Omega^c) < r_0, \end{cases} \quad (2.4)$$

$$\chi_2(x) = \begin{cases} 1 & \text{if } d(x, F) < 2r_0, \\ 0 & \text{if } x \notin \Omega, \end{cases} \quad (2.5)$$

for all $x \in \mathbb{R}^d$. Let $M' = 2\|\chi_2\|_{W^{1,\infty}(\mathbb{R}^d)}M + \|\chi_1\|_{W^{1,\infty}(\mathbb{R}^d)} + 1$. Let $C \in \mathcal{E}\mathcal{H}^\nu(\Omega, \theta, \mu, M)$. Define $C' = \chi_2 C + \chi_1 I$. Then $C' \in \mathcal{E}\mathcal{H}^\nu(\mathbb{R}^d, \theta, \mu \wedge 1, M')$. Since the operator $A_{C'}$ is a strongly elliptic operator on \mathbb{R}^d with $C^{0,\nu}$ -coefficients, it satisfies various kinds of De Giorgi estimates. On bounded open sets these are proved by Giaquinta [11, 12], or Xu–Zuily [21]. The global estimates follow from [9, Proposition 3.5] and [8, Proposition 2.6]. Precisely, there exist $c'_{DG}, c''_{DG} > 0$ and for all $\nu' \in (0, 1)$ a $c_{DG} > 0$, depending only on μ, M', ν and ν' , such that

$$\int_{B(x,r)} |\nabla u|^2 \leq c_{DG} \left(\frac{r}{R}\right)^{d-2+2\nu'} \int_{B(x,R)} |\nabla u|^2 \quad (2.6)$$

and

$$\begin{aligned} & \sum_{k=1}^d \int_{B(x,r)} |\partial_k u - \langle \partial_k u \rangle_{x,r}|^2 \\ & \leq c'_{DG} \left(\frac{r}{R}\right)^{d+2\tilde{\nu}} \sum_{k=1}^d \int_{B(x,R)} |\partial_k u - \langle \partial_k u \rangle_{x,R}|^2 + c''_{DG} R^{2\nu} \sum_{k=1}^d \int_{B(x,R)} |\nabla u|^2 \end{aligned} \quad (2.7)$$

for all $R \in (0, 1], r \in (0, R]$ and $u \in W^{1,2}(B(x, R))$ satisfying $A_{C'}u = 0$ weakly on $B(x, R)$, where $\tilde{\nu} = \frac{1}{2}(1 + \nu)$.

Write $\mathfrak{a} = \mathfrak{a}_C$, $A = A_C$ and $S = S^C$. It is well known that the semigroup generated by S has a kernel K^S satisfying continuity, holomorphy and Gaussian properties similar to (a)–(d) in the theorem. Define $K_z(x, y) = \chi(x) K_z^S(x, y) \chi(y)$. Then K_z is the kernel of $M_\chi S_z M_\chi$. Moreover, K satisfies Properties (a)–(c).

Note that $\mathfrak{a}(u, v) = \mathfrak{a}_{C'}(u, v)$ for all $u, v \in W^{1,2}(\mathbb{R}^d)$ with $\text{supp } u \subset F$. For all $\gamma \in [0, d]$ let $M_{2,\gamma}(\mathbb{R}^d)$ be the Morrey space and for all $\gamma \in [0, d + 2)$ let $\mathcal{M}_{2,\gamma}(\mathbb{R}^d)$ be the Campanato space as defined in [5] Section 2.

For all $\gamma \in [0, d)$ let $P(\gamma)$ be the following hypothesis:

For all $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\text{supp } \chi \subset F$ there exist $a_1, \omega_1 > 0$, depending only on $\chi, \Omega, \theta, \mu, M$ and ν , such that

$$\|M_\chi S_t^{(\rho)} u\|_{M_{2,\gamma}(\mathbb{R}^d)} \leq a_1 t^{-\gamma/4} e^{\omega_1(1+\rho^2)t} \|u\|_2$$

and

$$\|\nabla M_\chi S_t^{(\rho)} u\|_{M_{2,\gamma}(\mathbb{R}^d)} \leq a_1 t^{-\gamma/4} t^{-1/2} e^{\omega_1(1+\rho^2)t} \|u\|_2$$

uniformly for all $t > 0, u \in L_2(\mathbb{R}^d), \rho \in \mathbb{R}$ and $\psi \in \mathcal{D}_1$.

Arguing as in the proof of Lemma 3.3 in [5] it follows from (2.3) and the De Giorgi estimates (2.6) that $P(\gamma)$ is valid for all $\gamma \in [0, d)$.

Next, for all $\gamma \in [0, d + 2\nu]$ let $P'(\gamma)$ be the following hypothesis:

For all $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\text{supp } \chi \subset F$ there exist $a_1, \omega_1 > 0$, depending only on $\chi, \Omega, \theta, \mu, M$ and ν , such that

$$\|M_\chi S_t^{(\rho)} u\|_{\mathcal{M}_{2,\gamma}(\mathbb{R}^d)} \leq a_1 t^{-\gamma/4} e^{\omega_1(1+\rho^2)t} \|u\|_2$$

and

$$\|\nabla M_\chi S_t^{(\rho)} u\|_{\mathcal{M}_{2,\gamma}(\mathbb{R}^d)} \leq a_1 t^{-\gamma/4} t^{-1/2} e^{\omega_1(1+\rho^2)t} \|u\|_2$$

uniformly for all $t > 0, u \in L_2(\mathbb{R}^d), \rho \in \mathbb{R}$ and $\psi \in \mathcal{D}_2$.

Since $M_{2,\gamma} \cap L_2 = \mathcal{M}_{2,\gamma} \cap L_2$ for all $\gamma \in [0, d)$, with equivalent norms, one deduces from $P(\gamma)$ that also $P'(\gamma)$ is valid for all $\gamma \in [0, d)$. But arguing as in [8], proof of Proposition 3.2 and the proof of Lemma 3.3 in [5] it follows from the De Giorgi estimates (2.7) that $P'(\gamma)$ is valid for all $\gamma \in [0, d + 2\nu]$. Hence there are $a, \omega > 0$, depending only on $\chi, \Omega, \theta, \mu, M$ and ν , such that

$$\|\partial^\alpha M_\chi S_t^{(\rho)} u\|_\infty \leq a t^{-d/4} t^{-|\alpha|/2} e^{\omega(1+\rho^2)t} \|u\|_2 \quad (2.8)$$

and

$$\|(I - L(h))\partial^\alpha M_\chi S_t^{(\rho)} u\|_\infty \leq a t^{-d/4} t^{-|\alpha|/2} \left(\frac{|h|}{\sqrt{t}}\right)^\nu e^{\omega(1+\rho^2)t} \|u\|_2$$

for all multi-indices α with $|\alpha| \leq 1, t > 0, u \in L_2(\mathbb{R}^d), \rho \in \mathbb{R}$ and $\psi \in \mathcal{D}_2$. Here $L(h)$ denotes left translation, defined by $(L(h)u)(x) = u(x - h)$. Next, $\|S_t^{(\rho)}\|_{2 \rightarrow 2} \leq e^{\omega_0 \rho^2 t}$ for all $t > 0$, where $\omega_0 = 3(1 + \tan \theta)^2(1 + d^2 M)$ by Lemma 2.1(a).

Then the bounds of Property (d) follow from Lemma A.1 in Appendix A uniformly for all $z \in \Sigma_\theta^\circ$ and $C \in \mathcal{E}\mathcal{H}^\nu(\Omega, \theta, \mu, M)$. This proves the theorem with $S^\nu(\Omega, \theta, \mu, M)$ replaced by $\mathcal{E}\mathcal{H}^\nu(\Omega, \theta, \mu, M)$.

Let $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\chi \neq 0$ and $d(\text{supp } \chi, \Omega^c) > 0$. Let $r_0 = \frac{1}{2}d(\text{supp } \chi, \Omega^c)$ and let $\Omega' = \{x \in \Omega : d(x, \text{supp } \chi) < r_0\}$. Then Ω' is open, $d(\text{supp } \chi, (\Omega')^c) > 0$ and $d(\Omega', \Omega^c) \geq r_0 > 0$.

Let α, β be multi-indices with $|\alpha|, |\beta| \leq 1$. Using the Cauchy representation formula on the sector $\Sigma_{\theta_a}^\circ$ one deduces that $\partial_t \partial_x^\alpha \partial_y^\beta K_z^C$ satisfies Hölder type Gaussian bounds uniformly for all $C \in \mathcal{E}\mathcal{H}^v(\Omega', \theta, \mu, M)$, where K^C is the kernel associated with $z \mapsto M_\chi S_z^C M_\chi$. Hence the set of functions

$$\{(z, x, y) \mapsto (\partial_x^\alpha \partial_y^\beta K_z^C)(x, y) : C \in \mathcal{E}\mathcal{H}^v(\Omega', \theta, \mu, M)\}$$

is equicontinuous on compact subsets of $\Sigma_{\theta_a}^\circ \times \mathbb{R}^d \times \mathbb{R}^d$.

Fix $\tau \in C_c^\infty(B(0, r_0))$ with $\tau \geq 0$ and $\int \tau = 1$. For all $n \in \mathbb{N}$ define $\tau_n \in C_c^\infty(\mathbb{R}^d)$ by $\tau_n(x) = n^d \tau(nx)$. Now let $C \in \mathcal{E}^v(\Omega, \theta, \mu, M)$. For all $n \in \mathbb{N}$ and $k, l \in \{1, \dots, d\}$ define $c_{kl}^{(n)} = c_{kl} * \tau_n$ and define $C^{(n)} = (c_{kl}^{(n)})$. Then $C^{(n)} \in \mathcal{E}\mathcal{H}^v(\Omega', \theta, \mu, M)$ for all $n \in \mathbb{N}$. Write $S^{(n)} = S^{C^{(n)}}$, etc. Since $\{(z, x, y) \mapsto (\partial_x^\alpha \partial_y^\beta K_z^{(n)})(x, y) : n \in \mathbb{N}\}$ is equicontinuous on compact subsets of $\Sigma_{\theta_a}^\circ \times \mathbb{R}^d \times \mathbb{R}^d$ for all $|\alpha|, |\beta| \leq 1$ it follows with a diagonal argument from the Arzela–Ascoli theorem that there exists a subsequence $(K^{(n_k)})_{k \in \mathbb{N}}$ of $(K^{(n)})_{n \in \mathbb{N}}$ such that $K^{(\alpha, \beta)} = \lim_{k \rightarrow \infty} \partial_x^\alpha \partial_y^\beta K^{(n_k)}$ exists uniformly on compact subsets of $\Sigma_{\theta_a}^\circ \times \mathbb{R}^d \times \mathbb{R}^d$ and every multi-index α, β with $|\alpha|, |\beta| \leq 1$. Then $(z, x, y) \mapsto K_z^{(\alpha, \beta)}(x, y)$ is continuous on $\Sigma_{\theta_a}^\circ \times \mathbb{R}^d \times \mathbb{R}^d$. Set $K = K^{(\alpha, \beta)}$ where $|\alpha| = |\beta| = 0$. Obviously for every $z \in \Sigma_{\theta_a}^\circ$ the function K_z is once differentiable in each variable and the derivative with respect to one variable is differentiable in the other variable. Also K_z satisfies all the Gaussian bounds from the theorem and the constants in the Gaussians depend only on $\chi, \Omega, \theta', \mu, M$ and v . Let $z \in \Sigma_{\theta_a}^\circ$. Let $u, v \in C_c^\infty(\mathbb{R}^d)$. We shall prove below in Lemma 2.4 that $\lim_{n \rightarrow \infty} S_z^{(n)} u = S_z^C u$ in $L_2(\mathbb{R}^d)$. Hence

$$\begin{aligned} (S_z^C u, v) &= \lim_{k \rightarrow \infty} (S_z^{(n_k)} u, v) \\ &= \lim_{k \rightarrow \infty} \iint K_z^{(n_k)}(x, y) u(y) \overline{v(x)} dx dy \\ &= \iint K_z(x, y) u(y) \overline{v(x)} dx dy. \end{aligned}$$

Since K satisfies Gaussian bounds it follows that K_z is the kernel of S_z^C . This proves the theorem with $\mathcal{S}^v(\Omega, \theta, \mu, M)$ replaced by $\mathcal{E}^v(\Omega, \theta, \mu, M)$.

Finally, let $C \in \mathcal{S}^v(\Omega, \theta, \mu, M)$. For all $n \in \mathbb{N}$ define $C_n = C + \frac{1}{n} I$. Then $C_n \in \mathcal{E}^v(\Omega, \theta, \mu, M+1)$ for all $n \in \mathbb{N}$. Since $\lim_{n \rightarrow \infty} S_z^{C_n} u = S_z^C u$ in $L_2(\mathbb{R}^d)$ for all $z \in \Sigma_{\theta_a}^\circ$ and $u \in L_2(\mathbb{R}^d)$ by [3] Corollary 3.9, a similar approximation argument as in the previous step completes the proof of Theorem 2.3. \square

It remains to show the next lemma.

Lemma 2.4. *Let $\theta \in [0, \frac{\pi}{2})$ and $M > 0$. Let $C \in \mathcal{E}(\theta, M)$ and $\tau \in C_c^\infty(\mathbb{R}^d)$ with $\tau \geq 0$ and $\int \tau = 1$. For all $n \in \mathbb{N}$ define $\tau_n \in C_c^\infty(\mathbb{R}^d)$ by $\tau_n(x) = n^d \tau(nx)$ and set $C^{(n)} = (c_{kl}^{(n)})$, where $c_{kl}^{(n)} = \tau_n * c_{kl}$ for all $k, l \in \{1, \dots, d\}$. Then $C^{(n)} \in \mathcal{E}(\theta, M)$ for all $n \in \mathbb{N}$. Write $S^{(n)} = S^{C^{(n)}}$, $S = S^C$, etc. Then $\lim_{n \rightarrow \infty} S_z^{(n)} = S_z$ strongly in $\mathcal{L}(L_2(\mathbb{R}^d))$ for all $z \in \Sigma_{\theta_a}^\circ$.*

Proof. Let $\eta \in L_\infty(\mathbb{R}^d)$ and $u \in L_2(\mathbb{R}^d)$. Then

$$\left((I - L(x))\eta \right) u = (I - L(x))(\eta u) - (L(x)\eta) (I - L(x))u$$

for all $x \in \mathbb{R}^d$. So

$$\|(\eta - \tau_n * \eta)u\|_2 \leq \|\eta u - \tau_n * (\eta u)\|_2 + \|\eta\|_\infty \int_{\mathbb{R}^d} \tau_n(x) \|(I - L(x))u\|_2 dx$$

for all $n \in \mathbb{N}$ and $\lim \|(\eta - \tau_n * \eta)u\|_2 = 0$.

There exists a $\mu_0 > 0$ such that $\operatorname{Re}\langle C(x)\xi, \xi \rangle \geq \mu_0 |\xi|^2$ for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$. Then $\operatorname{Re}\langle C^{(n)}(x)\xi, \xi \rangle \geq \mu_0 |\xi|^2$ and $\|C^{(n)}(x)\| \leq M$ for all $x \in \mathbb{R}^d$, $\xi \in \mathbb{C}^d$ and $n \in \mathbb{N}$. Hence there exists a $c > 0$ such that $\|\nabla S_t^{(n)} u\|_2 \leq c t^{-1/2} \|u\|_2$ for all $t > 0$, $u \in L_2(\mathbb{R}^d)$ and $n \in \mathbb{N}$ (cf. the proof of (13) in [5]). By increasing c if necessary, it follows similarly that $\|\nabla S_t u\|_2 \leq c t^{-1/2} \|u\|_2$, and $\|\nabla S_t^{(n)*} u\|_2 \leq c t^{-1/2} \|u\|_2$ for all $t > 0$ and $u \in L_2(\mathbb{R}^d)$.

Without loss of generality we may assume that $z \in (0, \infty)$. Next, let $u, v \in L_2(\mathbb{R}^d)$. Then

$$\begin{aligned} ((S_z^{(n)} - S_z)u, v) &= \int_0^z \frac{d}{ds} (S_{z-s} u, S_s^{(n)*} v) ds \\ &= \int_0^z (A S_{z-s} u, S_s^{(n)*} v) - (S_{z-s} u, A_n^* S_s^{(n)*} v) ds \\ &= \int_0^z \mathfrak{a}((S_{z-s} u, S_s^{(n)*} v) - \mathfrak{a}_n((S_{z-s} u, S_s^{(n)*} v) ds \\ &= \sum_{k,l=1}^d \int_0^z ((c_{kl} - c_{kl}^{(n)}) \partial_k S_{z-s} u, \partial_l S_s^{(n)*} v) ds \end{aligned}$$

for all $n \in \mathbb{N}$. So

$$\|(S_z^{(n)} - S_z)u\|_2 \leq c \sum_{k,l=1}^d \int_0^z \|(c_{kl} - c_{kl}^{(n)}) \partial_k S_{z-s} u\|_2 s^{-1/2} ds$$

and the lemma follows from the Lebesgue dominated convergence theorem. \square

The proof of Theorem 2.3 together with Lemma 2.4 gives estimates which we need in the proof of Proposition 3.7.

Lemma 2.5. *Let $\Omega \subsetneq \mathbb{R}^d$ be open, $\theta \in [0, \frac{\pi}{2})$, $\mu, M > 0$, $\nu \in (0, 1)$ and $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\chi \neq 0$ and $d(\text{supp } \chi, \Omega^c) > 0$. Then there exist $a, \omega > 0$ such that for every $C \in \mathcal{S}^\nu(\Omega, \theta, \mu, M)$ one has $M_\chi S_t u \in W^{1,\infty}(\mathbb{R}^d)$ and*

$$\|\partial^\alpha M_\chi S_t u\|_\infty \leq a t^{-d/4} t^{-|\alpha|/2} e^{\omega t} \|u\|_2$$

for all multi-indices α with $|\alpha| \leq 1$, $t > 0$ and $u \in L_2(\mathbb{R}^d)$.

Proof. Let $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\chi \neq 0$ and $d(\text{supp } \chi, \Omega^c) > 0$. Let $r_0 = \frac{1}{2}d(\text{supp } \chi, \Omega^c)$ and $\Omega' = \{x \in \Omega : d(x, \text{supp } \chi) < r_0\}$. By (2.8) there exist $a, \omega > 0$ such that

$$\|\partial^\alpha M_\chi S_t^C u\|_\infty \leq a t^{-d/4} t^{-|\alpha|/2} e^{\omega t} \|u\|_2 \quad (2.9)$$

for all $C \in \mathcal{E}\mathcal{H}^\nu(\Omega, \theta, \mu, M)$, multi-indices α with $|\alpha| \leq 1$, $t > 0$ and $u \in L_2(\mathbb{R}^d)$. Let $C \in \mathcal{E}^\nu(\Omega, \theta, \mu, M)$. Let $\tau \in C_c^\infty(B(0, r_0))$ with $\tau \geq 0$ and $\int \tau = 1$. For all $n \in \mathbb{N}$ let $C^{(n)}$ be as in Lemma 2.4. Then $C^{(n)} \in \mathcal{E}\mathcal{H}^\nu(\Omega, \theta, \mu, M)$. Let $t > 0$ and $u \in L_2(\mathbb{R}^d)$. Then

$$|(S_t^{C^{(n)}} u, M_\chi \partial^\alpha v)| = |(\partial^\alpha M_\chi S_t^{C^{(n)}} u, v)| \leq a t^{-d/4} t^{-|\alpha|/2} e^{\omega t} \|u\|_2 \|v\|_1$$

for all multi-indices α with $|\alpha| \leq 1$, $v \in C_c^\infty(\mathbb{R}^d)$ and $n \in \mathbb{N}$. Now take the limit $n \rightarrow \infty$ and use Lemma 2.4. It follows that

$$|(M_\chi S_t^C u, \partial^\alpha v)| \leq a t^{-d/4} t^{-|\alpha|/2} e^{\omega t} \|u\|_2 \|v\|_1 \quad (2.10)$$

for all $v \in C_c^\infty(\mathbb{R}^d)$ and $|\alpha| \leq 1$. Choosing $|\alpha| = 0$, it follows that $M_\chi S_t u \in L_\infty(\mathbb{R}^d)$. Next $|\alpha| = 1$ and the density of $C_c^\infty(\mathbb{R}^d)$ in $W^{1,1}(\mathbb{R}^d)$ give that (2.10) is valid for all $v \in W^{1,1}(\mathbb{R}^d)$ and $|\alpha| = 1$. Hence $M_\chi S_t u \in W^{1,\infty}(\mathbb{R}^d)$ and (2.9) is valid.

Finally, if $C \in \mathcal{S}^\nu(\Omega, \theta, \mu, M)$ use the approximation $C_n = C + \frac{1}{n} I$ as at the end of the proof of Theorem 2.3 and argue similarly. \square

3. Real $W^{1,\infty}$ -coefficients

Let $\Omega \subset \mathbb{R}^d$ be open, $\theta \in [0, \frac{\pi}{2})$ and $\mu, M > 0$. Define $\mathcal{S}^1(\Omega, \theta, \mu, M, \text{real})$ to be the set of all $C \in \mathcal{S}^1(\Omega, \theta, \mu, M)$ such that c_{kl} is real valued for all $k, l \in \{1, \dots, d\}$. If $C \in \mathcal{S}^1(\Omega, \theta, \mu, M, \text{real})$ then \mathfrak{a}_C is closable. Moreover, if $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\text{supp } \chi \subset \Omega$ then there exists an $a > 0$ such that $\|M_\chi u\|_{W^{1,2}(\mathbb{R}^d)} \leq a \|u\|_{D(\mathfrak{a}_C)}$ for all $u \in W^{1,2}(\mathbb{R}^d)$. Hence $M_\chi(D(\overline{\mathfrak{a}_C})) \subset W^{1,2}(\mathbb{R}^d)$ and $\|M_\chi u\|_{W^{1,2}(\mathbb{R}^d)} \leq a \|u\|_{D(\overline{\mathfrak{a}_C})}$ for all $u \in D(\overline{\mathfrak{a}_C})$. In particular, $M_\chi S_t u \in W^{1,2}(\mathbb{R}^d)$ for all $t > 0$ and $u \in L_2(\mathbb{R}^d)$.

Throughout the remaining of this paper we set

$$\tilde{c}_{kl} = c_{kl} + c_{lk}$$

for all $k, l \in \{1, \dots, d\}$, whenever $C \in \mathcal{S}(\theta, M)$.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^d$ be open, $\theta \in [0, \frac{\pi}{2})$ and $\mu, M > 0$. Let $C \in \mathcal{E}^1(\Omega, \theta, \mu, M, \text{real})$. Let $\omega = 3d^2M$. Then*

$$\|S_t^{(\rho)}\|_{\infty \rightarrow \infty} \leq e^{\omega(|\rho| + \rho^2)t}$$

for all $t > 0, \rho \in \mathbb{R}$ and $\psi \in \mathcal{D}_2$.

Proof. Let $\rho \in \mathbb{R}$ and $\psi \in \mathcal{D}_2$. Obviously the form a_ρ is real. Integration by parts gives

$$\begin{aligned} \mathfrak{a}_\rho(u, v) &= \mathfrak{a}(u, v) - \rho \int \sum \tilde{c}_{kl} (\partial_k u) (\partial_l \psi) \bar{v} - \rho \int \sum (\partial_l c_{kl}) (\partial_k \psi) u \bar{v} \\ &\quad - \rho \int \sum c_{kl} (\partial_k \partial_l \psi) u \bar{v} - \rho^2 \int \sum c_{kl} (\partial_k \psi) (\partial_l \psi) u \bar{v} \end{aligned}$$

for all $u, v \in W^{1,2}(\mathbb{R}^d, \mathbb{R})$. Then the lemma follows from [17] Corollary 4.10. \square

It follows from Lemma 3.1 that the conditions of the next lemma are valid for $p = 1$.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^d$ be open, $\theta \in [0, \frac{\pi}{2})$, $\mu, M > 0$ and $p \in [1, \infty)$. Suppose that for all $\varepsilon > 0$ and $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\chi \neq 0$ and $d(\text{supp } \chi, \Omega^c) > 0$ there exist $a, \omega > 0$ such that*

$$\|M_\chi S_t^{(\rho)} u\|_p \leq a t^{-\frac{d}{2}(1-\frac{1}{p})} e^{\omega \rho^2 t} e^{\varepsilon t} \|u\|_1$$

for all $C \in \mathcal{E}^1(\Omega, \theta, \mu, M, \text{real})$, $t > 0$, $u \in L_1 \cap L_2$, $\rho \in \mathbb{R}$ and $\psi \in \mathcal{D}_2$. Then for all $q \in [p, \infty]$, $v \in (0, 1)$, $\varepsilon > 0$ and $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\frac{1}{p} - \frac{1}{q} < \frac{1-v}{d}$, $\chi \neq 0$ and $d(\text{supp } \chi, \Omega^c) > 0$ there exist $a, \omega > 0$ such that

$$\begin{aligned} \|M_\chi S_t^{(\rho)} u\|_q &\leq a t^{-\frac{d}{2}(1-\frac{1}{q})} e^{\omega \rho^2 t} e^{\varepsilon t} \|u\|_1 \quad \text{and} \\ \|(I - L(h))M_\chi S_t^{(\rho)} u\|_q &\leq a |h|^v t^{-\frac{d}{2}(1-\frac{1}{q})} t^{-\frac{v}{2}} e^{\omega \rho^2 t} e^{\varepsilon t} \|u\|_1 \end{aligned}$$

for all $C \in \mathcal{E}^1(\Omega, \theta, \mu, M, \text{real})$, $t > 0$, $u \in L_1 \cap L_2$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}_2$ and $h \in \mathbb{R}^d$.

Proof. Let $C \in \mathcal{E}^1(\Omega, \theta, \mu, M)$, $\varepsilon > 0$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}_2$, $\chi \in C_b^\infty(\mathbb{R}^d)$ and suppose $\chi \neq 0$ and $d(\text{supp } \chi, \Omega^c) > 0$. Then

$$\begin{aligned} \mathfrak{a}_\rho(M_\chi u, v) - \mathfrak{a}_\rho(u, M_\chi v) &= -\sum (M_{\partial_l \chi} \partial_k M_{\tilde{c}_{kl}} u, v) + \sum (M_{(\partial_l \chi)(\partial_k c_{kl})} u, v) \\ &\quad - \sum (M_{\partial_l \partial_k \chi} M_{c_{kl}} u, v) \\ &\quad - \rho \sum (M_{\tilde{c}_{kl}} M_{\partial_k \psi} M_{\partial_l \chi} u, v) \end{aligned} \tag{3.1}$$

for all $u, v \in W^{1,2}(\mathbb{R}^d)$. Note that $(\partial_l \chi)(\partial_k c_{kl})$ is a bounded function on \mathbb{R}^d since $\text{supp } \chi \subset \Omega$ and $c_{kl} \in W^{1,\infty}(\Omega)$. Since C is elliptic for all $u, v \in L_2(\mathbb{R}^d)$ one deduces

$$\begin{aligned}
& (M_\chi S_t^{(\rho)} u - S_t^{(\rho)} M_\chi u, v) \\
&= - \int_0^t \frac{d}{ds} (M_\chi S_{t-s}^{(\rho)} u, S_s^{(\rho)*} v) ds \\
&= \int_0^t \left(- (M_\chi A_\rho S_{t-s}^{(\rho)} u, S_s^{(\rho)*} v) + (M_\chi S_{t-s}^{(\rho)} u, A_\rho^* S_s^{(\rho)*} v) \right) ds \\
&= \int_0^t \alpha_\rho (M_\chi S_{t-s}^{(\rho)} u, S_s^{(\rho)*} v) - \alpha_\rho (S_{t-s}^{(\rho)} u, M_\chi S_s^{(\rho)*} v) ds \\
&= - \sum_{k,l=1}^d \int_0^t (S_s^{(\rho)} M_{\partial_l \chi} \partial_k M_{\tilde{c}_{kl}} S_{t-s}^{(\rho)} u, v) ds + R,
\end{aligned}$$

where R is the contribution of the last three terms in (3.1), which do not have a derivative. Therefore

$$\begin{aligned}
M_{\chi^2} S_t^{(\rho)} - M_\chi S_t^{(\rho)} M_\chi &= M_\chi [M_\chi, S_t^{(\rho)}] \\
&= - \sum_{k,l=1}^d \int_0^t M_\chi S_s^{(\rho)} M_{\partial_k \chi} \partial_l M_{\tilde{c}_{kl}} S_{t-s}^{(\rho)} ds + R', \quad (3.2)
\end{aligned}$$

with R' the contribution of the last three terms in (3.1).

Fix $k, l \in \{1, \dots, d\}$. Then

$$\begin{aligned}
& \int_0^t M_\chi S_s^{(\rho)} M_{\partial_k \chi} \partial_l M_{\tilde{c}_{kl}} S_{t-s}^{(\rho)} ds \\
&= \int_0^{t/2} M_\chi S_s^{(\rho)} M_{\partial_k \chi} \partial_l M_{\tilde{c}_{kl}} S_{t-s}^{(\rho)} ds + \int_{t/2}^t M_\chi S_s^{(\rho)} M_{\partial_k \chi} \partial_l M_{\tilde{c}_{kl}} S_{t-s}^{(\rho)} ds
\end{aligned} \quad (3.3)$$

for all $t > 0$. By Theorem 2.3 there are $a, \omega > 0$ such that

$$\begin{aligned}
\|M_\chi S_s^{(\rho)} M_{\partial_k \chi} \partial_l\|_{1 \rightarrow q} &\leq a s^{-\frac{d}{2}(1-\frac{1}{q})} s^{-1/2} e^{\omega \rho^2 s} e^{\varepsilon s} \quad \text{and} \\
\|(I - L(h)) M_\chi S_s^{(\rho)} M_{\partial_k \chi} \partial_l\|_{1 \rightarrow q} &\leq a |h|^v s^{-\frac{d}{2}(1-\frac{1}{q})} s^{-\frac{1+v}{2}} e^{\omega \rho^2 s} e^{\varepsilon s}
\end{aligned}$$

for all $s > 0, \rho \in \mathbb{R}, \psi \in \mathcal{D}_2$ and $h \in \mathbb{R}^d$. Suppose from now on that C is real valued. Since the matrix of coefficients is real it follows from Lemma 3.1 that there exists an $\omega' > 0$ such that $\|S_t^{(\rho)}\|_{1 \rightarrow 1} \leq e^{\omega' \rho^2 t} e^{\varepsilon t}$ for all $t > 0, \rho \in \mathbb{R}$ and $\psi \in \mathcal{D}_2$.

Then

$$\begin{aligned}
& \left\| \int_{t/2}^t M_\chi S_s^{(\rho)} M_{\partial_k \chi} \partial_t M_{\tilde{c}_{kl}} S_{t-s}^{(\rho)} u \, ds \right\|_q \\
& \leq 2a M \int_{t/2}^t s^{-\frac{d}{2}(1-\frac{1}{q})} s^{-1/2} e^{\omega \rho^2 s} e^{\varepsilon s} e^{\omega' \rho^2 (t-s)} e^{\varepsilon(t-s)} \|u\|_1 \, ds \\
& \leq 2^{d+1} a M t^{-\frac{d}{2}(1-\frac{1}{q})} t^{1/2} e^{(\omega+\omega')\rho^2 t} e^{\varepsilon t} \|u\|_1 \\
& \leq 2^{d+1} a \varepsilon^{-1/2} M t^{-\frac{d}{2}(1-\frac{1}{q})} e^{(\omega+\omega')\rho^2 t} e^{2\varepsilon t} \|u\|_1
\end{aligned}$$

for all $t > 0, u \in L_1 \cap L_2, \rho \in \mathbb{R}$ and $\psi \in \mathcal{D}_2$. Similarly,

$$\begin{aligned}
& \left\| \int_{t/2}^t (I - L(h)) M_\chi S_s^{(\rho)} M_{\partial_k \chi} \partial_t M_{\tilde{c}_{kl}} S_{t-s}^{(\rho)} u \, ds \right\|_q \\
& \leq 2^{d+1} a \varepsilon^{-1/2} M |h|^\nu t^{-\frac{d}{2}(1-\frac{1}{q})} t^{-\frac{\nu}{2}} e^{(\omega+\omega')\rho^2 t} e^{2\varepsilon t} \|u\|_1
\end{aligned}$$

for all $t > 0, u \in L_1 \cap L_2, \rho \in \mathbb{R}, \psi \in \mathcal{D}_2$ and $h \in \mathbb{R}^d$.

Next we estimate the $L_1 \rightarrow L_q$ norm of the first term in (3.3). There exists a $\tilde{\chi} \in C_b^\infty(\mathbb{R}^d)$ such that $\tilde{\chi}(x) = 1$ for all $x \in \text{supp } \chi$ and $d(\text{supp } \tilde{\chi}, \Omega^c) > 0$. Then by assumption there exist $a, \omega > 0$ such that

$$\|\tilde{\chi} S_t^{(\rho)} u\|_p \leq a t^{-\frac{d}{2}(1-\frac{1}{p})} e^{\omega \rho^2 t} e^{\varepsilon t} \|u\|_1$$

for all $t > 0, u \in L_1 \cap L_2, \rho \in \mathbb{R}$ and $\psi \in \mathcal{D}_2$. By Theorem 2.3 there are $a', \omega' > 0$ such that

$$\begin{aligned}
\|M_\chi S_s^{(\rho)} M_{\partial_k \chi} \partial_t\|_{p \rightarrow q} & \leq a' s^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} s^{-1/2} e^{\omega' \rho^2 s} e^{\varepsilon s} \quad \text{and} \\
\|(I - L(h)) M_\chi S_s^{(\rho)} M_{\partial_k \chi} \partial_t\|_{p \rightarrow q} & \leq a' |h|^\nu s^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} s^{-\frac{1+\nu}{2}} e^{\omega' \rho^2 s} e^{\varepsilon s}
\end{aligned}$$

for all $s > 0, \rho \in \mathbb{R}, \psi \in \mathcal{D}_2$ and $h \in \mathbb{R}^d$. Then

$$\begin{aligned}
& \left\| \int_0^{t/2} M_\chi S_s^{(\rho)} M_{\partial_k \chi} \partial_t M_{\tilde{c}_{kl}} S_{t-s}^{(\rho)} u \, ds \right\|_q \\
& \leq \int_0^{t/2} \|M_\chi S_s^{(\rho)} M_{\partial_k \chi} \partial_t\|_{p \rightarrow q} \|M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_{t-s}^{(\rho)} u\|_p \, ds \\
& \leq 2a a' M \int_0^{t/2} s^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} s^{-1/2} e^{\omega' \rho^2 s} e^{\varepsilon s} (t-s)^{-\frac{d}{2}(1-\frac{1}{q})} e^{\omega \rho^2 (t-s)} e^{\varepsilon(t-s)} \|u\|_1 \, ds \\
& \leq \frac{2^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})+1} a a' \varepsilon^{-1/2} M}{\frac{1}{2} - \frac{d}{2} \left(\frac{1}{p} - \frac{1}{q}\right)} t^{-\frac{d}{2}g(1-\frac{1}{p})} e^{(\omega+\omega')\rho^2 t} e^{2\varepsilon t} \|u\|_1
\end{aligned}$$

for all $t > 0, u \in L_1 \cap L_2, \rho \in \mathbb{R}$ and $\psi \in \mathcal{D}_2$. Similarly

$$\begin{aligned} & \left\| \int_0^{t/2} (I - L(h)) M_\chi S_s^{(\rho)} M_{\partial_k \chi} \partial_t M_{\tilde{c}_{kl}} S_{t-s}^{(\rho)} u \, ds \right\|_q \\ & \leq \frac{2^{\frac{d}{2}(\frac{1}{p} - \frac{1}{q}) + 1} a a' \varepsilon^{-\frac{1}{2}} M |h|^v}{\frac{1-v}{2} - \frac{d}{2} \left(\frac{1}{p} - \frac{1}{q} \right)} t^{-\frac{d}{2}(1-\frac{1}{p})} t^{-\frac{v}{2}} e^{(\omega+\omega')\rho^2 t} e^{2\varepsilon t} \|u\|_1 \end{aligned}$$

for all $t > 0, u \in L_1 \cap L_2, \rho \in \mathbb{R}, \psi \in \mathcal{D}_2$ and $h \in \mathbb{R}^d$.

The term R' in (3.2) can be estimated similarly. Using Theorem 2.3 to estimate the $L_1 \rightarrow L_q$ norms of $M_\chi S_t^{(\rho)} M_\chi$ and $(I - L(h)) M_\chi S_t^{(\rho)} M_\chi$ one deduces that there are $a, \omega > 0$ such that

$$\begin{aligned} \|M_{\chi^2} S_t^{(\rho)} u\|_q & \leq a t^{-\frac{d}{2}(1-\frac{1}{q})} e^{\omega\rho^2 t} e^{\varepsilon t} \|u\|_1 \quad \text{and} \\ \|(I - L(h)) M_{\chi^2} S_t^{(\rho)} u\|_q & \leq a |h|^v t^{-\frac{d}{2}(1-\frac{1}{q})} t^{-\frac{v}{2}} e^{\omega\rho^2 t} e^{\varepsilon t} \|u\|_1 \end{aligned}$$

for all $t > 0, u \in L_1 \cap L_2, \rho \in \mathbb{R}, \psi \in \mathcal{D}_2$ and $h \in \mathbb{R}^d$. Then the lemma follows. \square

Lemma 3.3. *Let $\Omega \subsetneq \mathbb{R}^d$ be open, $\theta \in [0, \frac{\pi}{2})$, $\mu, M > 0, v \in (0, 1), \varepsilon > 0$ and $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\chi \neq 0$ and $d(\text{supp } \chi, \Omega^c) > 0$. Then there exist $a > 0$ and $\omega \geq 0$ such that*

$$\begin{aligned} \|M_\chi S_t^{(\rho)} u\|_\infty & \leq a t^{-d/2} e^{\omega\rho^2 t} e^{\varepsilon t} \|u\|_1 \quad \text{and} \\ \|(I - L(h)) M_\chi S_t^{(\rho)} u\|_\infty & \leq a |h|^v t^{-d/2} t^{-v/2} e^{\omega\rho^2 t} e^{\varepsilon t} \|u\|_1 \end{aligned}$$

for all $C \in \mathcal{S}^1(\Omega, \theta, \mu, M, \text{real}), t > 0, u \in L_1 \cap L_2, \rho \in \mathbb{R}, \psi \in \mathcal{D}_2$ and $h \in \mathbb{R}^d$. In particular, $M_\chi S_t u \in C^{0,v}(\mathbb{R}^d)$ for all $t > 0$ and $u \in L_1(\mathbb{R}^d)$.

Proof. It follows by induction from Lemmas 3.1 and 3.2 that the current lemma is valid if $\mathcal{S}^1(\Omega, \theta, \mu, M, \text{real})$ is replaced by $\mathcal{E}^1(\Omega, \theta, \mu, M, \text{real})$. Then by approximating C by $C + \frac{1}{n} I$ the lemma follows. \square

Next we turn to derivatives of the semigroup.

Lemma 3.4. *Let $u \in W^{1,2}(\mathbb{R}^d), v \in L_2(\mathbb{R}^d)$ and $k \in \{1, \dots, d\}$. Then*

$$\begin{aligned} (\partial_k u, v) & = \frac{1}{c_0} \int_0^\infty \frac{((I - L(r e_k))^2 u, v)}{r} \frac{dr}{r} \\ & = \frac{1}{c_0} \int_0^\infty \frac{(I - L(r e_k))u, (I - L(-r e_k))v}{r} \frac{dr}{r}, \end{aligned}$$

where $c_0 = \int_0^\infty \frac{(1-e^{-r})^2}{r} \frac{dr}{r}$.

Proof. Write $D = \frac{d}{dx_k}$, the skew-adjoint operator in $L_2(\mathbb{R}^d)$.

Then $((I - L(r e_k))^2 u, v) = ((I - e^{-rD})^2 u, v)$. Now the lemma follows from Fourier theory (or spectral theory). \square

Lemma 3.5. *Let $\Omega \subsetneq \mathbb{R}^d$ be open, $\theta \in [0, \frac{\pi}{2})$, $\mu, M > 0$, $\nu \in (0, 1)$, $\varepsilon \in (0, 1]$ and $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\chi \neq 0$ and $d(\text{supp } \chi, \Omega^c) > 0$. Then there exists an $a > 0$ such that*

$$\begin{aligned} \|\partial_m M_\chi S_t\|_{1 \rightarrow \infty} &\leq a t^{-d/2} t^{-1/2} e^{\varepsilon t} \quad \text{and} \\ \|(I - L(h))\partial_m M_\chi S_t\|_{1 \rightarrow \infty} &\leq a t^{-d/2} t^{-1/2} \left(\frac{|h|}{\sqrt{t}}\right)^\nu e^{\varepsilon t} \end{aligned}$$

for all $C \in \mathcal{E}^1(\Omega, \theta, \mu, M, \text{real})$, $t > 0$, $m \in \{1, \dots, d\}$ and $h \in \mathbb{R}^d$.

Proof. We only prove the second estimate, the proof of the first one is similar. We argue as in the proof of Lemma 3.2 and use the commutator (3.2). There exists a $\tilde{\chi} \in C_b^\infty(\mathbb{R}^d)$ such that $\tilde{\chi}(x) = 1$ for all $x \in \text{supp } \chi$ and $d(\text{supp } \tilde{\chi}, \Omega^c) > 0$. Now we have

$$\begin{aligned} &(I - L(h))\partial_m \left(M_{\chi^2} S_t - M_\chi S_t M_\chi \right) \\ &= - \sum_{k,l=1}^d \int_0^t (I - L(h))\partial_m M_\chi S_s M_{\partial_k \chi} \partial_l M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_{t-s} ds + R, \end{aligned} \tag{3.4}$$

where R is the contribution of the second and third term in (3.1), which do not have a derivative. Note that the last term in (3.1) vanishes since $\rho = 0$. Again we split the integral in two parts. Fix $k, l \in \{1, \dots, d\}$. Then

$$\begin{aligned} &\left\| \int_{t/2}^t (I - L(h))\partial_m M_\chi S_s M_{\partial_k \chi} \partial_l M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_{t-s} ds \right\|_{1 \rightarrow \infty} \\ &\leq \int_{t/2}^t \|(I - L(h))\partial_m M_\chi S_s M_{\partial_k \chi} \partial_l\|_{1 \rightarrow \infty} \|M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_{t-s}\|_{1 \rightarrow 1} ds \\ &\leq a t^{-d/2} \left(\frac{|h|}{\sqrt{t}}\right)^\nu e^{\varepsilon t} \end{aligned}$$

for a suitable $a > 0$, by the estimates of Theorem 2.3 and Lemma 3.1.

For the integral over $(0, \frac{t}{2})$ we use Lemma 3.4 and write

$$\begin{aligned}
& \left\| \int_0^{t/2} (I - L(h)) \partial_m M_\chi S_s M_{\partial_k \chi} \partial_l M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_{t-s} ds \right\|_{1 \rightarrow \infty} \\
&= c_0^{-1} \sup_{\substack{u, v \in W^{1,2}(\mathbb{R}^d) \cap L_1(\mathbb{R}^d) \\ \|u\|_1, \|v\|_1 \leq 1}} \left| \int_0^{t/2} \int_0^\infty ((I - L(re_l)) M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_{t-s} u, \right. \\
&\quad \left. (I - L(-re_l)) M_{\partial_k \chi} S_s^* M_\chi \partial_m (I - L(-h)) v) \frac{dr}{r^2} ds \right| \\
&\leq c_0^{-1} \sup_{\substack{u, v \in W^{1,2}(\mathbb{R}^d) \cap L_1(\mathbb{R}^d) \\ \|u\|_1, \|v\|_1 \leq 1}} \int_0^{t/2} \int_0^\infty \|(I - L(re_l)) M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_{t-s} u\|_\infty \\
&\quad \times \|(I - L(-re_l)) M_{\partial_k \chi} S_s^* M_\chi \partial_m (I - L(-h)) v\|_1 \frac{dr}{r^2} ds,
\end{aligned}$$

where c_0 is as in Lemma 3.4. Next split the integral over $(0, \infty)$ in two parts: $(0, 1]$ and $[1, \infty)$. There exist $\nu_1, \nu_2 \in (0, 1)$ such that $\nu_2 + \nu < 1$ and $\nu_1 + \nu_2 > 1$. By Theorem 2.3 and Lemma 3.3 there exists an $a > 0$ such that

$$\begin{aligned}
& \|(I - L(h_1)) M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_s\|_{1 \rightarrow \infty} \leq a s^{-d/2} \left(\frac{|h_1|}{\sqrt{s}} \right)^{\nu_1} e^{\varepsilon s}, \\
& \|M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_s\|_{1 \rightarrow \infty} \leq a s^{-d/2} e^{\varepsilon s}, \\
& \|(I - L(h_1)) M_{\partial_k \chi} S_s^* M_\chi \partial_m (I - L(h_2))\|_{1 \rightarrow 1} \leq a s^{-1/2} \left(\frac{|h_1|}{\sqrt{s}} \right)^{\nu_2} \left(\frac{|h_2|}{\sqrt{s}} \right)^\nu e^{\varepsilon s} \quad \text{and} \\
& \|M_{\partial_k \chi} S_s^* M_\chi \partial_m (I - L(h_2))\|_{1 \rightarrow 1} \leq a s^{-1/2} \left(\frac{|h_2|}{\sqrt{s}} \right)^\nu e^{\varepsilon s}
\end{aligned}$$

for all $s > 0$ and $h_1, h_2 \in \mathbb{R}^d$. Let $u, v \in W^{1,2}(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$. Then

$$\begin{aligned}
& \int_0^{t/2} \int_0^1 \|(I - L(re_l)) M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_{t-s} u\|_\infty \|(I - L(-re_l)) M_{\partial_k \chi} S_s^* M_\chi \partial_m (I - L(-h)) v\|_1 \\
& \quad \times \frac{dr}{r^2} ds \\
& \leq a^2 \int_0^{t/2} \int_0^1 (t-s)^{-d/2} \left(\frac{r}{\sqrt{t-s}} \right)^{\nu_1} e^{\varepsilon(t-s)} s^{-1/2} \left(\frac{|h_1|}{\sqrt{s}} \right)^\nu \left(\frac{r}{\sqrt{s}} \right)^{\nu_2} e^{\varepsilon s} \|u\|_1 \|v\|_1 \\
& \quad \times \frac{dr}{r^2} ds \\
& = a' t^{-d/2} t^{-1/2} \left(\frac{|h_1|}{\sqrt{t}} \right)^\nu e^{\varepsilon t} t^{\frac{2-(\nu_1+\nu_2)}{2}} \|u\|_1 \|v\|_1 \\
& \leq a' \varepsilon^{-\frac{2-(\nu_1+\nu_2)}{2}} t^{-d/2} t^{-1/2} \left(\frac{|h_1|}{\sqrt{t}} \right)^\nu e^{2\varepsilon t} \|u\|_1 \|v\|_1
\end{aligned}$$

for a suitable $a' > 0$. Similarly, since left translations are isometries, one deduces

$$\begin{aligned}
 & \int_0^{t/2} \int_1^\infty \|(I - L(re_l))M_{\tilde{c}_{kl}}M_{\tilde{\chi}}S_{t-s}u\|_\infty \|(I - L(-re_l))M_{\partial_k \chi}S_s^*M_\chi \partial_m(I - L(-h))v\|_1 \\
 & \quad \times \frac{dr}{r^2} ds \\
 & \leq 4a^2 \int_0^{t/2} \int_1^\infty (t-s)^{-d/2} e^{\varepsilon(t-s)} s^{-1/2} \left(\frac{|h|}{\sqrt{s}}\right)^v e^{\varepsilon s} \|u\|_1 \|v\|_1 \frac{dr}{r^2} ds \\
 & = a'' t^{-d/2} t^{1/2} \left(\frac{|h|}{\sqrt{t}}\right)^v e^{\varepsilon t} \|u\|_1 \|v\|_1 \\
 & \leq a'' \varepsilon^{-1} t^{-d/2} t^{-1/2} \left(\frac{|h|}{\sqrt{t}}\right)^v e^{2\varepsilon t} \|u\|_1 \|v\|_1
 \end{aligned}$$

for a suitable $a'' > 0$. As before, the contribution of the term R in (3.4) can be estimated similarly and we leave the rest of the proof to the reader. \square

We next replace $\mathcal{E}^1(\Omega, \theta, \mu, M, \text{real})$ by $\mathcal{S}^1(\Omega, \theta, \mu, M, \text{real})$ in Lemma 3.5.

Lemma 3.6. *Let $\Omega \subsetneq \mathbb{R}^d$ be open, $\theta \in [0, \frac{\pi}{2})$, $\mu, M > 0$, $v \in (0, 1)$, $\varepsilon \in (0, 1]$ and $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\chi \neq 0$ and $d(\text{supp } \chi, \Omega^c) > 0$. Then there exists an $a > 0$ such that*

$$\begin{aligned}
 \|\partial_m M_\chi S_t\|_{1 \rightarrow \infty} & \leq a t^{-d/2} t^{-1/2} e^{\varepsilon t} \quad \text{and} \\
 \|(I - L(h))\partial_m M_\chi S_t\|_{1 \rightarrow \infty} & \leq a t^{-d/2} t^{-1/2} \left(\frac{|h|}{\sqrt{t}}\right)^v e^{\varepsilon t}
 \end{aligned}$$

for all $C \in \mathcal{S}^1(\Omega, \theta, \mu, M, \text{real})$, $t > 0$, $m \in \{1, \dots, d\}$ and $h \in \mathbb{R}^d$. In particular, $M_\chi S_t u \in W^{1+v, \infty}(\mathbb{R}^d)$ for all $t > 0$ and $u \in L_1(\mathbb{R}^d)$.

Proof. By Lemma 3.5 there exists an $a > 0$ such that

$$\|\partial_m M_\chi S_t\|_{1 \rightarrow \infty} \leq a t^{-d/2} t^{-1/2} e^{\varepsilon t} \tag{3.5}$$

for all $C \in \mathcal{E}^1(\Omega, \theta, \mu, M + 1, \text{real})$, $t > 0$ and $m \in \{1, \dots, d\}$. Fix $C \in \mathcal{S}^1(\Omega, \theta, \mu, M, \text{real})$. For all $n \in \mathbb{N}$ define $C^{(n)} = C + \frac{1}{n}I$. Let $t > 0$, $m \in \{1, \dots, d\}$ and $u, v \in C_c^\infty(\mathbb{R}^d)$. Then it follows from (3.5) that

$$|(M_\chi S_t^{C^{(n)}} u, \partial_m v)| \leq a t^{-d/2} t^{-1/2} e^{\varepsilon t} \|u\|_1 \|v\|_1$$

for all $n \in \mathbb{N}$. Since $\lim S_t^{C^{(n)}} = S_t^C$ strongly in $L_2(\mathbb{R}^d)$ by [3] Corollary 3.9 it follows that

$$|(M_\chi S_t^C u, \partial_m v)| \leq a t^{-d/2} t^{-1/2} e^{\varepsilon t} \|u\|_1 \|v\|_1 \tag{3.6}$$

for all $u, v \in C_c^\infty(\mathbb{R}^d)$. By continuity it then follows that (3.6) is valid for all $u \in L_1(\mathbb{R}^d)$ and $v \in C_c^\infty(\mathbb{R}^d)$. This implies that $M_\chi S_t^C u \in W^{1, \infty}(\mathbb{R}^d)$ and the first estimate of the lemma is valid. The second one follows similarly. \square

It follows from Lemma 3.3 that for each $t > 0$ there exists a measurable function $L_t: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that L_t is a kernel of $M_\chi S_t$ and L_t satisfies Gaussian bounds. But then it is unclear whether $x \mapsto L_t(x, y)$ is Hölder continuous or differentiable for some $y \in \mathbb{R}^d$. Even worse, there is no reason that the combined map $(t, x, y) \mapsto L_t(x, y)$ from $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ into \mathbb{R} is measurable. In order to circumvent this measurability problem with the uncountable many null sets, we first obtain a measurable map on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ for the kernels of $M_\chi S_t$ and its derivatives $\partial_m M_\chi S_t$ and then consider continuity and differential properties in Theorem 3.9.

Proposition 3.7. *Let $\Omega \subsetneq \mathbb{R}^d$ be open, $\theta \in [0, \frac{\pi}{2})$, $\mu, M > 0$ and $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\chi \not\equiv 0$, $\varepsilon > 0$ and $d(\text{supp } \chi, \Omega^c) > 0$. Then there exist $a, b > 0$ such that for all $C \in \mathcal{S}^1(\Omega, \theta, \mu, M, \text{real})$ and $m \in \{1, \dots, d\}$ there exist measurable functions $(t, x, y) \mapsto K_t(x, y)$ and $(t, x, y) \mapsto K_t^{(m)}(x, y)$ from $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ into \mathbb{R} such that K_t is a kernel of the operator $M_\chi S_t$ and $K_t^{(m)}$ is a kernel of the operator $\partial_m M_\chi S_t$ for all $t > 0$. Moreover,*

$$|K_t(x, y)| \leq a t^{-d/2} e^{\varepsilon t} e^{-b \frac{|x-y|^2}{t}} \quad \text{and}$$

$$|K_t^{(m)}(x, y)| \leq a t^{-d/2} t^{-1/2} e^{\varepsilon t} e^{-b \frac{|x-y|^2}{t}}$$

for all $t > 0$ and $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$.

Proof. Let a, ω be as in Lemma 3.3 (with $\nu = \frac{1}{2}$). Let $C \in \mathcal{S}^1(\Omega, \theta, \mu, M, \text{real})$. Then $M_\chi S_t$ is a continuous operator from $L_2(\mathbb{R}^d)$ into $L_\infty(\mathbb{R}^d)$ for all $t > 0$ by Lemma 2.5. Moreover, $z \mapsto S_z$ is a holomorphic contraction semigroup on $\Sigma_{\theta_a}^\circ$. Hence $M_\chi S_z$ maps $L_2(\mathbb{R}^d)$ continuously into $L_\infty(\mathbb{R}^d)$ for all $z \in \Sigma_{\theta_a}^\circ$ and $z \mapsto M_\chi S_z u$ is holomorphic for all $u \in L_2(\mathbb{R}^d)$. It follows from the discussion after Definition 1.8 in [2] and [2] Theorem 3.1 that there exists a measurable function $(z, x, y) \mapsto K_z(x, y)$ from $\Sigma_{\theta_a}^\circ \times \mathbb{R}^d \times \mathbb{R}^d$ into \mathbb{C} such that $z \mapsto K_z(x, y)$ is holomorphic for all $x, y \in \mathbb{R}^d$ and K_z is the kernel of $M_\chi S_z$ for all $z \in \Sigma_{\theta_a}^\circ$. In particular, K_t is a kernel of $M_\chi S_t$ for all $t > 0$. By the usual minimising argument the estimates of Lemma 3.3 give Gaussian bounds for the kernel K_t for each $t > 0$. Precisely, for each $t > 0$ one has $|K_t| \leq G_t$ a.e. on $\mathbb{R}^d \times \mathbb{R}^d$, where $G_t(x, y) = a t^{-d/2} e^{\varepsilon t} e^{-b \frac{|x-y|^2}{t}}$ for all $t > 0$ and $x, y \in \mathbb{R}^d$, where b depends only on ω and d . Obviously $(t, x, y) \mapsto G_t(x, y)$ is a continuous function from $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ into \mathbb{R} , therefore it is measurable. Then $(-G_t) \vee K_t \wedge G_t$ is also a kernel of $M_\chi S_t$ for each $t > 0$. Now the proposition follows for $M_\chi S_t$ with K_t replaced by $(-G_t) \vee K_t \wedge G_t$.

The argument for $\partial_m M_\chi S_t$ is similar. □

The next lemma is also valid for complex coefficients. The complex version will be used in the proof of Proposition 4.11.

Lemma 3.8. *Let $\Omega \subsetneq \mathbb{R}^d$ be open, $\theta \in [0, \frac{\pi}{2})$, $\mu, M > 0$, $C \in \mathcal{S}^1(\Omega, \theta, \mu, M)$ and $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\chi \neq 0$ and $d(\text{supp } \chi, \Omega^c) > 0$. Let $\tilde{\chi} \in C_b^\infty(\mathbb{R}^d)$ be such that $d(\text{supp } \tilde{\chi}, \Omega^c) > 0$ and $\tilde{\chi}(x) = 1$ for all $x \in \text{supp } \chi$. Then*

$$\begin{aligned} M_\chi^2 S_t &= M_\chi S_t M_\chi - \sum_{k,l=1}^d \int_0^t M_\chi S_s M_{\partial_k \chi} \partial_l M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_{t-s} ds \\ &\quad + \sum_{k,l=1}^d \int_0^t M_\chi S_s \left(M_{\partial_l \tilde{c}_{kl}} M_{\partial_k \chi} S_{t-s} - M_{c_{kl}} M_{\partial_k \chi} S_{t-s} \right) ds \end{aligned} \quad (3.7)$$

for all $t > 0$, where the operators act in $L_2(\mathbb{R}^d)$.

Proof. It follows from Lemma 2.1(d) that the integrals on the right hand side of (3.7) are convergent.

If $C \in \mathcal{E}^1(\Omega, \theta, \mu, M)$ then (3.7) follows from (3.2). For all $n \in \mathbb{N}$ let $C_n = C + \frac{1}{n}I$. Then $C_n \in \mathcal{E}^1(\Omega, \theta, \mu, M + 1)$. We use (3.7) with respect to $C^{(n)}$. Write $S = S^C$ and $S^{(n)} = S^{C_n}$. It follows from Lemma 2.1(d) that there exist $a, \omega > 0$ such that

$$\|M_\chi S_s^{(n)} M_{\partial_k \chi} \partial_l\|_{2 \rightarrow 2} \leq a s^{-1/2} e^{\omega s}$$

for all $s > 0$ and $n \in \mathbb{N}$. Obviously $\|S_s^{(n)}\|_{2 \rightarrow 2} \leq 1$ for all $s > 0$ and $n \in \mathbb{N}$.

Let $u, v \in L_2(\mathbb{R}^d)$. Then $\lim_{n \rightarrow \infty} S_s^{(n)} u = S_s u$ in $L_2(\mathbb{R}^d)$ for all $s \in (0, t]$ by [3, Corollary 3.9]. Note that $M_{\tilde{\chi}} S_s u \in W^{1,2}(\mathbb{R}^d)$ for all $s > 0$. Let $k, l \in \{1, \dots, d\}$. Then

$$\begin{aligned} &|(M_\chi S_s^{(n)} M_{\partial_k \chi} \partial_l M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_{t-s}^{(n)} u, v) - (M_\chi S_s M_{\partial_k \chi} \partial_l M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_{t-s} u, v)| \\ &\leq \frac{1}{n} |(M_\chi S_s^{(n)} M_{\partial_k \chi} \partial_l S_{t-s}^{(n)} u, v)| \\ &\quad + |(M_{\tilde{c}_{kl}} (S_{t-s}^{(n)} u - S_{t-s} u), \partial_l M_{\partial_k \chi} S_s^{(n)*} M_\chi v)| \\ &\quad + |(\partial_l M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_{t-s} u, M_{\partial_k \chi} (S_s^{(n)*} M_\chi v - S_s^* M_\chi v))| \end{aligned}$$

for all $s \in (0, t)$ and $n \in \mathbb{N}$. So

$$\lim_{n \rightarrow \infty} (M_\chi S_s^{(n)} M_{\partial_k \chi} \partial_l M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_{t-s}^{(n)} u, v) = (M_\chi S_s M_{\partial_k \chi} \partial_l M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_{t-s} u, v)$$

for all $s \in (0, t)$. Moreover,

$$|(M_\chi S_s^{(n)} M_{\partial_k \chi} \partial_l M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_{t-s}^{(n)} u, v)| \leq a s^{-1/2} e^{\omega s} \|u\|_2 \|v\|_2$$

for all $s \in (0, t)$ and $n \in \mathbb{N}$. So

$$\begin{aligned} &\lim_{n \rightarrow \infty} \int_0^t (M_\chi S_s^{(n)} M_{\partial_k \chi} \partial_l M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_{t-s}^{(n)} u, v) ds \\ &= \int_0^t (M_\chi S_s M_{\partial_k \chi} \partial_l M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_{t-s} u, v) ds \end{aligned}$$

by the Lebesgue dominated convergence theorem.

One can treat similarly the last term on the right hand side of (3.7) and the lemma follows. \square

The next theorem is a uniform version of Theorem 1.2.

Theorem 3.9. *Let $\Omega \subsetneq \mathbb{R}^d$ be open, $\theta \in [0, \frac{\pi}{2})$, $\mu, M > 0$, $\nu \in (0, 1)$, $\kappa > 0$, $\tau \in [0, 1)$, $\varepsilon \in (0, 1]$ and $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\chi \neq 0$ and $d(\text{supp } \chi, \Omega^c) > 0$. Then there exist $a, b > 0$ such that for all $C \in \mathcal{S}^1(\Omega, \theta, \mu, M, \text{real})$ there exists a measurable function $(t, x, y) \mapsto K_t(x, y)$ from $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ into \mathbb{R} such that the following is valid.*

- (a) *For all $t > 0$ the function K_t is a kernel of $M_\chi S_t$.*
- (b) *For all $t > 0$ and $y \in \mathbb{R}^d$ the function $x \mapsto K_t(x, y)$ is continuously differentiable on \mathbb{R}^d .*
- (c) *The function $t \mapsto (\partial_x^\alpha K_t)(x, y)$ is continuous for all $x, y \in \mathbb{R}^d$ and multi-index α with $|\alpha| \leq 1$.*
- (d) *For every multi-index α with $|\alpha| \leq 1$ one has*

$$|(\partial_x^\alpha K_t)(x, y)| \leq a t^{-d/2} t^{-|\alpha|/2} e^{\varepsilon t} e^{-b \frac{|x-y|^2}{t}} \quad (3.8)$$

and

$$|(\partial_x^\alpha K_t)(x+h, y) - (\partial_x^\alpha K_t)(x, y)| \leq a t^{-d/2} t^{-|\alpha|/2} \left(\frac{|h|}{|x-y| + \sqrt{t}} \right)^\nu e^{\varepsilon t} e^{-b \frac{|x-y|^2}{t}}$$

for all $t > 0$ and $x, y, h \in \mathbb{R}^d$ with $|h| \leq \tau |x-y| + \kappa \sqrt{t}$.

- (e) $(\partial^\alpha M_\chi S_t u)(x) = \int_{\mathbb{R}^d} (\partial_x^\alpha K_t)(x, y) u(y) dy$ for all $t > 0$, $u \in L_1(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and multi-index α with $|\alpha| \leq 1$.

Proof. Without loss of generality we may prove the proposition with $M_\chi S_t$ replaced by $M_\chi^2 S_t$. We consider each of the operators in the terms on the right hand side of (3.7). It is possible to differentiate all terms in (3.7) at least once in L_2 -sense. Let α be a multi-index with $|\alpha| \leq 1$. Clearly $\partial^\alpha M_\chi S_t M_\chi$ has a kernel satisfying the stated requirements by Theorem 2.3. Let $k, l \in \{1, \dots, d\}$. For all $t > 0$ consider the operator

$$\int_0^t \partial^\alpha M_\chi S_s M_{\partial_k \chi} \partial_l M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_{t-s} ds. \quad (3.9)$$

By Theorem 2.3 there exist a continuous function $(t, x, y) \mapsto L_t^{(1, \alpha)}(x, y)$ from $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ into \mathbb{R} and suitable constants $a_1, b_1 > 0$ such that for all $t > 0$ the function $L_t^{(1, \alpha)}$ is a kernel of the operator $\partial^\alpha M_\chi S_t M_{\partial_k \chi}$,

$$|L_t^{(1, \alpha)}(x, y)| \leq a_1 t^{-d/2} t^{-|\alpha|/2} e^{\varepsilon t} e^{-b_1 \frac{|x-y|^2}{t}} \text{ and}$$

$$|L_t^{(1, \alpha)}(x+h, y) - L_t^{(1, \alpha)}(x, y)| \leq a_1 t^{-d/2} t^{-|\alpha|/2} \left(\frac{|h|}{\sqrt{t}} \right)^\nu e^{\varepsilon t} e^{-b_1 \frac{|x-y|^2}{t}}$$

for all $t > 0$ and $x, y, h \in \mathbb{R}^d$ with $|h| \leq \frac{1}{2}|x - y| + \sqrt{t}$. Next, by Proposition 3.7 there exist a measurable function $(t, x, y) \mapsto L_t^{(2)}(x, y)$ from $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ into \mathbb{R} and suitable constants $a_2, b_2 > 0$ such that for all $t > 0$ the function $L_t^{(2)}$ is a kernel of the operator $\partial_t M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_t$ and

$$|L_t^{(2)}(x, y)| \leq a_2 t^{-d/2} t^{-1/2} e^{\varepsilon t} e^{-b_2 \frac{|x-y|^2}{t}}$$

for all $t > 0$ and $x, y \in \mathbb{R}^d$. Define the function $(t, x, y) \mapsto L_t^{(3, \alpha)}(x, y)$ from $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ into \mathbb{R} by

$$L_t^{(3, \alpha)}(x, y) = \int_0^t \int_{\mathbb{R}^d} L_s^{(1, \alpha)}(x, z) L_{t-s}^{(2)}(z, y) dz ds.$$

Since the convolution of two Gaussians is a Gaussian, it follows that the integral is convergent and $L^{(3, \alpha)}$ has appropriate Gaussian bounds. Moreover, $(t, x, y) \mapsto L_t^{(3, \alpha)}(x, y)$ is measurable and $L_t^{(3, \alpha)}$ is a kernel of the operator (3.9) for all $t > 0$. Define $L_t^{(3)} = L^{(3, \beta)}$ if $|\beta| = 0$. It is an elementary exercise in integration theory to prove that $x \mapsto L_t^{(3, \alpha)}(x, y)$ is continuous for all $t > 0$ and $y \in \mathbb{R}^d$, that $t \mapsto L_t^{(3, \alpha)}(x, y)$ is continuous for all $x, y \in \mathbb{R}^d$ and that for all $t > 0$ and $y \in \mathbb{R}^d$ the function $x \mapsto L_t^{(3)}(x, y)$ is differentiable and $(\partial_x^\alpha L_t^{(3)})(x, y) = L_t^{(3, \alpha)}(x, y)$ for all $x \in \mathbb{R}^d$.

The last term in (3.7) can be treated in a similar way and Statements (a)–(c) and the first part of Statement (d) follow. Let K be the so obtained kernel.

Next, let $t > 0$ and $u \in L_1(\mathbb{R}^d)$. Since $(x, y) \mapsto (\partial_x^\alpha K_t)(x, y)$ is a kernel of $\partial^\alpha M_\chi S_t$ it follows that $(\partial^\alpha M_\chi S_t u)(x) = \int_{\mathbb{R}^d} (\partial_x^\alpha K_t)(x, y) u(y) dy$ for a.e. $x \in \mathbb{R}^d$. Hence this is valid for all $x \in \mathbb{R}^d$ since $\partial^\alpha M_\chi S_t u$ is continuous by Lemmas 3.3 and 3.6. This proves Statement (e).

Finally we establish the Hölder Gaussian bounds of Statement (d). Set $\tilde{\nu} = \frac{1}{2}(1 + \nu)$. By Lemma 3.6 there exists an $a_1 > 0$, depending only on $\Omega, \theta, \mu, M, \nu, \varepsilon$ and χ , such that

$$\|(I - L(h))\partial_m M_\chi S_t\|_{1 \rightarrow \infty} \leq a_1 t^{-d/2} t^{-1/2} \left(\frac{|h|}{\sqrt{t}} \right)^{\tilde{\nu}} e^{\varepsilon t}$$

for all $t > 0, m \in \{1, \dots, d\}$ and $h \in \mathbb{R}^d$. Let $x, h \in \mathbb{R}^d, t > 0$ and $m \in \{1, \dots, d\}$. Then it follows from Statement (e) that

$$\begin{aligned} \left| \int \left((\partial_{x,m} K_t)(x, y) - (\partial_{x,m} K_t)(x - h, y) \right) u(y) dy \right| &= \left| \left((I - L(h))\partial_m M_\chi S_t u \right)(x) \right| \\ &\leq a_1 t^{-d/2} t^{-1/2} \left(\frac{|h|}{\sqrt{t}} \right)^{\tilde{\nu}} e^{\varepsilon t} \|u\|_1 \end{aligned}$$

for all $u \in L_1(\mathbb{R}^d)$. Hence there exists a null set $N_{x,h,t,m} \subset \mathbb{R}^d$ such that

$$|(\partial_{x,m} K_t)(x, y) - (\partial_{x,m} K_t)(x - h, y)| \leq a_1 t^{-d/2} t^{-1/2} \left(\frac{|h|}{\sqrt{t}} \right)^{\tilde{\nu}} e^{\varepsilon t} \quad (3.10)$$

for all $y \in \mathbb{R}^d \setminus N_{x,h,t,m}$. Then

$$N = \bigcup_{x,h \in \mathbb{Q}^d} \bigcup_{t \in (0, \infty) \cap \mathbb{Q}} \bigcup_{m=1}^d N_{x,h,t,m}$$

is a null set in \mathbb{R}^d and (3.10) is valid for all $y \in \mathbb{R}^d \setminus N$, $x, h \in \mathbb{Q}^d$, $t \in (0, \infty) \cap \mathbb{Q}$ and $m \in \{1, \dots, d\}$. Then by density and continuity (Statements (b) and (c)) one deduces that (3.10) is valid for all $y \in \mathbb{R}^d \setminus N$, $x, h \in \mathbb{R}^d$, $t > 0$ and $m \in \{1, \dots, d\}$.

Let a, b be as in (3.8). Then it follows as in the proof of Step A of Lemma A.1 that there are $a_2, b_2 > 0$, depending only on $a, b, a_1, \kappa, \tau, \varepsilon, \nu$ and $\tilde{\nu}$, such that

$$\begin{aligned} & |(\partial_{x,m} K_t)(x, y) - (\partial_{x,m} K_t)(x - h, y)| \\ & \leq a_2 t^{-d/2} t^{-|\alpha|/2} \left(\frac{|h|}{|x - y| + \sqrt{t}} \right)^{\nu} e^{\varepsilon t} e^{-b_2 \frac{|x-y|^2}{t}} \end{aligned}$$

for all $y \in \mathbb{R}^d \setminus N$, $x, h \in \mathbb{R}^d$, $t > 0$ and $m \in \{1, \dots, d\}$ with $|h| \leq \tau |x - y| + \kappa \sqrt{t}$. Now the theorem follows by replacing $(t, x, y) \mapsto K_t(x, y)$ by the function $(t, x, y) \mapsto K_t(x, y) \mathbb{1}_{\mathbb{R}^d \setminus N}(y)$. \square

Corollary 3.10. *Let $\Omega \subsetneq \mathbb{R}^d$ be open, $\theta \in [0, \frac{\pi}{2})$, $\mu, M > 0$ and $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\chi \neq 0$ and $d(\text{supp } \chi, \Omega^c) > 0$. Then there exists a $c > 0$ such that $M_\chi (I + A)^{-1} L_p(\mathbb{R}^d) \subset W^{1,p}(\mathbb{R}^d)$ and*

$$\|\partial_m M_\chi u\|_p \leq c \|(I + A)u\|_p$$

for all $C \in \mathcal{S}^1(\Omega, \theta, \mu, M, \text{real})$, $p \in [1, \infty]$, $u \in D(A)$ and $m \in \{1, \dots, d\}$.

Proof. The Gaussian bounds of Theorem 3.9 imply bounds $\|\partial_m M_\chi S_t\|_{p \rightarrow p} \leq c t^{-1/2} e^{t/2}$. Then the corollary follows by a Laplace transform. \square

4. Riesz transforms

In this section we shall prove that various Riesz transforms like $\partial_k M_\chi (I + A)^{-1/2}$ and $\partial_k M_\chi (I + A)^{-1/2} M_\chi$ are bounded on L_2 or L_p . The first results on L_2 for the Riesz transform merely use that $\chi \in W^{1,\infty}(\mathbb{R}^d)$.

If $\eta \in W^{1,\infty}(\mathbb{R}^d)$, then

$$\text{Re } a_C(\eta u) \leq 2\|\eta\|_\infty \text{Re } a_C(u) + 2M \|\nabla \eta\|_\infty^2 \|u\|_2^2$$

for all $u \in W^{1,2}(\mathbb{R}^d)$, $\theta \in [0, \frac{\pi}{2})$, $M > 0$ and $C \in \mathcal{S}(\theta, M)$. Therefore for self-adjoint operators the boundedness of the Riesz transforms on $L_2(\mathbb{R}^d)$ is trivial. Throughout this section let $L = -\sum_{k=1}^d \partial_k^2$ be the Laplacian and let $H = A_{\mathfrak{R}C}$.

The first lemma is a variation of Lemma 2.1(c).

Lemma 4.1. *Let $\theta \in [0, \frac{\pi}{2})$, $\mu, M > 0$, $\chi \in W^{1,\infty}(\mathbb{R}^d)$ and $C \in \mathcal{S}(\text{supp } \chi, \theta, \mu, M)$. Then $M_\chi D(H^{1/2}) \subset W^{1,2}(\mathbb{R}^d)$ and*

$$\|\partial_k M_\chi u\|_2 \leq \|(I + L)^{1/2} M_\chi u\|_2 \leq c_1 \|(I + H)^{1/2} u\|_2$$

for all $u \in D(H^{1/2})$ and $k \in \{1, \dots, d\}$, where

$$c_1 = \left(\|\chi\|_\infty^2 + 2 \frac{\|\chi\|_\infty^2}{\mu} + 2 \|\nabla \chi\|_\infty^2 \right)^{1/2}.$$

Proof. We only have to prove the last estimate. It follows from Lemma 2.1(c) that

$$\begin{aligned} \|(I+L)^{1/2} M_\chi u\|_2^2 &= \|M_\chi u\|_2^2 + \sum_{k=1}^d \|\partial_k M_\chi u\|_2^2 \\ &\leq \|\chi\|_\infty^2 \|u\|_2^2 + 2 \sum_{k=1}^d \|(\partial_k M_\chi - M_{\partial_k \chi})u\|_2^2 + 2 \sum_{k=1}^d \|M_{\partial_k \chi} u\|_2^2 \\ &\leq \|\chi\|_\infty^2 \|u\|_2^2 + 2 \frac{\|\chi\|_\infty^2}{\mu} \|H^{1/2} u\|_2^2 + 2 \|(\nabla \chi)u\|_2^2 \\ &\leq c_1^2 \|(I + H)^{1/2} u\|_2^2 \end{aligned}$$

as required. □

Lemma 4.2. *Let $\theta \in [0, \frac{\pi}{2})$, $\mu, M > 0$, $\gamma \in (0, 1)$ and $\chi \in W^{1,\infty}(\mathbb{R}^d)$. Then there exists a $c_2 > 0$ such that $M_\chi D(A^{\gamma/2}) \subset D(L^{\gamma/2})$ and*

$$\|(I + L)^{\gamma/2} M_\chi u\|_2 \leq c_2 \|(I + A)^{\gamma/2} u\|_2$$

for all $C \in \mathcal{S}(\text{supp } \chi, \theta, \mu, M)$ and $u \in D((I + A)^{\gamma/2})$.

Proof. Let c_1 be as in Lemma 4.1. Then M_χ is continuous from $D((I + H)^{1/2})$ into $D((I + L)^{1/2})$ with norm bounded by c_1 . Then by interpolation, Proposition G together with Theorem G in [1], it follows that $M_\chi D((I + H)^{\gamma/2}) \subset D((I + L)^{\gamma/2})$ and

$$\|(I + L)^{\gamma/2} M_\chi u\|_2 \leq c'_1 \|(I + H)^{\gamma/2} u\|_2$$

for all $u \in D((I + L)^{\gamma/2})$, where $c'_1 = c_1^\gamma \|\chi\|_\infty^{1-\gamma}$. But $D((I + H)^{\gamma/2}) = D((I + A)^{\gamma/2})$ with equivalent norms by [13] Theorem 3.1. Explicitly,

$$\|(I + H)^{\gamma/2} u\|_2 \leq \frac{1}{1 - \tan \frac{\pi \gamma}{4}} \|(I + A)^{\gamma/2} u\|_2$$

for all $u \in D((I + A)^{\gamma/2})$. Then the lemma follows. □

Lemma 4.3. *Let $\theta \in [0, \frac{\pi}{2})$, $\mu, M > 0$, $\chi \in W^{1,\infty}(\mathbb{R}^d)$ and $C \in \mathcal{E}(\text{supp } \chi, \theta, \mu, M)$. Then*

$$\|(I + H)^{1/2} M_\chi u\|_2 \leq 2\|(I + H)^{-1/2} M_\chi (I + A)u\|_2 + 4M^{1/2} \|\nabla \chi\|_\infty \|u\|_2$$

for all $u \in D(A)$.

Proof. The proof is a variation of the proof of Lemma 1 in [14]. If $u \in D(A)$ then

$$\begin{aligned} \|(I + H)^{1/2} M_\chi u\|_2^2 &= \text{Re } \mathfrak{a}(M_\chi u) + \|M_\chi u\|_2^2 \\ &= \text{Re } \mathfrak{a}(u, M_\chi^2 u) - \text{Re} \sum (c_{kl} \partial_k (\chi u), (\partial_l \chi) u) \\ &\quad + \text{Re} \sum (c_{kl} (\partial_k \chi) u, (\partial_l \chi) u) \\ &\quad + \text{Re} \sum (c_{kl} (\partial_k \chi) u, \partial_l (\chi u)) + \|M_\chi u\|_2^2. \end{aligned}$$

But

$$\begin{aligned} \text{Re } \mathfrak{a}(u, M_\chi^2 u) + \|M_\chi u\|_2^2 &= \text{Re}(Au, M_\chi^2 u) + \|M_\chi u\|_2^2 \\ &= \text{Re}((I + H)^{-1/2} M_\chi (I + A)u, (I + H)^{1/2} M_\chi u) \\ &\leq \|(I + H)^{-1/2} M_\chi (I + A)u\|_2 \|(I + H)^{1/2} M_\chi u\|_2 \\ &\leq \frac{1}{4} \|(I + H)^{1/2} M_\chi u\|_2^2 + \|(I + H)^{-1/2} M_\chi (I + A)u\|_2^2 \end{aligned}$$

and

$$\begin{aligned} & - \text{Re} \sum (c_{kl} \partial_k (\chi u), (\partial_l \chi) u) + \text{Re} \sum (c_{kl} (\partial_k \chi) u, (\partial_l \chi) u) \\ & \quad + \text{Re} \sum (c_{kl} (\partial_k \chi) u, \partial_l (\chi u)) \\ & \leq 2(\text{Re } \mathfrak{a}(M_\chi u))^{1/2} \left(\text{Re} \sum \int c_{kl} (\partial_k \chi) (\partial_l \chi) |u|^2 \right)^{1/2} \\ & \quad + \text{Re} \sum \int c_{kl} (\partial_k \chi) (\partial_l \chi) |u|^2 \\ & \leq 2(\text{Re } \mathfrak{a}(M_\chi u))^{1/2} M^{1/2} \|\nabla \chi\|_\infty \|u\|_2 + M \|\nabla \chi\|_\infty^2 \|u\|_2^2 \\ & \leq \frac{1}{4} \|(I + H)^{1/2} M_\chi u\|_2^2 + 5M \|\nabla \chi\|_\infty^2 \|u\|_2^2. \end{aligned}$$

So

$$\|(I + H)^{1/2} M_\chi u\|_2^2 \leq 2\|(I + H)^{-1/2} M_\chi (I + A)u\|_2^2 + 10M \|\nabla \chi\|_\infty^2 \|u\|_2^2$$

and the lemma follows. \square

The next lemma is well known, but we need uniform constants.

Lemma 4.4. *Let $0 < \gamma < \nu < 1$ and $\eta \in C^{0,\nu}(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$. Let $u \in D(L^{\gamma/2})$. Then $M_\eta u \in D(L^{\gamma/2})$ and*

$$\|(I + L)^{\gamma/2} M_\eta u\|_2 \leq c_4 (\|\eta\|_\infty \vee \|\eta\|_{C^{0,\nu}}) \|(I + L)^{\gamma/2} u\|_2,$$

where $c_4 = 1 + c_\gamma^{-1} 2^{d+3} \Gamma(\frac{\nu-\gamma}{2})$ and $c_\gamma = \int_0^\infty t^{-1-\frac{\gamma}{2}} (1 - e^{-t}) dt$.

Proof. Let T be the semigroup generated by $-L$ and for all $t > 0$ let G_t be the kernel of T_t . If $u \in L_2(\mathbb{R}^d)$ and $t > 0$ then

$$([T_t, M_\eta]u)(x) = \int G_t(y) (\eta(x - y) - \eta(x)) u(x - y) dy$$

for all $x \in \mathbb{R}^d$. But

$$\left| G_t(y) (\eta(x - y) - \eta(x)) \right| \leq 2M G_t(y) |y|^\nu \leq 2^{d+3} M t^{\nu/2} G_{2t}(y)$$

for all $x, y \in \mathbb{R}^d$, where $M = \|\eta\|_\infty \vee \|\eta\|_{C^{0,\nu}}$. Therefore $\|[T_t, M_\eta]u\|_2 \leq 2^{d+3} M t^{\nu/2} \|u\|_2$. Next,

$$(I + L)^{\gamma/2} = \frac{1}{c_\gamma} \int_0^\infty t^{-1-\frac{\gamma}{2}} (I - e^{-t} T_t) dt.$$

So for all $u \in C_c^\infty(\mathbb{R}^d)$ one obtains

$$[(I + L)^{\gamma/2}, M_\eta]u = -\frac{1}{c_\gamma} \int_0^\infty t^{-1-\frac{\gamma}{2}} e^{-t} [T_t, M_\eta]u dt.$$

Therefore

$$\|[(I + L)^{\gamma/2}, M_\eta]u\|_2 \leq \frac{2^{d+3} M}{c_\gamma} \int_0^\infty t^{-1+\frac{\nu-\gamma}{2}} e^{-t} dt \|u\|_2 = \frac{2^{d+3} M \Gamma(\frac{\nu-\gamma}{2})}{c_\gamma} \|u\|_2$$

and

$$\|(I + L)^{\gamma/2} M_\eta u\|_2 \leq c_4 \|(I + L)^{\gamma/2} u\|_2.$$

Then by density the lemma follows. □

Lemma 4.5. *Let $\theta \in [0, \frac{\pi}{2})$, $\mu, M > 0$, $\nu \in (0, 1)$ and $\chi \in W^{1,\infty}(\mathbb{R}^d)$. Set $\gamma = \frac{\nu}{2}$. Then there exists a $c_5 > 0$ such that $M_\chi^2 D(L^{(1+\gamma)/2}) \subset D(A^{(1+\gamma)/2})$ and*

$$\|(I + A)^{(1+\gamma)/2} M_\chi^2 u\|_2 \leq c_5 \|(I + L)^{(1+\gamma)/2} u\|_2$$

for all $C \in \mathcal{E}^\nu(\text{supp } \chi, \theta, \mu, M)$ and $u \in D(L^{(1+\gamma)/2})$.

Proof. The proof is a variation of the proof in [7]. Let $u \in D(L^{(1+\gamma)/2}) \subset W^{1,2}(\mathbb{R}^d)$ and $v \in D(A^*) \subset W^{1,2}(\mathbb{R}^d)$. Then

$$\begin{aligned}
& (M_\chi^2 u, (I + A^*)^{(1+\gamma)/2} v) \\
&= (M_\chi^2 u, (I + A^*)(I + A^*)^{-(1-\gamma)/2} v) \\
&= \sum (c_{kl} \partial_k M_\chi^2 u, \partial_l (I + A^*)^{-(1-\gamma)/2} v) + (M_\chi^2 u, (I + A^*)^{-(1-\gamma)/2} v) \\
&= \sum (c_{kl} M_\chi \partial_k u, \partial_l M_\chi (I + A^*)^{-(1-\gamma)/2} v) \tag{4.1} \\
&\quad - \sum (c_{kl} M_\chi \partial_k u, (\partial_l \chi) (I + A^*)^{-(1-\gamma)/2} v) \\
&\quad + 2 \sum (c_{kl} (\partial_k \chi) u, \partial_l M_\chi (I + A^*)^{-(1-\gamma)/2} v) \\
&\quad - 2 \sum (c_{kl} (\partial_k \chi) u, (\partial_l \chi) (I + A^*)^{-(1-\gamma)/2} v) \\
&\quad + (M_\chi u, M_\chi (I + A^*)^{-(1-\gamma)/2} v).
\end{aligned}$$

Fix $k, l \in \{1, \dots, d\}$. Then $\chi c_{kl} \in C^{0,\nu}(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$ and $\|\chi c_{kl}\|_{C^{0,\nu}(\mathbb{R}^d)} \leq 2M \|\chi\|_{W^{1,\infty}}$. So

$$\begin{aligned}
& |(c_{kl} M_\chi \partial_k u, \partial_l M_\chi (I + A^*)^{-(1-\gamma)/2} v)| \\
&= |((I + L)^{\gamma/2} M_{\chi c_{kl}} \partial_k u, (I + L)^{-\gamma/2} \partial_l M_\chi (I + A^*)^{-(1-\gamma)/2} v)| \\
&\leq \|(I + L)^{\gamma/2} M_{\chi c_{kl}} \partial_k u\|_2 \|(I + L)^{-\gamma/2} \partial_l M_\chi (I + A^*)^{-(1-\gamma)/2} v\|_2.
\end{aligned}$$

If c_4 is as in Lemma 4.4, then

$$\begin{aligned}
\|(I + L)^{\gamma/2} M_{\chi c_{kl}} \partial_k u\|_2 &\leq 2c_4 M \|\chi\|_{W^{1,\infty}} \|(I + L)^{\gamma/2} \partial_k u\|_2 \\
&\leq 2c_4 M \|\chi\|_{W^{1,\infty}} \|(I + L)^{(1+\gamma)/2} u\|_2.
\end{aligned}$$

Alternatively,

$$\|(I + L)^{-\gamma/2} \partial_l M_\chi (I + A^*)^{-(1-\gamma)/2} v\|_2 \leq \|(I + L)^{(1-\gamma)/2} M_\chi (I + A^*)^{-(1-\gamma)/2} v\|_2.$$

By Lemma 4.2 there exists a $c_2 > 0$, depending only on θ, μ, M, ν and χ , such that

$$\|(I + L)^{(1-\gamma)/2} M_\chi w\|_2 \leq c_2 \|(I + A^*)^{(1-\gamma)/2} w\|_2$$

for all $w \in D((I + L)^{(1-\gamma)/2})$. Hence

$$\begin{aligned}
\|(I + L)^{-\gamma/2} \partial_l M_\chi (I + A^*)^{-(1-\gamma)/2} v\|_2 &\leq c_2 \|(I + A^*)^{(1-\gamma)/2} (I + A^*)^{-(1-\gamma)/2} v\|_2 \\
&= c_2 \|v\|_2.
\end{aligned}$$

The other four terms in (4.1) can be estimated similarly.

Combining the contributions, it follows that there exists a $c > 0$, depending only on θ, μ, M, ν and χ , such that

$$|(M_\chi^2 u, (I + A^*)^{(1+\gamma)/2} v)| \leq c \|(I + L)^{(1+\gamma)/2} u\|_2 \|v\|_2$$

for all $v \in D(A^*)$. Since $D(A^*)$ is a core for $(I + A^*)^{(1+\gamma)/2}$, one deduces that $M_\chi^2 u \in D(A^{(1+\gamma)/2})$ and

$$\|(I + A)^{(1+\gamma)/2} M_\chi^2 u\|_2 \leq c \|(I + L)^{(1+\gamma)/2} u\|_2$$

as required. □

Now we are able to prove a uniform version of Theorem 1.3(c).

Theorem 4.6. *Let $\theta \in [0, \frac{\pi}{2})$, $\mu, M > 0$, $\nu \in (0, 1)$ and $\chi \in W^{1,\infty}(\mathbb{R}^d)$. Then there exists a $c > 0$ such that $M_\chi^3 u \in W^{1,2}(\mathbb{R}^d)$ and*

$$\|\partial_k M_\chi^3 u\|_2 \leq c \|(I + A)^{1/2} u\|_2$$

for all $C \in \mathcal{S}^\nu(\text{supp } \chi, \theta, \mu, M)$, $u \in D(A^{1/2})$ and $k \in \{1, \dots, d\}$.

Proof. First suppose that $C \in \mathcal{E}^\nu(\Omega, \theta, \mu, M)$. Let $c_5 > 0$ be as in Lemma 4.5. Then by interpolation, Proposition G together with Theorem G in [1], one establishes that $M_\chi^2 D((I + L)^{1/2}) \subset D((I + A)^{1/2})$ and

$$\|(I + A)^{1/2} M_\chi^2 u\|_2 \leq c'_5 \|(I + L)^{1/2} u\|_2$$

for all $u \in W^{1,2}(\mathbb{R}^d) = D(L^{1/2})$, where $c'_5 = c_5^{\frac{1}{1+\nu}} \|\chi\|_\infty^{\frac{\nu}{1+\nu}}$. Hence if $c_1 > 0$ is as in Lemma 4.1 then

$$\|(I + A)^{1/2} M_\chi^3 u\|_2 \leq c'_5 \|(I + L)^{1/2} M_\chi u\|_2 \leq c_1 c'_5 \|(I + H)^{1/2} u\|_2.$$

So

$$\|(I + A)^{1/2} M_\chi^3 (I + H)^{-1/2}\|_{2 \rightarrow 2} \leq c_1 c'_5$$

and then by duality

$$\|(I + H)^{-1/2} M_\chi^3 (I + A^*)^{1/2}\|_{2 \rightarrow 2} \leq c_1 c'_5. \tag{4.2}$$

Since $\mathcal{E}^\nu(\Omega, \theta, \mu, M)$ is invariant under taking adjoints, one may replace A^* by A in (4.2). Then Lemma 4.3 gives

$$\begin{aligned} \|(I + H)^{1/2} M_\chi^3 u\|_2 &\leq 2\|(I + H)^{-1/2} M_\chi^3 (I + A)u\|_2 + c_6 \|u\|_2 \\ &\leq 2c_1 c'_5 \|(I + A)^{1/2} u\|_2 + c_6 \|u\|_2 \\ &\leq (2c_1 c'_5 + c_6) \|(I + A)^{1/2} u\|_2 \end{aligned}$$

for all $u \in D(A)$, where $c_6 = 4M \|\nabla(\chi^3)\|_\infty$. Therefore

$$\|\partial_k M_\chi^3 u\|_2 \leq \mu^{-1/2} \|(I + H)^{1/2} M_\chi^3 u\|_2 \leq \mu^{-1/2} (2c_1 c'_5 + c_6) \|(I + A)^{1/2} u\|_2.$$

This extends to all $u \in D(A^{1/2})$ by density. It follows that

$$|(M_\chi^3 (I + A)^{-1/2} u, \partial_k v)| \leq \mu^{-1/2} (2c_1 c'_5 + c_6) \|u\|_2 \|v\|_2 \quad (4.3)$$

for all $u \in L_2(\mathbb{R}^d)$, $v \in W^{1,2}(\mathbb{R}^d)$, $k \in \{1, \dots, d\}$ and $C \in \mathcal{E}^v(\text{supp } \chi, \theta, \mu, M)$. By approximating C by $C + \frac{1}{n} I$ it follows as before that (4.3) extends to all $C \in \mathcal{S}^v(\text{supp } \chi, \theta, \mu, M)$ and the theorem follows. \square

The theorem has many corollaries.

Corollary 4.7. *Let $\Omega \subsetneq \mathbb{R}^d$ be open, $\theta \in [0, \frac{\pi}{2})$, $\mu, M > 0$, $\nu \in (0, 1)$, $p \in (1, \infty)$ and $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\chi \neq 0$ and $d(\text{supp } \chi, \Omega^c) > 0$. Then there exists a $c > 0$ such that*

$$\|\nabla M_\chi (I + A)^{-1/2} M_\chi\|_{p \rightarrow p} \leq c$$

for all $C \in \mathcal{S}^v(\Omega, \theta, \mu, M)$.

Proof. For all $t > 0$ let K_t be the continuous kernel of the operator $M_\chi S_t M_\chi$. Fix $m \in \{1, \dots, d\}$. By Theorem 2.3 there are $a > 0$ and $b \in (0, 1)$ such that

$$|(\partial_{x,m} K_t)(x, y)| \leq a t^{-d/2} t^{-1/2} e^{t/2} e^{-b \frac{|x-y|^2}{t}}$$

and

$$|(\partial_{x,m} K_t)(x+h, y+k) - (\partial_{x,m} K_t)(x, y)| \leq a t^{-d/2} t^{-|\alpha|/2} \left(\frac{|h| + |k|}{\sqrt{t}} \right)^\nu e^t e^{-b \frac{|x-y|^2}{t}}$$

for all $t > 0$ and $x, y, h, k \in \mathbb{R}^d$ with $|h| + |k| \leq \frac{1}{2} |x - y|$. For all $x, y \in \mathbb{R}^d$ with $x \neq y$ define

$$L(x, y) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} e^{-t} (\partial_{x,m} K_t)(x, y) dt.$$

Then

$$(\partial_m M_\chi (I + A)^{-1/2} M_\chi u)(x) = \int_{\mathbb{R}^d} L(x, y) u(y) dy$$

for all $u \in L_2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d \setminus \text{supp } u$. Moreover, for all $x, y, h \in \mathbb{R}^d$ with $|h| \leq \frac{1}{2} |x - y|$ one has

$$\begin{aligned} |L(x+h, y) - L(x, y)| &\leq \frac{a}{\sqrt{\pi}} \int_0^\infty t^{-d/2} t^{-1} \left(\frac{|h|}{\sqrt{t}} \right)^\nu e^{-b \frac{|x-y|^2}{t}} dt \\ &= c \left(\frac{|h|}{|x-y|} \right)^\nu \frac{1}{|x-y|^d}, \end{aligned}$$

where $c = \frac{a}{\sqrt{\pi}} \int_0^\infty t^{-1-\frac{d+v}{2}} e^{-\frac{b}{t}} dt$. Similarly,

$$|L(x, y + k) - L(x, y)| \leq c \left(\frac{|h|}{|x - y|} \right)^v \frac{1}{|x - y|^d}$$

for all $x, y, k \in \mathbb{R}^d$ with $|k| \leq \frac{1}{2} |x - y|$. In addition,

$$|L(x, y)| \leq \frac{a}{\sqrt{\pi}} \frac{1}{|x - y|^d} \int_0^\infty t^{-1-\frac{d+v}{2}} e^{-\frac{b}{t}} dt$$

for all $x, y \in \mathbb{R}^d$ with $x \neq y$. Therefore $\partial_m M_\chi (I + A)^{-1/2} M_\chi$ is a Calderón–Zygmund operator. Since it is bounded on $L_2(\mathbb{R}^d)$ by Theorem 4.6, it is also bounded in $L_p(\mathbb{R}^d)$ for all $p \in (1, \infty)$ by Theorem L in [1], or [20]. □

Corollary 4.8. *Let $\Omega \subsetneq \mathbb{R}^d$ be open, $\theta \in [0, \frac{\pi}{2})$, $\mu, M > 0$, $v \in (0, 1)$, $p \in (1, \infty)$, $k, l \in \{1, \dots, d\}$ and $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\chi \neq 0$ and $d(\text{supp } \chi, \Omega^c) > 0$. Then there exists a $c > 0$ such that*

$$\|\partial_k M_\chi (I + A)^{-1} M_\chi \partial_l\|_{p \rightarrow p} \leq c$$

for all $C \in \mathcal{S}^v(\Omega, \theta, \mu, M)$.

Proof. Note that in L_2 the operator $\partial_k M_\chi (I + A)^{-1/2}$ is bounded by Theorem 4.6. So by duality the theorem follows for $p = 2$. Then the rest of the proof is similar to the proof of Corollary 4.7. □

Corollary 4.9. *Let $\Omega \subsetneq \mathbb{R}^d$ be open, $\theta \in [0, \frac{\pi}{2})$, $\mu, M > 0$, $p \in (1, \infty)$, $m \in \{1, \dots, d\}$ and $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\chi \neq 0$ and $d(\text{supp } \chi, \Omega^c) > 0$. Then there exists a $c > 0$ such that*

$$\|\partial_m M_\chi (I + A)^{-1/2}\|_{p \rightarrow p} \leq c$$

for all $C \in \mathcal{S}^1(\Omega, \theta, \mu, M, \text{real})$.

Proof. It suffices to prove the corollary with M_χ replaced by M_χ^2 . We use again a commutator. It follows from Lemma 3.8 that on $L_2(\mathbb{R}^d)$ one has

$$\begin{aligned} & \partial_m M_\chi^2 (I + A)^{-1/2} - \partial_m M_\chi (I + A)^{-1/2} M_\chi \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} e^{-t} \partial_m M_\chi [M_\chi, S_t] dt \\ &= -\frac{1}{\sqrt{\pi}} \sum_{k,l=1}^d \int_0^\infty \int_0^t t^{-1/2} e^{-t} \partial_m M_\chi S_s M_{\partial_k \chi} \partial_l M_{\tilde{c}_{kl}} M_\chi^{-1} S_{t-s} ds dt + R, \end{aligned}$$

where R is the contribution of the last term in (3.7) and $\tilde{\chi} \in C_b^\infty(\mathbb{R}^d)$ is such that $d(\text{supp } \tilde{\chi}, \Omega^c) > 0$ and $\tilde{\chi}(x) = 1$ for all $x \in \text{supp } \chi$. Using the Gaussian bounds of Theorem 3.9 it follows that there exists a $c > 0$ such that

$$\begin{aligned} \|\partial_m M_\chi S_t M_{\partial_k \chi}\|_{p \rightarrow p} &\leq c t^{-1/2} e^{t/2} \quad \text{and} \\ \|\partial_l M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_t\|_{p \rightarrow p} &\leq c t^{-1/2} e^{t/2} \end{aligned}$$

for all $k, l \in \{1, \dots, d\}$ and $t > 0$. Then

$$\begin{aligned} &\left\| \int_0^\infty \int_0^t t^{-1/2} e^{-t} \partial_m M_\chi S_s M_{\partial_k \chi} \partial_l M_{\tilde{c}_{kl}} M_{\tilde{\chi}} S_{t-s} ds dt \right\|_{p \rightarrow p} \\ &\leq c^2 \int_0^\infty \int_0^t t^{-1/2} e^{-t/2} s^{-1/2} (t-s)^{-1/2} ds dt \\ &= c^2 \pi \sqrt{2\pi}. \end{aligned}$$

The contribution of R can be estimated similarly and the current corollary follows from Corollary 4.7. \square

We end with two propositions on second-order Riesz transforms.

Proposition 4.10. *Let $\Omega \subsetneq \mathbb{R}^d$ be open, $\theta \in [0, \frac{\pi}{2})$, $\mu, M > 0$, $p \in (1, \infty)$, $m, n \in \{1, \dots, d\}$ and $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\chi \neq 0$ and $d(\text{supp } \chi, \Omega^c) > 0$. Then there exists a $c > 0$ such that $M_\chi (I + A)^{-1} L_p(\mathbb{R}^d) \subset W^{2,p}(\mathbb{R}^d)$ and*

$$\|\partial_m \partial_n M_\chi (I + A)^{-1}\|_{p \rightarrow p} \leq c$$

for all $C \in S^1(\Omega, \theta, \mu, M, \text{real})$.

Proof. First let $C \in \mathcal{E}\mathcal{H}^1(\Omega, \theta, \mu, M, \text{real})$. Let $F = \text{supp } \chi$ and define χ_1, χ_2 as in (2.4) and (2.5). Let $M' = 2\|\chi_2\|_{W^{1,\infty}(\mathbb{R}^d)} M + \|\chi\|_{W^{1,\infty}(\mathbb{R}^d)} + 1$ and $C' = \chi_2 C + \chi_1 I$. Then $C' \in \mathcal{E}\mathcal{H}^1(\mathbb{R}^d, \theta, \mu \wedge 1, M')$. By [8] Proposition 5.1 there exists a suitable $c > 0$ such that $(I + A')^{-1} L_p(\mathbb{R}^d) \subset W^{2,p}(\mathbb{R}^d)$ and

$$\|\partial_m \partial_n u\|_{p \rightarrow p} \leq c \|(I + A')u\|_p$$

for all $u \in W^{2,p}(\mathbb{R}^d)$, where $A' = A_{C'}$. Then

$$\|\partial_m \partial_n M_\chi u\|_{p \rightarrow p} \leq c \|(I + A') M_\chi u\|_p = c \|(I + A) M_\chi u\|_p.$$

But

$$(I + A) M_\chi u = M_\chi (I + A)u - \sum_{k,l=1}^d \left((\partial_l \chi) c_{kl} \partial_k u + \partial_l c_{kl} (\partial_k \chi) u \right).$$

Hence it follows from Corollary 3.10 that there exists a suitable $c' > 0$ such that

$$\|\partial_m \partial_n M_\chi u\|_{p \rightarrow p} \leq c' \|(I + A)u\|_p$$

for all $u \in W^{2,p}(\mathbb{R}^d) = D(A)$. Then the proposition follows by approximation. \square

On L_2 the same argument works for operators with complex coefficients.

Proposition 4.11. *Let $\Omega \subsetneq \mathbb{R}^d$ be open, $\theta \in [0, \frac{\pi}{2})$, $\mu, M > 0$, $p \in (1, \infty)$, $m, n \in \{1, \dots, d\}$ and $\chi \in C_b^\infty(\mathbb{R}^d)$ with $\chi \neq 0$ and $d(\text{supp } \chi, \Omega^c) > 0$. Then there exists a $c > 0$ such that $M_\chi (I + A)^{-1} L_2(\mathbb{R}^d) \subset W^{2,2}(\mathbb{R}^d)$ and*

$$\|\partial_m \partial_n M_\chi (I + A)^{-1}\|_{2 \rightarrow 2} \leq c$$

for all $C \in \mathcal{S}^1(\Omega, \theta, \mu, M)$.

A. Gaussian bounds

The main aim of this appendix is to transfer weighted semigroup bounds into Gaussian kernel bounds, with optimal large time behaviour and optimal control of the constant in the Gaussian, including the Hölder bounds. Throughout this appendix we write $C^\nu = C^{0,\nu}(\mathbb{R}^d)$ for all $\nu \in (0, 1)$.

Lemma A.1. *Let $N, N^* \in \mathbb{N}_0$ and $\nu, \nu^* \in (0, 1)$. Let S be a C_0 -semigroup on $L_2(\mathbb{R}^d)$ and $T_1, T_2 \in \mathcal{L}(L_2(\mathbb{R}^d))$ such that $T_1 S_t(C_c^\infty(\mathbb{R}^d)) \subset W^{N+\nu, \infty}(\mathbb{R}^d)$ and $T_2^* S_t^*(C_c^\infty(\mathbb{R}^d)) \subset W^{N^*+\nu^*, \infty}(\mathbb{R}^d)$ for all $t > 0$. Assume that $[T_1, U_\rho] = [T_2, \dot{U}_\rho] = 0$ for all $\rho \in \mathbb{R}$ and $\psi \in \mathcal{D}_1$. Let $a_0, a_1, \omega_1, \omega > 0$ and suppose that*

$$\|U_\rho S_t U_{-\rho}\|_{2 \rightarrow 2} \leq a_0 e^{\omega \rho^2 t}$$

for all $t > 0$, $\rho \in \mathbb{R}$ and $\psi \in \mathcal{D}_1$. Moreover, suppose that

$$\begin{aligned} \|\partial^\alpha U_\rho T_1 S_t U_{-\rho} u\|_{C^\nu} &\leq a_1 t^{-d/4} t^{-|\alpha|/2} t^{-\nu/2} e^{\omega_1(1+\rho^2)t} \|u\|_2, \\ \|\partial^\beta U_\rho T_2^* S_t^* U_{-\rho} u\|_{C^{\nu^*}} &\leq a_1 t^{-d/4} t^{-|\beta|/2} t^{-\nu^*/2} e^{\omega_1(1+\rho^2)t} \|u\|_2 \end{aligned}$$

for all $t > 0$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}_{|\alpha| \vee |\beta| \vee 1}$, $u \in C_c^\infty(\mathbb{R}^d)$ and multi-indices α, β with $|\alpha| \leq N$ and $|\beta| \leq N^*$. Let $c > 0$ be such that

$$\sup\{\psi(x) - \psi(y) : \psi \in \mathcal{D}_{N \vee N^* \vee 1}\} \geq c |x - y| \quad (\text{A.1})$$

for all $x, y \in \mathbb{R}^d$ and set $b = \frac{c^2}{4\omega}$.

Then for all $t > 0$ the operator $T_1 S_t T_2$ has a continuous kernel K_t which is N_1 -times differentiable in the first variable and N_2 -times in the second one, in any order. Moreover, there exists an $a > 0$, depending only on $a_0, a_1, \omega, \omega_1, N, N^*, \nu, \nu^*, \|T_1\|$ and $\|T_2\|$ such that

$$|(\partial_x^\alpha \partial_y^\beta K_t)(x, y)| \leq a t^{-\frac{d+|\alpha|+|\beta|}{2}} \left(1 + t + \frac{|x-y|^2}{t}\right)^{\frac{d+|\alpha|+|\beta|}{2}} e^{-b \frac{|x-y|^2}{t}} \quad (\text{A.2})$$

for all $x, y \in \mathbb{R}^d$, $t > 0$ and multi-indices α, β with $|\alpha| \leq N$ and $|\beta| \leq N^*$.

Finally, let $\gamma, \gamma^* \in (0, 1)$, $\kappa > 0$, $\tau \in [0, 1)$,

$$0 < b_1 < b(1 - \tau)^2 \frac{c}{c + 2\tau}$$

and α, β multi-indices with $|\alpha| \leq N$, $|\beta| \leq N^*$, $|\alpha| + \gamma \leq N + \nu$ and $|\beta| + \gamma^* \leq N^* + \nu^*$. Then there exists an $a > 0$, depending only on $a_0, a_1, b_1, N, N^*, \nu, \nu^*, \gamma, \gamma^*, \kappa, \tau, \|T_1\|$ and $\|T_2\|$ such that

$$\begin{aligned} & |(\partial_x^\alpha \partial_y^\beta K_t)(x + h, y + k) - (\partial_x^\alpha \partial_y^\beta K_t)(x, y)| \\ & \leq a t^{-\frac{d+|\alpha|+|\beta|}{2}} (1+t)^{\frac{d+|\alpha|+|\beta|+\gamma+\gamma^*}{2}} \left(\left(\frac{|h|}{\sqrt{t}+|x-y|} \right)^\gamma + \left(\frac{|k|}{\sqrt{t}+|x-y|} \right)^{\gamma^*} \right) e^{-b_1 \frac{|x-y|^2}{t}} \end{aligned} \quad (\text{A.3})$$

for all $x, y, h, k \in \mathbb{R}^d$ and $t > 0$ with $|h| + |k| \leq \kappa \sqrt{t} + \tau |x - y|$.

In the proof of Lemma A.1 we need some estimates which are of independent interest.

Lemma A.2. Let $\nu \in (0, 1)$.

(a) If $u \in C^\nu \cap L_2$ then $u \in L_\infty$ and

$$\|u\|_\infty \leq \frac{d}{d+\nu} \varepsilon^\nu \|u\|_{C^\nu} + |B(1)|^{-1/2} \varepsilon^{-d/2} \|u\|_2$$

for all $\varepsilon \in (0, 1]$.

(b) If $k \in \{1, \dots, d\}$ and $u \in W^{1+\nu, \infty}(\mathbb{R}^d)$ then

$$\|\partial_k u\|_\infty \leq \frac{1}{1+\nu} \varepsilon^\nu \|\partial_k u\|_{C^\nu} + \varepsilon^{-(1-\nu)} \|u\|_{C^\nu}$$

for all $\varepsilon \in (0, 1]$.

(c) If $u \in W^{1, \infty}(\mathbb{R}^d)$, then

$$\|u\|_{C^\gamma} \leq 2 \|\nabla u\|_\infty^\gamma \|u\|_\infty^{1-\gamma}$$

for all $\gamma \in (0, 1)$.

(d) If $0 < \gamma < \nu$, then

$$\|u\|_{C^\gamma} \leq 2 \|u\|_{C^\nu}^{\frac{\gamma}{\nu}} \|u\|_\infty^{1-\frac{\gamma}{\nu}}$$

for all $u \in C^\nu \cap L_\infty$.

Proof. Let $x \in \mathbb{R}^d$ and $h \in B(\varepsilon)$. Then $|u(x)| \leq |u(x) - u(x+h)| + |u(x+h)| \leq |h|^\nu \|u\|_{C^\nu} + |u(x+h)|$. Integration over h gives

$$\begin{aligned} \varepsilon^d |B(1)| |u(x)| & \leq \|u\|_{C^\nu} \int_{B(\varepsilon)} |h|^\nu dh + \int_{B(\varepsilon)} |u(x+h)| dh \\ & \leq \|u\|_{C^\nu} d |B(1)| \int_0^\varepsilon \rho^\nu \rho^{d-1} d\rho + |B(\varepsilon)|^{1/2} \|u\|_2 \\ & = \|u\|_{C^\nu} \frac{d}{d+\nu} |B(1)| \varepsilon^{d+\nu} + |B(1)|^{1/2} \varepsilon^{d/2} \|u\|_2 \end{aligned}$$

from which Statement (a) follows.

For Statement (b) note that

$$\begin{aligned} u(x + \varepsilon e_k) - u(x) &= \int_0^\varepsilon (\partial_k u)(x + t e_k) dt \\ &= \varepsilon (\partial_k u)(x) + \int_0^\varepsilon \left((\partial_k u)(x + t e_k) - (\partial_k u)(x) \right) dt. \end{aligned}$$

So

$$\begin{aligned} |(\partial_k u)(x)| &\leq \frac{1}{\varepsilon} \left| u(x + \varepsilon e_k) - u(x) \right| + \frac{1}{\varepsilon} \int_0^\varepsilon \left| (\partial_k u)(x + t e_k) - (\partial_k u)(x) \right| dt \\ &\leq \varepsilon^{-(1-\nu)} \|u\|_{C^\nu} + \frac{1}{\varepsilon} \int_0^\varepsilon t^\nu \| \partial_k u \|_{C^\nu} dt \\ &= \varepsilon^{-(1-\nu)} \|u\|_{C^\nu} + \frac{1}{1+\nu} \varepsilon^\nu \| \partial_k u \|_{C^\nu} \end{aligned}$$

for all $x \in \mathbb{R}^d$.

The proof of Statements (c) and (d) is easy. □

Proof of Lemma A.1. We follow arguments as in [9, 10] and [18]. Set $N_0 = N \vee N^* \vee 1$.

Step 1. Since T_1 and T_2 commute with U_ρ one has estimates

$$\|U_\rho T_1 S_t U_{-\rho}\|_{2 \rightarrow 2} \leq a_0 \|T_1\| e^{\omega \rho^2 t}$$

and similarly for T_2 . It then follows from the first two statements of Lemma A.2 with $\varepsilon = t^{1/2} e^{-t}$ that

$$\begin{aligned} \|\partial^\alpha U_\rho T_1 S_t U_{-\rho} u\|_\infty &\leq \frac{1}{2} a_2 t^{-d/4} t^{-|\alpha|/2} e^{\omega_2(1+\rho^2)t} \|u\|_2, \\ \|\partial^\beta U_\rho T_2^* S_t^* U_{-\rho} u\|_\infty &\leq \frac{1}{2} a_2 t^{-d/4} t^{-|\beta|/2} e^{\omega_2(1+\rho^2)t} \|u\|_2 \end{aligned}$$

for all $t > 0$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}_{N_0}$, $u \in C_c^\infty(\mathbb{R}^d)$ and multi-indices α, β with $|\alpha| \leq N$ and $|\beta| \leq N^*$, where $a_2 = 4a_1 + 2|B(1)|^{-1/2} (\|T_1\| \vee \|T_2\|) a_0$ and $\omega_2 = (\omega_1 + 1 - \nu) \vee (\omega + \frac{d}{2})$. Then using the last two statements of Lemma A.2 one deduces that

$$\begin{aligned} \| \partial^\alpha U_\rho T_1 S_t U_{-\rho} u \|_{C^\gamma} &\leq a_2 t^{-d/4} t^{-|\alpha|/2} t^{-\gamma/2} e^{\omega_2(1+\rho^2)t} \|u\|_2, \\ \| \partial^\beta U_\rho T_2^* S_t^* U_{-\rho} u \|_{C^\gamma} &\leq a_2 t^{-d/4} t^{-|\beta|/2} t^{-\gamma^*/2} e^{\omega_2(1+\rho^2)t} \|u\|_2 \end{aligned}$$

for all $t > 0$, $\rho \in \mathbb{R}$, $\psi \in \mathcal{D}_{N_0}$, $u \in C_c^\infty(\mathbb{R}^d)$, multi-indices α, β and $\gamma, \gamma^* \in (0, 1)$ with $|\alpha| + \gamma \leq N + \nu$ and $|\beta| + \gamma^* \leq N^* + \nu^*$.

Step 2. Let $\gamma \in (0, 1)$ and α be a multi-index with $|\alpha| + \gamma \leq N + \nu$. Note that $U_\rho \partial_k = \partial_k U_\rho + \rho M_{\partial_k \psi} U_\rho$ and $|\rho| t^{(n+1)/2} \leq n! e^{(1+\rho^2)t}$ for all $n \in \mathbb{N}_0$. Hence it follows by induction to $|\alpha''|$ that

$$\begin{aligned} & \| |\partial^{\alpha'} U_\rho \partial^{\alpha''} T_1 S_t U_{-\rho} u| \|_{C^\gamma} \\ & \leq (1 + N! 2^N)^{|\alpha''|} a_2 t^{-d/4} t^{-(|\alpha'|+|\alpha''|)/2} t^{-\gamma/2} e^{(\omega_2+|\alpha''|)(1+\rho^2)t} \|u\|_2 \end{aligned}$$

for all $t > 0, \rho \in \mathbb{R}, \psi \in \mathcal{D}_{N_0}, u \in L_2$ and multi-indices α', α'' with $|\alpha'| + |\alpha''| \leq |\alpha|$. In particular,

$$\| |U_\rho \partial^\alpha T_1 S_t U_{-\rho} u| \|_{C^\gamma} \leq a_2 c_1 t^{-d/4} t^{-|\alpha|/2} t^{-\gamma/2} e^{(\omega_2+N)(1+\rho^2)t} \|u\|_2,$$

where $c_1 = (1 + N! 2^N)^N$. Similarly,

$$\| |U_\rho \partial^\alpha T_1 S_t U_{-\rho} u| \|_\infty \leq a_2 c_1 t^{-d/4} t^{-|\alpha|/2} e^{(\omega_2+N)(1+\rho^2)t} \|u\|_2.$$

If $h \in \mathbb{R}^d$ with $|h| \geq 1$ then

$$\begin{aligned} & \| (I - L(h)) U_\rho \partial^\alpha T_1 S_t U_{-\rho} u \|_\infty \\ & \leq 2a_2 c_1 t^{-d/4} t^{-|\alpha|/2} e^{(\omega_2+N)(1+\rho^2)t} \|u\|_2 \\ & \leq 2a_2 c_1 t^{-d/4} t^{-|\alpha|/2} \left(\frac{|h|}{\sqrt{t}} \right)^\gamma e^{(\omega_2+N+1)(1+\rho^2)t} \|u\|_2. \end{aligned}$$

So

$$\| (I - L(h)) U_\rho \partial^\alpha T_1 S_t U_{-\rho} u \|_\infty \leq a_3 t^{-d/4} t^{-|\alpha|/2} \left(\frac{|h|}{\sqrt{t}} \right)^\gamma e^{\omega_3(1+\rho^2)t} \|u\|_2$$

for all $t > 0, h \in \mathbb{R}^d, \rho \in \mathbb{R}, \psi \in \mathcal{D}_{N_0}$ and $u \in L_2$, where $a_3 = 2a_2 c_1$ and $\omega_3 = \omega_2 + N + 1$.

Step 3. Let $\gamma \in (0, 1)$ and α be a multi-index with $|\alpha| + \gamma \leq N + \nu$. Let $h \in \mathbb{R}^d, \psi \in \mathcal{D}_{N_0}, \rho \in \mathbb{R}$ and $u \in L_2$. Set $t_0 = \frac{1}{\omega_3(1+\rho^2)}$. If $t \in (0, t_0]$ then

$$\| (I - L(h)) U_\rho \partial^\alpha T_1 S_t U_{-\rho} u \|_\infty \leq a_3 e t^{-d/4} t^{-|\alpha|/2} \left(\frac{|h|}{\sqrt{t}} \right)^\gamma \|u\|_2$$

and if $t \in (t_0, \infty)$ then

$$\begin{aligned} & \| (I - L(h)) U_\rho \partial^\alpha T_1 S_t U_{-\rho} u \|_\infty \\ & = \| (I - L(h)) U_\rho \partial^\alpha T_1 S_{t_0} U_{-\rho} U_\rho S_{t-t_0} U_{-\rho} u \|_\infty \\ & \leq a_3 e t_0^{-d/4} t_0^{-|\alpha|/2} \left(\frac{|h|}{\sqrt{t_0}} \right)^\gamma \|U_\rho S_{t-t_0} U_{-\rho} u\|_2 \\ & \leq a_0 a_3 e \left(\omega_3(1+\rho^2)t \right)^{\frac{d}{4} + \frac{|\alpha|+\gamma}{2}} t^{-d/4} t^{-|\alpha|/2} \left(\frac{|h|}{\sqrt{t}} \right)^\gamma e^{\omega \rho^2(t-t_0)} \|u\|_2. \end{aligned}$$

Hence

$$\begin{aligned} & \|(I - L(h))U_\rho \partial^\alpha T_1 S_t U_{-\rho} u\|_\infty \\ & \leq a_4 \left(1 + \omega_3(1 + \rho^2)t\right)^{\frac{d}{4} + \frac{|\alpha| + \gamma}{2}} t^{-d/4} t^{-|\alpha|/2} \left(\frac{|h|}{\sqrt{t}}\right)^\gamma e^{\omega\rho^2 t} \|u\|_2 \end{aligned}$$

for all $t > 0$, where $a_4 = a_3(1 \vee a_0)e$. Similarly,

$$\|U_\rho \partial^\alpha T_1 S_t U_{-\rho} u\|_\infty \leq a_4 \left(1 + \omega_3(1 + \rho^2)t\right)^{\frac{d}{4} + \frac{|\alpha|}{2}} t^{-d/4} t^{-|\alpha|/2} e^{\omega\rho^2 t} \|u\|_2$$

for all $\psi \in \mathcal{D}_{N_0}$, $\rho \in \mathbb{R}$, $t > 0$, $|\alpha| \leq N$ and $u \in L_2$. Also similar bounds are valid with T_2^* and S_t^* .

Step 4. Let $h, k \in \mathbb{R}^d$, $\psi \in \mathcal{D}_{N_0}$, $\rho \in \mathbb{R}$, $t > 0$, $\gamma, \gamma^* \in (0, 1)$ and α, β be multi-indices with $|\alpha| + \gamma \leq N + \nu$ and $|\beta| + \gamma^* \leq N^* + \nu^*$. Then

$$\begin{aligned} & \|L(h)U_\rho \partial^\alpha T_1 S_t T_2 \partial^\beta U_{-\rho} L(-k) - U_\rho \partial^\alpha T_1 S_t T_2 \partial^\beta U_{-\rho}\|_{1 \rightarrow \infty} \\ & \leq \|(I - L(h))U_\rho \partial^\alpha T_1 S_{t/2} U_{-\rho}\|_{2 \rightarrow \infty} \|(I - L(k))U_\rho \partial^\beta T_2^* S_{t/2}^* U_{-\rho}\|_{2 \rightarrow \infty} \\ & \quad + \|(I - L(h))U_\rho \partial^\alpha T_1 S_{t/2} U_{-\rho}\|_{2 \rightarrow \infty} \|U_\rho \partial^\beta T_2^* S_{t/2}^* U_{-\rho}\|_{2 \rightarrow \infty} \\ & \quad + \|U_\rho \partial^\alpha T_1 S_{t/2} U_{-\rho}\|_{2 \rightarrow \infty} \|(I - L(k))U_\rho \partial^\beta T_2^* S_{t/2}^* U_{-\rho}\|_{2 \rightarrow \infty} \quad (\text{A.4}) \\ & \leq a_5 \left(1 + \omega_3(1 + \rho^2)t\right)^{\frac{d + |\alpha| + |\beta| + \gamma + \gamma^*}{2}} t^{-\frac{d + |\alpha| + |\beta|}{2}} e^{\omega\rho^2 t} \\ & \quad \times \left(\left(\frac{|h|}{\sqrt{t}}\right)^\gamma \left(\frac{|k|}{\sqrt{t}}\right)^{\gamma^*} + \left(\frac{|h|}{\sqrt{t}}\right)^\gamma + \left(\frac{|k|}{\sqrt{t}}\right)^{\gamma^*} \right), \end{aligned}$$

where $a_5 = 2^{(d + |\alpha| + |\beta| + 2)/2} a_4^2$. Similarly,

$$\|U_\rho \partial^\alpha T_1 S_t T_2 \partial^\beta U_{-\rho}\|_{1 \rightarrow \infty} \leq a_5 \left(1 + \omega_3(1 + \rho^2)t\right)^{\frac{d + |\alpha| + |\beta|}{2}} t^{-\frac{d + |\alpha| + |\beta|}{2}} e^{\omega\rho^2 t}. \quad (\text{A.5})$$

Step 5. Let $t > 0$ and α, β be multi-indices with $|\alpha| \leq N$ and $|\beta| \leq N^*$. Choosing $\rho = 0$ it follows from (A.5) and the Dunford–Pettis theorem that the operator $\partial^\alpha T_1 S_t T_2 \partial^\beta$ has a kernel $K_t^{(\alpha, \beta)} \in L_\infty(\mathbb{R}^d \times \mathbb{R}^d)$. If \tilde{L} denotes the w*-continuous left regular representation of $\mathbb{R}^d \times \mathbb{R}^d$ in $L_\infty(\mathbb{R}^d \times \mathbb{R}^d)$, then it follows from (A.4) that

$$\begin{aligned} & \|(I - \tilde{L}(h, k))K_t^{(\alpha, \beta)}\|_\infty \\ & \leq a_5 t^{-\frac{d + |\alpha| + |\beta|}{2}} (1 + \omega_3 t)^{\frac{d + |\alpha| + |\beta| + \nu + \nu^*}{2}} \left(\left(\frac{|h|}{\sqrt{t}}\right)^\nu \left(\frac{|k|}{\sqrt{t}}\right)^{\nu^*} + \left(\frac{|h|}{\sqrt{t}}\right)^\nu + \left(\frac{|k|}{\sqrt{t}}\right)^{\nu^*} \right) \end{aligned}$$

for all $(h, k) \in \mathbb{R}^d \times \mathbb{R}^d$. So $\lim_{(h, k) \rightarrow (0, 0)} \|(I - \tilde{L}(h, k))K_t^{(\alpha, \beta)}\|_\infty = 0$ and $K_t^{(\alpha, \beta)}$ is uniformly continuous on $\mathbb{R}^d \times \mathbb{R}^d$.

Define $K_t = K_t^{(\alpha, \beta)}$ if $|\alpha| = |\beta| = 0$. Thus K_t is the kernel of $T_1 S_t T_2$. Let $|\alpha| \leq N, |\beta| \leq N^*$ and $t > 0$. Then for all $u, v \in C_c^\infty(\mathbb{R}^d)$ one has

$$\begin{aligned} & (-1)^{|\alpha|+|\beta|} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K_t(x, y) (\partial^\alpha u)(x) (\partial^\beta v)(y) dx dy \\ &= (-1)^{|\alpha|+|\beta|} (T_1 S_t T_2 \partial^\beta v, \partial^\alpha \bar{u}) \\ &= (-1)^{|\beta|} (\partial^\alpha T_1 S_t T_2 \partial^\beta v, \bar{u}) \\ &= (-1)^{|\beta|} \int_{\mathbb{R}^d} u(x) (\partial^\alpha T_1 S_t T_2 \partial^\beta v)(x) dx \\ &= (-1)^{|\beta|} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K_t^{(\alpha, \beta)}(x, y) u(x) v(y) dx dy. \end{aligned}$$

So by density

$$\begin{aligned} & (-1)^{|\alpha|+|\beta|} \int_{\mathbb{R}^d \times \mathbb{R}^d} K_t(x, y) (\partial_x^\alpha \partial_y^\beta w)(x, y) d(x, y) \\ &= (-1)^{|\beta|} \int_{\mathbb{R}^d \times \mathbb{R}^d} K_t^{(\alpha, \beta)}(x, y) w(x, y) d(x, y) \end{aligned}$$

for all $w \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and the $(-1)^{|\beta|} K_t^{(\alpha, \beta)}$ are the successive distributional derivatives of K_t . Since the $K_t^{(\alpha, \beta)}$ are continuous one deduces from the lemma of Du Bois–Reymond that K_t is N times differentiable in the first variable, the derivatives are N^* -times differentiable in the second variable and all derivatives are continuous.

Step 6. Let $|\alpha| \leq N, |\beta| \leq N^*, t > 0$ and $x, y \in \mathbb{R}^d$. Then it follows from (A.5) that

$$|K_t^{(\alpha, \beta)}(x, y)| \leq a_5 \left(1 + \omega_3(1 + \rho^2)t\right)^{\frac{d+|\alpha|+|\beta|}{2}} t^{-\frac{d+|\alpha|+|\beta|}{2}} e^{\omega\rho^2 t} e^{-\rho(\psi(x)-\psi(y))}$$

for all $\rho \geq 0$ and $\psi \in \mathcal{D}_{N_0}$. Minimizing over ψ and using (A.1) gives

$$|K_t^{(\alpha, \beta)}(x, y)| \leq a_5 \left(1 + \omega_3(1 + \rho^2)t\right)^{\frac{d+|\alpha|+|\beta|}{2}} t^{-\frac{d+|\alpha|+|\beta|}{2}} e^{\omega\rho^2 t} e^{-\rho c|x-y|}$$

and with the choice $\rho = \frac{c|x-y|}{2\omega t}$ one deduces that

$$|K_t^{(\alpha, \beta)}(x, y)| \leq a_5 \left(1 + \omega_3 \left(t + \frac{c^2|x-y|^2}{4\omega^2 t}\right)\right)^{\frac{d+|\alpha|+|\beta|}{2}} t^{-\frac{d+|\alpha|+|\beta|}{2}} e^{-b\frac{|x-y|^2}{t}}. \quad (\text{A.6})$$

This proves the bounds (A.2).

Step 7. Let α, β be multi-indices, $\gamma, \gamma^* \in (0, 1)$ and suppose that $|\alpha| + |\gamma| \leq N + \nu$ and $|\beta| + \gamma^* \leq N^* + \nu^*$. Let $\kappa > 0$ and $\tau \in [0, 1)$. There exists a $\tau_1 \in (\tau, 1)$ such that

$$b_1 = b(1 - \tau_1)^2 \frac{c}{c + 2\tau}.$$

Set $\lambda = \frac{b}{b_1} \geq 1$. Further, let $\varepsilon, \eta \in (0, 1)$. Let $x, y, h, k \in \mathbb{R}^d, t > 0$ and suppose that $|h| + |k| \leq \kappa \sqrt{t} + \tau |x - y|$. Let $\rho \in \mathbb{R}$ and $\psi \in \mathcal{D}_{N_0}$.

If $RHS = a_5 \dots$ denotes the right hand side of (A.4), then it follows from (A.4) that

$$\left| e^{-\rho\psi(x)} K_t^{(\alpha, \beta)}(x, y) e^{\rho\psi(y)} - e^{-\rho\psi(x-h)} K_t^{(\alpha, \beta)}(x-h, y-k) e^{\rho\psi(y-k)} \right| \leq RHS.$$

So

$$\begin{aligned} & \left| K_t^{(\alpha, \beta)}(x, y) - e^{\rho(\psi(x) - \psi(x-h))} K_t^{(\alpha, \beta)}(x-h, y-k) e^{-\rho(\psi(y) - \psi(y-k))} \right| \\ & \leq RHS \cdot e^{\rho(\psi(x) - \psi(y))} \end{aligned}$$

and

$$\begin{aligned} & \left| K_t^{(\alpha, \beta)}(x, y) - K_t^{(\alpha, \beta)}(x-h, y-k) \right| \\ & \leq RHS \cdot e^{\rho(\psi(x) - \psi(y))} \\ & \quad + \left| 1 - e^{\rho(\psi(x) - \psi(x-h))} e^{-\rho(\psi(y) - \psi(y-k))} \right| \left| K_t^{(\alpha, \beta)}(x-h, y-k) \right| \\ & \leq RHS \cdot e^{\rho(\psi(x) - \psi(y))} + |\rho| (|h| + |k|) e^{|\rho|(|h| + |k|)} \left| K_t^{(\alpha, \beta)}(x-h, y-k) \right|. \end{aligned}$$

Suppose $\rho \geq 0$. Optimizing over ψ gives

$$\begin{aligned} & \left| K_t^{(\alpha, \beta)}(x, y) - K_t^{(\alpha, \beta)}(x-h, y-k) \right| \\ & \leq RHS \cdot e^{-c\rho|x-y|} + \rho (|h| + |k|) e^{\rho(|h| + |k|)} \left| K_t^{(\alpha, \beta)}(x-h, y-k) \right| \\ & \leq a_5 \left(1 + \omega_3(1 + \rho^2)t \right)^E t^{-\tilde{E}} e^{\omega\rho^2 t} e^{-c\rho|x-y|} \\ & \quad \times \left(\left(\frac{|h|}{\sqrt{t}} \right)^\gamma \left(\frac{|k|}{\sqrt{t}} \right)^{\gamma^*} + \left(\frac{|h|}{\sqrt{t}} \right)^\gamma + \left(\frac{|k|}{\sqrt{t}} \right)^{\gamma^*} \right) \\ & \quad + \rho (|h| + |k|) e^{\rho(|h| + |k|)} \left| K_t^{(\alpha, \beta)}(x-h, y-k) \right|, \end{aligned}$$

where for brevity we set $E = \frac{d+|\alpha|+|\beta|+\gamma+\gamma^*}{2}$ and $\tilde{E} = \frac{d+|\alpha|+|\beta|}{2}$. Choose $\rho = \frac{c|x-y|}{2\lambda\omega t}$. Before we estimate both terms in (A.7) we need one more estimate to replace the denominator \sqrt{t} by $\sqrt{t} + |x - y|$.

Since $\frac{|x-y|}{\sqrt{t}} \leq \frac{1}{\sqrt{\varepsilon}} e^{\varepsilon \frac{|x-y|^2}{t}}$ it follows that

$$t^{-1/2}(\sqrt{t} + |x-y|) = 1 + \frac{|x-y|}{\sqrt{t}} \leq \frac{2}{\sqrt{\varepsilon}} e^{\varepsilon \frac{|x-y|^2}{t}}$$

and

$$t^{-1/2} \leq \frac{2}{\sqrt{\varepsilon}} \frac{1}{\sqrt{t} + |x-y|} e^{\varepsilon \frac{|x-y|^2}{t}}.$$

Therefore

$$\left(\frac{|h|}{\sqrt{t}}\right)^\gamma + \left(\frac{|k|}{\sqrt{t}}\right)^{\gamma^*} \leq \frac{2}{\sqrt{\varepsilon}} \left(\left(\frac{|h|}{\sqrt{t} + |x-y|}\right)^\gamma + \left(\frac{|k|}{\sqrt{t} + |x-y|}\right)^{\gamma^*} \right) e^{\varepsilon \frac{|x-y|^2}{t}}$$

and

$$\begin{aligned} \left(\frac{|h|}{\sqrt{t}}\right)^\gamma \left(\frac{|k|}{\sqrt{t}}\right)^{\gamma^*} &\leq \frac{4}{\varepsilon} \left(\frac{|h|}{\sqrt{t} + |x-y|}\right)^\gamma \left(\frac{|k|}{\sqrt{t} + |x-y|}\right)^{\gamma^*} e^{2\varepsilon \frac{|x-y|^2}{t}} \\ &\leq \frac{4}{\varepsilon} (\kappa + \tau)^{\gamma^*} \left(\frac{|h|}{\sqrt{t} + |x-y|}\right)^\gamma e^{2\varepsilon \frac{|x-y|^2}{t}}. \end{aligned}$$

We estimate both terms in (A.7) separately.

For the first term note that

$$\omega \rho^2 t - c \rho |x-y| = -\frac{c^2 |x-y|^2}{4\omega t} \frac{2\lambda-1}{\lambda^2} \leq -\frac{b}{\lambda} \frac{|x-y|^2}{t} = -b_1 \frac{|x-y|^2}{t}.$$

Therefore the first term in (A.7) can be estimated by

$$\begin{aligned} \frac{6}{\varepsilon} a_5 (1 + \kappa) \left(1 + \omega_3 t + \frac{\omega_3 c^2 |x-y|^2}{4\lambda^2 \omega^2 t}\right)^E t^{-\tilde{E}} \left(\left(\frac{|h|}{\sqrt{t} + |x-y|}\right)^\gamma \right. \\ \left. + \left(\frac{|k|}{\sqrt{t} + |x-y|}\right)^{\gamma^*} \right) e^{-(b_1 - 2\varepsilon) \frac{|x-y|^2}{t}}. \end{aligned}$$

For the second term we use (A.6) to estimate

$$|K_t^{(\alpha, \beta)}(x-h, y-k)| \leq a_5 \left(1 + \omega_3 \left(t + \frac{c^2 |x-y-h+k|^2}{4\omega^2 t}\right)\right)^{\tilde{E}} t^{-\tilde{E}} e^{-b \frac{|x-y-h+k|^2}{t}}.$$

Clearly

$$|x-y-h+k|^2 \leq 2|x-y|^2 + 2(|h| + |k|)^2 \leq 6|x-y|^2 + 4\kappa^2 t.$$

For the exponential set $\eta = \frac{\tau_1}{1-\tau_1} > 0$ and $\delta = \frac{\tau_1^2}{\tau^2} - 1 > 0$. Then

$$\begin{aligned} -b \frac{|x-y-h+k|^2}{t} &\leq -b \frac{|x-y|^2}{(1+\eta)t} + b \frac{|h-k|^2}{\eta t} \\ &\leq -b \frac{|x-y|^2}{(1+\eta)t} + \frac{b}{\eta t} \left((1+\delta) \tau^2 |x-y|^2 + (1+\delta^{-1}) \kappa^2 t \right) \\ &= -b(1-\tau_1)^2 \frac{|x-y|^2}{t} + b \kappa^2 \frac{\tau_1 - \tau_1^2}{\tau_1^2 - \tau^2}. \end{aligned}$$

So

$$\begin{aligned} &|K_t^{(\alpha, \beta)}(x-h, y-k)| \\ &\leq a_5 \left(1 + \frac{\omega_3 c^2 \kappa^2}{\omega^2} + \omega_3 t + \frac{3\omega_3 c^2 |x-y|^2}{2\omega^2 t} \right)^{\tilde{E}} t^{-\tilde{E}} e^{-b(1-\tau_1)^2 \frac{|x-y|^2}{t}} \exp\left(b \kappa^2 \frac{\tau_1 - \tau_1^2}{\tau_1^2 - \tau^2} \right). \end{aligned}$$

Next we estimate the factor $\rho(|h| + |k|) e^{\rho(|h|+|k|)}$. One has

$$\rho(|h| + |k|) \leq \frac{c|x-y|}{2\lambda\omega t} (\kappa\sqrt{t} + \tau|x-y|) \leq \frac{c(\tau + \eta\kappa)}{2\lambda\omega} \frac{|x-y|^2}{t} + \frac{c\kappa}{2\eta\lambda\omega}.$$

and alternatively

$$\begin{aligned} \rho(|h| + |k|) &\leq \frac{c|x-y|}{2\lambda\omega\sqrt{t}} \frac{|h| + |k|}{\sqrt{t}} \\ &\leq \frac{c|x-y|}{\sqrt{\varepsilon}\lambda\omega\sqrt{t}} \frac{|h| + |k|}{\sqrt{t} + |x-y|} e^{\varepsilon \frac{|x-y|^2}{t}} \\ &\leq \frac{c(1+\kappa)^2}{\varepsilon\lambda\omega} e^{2\varepsilon \frac{|x-y|^2}{t}} \left(\left(\frac{|h|}{\sqrt{t} + |x-y|} \right)^\gamma + \left(\frac{|k|}{\sqrt{t} + |x-y|} \right)^{\gamma^*} \right). \end{aligned}$$

So

$$\begin{aligned} \rho(|h| + |k|) e^{\rho(|h|+|k|)} &\leq \frac{c(1+\kappa)^2}{\varepsilon\lambda\omega} \exp\left(\frac{c\kappa}{2\eta\lambda\omega} \right) \exp\left(\left(\frac{c(\tau + \eta\kappa)}{2\lambda\omega} + 2\varepsilon \right) \frac{|x-y|^2}{t} \right) \\ &\quad \times \left(\left(\frac{|h|}{\sqrt{t} + |x-y|} \right)^\gamma + \left(\frac{|k|}{\sqrt{t} + |x-y|} \right)^{\gamma^*} \right). \end{aligned}$$

Using the identity $b(1 - \tau_1)^2 - \frac{c\tau}{2\lambda\omega} = b_1$ one deduces that the second term in (A.7) can be estimated by

$$\frac{a_5 c(1+\kappa)^2}{\varepsilon\lambda\omega} \exp\left(\frac{c\kappa}{2\eta\lambda\omega}\right) \exp\left(b\kappa^2 \frac{\tau_1 - \tau_1^2}{\tau_1^2 - \tau^2}\right) \left(1 + \frac{\omega_3 c^2 \kappa^2}{\omega^2} + \omega_3 t + \frac{3\omega_3 c^2 |x-y|^2}{2\omega^2 t}\right)^{\tilde{E}} \\ \times t^{-\tilde{E}} \left(\left(\frac{|h|}{\sqrt{t} + |x-y|}\right)^\gamma + \left(\frac{|k|}{\sqrt{t} + |x-y|}\right)^{\gamma^*} \right) e^{-(b_1 - 2\varepsilon - \frac{c\kappa\eta}{2\lambda\omega}) \frac{|x-y|^2}{t}}$$

Then (A.3) follows. \square

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