

Isothermalisation for a non-local heat equation

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Abstract. In this paper we study the asymptotic behavior for a nonlocal heat equation in an inhomogenous medium:

$$\rho(x)u_t = J * u - u \text{ in } \mathbb{R}^N \times (0, \infty),$$

where ρ is a continuous positive function, u is non-negative and J is a probability measure having finite second-order momentum. Depending on integrability conditions on the initial data u_0 and ρ , we prove various isothermalisation results, *i.e.*, $u(t)$ converges to a constant state in the whole space.

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1. Introduction

The aim of this paper is to study the asymptotic behavior for a nonlocal heat equation in an inhomogenous medium:

$$\begin{cases} \rho(x)u_t = J * u - u, & (x, t) \in \mathbb{R}^N \times (0, \infty), \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \quad (1.1)$$

Here, u_0 is a non-negative continuous function in \mathbb{R}^N and $*$ denotes the convolution with a kernel $J : \mathbb{R}^N \rightarrow \mathbb{R}$, which is a radial, continuous probability density having finite second-order momentum:

$$\int_{\mathbb{R}^N} J(s) ds = 1, \quad \mathbb{E}(J) = \int_{\mathbb{R}^N} sJ(s) ds = 0, \quad \mathbb{V}(J) = \int_{\mathbb{R}^N} s^2 J(s) ds < +\infty.$$

Typical examples of kernel that we consider are the gaussian law, the exponential law or any compactly supported kernels. We also assume that ρ is a positive, continuous function in \mathbb{R}^N , whether integrable or not.

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The operator $J * u - u$ can be interpreted as a non-local diffusion operator. Indeed, if $u(x, t)$ represents the density of a single population and $J(x - y)$ is the probability to jump from y to x then the term $(J * u)(x)$ is the rate at which individuals arrive to x and $-u(x)$ is the rate at which individuals leave from x , see for instance [6]. In the case of heat propagation, u stands for a temperature and $\rho(x)$ represents the density of the medium.

Problem (1.1) is called *non-local* because the diffusion at $u(x, t)$ depends on all the values of u in the support of J and not only of the value of $u(x, t)$, as it is the case for the local diffusion problem

$$\begin{cases} \rho(x)u_t = \Delta u, & (x, t) \in \mathbb{R}^N \times (0, \infty), \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

For this local problem it is well known that for dimension $N = 1, 2$ there exists a unique solution in the class of bounded solutions, see [7] and [4]. Moreover, if $\rho \in L^1(\mathbb{R}^N)$ and u_0 is bounded, then as $t \rightarrow \infty$ the solution converges on compact sets to $\mathbb{E}_\rho(u_0)$, the mean of u_0 with respect to ρ :

$$\mathbb{E}_\rho(u_0) := \frac{\int_{\mathbb{R}^N} u_0(y)\rho(y) dy}{\int_{\mathbb{R}^N} \rho(y) dy} \in \mathbb{R}_+.$$

This phenomenon is called *isothermalisation*, since the heat distribution converges to a non-trivial isothermal state in all the space. However, for dimension $N \geq 3$ uniqueness is lost in the class of bounded solutions and some solutions decrease to zero as $t \rightarrow \infty$, so that isothermalisation does not take place, see [8] and [5].

We are also facing the influence of the space dimension in the nonlocal case. More precisely, if J is compactly supported and $N = 1, 2$, we prove uniqueness of bounded solutions. For dimension $N \geq 3$, we need (as in the local case) additional conditions on the behavior of ρ at infinity to get uniqueness of bounded solutions.

The case when ρ is not integrable is also considered, which is more related to the study of the homogeneous case ($\rho \equiv 1$), see [3] and [1]. For bounded solutions, the flux at infinity is so big that solutions go down to zero asymptotically while if the data is unbounded, the solution may go to infinity asymptotically as $t \rightarrow \infty$.

Organization and main results

We first prove in Section 2 a comparison result which implies uniqueness for problem (1.1) in some class. This comparison states that if \bar{u} is a supersolution of the equation, and \underline{u} is a subsolution, (and both are ordered initially), then we have $\bar{u} \geq \underline{u}$ in $\mathbb{R}^N \times [0, \infty)$. Such a result is obtained by standard arguments once we are able to construct a suitable strict supersolution (see Lemma 2.5 and Lemma 2.7). But this requires some assumptions on ρ . If ρ degenerates too fast at infinity, the maximum principle may be violated.

In Section 3 we study the existence in the class of bounded solutions, which is obtained by approximation with Neuman problems in bounded domains. Our first main result is the following:

Theorem 1.1. *Let $\rho > 0$, be continuous, and u_0 be a bounded non-negative continuous function. Then there exists a non-negative classical bounded solution of problem (1.1). Moreover, if either*

$$\rho(x) \geq \frac{\eta}{1 + |x|^2},$$

or $N = 1, 2$ and J has compact support then uniqueness holds in this class.

We denote by $L^1(\rho)$ the space of measurable functions $f : \mathbb{R}^N \rightarrow \mathbb{R}$ such that $\rho f \in L^1(\mathbb{R}^N)$. Section 4 is devoted to study the isothermalisation phenomenon for bounded initial data in this space. There are two major tools we use to derive the isothermalisation effect for $u_0 \in L^1(\rho)$: (i) an estimate of the solution in $L^1(\rho)$; (ii) some weighted energy estimates in $L^2(\mathbb{R}^N)$. Then the main theorem is as follows:

Theorem 1.2. *Let $\rho > 0$, be continuous, integrable and u_0 be a bounded non-negative continuous function. Then, the solution satisfies the following mass conservation*

$$\int_{\mathbb{R}^N} \rho(x)u(x, t) \, dx = \int_{\mathbb{R}^N} \rho(x)u_0(x) \, dx. \quad (1.2)$$

Moreover, $u(x, t) \rightarrow \mathbb{E}_\rho(u_0)$ as $t \rightarrow \infty$ in $L^p_{\text{loc}}(\mathbb{R}^N)$ for any $1 \leq p < \infty$; the convergence also holds in $L^1(\rho)$:

$$\lim_{t \rightarrow \infty} \int_{\mathbb{R}^N} |u(x, t) - \mathbb{E}_\rho(u_0)| \rho(x) \, dx = 0.$$

Notice that since $\rho \in L^1(\mathbb{R}^N)$ and u_0 is bounded then $u_0 \in L^1(\rho)$. If ρ is not integrable but still $u_0 \in L^1(\rho)$ the flux at infinity forces the solution to go to zero:

Theorem 1.3. *Let $\rho > 0$ and u_0 be a bounded non-negative continuous function such that $u_0 \in L^1(\rho)$. If ρ is not integrable in \mathbb{R}^N , then $u(x, t) \rightarrow 0$ as $t \rightarrow \infty$ in $L^p_{\text{loc}}(\mathbb{R}^N)$ for any $1 \leq p < \infty$.*

Finally, in Section 5 we investigate the case of unbounded initial data. We first prove an existence result in the class of unbounded solutions provided ρ does not degenerate too rapidly at infinity. More precisely, if

$$\rho(x) \geq \frac{\eta}{1 + |x|^\gamma} \quad \gamma \leq 2, \quad (1.3)$$

then we have an existence result for quadratic initial data:

Theorem 1.4. *Let u_0 be a positive continuous function with at most quadratic growth at infinity. If the function ρ satisfies (1.3) then there exists a solution of problem (1.1).*

More generally, if ρ is integrable, we prove similar isothermalisation results for the minimal solution:

Theorem 1.5. *We assume that ρ is a continuous positive integrable function in \mathbb{R}^N and that u_0 is a continuous non-negative function, possibly unbounded, such that there exists a solution u . Noting \underline{u} the minimal solution, the following holds:*

i) *If $u_0 \in L^1(\rho)$ the isothermalisation takes place in $L^1(\rho)$,*

$$\lim_{t \rightarrow \infty} \int_{\mathbb{R}^N} |\underline{u}(x, t) - \mathbb{E}_\rho(u_0)| \rho(x) \, dx = 0.$$

ii) *If $u_0 \notin L^1(\rho)$, we have that for all $1 \leq p < \infty$,*

$$\lim_{t \rightarrow \infty} \underline{u}(x, t) = \infty \quad \text{in } L^p_{\text{loc}}(\mathbb{R}^N).$$

In the case of nonintegrable ρ 's with $u_0 \notin L^1(\rho)$ the asymptotic behavior is more difficult to treat. For instance, if $\rho \equiv 1$, the solutions

$$u(x, t) = |x|^2 + \mathbb{V}(J)t, \quad \text{and} \quad u(x, t) = 1$$

have different behavior. Thus, there is a balance between ρ , J , and the initial data u_0 which is not easy to handle and the question remains open.

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2. Preliminaries

Let us specify first what is the notion of solution that we use:

Definition 2.1. Let $u_0 \in L^1_{\text{loc}}(\mathbb{R}^N)$. By a strong solution of (1.1) we mean a function $u \in C^0([0, \infty); L^1_{\text{loc}}(\mathbb{R}^N))$ such that $u_t, J * u \in L^1_{\text{loc}}(\mathbb{R}^N \times (0, \infty))$, the equation is satisfied in the L^1_{loc} -sense and such that $u(x, 0) = u_0(x)$ almost everywhere in \mathbb{R}^N .

We shall consider also solutions with more regularity:

Definition 2.2. A classical solution of (1.1) is a strong solution such that moreover $u, u_t, J * u \in C^0(\mathbb{R}^N \times [0, \infty))$ and the equation holds in the classical sense everywhere in $\mathbb{R}^N \times [0, \infty)$.

A classical sub or supersolution is defined as usual with inequalities instead of equalities in the equation.

Let us recall some basic facts for (1.1) with $\rho \equiv 1$ (which are more or less valid for a suitable class of functions ρ as we shall see below):

maximum principle If u reaches a max at x_0 , then the convolution $J * u - u$ is necessarily non-negative at x_0 (this is a consequence of the fact that J is a probability density). And moreover, if $(J * u - u)(x_0) = 0$, then $u(x) = u(x_0)$ in the support of J , translated at x_0 .

comparison result This kind of nonlocal equation enjoys various comparison principles between sub and supersolutions (whether in \mathbb{R}^N or in bounded domains), which rely on the maximum principle above.

existence and uniqueness Both are typically obtained through fixed-point theorems, but uniqueness is also (of course) a consequence of the comparison between sub/supersolutions.

We refer to [1–3] for more details and references.

Now let us state a simple regularity Lemma, which contains a technical trick that we shall use several times in the sequel:

Lemma 2.3. *Let u be a strong solution of (1.1). We assume moreover that u_0 is continuous in \mathbb{R}^N and that the convolution term $J * u$ is continuous in $\mathbb{R}^N \times [0, \infty)$. Then u and u_t are also continuous in $\mathbb{R}^N \times [0, \infty)$ and u is a classical solution.*

Proof. We introduce the following transform:

$$\mathcal{T}_\rho[u](x, t) := e^{t/\rho(x)} u(x, t). \quad (2.1)$$

A straightforward calculus show that $v = \mathcal{T}_\rho[u]$ satisfies

$$v_t = \frac{e^{t/\rho(x)}}{\rho(x)} (J * u)(x, t),$$

which is a continuous function in $\mathbb{R}^N \times [0, \infty)$. Integrating between 0 and t we get:

$$v(x, t) = \int_0^t \partial_t v(x, s) ds + v(x, 0) = \int_0^t \frac{e^{s/\rho(x)}}{\rho(x)} (J * u)(x, s) ds + u(x, 0),$$

hence v is continuous in $\mathbb{R}^N \times [0, \infty)$. This implies that u is also continuous in $\mathbb{R}^N \times [0, \infty)$, and the equation holds in the classical sense. \square

Remark 2.4. It is well-known in convolution theory that under one of the following assumptions, the convolution term is continuous:

- (i) u bounded (since J is integrable);
- (ii) J compactly supported and u locally integrable.

The following lemma concerns the comparison of classical sub/supersolutions of the problem. In order to prove that we need to find a strict supersolution ψ , which satisfies

$$\begin{aligned} \psi &\in C^0(\mathbb{R}^N \times [0, \infty)), \psi \geq 0, \\ \psi(x, t) &\rightarrow +\infty \text{ as } |x| \rightarrow \infty \text{ uniformly for } t \in [0, \infty), \\ \rho\psi_t &> J * \psi - \psi. \end{aligned} \tag{2.2}$$

Lemma 2.5. *In each of the following three cases, if $A > 0$ and λ is large enough, the function ψ satisfies (2.2):*

(i) *If function ρ satisfies*

$$\rho(x) \geq \frac{\eta}{1 + |x|^2},$$

then

$$\psi(x, t) = Ae^{\lambda t} (1 + |x|^2).$$

(ii) *If J has compact support and $N = 1$,*

$$\psi(x, t) = Ae^{\lambda t} (1 + |x|).$$

(iii) *If J has compact support and $N = 2$,*

$$\psi(x, t) = Ae^{\lambda t} (1 + (\ln |x|)_+).$$

Proof. The regularity and the behaviour at infinity of ψ is clear, so we only need to prove that ψ is a strictly supersolution.

In the first case a direct computation gives us that

$$J * \psi - \psi = Ae^{\lambda t} \nabla(J).$$

On the other hand,

$$\rho\psi_t = \rho A \lambda e^{\lambda t} (1 + |x|^2) \geq A \lambda \eta e^{\lambda t}.$$

Then, for $\lambda > \nabla(J)/\eta$, we are done.

In the other two cases, we notice first that ψ is harmonic outside the origin. Since J is a radial probability density supported in $B(0, R)$ (then $\mathbb{E}(J) = 0$), this implies that

$$J * \psi - \psi = 0, \quad \text{for } |x| \geq 2R.$$

On the other hand, since ψ is continuous in $B(0, 2R)$, there exists $K > 0$ such that

$$J * \psi - \psi \leq Ke^{\lambda t} \quad \text{for } |x| < 2R.$$

Moreover, if $|x| \leq 2R$ the function ρ is bounded from below, so there exists a constant \tilde{K} such that

$$\rho\psi_t = \rho\lambda\psi \geq \tilde{K}\lambda e^{\lambda t}.$$

Therefore, taking $\lambda \geq K/\tilde{K}$, we get that ψ is a strictly supersolution. □

Remark 2.6. In dimension $N \geq 3$ and J compactly supported, the function $\psi(x, t) = e^{\lambda t} \min(1, |x|^{2-N})$ would give a strict supersolution, however it goes to zero as $|x| \rightarrow \infty$. Hence it cannot be used to obtain a comparison principle for bounded solutions.

Lemma 2.7. *Let ψ satisfy (2.2). Let \underline{u} be a classical subsolution of (1.1) and \bar{u} a classical supersolution of (1.1) such that $\underline{u}(x, 0) \leq \bar{u}(x, 0)$. If for all $t \geq 0$,*

$$\limsup_{|x| \rightarrow \infty} \frac{u - \bar{u}}{\psi} \leq 0 \tag{2.3}$$

then $\bar{u} \geq \underline{u}$ in $\mathbb{R}^N \times \mathbb{R}_+$.

Proof. We consider the function

$$w_\delta = \underline{u} - \bar{u} - \delta\psi,$$

which satisfies the inequality

$$\rho(x)(w_\delta)_t(x, t) < (J * w_\delta)(x, t) - w_\delta(x, t). \tag{2.4}$$

From (2.3), we deduce that on $\mathbb{R}^N \times [0, T]$, w_δ attains its maximum at a point (x_0, t_0) . Observe that at this point,

$$\rho(x_0)(w_\delta)_t(x_0, t_0) < (J * w_\delta)(x_0, t_0) - w_\delta(x_0, t_0) \leq 0$$

but if $t_0 \in (0, T)$, $\partial_t w_\delta(x_0, t_0) = 0$ and if $t_0 = T$, $\partial_t w_\delta(x_0, t_0) \geq 0$. Hence in each of these cases we reach a contradiction. We are left with the last possibility, $t_0 = 0$, which implies $\max w_\delta \leq 0$. Passing to the limit as $\delta \rightarrow 0$, since $T > 0$ is arbitrary we obtain the comparison

$$\underline{u} - \bar{u} \leq 0 \quad \text{for all } x \in \mathbb{R}^N, t \geq 0. \quad \square$$

As direct corollary, we have the following uniqueness result, which is valid only in a suitable class of solutions, as it is the case for the local heat equation.

Corollary 2.8. *Let ψ satisfy (2.2). Let u_0 be a continuous function which grows strictly less than ψ . Then there exists at most one classical solution u of (1.1) such that $|u|$ grows strictly less than ψ .*

3. Existence and uniqueness of bounded solutions

We obtain an existence result by approximation. We consider the following Neumann problem, where χ_n denotes the indicator of the ball B_n :

$$\begin{cases} \rho(x)\partial_t u_n = \int_{B_n} (u_n(y, t) - u_n(x, t))J(x - y) dy, & x \in B_n, \\ u_n(x, 0) = u_0\chi_n(x), \end{cases} \tag{3.1}$$

First, let us mention that this problem also enjoys a comparison result:

Lemma 3.1. *Let $\bar{u} \in C^1([0, \infty); C^0(\bar{B}_n))$ and $\underline{u} \in C^1([0, \infty); C^0(\bar{B}_n))$ be a supersolution and a subsolution of (3.1) such that $\bar{u}(x, 0) \geq \underline{u}(x, 0)$ in \bar{B}_n . Then $\bar{u} \geq \underline{u}$ in $\bar{B}_n \times [0, \infty)$.*

Proof. The strategy is standard, and follows Lemma 2.7 except that we are in a bounded domain so we need not worry about localizing the supremum with a function ψ .

Take $c, T > 0$ and consider

$$w_{c,T}(x, t) := \underline{u}(x, t) - \bar{u}(x, t) - \frac{c}{T - t}$$

in $\bar{B}_n \times [0, T]$. This function satisfies the inequality

$$\rho(x)\partial_t w_{c,T} \leq \int_{B_n} (w_{c,T}(y, t) - w_{c,T}(x, t))J(x - y) \, dy - \frac{c\rho(x)}{(T - t)^2}.$$

Notice that this inequality is still valid if $x_0 \in \partial B_n$, this is just using the Dominated Convergence Theorem with a sequence $x_n \rightarrow x_0, x_n \in B_n$.

We shall prove that for any $c, T > 0$, we have $\max w_{c,T} \leq 0$. Of course $w_{c,T}$ reaches its maximum in $\bar{B}_n \times [0, T]$ because it is continuous, and obviously it cannot be attained at $t = T$. If this max was attained at $t = 0$, we would deduce that $\max w_{c,T} \leq 0$ which is what we want.

So, let us assume that the maximum point (x_0, t_0) is such that $0 < t < T$. Hence, by the above inequality we get the following contradiction

$$\begin{aligned} 0 &= \rho(x_0)\partial_t w_{c,T}(x_0, t_0) \\ &\leq \int_{B_n} (w_{c,T}(y, t_0) - w_{c,T}(x_0, t_0))J(x_0 - y) \, dy - \frac{c\rho(x_0)}{(T - t_0)^2} < 0. \end{aligned}$$

So we are left with $t_0 = 0$ and $\max w_{c,T} \leq 0$.

Passing to the limit as $c \rightarrow 0$ we deduce that $\underline{u} \leq \bar{u}$ in $B_n \times [0, T]$, and then we send $T \rightarrow \infty$ to finish the proof. \square

Lemma 3.2. *Let $\rho > 0$ be a continuous function in \mathbb{R}^N and u_0 be a non-negative bounded continuous function. Then for any $n \geq 0$, there exists a unique solution $u_n \in C^1([0, \infty); C^0(\bar{B}_n))$, which satisfies $0 \leq u_n \leq \|u_0\|_\infty$. Moreover, the following conservation law holds:*

$$\int_{B_n} u_n(x, t)\rho(x) \, dx = \int_{B_n} u_0(x)\rho(x) \, dx. \tag{3.2}$$

Proof. Following [3], we consider $t_0 > 0$ to be fixed later on and the Banach space $X_{t_0} = C^0([0, t_0] \times \bar{B}_n)$ equipped with the norm $\|w\| = \max \{ \|w(\cdot, t)\|_{L^\infty(\bar{B}_n)}, 0 \leq t \leq t_0 \}$. Then we define an operator $T : X_{t_0} \rightarrow X_{t_0}$ as follows:

$$T_{w_0}(w)(x, t) := w_0(x) + \frac{1}{\rho} \int_0^t \int_{B_n} J(x - y)(w(y, s) - w(x, s)) \, dy \, ds.$$

A straightforward calculus, using that $\rho \geq \rho_0$ in B_n shows that:

$$\|T_{w_0}(w) - T_{z_0}(z)\| \leq \|w_0 - z_0\|_{L^\infty(\overline{B_n})} + \frac{2t_0}{\rho_0} \|w - z\|.$$

We deduce that for any fixed initial data $u_0 \in L^\infty(\overline{B_n})$, if $t_0 > 0$ is sufficiently small (say $t_0 < \rho_0/4$), T_{u_0} is a strict contraction in the Banach space X_{t_0} . Hence there exists a unique function $u_n \in X_{t_0}$ such that $T_{u_0}(u_n) = u_n$, which means that for any $x \in \overline{B_n}$ and $t \in [0, t_0]$, one has

$$u_n(x, t) = u_0(x) + \frac{1}{\rho} \int_0^t \int_{B_n} J(x - y)(u_n(y, s) - u_n(x, s)) \, dy \, ds.$$

This in particular implies that $u(0) = u_0$ and moreover, that the weak derivative $\partial_t u$ exists, which is given by

$$\partial_t u_n = \frac{1}{\rho} \int_{B_n} J(x - y)(u_n(y, s) - u_n(x, s)) \, dy.$$

Since the right hand side of the equation is a continuous function in $\overline{B_n} \times [0, t_0]$, we have that $\partial_t u$ is also continuous in $\overline{B_n} \times [0, t_0]$. To extend the solution to $[0, \infty)$ we take as initial data $u(x, t_0) \in C^0(\overline{B_n})$ and obtain a solution up $[0, 2t_0]$. We then iterate the procedure to construct a solution for all time $t > 0$.

On the other hand, we note that the constants functions $\bar{u} = \|u_0\|_\infty$ and $\underline{u} = 0$ are a supersolution and subsolution respectively, then by comparison

$$0 \leq u_n \leq \|u_0\|_\infty.$$

Finally, we observe that

$$\rho(x)u_n(x, t) - \rho(x)u_0(x) = \int_0^t \int_{B_n} J(x - y)(u_n(y, s) - u_n(x, s)) \, dy \, ds.$$

Then, integrating in x and apply Fubini's theorem in the right hand side of the equation, we get de desired conservation law. \square

Proposition 3.3. *Let $\rho > 0$ be a continuous function in \mathbb{R}^N and u_0 be a non-negative continuous bounded function and for any integer $n > 0$, let u_n be the solution constructed in Lemma 3.2. Then as $n \rightarrow \infty$, along a subsequence $u_n \rightarrow u$ in $L^1_{loc}(\mathbb{R}^N \times [0, \infty))$ where u is a non-negative classical bounded solution of (1.1).*

Proof. Since the sequence u_n is bounded, it converges (along a subsequence still denoted u_n) in L^∞ -weak* to some non-negative and bounded function u . Since $J \in L^1$, this implies that $J * u_n \rightarrow J * u$ strongly and that $\partial_t u_n \rightarrow \partial_t u$ in the sense of distributions. We can then pass to the limit in the sense of distributions but here also we want a better convergence.

Let us introduce a modified version of transform \mathcal{T} as follows:

$$v_n(x, t) := e^{t(J*\chi_n)(x)/\rho(x)} u_n(x, t).$$

Since (3.1) can be written as

$$\rho(x)\partial_t(u_n)(x, t) = [J * (u_n \chi_n)](x, t) - (J * \chi_n)(x)u_n(x),$$

it follows immediately that v_n satisfies the equation

$$\partial_t v_n = \frac{e^{t(J*\chi_n)(x)/\rho(x)}}{\rho(x)} J * u_n.$$

This implies that $\partial_t v_n$ converges strongly on compact sets of $\mathbb{R}^N \times [0, \infty)$, and so does $v_n(x, t) = u_0(x)\chi_n(x) + \int_0^t (v_n)_t(x, s) ds$. Then u_n also converges strongly on compact sets of $\mathbb{R}^N \times [0, \infty)$ to its limit u .

Passing to the limit in the equation, we see that u is a strong solution of (1.1), which implies that it is a classical solution of this equation – this follows from Lemma 2.3. □

Proof of Theorem 1.1. It is just the combination of Proposition 3.3, Corollary 2.8 and Lemma 2.5. □

Now, we pass to the limit in the conservation law, to obtain:

Theorem 3.4. *Let $\rho > 0$, be continuous and $u_0 \in C^0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap L^1(\rho)$. Then:*

(i) *for any $t > 0$, $u(\cdot, t) \in L^1(\rho)$ and*

$$\int_{\mathbb{R}^N} \rho(x)u(x, t) dx \leq \int_{\mathbb{R}^N} \rho(x)u_0(x) dx.$$

(ii) *if moreover we assume that $\rho \in L^1(\mathbb{R}^N)$ then the conservation law holds:*

$$\int_{\mathbb{R}^N} \rho(x)u(x, t) dx = \int_{\mathbb{R}^N} \rho(x)u_0(x) dx.$$

Proof. Since $u_0 \in L^\infty \cap C^0$, the approximating sequence $\{u_n\}$ of Proposition 3.3 satisfies that (up to a subsequence) $u_n \rightarrow u$ almost everywhere, with $0 \leq u_n \leq \|u_0\|$ and

$$\int_{\mathbb{R}^N} u_n(x, t)\chi_n(x)\rho(x) dx = \int_{\mathbb{R}^N} u_0(x)\chi_n(x)\rho(x) dx. \tag{3.3}$$

Since $u_0 \in L^1(\rho)$, the dominated convergence theorem yields the convergence of the right-hand side integral as $n \rightarrow \infty$. For the left hand side, using Fatou’s Lemma we obtain that

$$\int_{\mathbb{R}^N} \rho(x)u(x, t) dx \leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \rho(x)u_n(x, t)\chi_n(x) dx = \int_{\mathbb{R}^N} \rho(x)u_0(x) dx,$$

which proves assumption (i).

Finally, if we assume that $\rho \in L^1(\mathbb{R}^N)$ we can use also the dominated convergence theorem for the sequence $u_n \chi_n \rho$, which is bounded by $\|u_0\|_\infty \rho \in L^1(\mathbb{R}^N)$. We then pass to the limit in the left-hand side of (3.3) and get (ii). \square

4. Asymptotic behaviour for bounded solutions

We shall now derive our main results concerning the asymptotic behaviour for (1.1). We divide the proof in several steps.

4.1. Weak limit

This first step is easy, it only comes from the fact that the solution is globally bounded:

Lemma 4.1. *Let $\rho > 0$, continuous and $u_0 \in C^0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Let u be a bounded classical solution. Then for any $s > 0$ there exists a subsequence $t_k \rightarrow +\infty$ such that the following limit exists in L^∞ -weak*:*

$$u_\infty(x, s) := \lim_{t_k \rightarrow \infty} u(x, s + t_k).$$

Proof. Since u is bounded, thus there exists a subsequence $t_k \rightarrow \infty$ such that $u(\cdot, s + t_k)$ converges in L^∞ -weak* to a function $u_\infty(\cdot, s) \in L^\infty(\mathbb{R}^N)$. \square

4.2. Lyapounov functional

We now want a stronger result, so we use a Lyapounov functional:

Lemma 4.2. *Let $t_0 > 0$. Assume the hypotheses of Lemma 4.1 and that $u_0 \in L^2(\rho)$. Then there exists a constant $C = C(u_0, \rho, t_0)$ such that for all $t \geq t_0 > 0$,*

$$\int_t^\infty \int_{\mathbb{R}^N} \rho(x) (u_t)^2(x, s) \, ds \leq C.$$

Proof. We first go back to the approximating scheme (3.1). Notice that the following quantity

$$F[u_n](t) = \int_{B_n} \int_{B_n} J(x-y) (u_n(x, t) - u_n(y, t))^2 \, dx \, dy$$

is a Lyapounov functional which is nonincreasing along the evolution orbits at the level n . Indeed, multiplying the equation by u and integrating in B_n we get

$$\begin{aligned} \int \rho(x) \partial_t u_n \cdot u_n \, dx &= \iint J(x-y) u_n(x) u_n(y) \, dx \, dy - \int u_n(x)^2 \, dx \\ &= \frac{1}{2} \iint J(x-y) (2u_n(x)u_n(y) - 2u_n(x)^2) \, dx \, dy \\ &= \frac{1}{2} \iint J(x-y) (2u_n(x)u_n(y) - u_n(x)^2 - u_n(y)^2) \, dx \, dy \\ &= -\frac{1}{2} F[u_n](t). \end{aligned}$$

Therefore,

$$F[u_n](t) = -\frac{d}{dt} \int_{B_n} \rho(x) u_n^2(x, t) \, dx. \quad (4.1)$$

In a similar way, multiplying the equation by $\partial_t u_n$ and integrating in space we obtain

$$\frac{d}{dt} F[u_n](t) = -4 \int_{B_n} \rho(x) ((u_n)_t)^2(x, t) \, dx. \quad (4.2)$$

Integrating (4.1) we have that for some $C' = C'(u_0, \rho)$,

$$\int_0^t F[u_n](s) \, ds = 2 \int_{B_n} \rho(x) u_0^2(x) \chi_n(x) \, dx - 2 \int_{B_n} \rho(x) u_n^2(x, t) \, dx \leq C'.$$

Indeed, as $u_0 \in L^2(\rho)$ we have by monotone convergence that

$$\int_{\mathbb{R}^N} \rho(x) u_0^2(x) \chi_n(x) \, dx \rightarrow \int_{\mathbb{R}^N} \rho(x) u_0^2(x) \, dx < \infty.$$

Using now the monotonicity in t of $F[u_n](t)$, we get

$$t F[u_n](t) \leq \int_0^t F[u_n](s) \, ds \leq C'.$$

Moreover, $F[u_n](t)$ is positive so that, integrating (4.2), we get for any $t \geq t_0$:

$$\int_t^\infty \int_{\mathbb{R}^N} \rho((u_n)_t)^2(x, s) \, dx \, ds \leq \frac{1}{4} F[u_n](t) \leq \frac{C'}{4t_0}.$$

Using Fatou's Lemma and the fact that $\rho(u_n)_t$ converges strongly to ρu_t , we obtain the desired result. \square

As an immediate consequence of this result we obtain:

Lemma 4.3. *Assume the hypotheses of Lemma 4.1 and that $u_0 \in L^2(\rho)$. For all sequence $t_k \rightarrow \infty$ and $s > 0$,*

$$\|\sqrt{\rho(\cdot)}u(\cdot, s + t_k) - \sqrt{\rho(\cdot)}u(\cdot, t_k)\|_{L^2(\mathbb{R}^N)}^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Hence, the limit function $u_\infty(x, s)$ does not depend on the variable $s > 0$.

Proof. Notice that for all sequence $t_k \rightarrow \infty$, we get

$$\begin{aligned} \|\sqrt{\rho(\cdot)}u(\cdot, s + t_k) - \sqrt{\rho(\cdot)}u(\cdot, t_k)\|_{L^2(\mathbb{R}^N)}^2 &= \int_{\mathbb{R}^N} \rho(x) \left(\int_{t_k}^{t_k+s} u_t(x, \sigma) \, d\sigma \right)^2 dx \\ &\leq s \int_{\mathbb{R}^N} \int_{t_k}^{t_k+s} \rho(x)(u_t)^2(x, \sigma) \, d\sigma \, dx, \end{aligned}$$

which goes to zero as $t_k \rightarrow +\infty$. □

4.3. The ω -limit set

We define the ω -limit set as follows:

$$\omega(u_0) = \{u_\infty \in C^0(\mathbb{R}^N) : \exists t_j \rightarrow \infty \text{ such that } u(\cdot, t_j) \rightarrow u_\infty(\cdot) \text{ in } L^\infty\text{-weak}^*\}.$$

Lemma 4.4. *Under the hypotheses of Lemma 4.1 and that $u_0 \in L^2(\rho)$, the ω -limit set is reduced to constants.*

Proof. Since $u(x, s + t_k)$ converges weakly in L^∞ -weak*, then as $t_k \rightarrow \infty$,

$$(J * u)(x, s + t_k) = \int J(x - y)u(y, s + t_k) \, dy \xrightarrow{\text{pointwise}} (J * u_\infty)(x, s).$$

Moreover, since u is bounded, the convergence of $J * u$ is also strong in L^1_{loc} . On the other hand, $\partial_t \rho u(s + t_k) \rightarrow \rho \partial_s u_\infty(s)$ in the sense of distributions. We then pass to the limit in the sense of distributions in the equation and get

$$\rho(x) \frac{\partial}{\partial s} u_\infty(x, s) = J * u_\infty(x, s) - u_\infty(x, s).$$

Using Lemma 4.3 we know that u_∞ is independent of s so that u_∞ is a bounded solution (in the sense of distributions) of

$$J * u_\infty - u_\infty = 0 \quad \text{in } \mathbb{R}^N.$$

We first deduce that u_∞ is continuous because the convolution term is continuous. Then we consider the Fourier transform in the space of tempered distributions:

$$\hat{J} \cdot \hat{u}_\infty - \hat{u}_\infty = 0.$$

Since $|\hat{J}| < 1$ except for $\hat{J}(0) = 1$, we see that the (generalized) Fourier transform of u_∞ has a support contained in $\{0\}$. This implies that u_∞ is a polynom but since it is bounded, it has to be a constant. □

4.4. Identification of the limit

We are now ready to identify the ω -limit set.

Lemma 4.5. *We assume the hypotheses of Lemma 4.1 and that $u_0 \in L^1(\rho)$. Then the following holds:*

- (i) if $\rho \in L^1(\mathbb{R}^N)$, $\omega(u_0) = \{\mathbb{E}_\rho(u_0)\}$;
- (ii) if $\rho \notin L^1(\mathbb{R}^N)$, $\omega(u_0) = \{0\}$.

Proof. Notice first that since $u_0 \in L^1(\rho) \cap L^\infty$, then $u_0 \in L^2(\rho)$, hence we may use Lemma 4.4. In the integrable case, $\rho \in L^1$, we observe that as u is uniformly bounded, the dominated convergence Theorem gives

$$\int_{\mathbb{R}^N} \rho(x)u_n(x, s + t_j) dx \rightarrow u_\infty \int_{\mathbb{R}^N} \rho(x) dx.$$

Therefore, by Theorem 3.4-(ii) we obtain $u_\infty = \mathbb{E}_\rho(u_0)$, so that the ω -limit set is reduced to $\{\mathbb{E}_\rho(u_0)\}$.

In the case $\rho \notin L^1(\mathbb{R}^N)$, we take a compact set K such that

$$\int_K \rho(x) dx > \int_{\mathbb{R}^N} \rho(x)u_0(x) dx ,$$

which is always possible since $u_0 \in L^1(\rho)$. Using Theorem 3.4-(i), Lemma 4.4 and Fatou’s Lemma, we obtain

$$\begin{aligned} u_\infty \int_K \rho(x) dx &\leq \liminf_{n \rightarrow \infty} \int_K \rho(x)u_n(x, s + t_j) dx \\ &\leq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \rho(x)u_n(x, s + t_j) dx \\ &\leq \int_{\mathbb{R}^N} \rho(x)u_0(x) dx. \end{aligned}$$

This implies that necessarily $u_\infty = 0$, hence the ω -limit set is reduced to $\{0\}$. □

4.5. Proofs of Theorems 1.2 and 1.3

As consequence of the fact that the ω -limit is given by only one function we can pass to the limit in the time variable without extracting any subsequence. Then it only remains to check that the convergence is better, which is done by using transform \mathcal{T}_ρ .

Proof of Theorem 1.3. Under the hypotheses of the Theorem, we have that u_0 is continuous, bounded, and ρ not integrable, but nevertheless $u_0 \in L^1(\rho) \cap L^2(\rho)$. Thus Lemmas 4.1 and 4.5 imply that for any $s > 0$, at least along a subsequence

$t_n \rightarrow \infty$ we have $u(x, s + t_n) \rightarrow 0$ in L^∞ -weak*. But the same arguments are valid for any other subsequence such that $u(x, s + t'_n)$ converges weakly. Since the limit is always zero, we deduce that for any $s > 0$,

$$u(x, s + t) \xrightarrow[t \rightarrow \infty]{L^\infty\text{-weak}^*} 0,$$

which implies that $(J * u(s + t))$ converges strongly in L^1_{loc} as $t \rightarrow \infty$. Then,

$$\rho(x)\partial_s u(x, s + t) = (J * u(s + t))(x) - u(x, s + t) \xrightarrow[t \rightarrow \infty]{L^\infty\text{-weak}^*} 0.$$

Even more, since $t \rightarrow +\infty$ we may assume that $t > t_0$ for some $t_0 > 0$ and from lemma 4.2 we obtain that for any compact set K ,

$$\begin{aligned} \int_t^\infty \left(\int_K \rho(x)|u_t|(x, s) dx \right)^2 ds &\leq \int_t^\infty \left(\int_K \rho(x)|u_t|^2(x, s) dx \right) \\ &\leq \left(\int_K \rho(x) dx \right) \leq C(K, u_0, \rho). \end{aligned}$$

Then, at least for some sequence $t_k \rightarrow +\infty$, we have $\rho(x)\partial_s u(x, s + t_k) \rightarrow 0$ in L^1_{loc} . Summing up, we obtain that

$$\lim_{t_k \rightarrow +\infty} u(x, s + t_k) = 0 \text{ in } L^1_{loc}.$$

Of course, if $t \mapsto u(x, s + t)$ were to converge in L^1_{loc} along another subsequence $t'_k \rightarrow \infty$, the limit would necessarily be zero, so that finally $u(\cdot, t) \rightarrow 0$ in $L^1_{loc}(\mathbb{R}^N)$ as $t \rightarrow \infty$. Moreover, since $t \mapsto u(\cdot, t)$ remains bounded in $L^\infty(\mathbb{R}^N)$, we deduce that the convergence holds in $L^p_{loc}(\mathbb{R}^N)$ for any $1 \leq p < \infty$. \square

Proof of Theorem 1.2. The first part is done exactly as in the proof of Theorem 1.3, except that ρ is integrable here so that the limit is not zero, but $\mathbb{E}_\rho(u_0)$. To end the proof in this case, it only remains to prove the $L^1(\rho)$ convergence. We fix $\varepsilon > 0$ and choose $R > 0$ big enough so that (remember that ρ is integrable):

$$\int_{|x|>R} \rho(x) dx \leq \varepsilon.$$

Then

$$\int_{\mathbb{R}^N} |u(x, t) - u_\infty| \rho(x) dx \leq 2\varepsilon \|u_0\|_\infty + \int_{|x| \leq R} |u(x, t) - u_\infty| \rho(x) dx,$$

and using the L^1_{loc} convergence we get:

$$\limsup_{t \rightarrow \infty} \int_{\mathbb{R}^N} |u(x, t) - u_\infty| \rho(x) dx \leq 2\varepsilon \|u_0\|_\infty.$$

Since ε is arbitrary, we get that the limit is zero, which ends the proof. \square

5. Unbounded solutions

In this section we derive some results for unbounded initial data and solutions. Let us mention that in the case $\rho \equiv 1$, further results are to be found in [1]. But here we still face the problem of the space inhomogeneity implied by ρ .

5.1. An existence result for unbounded solutions

Proposition 5.1. *Let us assume that (2.2) holds. Then for any non-negative $u_0 \in C^0(\mathbb{R}^N)$, satisfying $u_0(x) \leq \psi(x, 0)$ there exist a classical solution u of (1.1) with $u(x, 0) = u_0(x)$.*

Proof. Let us first consider an approximation $u_{0n} = u_0 \cdot \chi_n$ where χ_n is smooth, non-negative, compactly supported and $\chi_n \nearrow 1$. Let u_n be the solution of (1.1) with initial data u_{0n} given by Proposition 3.3, then by applying the comparison result for bounded solutions, the sequence u_n is nondecreasing.

Notice that ψ is not bounded, but this is allowed in Lemma 2.7, which gives:

$$u_n(x, t) \leq \psi(x, t).$$

Hence the sequence u_n converges to some u and we are able to pass to the limit in $J * u_n$ by dominated convergence, using that ψ is integrable with respect to translations of J .

Using now Lemma 2.3, we deduce that u is a classical solution of (1.1) and the initial data of u is u_0 . \square

Remark 5.2. This construction does in fact give a minimal solution: if u_1 is any other solution, then it can be used as a supersolution for any u_n and passing to the limit shows that $u \leq u_1$. One can think that if we restrict the initial data to grow strictly less than ψ , then uniqueness holds because the comparison argument is valid in this class. However, it is not clear whether the constructed solution enters this class unless we know more about u_0 , see [1].

5.2. Asymptotic behaviour for unbounded solutions when ρ is integrable

We prove now that if u_0 is integrable with respect to ρ , the isothermisation phenomenon occurs (whether infinite or not). Notice that we gave sufficient conditions for existence of a minimal solution in the previous section. The first result is the following:

Proposition 5.3. *Let $u_0 \in C^0(\mathbb{R}^N) \cap L^1(\rho)$, $\rho \in L^1(\mathbb{R}^N)$ and assume there exists a solution u such that $u(x, 0) = u_0(x)$. Then, if \underline{u} denotes the minimal solution, we have*

$$\lim_{t \rightarrow \infty} \int_{\mathbb{R}^N} |\underline{u}(x, t) - \mathbb{E}_\rho(u_0)| \rho(x) dx = 0.$$

Proof. Let \underline{u} be the minimal solution and let us use the same monotone approximations that were used in Proposition 5.1. Since u_n is bounded, Lemma 2.7 implies that we have a bound from above:

$$u(x, t) \geq u_n(x, t) \text{ in } \mathbb{R}^N \times [0, \infty).$$

But since $u_n(x, 0) \in L^\infty(\mathbb{R}^N)$, Theorem 3.4 implies

$$\int_{\mathbb{R}^N} u_n(x, t)\rho(x) \, dx = \int_{\mathbb{R}^N} u_n(x, 0)\rho(x) \, dx.$$

Moreover, the convergence of u_n to \underline{u} is monotone, so we can pass to the limit in the above equation to obtain

$$\int_{\mathbb{R}^N} \underline{u}(x, t)\rho(x) \, dx = \int_{\mathbb{R}^N} \underline{u}(x, 0)\rho(x) \, dx.$$

Using the above three equations we get that

$$\int_{\mathbb{R}^N} |\underline{u}(x, t) - \mathbb{E}_\rho(u_0)|\rho(x) \, dx \leq I_1 + I_2 + I_3,$$

where,

$$I_1 = \int_{\mathbb{R}^N} |\underline{u}(x, t) - u_n(x, t)|\rho(x) \, dx = \int_{\mathbb{R}^N} (\underline{u}(x, 0) - u_n(x, 0))\rho(x) \, dx.$$

$$I_2 = \int_{\mathbb{R}^N} |u_n(x, t) - \mathbb{E}_\rho(u_0\chi_n)|\rho(x) \, dx,$$

$$I_3 = \int_{\mathbb{R}^N} |\mathbb{E}_\rho(u_0\chi_n) - \mathbb{E}_\rho(u_0)|\rho(x) \, dx.$$

Observe that I_1 and I_3 are independents of t and tend to zero as $n \rightarrow \infty$. Moreover, u_n satisfies the hypothesis of Theorem 1.2 so that I_2 tends to zero as $t \rightarrow \infty$. Therefore, we first have

$$\limsup_{t \rightarrow +\infty} \int_{\mathbb{R}^N} |\underline{u}(x, t) - \mathbb{E}_\rho(u_0)|\rho(x) \, dx \leq I_1 + I_3,$$

so that taking the limit as $n \rightarrow \infty$ yields the desired result. □

In the case when $u_0 \notin L^1(\rho)$, then infinite isothermalisation occurs:

Proposition 5.4. *Let $\rho \in L^1(\mathbb{R}^N)$ and $u_0 \in C^0(\mathbb{R}^N)$ be such that $u_0 \notin L^1(\rho)$. Then for any solution u with initial data u_0 and any $1 \leq p < \infty$, the following asymptotic behaviour holds:*

$$\lim_{t \rightarrow +\infty} u(x, t) = +\infty \text{ in } L^p_{\text{loc}}(\mathbb{R}^N).$$

Proof. As before, if there exists a solution, then we can approximate the minimal solution \underline{u} by the family u_n used in Proposition 5.1. Since this approximation is monotone,

$$\underline{u}(x, t) \geq u_n(x, t) \text{ in } \mathbb{R}^N \times [0, \infty).$$

But $u_n(x, 0)$ satisfies the hypotheses of Theorem 1.2 so that

$$\liminf_{t \rightarrow +\infty} \underline{u}(x, t) \geq \lim_{t \rightarrow +\infty} u_n(x, t) = c_n,$$

where $c_n = \mathbb{E}_\rho(u_n(x, 0))$, the limit holding in all $L^p_{\text{loc}}(\mathbb{R}^N)$. Hence passing to the limit as $n \rightarrow +\infty$, we obtain the result for \underline{u} since $c_n \rightarrow \mathbb{E}_\rho(u_0) = +\infty$, thus the same holds for any other solution. \square

Theorems 1.4 and 1.5 follow from the conjunction of Propositions 5.1, 5.3 and 5.4.

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