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L^p estimates for the wave equation associated to the Grushin operator

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Abstract. We prove that the solution of the wave equation associated to the Grushin operator $G = -\Delta - |x|^2\partial_t^2$ is bounded on $L^p(\mathbb{R}^{n+1})$, with $1 < p < \infty$, when $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{n+2}$.

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1. Introduction

Consider the solution of the initial value problem

$$\partial_t^2 u(x,t) = \Delta u(x,t), \; u(x,0) = 0, \; \partial_t u(x,0) = f(x)$$

for the standard wave equation associated to the Laplacian on $\mathbb{R}^n$. Representing the solution as $u(x,t) = \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}} f(x)$ one can investigate the L^p mapping properties of $\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}$. This problem has been studied by several authors: Peral [9] and Miyachi [5] have obtained the sharp range of p, viz. $\left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{1}{n-1}$, for which $\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}$ is bounded on $L^p(\mathbb{R}^n)$. Other L^p-L^q estimates were considered, *e.g.*, by Strichartz [12]. The case of the Hermite operator $-\Delta + |x|^2$ has been treated by one of us [7], and more general operators of the form $-\Delta + V$ by Zhong [16]. In all these cases the optimal range of p for which the solution operator is bounded on $L^p(\mathbb{R}^n)$ is known.

All the operators mentioned above are elliptic, but results for operators from the subelliptic case are also available. The wave equation associated to the sub-laplacian $\mathcal{L}$ on the Heisenberg group $\mathbb{H}^n$ has been studied by Müller and Stein [6]. They have shown that the solution operator $\frac{\sin t\sqrt{\mathcal{L}}}{\sqrt{\mathcal{L}}}$ is bounded on $L^p(\mathbb{H}^n)$ for all p satisfying $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{d-1}$ where $d = 2n+1$ is the Euclidean dimension of $\mathbb{H}^n$. The

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interesting point to note here is the appearance of $d-1$ rather that $Q-1$, where $Q=2n+2$ is the homogeneous dimension. The weaker result with $Q-1$ in place of $d-1$ is known from earlier works. Also when one considers only functions on the Heisenberg group which are band limited in the central variable, the range can be further extended to $\left|\frac{1}{p}-\frac{1}{2}\right|<\frac{1}{2n-1}$, as was shown in [8].

In this article we are interested in the wave equation associated to the Grushin operator $G=-\Delta-|x|^2\partial_t^2$ on $\mathbb{R}^{n+1}$. Though this operator is very similar to the sublaplacian with a very explicit spectral decomposition, the study of spectral multipliers poses formidable problems due to the lack of a group structure on $\mathbb{R}^{n+1}$ compatible with the operator. However, G can be obtained from $\mathcal{L}$ on the Heisenberg group via a certain representation and hence in principle transference techniques can be used to prove weaker versions of multiplier theorems. As the dimension of $\mathbb{H}^n$ is $2n+1$ whereas G lies on an $(n+1)$-dimensional space, results obtained via transference are far from optimal. In a recent work [3] the authors have studied multipliers associated to G using operator-valued Fourier multipliers.

The study of the wave equation associated to the Grushin operator in one dimension has been initiated by Ralf Meyer [4]. In his unpublished thesis written under the guidance of Detlef Müller, he has proved the following theorem. He considers the class of functions which are supported in $S_{\mathcal{C}_1}=\{(x,t):|x|\le\mathcal{C}_1\}$.

Theorem 1.1. *For every $\mathcal{C}_1$, $s>0$, $1\le p\le\infty$ and $\alpha>\left|\frac{1}{p}-\frac{1}{2}\right|$ there exists a constant $C=C^{\alpha}_{p,s,\mathcal{C}_1}$ such that for all $f\in L^p(\mathbb{R}^2)$ with support contained in $S_{\mathcal{C}_1}$ the estimates*

$$\left\|\frac{\cos s\sqrt{G}}{(1+G)^{\alpha/2}}f\right\|_{L^p(\mathbb{R}^2)}\le C\|f\|_{L^p(\mathbb{R}^2)},$$

and

$$\left\|\frac{\sin s\sqrt{G}}{\sqrt{G}(1+G)^{(\alpha-1)/2}}f\right\|_{L^p(\mathbb{R}^2)}\le C\|f\|_{L^p(\mathbb{R}^2)}$$

are valid.

Meyer has proved the above theorem following the approach used by Müller and Stein for the Heisenberg group. By a very careful analysis of certain kernels he obtained extremely delicate estimates which were possible only under some assumptions on the support. It is almost impossible either to get rid of this assumption or to use the same method in higher dimensions. Fortunately, there is an alternative approach that we have used elsewhere in studying multipliers for the Grushin operator. This approach allows us to prove L^p estimates for the wave equation associated to higher-dimensional Grushin operators. The idea is to consider multipliers for G as operator-valued multipliers for the one-dimensional Euclidean Fourier transform. To elaborate on this let us consider the spectral decomposition of G. Let

$$f^\lambda(x)=\int_{-\infty}^{\infty}f(x,t)e^{i\lambda t}\,dt$$

stand for the inverse Fourier transform of $f(x,t)$ in the variable t. Then by applying G to the inversion formula

$$f(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda t} f^{\lambda}(x)d\lambda$$

we see that

$$Gf(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda t} H(\lambda) f^{\lambda}(x)d\lambda$$

where $H(\lambda) = -\Delta + \lambda^2|x|^2$ is the scaled Hermite operator on $\mathbb{R}^n$. The spectral decomposition of $H(\lambda)$ is explicitly known and given by

$$H(\lambda) = \sum_{k=0}^{\infty}(2k+n)|\lambda| P_k(\lambda)$$

where the $P_k(\lambda)$'s are the Hermite projections, see [13]. (We will say more about these projections later in the paper.) Consequently, the spectral decomposition of G is written as

$$Gf(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda t}\left(\sum_{k=0}^{\infty}(2k+n)|\lambda| P_k(\lambda) f^{\lambda}(x)\right) d\lambda.$$

Given a bounded function m on the spectrum of G, which is just the half-line $[0,\infty)$, we can define $m(G)$ by the spectral theorem. In view of the above decomposition we see that

$$m(G)f(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda t} m(H(\lambda)) f^{\lambda}(x)d\lambda$$

where the Hermite multiplier $m(H(\lambda))$ is given by

$$m(H(\lambda)) = \sum_{k=0}^{\infty} m((2k+n)|\lambda|) P_k(\lambda).$$

Set $X = L^p(\mathbb{R}^n)$ and identify $L^p(\mathbb{R}^{n+1})$ to $L^p(\mathbb{R}, X)$, the L^p space of Banach space-valued functions on $\mathbb{R}$. With this identification we see that $m(G)$ can be considered as a Fourier multiplier on $\mathbb{R}$ for X-valued functions, the multiplier being given by $m(H(\lambda))$. Of course, we need to assume that the $m(H(\lambda))$'s are uniformly bounded on $X = L^p(\mathbb{R}^n)$ even for the boundedness of $m(G)$ on $L^2(\mathbb{R}, X)$. Further conditions are needed to guarantee the boundedness of $m(G)$ on $L^p(\mathbb{R}, X)$.

Fortunately for us the problem of operator-valued multipliers has been studied by L. Weis [15], who has obtained some sufficient conditions. The following theorem has been proved in a slightly more general set-up. Given a function m taking values in $B(X, Y)$, the space of bounded linear operators from X into Y, one can define

$$T_m f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda t} m(\lambda)\hat{f}(\lambda)d\lambda$$

for all $f \in L^2(\mathbb{R}, X)$. This operator is clearly bounded from $L^2(\mathbb{R}, X)$ into $L^2(\mathbb{R}, Y)$ provided $m(\lambda)$ is uniformly bounded. For such operators we have the following result:

Theorem 1.2. *Let X and Y be UMD spaces. Let $m : \mathbb{R}^* \to B(X, Y)$ be a differentiable function such that the families $\{m(\lambda) : \lambda \in \mathbb{R}^*\}$ and $\{\lambda \frac{d}{d\lambda} m(\lambda) : \lambda \in \mathbb{R}^*\}$ are R-bounded. Then m defines a Fourier multiplier which is bounded from $L^p(\mathbb{R}, X)$ into $L^p(\mathbb{R}, Y)$ for all $1 < p < \infty$.*

Note that mere uniform boundedness of $m(\lambda)$ and $\lambda \frac{d}{d\lambda} m(\lambda)$ are not enough to guarantee the L^p boundedness of the Fourier multiplier. However, as the reader may recall, they are sufficient in the scalar case. In most applications of the above theorem, the crux of the matter lies in proving the R-boundedness of these families. For our main result we only need to use this theorem when $X = Y = L^p(\mathbb{R}^n)$, in which case the R-boundedeness is equivalent to vector-valued inequalities for $m(\lambda)$ and $\lambda \frac{d}{d\lambda} m(\lambda)$. Indeed, the R-boundedness of a family of operators $T(\lambda)$ is equivalent to the inequality

$$\left\| \left(\sum_{j=1}^{\infty} |T(\lambda_j) f_j|^2 \right)^{\frac{1}{2}} \right\|_p \le C \left\| \left(\sum_{j=1}^{\infty} |f_j|^2 \right)^{\frac{1}{2}} \right\|_p$$

for all possible choices of $\lambda_j \in \mathbb{R}^*$ and $f_j \in L^p(\mathbb{R}^n)$. Thus we only need to verify this vector-valued inequality for the two families in the theorem.

We consider the following initial value problem for the wave equation:

$$\begin{aligned} &\partial_s^2 u(x, t; s) + G u(x, t; s) = 0 \\ &u(x, t; 0) = 0, \ \partial_s u(x, t; 0) = f(x, t). \end{aligned} \tag{1.1}$$

Using the functional calculus for G, it is easy to see that the solution of the above equation is given by

$$u(x, t; s) = \frac{\sin s\sqrt{G}}{\sqrt{G}} f(x, t).$$

Since G is a homogeneous operator of degree $n+2$ under the nonisotropic dilation $D_s f(x, t) = f(sx, s^2 t)$, it is enough to consider the case $s = 1$. Our main result is the following theorem:

Theorem 1.3. *Let $n \ge 2$. The operator $\frac{\sin\sqrt{G}}{\sqrt{G}}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for all p satisfying $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{n+2}$.*

Note that in the above theorem the homogeneous dimension $n+2$ occurs. We believe that the optimal result is that in which $n+2$ is replaced by n. The Fourier multiplier corresponding to $\frac{\sin s\sqrt{G}}{\sqrt{G}}$ is given by $\frac{\sin s\sqrt{H(\lambda)}}{\sqrt{H(\lambda)}}$, which is precisely the solution operator for the wave equation associated to the Hermite operator. For a fixed λ, boundedness of this operator on $L^p(\mathbb{R}^n)$ is known for the range $|\frac{1}{p}-\frac{1}{2}| < \frac{1}{n}$, see [7,16]. What we need to prove is the R-boundedness of the above family as well as the same for λ times its derivative. The major part of this paper is concerned with this problem.

To prove Theorem 1.3 we consider a more general class of oscillatory multipliers of G, viz. $\frac{J_\alpha(\sqrt{G})}{\sqrt{G}^\alpha}$ for $\Re\alpha \geq -1/2$, where J_α is the Bessel function of order α. This is a densely defined analytic family of operators acting on $L^p(\mathbb{R}^{n+1})$. When $\alpha = 1/2$ we get back the solution operator of the wave equation and hence Theorem 1.3 descends once we prove the following:

Theorem 1.4. *Let $n \geq 2$ and $1 < p < \infty$. Then $\frac{J_\alpha(\sqrt{G})}{\sqrt{G}^\alpha}$ is bounded on $L^p(\mathbb{R}^{n+1})$ whenever $\Re(\alpha) > (n+2)|\frac{1}{p} - \frac{1}{2}| - \frac{1}{2}$.*

Recall that the Bessel functions $J_\alpha(t)$ are defined even for complex values of α. In fact the Poisson integral representation

$$J_\alpha(t) = \frac{(t/2)^\alpha}{\Gamma((2\alpha+1)/2)\Gamma(1/2)} \int_{-1}^{1} e^{its}(1-s^2)^{(2\alpha-1)/2} ds$$

is valid as long as $\Re(\alpha) > -1/2$. Moreover, when $\alpha = \beta + \delta + i\gamma$ with $\beta > -1/2, \delta > 0, \gamma \in \mathbb{R}$, we have the identity

$$\frac{J_\alpha(t)}{t^\alpha} = \frac{2^{1-\delta-i\gamma}}{\Gamma(\delta+i\gamma)} \int_0^1 \frac{J_\beta(st)}{(st)^\beta}(1-s^2)^{\delta+i\gamma-1} s^{2\beta+1} ds.$$

Thus we see that $\frac{J_\alpha(\sqrt{G})}{\sqrt{G}^\alpha}$ is an analytic family of operators which is bounded on $L^2(\mathbb{R}^{n+1})$ whenever $\Re(\alpha) \geq -\frac{1}{2}$. Using the above formula we can also check that the family is admissible. Hence we can appeal to Stein's analytic interpolation theorem to obtain Theorem 1.4 as soon as we get the following:

Theorem 1.5. *Let $n \geq 2$. Then $\frac{J_\alpha(\sqrt{G})}{\sqrt{G}^\alpha}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for all $1 < p < \infty$ provided $\Re(\alpha) > \frac{n+1}{2}$.*

Thus by setting $m_\alpha(u) = \frac{J_\alpha(\sqrt{u})}{\sqrt{u}^\alpha}$ we study the R-boundedness of the family $m_\alpha(H(\lambda))$ when $\Re(\alpha) > \frac{n+1}{2}$. We also need to study the R-boundedness of $\lambda \frac{d}{d\lambda} m_\alpha(H(\lambda))$. We address these problems in the next two sections.

We conclude this introduction with the following remarks. In all the theorems stated above we have assumed $n \geq 2$. The reason is the following: in the proof of Proposition 2.2 below, which is used Theorem 2.1, we need to use the estimate $\Phi_k(x,x) \leq C(2k+n)^{n/2-1}$, for $x \in \mathbb{R}^n$, which is valid only when $n \geq 2$. Here $\Phi_k(x,y)$ is the kernel of the projection P_k associated to the Hermite operator H. In the one-dimensional case $\Phi_k(x,x) = (h_k(x))^2$, where h_k is the

k-th Hermite function on $\mathbb{R}$, behaves like $k^{-1/6}$ and hence we do not get an analogue of Proposition 2.2. However, when B is a compact subset of $\mathbb{R}$ we do have $\sup_{x\in B}(h_k(x))^2 \le C(2k+1)^{-1/2}$ and hence it is possible to prove a version of Theorem 1.3 for the operator $\chi_B \frac{\sin\sqrt{G}}{\sqrt{G}} \chi_B$. We do not pursue this here as the result of Meyer is stronger than what we can prove.

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2. A maximal theorem for $m_\alpha(H(\lambda))$

As we mentioned at the end of the introduction we are interested in proving vector-valued inequalities for the families $T_\alpha(\lambda) = m_\alpha(H(\lambda))$ and $\lambda \frac{d}{d\lambda} T_\alpha(\lambda)$. In order to do so we need a maximal theorem for the family $T_\alpha(\lambda)$, which means that we have to get estimates for the maximal function $T_\alpha^* f(x) = \sup_{\lambda\in\mathbb{R}^*} |T_\alpha(\lambda) f(x)|$. For $1 \le p < \infty$ let $M_p f(x) = (M|f|^p(x))^{1/p}$ where Mf is the Hardy-Littlewood maximal function. Let $\alpha = x + iy$. We call $c(\alpha)$ an admissible function (or function of admissible growth) if

$$\sup_{y\in\mathbb{R}} e^{-b|y|} \log(|c(\alpha)|) < \infty$$

for some $b < \pi$. With this terminolgy we have the following:

Theorem 2.1. *Let $n \ge 2$. Then:*

(i) *For $\Re(\alpha) > \frac{(n-1)}{2}$ we have*

$$T_\alpha^* f(x) \le C_2(\alpha) M_2 f(x);$$

(ii) *For $\Re(\alpha) > n - \frac{1}{2}$ we have $T_\alpha^* f(x) \le C_1(\alpha) M f(x)$, where the functions C_1 and C_2 are admissible.*

This theorem will be proved by obtaining good estimates for the kernel of $T_\alpha(\lambda)$. We briefly recall some details from the spectral theory of the Hermite operator $H(\lambda)$. Let Φ_α, with $\alpha \in \mathbb{N}^n$, stand for the normalised Hermite functions on $\mathbb{R}^n$, which are eigenfunctions of $H = H(1)$ with eigenvalues $2|\alpha| + n$ and form an orthonormal basis for $L^2(\mathbb{R}^n)$. It follows that for $\lambda \in \mathbb{R}^*$ the functions $\Phi_\alpha^\lambda(x) = |\lambda|^{n/4} \Phi_\alpha(|\lambda|^{1/2} x)$ satisfy $H(\lambda)\Phi_\alpha^\lambda = (2|\alpha| + n)|\lambda|\Phi_\alpha^\lambda$. The spectral projections $P_k(\lambda)$ of $H(\lambda)$ are defined by

$$P_k(\lambda) f = \sum_{|\alpha|=k} (f, \Phi_\alpha^\lambda)\Phi_\alpha^\lambda.$$

It follows that $P_k(\lambda) = \delta_\lambda P_k \delta_\lambda^{-1}$ where $\delta_\lambda f(x) = f(|\lambda|^{\frac{1}{2}} x)$ and $P_k = P_k(1)$. Therefore, $m(H(\lambda)) = \delta_\lambda m(|\lambda| H)\delta_\lambda^{-1}$ for any multiplier m.

The above remarks imply that

$$\frac{J_\alpha\left(\sqrt{H(\lambda)}\right)}{\sqrt{H(\lambda)}^\alpha} f(x) = \delta_\lambda \frac{J_\alpha\left(\sqrt{|\lambda| H}\right)}{\left(\sqrt{|\lambda| H}\right)^\alpha} \delta_\lambda{}^{-1} f(x).$$

In view of this relation, a moment's thought reveals that it is enough to consider the maximal function

$$\sup_{t>0} \left| \frac{J_\alpha\left(t\sqrt{H}\right)}{\left(t\sqrt{H}\right)^\alpha} f(x) \right|$$

and establish the estimates stated in the theorem above.

By the definition

$$\frac{J_\alpha\left(t\sqrt{H}\right)}{\left(t\sqrt{H}\right)^\alpha} f = \sum_{k=0}^{\infty} \frac{J_\alpha\left(t\sqrt{2k+n}\right)}{\left(t\sqrt{2k+n}\right)^\alpha} P_k f$$

and hence it follows that $\frac{J_\alpha\left(t\sqrt{H}\right)}{\left(t\sqrt{H}\right)^\alpha}$ is an integral operator whose kernel $K_t^\alpha(x, y)$ is given by

$$K_t^\alpha(x, y) = \sum_{k=0}^{\infty} \frac{J_\alpha\left(t\sqrt{2k+n}\right)}{\left(t\sqrt{2k+n}\right)^\alpha} \Phi_k(x, y),$$

where $\Phi_k(x, y) = \sum_{|\beta|=k} \Phi_\beta(x)\Phi_\beta(y)$ is the kernel of P_k. We require the following estimates on the kernel K_t^α.

Proposition 2.2. *Let $n \geq 2$. Then:*

(i) *For $\Re(\alpha) > \frac{n-1}{2}$ we have*

$$\int_{|x-y|>r} |K_t^\alpha(x, y)|^2 dy \leq C_2(\alpha) t^{-n} (1 + rt^{-1})^{-2\Re(\alpha)-1};$$

(ii) *For $\Re(\alpha) > n - \frac{1}{2}$ we have*

$$\sup_{|x-y|>r} |K_t^\alpha(x, y)| \leq C_1(\alpha) t^{-n} (1 + rt^{-1})^{-\Re(\alpha)-1/2}$$

where C_1 and C_2 are functions of admissible growth.

Assuming the proposition for a moment, we complete the proof of Theorem 2.1. For $x \in \mathbb{R}^n$ we define $f_k(y) = \chi_{\{y:2^k<|x-y|\le 2^{k+1}\}}(y)f(y)$, for $k \in \mathbb{Z}$, so that $f = \sum_{k=-\infty}^{\infty} f_k$ and

$$\frac{J_\alpha(t\sqrt{H})}{(t\sqrt{H})^\alpha}f(x) = \sum_{k=-\infty}^{\infty}\int_{\mathbb{R}^n} K_t^\alpha(x,y)f_k(y)dy.$$

After applying the Cauchy-Schwarz inequality to each term in the sum we see that $\frac{J_\alpha(t\sqrt{H})}{(t\sqrt{H})^\alpha}f(x)$ is bounded by

$$\sum_{k=-\infty}^{\infty} 2^{(k+1)n/2}\left(\int_{|x-y|>2^k}|K_t^\alpha(x,y)|^2dy\right)^{\frac{1}{2}}\left(\frac{1}{2^{(k+1)n}}\int_{|x-y|\le 2^{k+1}}|f(y)|^2dy\right)^{\frac{1}{2}}.$$

As the second factor inside the summation is bounded by $M_2 f(x)$, in view of Proposition 2.2 we have, whenever $\Re(\alpha) > \frac{n-1}{2}$, the estimate

$$\left|\frac{J_\alpha(t\sqrt{H})}{(t\sqrt{H})^\alpha}f(x)\right| \le C(\alpha)\left(\sum_{k=-\infty}^{\infty}(2^kt^{-1})^{n/2}(1+2^kt^{-1})^{-\Re\alpha-1/2}\right)M_2 f(x).$$

Thus we are left to show that

$$G(t) := \sum_{k=-\infty}^{\infty}(2^kt^{-1})^{n/2}(1+2^kt^{-1})^{-\Re\alpha-1/2}$$

is a uniformly bounded function of $t > 0$. Note that $G(2^it) = G(t)$, for all $i \in \mathbb{Z}$ and hence it is enough to prove the boundedness of G on the interval $[1,2]$. But G is a continuous function on $[1,2]$ as the series converges uniformly on this interval when $\Re(\alpha) > \frac{n-1}{2}$.

This proves part (i) of Theorem 2.1. To prove (ii) we proceed as above and use the second estimate of Proposition 2.2, which is valid when $\Re(\alpha) > n - \frac{1}{2}$. The details are left to the reader. Theorem 2.1 is established.

We now turn our attention to the proof of Proposition 2.2. We will treat the cases $rt^{-1} \le 1$ and $rt^{-1} > 1$ separately. In the first case we only need to show that

$$\int_{|x-y|>r}|K_t^\alpha(x,y)|^2dy \le C(\alpha)t^{-n}.$$

Since

$$\int_{|x-y|>r}|K_t^\alpha(x,y)|^2dy \le \int_{\mathbb{R}^n}|K_t^\alpha(x,y)|^2dy,$$

we will actually estimate the second integral in the above inequality. Recalling the definition of $K_t^\alpha(x,y)$ and using the orthogonality of the Hermite functions we see that

$$\int_{\mathbb{R}^n} |K_t^\alpha(x,y)|^2 dy = \sum_{k=0}^{\infty} \left| \frac{J_\alpha\left(t\sqrt{2k+n}\right)}{\left(t\sqrt{2k+n}\right)^\alpha} \right|^2 \Phi_k(x,x).$$

Splitting the sum into two parts we first consider

$$\sum_{j=1}^{\infty} \sum_{2^{-j}<t\sqrt{2k+n}\le 2^{-j+1}} \left| \frac{J_\alpha\left(t\sqrt{2k+n}\right)}{\left(t\sqrt{2k+n}\right)^\alpha} \right|^2 \Phi_k(x,x).$$

As $\left|\frac{J_\alpha(s)}{s^\alpha}\right| \le c(\alpha)$ [14], where $c(\alpha)$ is an admissible function of α, for all $s \ge 0$ the above sum is bounded by

$$c(\alpha)^2 \sum_{j=1}^{\infty} \sum_{2^{-j}<t\sqrt{2k+n}\le 2^{-j+1}} \Phi_k(x,x).$$

Finally, we make use of the estimate $\Phi_k(x,x) \le C(2k+n)^{n/2-1}$ proved in [13, Lemma 3.2.2, Chapter 3], valid for $n \ge 2$, to see that the above sum is bounded by

$$\sum_{j=1}^{\infty} \left(2^{-2j}t^{-2}\right)^{n/2} \le Ct^{-n}$$

which takes care of the first sum.

To estimate the second sum, namely

$$\sum_{j=0}^{\infty} \sum_{t\sqrt{2k+n}\sim 2^j} \left| \frac{J_\alpha(t\sqrt{2k+n})}{\left(t\sqrt{2k+n}\right)^\alpha} \right|^2 \Phi_k(x,x)$$

we make use of the estimate

$$\left| \frac{J_\alpha\left(t\sqrt{2k+n}\right)}{\left(t\sqrt{2k+n}\right)^\alpha} \right| \le c(\alpha)\left(t\sqrt{2k+n}\right)^{-\Re(\alpha)-1/2},$$

when $t\sqrt{2k+n} \ge 1$. As before this leads to the estimate

$$c(\alpha)^2 \sum_{j=0}^{\infty} \sum_{2^j<t\sqrt{2k+n}\le 2^{j+1}} \left(t\sqrt{2k+n}\right)^{-2\Re(\alpha)-1} (2k+n)^{\frac{n}{2}-1}.$$

After simplifying further this sum we get the estimate

$$c(\alpha)^2 t^{-n} \sum_{j=0}^{\infty} 2^{-2j(\Re(\alpha)+\frac{1-n}{2})}.$$

The sum over j converges if and only if $\Re(\alpha) > \frac{n-1}{2}$. This takes care of the second sum. Thus we have proved the required estimate when $rt^{-1} \leq 1$.

We now treat the second case, namely $rt^{-1} > 1$. We estimate the integral when $\Re(\alpha) > \frac{n-1}{2}$ first. Note that it is enough to prove the estimate

$$\int_{|x-y|>r} |K_t^\alpha(x,y)|^2 dy \leq c(\alpha) t^{-n+2m} r^{-2m}$$

for some integer $m > \Re(\alpha) + \frac{1}{2}$. Since

$$\int_{|x-y|>r} |K_t^\alpha(x,y)|^2 dy \leq r^{-2m} \int ||x-y|^m K_t^\alpha(x,y)|^2 dy,$$

it is enough to prove

$$\int |(x-y)^\beta K_t^\alpha(x,y)|^2 dy \leq c(\alpha) t^{-n+2m}$$

for all $\beta \in \mathbb{N}^n$ with $|\beta| = m$. In order to do this we make use of [13, Lemma 3.2.3], that we state below for the convenience of the reader.

Given a function ψ defined on $[0,\infty)$ consider the kernel M_ψ defined by

$$M_\psi(x,y) = \sum_{\mu \in \mathbb{N}^n} \psi(|\mu|) \Phi_\mu(x) \Phi_\mu(y).$$

Let $\Delta\psi(s) = \psi(s+1) - \psi(s)$ be the forward finite difference and let $\Delta^k \psi$ be defined inductively. Let $\Delta^k M_\psi$ stand for the kernel $M_{\Delta^k \psi}$. We also define $B_j = -\partial_{y_j} + y_j$, and $A_j = -\partial_{x_j} + x_j$ for $j = 1, 2, \cdots, n$. For multi-indices μ, we define A^μ, B^μ in the usual manner. With this notation we have the following:

Lemma 2.3. *For any multi-index $\beta \in \mathbb{N}^n$ we have*

$$(x-y)^\beta M_\psi(x,y) = \sum_{\gamma,\mu} C_{\gamma,\mu} (B-A)^\gamma \Delta^{|\mu|} M_\psi(x,y),$$

where the sum extends over all multi-indices μ and γ satisfying $2\mu_j - \gamma_j = \beta_j$, and $\mu_j \leq \beta_j$.

Let us fix $\beta \in \mathbb{N}^n$ with $|\beta| = m$. In view of the above lemma we have

$$(x-y)^\beta K_t^\alpha(x,y) = \sum_{\gamma,k} c_{\gamma k} (B-A)^\gamma \Delta^k M_\psi(x,y)$$

where $\psi(|\mu|) = \frac{J_\alpha(t\sqrt{2|\mu|+n})}{(t\sqrt{2|\mu|+n})^\alpha}$ and the sum extends over all γ and k with $|\gamma| = 2k - m$, and $k \leq m$. After expanding $(B-A)^\gamma$ the above becomes a finite linear combination of terms of the form

$$\sum_\mu \Delta^k \psi(|\mu|) A^\tau \Phi_\mu(x) B^\sigma \Phi_\mu(y)$$

where $|\tau| + |\sigma| = |\gamma|$. By the mean value theorem we can write

$$\triangle^k \psi(|\mu|) = \int_0^1 \cdots \int_0^1 \psi^{(k)}(|\mu| + s_1 + \cdots + s_k) ds_1 ds_2 \cdots ds_k,$$

and hence it is enough to prove that

$$\int_{\mathbb{R}^n} \left| \sum_\mu \psi^{(k)}(|\mu|) A^\tau \Phi_\mu(x) B^\sigma \Phi_\mu(y) \right|^2 dy \le C t^{-n+2m}$$

for each τ, σ and k as above.

We make use of the facts

$$A_j \Phi_\mu(x) = (2|\mu_j| + 2)^{\frac{1}{2}} \Phi_{\mu+e_j}(x), \quad B_j \Phi_\mu(y) = (2|\mu_j| + 2)^{\frac{1}{2}} \Phi_{\mu+e_j}(y)$$

(see [13]) where e_j are the coordinate vectors. In view of this the above integral is dominated by

$$\sum_{N=0}^\infty |\psi^{(k)}(N)|^2 (2N + n)^{|\tau|+|\sigma|} \Phi_{N+|\tau|}(x, x).$$

Again, if we use the estimate $\Phi_N(x, x) \le C(2N + n)^{\frac{n}{2}-1}$ and the fact that $|\tau| + |\sigma| = 2k - m$, the above quantity is dominated by

$$\sum_{N=0}^\infty |\psi^k(N)|^2 (2N + n)^{2k-m+\frac{n}{2}-1}.$$

Now recall that $\psi(N) = \frac{J_\alpha(t\sqrt{2N+n})}{(t\sqrt{2N+n})^\alpha}$, so that $\psi^{(k)}(N) = \frac{d^k}{d\lambda^k} \frac{J_\alpha(t\sqrt{\lambda})}{(t\sqrt{\lambda})^\alpha}|_{\lambda=2N+n}$. Making use of the well-known relation

$$\frac{d}{d\lambda} \frac{J_\alpha\left(\sqrt{\lambda}\right)}{\left(\sqrt{\lambda}\right)^\alpha} = -\frac{1}{2} \frac{J_{\alpha+1}\left(\sqrt{\lambda}\right)}{\left(\sqrt{\lambda}\right)^{\alpha+1}},$$

(see [14]) we get

$$\psi^{(k)}(N) = t^{2k} \frac{J_{\alpha+k}\left(t\sqrt{2N+n}\right)}{\left(t\sqrt{2N+n}\right)^{\alpha+k}}.$$

Plugging this in the above expression we get

$$\begin{aligned} &\int \left| \sum_\mu \psi^{(k)}(|\mu|) A^\tau \Phi_\mu(x) B^\sigma \Phi_\mu(y) \right|^2 dy \\ &\le C \sum_{N=0}^\infty \left| t^{2k} \frac{J_{\alpha+k}\left(t\sqrt{2N+n}\right)}{(t\sqrt{2N+n})^{\alpha+k}} \right|^2 (2N + n)^{2k-m+\frac{n}{2}-1}. \end{aligned}$$

As before we estimate the above sum by splitting it into two parts. For the part

$$\sum_{j=1}^{\infty} \sum_{t\sqrt{2N+n}\sim 2^{-j}} \left| t^{2k} \frac{J_{\alpha+k}\left(t\sqrt{2N+n}\right)}{\left(t\sqrt{2N+n}\right)^{\alpha+k}} \right|^2 (2N+n)^{2k-m+\frac{n}{2}-1}$$

we use the boundedness of the Bessel function which results in the estimate

$$c_k(\alpha)^2 \sum_{j=1}^{\infty} t^{4k} \left(\frac{2^{-2j}}{t^2}\right)^{2k-m+\frac{n}{2}} = c_k(\alpha)^2 t^{-n+2m} \sum_{j=1}^{\infty} 2^{-2j(2k-m)} 2^{-nj}.$$

Since $2k - m = |\gamma| \geq 0$ the above sum clearly converges. To treat the sum

$$\sum_{j=0}^{\infty} \sum_{t\sqrt{2N+n}\sim 2^{j}} \left| t^{2k} \frac{J_{\alpha+k}\left(t\sqrt{2N+n}\right)}{\left(t\sqrt{2N+n}\right)^{\alpha+k}} \right|^2 (2N+n)^{2k-m+\frac{n}{2}-1}$$

we make use of the estimate

$$\left| \frac{J_{\alpha+k}(t\sqrt{2N+n})}{\left(t\sqrt{2N+n}\right)^{\alpha+k}} \right| \leq c_k(\alpha) \left(t\sqrt{2N+n}\right)^{-\Re(\alpha+k)-\frac{1}{2}}.$$

Using the above estimate and simplifying we get

$$\begin{aligned} c_k(\alpha)^2 \sum_{j=0}^{\infty} t^{4k} 2^{-2j(\Re(\alpha+k)+\frac{1}{2})} \left(\frac{2^{2j}}{t^2}\right)^{2k-m+\frac{n}{2}} \\ = c_k(\alpha)^2 t^{-n+2m} \sum_{j=0}^{\infty} 2^{-2j(\Re(\alpha)+\frac{1-n}{2}))} 2^{2j(k-m)}. \end{aligned}$$

As $k \leq m$, the sum over j is finite as soon as $\Re(\alpha) > \frac{n-1}{2}$.

Thus Proposition 2.2 (i) is completely proved when $\Re(\alpha) > \frac{n-1}{2}$. What remains to be considered is the second part for $\Re(\alpha) > n - \frac{1}{2}$. Here again we consider two cases, namely $rt^{-1} \leq 1$ and $rt^{-1} > 1$. When $rt^{-1} \leq 1$ it is enough to show that

$$\sup_{|x-y|>r} |K_t^\alpha(x, y)| \leq c(\alpha) t^{-n}$$

for $\Re(\alpha) > n - \frac{1}{2}$. Clearly,

$$\sup_{|x-y|>r} |K_t^\alpha(x, y)| \leq \sup_{x,y\in\mathbb{R}^n} |K_t^\alpha(x, y)|.$$

So it suffices to show that

$$\sup_{x,y\in\mathbb{R}^n} |K_t^\alpha(x, y)| \leq c(\alpha) t^{-n}.$$

Using the definition of K_t^α we get that

$$|K_t^\alpha(x,y)| \leq \sum_k \left| \frac{J_\alpha\left(t\sqrt{2k+n}\right)}{\left(t\sqrt{2k+n}\right)^\alpha} \right| \cdot |\Phi_k(x,y)|.$$

Since $\Phi_k(x,y) = \sum_{|\beta|=k} \Phi_\beta(x)\Phi_\beta(y)$, an application of the Cauchy-Schwarz inequality gives us

$$\left| \sum_{|\beta|=k} \Phi_\beta(x)\Phi_\beta(y) \right| \leq \sqrt{\Phi_k(x,x)\Phi_k(y,y)}.$$

For $n \geq 2$ we know from [13] that $\sup_{x\in\mathbb{R}^n} \Phi_k(x,x) \leq (2k+n)^{\frac{n}{2}-1}$. Using this estimate we get that

$$|K_t^\alpha(x,y)| \leq \sum_k \left| \frac{J_\alpha\left(t\sqrt{2k+n}\right)}{\left(t\sqrt{2k+n}\right)^\alpha} \right| (2k+n)^{\frac{n}{2}-1}.$$

Now, proceeding as in the previous part, *i.e.*, splitting the sum into two parts and using the estimates on the Bessel function, we get the desired inequality for $\Re(\alpha) > n - \frac{1}{2}$.

When $rt^{-1} > 1$ it is enough to show that

$$\sup_{|x-y|>r} |K_t^\alpha(x,y)| \leq c(\alpha) t^{-n+\Re(\alpha)+\frac{1}{2}} r^{-\Re(\alpha)-\frac{1}{2}}$$

for $\Re(\alpha) > n - \frac{1}{2}$. As before we only need to show that

$$\sup_{|x-y|>r} |K_t^\alpha(x,y)| \leq c(\alpha) t^{-n+m} r^{-m}$$

for some $m > \Re(\alpha) + \frac{1}{2}$ which in turn will follow once we show that

$$\sum_{|\beta|=m} \sup_{x,y\in\mathbb{R}^n} |(x-y)^\beta K_t^\alpha(x,y)| \leq c(\alpha) t^{-n+m}.$$

Keeping the same notation as in the previous part, the above estimate will follow from the estimates

$$\sup_{x,y\in\mathbb{R}^n} \left| \sum_\mu \psi^{(k)}(|\mu|) A^\tau \Phi_\mu(x) B^\sigma \Phi_\mu(y) \right| \leq c(\alpha) t^{-n+m},$$

where $k \leq m$ and $|\tau| + |\sigma| = 2k - m \geq 0$.

Recalling the action of A_j and B_j on Hermite functions we see that

$$\sum_{|\mu|=N} |A^\tau \Phi_\mu(x) B^\sigma \Phi_\mu(y)| \le C(2N+n)^{\frac{1}{2}(|\tau|+|\sigma|)} \sqrt{\Phi_{N+m}(x,x)\Phi_{N+m}(y,y)}.$$

Using the fact that $|\tau|+|\sigma| = 2k-m$ the estimates on $\Phi_k(x,x)$ lead to

$$\sum_{|\mu|=N} |A^\tau \Phi_\mu(x) B^\sigma \Phi_\mu(y)| \le C(2N+n)^{k-m/2+\frac{n}{2}-1}.$$

Recalling that $\psi^{(k)}(N) = t^{2k} \frac{J_{\alpha+k}(t\sqrt{2N+n})}{(t\sqrt{2N+n})^{\alpha+k}}$ we need to estimate

$$\sum_{N=0}^{\infty} \left| t^{2k} \frac{J_{\alpha+k}\left(t\sqrt{2N+n}\right)}{\left(t\sqrt{2N+n}\right)^{\alpha+k}} \right| (2N+n)^{k-\frac{m}{2}+\frac{n}{2}-1}.$$

As before splitting the above sum into two parts and using the estimates on Bessel functions we get the required estimate for $\Re(\alpha) > n - \frac{1}{2}$. Thus Proposition 2.2 is completely proved.

In the next section when trying to prove the R-boundedness of $\lambda \frac{d}{d\lambda} T_\alpha(\lambda)$ we will encounter the family $H(\lambda) m_{\alpha+1}(H(\lambda))$. Hence we require the following maximal theorem for this family:

Theorem 2.4. *Let $n \ge 2$.*

(i) *For $\Re(\alpha) > \frac{(n+1)}{2}$,*

$$\sup_{\lambda \in \mathbb{R}^*} |H(\lambda) m_{\alpha+1}(H(\lambda)) f(x)| \le C_2(\alpha) M_2 f(x);$$

(ii) *For $\Re(\alpha) > n + \frac{1}{2}$,*

$$\sup_{\lambda \in \mathbb{R}^*} |H(\lambda) m_{\alpha+1}(H(\lambda)) f(x)| \le C_1(\alpha) M f(x)$$

where the functions C_1 and C_2 are admissible.

In order to prove this theorem we need an analogue of Proposition 2.2 for the kernel

$$\tilde{K_t}^\alpha(x,y) = \sum_{k=0}^{\infty} t^2(2k+n) \frac{J_{\alpha+1}\left(t\sqrt{2k+n}\right)}{\left(t\sqrt{2k+n}\right)^{\alpha+1}} \Phi_k(x,y).$$

This kernel is estimated just like the kernel K_t^α. Note that when $t\sqrt{2k+n} \le 1$ both $\frac{J_\alpha(t\sqrt{2k+n})}{(t\sqrt{2k+n})^\alpha}$ and $t^2(2k+n)\frac{J_{\alpha+1}(t\sqrt{2k+n})}{(t\sqrt{2k+n})^{\alpha+1}}$ are bounded. On the other hand when $t\sqrt{2k+n} \ge 1$

$$t^2(2k+n) \left| \frac{J_{\alpha+1}\left(t\sqrt{2k+n}\right)}{\left(t\sqrt{2k+n}\right)^{\alpha+1}} \right| \le C(\alpha) \left(t\sqrt{2k+n}\right)^{-\Re(\alpha)+1/2}$$

and since we are assuming $\Re(\alpha) > \frac{n+1}{2}$ the same estimates as in Proposition 2.2 are satisfied by $\tilde{K_t}^{\alpha}(x, y)$. This takes care of the part when $rt^{-1} \leq 1$. Recall that in the proof of Proposition 2.2 when $rt^{-1} > 1$ we needed some estimates on the derivative of the multiplier. The k-th derivative of the function $t^2u\ m_{\alpha+1}(t^2u)$ at $|\mu|$ is given by

$$t^{2k}\left[k\frac{J_{\alpha+k}\left(t\sqrt{2|\mu|+n}\right)}{\left(t\sqrt{2|\mu|+n}\right)^{\alpha+k}} + t^2(2|\mu|+n)\frac{J_{\alpha+1+k}\left(t\sqrt{2|\mu|+n}\right)}{\left(t\sqrt{2|\mu|+n}\right)^{\alpha+1+k}}\right]$$

which can be estimated in a similar way as in the case of K_t^{α}. We leave the details to the reader.

3. The R-boundedness of $T_\alpha(\lambda)$ and $\frac{d}{d\lambda}T_\alpha(\lambda)$

3.1. The R-boundedness of $m_\alpha(H(\lambda))$

Making use of the maximal theorem proved in the previous section we will now prove the required vector-valued inequalities for the family $T_\alpha(\lambda) = m_\alpha(H(\lambda))$. Using Proposition 2.2 it is possible to get the estimate

$$\left(\int_{|x-y|>r} |K_t^{\alpha}(x, y)|^p dy\right)^{1/p} \leq C_2(\alpha)t^{-n/(2p')}\left(1+rt^{-1}\right)^{-\Re(\alpha)-1/2+n(\frac{1}{p}-\frac{1}{2})}$$

for $1 < p \leq 2$ and $\Re(\alpha) > \frac{n-1}{2}$. This will lead as before to the estimate

$$\sup_{\lambda\in\mathbb{R}^*} |T_\alpha(\lambda)f(x)| \leq C(\alpha)M_p f(x)$$

whenever $p \geq 2$. Unfortunately, this estimate is not good enough to yield the required vector-valued inequality for the family $T_\alpha(\lambda)$. What we can prove is the inequality

$$\left\|\left(\sum_{j=1}^{\infty}|T_\alpha(\lambda_j)f_j|^r\right)^{1/r}\right\|_p \leq C\left\|\left(\sum_{j=1}^{\infty}|f_j|^r\right)^{1/r}\right\|_p$$

for all $r > p \geq 2$. As we need the case $r = 2$ we have to proceed in a different way using analytic interpolation. We first prove the following:

Proposition 3.1. *For any $\varphi \in C_c^{\infty}(\mathbb{R}^n)$ the operator $T_\alpha(\lambda)$ satisfies the following*

$$\int_{\mathbb{R}^n} |T_\alpha(\lambda)f(x)|^2|\varphi(x)|dx \leq c(\alpha)\int_{\mathbb{R}^n} |f(x)|^2 M\varphi(x)dx$$

for $\Re(\alpha) > \frac{n-1}{2}$. Moreover, $c(\alpha)$ is an admissible function of α, and it is independent of the choice of φ and λ.

Proof. We make use of Theorem 2.1 along with a lemma due to Fefferman and Stein [1, Section 3, Lemma 1] which states that

$$\int_{\mathbb{R}^n} Mf(x)^r |\varphi(x)| dx \leq C_r \int |f(x)|^r M\varphi(x) dx,$$

for all $r > 1$ with C_r independent of f and φ. Therefore, for $\Re(\alpha) > \frac{n-1}{2}$ we have, from Theorem 2.1,

$$\int_{\mathbb{R}^n} |T_\alpha(\lambda) f(x)|^p |\varphi(x)| dx \leq C_2(\alpha) \int (M|f|^2(x))^{p/2} |\varphi(x)| dx$$

which using the Fefferman-Stein Lemma yields

$$\int_{\mathbb{R}^n} |T_\alpha(\lambda) f(x)|^p |\varphi(x)| dx \leq C_3(\alpha) \int |f(x)|^p M\varphi(x) dx$$

for all $p > 2$. Similarly, when $\Re(\alpha) > n - \frac{1}{2}$ we have

$$\int_{\mathbb{R}^n} |T_\alpha(\lambda) f(x)|^p |\varphi(x)| dx \leq C_4(\alpha) \int_{\mathbb{R}^n} |f(x)|^p M\varphi(x) dx$$

for all $p > 1$. Thus we see that

$$T_\alpha(\lambda) : L^p(\mathbb{R}^n, M\varphi(x)\ \ dx) \longrightarrow L^p(\mathbb{R}^n, |\varphi(x)|\ \ dx)$$

is bounded for all $p > 2$ if $\Re(\alpha) > \frac{n-1}{2}$ and for all $p > 1$ if $\Re(\alpha) > n - \frac{1}{2}$.

We want to apply Stein's anaytic interpolation theorem [10, Chapter 5, Theorem 4.1] to the family $T_\alpha(\lambda)$. It is easy to see that the norm of $T_\alpha(\lambda)$ is independent of φ and λ in both cases and is an admissible family of operators in α. Fix $\alpha \in \mathbb{C}$ such that $\Re\alpha > \frac{n-1}{2}$ and $\lambda \neq 0$. Let $\delta > 0$ be chosen so that $\Re\alpha = \frac{n-1}{2} + \delta$. Define an analytic family of operators S_z on the strip $S = \{z \in \mathbb{C} : 0 \leq \Re(z) \leq 1\}$ by setting $S_z f = T_{(nz+n-1+\delta)/2}(\lambda)$. Let $\epsilon = \frac{1}{2}(-1 + \sqrt{1 + (8\delta)/n})$ and take $p_0 = 2 + \epsilon$ and $p_1 = 1 + \epsilon$. Then it is clear, from Theorem 2.1, that

$$\int_{\mathbb{R}^n} |S_{iy} f(x)|^{p_0} |\varphi(x)| \quad dx \leq C_1(iy) \int_{\mathbb{R}^n} |f(x)|^{p_0} M\varphi(x) \quad dx$$

and

$$\int_{\mathbb{R}^n} |S_{1+iy} f(x)|^{p_1} |\varphi(x)| \quad dx \leq C_2(1+iy) \int_{\mathbb{R}^n} |f(x)|^{p_1} M\varphi(x) \quad dx,$$

where $C_1(iy)$ and $C_2(1 + iy)$ are admissible functions independent of φ and λ. By interpolation, it follows that $S_{\delta/n}$ is bounded from $L^p(\mathbb{R}^n, M\varphi(x)\ \ dx)$ into $L^p(\mathbb{R}^n, |\varphi(x)|\ \ dx)$ where $\frac{1}{p} = \frac{1-\delta/n}{2+\epsilon} + \frac{\delta/n}{1+\epsilon}$. A simple calculation recalling the definition of ϵ, shows that $p = 2$ and hence $S_{\delta/n} = T_{\frac{n-1}{2}+\delta}(\lambda)$ is bounded from

$L^2(\mathbb{R}^n, M\varphi(x)\ \ dx)$ into $L^2(\mathbb{R}^n, |\varphi(x)|\ \ dx)$ which proves the theorem when α is real.

When α is not real we write $\alpha = \beta + \delta + i\gamma$ where $\beta > \frac{n-1}{2}$, $\delta > 0$, and we make use of the identity

$$\frac{J_{\beta+\delta+i\gamma}\left(\sqrt{H(\lambda)}\right)}{\left(\sqrt{H(\lambda)}\right)^{\beta+\delta+i\gamma}} = \frac{2^{1-\delta-i\gamma}}{\Gamma(\delta+i\gamma)} \int_0^1 \frac{J_\beta\left(s\sqrt{H(\lambda)}\right)}{\left(s\sqrt{H(\lambda)}\right)^\beta} \left(1-s^2\right)^{\delta+i\gamma-1} s^{2\beta+1} ds. \ \square$$

We can now prove the vector-valued inequality for $\{T_\alpha(\lambda)\}_{\lambda\in\mathbb{R}^*}$ thus proving the R-boundedness of $m(H(\lambda))$.

Theorem 3.2. *Let $T_\alpha(\lambda)$ be as defined before. Then for any choice of $\lambda_j \in \mathbb{R}^*$ and $f_j \in L^p(\mathbb{R}^n)$ we have*

$$\left\| \left(\sum_{j=1}^{\infty} |T_\alpha(\lambda_j) f_j|^2 \right)^{\frac{1}{2}} \right\|_p \leq C \left\| \left(\sum_{j=1}^{\infty} |f_j|^2 \right)^{\frac{1}{2}} \right\|_p$$

for all $1 < p < \infty$ provided $\Re(\alpha) > \frac{n-1}{2}$.

Proof. When $p = 2$ the vector-valued inequality follows trivially as $\frac{J_\alpha(\sqrt{(2k+n)|\lambda_j|})}{(\sqrt{(2k+n)|\lambda_j|})^\alpha}$ is uniformly bounded independently of j and k for any α with $\Re\alpha \geq -\frac{1}{2}$. So it follows that $\|T_\alpha(\lambda_j) f_j\|_2 \leq C\|f_j\|_2$, where C is independent of j and hence we get the vector-valued inequality for $p = 2$.

We will now deal with the case $p \neq 2$. Without loss of generality we can assume that $p > 2$, as the case $1 < p < 2$ can be treated using a duality argument. Let $\frac{p}{2} = q$. Clearly,

$$\left\| \left(\sum_{j=1}^{\infty} |T_\alpha(\lambda_j) f_j|^2 \right)^{\frac{1}{2}} \right\|_p = \left\| \sum_{j=1}^{\infty} |T_\alpha(\lambda_j) f_j|^2 \right\|_q^{\frac{1}{2}}.$$

So, it is sufficient to deal with $\| \sum_{j=1}^{\infty} |T_\alpha(\lambda_j) f_j|^2 \|_q$. As we know

$$\left\| \sum_{j=1}^{\infty} |T_\alpha(\lambda_j) f_j|^2 \right\|_q = \sup_{\|\varphi\|_{q'} \leq 1, \varphi \in C_c^\infty(\mathbb{R}^n)} \left| \int_{\mathbb{R}^n} \sum_{j=1}^{\infty} |T_\alpha(\lambda_j) f_j(x)|^2 \varphi(x) dx \right|$$

it is enough to estimate the integral on the right-hand side. In view of Proposition 3.1, for $\Re(\alpha) > \frac{n-1}{2}$ we have

$$\int_{\mathbb{R}^n} |T_\alpha(\lambda_j) f_j(x)|^2 |\varphi(x)| dx \leq C \int_{\mathbb{R}^n} |f_j(x)|^2 M\varphi(x)\ \ dx,$$

where C is independent of φ and λ_j. Therefore,

$$\left|\int_{\mathbb{R}^n}\sum_{j=1}^{\infty}|T_\alpha(\lambda_j)f_j(x)|^2\varphi(x)dx\right| \le \sum_{j=1}^{\infty}\int_{\mathbb{R}^n}|T_\alpha(\lambda_j)f_j(x)|^2|\varphi(x)|dx$$
$$\le C\int_{\mathbb{R}^n}\sum_{j=1}^{\infty}|f_j(x)|^2 M\varphi(x)dx.$$

Applying Hölder's inequality to the right-hand side of the above we get

$$\left|\int_{\mathbb{R}^n}\sum_{j=1}^{\infty}|T_\alpha(\lambda_j)f_j(x)|^2\varphi(x)dx\right| \le C\|\sum_{j=1}^{\infty}|f_j|^2\|_q\|M\varphi\|_{q'},$$

for $\Re(\alpha) > \frac{n-1}{2}$. Since $q' > 1$ and $\|\varphi\|_{q'} \le 1$, by the boundedness of the Hardy-Littlewood maximal function on $L^{q'}(\mathbb{R}^n)$, we get

$$\sup_{\|\varphi\|_{q'}\le 1,\varphi\in C_c^\infty(\mathbb{R}^n)}\left|\int_{\mathbb{R}^n}\sum_{j=1}^{\infty}|T_\alpha(\lambda_j)f_j(x)|^2\varphi(x)\ dx\right| \le C\|\sum_{j=1}^{\infty}|f_j|^2\|_q,$$

for $\Re(\alpha) > \frac{n-1}{2}$. Hence we get the required vector-valued inequality for $T_\alpha(\lambda)$. □

3.2. The R-boundedness of $\lambda\frac{d}{d\lambda}m_\alpha(H(\lambda))$

In this subsection we prove the vector-valued inequality required to establish the R-boundedness of the family $\lambda\frac{d}{d\lambda}m_\alpha(H(\lambda))$. Without loss of generality we assume that $\lambda > 0$ as the case of $\lambda < 0$ follows in a similar fashion with λ replaced by $-\lambda$. The derivative of $m_\alpha(H(\lambda))$ has been calculated in our earlier work [3, Lemma 3.4]. It has been shown that $\lambda\frac{d}{d\lambda}m_\alpha(H(\lambda))$ is a linear combination of terms of the form

$$A_j^2(\lambda)\int_0^1 m'_\alpha(H(\lambda)+2s\lambda)ds, \quad A_j^{*2}(\lambda)\int_0^1 m'_\alpha(H(\lambda)+2s\lambda)ds$$

and $H(\lambda)m'_\alpha(H(\lambda))$.

Theorem 3.3. *The family $S_\alpha(\lambda) = \lambda\frac{d}{d\lambda}T_\alpha(\lambda)$ satisfies the inequality*

$$\left\|\left(\sum_{j=1}^{\infty}|S_\alpha(\lambda_j)f_j|^2\right)^{\frac{1}{2}}\right\|_p \le C\left\|\left(\sum_{j=1}^{\infty}|f_j|^2\right)^{\frac{1}{2}}\right\|_p$$

for $1 < p < \infty$ when $\Re(\alpha) > \frac{n+1}{2}$ for all choices of $\lambda_j \in \mathbb{R}^*$ and $f_j \in L^p(\mathbb{R}^n)$.

As we have noted above, $S_\alpha(\lambda)$ is a linear combination of several terms. We will show that each term satisfies the above vector-valued inequality. Since we have already taken care of $m_\alpha(H(\lambda))$ we will begin with the term $H(\lambda)m'_\alpha(H(\lambda))$. Recalling that $m_\alpha(u) = \frac{J_\alpha(\sqrt{u})}{\sqrt{u}^\alpha}$, in view of the relation $\frac{d}{dt}\frac{J_\alpha(\sqrt{t})}{\sqrt{t}^\alpha} = \frac{-1}{2}\frac{J_{\alpha+1}(\sqrt{t})}{\sqrt{t}^{\alpha+1}}$ we get

$$H(\lambda)m'_\alpha(H(\lambda)) = -\frac{1}{2}H(\lambda)m_{\alpha+1}(H(\lambda)).$$

The required maximal theorem for this family has been proved at the end of the previous section. The R-boundedness of this family can now be proved repeating the proofs of Proposition 3.1 and Theorem 3.2.

We will now sketch the proof of the vector-valued inequality for the remaining terms. We will only consider the term

$$A_j^2(\lambda)\int_0^1 m'_\alpha(H(\lambda)+2s\lambda)ds,$$

as the other one can be treated similarly. As observed above

$$m'_\alpha(H(\lambda)+2s\lambda) = -\frac{1}{2}m_{\alpha+1}(H(\lambda)+2s\lambda)$$

and hence we have to consider

$$A_j^2(\lambda)H(\lambda)^{-1}\int_0^1 H(\lambda)m_{\alpha+1}(H(\lambda)+2s\lambda)ds.$$

As was shown in [3] the operator $A_j^2(\lambda)H(\lambda)^{-1}$turns out to be a Calderon-Zygmund singular integral operator whose CZ constants are uniform in λ. Hence by a theorem of Cordoba and Fefferman [2] the family $A_j^2(\lambda)H(\lambda)^{-1}$ satisfies a vector-valued inequality. (See [3, Theorem 2.1].)

Finally we are left with the operator family

$$\int_0^1 H(\lambda)m_{\alpha+1}(H(\lambda)+2s\lambda)ds$$

and hence it is enough to show that the family

$$H(\lambda)m_{\alpha+1}(H(\lambda)+2s\lambda)$$

is R-bounded uniformly in $s \in (0, 1)$. But the treatment of this is very similar to that of $H(\lambda)m_{\alpha+1}(H(\lambda))$ which we considered before using the maximal Theorem 2.3. Once again we leave the details to the reader. This completes the proof.

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