Bubbling solutions for an elliptic equation with exponential Neumann data in $\mathbb{R}^2$

SHENGBING DENG AND MONICA MUSSO

Abstract. Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary; we study the following Neumann problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial v} = \lambda u^{p-1} e^{u^p} & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

where v is the outer normal vector of $\partial\Omega$, $\lambda > 0$ is a small parameter and $0 < p < 2$. We construct bubbling solutions to problem (0.1) by a Lyapunov-Schmidt reduction procedure.

Mathematics Subject Classification (2010): 35B10 (primary); 35B33, 35J08, 58J05 (secondary).

1. Introduction

In this paper we consider the boundary-value problem

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial v} = \lambda u^{p-1} e^{u^p} & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, v is the outer normal vector of $\partial\Omega$, $\lambda > 0$ is a small parameter, and $0 < p < 2$.

In [6] Dávila-del Pino-Musso analyzed the asymptotic behavior of solutions to problem (1.1) when $p = 1$. Namely, they considered the following problem:

$$\begin{cases} -\Delta u + u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial v} = \lambda e^u & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Suppose that u_λ is a family of solutions to (1.2), with the property that $\lambda \int_{\partial\Omega} e^{u_\lambda}$ is bounded as $\lambda \to 0$; then there is an integer $k \geq 1$ such that, up to subsequences,

$$\lim_{\lambda \to 0} \lambda \int_{\partial\Omega} e^{u_\lambda} = 2k\pi. \tag{1.3}$$

Moreover, there are k distinct points ξ_j, with $j = 1, \ldots, k$, on the boundary of Ω, such that λe^{u_λ} approaches the sum of k Dirac masses centered at these points ξ_j. The location of such points can be characterized as that of the critical points of the function given by

$$\varphi_k(\xi_1, \ldots, \xi_k) = -\left[\sum_{j=1}^{k} H(\xi_j, \xi_j) + \sum_{l \neq j} G(\xi_l, \xi_j) \right], \tag{1.4}$$

where $G(x, y)$ is the Green function of the problem

$$\begin{cases} -\Delta_x G(x, y) + G(x, y) = 0 & x \in \Omega \\ \dfrac{\partial G(x, y)}{\partial \nu_x} = 2\pi \delta_y(x) & x \in \partial\Omega, \end{cases} \tag{1.5}$$

and H is its regular part defined by

$$H(x, y) = G(x, y) - 2 \log \frac{1}{|x - y|}. \tag{1.6}$$

The authors in [6] also proved the existence of solutions with the above properties. More precisely, they showed that, for any $k \geq 1$, Problem (1.2) has a solution u_λ, for all λ small enough, with the property that

$$\lambda \int_{\partial\Omega} e^{u_\lambda} \, dx \to 2k\pi \quad \text{as} \quad \lambda \to 0.$$

These solutions develop k peaks, as $\lambda \to 0$, around k points of the boundary of Ω, corresponding to the critical points of the function φ_k defined in (1.4).

In this paper we will extend this result and we will consider the problem of the existence of solutions to (1.1) for any value of p with $0 < p < 2$. In particular we will recover the result in [6], for $p = 1$. Problem (1.1) is the Euler-Lagrange equation for the functional $J_\lambda : H^1(\Omega) \to \mathbb{R}$ defined by

$$J_\lambda(u) = \frac{1}{2} \int_\Omega \left(|\nabla u|^2 + u^2 \right) - \frac{\lambda}{p} \int_{\partial\Omega} e^{u^p}.$$

By the Young and Hölder inequalities, we know that J_λ corresponds to the critical Trudinger-Moser trace embedding

$$H^1(\Omega) \ni u \mapsto e^{u^2} \in L^q(\partial\Omega) \quad \forall q \geq 1,$$

which is connected to the following critical Trudinger-Moser trace inequalities

$$S_\alpha := \sup\left\{\int_{\partial\Omega} e^{\alpha u^2} \; : \; u \in H^1(\Omega)\backslash\{0\}, \;\; \|u\|_{H^1} \le 1, \; \int_{\partial\Omega} u = 0\right\} < \infty \quad (1.7)$$

for any $\alpha \le \pi$, see [1]. Multiplying by a suitable test function, we can find that smallness of λ is necessary for the existence of a solution. From (1.7), there is a minimizer solution near zero. On the other hand, there is a second solution for (1.1) by the Mountain Pass Theorem.

Let ε be the parameter, which depends on λ, defined by the relation

$$p\lambda \left(-\frac{2}{p}\log\varepsilon\right)^{\frac{2(p-1)}{p}} \varepsilon^{\frac{p-2}{p}} = 1. \quad (1.8)$$

Observe that, as $\lambda \to 0$, we have $\varepsilon \to 0$. Furthermore $\varepsilon = \lambda$ when $p = 1$.

Our main result is the following:

Theorem 1.1. *Assume* $0 < p < 2$. *For any integer* $k \ge 1$ *problem* (1.1) *has two families of solutions* $u_{i,\lambda}$, *for* $i = 1, 2$ *and for all* λ *small enough, such that*

$$\lim_{\lambda\to 0} \varepsilon^{\frac{2-p}{p}} \int_{\partial\Omega} e^{u_{i,\lambda}^p} = 2k\pi, \quad (1.9)$$

where ε *is given by* (1.8). *Furthermore, there exist two families of points* $\xi_\lambda^i = (\xi_{1,\lambda}^i, \ldots, \xi_{k,\lambda}^i) \in (\partial\Omega)^k$, *for* $i = 1, 2$, *with*

$$|\xi_{l,\lambda}^i - \xi_{j,\lambda}^i| > \delta \quad for \quad l \ne j \quad and \quad i = 1, 2$$

for some small but fixed number $\delta > 0$, *such that* $\xi_\lambda^i \to \xi^i$ *as* $\lambda \to 0$, *with*

$$\nabla\varphi_k(\xi_1^i, \ldots, \xi_k^i) = 0$$

and

$$u_{i,\lambda}(x) = p^{-\frac{1}{2}} \sqrt{\lambda}\, \varepsilon^{\frac{p-2}{2p}} \left[\sum_{j=1}^{k} G(x, \xi_{j,\lambda}^i) + o(1)\right] \quad (1.10)$$

where $o(1) \to 0$ *on each compact subset of* $\bar\Omega\backslash\{\xi_1^i, \ldots, \xi_k^i\}$. *Moreover,*

$$J_\lambda(u_{i,\lambda}) = \lambda\varepsilon^{\frac{p-2}{p}} \left[-\frac{2k\pi}{p} + \frac{2k\pi}{p}\log\frac{1}{\varepsilon} + \frac{\pi}{2-p}\varphi_k(\xi_\lambda^i) + |\log\varepsilon|^{-1}O(1)\right] \quad (1.11)$$

where $O(1)$ *is uniformly bounded as* $\lambda \to 0$.

The proof of our result relies on a very well known Lyapunov-Schmidt reduction procedure, introduced in [2, 15] and used in many different contexts, see for instance [5–14, 17]. A key step in this procedure is to find a good first approximation for the solution. Usually, this ansatz is built as a sum of terms, each one of which turns out to solve an associate limit problem, properly scaled and translated. For our problem, the limit problem is

$$
\begin{cases}
\Delta v = 0 & \text{in } \mathbb{R}^2_+ \\[2mm]
\dfrac{\partial v}{\partial \nu} = e^v & \text{on } \partial \mathbb{R}^2_+ \\[2mm]
\displaystyle\int_{\partial \mathbb{R}^2_+} e^v < \infty.
\end{cases}
\tag{1.12}
$$

A family of solutions to (1.12) is given by

$$
w_{t,\mu}(x) = w_{t,\mu}(x_1, x_2) = \log \frac{2\mu}{(x_1 - t)^2 + (x_2 + \mu)^2}, \tag{1.13}
$$

where $t \in \mathbb{R}$ and $\mu > 0$ are parameters. Set

$$
w_\mu(x) := w_{0,\mu}(x) = \log \frac{2\mu}{x_1^2 + (x_2 + \mu)^2}. \tag{1.14}
$$

If we use the above solution, properly scaled, and centered at several points on the boundary of the domain, as our approximate solution, we get a very good approximation of a solution in a region far away from the points. In other words, the error is relatively small far away from these points. Close to the concentration points on the boundary, this approximation turns out to be good enough for our construction only when $p = 1$. We refer to [6]. Unfortunately this is not the case when $p \in (0, 2)$ and $p \neq 1$. In fact, in this case, we need to further improve the approximation near the concentration points. We do this by adding two other terms in the expansion of the solution: this can be done in a very natural way, which has first been used, for instance, in [12] for studying the problem

$$
\begin{cases}
\Delta u + u^q = 0, \ u > 0 & \text{in } \Omega \\[2mm]
u = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{1.15}
$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$, and q is a large exponent. Later on, this method has been applied in other contexts, see [5, 13, 14, 17]. In particular, H. Castro in [5] used this method to study the Neumann problem

$$
\begin{cases}
-\Delta u + u = 0, \ u > 0 & \text{in } \Omega \\[2mm]
\dfrac{\partial u}{\partial \nu} = u^q & \text{on } \partial\Omega,
\end{cases}
\tag{1.16}
$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, ν is the outer normal vector to $\partial\Omega$, and q is a large exponent. He showed that, if $q > 1$ is a large parameter, for any integer $k \geq 1$, there exists at least two families of solutions u_q, which develop exactly k peaks $\xi_j \in \partial\Omega$, and in the sense that $q u_q^q \rightharpoonup 2e\pi \sum_{j=1}^k \delta_{\xi_j}$ as $q \to +\infty$.

It is important to remark about the analogy existing between our result and the Dirchlet problem

$$\begin{cases} \Delta u + \lambda u^{p-1} e^{u^p} = 0, \ \ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.17}$$

For $p = 1$, the asymptotic behaviour of solutions to (1.17) for which $\lambda \int_\Omega e^u$ remains uniformly bounded as $\lambda \to 0$ is well understood after the works [4, 16, 18]: indeed, λe^u approaches a superposition of Dirac deltas centered at points in the interior of Ω. On the other hand, the construction of solutions with this behaviour has been achieved in [3, 8, 11]. In our previous paper [10] we constructed bubbling solutions to (1.17) for the whole range $0 < p < 2$. We showed that *If Ω is not simply connected and if $k \geq 1$ is any integer, then for all small λ Problem* (1.17) *has a solution concentrating around k points in the interior of Ω.* In fact this result was shown first in [8] for $p = 1$. Construction of bubbling solutions for Problem (1.17) and $p = 2$ has been treated in [9]: in this case, the existence of such a solution is not only related to the location of the concentration points, but also to the rate of concentration at these points.

This paper is organized as follows: Section 2 is devoted to describing a first approximation solution to problem (1.1) and to estimating the error. Furthermore, problem (1.1) is rewritten as a fixed-point problem, involving a linear operator. In Section 3 we study the invertibility of the linear problem. In Section 4 we study the nonlinear problem. In Section 5 we study the variational reduction, and we prove the main Theorem 1.1 in Section 6. We will give some estimates in the Appendix, Section 7.

In this paper, the symbol C denotes a generic positive constant independent of λ, that can change from one line to another. The symbols $O(t)$ (respectively $o(t)$) will denote quantities for which $\frac{O(t)}{|t|}$ stays bounded (respectively, $\frac{o(t)}{|t|}$ tends to zero) as the parameter t goes to zero. In particular, we will often use the notation $o(1)$ for a quantity which tends to zero as $t \to 0$.

2. Preliminaries and ansatz for the solution

For any parameter $\varepsilon > 0$, we can produce a solution to

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}_+^2 \\ \dfrac{\partial u}{\partial \nu} = \varepsilon e^u & \text{on } \partial\mathbb{R}_+^2, \end{cases} \tag{2.1}$$

by taking

$$u(x) = w_\mu(x/\varepsilon) - 2\log\varepsilon = \log\frac{2\mu}{x_1^2 + (x_2 + \varepsilon\mu)^2},$$

where w_μ is defined by (1.14). Based on this observation, we choose a sufficiently small but fixed number $\delta > 0$ and assume to be given k points ξ_j, $j = 1, \ldots, k$, on $\partial\Omega$, satisfying

$$|\xi_l - \xi_j| > \delta, \quad \text{for } l \neq j. \tag{2.2}$$

Furthermore, we consider k positive numbers μ_j such that

$$\delta < \mu_j < \delta^{-1}, \quad \text{for all} \quad j = 1, \ldots, k. \tag{2.3}$$

We define

$$u_j(x) = \log\frac{2\mu_j}{|x - \xi_j - \varepsilon\mu_j\nu(\xi_j)|^2}.$$

We define the first approximate solution by

$$U(x) = \frac{1}{p\gamma^{p-1}}\sum_{j=1}^{k}\left[u_j(x) + H_j^\varepsilon(x)\right],$$

with some number γ, to be defined later on, where H_j^ε is a correction term given as the solution of

$$\begin{cases} -\Delta H_j^\varepsilon + H_j^\varepsilon = -u_j & \text{in } \Omega \\ \dfrac{\partial H_j^\varepsilon}{\partial\nu} = \varepsilon e^{u_j} - \dfrac{\partial u_j}{\partial\nu} & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

We have the validity of the following lemma.

Lemma 2.1. *Assume* (2.2) *and* (2.3). *For any* $0 < \alpha < 1$, *one has*

$$H_j^\varepsilon(x) = H(x, \xi_j) - \log(2\mu_j) + O(\varepsilon^\alpha) \tag{2.5}$$

uniformly in $\bar{\Omega}$, *where* H *is the regular part of Green's function defined* (1.5).

We will give the proof of this lemma in the Appendix, Section 7.

We shall show later on that $U(x)$ is a good approximation for a solution to (1.1) far from the points ξ_j, but unfortunately, when $p \neq 1$, it is not good enough for our construction close to the points ξ_j. This is the reason why we need to further adjust this ansatz. In order to do this, let us first introduce the following result, whose proof is given in [6].

Proposition 2.2. *Any bounded solution of the following problem*

$$\begin{cases} \Delta\phi = 0 & \text{in } \mathbb{R}^2_+ \\ \dfrac{\partial\phi}{\partial\nu} - e^{w_\mu}\phi = 0 & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{2.6}$$

is a linear combination of

$$z_{0\mu}(x) = -\frac{1}{\mu}\left(x \cdot \nabla w_\mu(x) + 1\right) = \frac{1}{\mu} - 2\frac{x_2 + \mu}{x_1^2 + (x_2 + \mu)^2}, \tag{2.7}$$

and

$$z_{1\mu}(x) = \frac{\partial w_\mu}{\partial x_1} = -2\frac{x_1}{x_1^2 + (x_2 + \mu)^2}. \tag{2.8}$$

Now, let us consider the following problem

$$\begin{cases} \Delta\phi = 0 & \text{in } \mathbb{R}^2_+ \\ \dfrac{\partial\phi}{\partial\nu} - e^{w_\mu}\phi = e^{w_\mu}g & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{2.9}$$

with w_μ defined in (1.14). In [5] it is showed that

Proposition 2.3. *Let g be a $C^1(\partial\mathbb{R}^2_+)$ function such that, for $\mu > 0, k \geq 0$, satisfies*

$$g(x) = O(\log^k(1 + |x|)) \quad \text{as } |x| \to \infty, \tag{2.10}$$

and

$$\int_{\partial\mathbb{R}^2_+} e^{w_\mu} g z_{0\mu} = 0 = \int_{\partial\mathbb{R}^2_+} e^{w_\mu} g z_{1\mu}. \tag{2.11}$$

Then (2.9) has a solution $\phi \in C^\alpha(\mathbb{R}^2_+)$. Moreover, for any $0 < \alpha < 1$, and $|x| \to \infty$,

$$|\phi(x)| \leq C\frac{1}{|x|^\alpha}, \quad |\nabla\phi(x)| \leq C\frac{1}{|x|^{1+\alpha}}, \quad |\nabla^2\phi(x)| \leq C\frac{1}{|x|^{2+\alpha}}, \tag{2.12}$$

where C is a positive constant, which depends on $\|g\|_{L^p(\partial\mathbb{R}^2_+)}$, for some $p = p(\alpha) > 1$.

Let us define ϕ_{1j} to be the solution of the problem

$$\begin{cases} \Delta\phi_{1j} = 0 & \text{in } \mathbb{R}^2_+ \\ \dfrac{\partial\phi_{1j}}{\partial\nu} - e^{w_{\mu_j}}\phi_{1j} = e^{w_{\mu_j}}g_1 & \text{on } \partial\mathbb{R}^2_+, \end{cases} \tag{2.13}$$

where $w_{\mu_j}(y) = \log \frac{2\mu_j}{y_1^2+(y_2+\mu_j)^2}$ and

$$g_1 = \alpha_{1j}(w_{\mu_j} - 1) + w_{\mu_j} + \frac{1}{2}(w_{\mu_j})^2$$

with α_{1j} is a constant to be fixed, in terms of μ_j. Observe first that, by definition, the function g_1 satisfies (2.10). We now choose α_{1j} such that the orthogonality condition (2.11) holds. First we observe that g_1 is a symmetric function for any choice of α_{1j}, hence

$$\int_{\partial \mathbb{R}^2_+} e^{w_{\mu_j}} g_1 z_{1\mu_j} = 0.$$

Next, we chose the parameter α_{1j} such that the other orthogonality condition is satisfied. Since

$$\begin{aligned}
&\int_{\partial \mathbb{R}^2_+} e^{w_{\mu_j}} g_1 z_{0\mu_j} \\
&= -\frac{1}{\mu_j} \int_{\partial \mathbb{R}^2_+} e^{w_{\mu_j}} \left(\alpha_{1j}(w_{\mu_j} - 1) + w_{\mu_j} + \frac{1}{2}(w_{\mu_j})^2 \right) (y \cdot \nabla w_{\mu_j}(y) + 1) \\
&= -\frac{1}{\mu_j} \alpha_{1j} \int_{-\infty}^{\infty} \left(e^{w_{\mu_j}(y_1,0)} \left(w_{\mu_j}(y_1, 0) - 1 \right) \frac{\partial w_{\mu_j}}{\partial y_1}(y_1, 0) y_1 \right. \\
&\qquad\qquad\qquad \left. + e^{w_{\mu_j}(y_1,0)} \left(w_{\mu_j}(y_1, 0) - 1 \right) \right) dy_1 \\
&\quad - \frac{1}{\mu_j} \int_{-\infty}^{\infty} \left(e^{w_{\mu_j}(y_1,0)} w_{\mu_j}(y_1, 0) \frac{\partial w_{\mu_j}}{\partial y_1}(y_1, 0) y_1 + e^{w_{\mu_j}(y_1,0)} w_{\mu_j}(y_1, 0) \right) dy_1 \\
&\quad - \frac{1}{\mu_j} \int_{-\infty}^{\infty} \left(e^{w_{\mu_j}(y_1,0)} \frac{(w_{\mu_j})^2}{2}(y_1,0) \frac{\partial w_{\mu_j}}{\partial y_1}(y_1,0) y_1 + e^{w_{\mu_j}(y_1,0)} \frac{(w_{\mu_j})^2}{2}(y_1,0) \right) dy_1 \\
&= -\frac{1}{\mu_j} \left[\alpha_{1j} \int_{-\infty}^{\infty} e^{w_{\mu_j}(y_1,0)} dy_1 + \int_{-\infty}^{\infty} e^{w_{\mu_j}(y_1,0)} w_{\mu_j}(y_1, 0) dy_1 \right].
\end{aligned}$$

Thus we define α_{1j} such that

$$\alpha_{1j} \int_{-\infty}^{\infty} e^{w_{\mu_j}(y_1,0)} dy_1 + \int_{-\infty}^{\infty} e^{w_{\mu_j}(y_1,0)} w_{\mu_j}(y_1, 0) dy_1 = 0.$$

Since

$$\int_{-\infty}^{\infty} e^{w_{\mu_j}(y_1,0)} dy_1 = \int_{-\infty}^{\infty} \frac{2\mu_j}{y_1^2 + \mu_j^2} dy_1 = 2 \int_{-\infty}^{\infty} \frac{1}{t^2 + 1} dt = 2\pi,$$

and

$$\int_{-\infty}^{\infty} e^{w_{\mu_j}(y_1,0)} w_{\mu_j}(y_1,0) dy_1$$

$$= \int_{-\infty}^{\infty} \frac{2\mu_j}{y_1^2 + \mu_j^2} \log \frac{2\mu_j}{y_1^2 + \mu_j^2} \, dy_1$$

$$= 2 \int_{-\infty}^{\infty} \frac{1}{t^2 + 1} \left[\log \frac{1}{t^2 + 1} + \log(2\mu_j^{-1}) \right] dt = -2\pi \log(2\mu_j).$$

Here we use the following fact (see the proof in the Appendix)

$$\int_{-\infty}^{\infty} \frac{1}{t^2 + 1} \log \frac{1}{t^2 + 1} dt = -2\pi \log 2. \tag{2.14}$$

Therefore, α_{1j} is given by

$$\alpha_{1j} = \log(2\mu_j). \tag{2.15}$$

Then we get the existence of ϕ_{1j} by Proposition 2.3. With this function, we define

$$w_{1j}(y) = \phi_{1j}(y) + \alpha_{1j} w_{\mu_j}(y).$$

We observe that w_{1j} satisfies

$$\begin{cases} \Delta w_{1j} = 0 & \text{in } \mathbb{R}_+^2 \\ \dfrac{\partial w_{1j}}{\partial \nu} - e^{w_{\mu_j}} w_{1j} = e^{w_{\mu_j}} \left(w_{\mu_j} + \dfrac{1}{2}(w_{\mu_j})^2 \right) & \text{on } \partial\mathbb{R}_+^2. \end{cases}$$

Next, assume for the moment that $p \neq 1$ and consider ϕ_{2j}, a solution of

$$\begin{cases} \Delta \phi_{2j} = 0 & \text{in } \mathbb{R}_+^2 \\ \dfrac{\partial \phi_{2j}}{\partial \nu} - e^{w_{\mu_j}} \phi_{2j} = e^{w_{\mu_j}} g_2 & \text{on } \partial\mathbb{R}_+^2, \end{cases} \tag{2.16}$$

where

$$g_2 = \alpha_{2j}(w_{\mu_j} - 1) + w_{1j} + \frac{p-2}{2(p-1)}(w_{\mu_j})^2 + \frac{1}{2}(w_{1j})^2$$

$$+ \frac{1}{8}(w_{\mu_j})^4 + w_{\mu_j} w_{1j} + \frac{1}{2}(w_{\mu_j})^3 + \frac{1}{2} w_{1j}(w_{\mu_j})^2.$$

Again, by definition, we observe that g_2 satisfies (2.10) and it is a symmetric function. Thus, arguing as before, we can find a constant α_{2j} such that the orthogonality conditions (2.11) are satisfied. Then we have the existence of function ϕ_{2j} by Proposition 2.3.

For $\xi_j \in \partial\Omega$, let $\delta > 0$ be a fixed small radius, depending only in the geometry of Ω, such that

$$F_j : B_\delta(0) \cap (\Omega - \xi_j) \to M \cap \mathbb{R}_+^2, \qquad (2.17)$$

is a C^2 diffeomorphism, and M an open neighborhood of the origin such that

$$F_j\left(B_\delta(0) \cap (\partial\Omega - \xi_j)\right) \subseteq M \cap \partial\mathbb{R}_+^2.$$

We can select F_j so that it preserves the area. For $i = 1, 2$, define

$$\begin{aligned}
\tilde{w}_{ij}(x) &= \phi_{ij}\left(\frac{F_j(x - \xi_j)}{\varepsilon}\right) + \alpha_{ij} w_{\mu_j}\left(\frac{x - \xi_j}{\varepsilon}\right) \\
&:= \tilde{\phi}_{ij}(y) + \alpha_{ij}\tilde{w}_j(y),
\end{aligned}$$

where

$$\tilde{w}_j(y) := w_{\mu_j}\left(y - \xi_j'\right) = \log \frac{2\mu_j}{|y - \xi_j' - \mu_j \nu(\xi_j')|^2},$$

with $\xi_j' = \xi_j/\varepsilon$ and where we will write ν for the exterior normal unit vector to $\partial\Omega$ and $\partial\Omega_\varepsilon$. Here $\Omega_\varepsilon = \varepsilon^{-1}\Omega$. Then, for any $0 < p < 2$, let us define the first approximation solution to (1.1) is

$$\begin{aligned}
U_\lambda(x) = \frac{1}{p\gamma^{p-1}} \sum_{j=1}^{k} \Bigg[& u_j(x) + H_j^\varepsilon(x) + \frac{p-1}{p}\frac{1}{\gamma^p}\left(\tilde{w}_{1j}(x) + H_{1j}^\varepsilon(x)\right) \\
& + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\left(\tilde{w}_{2j}(x) + H_{2j}^\varepsilon(x)\right) \Bigg]
\end{aligned} \qquad (2.18)$$

where H_{ij}^ε, $i = 1, 2$, is a new correction term, given by the solution of

$$\begin{cases}
-\Delta H_{ij}^\varepsilon + H_{ij}^\varepsilon = -\alpha_{ij}\tilde{w}_j(x/\varepsilon) & \text{in } \Omega \\
\dfrac{\partial H_{ij}^\varepsilon}{\partial\nu} = \alpha_{ij}\left(\varepsilon e^{u_j} - \dfrac{\partial u_j}{\partial\nu}\right) & \text{on } \partial\Omega.
\end{cases} \qquad (2.19)$$

By the same arguments used in Lemma 2.1, we have the following result, whose proof is postponed to the Appendix, Section 7.

Lemma 2.4. *For any $0 < \alpha < 1$, for $i = 1, 2$, one has*

$$H_{ij}^\varepsilon(x) = \alpha_{ij}H(x, \xi_j) - \alpha_{ij}\log(2\mu_j) - 2\alpha_{ij}\log\varepsilon + O(\varepsilon^\alpha) \qquad (2.20)$$

uniformly in $\bar{\Omega}$, where H is the regular part of Green's function defined (1.5).

Consider now the change of variables

$$v(y) = p\gamma^{p-1}u(\varepsilon y) - p\gamma^p, \quad \text{with } \gamma^p = -\frac{2}{p}\log\varepsilon.$$

Then under the choice of ε in (1.8), problem (1.1) reduces to

$$\begin{cases} -\Delta v + \varepsilon^2 v = 2\varepsilon^2\log\varepsilon & \text{in } \Omega_\varepsilon \\ \dfrac{\partial v}{\partial \nu} = f(v) & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{2.21}$$

where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$, and

$$f(v) = \left(1 + \frac{v}{p\gamma^p}\right)^{p-1} e^{\gamma^p[(1+\frac{v}{p\gamma^p})^p - 1]}. \tag{2.22}$$

Let us define the first approximation solution to (2.21) as

$$V_\lambda(y) = p\gamma^{p-1}U_\lambda(\varepsilon y) - p\gamma^p, \tag{2.23}$$

with U_λ defined by (2.18).

We next describe V_λ close to the points ξ'_j. We write $y = \varepsilon^{-1}x$, $\xi'_j = \varepsilon^{-1}\xi_j$. For $|x - \xi_j| < \delta$ with δ sufficiently small but fixed, by Lemmas 2.1 and 2.4, and the fact $u_j(\varepsilon y) - p\gamma^p = \tilde{w}_j(y)$, we have

$V_\lambda(y)$

$$= u_j(\varepsilon y) + H_j^\varepsilon(\varepsilon y) + \frac{p-1}{p}\frac{1}{\gamma^p}\left(\tilde{w}_{1j}(\varepsilon y) + H_{1j}^\varepsilon(\varepsilon y)\right)$$

$$+ \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\left(\tilde{w}_{2j}(\varepsilon y) + H_{2j}^\varepsilon(\varepsilon y)\right) - p\gamma^p$$

$$+ \sum_{l\neq j}^k \left(\tilde{w}_l(\varepsilon y) + H_l^\varepsilon(\varepsilon y) + \frac{p-1}{p}\frac{1}{\gamma^p}\left(\tilde{w}_{1l}(\varepsilon y) + H_{1l}^\varepsilon(\varepsilon y)\right)\right.$$

$$\left.+ \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\left(\tilde{w}_{2l}(\varepsilon y) + H_{2l}^\varepsilon(\varepsilon y)\right)\right) \tag{2.24}$$

$$= \tilde{w}_j(y) + \frac{p-1}{p}\frac{1}{\gamma^p}\tilde{w}_{1j}(\varepsilon y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\tilde{w}_{2j}(\varepsilon y) + O(\varepsilon|y-\xi'_j|) + O(\varepsilon^\alpha)$$

$$- \log(2\mu_j) + \left[1 + \alpha_{1j}\frac{p-1}{p}\frac{1}{\gamma^p} + \alpha_{2j}\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\right]$$

$$\times \left(H(\xi_j,\xi_j) + \sum_{l\neq j}^k G(\xi_l,\xi_j)\right)$$

$$- \left[\alpha_{1j}\frac{p-1}{p}\frac{1}{\gamma^p} + \alpha_{2j}\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\right]\left(\log(2\mu_j) + 2\log\varepsilon\right).$$

We now choose the parameters μ_j: we assume they are defined by the relation

$$
\begin{aligned}
\log(2\mu_j) &= \left(H(\xi_j, \xi_j) + \sum_{l \neq j}^{k} G(\xi_l, \xi_j) \right) + (p-1)\alpha_{1j} \\
&\quad + \alpha_{1j} \frac{p-1}{p} \frac{1}{\gamma^p} \left(H(\xi_j, \xi_j) + \sum_{l \neq j}^{k} G(\xi_l, \xi_j) \right. \\
&\qquad\qquad \left. - \log(2\mu_j) + (p-1)\frac{\alpha_{2j}}{\alpha_{1j}} \right) \\
&\quad + \alpha_{2j} \left(\frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \left(H(\xi_j, \xi_j) + \sum_{l \neq j}^{k} G(\xi_l, \xi_j) - \log(2\mu_j) \right).
\end{aligned}
\tag{2.25}
$$

Taking into account the explicit expression (2.15) of the constant α_{1j}, we observe that μ_j bifurcates, as λ goes to zero, from the value

$$
\bar{\mu}_j = \frac{1}{2} e^{\frac{1}{2-p} \left[H(\xi_j, \xi_j) + \sum_{l \neq j}^{k} G(\xi_l, \xi_j) \right]}
\tag{2.26}
$$

solution of equation

$$
\log(2\mu_j) = \left(H(\xi_j, \xi_j) + \sum_{l \neq j}^{k} G(\xi_l, \xi_j) \right) + (p-1)\alpha_{1j}.
\tag{2.27}
$$

Thus, μ_j is a perturbation of order $\frac{1}{\gamma^p}$ of the value $\bar{\mu}_j$, namely

$$
\log(2\mu_j) = \frac{1}{2-p} \left(H(\xi_j, \xi_j) + \sum_{l \neq j}^{k} G(\xi_l, \xi_j) \right) \left(1 + O\left(\frac{1}{\gamma^p} \right) \right).
\tag{2.28}
$$

Then, by this choice of the parameters μ_j, we deduce that, if $|y - \xi_j'| < \delta/\varepsilon$ with δ sufficiently small but fixed, we can rewrite

$$
V_\lambda(y) = \tilde{w}_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} \tilde{w}_{1j}(\varepsilon y) + \left(\frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \tilde{w}_{2j}(\varepsilon y) + \theta(y),
\tag{2.29}
$$

with

$$\theta(y) = O(\varepsilon|y - \xi_j'|) + O(\varepsilon^\alpha).$$

We will look for solutions to (2.21) of the form

$$v = V_\lambda + \phi,$$

where V_λ is defined as in (2.23), and ϕ represents a lower order correction. We aim at finding a solution for small ϕ provided that the points ξ_j are suitably chosen. For small ϕ, we can rewrite problem (2.21) as a nonlinear perturbation of its linearization, namely,

$$\begin{cases} -\Delta\phi + \varepsilon^2\phi = 0 & x \in \Omega_\varepsilon \\ L(\phi) = E_\lambda + N(\phi) & x \in \partial\Omega_\varepsilon, \end{cases} \tag{2.30}$$

where

$$L(\phi) := \frac{\partial\phi}{\partial\nu} - f'(V_\lambda)\phi, \tag{2.31}$$

$$E_\lambda := f(V_\lambda) - \frac{\partial V_\lambda}{\partial\nu}, \tag{2.32}$$

$$N(\phi) := f(V_\lambda + \phi) - f(V_\lambda) - f'(V_\lambda)\phi. \tag{2.33}$$

We recall that $f(t) = (1 + \frac{t}{p\gamma^p})^{p-1}e^{\gamma^p[(1+\frac{t}{p\gamma^p})^p - 1]}$.

In order to solve the problem (2.30), first we have to study the invertibility properties of the linear operator L. In order to do this, we introduce a weighted L^∞-norm defined as

$$\|h\|_{*,\partial\Omega_\varepsilon} := \sup_{y\in\Omega_\varepsilon}\left(\sum_{j=1}^k (1 + |y - \xi_j'|)^{-1-\sigma} + \varepsilon\right)^{-1}|h(y)| \tag{2.34}$$

for any $h \in L^\infty(\partial\Omega_\varepsilon)$, where we fix $0 < \sigma < 1$ later on. With respect to this norm, the error term E_λ given in (2.32) can be estimated in the following way:

Lemma 2.5. *Let $\delta > 0$ be a small but fixed number, assume* (2.2) *and* (2.3). *Then there exists $C > 0$ such that*

$$\|E_\lambda\|_{*,\partial\Omega_\varepsilon} \leq \frac{C}{\gamma^{3p}} = \frac{C}{|\log\varepsilon|^3} \tag{2.35}$$

for all λ small enough.

Proof. Far away from the points ξ_j, namely for $|x - \xi_j| > \delta$, *i.e.* $|y - \xi'_j| > \frac{\delta}{\varepsilon}$, for all $j = 1, \ldots, k$, we have that

$$\frac{\partial V_\lambda}{\partial \nu} = p\gamma^{p-1}\varepsilon \frac{\partial U_\lambda(\varepsilon y)}{\partial \nu} = O(\gamma^{p-1}\varepsilon^2).$$

On the other hand, in this region we have

$$1 + \frac{V_\lambda(y)}{p\gamma^p} = 1 + \frac{2\log \varepsilon + O(1)}{p\gamma^p} = \frac{O(1)}{|\log \varepsilon|}$$

where $O(1)$ denotes a smooth function, uniformly bounded, as $\varepsilon \to 0$, in the considered region. Hence

$$f(V_\lambda) = \left(1 + \frac{V_\lambda}{p\gamma^p}\right)^{p-1} e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p - 1]} = \frac{\varepsilon^{\frac{2}{p}}}{|\log \varepsilon|^{p-1}} O(1).$$

Hence if we are far away from the points ξ_j, or equivalently for $|y - \xi'_j| > \frac{\delta}{\varepsilon}$, the size of the error, measured with respect to the $\| \cdot \|_{*,\partial\Omega_\varepsilon}$-norm, is relatively small. Namely, if we denote by 1_{outer} the characteristic function of the set $\{y : |y - \xi'_j| > \frac{\delta}{\varepsilon}, \ j = 1, \ldots, k\}$, then in this region we have

$$\|E_\lambda 1_{\text{outer}}\|_{*,\partial\Omega_\varepsilon} \leq C \frac{\varepsilon^{\frac{2-p}{p}}}{|\log \varepsilon|^{p-1}}. \tag{2.36}$$

Let us now fix the index j in $\{1, \ldots, k\}$, for $|y - \xi'_j| < \frac{\delta}{\varepsilon}$, we have

$$\frac{\partial V_\lambda}{\partial \nu} = e^{\tilde{w}_j(y)}$$
$$+ \frac{p-1}{p}\frac{1}{\gamma^p}\frac{\partial \tilde{w}_{1j}(x)}{\partial \nu} + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\frac{\partial \tilde{w}_{2j}(x)}{\partial \nu} + O(\varepsilon^2)$$
$$= e^{\tilde{w}_j(y)} + \frac{p-1}{p}\frac{1}{\gamma^p}\left(\frac{\partial \tilde{\phi}_{1j}(y)}{\partial \nu} + \alpha_{1j}e^{\tilde{w}_j}\right)$$
$$+ \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\left(\frac{\partial \tilde{\phi}_{2j}(y)}{\partial \nu} + \alpha_{2j}e^{\tilde{w}_j}\right) + O(\varepsilon^2).$$

On the other hand, for any $R > 0$ large but fixed, in the ball $|y - \xi'_j| < R_\varepsilon := R|\log\varepsilon|^\alpha$, with $\alpha \geq 3$, we can use the Taylor expansion to obtain

$$f(V_\lambda)$$

$$= \left(1 + \frac{1}{p\gamma^p}\left(\tilde{w}_j(y) + \frac{p-1}{p}\frac{1}{\gamma^p}\tilde{w}_{1j}(\varepsilon y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\tilde{w}_{2j}(\varepsilon y) + \theta(y)\right)\right)^{p-1}$$

$$\times e^{\gamma^p\left[\left(1 + \frac{1}{p\gamma^p}\left(\tilde{w}_j(y) + \frac{p-1}{p}\frac{1}{\gamma^p}\tilde{w}_{1j}(\varepsilon y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\tilde{w}_{2j}(\varepsilon y) + \theta(y)\right)\right)^p - 1\right]}$$

$$= \left(1 + \frac{p-1}{p}\frac{1}{\gamma^p}\tilde{w}_j(y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\tilde{w}_{1j}(\varepsilon y)\right.$$

$$\left. + \left(\frac{p-1}{p}\right)^3\frac{1}{\gamma^{3p}}\tilde{w}_{2j}(\varepsilon y) + \frac{p-1}{p}\frac{1}{\gamma^p}\theta(y)\right)$$

$$\times e^{\tilde{w}_j(y)}e^{\frac{p-1}{p}\frac{1}{\gamma^p}\tilde{w}_{1j}(\varepsilon y)}e^{\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\tilde{w}_{2j}(\varepsilon y)}e^{\theta(y)}$$

$$\times e^{\frac{1}{2}\frac{p-1}{p}\frac{1}{\gamma^p}\left[\tilde{w}_j(y) + \frac{p-1}{p}\frac{1}{\gamma^p}\tilde{w}_{1j}(\varepsilon y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\tilde{w}_{2j}(\varepsilon y) + \theta(y)\right]^2}$$

$$= e^{\tilde{w}_j(y)} + \frac{p-1}{p}\frac{1}{\gamma^p}\left\{e^{\tilde{w}_j}\tilde{w}_{1j}(\varepsilon y) + e^{\tilde{w}_j}\left[\tilde{w}_j + \frac{1}{2}(\tilde{w}_j)^2\right]\right\} + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}$$

$$\times \left\{e^{\tilde{w}_j}\tilde{w}_{2j}(\varepsilon y) + e^{\tilde{w}_j}\left[\tilde{w}_{1j} + \frac{p-2}{2(p-1)}(\tilde{w}_j)^2 + \frac{1}{2}(\tilde{w}_{1j})^2 + \frac{1}{8}(\tilde{w}_j)^4\right.\right.$$

$$\left.\left. + \tilde{w}_j\tilde{w}_{1j} + \frac{1}{2}(\tilde{w}_j)^3 + \frac{1}{2}\tilde{w}_{1j}(\tilde{w}_j)^2\right]\right\}$$

$$+ \frac{p-1}{p}\frac{1}{\gamma^p}e^{\tilde{w}_j}\theta(y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}e^{\tilde{w}_j}\left[\tilde{w}_j + \tilde{w}_{1j}(\varepsilon y)\right]\theta(y)$$

$$+ \left(\frac{p-1}{p}\right)^3\frac{1}{\gamma^{3p}}\left\{e^{\tilde{w}_j}\tilde{w}_{2j}(\varepsilon y) + e^{\tilde{w}_j}\left[\frac{1}{6}(\tilde{w}_{1j})^3(\varepsilon y) + \frac{1}{48}(\tilde{w}_j)^6 + \tilde{w}_j\tilde{w}_{1j}(\varepsilon y)\right.\right.$$

$$+ \frac{1}{2}(\tilde{w}_{1j})^2(\varepsilon y) + 2\tilde{w}_j\tilde{w}_{2j}(\varepsilon y) + \frac{1}{2}\tilde{w}_{1j}(\varepsilon y)(\tilde{w}_j)^3$$

$$+ \frac{2p-3}{2(p-1)}\tilde{w}_j(\tilde{w}_{1j})^2(\varepsilon y) + \frac{1}{8}(\tilde{w}_j)^5 + (\tilde{w}_{1j})^2(\varepsilon y)$$

$$+ \frac{1}{2}\tilde{w}_{1j}(\varepsilon y)(\tilde{w}_j)^2 + \frac{p-2}{4(p-1)}(\tilde{w}_j)^4 + \tilde{w}_{1j}(\varepsilon y)\tilde{w}_{2j}(\varepsilon y)$$

$$\left.\left. + \frac{1}{8}\tilde{w}_{1j}(\varepsilon y)(\tilde{w}_j)^4 + \frac{1}{4}(\tilde{w}_j\tilde{w}_{1j}(\varepsilon y))^2 + \frac{1}{2}\tilde{w}_{2j}(\varepsilon y)(\tilde{w}_j)^2\right]\right\}$$

$$+ O\left(\frac{\log|y - \xi'_j|}{\gamma^{3p}}\right).$$

Thus, thanks to the fact that we have improved our original approximation with the terms $\tilde{w}_{1j}(\varepsilon y)$ and $\tilde{w}_{2j}(\varepsilon y)$, and the definition of $*$-norm, we get

$$\|E_\lambda 1_{B(\xi_j', R_\varepsilon)}\|_{*,\partial\Omega_\varepsilon} \leq \frac{C}{\gamma^{3p}} = \frac{C}{|\log\varepsilon|^3}. \tag{2.37}$$

Here $1_{B(\xi_j', R_\varepsilon)}$ denotes the characteristic function of $B(\xi_j', R_\varepsilon)$. Finally, in the remaining region, namely where $R_\varepsilon < |y - \xi_j'| < \frac{\delta}{\varepsilon}$, for any $j = 1, \ldots, k$, we have from one hand that $|\frac{\partial V_\lambda(y)}{\partial \nu}| \leq Ce^{\tilde{w}_j(y)}$, and also $|f(V_\lambda(y))| \leq Ce^{\tilde{w}_j(y)}$ as consequence of (2.24). These facts, together with (2.36) and (2.37) give estimate (2.35). □

In the next two lemmas we prove some estimates that will be of use later on in our construction.

Lemma 2.6. *Let $\delta > 0$ be a small but fixed number, assume (2.2) and (2.3). Then there exists $C > 0$ such that, for all small λ,*

$$\|f'(V_\lambda) - e^{\tilde{w}_j}\|_{*,\partial\Omega_\varepsilon} \leq \frac{C}{|\log\varepsilon|} \tag{2.38}$$

and there exists some positive constant D_0 such that

$$f'(V_\lambda) \leq D_0 \sum_{j=1}^k e^{\tilde{w}_j}. \tag{2.39}$$

Proof. We have

$$f'(V_\lambda) = \frac{p-1}{p}\frac{1}{\gamma^p}\left(1 + \frac{V_\lambda}{p\gamma^p}\right)^{p-2} e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p - 1]}$$
$$+ \left(1 + \frac{V_\lambda}{p\gamma^p}\right)^{2(p-1)} e^{\gamma^p[(1+\frac{V_\lambda}{p\gamma^p})^p - 1]} := I_a + I_b.$$

As in the proof of Lemma 2.5, far away from the points ξ_j, namely for $|x - \xi_j| > \delta$, i.e. $|y - \xi_j'| > \frac{\delta}{\varepsilon}$, for all $j = 1, \ldots, k$, we have

$$1 + \frac{V_\lambda(y)}{p\gamma^p} = 1 + \frac{2\log\varepsilon + O(1)}{p\gamma^p} = \frac{O(1)}{|\log\varepsilon|}.$$

Therefore

$$I_a = \frac{\varepsilon^{\frac{2}{p}}}{|\log\varepsilon|^{p-1}}O(1), \quad \text{and} \quad I_b = \frac{\varepsilon^{\frac{2}{p}}}{|\log\varepsilon|^{2(p-1)}}O(1).$$

Then we have

$$f'(V_\lambda)1_{\text{outer}} = \frac{\varepsilon^{\frac{2}{p}}}{|\log \varepsilon|^{p-1}} O(1). \tag{2.40}$$

On the other hand, fix the index j in $\{1, \ldots, k\}$, for $|y - \xi'_j| < R_\varepsilon$ with $R_\varepsilon = R|\log \varepsilon|$, for any $R > 0$ large but fixed; we use the Taylor expansion to get

$$
\begin{aligned}
I_a &= \frac{p-1}{p} \frac{1}{\gamma^p} \left(1 + \frac{1}{p\gamma^p} \left(\tilde{w}_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} \tilde{w}_{1j}(y) \right.\right. \\
&\qquad\qquad \left.\left. + \left(\frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \tilde{w}_{2j}(y) + \theta(y) \right) \right)^{p-2} \\
&\quad \times e^{\gamma^p \left[\left(1 + \frac{1}{p\gamma^p} \left(\tilde{w}_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} \tilde{w}_{1j}(y) + \left(\frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \tilde{w}_{2j}(y) + \theta(y) \right) \right)^p - 1 \right]} \\
&= \frac{p-2}{p} \frac{1}{\gamma^p} \left[\frac{p-1}{p-2} + \frac{p-1}{p} \frac{1}{\gamma^p} \tilde{w}_j(y) + \left(\frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \tilde{w}_{1j}(y) \right. \\
&\qquad\qquad \left. + \left(\frac{p-1}{p} \right)^3 \frac{1}{\gamma^{3p}} \tilde{w}_{2j}(y) + \frac{p-1}{p} \frac{1}{\gamma^p} \theta(y) \right] \\
&\quad \times e^{\tilde{w}_j(y)} e^{\frac{p-1}{p} \frac{1}{\gamma^p} \tilde{w}_{1j}(y)} e^{\left(\frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \tilde{w}_{2j}(y)} e^{\theta(y)} \\
&\quad \times e^{\frac{1}{2} \frac{p-1}{p} \frac{1}{\gamma^p} \left[\tilde{w}_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} \tilde{w}_{1j}(y) + \left(\frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \tilde{w}_{2j}(y) + \theta(y) \right]^2};
\end{aligned}
$$

and

$$
\begin{aligned}
I_b &= \left(1 + \frac{1}{p\gamma^p} \left(\tilde{w}_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} \tilde{w}_{1j}(y) \right.\right. \\
&\qquad\qquad \left.\left. + \left(\frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \tilde{w}_{2j}(y) + \theta(y) \right) \right)^{2(p-1)} \\
&\quad \times e^{\gamma^p \left[\left(1 + \frac{1}{p\gamma^p} \left(\tilde{w}_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} \tilde{w}_{1j}(y) + \left(\frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \tilde{w}_{2j}(y) + \theta(y) \right) \right)^p - 1 \right]} \\
&= \left[1 + \frac{2(p-1)}{p} \frac{1}{\gamma^p} \tilde{w}_j(y) + 2 \left(\frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \tilde{w}_{1j}(y) \right. \\
&\qquad\qquad \left. + 2 \left(\frac{p-1}{p} \right)^3 \frac{1}{\gamma^{3p}} \tilde{w}_{2j}(y) + \frac{2(p-1)}{p} \frac{1}{\gamma^p} \theta(y) \right] \\
&\quad \times e^{\tilde{w}_j(y)} e^{\frac{p-1}{p} \frac{1}{\gamma^p} \tilde{w}_{1j}(y)} e^{\left(\frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \tilde{w}_{2j}(y)} e^{\theta(y)} \\
&\quad \times e^{\frac{1}{2} \frac{p-1}{p} \frac{1}{\gamma^p} \left[\tilde{w}_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} \tilde{w}_{1j}(y) + \left(\frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \tilde{w}_{2j}(y) + \theta(y) \right]^2}.
\end{aligned}
$$

By the definition of $\tilde{w}_{1j}$ and $\tilde{w}_{2j}$ we get

$$I_a 1_{B(\xi_j', R_\varepsilon)} = \frac{O(1)}{|\log \varepsilon|}, \qquad I_b 1_{B(\xi_j', R_\varepsilon)} - e^{\tilde{w}_j(y)} = \frac{O(1)}{|\log \varepsilon|}. \tag{2.41}$$

Finally, in the remaining region, namely where for any $j = 1, \ldots, k$ we have $R_\varepsilon < |y - \xi_j'| < \frac{\delta}{\varepsilon}$, we have

$$|I_a| \leq \frac{C}{|\log \varepsilon|} e^{\tilde{w}_j(y)}, \qquad |I_b| \leq C e^{\tilde{w}_j(y)}. \tag{2.42}$$

Then, from (2.41) and the definition of $*$−norm, we find that very close to the point ξ_j on $\partial\Omega$, we have

$$\|f'(V_\lambda) - e^{\tilde{w}_j}\|_{*,\partial\Omega_\varepsilon} = \frac{O(1)}{|\log \varepsilon|}$$

which implies (2.38). Combining (2.40), (2.41) with (2.42) we obtain the estimate (2.39). □

Lemma 2.7. *We have*

$$\|f''(V_\lambda)\|_{*,\partial\Omega_\varepsilon} \leq C \tag{2.43}$$

for some positive constant.

Proof. We have

$$f''(V_\lambda) = \frac{(p-1)(p-2)}{p^2} \frac{1}{\gamma^{2p}} \left(1 + \frac{V_\lambda}{p\gamma^p}\right)^{p-3} e^{\gamma^p [(1 + \frac{V_\lambda}{p\gamma^p})^p - 1]}$$

$$+ \frac{3(p-1)}{p} \frac{1}{\gamma^p} \left(1 + \frac{V_\lambda}{p\gamma^p}\right)^{2p-3} e^{\gamma^p [(1 + \frac{V_\lambda}{p\gamma^p})^p - 1]}$$

$$+ \left(1 + \frac{V_\lambda}{p\gamma^p}\right)^{3(p-1)} e^{\gamma^p [(1 + \frac{V_\lambda}{p\gamma^p})^p - 1]} = I_c + I_d + I_e,$$

by a similar computation as before. Far away from the points ξ_j, namely for $|x - \xi_j| > \delta$, *i.e.* $|y - \xi_j'| > \frac{\delta}{\varepsilon}$, for all $j = 1, \ldots, k$, we have

$$I_c = \frac{\varepsilon^{\frac{2}{p}}}{|\log \varepsilon|^{p-1}} O(1), \quad I_d = \frac{\varepsilon^{\frac{2}{p}}}{|\log \varepsilon|^{2(p-1)}} O(1), \quad \text{and } I_e = \frac{\varepsilon^{\frac{2}{p}}}{|\log \varepsilon|^{3(p-1)}} O(1).$$

Then

$$f''(V_\lambda) 1_{\text{outer}} = \frac{\varepsilon^{\frac{2}{p}}}{|\log \varepsilon|^{p-1}} O(1), \tag{2.44}$$

where again $O(1)$ denotes a function which is uniformly bounded, as $\varepsilon \to 0$, in the considered region. Let us now fix the index j in $\{1, \ldots, k\}$; for $|y - \xi'_j| < R_\varepsilon$ with any $R_\varepsilon := R|\log \varepsilon|$ for some $R > 0$ large but fixed, by the Taylor expansion, we have

$$
I_c = \frac{(p-1)(p-2)}{p^2} \frac{1}{\gamma^{2p}} \left(1 + \frac{1}{p\gamma^p} \left(\tilde{w}_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} \tilde{w}_{1j}(y) \right.\right.
$$
$$
\left.\left. + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}} \tilde{w}_{2j}(y) + \theta(y) \right) \right)^{p-3}
$$
$$
\times\, e^{\gamma^p \left[\left(1 + \frac{1}{p\gamma^p} \left(\tilde{w}_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} \tilde{w}_{1j}(y) + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}} \tilde{w}_{2j}(y) + \theta(y) \right) \right)^p - 1 \right]}
$$
$$
= \frac{(p-2)(p-3)}{p^2} \frac{1}{\gamma^{2p}} \left[\frac{p-1}{p-3} + \frac{p-1}{p} \frac{1}{\gamma^p} \tilde{w}_j(y) + \frac{(p-1)^2}{p^2} \frac{1}{\gamma^{2p}} \tilde{w}_{1j}(y) \right.
$$
$$
\left. + \frac{(p-1)^3}{p^3} \frac{1}{\gamma^{3p}} \tilde{w}_{2j}(y) + \frac{p-1}{p} \frac{1}{\gamma^p} \theta(y) \right]
$$
$$
\times\, e^{\tilde{w}_j(y)} e^{\frac{p-1}{p} \frac{1}{\gamma^p} \tilde{w}_{1j}(y)} e^{\left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}} \tilde{w}_{2j}(y)} e^{\theta(y)}
$$
$$
\times\, e^{\frac{1}{2} \frac{p-1}{p} \frac{1}{\gamma^p} \left[\tilde{w}_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} \tilde{w}_{1j}(y) + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}} \tilde{w}_{2j}(y) + \theta(y) \right]^2} ;
$$

$$
I_d = \frac{3(p-1)}{p} \frac{1}{\gamma^p} \left(1 + \frac{1}{p\gamma^p} \left(\tilde{w}_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} \tilde{w}_{1j}(y) \right.\right.
$$
$$
\left.\left. + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}} \tilde{w}_{2j}(y) + \theta(y) \right) \right)^{2p-3}
$$
$$
\times\, e^{\gamma^p \left[\left(1 + \frac{1}{p\gamma^p} \left(\tilde{w}_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} \tilde{w}_{1j}(y) + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}} \tilde{w}_{2j}(y) + \theta(y) \right) \right)^p - 1 \right]}
$$
$$
= \frac{3(2p-3)}{p} \frac{1}{\gamma^p} \left[\frac{p-1}{2p-3} + \frac{p-1}{p} \frac{1}{\gamma^p} \tilde{w}_j(y) + \frac{(p-1)^2}{p^2} \frac{1}{\gamma^{2p}} \tilde{w}_{1j}(y) \right.
$$
$$
\left. + \frac{(p-1)^3}{p^3} \frac{1}{\gamma^{3p}} \tilde{w}_{2j}(y) + \frac{p-1}{p} \frac{1}{\gamma^p} \theta(y) \right]
$$
$$
\times\, e^{\tilde{w}_j(y)} e^{\frac{p-1}{p} \frac{1}{\gamma^p} \tilde{w}_{1j}(y)} e^{\left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}} \tilde{w}_{2j}(y)} e^{\theta(y)}
$$
$$
\times\, e^{\frac{1}{2} \frac{p-1}{p} \frac{1}{\gamma^p} \left[\tilde{w}_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} \tilde{w}_{1j}(y) + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}} \tilde{w}_{2j}(y) + \theta(y) \right]^2} ;
$$

and

$$
\begin{aligned}
I_e &= \left(1 + \frac{1}{p\gamma^p}\left(\tilde{w}_j(y) + \frac{p-1}{p}\frac{1}{\gamma^p}\tilde{w}_{1j}(y)\right.\right. \\
&\qquad \left.\left. + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\tilde{w}_{2j}(y) + \theta(y)\right)\right)^{3(p-1)} \\
&\quad \times e^{\gamma^p\left[\left(1 + \frac{1}{p\gamma^p}\left(\tilde{w}_j(y) + \frac{p-1}{p}\frac{1}{\gamma^p}\tilde{w}_{1j}(y) + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}}\tilde{w}_{2j}(y) + \theta(y)\right)\right)^p - 1\right]}
\end{aligned}
$$

$$
\begin{aligned}
&= \left[1 + \frac{3(p-1)}{p}\frac{1}{\gamma^p}\tilde{w}_j(y) + \frac{3(p-1)^2}{p^2}\frac{1}{\gamma^{2p}}\tilde{w}_{1j}(y)\right. \\
&\qquad \left. + \frac{3(p-1)^3}{p^3}\frac{1}{\gamma^{3p}}\tilde{w}_{2j}(y) + \frac{3(p-1)}{p}\frac{1}{\gamma^p}\theta(y)\right] \\
&\quad \times e^{\tilde{w}_j(y)}e^{\frac{p-1}{p}\frac{1}{\gamma^p}\tilde{w}_{1j}(y)}e^{\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\tilde{w}_{2j}(y)}e^{\theta(y)} \\
&\quad \times e^{\frac{1}{2}\frac{p-1}{p}\frac{1}{\gamma^p}\left[\tilde{w}_j(y) + \frac{p-1}{p}\frac{1}{\gamma^p}\tilde{w}_{1j}(y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\tilde{w}_{2j}(y) + \theta(y)\right]^2}.
\end{aligned}
$$

Therefore, we get

$$
I_c 1_{B(\xi_j', R_\varepsilon)} = \frac{O(1)}{|\log \varepsilon|}, \quad I_d 1_{B(\xi_j', R_\varepsilon)} = \frac{O(1)}{|\log \varepsilon|^2}, \quad I_e 1_{B(\xi_j', R_\varepsilon)} = O(1). \quad (2.45)
$$

Finally, for $R_\varepsilon < |y - \xi_j'| < \frac{\delta}{\varepsilon}$, for any j, we have

$$
|I_c| \le \frac{C}{|\log \varepsilon|}, \quad |I_d| \le \frac{C}{|\log \varepsilon|^2}, \quad |I_e| = O(1) + Ce^{\tilde{w}_j(y)}. \quad (2.46)
$$

From (2.44) and (2.45) with (2.46), by the definition of $*$-norm, we obtain that (2.43) holds. □

3. The linearized problem

In this section we prove the bounded invertibility of the operator L. First of all, we will solve the following linear problem: given $h \in C(\partial\Omega_\varepsilon)$, find a function ϕ such that

$$
\begin{cases}
-\Delta\phi + \varepsilon^2\phi = 0 & \text{in } \Omega_\varepsilon \\
L(\phi) = h + \sum_{j=1}^{k} c_j \chi_j Z_{1j} & \text{on } \partial\Omega_\varepsilon \\
\int_{\Omega_\varepsilon} \chi_j Z_{1j}\phi = 0 & \text{for } j = 1, \dots, k,
\end{cases} \quad (3.1)
$$

for certain scalars c_j, where the operator L is defined as in (2.31), and Z_{1j}, χ_j are defined as follows: set

$$F_j^\varepsilon(y) = \frac{1}{\varepsilon} F_j(\varepsilon y), \tag{3.2}$$

with F_j given by (2.17); then

$$Z_{ij}(y) = z_{ij}(F_j^\varepsilon(y)), \quad i = 0, 1, \quad j = 1, \ldots, k,$$

with z_{0j} and z_{1j} defined in (2.7) and (2.8) respectively.

Next, let us consider a large but fixed number $R_0 > 0$ and a non-negative radial and smooth cut-off function χ with $\chi(r) = 1$ if $r < R_0$ and $\chi(r) = 0$ if $r > R_0 + 1$, $0 \leq \chi \leq 1$. Then set

$$\chi_j(y) = \chi(|F_j^\varepsilon(y)|).$$

Equation (3.1) is solved in the following proposition.

Proposition 3.1. *Let $\delta > 0$ be a small but fixed number, assume (2.2) and (2.3), and let μ_j be given by (2.28). Then there exist positive numbers λ_0 and C such that problem (3.1) has a unique solution $\phi = T_\lambda(h)$ which satisfies*

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C \left(\log \frac{1}{\varepsilon} \right) \|h\|_{*, \partial \Omega_\varepsilon} \tag{3.3}$$

for all $\lambda < \lambda_0$.

We carry out the proof in the following steps.

Step 1: Constructing a suitable barrier.

Lemma 3.2. *There exist positive constants R_1 and C, independent of λ, such that if λ is small enough, there exists $\psi : \Omega_\varepsilon \setminus \cup_{j=1}^k B_{R_1}(\xi_j') \to \mathbb{R}$, smooth and positive, satisfying*

$$\begin{cases} -\Delta \psi + \varepsilon^2 \psi \geq \sum_{j=1}^k \dfrac{1}{|y - \xi_j'|^{2+\sigma}} + \varepsilon^2 & \text{in } \Omega_\varepsilon \setminus \cup_{j=1}^k B_{R_1}(\xi_j') \\[2mm] \dfrac{\partial \psi}{\partial \nu} - f'(V_\lambda)\psi \geq \sum_{j=1}^k \dfrac{1}{|y - \xi_j'|^{1+\sigma}} + \varepsilon & \text{on } \partial \Omega_\varepsilon \setminus \cup_{j=1}^k B_{R_1}(\xi_j') \\[2mm] \psi \geq 1 & \text{on } \Omega_\varepsilon \cap (\cup_{j=1}^k \partial B_{R_1}(\xi_j')). \end{cases}$$

Moreover, we have a uniform bound

$$0 < \psi \leq C \quad \text{in } \Omega_\varepsilon \setminus \cup_{j=1}^k B_{R_1}(\xi_j').$$

Proof. Let $\eta_j \in C_0^\infty(\mathbb{R}^2)$ be such that $0 \le \eta_j \le 1, \eta_j \equiv 1$ in $\Omega_\varepsilon \cap B_{\delta/2\varepsilon}(\xi_j'), \eta_j \equiv 0$ in $\Omega_\varepsilon \backslash B_{\delta/\varepsilon}(\xi_j'), |\nabla\eta_j| \le C\varepsilon$ in $\Omega_\varepsilon, |\Delta\eta_j| \le C\varepsilon^2$ in Ω. Let $\psi_0(y) = \tilde{\psi}(\varepsilon y)$, where $\tilde{\psi}$ is the solution to

$$\begin{cases} -\Delta\tilde{\psi} + \tilde{\psi} = 1 & \text{in } \Omega \\ \dfrac{\partial\tilde{\psi}}{\partial\nu} = 1 & \text{on } \partial\Omega, \end{cases}$$

so that

$$-\Delta\psi_0 + \varepsilon^2\psi = \varepsilon^2 \quad \text{in } \Omega_\varepsilon, \quad \text{and} \quad \frac{\partial\psi_0}{\partial\nu} = \varepsilon \quad \text{on } \partial\Omega_\varepsilon.$$

In particular, ψ_0 is uniformly bounded in Ω_ε. Take the function

$$\psi = \sum_{j=1}^k \eta_j \left[\frac{(y - \xi_j') \cdot \nu(\xi_j')}{r^{1+\sigma}} + C\frac{1}{r^\sigma} \right] + C\psi_0,$$

where $r = |y - \xi_j' - \mu_j\nu(\xi_j')|$. It is directly checked that ψ satisfies the required condition. $\qquad\square$

Step 2: Transferring a linear equation.
We first study the linear equation

$$\begin{cases} -\Delta\phi + \varepsilon^2\phi = h_1 & \text{in } \Omega_\varepsilon \\ \dfrac{\partial\phi}{\partial\nu} - f'(V_\lambda)\phi = h & \text{on } \partial\Omega_\varepsilon, \end{cases} \qquad (3.4)$$

where h_1, h are in suitable weight spaces: we consider for h the norm defined in (2.34) and for h_1

$$\|h_1\|_{**,\Omega_\varepsilon} := \sup_{y\in\Omega_\varepsilon} \left(\sum_{j=1}^k (1 + |y - \xi_j'|)^{-2-\sigma} + \varepsilon^2 \right)^{-1} |h_1(y)|. \qquad (3.5)$$

For the solution of (3.4) under some orthogonality conditions, we have an a priori estimate:

Lemma 3.3. *There are $R_0 > 0$ and $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$ and any solution of (3.4) with the orthogonality conditions*

$$\int_{\Omega_\varepsilon} \chi_j Z_{ij}\phi = 0 \quad \forall\, i = 0, 1, \quad j = 1, \ldots, k, \qquad (3.6)$$

we have

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \le C \left(\|h\|_{*,\partial\Omega_\varepsilon} + \|h_1\|_{**,\Omega_\varepsilon} \right) \qquad (3.7)$$

where C is independent of λ.

Proof. We take $R_0 = 2R_1$, with R_1 being the constant of Lemma 3.2. Thanks to the barrier ψ of that lemma we deduce that the following maximum principle holds in $\Omega_\varepsilon \setminus \cup_{j=1}^k B_{R_1}(\xi_j')$: if $\phi \in H^1(\Omega_\varepsilon \setminus \cup_{j=1}^k B_{R_1}(\xi_j'))$ satisfies

$$\begin{cases} -\Delta\phi + \varepsilon^2\phi \geq 0 & \text{in } \Omega_\varepsilon \setminus \cup_{j=1}^k B_{R_1}(\xi_j') \\ \dfrac{\partial\phi}{\partial\nu} - f'(V_\lambda)\phi \geq 0 & \text{on } \partial\Omega_\varepsilon \setminus \cup_{j=1}^k B_{R_1}(\xi_j') \\ \phi \geq 0 & \text{on } \Omega_\varepsilon \cap (\cup_{j=1}^k \partial B_{R_1}(\xi_j')), \end{cases}$$

then $\phi \geq 0$ in $\Omega_\varepsilon \setminus \cup_{j=1}^k B_{R_1}(\xi_j')$.

Let h_1, h be bounded and ϕ be a solution to (3.4) satisfying (3.6). Define the inner norm of ϕ as

$$\|\phi\|_i = \sup_{\Omega_\varepsilon \cap \left(\cup_{j=1}^k B_{R_1}(\xi_j')\right)} |\phi|,$$

and set

$$\tilde{\phi} = C_1\psi \left(\|\phi\|_i + \|h\|_{*,\partial\Omega_\varepsilon} + \|h_1\|_{**,\Omega_\varepsilon}\right)$$

with C_1 a constant independent of λ and ψ the function given in Lemma 3.2. By the above maximum principle we deduce that $\phi \leq \tilde{\phi}$ and $-\phi \leq \tilde{\phi}$ in $\Omega_\varepsilon \setminus \cup_{j=1}^k B_{R_1}(\xi_j')$. Since ψ is uniformly bounded, we have

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C \left(\|\phi\|_i + \|h\|_{*,\partial\Omega_\varepsilon} + \|h_1\|_{**,\Omega_\varepsilon}\right) \tag{3.8}$$

for some constant C independent of ϕ and λ.

We prove the lemma by contradiction. Assume that there exist a sequence $\lambda_n \to 0$, points $\xi_1^n, \ldots, \xi_k^n$ on $\partial\Omega$ satisfying (2.2) and functions ϕ_n, f_n and h_n with $\|\phi_n\|_{L^\infty(\Omega_{\varepsilon_n})} = 1$, $\|h_1\|_{**,\Omega_{\varepsilon_n}} \to 0$, $\|h\|_{*,\partial\Omega_{\varepsilon_n}} \to 0$, such that, for each n, ϕ_n solves (3.4) satisfying (3.6). By (3.8) we see that $\|\phi_n\|_i$ stays away from zero. For some of the indices, say j, we can assume that $\sup_{B_{R_1}(\xi_j')} |\phi_n| \geq c > 0$ for all n. Consider $\hat{\phi}_n = \phi_n(z - \xi_j')$, and let us translate and rotate Ω_{ε_n} such that Ω_{ε_n} approaches the upper half-plane and $\xi_j' = 0$. Then by an elliptic estimate $\hat{\phi}_n$ converges uniformly on compact sets to a nontrivial solution $\hat{\phi}$ of (2.6). By Proposition 2.2 $\hat{\phi}$ is a linear combination of z_{0j} and z_{1j}. On the other hand, taking limit in the orthogonality relation (3.6), we find that $\int_{\mathbb{R}_+^2} \chi\hat{\phi}z_{ij} = 0$ for $i = 0, 1$. This contradicts the fact that $\hat{\phi} \not\equiv 0$. □

Step 3: Establishing an a priori estimate.
In what follows, we will establish an a priori estimate for a solution to (3.4) with the orthogonality condition $\int_{\Omega_\varepsilon} \chi_j Z_{1j}\phi = 0$ only.

Lemma 3.4. *For small enough* λ, *if* ϕ *is a solution of* (3.4) *and satisfies*

$$\int_{\Omega_\varepsilon} \chi_j Z_{1j} \phi = 0 \quad \forall\, j = 1, \dots, k, \tag{3.9}$$

then there holds

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \le C |\log \varepsilon| \left(\|h\|_{*,\partial\Omega_\varepsilon} + \|h_1\|_{**,\Omega_\varepsilon} \right), \tag{3.10}$$

where C is independent of λ.

Proof. Let ϕ satisfy (3.4) and (3.9). In order to use Lemma 3.3, we will modify ϕ to $\tilde{\phi}$ so to satisfy all orthogonality relations with respect to Z_{ij} for $i = 0, 1$. Let $R > R_0 + 1$ be large but fixed, $\delta > 0$ be small and fixed. Set

$$\hat{Z}_{0j}(y) = \psi Z_{0j}(y),$$

where

$$\psi(y) = \tilde{h}(|F_j^\varepsilon(y)|), \qquad \tilde{h}(x) = \frac{\log(\delta/\varepsilon) - \log|x|}{\log(\delta/\varepsilon) - \log R}$$

with F_j^ε the change of variables defined in (3.2). We observe that $\tilde{h}$ is just the solution to

$$\begin{cases} \Delta \tilde{h} = 0 & \text{in } B_{\delta/\varepsilon}(0) \setminus \bar{B}_R(0) \\ \tilde{h} = 1 & |x| = R \\ \tilde{h} = 0 & |x| = \delta/\varepsilon. \end{cases}$$

Let $\bar{\eta}_{1j}, \bar{\eta}_{2j}$ be radial smooth cut-off functions on $\mathbb{R}^2$ such that

$$0 \le \bar{\eta}_{1j} \le 1,\ |\nabla \bar{\eta}_{1j}| \le C \quad \text{in } \mathbb{R}^2,$$
$$\bar{\eta}_{1j} \equiv 1 \quad \text{in } B_R(0), \quad \bar{\eta}_{1j} = 0 \quad \text{in } \mathbb{R}^2 \setminus B_{R+1}(0),$$

and

$$0 \le \bar{\eta}_{2j} \le 1,\ |\nabla \bar{\eta}_{2j}| \le C\varepsilon/\delta,\ |\nabla^2 \bar{\eta}_{2j}| \le C\varepsilon^2/\delta^2 \quad \text{in } \mathbb{R}^2,$$
$$\bar{\eta}_{2j} \equiv 1 \quad \text{in } B_{\frac{\delta}{4\varepsilon}}(0), \quad \bar{\eta}_{2j} = 0 \quad \text{in } \mathbb{R}^2 \setminus B_{\frac{\delta}{3\varepsilon}}(0).$$

Now, we write

$$\eta_{1j}(y) = \bar{\eta}_{1j}(F_j^\varepsilon(y)), \quad \eta_{2j}(y) = \bar{\eta}_{2j}(F_j^\varepsilon(y)). \tag{3.11}$$

Define

$$\tilde{Z}_{0j} = \eta_{1j} Z_{0j} + (1 - \eta_{1j})\eta_{2j} \hat{Z}_{0j}.$$

Given ϕ satisfying (3.4) and (3.9), we set

$$\tilde{\phi} = \phi + \sum_{j=1}^{k} d_j \tilde{Z}_{0j}, \quad \text{where } d_j = -\frac{\int_{\Omega_\varepsilon} \chi_j Z_{0j} \phi}{\int_{\Omega_\varepsilon} Z_{0j}^2 \chi_j}.$$

Therefore, our result is a direct consequence of the following:
Claim:

$$|d_j| \leq C |\log \varepsilon| \left(\|h\|_{*,\partial\Omega_\varepsilon} + \|h_1\|_{**,\Omega_\varepsilon} \right) \quad \forall \, j = 1, \ldots, k. \qquad (3.12)$$

First, using the notation $\widetilde{L} = -\Delta + \varepsilon^2 I$, we observe that $\tilde{\phi}$ satisfies

$$\begin{cases} \widetilde{L}(\tilde{\phi}) = h_1 + \sum_{j=1}^{k} d_j \widetilde{L}(\tilde{Z}_{0j}) & \text{in } \Omega_\varepsilon \\ \dfrac{\partial \tilde{\phi}}{\partial \nu} - f'(V_\lambda)\tilde{\phi} = h + \sum_{j=1}^{k} d_j \left(\dfrac{\partial \tilde{Z}_{0j}}{\partial \nu} - f'(V_\lambda)\tilde{Z}_{0j} \right) & \text{on } \partial\Omega_\varepsilon. \end{cases} \qquad (3.13)$$

Thus, by Lemma 3.3, we have

$$\|\tilde{\phi}\|_{L^\infty(\Omega_\varepsilon)} \leq C \sum_{j=1}^{k} |d_j| \left(\left\| \frac{\partial \tilde{Z}_{0j}}{\partial \nu} - f'(V_\lambda)\tilde{Z}_{0j} \right\|_{*,\partial\Omega_\varepsilon} + \|\widetilde{L}(\tilde{Z}_{0j})\|_{**,\Omega_\varepsilon} \right) \qquad (3.14)$$
$$+ C\|h\|_{*,\partial\Omega_\varepsilon} + C\|h_1\|_{**,\Omega_\varepsilon}.$$

Multiplying the first equation in (3.13) by $\tilde{Z}_{0l}$, integrating by parts and using the second equation in (3.13), we find

$$d_l \left[\int_{\Omega_\varepsilon} \widetilde{L}(\tilde{Z}_{0l})\tilde{Z}_{0l} + \int_{\partial\Omega_\varepsilon} \tilde{Z}_{0l} \left(\frac{\partial \tilde{Z}_{0l}}{\partial \nu} - f'(V_\lambda)\tilde{Z}_{0l} \right) \right]$$
$$= -\int_{\partial\Omega_\varepsilon} h\tilde{Z}_{0l} - \int_{\Omega_\varepsilon} h_1\tilde{Z}_{0l} + \int_{\partial\Omega_\varepsilon} \tilde{\phi} \left(\frac{\partial \tilde{Z}_{0l}}{\partial \nu} - f'(V_\lambda)\tilde{Z}_{0l} \right) + \int_{\Omega_\varepsilon} \tilde{\phi}\widetilde{L}(\tilde{Z}_{0l}). \qquad (3.15)$$

Thus by (3.14), we deduce that

$$d_l \left[\int_{\Omega_\varepsilon} \widetilde{L}(\tilde{Z}_{0l})\tilde{Z}_{0l} + \int_{\partial\Omega_\varepsilon} \tilde{Z}_{0l} \left(\frac{\partial \tilde{Z}_{0l}}{\partial \nu} - f'(V_\lambda)\tilde{Z}_{0l} \right) \right]$$
$$\leq C\|h\|_{*,\partial\Omega_\varepsilon} + C\|h_1\|_{**,\Omega_\varepsilon} + \|\tilde{\phi}\|_{L^\infty(\Omega_\varepsilon)} \left\| \frac{\partial \tilde{Z}_{0l}}{\partial \nu} - f'(V_\lambda)\tilde{Z}_{0l} \right\|_{*,\partial\Omega_\varepsilon}$$
$$+ \|\tilde{\phi}\|_{L^\infty(\Omega_\varepsilon)}\|\widetilde{L}(\tilde{Z}_{0l})\|_{**,\Omega_\varepsilon} \qquad (3.16)$$
$$\leq C\left(\|h\|_{*,\partial\Omega_\varepsilon} + \|h_1\|_{**,\Omega_\varepsilon} \right) \left(1 + \left\| \frac{\partial \tilde{Z}_{0l}}{\partial \nu} - f'(V_\lambda)\tilde{Z}_{0l} \right\|_{*,\partial\Omega_\varepsilon} + \|\widetilde{L}(\tilde{Z}_{0l})\|_{**,\Omega_\varepsilon} \right)$$
$$+ C \sum_{j=1}^{k} \left(\left\| \frac{\partial \tilde{Z}_{0j}}{\partial \nu} - f'(V_\lambda)\tilde{Z}_{0j} \right\|_{*,\partial\Omega_\varepsilon} + \|\widetilde{L}(\tilde{Z}_{0j})\|_{**,\Omega_\varepsilon} \right).$$

We achieve the claim by proving the following estimates: for some constant $C > 0$, independent of λ,

$$\int_{\Omega_\varepsilon} \widetilde{L}(\tilde{Z}_{0j})\tilde{Z}_{0j} + \int_{\partial\Omega_\varepsilon} \tilde{Z}_{0j}\left(\frac{\partial\tilde{Z}_{0j}}{\partial\nu} - f'(V_\lambda)\tilde{Z}_{0j}\right) \geq \frac{C}{|\log\varepsilon|}, \qquad (3.17)$$

$$\|\widetilde{L}(\tilde{Z}_{0j})\|_{**,\Omega_\varepsilon} \leq \frac{C}{|\log\varepsilon|}, \qquad (3.18)$$

$$\left\|\frac{\partial\tilde{Z}_{0j}}{\partial\nu} - f'(V_\lambda)\tilde{Z}_{0j}\right\|_{*,\partial\Omega_\varepsilon} \leq \frac{C}{|\log\varepsilon|}. \qquad (3.19)$$

In [6] it is showed that estimates (3.17), (3.18) and (3.19) hold. $\qquad \square$

Step 4: In proving the solvability of (3.1), we may first solve the following problem: given $h \in L^\infty(\Omega_\varepsilon)$ find $\phi \in L^\infty(\Omega_\varepsilon)$ and $d_1, \ldots, d_k \in \mathbb{R}$ such that

$$\begin{cases} -\Delta\phi + \varepsilon^2\phi = \sum_{j=1}^{k} d_j\chi_j Z_{1j} & \text{in } \Omega_\varepsilon \\ \dfrac{\partial\phi}{\partial\nu} - f'(V_\lambda)\phi = h & \text{on } \partial\Omega_\varepsilon \\ \displaystyle\int_{\Omega_\varepsilon} \chi_j Z_{1j}\phi = 0 & \text{for } j = 1, \ldots, k. \end{cases} \qquad (3.20)$$

We first prove that for any $\phi, d_1, \ldots, d_k$ solution to (3.20) the bound

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C|\log\varepsilon|\|h\|_{*,\partial\Omega_\varepsilon} \qquad (3.21)$$

holds. In fact, by Lemma 3.4, we have

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C|\log\varepsilon|\left(\|h\|_{*,\partial\Omega_\varepsilon} + \sum_{j=1}^{k}|d_j|\right) \qquad (3.22)$$

and therefore it is enough to prove that $|d_j| \leq C\|h\|_{*,\partial\Omega_\varepsilon}$.

Let η_{2j} be the cut-off function defined in (3.11), and multiply (3.20) by $\eta_{2l}Z_{1l}$. Integrating by parts we get

$$\begin{aligned} d_l\int_{\Omega_\varepsilon}\chi_l Z_{1l}^2 = &-\int_{\partial\Omega_\varepsilon} h\eta_{2l}Z_{1l} + \int_{\partial\Omega_\varepsilon}\phi\frac{\partial\eta_{2l}}{\partial\nu}Z_{1l} \\ &+ \int_{\partial\Omega_\varepsilon}\phi\eta_{2l}Z_{1l}\left(\frac{\partial Z_{1l}}{\partial\nu} - f'(V_\lambda)Z_{1l}\right) \\ &+ \int_{\Omega_\varepsilon}\phi(-\Delta(\eta_{2l}Z_{1l}) + \varepsilon^2\eta_{2l}Z_{1l}). \end{aligned} \qquad (3.23)$$

Since, $Z_{1l} = O\left(\frac{1}{1+r}\right)$ and $\nabla \eta_{2l} = O(\varepsilon)$, we have

$$\left| \int_{\partial \Omega_\varepsilon} \phi \frac{\partial \eta_{2l}}{\partial \nu} Z_{1l} \right| \leq C\varepsilon \log \frac{1}{\varepsilon}.$$

On the other hand, we can estimate that

$$\frac{\partial Z_{1l}}{\partial \nu} - f'(V_\lambda)Z_{1l} = O\left(\frac{\varepsilon}{1+r}\right) + O\left(\frac{\varepsilon^\alpha}{1+r^2}\right), \quad |y| < \frac{\delta}{\varepsilon}, \quad y \in \partial \Omega_\varepsilon,$$

which implies that

$$\int_{\partial \Omega_\varepsilon} \left| \frac{\partial Z_{1l}}{\partial \nu} - f'(V_\lambda)Z_{1l} \right| = O(\varepsilon^\alpha). \tag{3.24}$$

Moreover, we deduce that

$$\int_{\partial \Omega_\varepsilon} \left| -\Delta(\eta_{2l}Z_{1l}) + \varepsilon^2 \eta_{2l}Z_{1l} \right| = O\left(\varepsilon \log \frac{1}{\varepsilon}\right). \tag{3.25}$$

Thus from (3.23)-(3.25), we conclude that

$$d_l \int_{\Omega_\varepsilon} \chi_l Z_{1l}^2 \leq C\|h\|_{*,\partial \Omega_\varepsilon} + C\varepsilon^\alpha \|\phi\|_{L^\infty(\Omega_\varepsilon)}. \tag{3.26}$$

Combining (3.22) and (3.26) we have

$$|d_l| \leq C\left(\|h\|_{*,\partial \Omega_\varepsilon} + C\varepsilon^\alpha \log \frac{1}{\varepsilon^2} \sum_{j=1}^k |d_j|\right).$$

This implies that

$$|d_l| \leq C\|h\|_{*,\partial \Omega_\varepsilon} \tag{3.27}$$

which proves (3.21).

Now consider the Hilbert space

$$\mathbb{H} = \left\{ \phi \in H^1(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} \chi_j Z_{1j}\phi = 0 \quad \forall j = 1, \ldots, k \right\},$$

endowed the norm $\|\phi\|_{H^1}^2 = \int_{\Omega_\varepsilon} |\nabla\phi|^2 + \varepsilon^2\phi^2$. Problem (3.20), expressed in a weak form, is equivalent to seeking $\phi \in \mathbb{H}$ such that

$$\int_{\Omega_\varepsilon} \nabla\phi\nabla\psi + \varepsilon^2\phi\psi - \int_{\partial \Omega_\varepsilon} f'(V_\lambda)\psi = \int_{\partial \Omega_\varepsilon} h\psi, \quad \text{for all } \psi \in \mathbb{H}.$$

Fredholm's alternative guarantees unique solvability of (3.20), which is satisfied by (3.21).

Step 5: In order to solve (3.1), let $Y_i \in L^\infty(\Omega_\varepsilon)$, $d_{ij} \in \mathbb{R}$ be the solution of (3.20) with $h = \chi_i Z_{1i}$, that is

$$
\begin{cases}
-\Delta Y_i + \varepsilon^2 Y_i = \sum_{j=1}^{k} d_j \chi_j Z_{1j} & \text{in } \Omega_\varepsilon \\[2mm]
\dfrac{\partial Y_i}{\partial \nu} - f'(V_\lambda) Y_i = \chi_i Z_{1i} & \text{on } \partial\Omega_\varepsilon \\[2mm]
\displaystyle\int_{\Omega_\varepsilon} \chi_j Z_{1j} Y_i = 0 & \text{for } j = 1, \ldots, k.
\end{cases}
\tag{3.28}
$$

From Step 4, there is a unique solution $Y_i \in L^\infty(\Omega_\varepsilon)$ of (3.28), and

$$
\|Y_i\|_{L^\infty(\Omega_\varepsilon)} \leq C |\log \varepsilon|, \qquad |d_{ij}| \leq C
\tag{3.29}
$$

for some constant C independent of λ.

Multiplying (3.28) by $\eta_{2j} Z_{1j}$, and integrates by parts, we have

$$
d_{ij} \int_{\Omega_\varepsilon} \chi_j Z_{1j}^2 + \delta_{ij} \int_{\partial\Omega_\varepsilon} \chi_j Z_{1j}^2 = \int_{\partial\Omega_\varepsilon} \left(\frac{\partial Z_{1j}}{\partial \nu} - f'(V_\lambda) Z_{1j} \right) \eta_{2j} Y_i
$$
$$
+ \int_{\partial\Omega_\varepsilon} \frac{\partial \eta_{2j}}{\partial \nu} Z_{1j} Y_i,
$$

where δ_{ij} is Kronecker's delta. From (3.24), (3.25) and (3.27) we obtain

$$
d_{ij} \int_{\Omega_\varepsilon} \chi_j Z_{1j}^2 + \delta_{ij} \int_{\partial\Omega_\varepsilon} \chi_j Z_{1j}^2 = O\left(\varepsilon^\alpha \log \frac{1}{\varepsilon} \right).
$$

Then we get

$$
d_{ij} = A\delta_{ij} + O\left(\varepsilon^\alpha \log \frac{1}{\varepsilon} \right)
\tag{3.30}
$$

with $A > 0$ is independent of ε. Hence the matrix D with entries d_{ij} in invertible for small ε and $\|D^{-1}\| \leq C$ uniformly in ε. Then, given $h \in L^\infty(\partial\Omega_\varepsilon)$ we find ϕ_1, $d_1, \ldots, d_k$, the solution to (3.28) and define

$$
\phi = \phi_1 + \sum_{i=1}^{k} c_i Y_i,
$$

where c_i satisfies

$$
\sum_{i=1}^{k} c_i d_{ij} = -d_j, \quad \forall\, j = 1, \ldots, k.
$$

Then ϕ satisfies (3.1) and we have

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq \|\phi_1\|_{L^\infty(\Omega_\varepsilon)} + \log\frac{1}{\varepsilon}\sum_{i=1}^{k}|c_i|$$

$$\leq C\log\frac{1}{\varepsilon}\|h\|_{*,\partial\Omega_\varepsilon} + \log\frac{1}{\varepsilon}\sum_{i=1}^{k}|d_i|$$

$$\leq C\log\frac{1}{\varepsilon}\|h\|_{*,\partial\Omega_\varepsilon}$$

by (3.27). This finishes the proof of Proposition 3.1.

Remark 3.5. A slight modification of the proof above also shows that for any $h \in L^\infty(\partial\Omega_\varepsilon)$ and $h_1 \in L^\infty(\Omega_\varepsilon)$, the equation

$$\begin{cases} -\Delta\phi + \varepsilon^2\phi = h_1 & \text{in } \Omega_\varepsilon \\ L(\phi) = h + \sum_{j=1}^{k} c_j\chi_j Z_{1j} & \text{on } \partial\Omega_\varepsilon \\ \int_{\Omega_\varepsilon}\chi_j Z_{1j}\phi = 0 & \text{for } j = 1,\ldots,k, \end{cases}$$

has a unique solution $\phi, c_1, \ldots, c_k$ satisfying

$$\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq C\log\frac{1}{\varepsilon}\left(\|h\|_{*,\partial\Omega_\varepsilon} + \|h_1\|_{**,\Omega_\varepsilon}\right),$$
$$|c_j| \leq C\left(\|h\|_{*,\partial\Omega_\varepsilon} + \|h_1\|_{**,\Omega_\varepsilon}\right), \quad \forall j = 1,\ldots,k$$

for some C independent of λ.

Proposition 3.1 implies that the unique solution $\phi = T_\lambda(h)$ of (3.1) defines a continuous linear map from the Banach space $\mathcal{C}_*$ of all functions h in L^∞ for which $\|h\|_* < \infty$ into L^∞, with norm bounded uniformly in λ.

Lemma 3.6. *The operator T_λ is differentiable with respect to the variables $\xi_1, \ldots, \xi_k$ on $\partial\Omega$ satisfying (2.2), and one has the estimate*

$$\|\partial_{\xi_l'}T_\lambda(h)\|_{L^\infty(\Omega_\varepsilon)} \leq C\left(\log\frac{1}{\varepsilon}\right)^2\|h\|_{*,\partial\Omega_\varepsilon} \quad \text{for } l = 1,\ldots,k, \qquad (3.31)$$

for a given positive C, independent of λ, and for all small enough λ.

Proof. Differentiating equation (3.1), formally $Z := \partial_{\xi_l'}\phi$ should satisfy in Ω_ε the equation

$$-\Delta Z + \varepsilon^2 Z = 0 \quad \text{in } \Omega_\varepsilon,$$

and on the boundary $\partial\Omega_\varepsilon$

$$L(Z) = -\partial_{\xi_l'}(f'(V_\lambda))\phi + c_l\partial_{\xi_l'}(\chi_l Z_{1l}) + \sum_{j=1}^{k} d_j Z_{1j}\chi_j$$

with $d_j = \partial_{\xi_l'} c_j$; moreover the orthogonality conditions now become

$$\int_{\Omega_\varepsilon} Z_{1j}\chi_j Z = 0 \quad \text{if } j \neq l$$

$$\int_{\Omega_\varepsilon} Z_{1l}\chi_l Z = -\int_{\Omega_\varepsilon} \partial_{\xi_l'}(Z_{1l}\chi_l)\phi.$$

We consider the constants b_l defined by

$$b_l \int_{\Omega_\varepsilon} \chi_l^2 |Z_{1l}^2 = \int_{\Omega_\varepsilon} \partial_{\xi_l'}(Z_{1l}\chi_l)\phi, \quad \text{for } l = 1, \ldots, k.$$

Define

$$\tilde{Z} = Z + b_l \chi_l Z_{1l}.$$

We then have

$$\begin{cases} -\Delta\tilde{Z} + \varepsilon^2\tilde{Z} = a & \text{in } \Omega_\varepsilon \\ L(\tilde{Z}) = b + \sum_{j=1}^{k} d_j Z_{1j}\chi_j & \text{on } \partial\Omega_\varepsilon \\ \int_{\Omega_\varepsilon} \chi_j Z_{1j}\tilde{Z} = 0 & \text{for } j = 1, \ldots, k, \end{cases}$$

where

$$a = b_l(-\Delta(\chi_l Z_{1l}) + \varepsilon^2\chi_l Z_{1l}),$$
$$b = -\partial_{\xi_l'}(f'(V_\lambda))\phi + c_l\partial_{\xi_l'}(Z_{1l}\chi_l) + L(\chi_l Z_{1l}),$$

and we have

$$\|a\|_{**,\Omega_\varepsilon} \leq C\log\frac{1}{\varepsilon}\|h\|_{*,\partial\Omega_\varepsilon}, \quad \|b\|_{*,\partial\Omega_\varepsilon} \leq C\log\frac{1}{\varepsilon}\|h\|_{*,\partial\Omega_\varepsilon}.$$

Hence, using Proposition 3.1 we obtain

$$\|\partial_{\xi_l'} T_\lambda(h)\|_{L^\infty(\Omega_\varepsilon)} \leq C\left(\log\frac{1}{\varepsilon}\right)^2 \|h\|_{*,\partial\Omega_\varepsilon} \quad \text{for } l = 1, \ldots, k. \qquad \square$$

4. The nonlinear problem

Let us now introduce the following auxiliary nonlinear problem

$$
\begin{cases}
-\Delta\phi + \varepsilon^2\phi = 0 & \text{in } \Omega_\varepsilon \\
L(\phi) = E_\lambda + N(\phi) + \sum_{j=1}^{k} c_j\chi_j Z_{1j} & \text{on } \partial\Omega_\varepsilon \\
\displaystyle\int_{\Omega_\varepsilon} \chi_j Z_{1j}\phi = 0 & \text{for } j = 1, \dots, k.
\end{cases}
\tag{4.1}
$$

We have the following result:

Proposition 4.1. *Under the condition of Proposition 3.1, there exist positive numbers λ_0 and C such that problem (4.1) has a unique solution ϕ which satisfies*

$$
\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq \frac{C}{|\log\varepsilon|^2},
\tag{4.2}
$$

for all $\lambda < \lambda_0$. Moreover, if we consider the map $\xi' \mapsto \phi$ into the space $C(\bar{\Omega}_\varepsilon)$, the derivative $D_{\xi'}\phi$ exists and defines a continuous function of ξ'. Besides, there is a constant $C > 0$ such that

$$
\|D_{\xi'}\phi\|_{L^\infty(\Omega_\varepsilon)} \leq \frac{C}{|\log\varepsilon|}.
\tag{4.3}
$$

Proof. In terms of the operator T_λ defined in Proposition 3.1, problem (4.1) becomes

$$
\phi = T_\lambda\left(N(\phi) + E_\lambda\right) := A(\phi).
\tag{4.4}
$$

For a given number $M > 0$, let us consider the region

$$
\mathcal{F}_M := \left\{\phi \in C(\bar{\Omega}_\varepsilon) \ : \ \|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq \frac{M}{|\log\varepsilon|^2}\right\}.
$$

From Proposition 3.1 we get

$$
\|A(\phi)\|_{L^\infty(\Omega_\varepsilon)} \leq C\left(\log\frac{1}{\varepsilon}\right)\left[\|N(\phi)\|_{*,\partial\Omega_\varepsilon} + \|E_\lambda\|_{*,\partial\Omega_\varepsilon}\right].
$$

From (2.35) and (2.43), by the definition of $N(\phi)$ in (2.33), we have

$$
\|A(\phi)\|_{L^\infty(\Omega_\varepsilon)} \leq C|\log\varepsilon|\left(C\|\phi\|_{L^\infty(\Omega_\varepsilon)}^2 + \frac{1}{|\log\varepsilon|^3}\right).
$$

We then get that $A(\mathcal{F}_M) \subset \mathcal{F}_M$ for a sufficiently large but fixed M and all small λ. Moreover, for any $\phi_1, \phi_2 \in \mathcal{F}_M$, one has

$$\|N(\phi_1) - N(\phi_2)\|_{*,\partial\Omega_\varepsilon} \le C \left(\max_{i=1,2} \|\phi_i\|_{L^\infty(\Omega_\varepsilon)} \right) \|\phi_1 - \phi_2\|_{L^\infty(\Omega_\varepsilon)}.$$

In fact,

$$
\begin{aligned}
N(\phi_1) - N(\phi_2) &= f(V_\lambda + \phi_1) - f(V_\lambda + \phi_2) - f'(V_\lambda)(\phi_1 - \phi_2) \\
&= \int_0^1 \left(\frac{d}{dt} f(V_\lambda + \phi_2 + t(\phi_1 - \phi_2)) \right) dt - f'(V_\lambda)(\phi_1 - \phi_2) \\
&= \int_0^1 \left(f'(V_\lambda + \phi_2 + t(\phi_1 - \phi_2)) - f'(V_\lambda) \right) dt \, (\phi_1 - \phi_2).
\end{aligned}
$$

Thus, for a certain $t^* \in (0, 1)$, and $s \in (0, 1)$

$$|N(\phi_1) - N(\phi_2)| \le C|f'(V_\lambda + \phi_2 + t^*(\phi_1 - \phi_2)) - f'(V_\lambda)|\|\phi_1 - \phi_2\|_{L^\infty(\Omega_\varepsilon)}$$
$$\le C|f''(V_\lambda + s\phi_2 + t^*(\phi_1 - \phi_2))|\left(\|\phi_1\|_{L^\infty(\Omega_\varepsilon)} + \|\phi_2\|_{L^\infty(\Omega_\varepsilon)} \right) \|\phi_1 - \phi_2\|_{L^\infty(\Omega_\varepsilon)}.$$

Thanks to (2.43) and the fact that $\|\phi_1\|_{L^\infty(\Omega_\varepsilon)}, \|\phi_2\|_{L^\infty(\Omega_\varepsilon)} \to 0$ as $\lambda \to 0$, we conclude that

$$
\begin{aligned}
&\|N(\phi_1) - N(\phi_2)\|_{*,\partial\Omega_\varepsilon} \\
&\le C\|f''(V_\lambda)\|_{*,\partial\Omega_\varepsilon} \left(\|\phi_1\|_{L^\infty(\Omega_\varepsilon)} + \|\phi_2\|_{L^\infty(\Omega_\varepsilon)} \right) \|\phi_1 - \phi_2\|_{L^\infty(\Omega_\varepsilon)} \\
&\le C(\|\phi_1\|_{L^\infty(\Omega_\varepsilon)} + \|\phi_2\|_{L^\infty(\Omega_\varepsilon)})\|\phi_1 - \phi_2\|_{L^\infty(\Omega_\varepsilon)}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\|A(\phi_1) - A(\phi_2)\|_{L^\infty(\Omega_\varepsilon)} &\le C|\log\varepsilon|\|N(\phi_1) - N(\phi_2)\|_{*,\partial\Omega_\varepsilon} \\
&\le C|\log\varepsilon| \left(\max_{i=1,2} \|\phi_i\|_{L^\infty(\Omega_\varepsilon)} \right) \|\phi_1 - \phi_2\|_{L^\infty(\Omega_\varepsilon)}.
\end{aligned}
$$

Thus the operator A has a small Lipschitz constant in $\mathcal{F}_M$ for all small λ, therefore a unique fixed point of A exists in this region.

We shall next analyze the differentiability of the map $\xi' = (\xi'_1, \ldots, \xi'_k) \mapsto \phi$. Assume for instance that the partial derivative $\partial_{\xi'_l}\phi$ exists, for $l = 1, \ldots, k$. Since $\phi = T_\lambda(N(\phi) + E_\lambda)$, formally we have

$$\partial_{\xi'_l}\phi = (\partial_{\xi'_l}T_\lambda)(N(\phi) + E_\lambda) + T_\lambda\left(\partial_{\xi'_l}N(\phi) + \partial_{\xi'_l}E_\lambda \right).$$

From (3.31), we have

$$\|\partial_{\xi'_l}T_\lambda(N(\phi) + E_\lambda)\|_{L^\infty(\Omega_\varepsilon)} \le C|\log\varepsilon|^2\|N(\phi) + E_\lambda\|_{*,\partial\Omega_\varepsilon} \le \frac{C}{|\log\varepsilon|}.$$

On the other hand,

$$\partial_{\xi'_m} N(\phi) = \left[f'(V_\lambda + \phi) - f'(V_\lambda) - f''(V_\lambda)\phi \right] \partial_{\xi'_l} V_\lambda + \partial_{\xi'_l} \left[f'(V_\lambda) - e^{w_{\mu j}} \right] \phi$$
$$+ \left[f'(V_\lambda + \phi) - f'(V_\lambda) \right] \partial_{\xi'_l} \phi + \left[f'(V_\lambda) - e^{w_{\mu j}} \right] \partial_{\xi'_l} \phi.$$

Then,

$$\|\partial_{\xi'_l} N(\phi)\|_{*,\partial\Omega_\varepsilon} \le C \left\{ \|\phi\|^2_{L^\infty(\Omega_\varepsilon)} + \frac{1}{|\log\varepsilon|} \|\phi\|_{L^\infty(\Omega_\varepsilon)} \right.$$
$$\left. + \|\partial_{\xi'_l}\phi\|_{L^\infty(\Omega_\varepsilon)} \|\phi\|_{L^\infty(\Omega_\varepsilon)} + \frac{1}{|\log\varepsilon|} \|\partial_{\xi'_l}\phi\|_{L^\infty(\Omega_\varepsilon)} \right\}.$$

Since $\|\partial_{\xi'_l} E_\lambda\|_{*,\partial\Omega_\varepsilon} \le \frac{C}{|\log\varepsilon|^3}$, by Proposition 3.1 we then have

$$\|\partial_{\xi'_l}\phi\|_{L^\infty(\Omega_\varepsilon)} \le \frac{C}{|\log\varepsilon|}$$

for all $l = 1, \dots, k$. Hence the regularity of the map $\xi' \mapsto \phi$ can be proved by standard arguments involving the implicit function theorem and the fixed point representation (4.4). This concludes proof of the proposition. $\quad\square$

5. Variational reduction

After problem (4.1) has been solved, in order to find a solution to the original problem we need to find ξ' such that

$$c_j(\xi') = 0 \quad \text{for all } j = 1, \dots, k. \tag{5.1}$$

This problem is indeed variational: it is equivalent to finding the critical points of a function of $\xi = \varepsilon\xi'$. Associated to (1.1), let us introduce the energy functional $J_\lambda : H^1(\Omega) \to \mathbb{R}$ given by

$$J_\lambda(u) = \frac{1}{2} \int_\Omega (|\nabla u|^2 + u^2) - \frac{\lambda}{p} \int_{\partial\Omega} e^{u^p}, \tag{5.2}$$

and the finite-dimensional restriction

$$F_\lambda(\xi) = J_\lambda \left(U_\lambda(\xi) + \tilde\phi(\xi) \right), \tag{5.3}$$

where $\tilde\phi = \tilde\phi(\xi) = \tilde\phi(x,\xi)$ is the function defined in Ω by the relation $\tilde\phi(x,\xi) = \phi(\frac{x}{\varepsilon}, \frac{\xi}{\varepsilon})$, with ϕ the unique solution to problem (4.1) given by Proposition 3.1.

Lemma 5.1. *The functional $F_\lambda(\xi)$ is of class C^1. Moreover, for all $\lambda > 0$ sufficiently small, if $D_\xi F_\lambda(\xi) = 0$, then ξ satisfies (5.1).*

Proof. A direct consequence of Proposition 4.1 and the definition of the function U_λ is the fact that the map $\xi \mapsto F_\lambda(\xi)$ is of class C^1. Define

$$I_\lambda(v) = \frac{1}{2}\int_{\Omega_\varepsilon}\left(|\nabla v|^2 + \varepsilon^2 v^2\right) - \int_{\partial\Omega_\varepsilon} e^{\gamma^p[(1+\frac{v}{p\gamma^p})^p - 1]}.$$

Let us differentiate the function $F_\lambda(\xi)$ with the respect to ξ. Since

$$J_\lambda\left(\left(U_\lambda + \tilde\phi\right)(x,\xi)\right) = \frac{1}{p^2\gamma^{2(p-1)}} I_\lambda\left((V_\lambda + \phi)\left(\frac{x}{\varepsilon},\frac{\xi}{\varepsilon}\right)\right), \qquad (5.4)$$

we can differentiate directly $I_\lambda(V_\lambda(\xi) + \phi(\xi))$ under the integral sign, so that

$$\partial_{\xi_l} F_\lambda(\xi) = \frac{1}{p^2\gamma^{2(p-1)}}\varepsilon^{-1} DI_\lambda(V_\lambda(\xi) + \phi(\xi))\left[\partial_{\xi_l'} V_\lambda(\xi) + \partial_{\xi_l'}\phi(\xi)\right]$$

$$= \frac{1}{p^2\gamma^{2(p-1)}}\varepsilon^{-1}\sum_{j=1}^{k}\int_{\partial\Omega_\varepsilon} c_j\chi_j Z_{1j}\left[\partial_{\xi_l'} V_\lambda(\xi) + \partial_{\xi_l'}\phi(\xi)\right]$$

$$= \frac{1}{p^2\gamma^{2(p-1)}}\varepsilon^{-1}\left[\sum_{j=1}^{k}\int_{\partial\Omega_\varepsilon} c_j\chi_j Z_{1j}\partial_{\xi_l'} V_\lambda(\xi) + \sum_{j=1}^{k}\int_{\partial\Omega_\varepsilon} c_j\chi_j Z_{1j}\partial_{\xi_l'}\phi(\xi)\right].$$

By the expansion of V_λ, we have

$$\partial_{\xi_l'} V_\lambda = \partial_{\xi_l'}\left(\tilde w_l(y) + \frac{p-1}{p}\frac{1}{\gamma^p}\tilde w_{1l}(\varepsilon y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\tilde w_{2l}(\varepsilon y) + \theta(y)\right)$$

$$= \partial_{\xi_l'}\tilde w_l(y) + \frac{p-1}{p}\frac{1}{\gamma^p}\partial_{\xi_l'}\tilde w_{1l}(\varepsilon y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\partial_{\xi_l'}\tilde w_{2l}(\varepsilon y) + \partial_{\xi_l'}\theta(y)$$

$$= -Z_{1l} + \frac{p-1}{p}\frac{1}{\gamma^p}\partial_{\xi_l'}\tilde w_{1l}(\varepsilon y) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\partial_{\xi_l'}\tilde w_{2l}(\varepsilon y) + \partial_{\xi_l'}\theta(y).$$

Hence, for $j \neq l$, we have

$$\int_{\partial\Omega_\varepsilon}\chi_j Z_{1j}\partial_{\xi_l'} V_\lambda(\xi) = -\left(\int_{\partial\Omega_\varepsilon}\chi_j Z_{1j} Z_{1l}\right)\left(1 + O\left(\frac{1}{|\log\varepsilon|}\right)\right) = O(\varepsilon)$$

while for $j = l$ we have

$$\int_{\partial\Omega_\varepsilon}\chi_j Z_{1j}\partial_{\xi_l'} V_\lambda(\xi) = -\left(\int_{\partial\Omega_\varepsilon}\chi_l Z_{1l}^2\right)\left(1 + O\left(\frac{1}{|\log\varepsilon|}\right)\right).$$

We thus conclude that

$$\sum_{j=1}^{k} c_j\int_{\partial\Omega_\varepsilon}\chi_j Z_{1j}\partial_{\xi_l'} V_\lambda(\xi) = -c_l\left(\int_{\partial\Omega_\varepsilon}\chi_l Z_{1l}^2\right)\left(1 + O\left(\frac{1}{|\log\varepsilon|}\right)\right) + \sum_{j\neq l} c_j O(\varepsilon).$$

On the other hand, taking into account (4.3), we get

$$\left| \sum_{j=1}^{k} c_j \int_{\partial \Omega_\varepsilon} \chi_j Z_{1j} \partial_{\xi_l'} \phi(\xi) \right| \le \frac{C}{|\log \varepsilon|} \sum_{j=1}^{k} |c_j|.$$

Then, if $D_\xi F_\lambda(\xi) = 0$, for $j = 1, 2, \ldots, k$, we then have

$$c_l \left(\int_{\partial \Omega_\varepsilon} \chi_l Z_{1l}^2 \right) \left(1 + O\left(\frac{1}{|\log \varepsilon|} \right) \right) + \sum_{j \ne l} c_j O(\varepsilon) = 0, \quad \text{for all } l = 1, \ldots, k. \quad (5.5)$$

This concludes the proof of the lemma. $\qquad\qquad\square$

Next, we will write the expansion of $F_\lambda(\xi)$ as λ goes to zero.

Lemma 5.2. *Let $\delta > 0$ be fixed and assume that the constants μ_j are given by (2.28). Then there exist positive numbers λ_0 and C such that the following expansion holds*

$$\lambda^{-1} \varepsilon^{\frac{2-p}{p}} F_\lambda(\xi) = -\frac{2k\pi}{p} + \frac{2k\pi}{p} \log \frac{1}{\varepsilon} + \frac{\pi}{2-p} \varphi_k(\xi) + |\log \varepsilon|^{-1} \Theta_\lambda(\xi) \quad (5.6)$$

where

$$\varphi_k(\xi) = \varphi_k(\xi_1, \ldots, \xi_k) = - \left[\sum_{j=1}^{k} H(\xi_j, \xi_j) + \sum_{l \ne j} G(\xi_l, \xi_j) \right]. \quad (5.7)$$

In (5.6) $\Theta_\lambda(\xi)$ denotes a generic smooth function, which is uniformly bounded, as $\lambda \to 0$, for points ξ satisfying (2.2).

Proof. We have

$$F_\lambda(\xi) = J_\lambda \left(U_\lambda(\xi) + \tilde{\phi}(\xi) \right)$$

$$= \frac{1}{2} \int_\Omega \left[\left| \nabla \left(U_\lambda + \tilde{\phi} \right) \right|^2 + \left(U_\lambda + \tilde{\phi} \right)^2 \right] - \frac{\lambda}{p} \int_{\partial \Omega} e^{\left(U_\lambda + \tilde{\phi} \right)^p}.$$

From (5.4) we have that

$$J_\lambda \left(U_\lambda(\xi) + \tilde{\phi}(\xi) \right) - J_\lambda \left(U_\lambda(\xi) \right) = \frac{1}{p^2 \gamma^{2(p-1)}} \left[I_\lambda(V_\lambda + \phi) - I_\lambda(V_\lambda) \right].$$

Since by construction $I_\lambda'(V_\lambda + \phi)[\phi] = 0$, we have

$$J_\lambda \left(U_\lambda(\xi) + \tilde{\phi}(\xi) \right) - J_\lambda \left(U_\lambda(\xi) \right) = \frac{1}{p^2 \gamma^{2(p-1)}} \int_0^1 D^2 I_\lambda(V_\lambda + t\phi)\phi^2 (1 - t) \, dt$$

$$= \frac{1}{p^2 \gamma^{2(p-1)}} \int_0^1 \left[\int_{\partial \Omega_\varepsilon} (E_\lambda + N(\phi))\phi + \int_{\partial \Omega_\varepsilon} [f_\lambda'(V_\lambda) - f_\lambda'(V_\lambda + t\phi)]\phi^2 \right] (1 - t) \, dt.$$

Since $\|E_\lambda\|_{*,\partial\Omega_\varepsilon} \leq \frac{C}{|\log\varepsilon|^3}$, $\|\phi\|_{L^\infty(\Omega_\varepsilon)} \leq \frac{C}{|\log\varepsilon|^2}$, $\|N(\phi)\|_{*,\partial\Omega_\varepsilon} \leq \frac{C}{|\log\varepsilon|^4}$ and (2.43), we get that

$$\left| J_\lambda\left(U_\lambda(\xi) + \tilde{\phi}(\xi)\right) - J_\lambda\left(U_\lambda(\xi)\right) \right| \leq \frac{C}{\gamma^{2(p-1)}|\log\varepsilon|^3}. \tag{5.8}$$

Next we expand

$$J_\lambda\left(U_\lambda(\xi)\right) = \frac{1}{2}\int_\Omega \left[|\nabla(U_\lambda(\xi))|^2 + U_\lambda(\xi)^2\right] - \frac{\lambda}{p}\int_{\partial\Omega} e^{(U_\lambda(\xi))^p}. \tag{5.9}$$

Now we write

$$U_j(x) := u_j(x) + H_j^\varepsilon(x), \quad U_{1j} := \tilde{w}_{1j}(x) + H_{1j}^\varepsilon(x), \quad U_{2j} := \tilde{w}_{2j}(x) + H_{2j}^\varepsilon(x).$$

By (2.18), we have

$$U_\lambda(x) = \frac{1}{p\gamma^{p-1}}\sum_{j=1}^k \left(U_j(x) + \frac{p-1}{p}\frac{1}{\gamma^p}U_{1j}(x) + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}U_{2j}(x)\right).$$

Then we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \left[|\nabla(U_\lambda(\xi))|^2 + U_\lambda(\xi)^2\right] \\
&= \frac{1}{p^2\gamma^{2(p-1)}}\left\{\frac{1}{2}\sum_{j=1}^k \int_\Omega (|\nabla U_j|^2 + U_j^2) + \sum_{l\neq j}\int_\Omega (\nabla U_l\nabla U_j + U_l U_j) \right. \\
&\quad + \frac{p-1}{p}\frac{1}{\gamma^p}\sum_{j=1}^k \int_\Omega (\nabla U_j\nabla U_{1j} + U_j U_{1j}) \\
&\quad + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\sum_{j=1}^k \int_\Omega (\nabla U_j\nabla U_{2j} + U_j U_{2j}) \\
&\quad + \left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\left[\frac{1}{2}\sum_{j=1}^k \int_\Omega (|\nabla U_{1j}|^2 + U_{1j}^2)\right. \\
&\quad\quad\quad \left. + \sum_{l\neq j}\int_\Omega (\nabla U_{1l}\nabla U_{1j} + U_{1l}U_{1j})\right] \\
&\quad + \left(\frac{p-1}{p}\right)^3\frac{1}{\gamma^{3p}}\sum_{j=1}^k \int_\Omega (\nabla U_{1j}\nabla U_{2j} + U_{1j}U_{2j}) \\
&\quad + \left(\frac{p-1}{p}\right)^4\frac{1}{\gamma^{4p}}\left[\frac{1}{2}\sum_{j=1}^k \int_\Omega (|\nabla U_{2j}|^2 + U_{2j}^2)\right. \\
&\quad\quad\quad \left.\left. + \sum_{l\neq j}\int_\Omega (\nabla U_{2l}\nabla U_{2j} + U_{2l}U_{2j})\right]\right\}.
\end{aligned} \tag{5.10}$$

Let us estimate the first two terms. We observe that the remaining terms are $O\left(\frac{1}{\gamma^{2(p-1)}\gamma^p}\right)$. We have

$$\int_\Omega (|\nabla U_j|^2 + U_j^2) = \int_\Omega |\nabla u_j|^2 + \int_\Omega u_j^2 + \int_\Omega |\nabla H_j^\varepsilon|^2 + \int_\Omega (H_j^\varepsilon)^2$$
$$+ 2\int_\Omega \nabla u_j \nabla H_j^\varepsilon + 2\int_\Omega u_j H_j^\varepsilon. \tag{5.11}$$

Multiplying (2.4) by H_j^ε yields

$$\int_\Omega |\nabla H_j^\varepsilon|^2 + \int_\Omega (H_j^\varepsilon)^2 = -\int_\Omega u_j H_j^\varepsilon + \int_{\partial\Omega} \frac{\partial H_j^\varepsilon}{\partial\nu} H_j^\varepsilon$$
$$= -\int_\Omega u_j H_j^\varepsilon + \varepsilon\int_{\partial\Omega} e^{u_j} H_j^\varepsilon - \int_{\partial\Omega} \frac{\partial u_j}{\partial\nu} H_j^\varepsilon.$$

Multiplying (2.4) by u_j^ε again, we find

$$\int_\Omega u_j^2 + \int_\Omega H_j^\varepsilon u_j = -\int_\Omega \nabla u_j \nabla H_j^\varepsilon + \varepsilon\int_{\partial\Omega} e^{u_j} u_j - \int_{\partial\Omega} \frac{\partial u_j}{\partial\nu} u_j.$$

Then we get

$$\int_\Omega (|\nabla U_j|^2 + U_j^2)$$
$$= \int_\Omega |\nabla u_j|^2 - \int_{\partial\Omega} \frac{\partial u_j}{\partial\nu} u_j + \int_\Omega \nabla u_j \nabla H_j^\varepsilon - \int_{\partial\Omega} \frac{\partial u_j}{\partial\nu} H_j^\varepsilon + \varepsilon\int_{\partial\Omega} e^{u_j}(u_j + H_j^\varepsilon)$$
$$= \varepsilon\int_{\partial\Omega} e^{u_j}(u_j + H_j^\varepsilon)$$
$$= \varepsilon\int_{\partial\Omega} \frac{2\mu_j}{|x - \xi_j - \varepsilon\mu_j\nu(\xi_j)|^2}\left(\log \frac{1}{|x - \xi_j - \varepsilon\mu_j\nu(\xi_j)|^2} + H(x, \xi_j) + O(\varepsilon^\alpha)\right).$$

Taking the change of variables $y = \frac{x - \xi_j}{\varepsilon\mu_j}$, we have

$$\int_\Omega (|\nabla U_j|^2 + U_j^2)$$
$$= \int_{\partial\Omega_{\varepsilon\mu_j}} \frac{2}{|y - \nu(0)|^2}\left(\log \frac{1}{|y - \nu(0)|^2} + H(\xi_j + \varepsilon\mu_j y, \xi_j) - 2\log(\mu_j\varepsilon)\right) \tag{5.12}$$
$$+ O(\varepsilon^\alpha).$$

Since

$$\int_{\partial\Omega_{\varepsilon\mu_j}} \frac{1}{|y - \nu(0)|^2} = \pi + O(\varepsilon),$$

$$\int_{\partial\Omega_{\varepsilon\mu_j}} \frac{1}{|y - \nu(0)|^2}\log \frac{1}{|y - \nu(0)|^2} = \int_{-\infty}^\infty \frac{1}{1 + t^2}\log \frac{1}{1 + t^2} \, dt + O(\varepsilon^\alpha)$$
$$= -2\pi\log 2 + O(\varepsilon^\alpha),$$

and

$$\int_{\partial\Omega_{\varepsilon\mu_j}} \frac{2}{|y-\nu(0)|^2} \left(H(\xi_j + \varepsilon\mu_j y, \xi_j) - H(\xi_j, \xi_j)\right)$$

$$= \int_{\partial\Omega_{\varepsilon\mu_j}} \frac{2}{|y-\nu(0)|^2} O(\varepsilon^\alpha |y|^\alpha) = O(\varepsilon^\alpha)$$

we obtain

$$\frac{1}{2} \sum_{j=1}^k \int_\Omega (|\nabla U_j|^2 + U_j^2)$$

$$= -2k\pi \log 2 + \pi \sum_{j=1}^k \left[H(\xi_j, \xi_j) - 2\log(\varepsilon\mu_j) \right] + O(\varepsilon^\alpha) \qquad (5.13)$$

$$= -2k\pi \log \varepsilon + \pi \sum_{j=1}^k \left[H(\xi_j, \xi_j) - 2\log(2\mu_j) \right] + O(\varepsilon^\alpha).$$

On the other hand, we have

$$\sum_{l\neq j} \int_\Omega (\nabla U_l \nabla U_j + U_l U_j)$$

$$= \sum_{l\neq j} \int_\Omega \nabla u_l \nabla u_j + 2 \int_\Omega \nabla u_l \nabla H_j^\varepsilon + \int_\Omega \nabla H_l^\varepsilon \nabla H_j^\varepsilon \qquad (5.14)$$

$$+ \int_\Omega u_l u_j + 2 \int_\Omega u_l H_j^\varepsilon + \int_\Omega H_l^\varepsilon H_j^\varepsilon.$$

Multiplying (2.4) by H_l^ε yields

$$\int_\Omega \nabla H_j^\varepsilon \nabla H_l^\varepsilon + \int_\Omega H_j^\varepsilon H_l^\varepsilon = -\int_\Omega u_j H_l^\varepsilon + \varepsilon \int_{\partial\Omega} e^{u_j} H_l^\varepsilon - \int_{\partial\Omega} \frac{\partial u_j}{\partial \nu} H_l^\varepsilon. \quad (5.15)$$

Multiplying (2.4) by u_l^ε again, we have

$$\int_\Omega \nabla u_j^\varepsilon \nabla u_l^\varepsilon + \int_\Omega H_j^\varepsilon u_l^\varepsilon = -\int_\Omega \nabla H_j^\varepsilon \nabla u_l + \varepsilon \int_{\partial\Omega} e^{u_j} u_l^\varepsilon - \int_{\partial\Omega} \frac{\partial u_j}{\partial \nu} u_l. \quad (5.16)$$

By (5.14)-(5.16) we find that

$$\sum_{l\neq j} \int_\Omega (\nabla U_l \nabla U_j + U_l U_j)$$

$$= \sum_{l\neq j} \int_\Omega \nabla u_l \nabla u_j + 2 \int_\Omega \nabla u_l \nabla H_j^\varepsilon + \int_\Omega \nabla H_l^\varepsilon \nabla H_j^\varepsilon$$

$$+ \int_\Omega u_l u_j + 2 \int_\Omega u_l H_j^\varepsilon + \int_\Omega H_l^\varepsilon H_j^\varepsilon \qquad (5.17)$$

$$= \pi \sum_{l\neq j} G(\xi_l, \xi_j) + O(\varepsilon^\alpha).$$

Therefore, by (5.10), (5.13) and (5.17), using the choice of μ_j in (2.28), we get

$$\frac{1}{2}\int_\Omega\left[|\nabla(U_\lambda(\xi))|^2+U_\lambda(\xi)^2\right]$$

$$=\frac{1}{p^2\gamma^{2(p-1)}}\left[-2k\pi\log\varepsilon-\frac{p}{2-p}\pi\left(\sum_{j=1}^k H(\xi_j,\xi_j)+\sum_{l\neq j}^k G(\xi_l,\xi_j)\right)\right. \tag{5.18}$$

$$\left.+O(|\log\varepsilon|^{-1})\right].$$

Finally, let use estimate the second term in (5.9). We have

$$\frac{\lambda}{p}\int_{\partial\Omega}e^{U_\lambda^p}=\frac{\lambda}{p}\int_{\partial\Omega}e^{\gamma^p\left(1+\frac{1}{p\gamma^p}V_\lambda(\frac{x}{\varepsilon})\right)^p}$$

$$=\frac{\lambda}{p}\sum_{j=1}^k\int_{\partial\Omega\cap B(\xi_j,\tilde\delta)}e^{\gamma^p\left(1+\frac{1}{p\gamma^p}V_\lambda(\frac{x}{\varepsilon})\right)^p} \tag{5.19}$$

$$+\frac{\lambda}{p}\int_{\partial\Omega\setminus\bigcup_{j=1}^k B(\xi_j,\tilde\delta)}e^{\gamma^p\left(1+\frac{1}{p\gamma^p}V_\lambda(\frac{x}{\varepsilon})\right)^p}$$

$$:=I_1+I_2.$$

First we observe that

$$I_2=\lambda\Theta_\lambda(\xi) \tag{5.20}$$

with $\Theta_\lambda(\xi)$ a function uniformly bounded as $\lambda\to 0$. On the other hand,

$$I_1=\frac{1}{p^2\gamma^{2(p-1)}}\sum_{j=1}^k\int_{\partial\Omega_\varepsilon\cap B(\xi_j',\tilde\delta/\varepsilon)}e^{\gamma^p\left[\left(1+\frac{1}{p\gamma^p}V_\lambda(y)\right)^p-1\right]}$$

$$=\frac{1}{p^2\gamma^{2(p-1)}}\sum_{j=1}^k\int_{\partial\Omega_\varepsilon\cap B(\xi_j',\tilde\delta/\varepsilon)}e^{\left\{\tilde w_j(y)+\frac{p-1}{p}\frac{1}{\gamma^p}\tilde w_{1j}(y)+\left(\frac{p-1}{p}\right)^2\frac{1}{\gamma^{2p}}\tilde w_{2j}(y)+\theta(y)\right\}}$$

$$\times\left(1+O\left(\frac{1}{\gamma^p}\right)\right) \tag{5.21}$$

$$=\frac{1}{p^2\gamma^{2(p-1)}}\sum_{j=1}^k\int_{\partial\Omega_\varepsilon\cap B(0,\frac{\tilde\delta}{\mu_j\varepsilon})}\frac{2}{|y-\nu(0)|^2}\left(1+O\left(\frac{1}{\gamma^p}\right)\right)$$

$$=\frac{1}{p^2\gamma^{2(p-1)}}2k\pi\left(1+|\log\varepsilon|^{-1}\Theta_\lambda(\xi)\right),$$

with $\Theta_\lambda(\xi)$ a function uniformly bounded as $\lambda\to 0$. From (5.19)-(5.21) we get

$$\frac{\lambda}{p}\int_{\partial\Omega}e^{U_\lambda^p}=\frac{1}{p^2\gamma^{2(p-1)}}\left[2k\pi\left(1+|\log\varepsilon|^{-1}\Theta_\lambda(\xi)\right)\right]. \tag{5.22}$$

By (1.8), (5.8), (5.9), (5.18) and (5.22), we can write the whole asymptotic expansion of $F_\lambda(\xi)$, namely (5.6) holds. $\square$

6. Proof of the main theorem

Proof of Theorem 1.1. From Lemma 5.1, the function

$$U_\lambda(\xi) + \tilde\phi(\xi) = \frac{1}{p\gamma^{p-1}}\left(p\gamma^p + (V_\lambda + \phi)\left(\frac{x}{\varepsilon}\right)\right)$$

where V_λ defined by (2.23) and $\phi(\xi)$ is the unique solution of problem (4.1), is a solution of problem (1.1) if we adjust ξ so that it is a critical point of $F_\lambda(\xi)$ defined by (5.3). This is equivalent to finding a critical point of

$$\widetilde{F}_\lambda(\xi) := A\lambda^{-1}\varepsilon^{\frac{2-p}{p}} F_\lambda(\xi) + B + C\log\varepsilon,$$

for suitable constants A, B and C. On the other hand, from Lemmas 5.2, for $\xi = (\xi_1, \ldots, \xi_k) \in \partial\Omega$ satisfying (2.2), we have that,

$$\widetilde{F}_\lambda(\xi) = \varphi_k(\xi) + O(|\log\varepsilon|^{-1})\Theta_\lambda(\xi), \tag{6.1}$$

where φ_k is given by (1.4), and $\Theta_\lambda(\xi)$ is uniformly bounded in the considered region as $\lambda \to 0$.

Next we show that φ_k has at least two critical points. Assume $\xi_1, \ldots, \xi_k$ belong to the same component of $\partial\Omega$. We denote by Γ such a component. Let D be the diagonal in $(\partial\Omega)^k$. Define $\hat{\Omega}_k = (\partial\Omega)^k \setminus D$ and

$$\tilde{\Omega}_k = \{\xi \in C^k \setminus D \,:\, |\xi_i - \xi_j| > \delta\}$$

for some small but fixed $\delta > 0$. We show that φ_k has at least two critical points in $\tilde{\Omega}_k$.

First observe that φ_k has an absolute maximum m in $\tilde{\Omega}_k$: indeed, φ_k is C^1, it is bounded from above in $\hat{\Omega}_k$ (and hence in $\tilde{\Omega}_k$) and

$$\varphi_k(\xi_1, \ldots, \xi_k) \to -\infty, \quad \text{as} \quad |\xi_i - \xi_j| \to 0 \quad \text{for some} \quad i \neq j.$$

On the other hand, in our setting the number of critical points of φ_k in $\tilde{\Omega}_k$ can be estimated from below by the Ljusternik-Schnirelman category of $\tilde{\Omega}_k$, namely the minimal number of closed and contractible sets in $\tilde{\Omega}_k$ whose union covers $\tilde{\Omega}_k$. In [6], it is proved that the Ljusternik-Schnirelman category of $\tilde{\Omega}_k$, $\mathrm{cat}(\tilde{\Omega}_k)$, is greater than or equal to two, for any $k \geq 1$. Define

$$c = \sup\Gamma \in \Xi \inf_{\xi \in \Gamma} \varphi_k(\xi) \tag{6.2}$$

where
$$\Xi = \{\Gamma \subset \tilde{\Omega}_k : \Gamma \quad \text{closed and} \quad \text{cat}(\Gamma) \geq 2\}.$$

Then c is a critical level. If $c \neq m$, then φ_k has at least two distinct critical points in $'\tilde{\Omega}_k$. If $c = m$, then (6.2) gives the existence of a set Γ with $\text{cat}(\Gamma) \geq 2$ where φ_k reaches its maximum. In this case we conclude that there are infinitely many critical points for φ_k in $\tilde{\Omega}_k$.

These critical points persist under small C^0-perturbation of the function, thus (6.1) gives that also the function $\widetilde{F}_\lambda(\xi)$ has two distinct critical points in $\tilde{\Omega}_k$, and hence problem (1.1) has at least two distinct solutions.

Expansion (1.9) follows from (1.8) and (5.22), while (1.10) holds as a direct consequence of the construction of U_λ. Expansion (1.11) is a consequence of (5.6). $\qquad \square$

Remark 6.1. Let us mention that one can get a stronger existence result assuming that the function φ_k has some other critical points in $\tilde{\Omega}_k$, provided it is a *topologically non trivial* critical point for φ_k. We refer to [6] for further reference.

7. Appendix

Proof of (2.14). Since $(\arctan t)' = \frac{1}{1+t^2}$, integration by parts gives

$$\int_0^\infty \frac{1}{t^2+1} \log \frac{1}{t^2+1} dt = \int_0^\infty \log \frac{1}{t^2+1} d(\arctan t)$$
$$= \left[\arctan t \log \frac{1}{1+t^2} \right] \Big|_0^{+\infty} + \int_0^{+\infty} \frac{2t \arctan t}{1+t^2} dt.$$

Setting $t = \tan x$ we have $1 + t^2 = \sec^2 t, dt = \sec^2 t dx$, hence

$$\int_0^{+\infty} \frac{2t \arctan t}{1+t^2} dt = 2 \int_0^{\frac{\pi}{2}} x \tan x dx$$
$$= -2 \int_0^{\frac{\pi}{2}} x d(\log(\cos x))$$
$$= \left[-2x \log(\cos x) \right] \Big|_0^{\frac{\pi}{2}} + 2 \int_0^{\frac{\pi}{2}} \log(\cos x) \, dx$$
$$= \left[-2 \arctan t \log[\cos(\arctan t)] \right] \Big|_0^{+\infty} + 2 \int_0^{\frac{\pi}{2}} \log(\cos x) \, dx$$
$$= \left[- \arctan t \log[\cos^2(\arctan t)] \right] \Big|_0^{+\infty} + 2 \int_0^{\frac{\pi}{2}} \log(\cos x) \, dx.$$

We have

$$\cos^2(\arctan t) = \cos^2 x = \frac{1}{\sec^2 x} = \frac{1}{1 + \tan^2 x} = \frac{1}{1 + t^2}.$$

On the other hand, we note that

$$\int_0^{\frac{\pi}{2}} \log(\cos x)\, dx = \int_0^{\frac{\pi}{2}} \log(\sin x)\, dx.$$

Then we have

$$2 \int_0^{\frac{\pi}{2}} \log(\cos x)\, dx$$

$$= \int_0^{\frac{\pi}{2}} \log(\cos x)\, dx + \int_0^{\frac{\pi}{2}} \log(\sin x)\, dx$$

$$= \int_0^{\frac{\pi}{2}} \log\left(\frac{\sin(2x)}{2}\right) dx = \frac{1}{2} \int_0^{\pi} \log\left(\frac{\sin(x)}{2}\right) dx$$

$$= \frac{1}{2} \int_0^{\pi} \log(\sin x)\, dx - \frac{\pi}{2} \log 2$$

$$= \int_0^{\frac{\pi}{2}} \log(\sin x)\, dx - \frac{\pi}{2} \log 2$$

$$= \int_0^{\frac{\pi}{2}} \log(\cos x)\, dx - \frac{\pi}{2} \log 2.$$

Hence we get

$$\int_{-\infty}^{\infty} \frac{1}{t^2+1} \log\frac{1}{t^2+1}\, dt = 2 \int_0^{\infty} \frac{1}{t^2+1} \log\frac{1}{t^2+1}\, dt$$

$$= 4 \int_0^{\frac{\pi}{2}} \log(\cos x)\, dx = -2\pi \log 2. \qquad \square$$

Proof of Lemma 2.1. On the boundary, we have

$$\frac{\partial H_j^{\varepsilon}}{\partial \nu} = \varepsilon e^{u_j} - \frac{\partial u_j}{\partial \nu} = 2\varepsilon \mu_j \frac{1 - \nu(\xi_j) \cdot \nu(x)}{|x - \xi_j - \varepsilon\mu_j \nu(\xi_j)|^2} + 2\frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j - \varepsilon\mu_j \nu(\xi_j)|^2}.$$

Thus,

$$\lim_{\varepsilon \to 0} \frac{\partial H_j^{\varepsilon}}{\partial \nu} = 2\frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2}, \quad \forall\, x \neq \xi_j.$$

Set $z_{\varepsilon}(x) = H_j^{\varepsilon}(x) + \log(2\mu_j) - H(x, \xi_j)$; then from the definition of $H(x, \xi_j)$ and H_j^{ε} we have

$$\begin{cases} -\Delta z_{\varepsilon} + z_{\varepsilon} = \log \dfrac{1}{|x - \xi_j|^2} - \log \dfrac{1}{|x - \xi_j - \varepsilon\mu_j \nu(\xi_j)|^2} & \text{in } \Omega \\[2ex] \dfrac{\partial z_{\varepsilon}}{\partial \nu} = \dfrac{\partial H_j^{\varepsilon}}{\partial \nu} - 2\dfrac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2} & \text{on } \partial\Omega. \end{cases}$$

First, we claim that there is a positive constant C such that

$$\left\| \frac{\partial H_j^\varepsilon}{\partial \nu} - 2\frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2} \right\|_{L^q(\partial\Omega)} \le C\varepsilon^{1/q}, \quad \forall q > 1. \tag{7.1}$$

In fact,

$$\frac{\partial H_j^\varepsilon}{\partial \nu} - 2\frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2}$$

$$= 2\varepsilon\mu_j \frac{1 - \nu(\xi_j) \cdot \nu(x)}{|x - \xi_j - \varepsilon\mu_j\nu(\xi_j)|^2} + 2\frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j - \varepsilon\mu_j\nu(\xi_j)|^2} - 2\frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2}$$

$$= 2\varepsilon\mu_j \frac{1 - \nu(\xi_j) \cdot \nu(x)}{|x - \xi_j - \varepsilon\mu_j\nu(\xi_j)|^2} + 2\varepsilon\mu_j \frac{(x - \xi_j) \cdot \nu(x)\left[2(x - \xi_j) \cdot \nu(x) - \varepsilon\mu_j\right]}{|x - \xi_j|^2|x - \xi_j - \varepsilon\mu_j\nu(\xi_j)|^2}.$$

Now, we observe that

$$|1 - \nu(\xi_j) \cdot \nu(x)| \le C|x - \xi_j|^2, \quad |(x - \xi_j) \cdot \nu(x)| \le C|x - \xi_j|^2, \quad \forall x \in \partial\Omega.$$

Hence,

$$\left| \frac{\partial H_j^\varepsilon}{\partial \nu} - 2\frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2} \right| \le C\varepsilon + C\frac{\varepsilon|2(x - \xi_j) \cdot \nu(\xi_j) - \varepsilon\mu_j|}{|x - \xi_j - \varepsilon\mu_j\nu(\xi_j)|^2}. \tag{7.2}$$

For small $\rho > 0$, we have

$$\left| \frac{\partial H_j^\varepsilon}{\partial \nu} - 2\frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2} \right| \le C\varepsilon, \quad \text{for } \forall |x - \xi_j| \ge \rho, \ x \in \partial\Omega. \tag{7.3}$$

Now take $q > 1$; we have

$$\int_{B_\rho(\xi_j) \cap \partial\Omega} \left| \frac{\varepsilon|2(x - \xi_j) \cdot \nu(\xi_j) - \varepsilon\mu_j|}{|x - \xi_j - \varepsilon\mu_j\nu(\xi_j)|^2} \right|^q dx$$

$$= C\varepsilon \int_{B_{\rho/\varepsilon}(0) \cap \partial\Omega_\varepsilon} \left| \frac{2y \cdot \nu(0) - \mu_j}{y - \mu_j\nu(0)} \right|^q dy \tag{7.4}$$

$$\le C\varepsilon \int_0^{\rho/\varepsilon} \frac{1}{(1 + s)^q} ds \le C\varepsilon.$$

Combining (7.2) with (7.3) and (7.4) we conclude that (7.1) holds.

Next, we show that

$$\left\| \log\frac{1}{|x - \xi_j|^2} - \log\frac{1}{|x - \xi_j - \varepsilon\mu_j\nu(\xi_j)|^2} \right\|_{L^q(\Omega)} \le C\varepsilon \tag{7.5}$$

$$\text{for any } 1 < q < 2.$$

In fact, for $q \geq 1$, we write

$$
\left\| \log \frac{1}{|x - \xi_j|^2} - \log \frac{1}{|x - \xi_j - \varepsilon \mu_j \nu(\xi_j)|^2} \right\|_{L^q(\Omega)}^q
$$

$$
= \int_{B_{10\varepsilon\mu_j}(\xi_j) \cap \Omega} \cdots + \int_{\Omega \setminus B_{10\varepsilon\mu_j}(\xi_j)} \cdots := I + II.
\tag{7.6}
$$

Next we estimate I and II. For I, we observe that

$$
\int_{B_{10\varepsilon\mu_j}(\xi_j) \cap \Omega} \left| \log \frac{1}{|x - \xi_j|^2} \right|^q dx \leq C \int_0^{C\varepsilon} |\log r|^q r \, dr \leq C\varepsilon^2 \left(\log \frac{1}{\varepsilon} \right)^q,
$$

and the same bound is true for the integral of $\left| \log \frac{1}{|x - \xi_j - \varepsilon \mu_j \nu(\xi_j)|^2} \right|^q$ in $B_{10\varepsilon\mu_j}(\xi_j) \cap \Omega$. Hence we have

$$
|I| \leq C\varepsilon^2 \left(\log \frac{1}{\varepsilon} \right)^q.
\tag{7.7}
$$

For II, if $|x - \xi_j| \geq 10\varepsilon\mu_j$, we have

$$
|x - \xi_j| \leq |x - \xi_j - t\varepsilon\mu_j\nu(\xi_j)| + \mu_j\varepsilon \leq |x - \xi_j - t\varepsilon\mu_j\nu(\xi_j)| + \frac{1}{10}|x - \xi_j|
$$

for any $t \in [0, 1]$, then we have $|x - \xi_j| \leq C|x - \xi_j - t\varepsilon\mu_j\nu(\xi_j)|$. Using this fact, we can obtain

$$
\left| \log \frac{1}{|x - \xi_j|^2} - \log \frac{1}{|x - \xi_j - \varepsilon\mu_j\nu(\xi_j)|^2} \right|
$$

$$
\leq C \sup_{0 \leq t \leq 1} \frac{C\varepsilon}{|x - \xi_j - \varepsilon\mu_j\nu(\xi_j)|} \leq \frac{C\varepsilon}{|x - \xi_j|}.
$$

Thus for $1 < q < 2$,

$$
|II| \leq C\varepsilon^q \int_{10\varepsilon\mu_j}^D r^{1-q} dr \leq C\varepsilon^q,
\tag{7.8}
$$

where D is the diameter of Ω. Thus, combining (7.6) with (7.7) and (7.8) we obtain that (7.5) holds.

Therefore by elliptic regularity theory, we obtain

$$
\|z_\varepsilon\|_{W^{1+s,q}(\Omega)} \leq \left(\left\| \frac{\partial z_\varepsilon}{\partial \nu} \right\|_{L^q(\partial\Omega)} + \|\Delta z_\varepsilon\|_{L^q(\Omega)} \right) \leq C\varepsilon^{1/q}
\tag{7.9}
$$

for any $0 < s < \frac{1}{q}$. By the Morrey embedding we obtain

$$
\|z_\varepsilon\|_{C^\beta(\bar{\Omega})} \leq C\varepsilon^{1/q}
$$

for any $0 < \beta < \frac{1}{2} + \frac{1}{q}$. This proves the lemma with $\alpha = \frac{1}{q}$. $\qquad\square$

Proof of Lemma 2.4. The proof follows from the same arguments as those used to prove Lemma 2.1. First, on the boundary, we have

$$\lim_{\varepsilon \to 0} \frac{\partial H_j^\varepsilon}{\partial \nu}(x) = 2\alpha_{ij} \frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2}, \quad \forall\, x \neq \xi_j.$$

The regular part of Green's function satisfies

$$\begin{cases} -\Delta_x H(x, \xi_j) + H(x, \xi_j) = -\log \dfrac{1}{|x - \xi_j|^2} & \text{in } \Omega \\[2mm] \dfrac{\partial H(x, \xi_j)}{\partial \nu_x} = 2\dfrac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2} & \text{on } \partial\Omega. \end{cases}$$

Setting $\tilde{z}_\varepsilon = H_j^\varepsilon(x) + \alpha_{ij} \log(2\mu_j \varepsilon^2) - \alpha_{ij} H(x, \xi_j)$, we have

$$\begin{aligned} -\Delta \tilde{z}_\varepsilon + \tilde{z}_\varepsilon &= -\Delta H_j^\varepsilon + H_j^\varepsilon + \alpha_{ij} \log(2\mu_j \varepsilon^2) - \alpha_{ij} \left[-\Delta H(x, \xi_j) + H(x, \xi_j) \right] \\ &= -\alpha_{ij} \tilde{w}_j + \alpha_{ij} \log(2\mu_j \varepsilon^2) - \alpha_{ij} \left[-\Delta H(x, \xi_j) + H(x, \xi_j) \right] \\ &= \alpha_{ij} \left[\log \frac{1}{|x - \xi_j|^2} - \log \frac{1}{|x - \xi_j - \varepsilon\mu_j \nu(\xi_j)|^2} \right] \quad \text{in } \Omega. \end{aligned}$$

On the other hand, on the boundary we have

$$\frac{\partial \tilde{z}_\varepsilon}{\partial \nu} = \alpha_{ij} \left[\frac{\partial H_j^\varepsilon}{\partial \nu} - 2\frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2} \right].$$

From (7.1) and (7.9), by the same procedure as in the proof of Lemma 2.1, we obtain that (2.20) holds. □

References

[1] ADIMURTHI and S. L. YADAVA, *Critical exponent problem in* $\mathbb{R}^2$ *with Neumann boundary condition*, Comm. Partial Differential Equations **15** (1990), 461–501.

[2] A. BAHRI and J. M. CORON, *On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of the topology of the domain*, Comm. Pure Appl. Math. **41** (1988), 255–294.

[3] S. BARAKET and F. PACARD, *Construction of singular limits for a semilinear elliptic equation in dimension* 2, Calc. Var. Partial Differential Equations **6** (1998), 1–38.

[4] H. BREZIS and F. MERLE, *Uniform estimates and blow-up behavior for solutions of* $-\Delta u = V(x)e^u$ *in two dimensions*, Comm. Partial Differential Equations **16** (1991), 1223–1253.

[5] H. CASTRO, *Solutions with spikes at the boundary for a 2D nonlinear Neumann problem with large exponent*, J. Differential Equations **246** (2009), 2991–3037.

[6] J. DÁVILA, M. DEL PINO and M. MUSSO, *Concentrating solutions in a two-dimensional elliptic problem with exponential Neumann data*, J. Funct. Anal. **227** (2005), 430–490.

[7] J. DÁVILA, M. DEL PINO, M. MUSSO and J. WEI, *Singular limits of a two-dimensional boundary value problem arising in corrosion modelling*, Arch. Ration. Mech. Anal. **182** (2006), 181–221.

[8] M. DEL PINO, M. KOWALCZYK and M. MUSSO, *Singular limits in Liouville-type equations*, Calc. Var. Partial Differential Equations **24** (2005), 47–81.

[9] M. DEL PINO, M. MUSSO and B. RUF, *New solutions for Trudinger-Moser critical equations in $\mathbb{R}^2$*, J. Funct. Anal. **258** (2010), 421–457.

[10] S. DENG and M. MUSSO, *Bubbling solutions for an exponential nonlinearity in $\mathbb{R}^2$*, J. Differential Equations (2014), DOI: 10.1016/j.jde.2014.05.034

[11] P. ESPOSITO, M. GROSSI and A. PISTOIA, *On the existence of blowing-up solutions for a mean field equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **22** (2005), 227–257.

[12] P. ESPOSITO, M. MUSSO and A. PISTOIA, *Concentrating solutions for a planar elliptic problem involving nonlinearities with large exponent*, J. Differential Equations **227** (2006), 29–68.

[13] P. ESPOSITO, M. MUSSO and A. PISTOIA, *On the existence and profile of nodal solutions for a two-dimensional elliptic problem with large exponent in nonlinearity*, Proc. Lond. Math. Soc. **94** (2007), 497–519.

[14] P. ESPOSITO, A. PISTOIA and J. WEI, *Concentrating solutions for the Hénon equation in $\mathbb{R}^2$*, J. Anal. Math. **100** (2006), 249–280.

[15] A. FLOER and A. WEINSTEIN, *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential*, J. Funct. Anal. **69** (1986), 397–408.

[16] Y. Y. LI and I. SHAFRIR, *Blow-up analysis for solutions of $-\Delta u = V e^u$ in dimension two*, Indiana Univ. Math. J. **43** (1994), 1255–1270.

[17] M. MUSSO and J. WEI, *Stationary solutions to a Keller-Segel chemotaxis system*, Asymptot. Anal. **49** (2006), 217–247.

[18] K. NAGASAKI and T. SUZUKI, *Asymptotic analysis for two-dimensional elliptic eigenvalue problems with exponentially dominated nonlinearities*, Asymptot. Anal. **3** (1990), 173–188.

Departamento de Ingeniería Matemática
and CMM
Universidad de Chile
Casilla 170 Correo 3
Santiago, Chile
shbdeng@yahoo.com.cn

Departamento de Matemática
Pontificia Universidad Catolica de Chile
Avda. Vicuña Mackenna
4860, Macul, Chile
mmusso@mat.puc.cl