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Symmetries of F-manifolds with eventual identities and special families of connections

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Abstract. We construct a duality for F-manifolds with eventual identities and certain special families of connections and we study its interactions with several well-known constructions from the theory of Frobenius and F-manifolds.

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1. Introduction

The concept of an F-manifold was introduced by Hertling and Manin [6].

Definition 1.1. Let $(M, \circ, e)$ be a manifold with a fiber-preserving commutative, associative, bilinear multiplication $\circ$ on TM, with unit field e. Then $(M, \circ, e)$ is an F-manifold if, for any vector fields $X, Y \in \mathcal{X}(M)$,

$$L_{X\circ Y}(\circ) = X \circ L_Y(\circ) + Y \circ L_X(\circ). \tag{1.1}$$

These F-manifolds were originally defined in the context of Frobenius manifolds (all Frobenius manifolds are examples of F-manifolds) and singularity theory, and have more recently found applications in other areas of mathematics. On writing $\tilde{\mathcal{C}}(X, Y) = X \circ Y$ the F-manifold condition may be expressed succinctly, in terms of the Schouten bracket, as $[\tilde{\mathcal{C}}, \tilde{\mathcal{C}}] = 0$. This provided the starting point for the construction of multi-field generalizations and the deformation theory of such objects [17]. Interestingly, such conditions date back to the work of Nijenhuis [19] and Yano and Ako [22].

Applications of F-manifolds within the theory of integrable systems, and more specifically, equations of hydrodynamic type, have also appeared [10–13]. In a

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sense an F-manifold is a more general and fundamental object than a Frobenius manifold, so it is not surprising that such applications have appeared, providing generalizations of ideas originally formulated for Frobenius manifolds.

Frobenius manifolds by definition come equipped with a metric, in fact a pencil of metrics, and the corresponding Levi-Civita connections play a central role in the theory of these manifolds. In [15] Manin dispensed with such metrics and considered F-manifolds with compatible flat connections. Many of the fundamental properties remain in this more general setting. Applications of F-manifolds with compatible flat connections have also recently appeared in the theory of integrable systems [12,13].

Given an F-manifold with an invertible vector field $\mathcal{E}$ (*i.e.* there is a vector field $\mathcal{E}^{-1}$ such that $\mathcal{E}^{-1} \circ \mathcal{E} = e$) one may define a new, dual or twisted, multiplication

$$X * Y := X \circ Y \circ \mathcal{E}^{-1}, \quad \forall X, Y \in \mathcal{X}(M). \tag{1.2}$$

This is clearly commutative and associative with $\mathcal{E}$ being the unit field. Such a multiplication was introduced by Dubrovin [3] in the special case when $\mathcal{E}$ is the Euler field of a Frobenius manifold and used it to define a so-called almost dual Frobenius manifold. The adjective 'almost' is used since, while the new objects satisfy most of the axioms of a Frobenius manifold, they crucially do not satisfy all of them. A question, raised by Manin [15], is the characterization of those vector fields – called eventual identities – for which $*$ defines an F-manifold. This question was answered by the authors in [4]. At the level of F-manifold structures one has a perfect duality: only when metrics are introduced with certain specified properties is this duality broken to almost-duality. The overall aim of this paper is to construct, by an appropriate twisting by an eventual identity, a duality for F-manifolds with eventual identities and certain special families of connections and to study various applications of such a duality.

1.1. Outline

This paper is structured as follows:

In Section 2 we review the basic facts we need about eventual identities and compatible connections on F-manifolds. For more details on these topics, see [4,7, 15].

In Section 3 we give examples of eventual identities. The most important class of eventual identities are (invertible) Euler fields and their powers on Frobenius, or, more generally, F-manifolds. We describe the eventual identities on semi-simple F-manifolds, and we construct a class of structures close to Frobenius manifolds, which admit an eventual identity, but no Euler field affine in flat coordinates.

The motivation for our treatment from Section 4 is the second structure connection of a Frobenius manifold $(M, \circ, e, E, \tilde{g})$ and the way it is related to the first structure connection (*i.e.* the Levi-Civita connection of $\tilde{g}$). The first structure connection is a compatible connection on the underlying F-manifold $(M, \circ, e)$ of the Frobenius manifold. The second structure connection is the Levi-Civita connection of the second metric $g(X, Y) = \tilde{g}(E^{-1} \circ X, Y)$ and is compatible with the dual multiplication $X * Y = X \circ Y \circ E^{-1}$. With this motivation, it is natural to ask if the dual $(M, *, \mathcal{E})$ of an F-manifold $(M, \circ, e, \mathcal{E})$ with an eventual identity $\mathcal{E}$ inherits a canonical compatible, torsion-free connection, from such a connection $\tilde{\nabla}$ on $(M, \circ, e)$. We prove that there is a canonical family, rather than a single connection. This family consists of all connections of the form

$$\nabla_X Y := \mathcal{E} \circ \tilde{\nabla}_X(\mathcal{E}^{-1} \circ X) + \mathcal{E} \circ \tilde{\nabla}_Y(\mathcal{E}^{-1}) \circ X + V \circ X \circ Y, \tag{1.3}$$

where V is an arbitrary vector field. It arises naturally by asking that it contains torsion-free connections, which are compatible with the dual multiplication $X * Y = X \circ Y \circ \mathcal{E}^{-1}$ and are related to $\tilde{\nabla}$ in a similar way as the first and second structure connections of a Frobenius manifold are related (see Theorem 4.1 and the comments before). Finally we define the second structure connection of $(M, \circ, e, \mathcal{E}, \tilde{\nabla})$ and we show that it belongs to the canonical family (1.3) (see Definition 4.4 and Proposition 4.6).

In Section 5 we develop our main result (see Theorem 5.3). Here we introduce the notion of special family of connections, which plays a key role throughout this paper, and we interpret Theorem 4.1 as a duality (or an involution) on the set of F-manifolds with eventual identities and special families of connections.

The following sections are devoted to applications of our main result. In Section 6 we construct a duality for F-manifolds with eventual identities and compatible, torsion-free connections preserving the unit fields (see Proposition 6.2 with $U = 0$). Therefore, while in the setting of Frobenius manifolds the almost duality is not symmetric (the unit field of a Frobenius manifold is parallel, but the unit field of a dual almost Frobenius manifold is not, in general), in the larger setting of F-manifolds there is a perfect symmetry at the level of compatible, torsion-free connections which preserve the unit fields. We also construct a duality for F-manifolds with eventual identities and second structure connections (see Proposition 6.3). These dualities follow from our main result (Theorem 5.3 of Section 5 mentioned above), by noticing that we can fix a connection from a special family by prescribing the covariant derivative of the unit field (see Lemma 6.1).

In Section 7 we consider compatible, torsion-free connections on an F-manifold $(M, \circ, e)$, which satisfy the curvature condition

$$V \circ R_{Z,Y}(X) + Y \circ R_{V,Z}(X) + Z \circ R_{Y,V}(X) = 0, \quad \forall X, Y, Z, V, \tag{1.4}$$

introduced and studied in [12], in connection with the theory of equations of hydrodynamic type. This condition serves as the compatibility condition for an overdetermined linear system for families of vector fields that generate the symmetries of a system of hydrodynamic type. In the case of a semi-simple F-manifold, when canonical coordinates exist, this curvature condition reduces to the well-known semi-Hamiltonian condition first introduced by Tsarev [21]. We show that condition (1.4), if true, is independent of the choice of connection in a special family and is preserved by the duality for F-manifolds with eventual identities and special families of connections (see Theorem 7.1).

In the same framework, in Section 8 we consider flat, compatible, torsion-free connections on F-manifolds and their behaviour under the duality. It is easy to see that a special family of connections always contains non-flat connections (see Lemma 8.1). Our main result in this section is a necessary and sufficient condition on the eventual identity, which insures that the dual of a special family which contains a flat connection also has this property (see Theorem 8.2). This condition generalizes the usual condition $\tilde{\nabla}^2(E) = 0$ on the Euler field of a Frobenius manifold. We end this section with various other relevant remarks and comments in this direction (see Remark 8.3).

In Section 9 we develop a method which produces F-manifolds with compatible, torsion-free connections from an external bundle with additional structures. Under flatness assumptions, this is a reformulation of [15, Theorem 4.3]. External bundles with additional structures can be used to construct Frobenius manifolds [20] and the so called CV or CDV-structures on manifolds [8], which are key notions in tt^*-geometry and share many properties in common with Frobenius structures. Treating the tangent bundle as an external bundle, we introduce the notion of Legendre (or primitive) field and Legendre transformation of a special family of connections (see Definition 9.3). They are closely related to the corresponding notions [2,16] from the theory of Frobenius manifolds (see Remark 9.4). We prove that any Legendre transformation of a special family of connections on an F-manifold $(M, \circ, e)$ is also a special family on $(M, \circ, e)$ and we show that various curvature properties of special families are preserved by the Legendre transformations (see Proposition 9.5). Our main result in this section states that our duality for F-manifolds with eventual identities and special families of connections (see Theorem 5.3) commutes with Legendre transformations (see Theorem 9.6).

2. Preliminary material

In this section we recall two notions we need from the theory of F-manifolds: eventual identities, recently introduced in [4], and compatible connections. We begin by fixing our conventions.

Conventions 2.1. All results from this paper are stated in the smooth category (except the examples from Section 3) but they also apply to the holomorphic setting. Along the paper $\mathcal{X}(M)$ is the sheaf of smooth vector fields on a smooth manifold M and, for a smooth vector bundle V over M, $\Omega^1(M, V)$ is the sheaf of smooth 1-forms with values in V. On an F-manifold $(M, \circ, e)$ we shall often use the notation X^k (where $X \in \mathcal{X}(M)$ and $k \geq 1$) for $X \circ \ldots \circ X$ (k-times), X^0 for e and X^{-1} for the inverse of e with respect to $\circ$ (when it exists); for $k \leq 0$, $X^k := (X^{-1})^{-k}$.

2.1. Eventual identities on F-manifolds

As already mentioned in the introduction, an invertible vector field $\mathcal{E}$ on an F-manifold $(M, \circ, e)$ is an eventual identity [15] if the multiplication

$$X * Y := X \circ Y \circ \mathcal{E}^{-1}$$

defines a new F-manifold structure on M. The following theorem proved in [4] will be used throughout this paper.

Theorem 2.2.

i) *Let* $(M, \circ, e)$ *be an* F*-manifold and* $\mathcal{E}$ *an invertible vector field. Then* $\mathcal{E}$ *is an eventual identity if and only if*

$$L_{\mathcal{E}}(\circ)(X, Y) = [e, \mathcal{E}] \circ X \circ Y, \quad \forall X, Y \in \mathcal{X}(M). \tag{2.1}$$

ii) *Assume that* (2.1) *holds and let*

$$X * Y := X \circ Y \circ \mathcal{E}^{-1}$$

be the new F*-manifold multiplication. Then* e *is an eventual identity on* $(M, *, \mathcal{E})$ *and the map*

$$(M, \circ, e, \mathcal{E}) \to (M, *, \mathcal{E}, e)$$

is an involution on the set of F*-manifolds with eventual identities.*

Using the characterization of eventual identities provided by Theorem 2.2 i), it may be shown that the eventual identities form a subgroup in the group of invertible vector fields on an F-manifold. Also, if $\mathcal{E}$ is an eventual identity then, for any $m, n \in \mathbb{Z}$,

$$[\mathcal{E}^n, \mathcal{E}^m] = (m - n)\mathcal{E}^{m+n-1} \circ [e, \mathcal{E}]. \tag{2.2}$$

Moreover, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are eventual identities and $[\mathcal{E}_1, \mathcal{E}_2]$ is invertible, then $[\mathcal{E}_1, \mathcal{E}_2]$ is also an eventual identity. Any eventual identity on a product F-manifold decomposes into a sum of eventual identities on the factors. It follows that the duality for F-manifolds with eventual identities, described in Theorem 2.2 ii), commutes with the decomposition of F-manifolds [7]. For proofs of these facts, see [4].

Remark 2.3. An Euler field on an F-manifold $(M, \circ, e)$ is a vector field E such that $L_E(\circ) = d\circ$, where d is a constant, called the weight of E. It is easy to check that $[e, E] = de$. From Theorem 2.2, any invertible Euler field is an eventual identity. It would be interesting to generalize to eventual identities the various existing interpretations of Euler fields in terms of extended connections, like *e.g.* in [15], where the F-manifold comes with a compatible flat structure and a vector field is shown to be Euler (of weight one) if and only if a certain extended connection is flat.

2.2. Compatible connections on F-manifolds

Let $(M, \circ, e)$ be a manifold with a fiber-preserving commutative, associative, bilinear multiplication $\circ$ on TM with unit field e. Let $\tilde{C}$ be the End(TM)-valued 1-form (the associated Higgs field) defined by

$$\tilde{C}_X(Y) = X \circ Y, \quad \forall X, Y \in \mathcal{X}(M).$$

Let $\tilde{\nabla}$ be a connection on TM, with torsion $T^{\tilde{\nabla}}$. The exterior derivative $d^{\tilde{\nabla}}\tilde{\mathcal{C}}$ is a 2-form with values in $\mathrm{End}(TM)$, defined by

$$(d^{\tilde{\nabla}}\tilde{\mathcal{C}})_{X,Y} := \tilde{\nabla}_X(\tilde{\mathcal{C}}_Y) - \tilde{\nabla}_Y(\tilde{\mathcal{C}}_X) - \tilde{\mathcal{C}}_{[X,Y]}, \quad \forall X, Y \in \mathcal{X}(M).$$

It is straightforward to check that for any $X, Y, Z \in \mathcal{X}(M)$,

$$(d^{\tilde{\nabla}}\tilde{\mathcal{C}})_{X,Y}(Z) = \tilde{\nabla}_X(\circ)(Y, Z) - \tilde{\nabla}_Y(\circ)(X, Z) + T^{\tilde{\nabla}}(X, Y) \circ Z, \tag{2.3}$$

where the $(3, 1)$-tensor field $\tilde{\nabla}(\circ)$ is defined by

$$\tilde{\nabla}_X(\circ)(Y, Z) := \tilde{\nabla}_X(Y \circ Z) - \tilde{\nabla}_X(Y) \circ Z - Y \circ \tilde{\nabla}_X(Z), \quad \forall X, Y, Z \in \mathcal{X}(M). \tag{2.4}$$

The connection $\tilde{\nabla}$ is called compatible with $\circ$ if $\tilde{\nabla}(\circ)$ is totally symmetric (as a vector valued $(3, 0)$-tensor field). Note, from the commutativity of $\circ$, that $\tilde{\nabla}_X(\circ)(Y, Z)$ is always symmetric in Y and Z and the total symmetry of $\tilde{\nabla}(\circ)$ is equivalent to

$$\tilde{\nabla}_X(\circ)(Y, Z) = \tilde{\nabla}_Y(\circ)(X, Z), \quad \forall X, Y, Z \in \mathcal{X}(M).$$

From (2.3), if $\tilde{\nabla}$ is torsion-free, then it is compatible with $\circ$ if and only if

$$d^{\tilde{\nabla}}\tilde{\mathcal{C}} = 0. \tag{2.5}$$

From [8, Lemma 4.3] (which holds both in the smooth and holomorphic settings), the existence of a connection $\tilde{\nabla}$ on TM (not necessarily torsion-free) such that relation (2.5) holds implies that $(M, \circ, e)$ is an F-manifold. In particular, the existence of a torsion-free connection, compatible with $\circ$, implies that $(M, \circ, e)$ is an F-manifold [7]. Moreover, if $\tilde{\nabla}$ is the Levi-Civita connection of a multiplication invariant metric $\tilde{g}$ (*i.e.* $\tilde{g}(X \circ Y, Z) = \tilde{g}(X, Y \circ Z)$ for any vector fields X, Y, Z), then the total symmetry of $\tilde{\nabla}(\circ)$ is equivalent to the F-manifold condition (1.1) and the closedness of the coidentity $\tilde{g}(e)$ (which is the 1-form $\tilde{g}$-dual to the unit field e), see [7, Theorem 2.15].

Finally, we need to recall the definition of Frobenius manifolds.

Definition 2.4. A Frobenius manifold is an F-manifold $(M, \circ, e, E, \tilde{g})$ together with an Euler field E of weight 1 and a multiplication invariant flat metric $\tilde{g}$, such that the following conditions are satisfied:

i) the unit field e is parallel with respect to the Levi-Civita connection of $\tilde{g}$;
ii) the Euler field E rescales the metric $\tilde{g}$ by a constant.

Since the coidentity $\tilde{g}(e)$ of a Frobenius manifold $(M, \circ, e, E, \tilde{g})$ is a parallel (hence closed) 1-form, our comments above imply that the Levi-Civita connection $\tilde{\nabla}$ of $\tilde{g}$ (sometimes called the first structure connection) is a compatible connection on the underlying F-manifold $(M, \circ, e)$. Moreover, since $\tilde{\nabla}$ is flat and E rescales $\tilde{g}$ by a constant, $\tilde{\nabla}^2 E = 0$, *i.e.*

$$\tilde{\nabla}^2_{X,Y}(E) := \tilde{\nabla}_X \tilde{\nabla}_Y E - \tilde{\nabla}_{\tilde{\nabla}_X Y} E = 0, \quad X, Y \in \mathcal{X}(M).$$

The above relation implies that E is affine in flat coordinates for $\tilde{g}$.

3. Examples of eventual identities

Most of the results in this paper are constructive in nature: given some geometric structure based around an F-manifold, the existence of an eventual identity enables one to study symmetries of the structure and hence to construct new examples of the structure under study. For this procedure to work requires the existence of such an eventual identity.

The existence of an eventual identity on an F-manifold is not a priori obvious since equation (2.1) in Theorem 2.2, seen as differential equations for the components of $\mathcal{E}$, is overdetermined: there are $n(n+1)/2$ equations for the n unknown components (where $n = \dim(M)$). However, if the multiplication is semi-simple then solutions do exist [4]:

Example 3.1. Let $(M, \circ, e)$ be a semi-simple F-manifold with canonical coordinates $(u^1, \cdots, u^n)$, *i.e.*

$$\frac{\partial}{\partial u^i} \circ \frac{\partial}{\partial u^j} = \delta_{ij} \frac{\partial}{\partial u^j}, \quad \forall i, j$$

and

$$e = \frac{\partial}{\partial u^1} + \cdots + \frac{\partial}{\partial u^n}.$$

Any eventual identity is of the form

$$\mathcal{E} = f_1(u^1) \frac{\partial}{\partial u^1} + \cdots + f_n(u^n) \frac{\partial}{\partial u^n},$$

where f_i are arbitrary non-vanishing functions depending only on u^i.

In the following example one has a structure close to that of a Frobenius manifold, but no Euler field exists which is affine in the flat coordinates. However one may construct an eventual identity for the underlying F-manifold structure.

Example 3.2. Consider the prepotential

$$F = \frac{1}{2} t_1^2 t_2 + \frac{1}{4} t_2^2 \log(t_2^2) - \frac{\kappa}{6} t_1^3 .$$

This may be regarded as a deformation of the two-dimensional Frobenius manifold given by $\kappa = 0$, but if $\kappa \neq 0$ one does not have an Euler field which is affine in flat coordinates, *i.e.* the resulting multiplication is not quasi-homogeneous. However, one can construct an eventual identity by deforming the vector field

$$E = t_1 \frac{\partial}{\partial t_1} + 2t_2 \frac{\partial}{\partial t_2}$$

which is the Euler field for the initial Frobenius manifold.

Consider the ansatz

$$\mathcal{E} = E + \kappa \left[f(t_2) \frac{\partial}{\partial t_1} + g(t_2) \frac{\partial}{\partial t_2} \right]$$

where for simplicity it has been assumed that f and g are functions of t_2 alone.

The eventual identity condition (2.1) is trivially satisfied if X or $Y = e$ so one only has to consider the case $X = Y = \frac{\partial}{\partial t_2}$, and the two components of the resulting eventual identity equation yields differential equations the functions f and g, namely (where $x = t_2$ and $g' = dg/dx$ etc.):

$$\begin{aligned} 2xg' - g - \kappa x f' &= 0\,, \\ 2x^2 f' + \kappa x g' - \kappa g &= x\,. \end{aligned}$$

To get an understanding of these equations one can construct a solution as a power series in the 'deformation' parameter κ:

$$\begin{aligned} f(t_2) &= \left(\frac{1}{2} \log t_2 \right) + \kappa \left(\frac{-1}{2t_2^{\frac{1}{2}}} \right) + \kappa^2 \left(\frac{1}{4t_2} \right) + \ldots \\ g(t_2) &= \left(\frac{t_2^{\frac{1}{2}}}{2} \right) + \kappa \left(\frac{-1}{2} \right) + \kappa^2 \left(\frac{-1}{8t_2^{\frac{3}{2}}} \right) + \ldots\,. \end{aligned}$$

Solving the differential equations exactly (and ignoring arbitrary constants) gives

$$f(x) = \frac{1}{2} \log x + \kappa \Delta(x)\,, \qquad g(x) = -2x\Delta(x)$$

where

$$\Delta(x) = \frac{\tanh^{-1} \left[\dfrac{\sqrt{\kappa^2 + 4x}}{\kappa} \right]}{\sqrt{\kappa^2 + 4x}}\,.$$

Note that the form of the ansatz ensure that $[e, \mathcal{E}] = e$.

This example is not isolated but belongs to a wider class. The A_N-Frobenius manifold may be constructed via a superpotential construction. With

$$\lambda(p) = p^{N+1} + s_N p^{N-1} + \ldots + s_2 p + s_1$$

the metric and multiplication for the A_N-Frobenius structure are given by the residue formulae

$$\tilde{g}(\partial_{s_i}, \partial_{s_j}) = -\sum \underset{d\lambda=0}{\text{res}} \left\{ \frac{\partial_{s_i}\lambda(p)\, \partial_{s_j}\lambda(p)}{\lambda'(p)}\, dp \right\},$$

$$\tilde{c}\,(\partial_{s_i}, \partial_{s_j}, \partial_{s_k}) = -\sum \underset{d\lambda=0}{\text{res}} \left\{ \frac{\partial_{s_i}\lambda(p)\, \partial_{s_j}\lambda(p)\, \partial_{s_k}\lambda(p)}{\lambda'(p)}\, dp \right\}$$

and the Euler and identity vector fields follow from the form of the superpotential. This construction may be generalized by taking λ to be any holomorphic map from a Riemann surface to $\mathbb{P}^1$, with the moduli space of such maps (or Hurwitz space) carrying the structure of a Frobenius manifold.

To move away from Frobenius manifolds one may consider different classes of superpotentials [9]. Taking

$$\lambda = \text{rational function} + \kappa \log(\text{rational function}) \tag{3.1}$$

results, via similar residue formulae as above, to a flat metric and a semi-simple solution of the WDVV equations (note, the above example falls, after a Legendre transformation, into this class). Since the multiplication is semi-simple, eventual identities will exist, but their form in the flat coordinates for $\tilde{g}$ is not obvious.

Remarkably, this type of superpotential appeared at the same time in two different areas of mathematics. Motivated by the theory of integrable systems Chang [1] constructed a two-dimensional solution to the WDVV equations from a so-called "water-bag" reduction of the Benny hierarchy, and this was generalized to arbitrary dimension in [9] by considering superpotential of the form (3.1). Mathematically, the same form of superpotential appeared in the work of Milanov and Tseng on the equivariant orbifold structure of the complex projective line [18]. We will return to the construction of such eventual identities for these classes of F-manifolds in a future paper.

4. Compatible connections on F-manifolds and dual F-manifolds

We begin with a short review, intended for motivation, of the second structure connection of a Frobenius manifold. Then we prove our main result, namely that the dual of an F-manifold $(M, \circ, e, \mathcal{E}, \tilde{\nabla})$ with an eventual identity $\mathcal{E}$ and a compatible, torsion-free connection $\tilde{\nabla}$, comes naturally equipped with a family of compatible, torsion-free connections - namely the connections ∇^A from Theorem 4.1, with A given by (4.4). Finally, we define the second structure connection for $(M, \circ, e, \mathcal{E}, \tilde{\nabla})$ and we show that it belongs to this family (see Section 4.3).

4.1. Motivation

Recall that the second structure connection $\widehat{\nabla}$ of a Frobenius manifold $(M, \circ, e, E, \tilde{g})$ is the Levi-Civita connection of the second metric

$$g(X, Y) = \tilde{g}(E^{-1} \circ X, Y)$$

(where we assume that E is invertible). Together with the first structure connection $\tilde{\nabla}$, it determines a pencil of flat connections, which plays a key role in the theory of Frobenius manifolds. The compatibility of $\widehat{\nabla}$ with the dual multiplication

$$X * Y = X \circ Y \circ E^{-1}$$

follows from a result of Hertling already mentioned in Section 2: $\widehat{\nabla}$ is the Levi-Civita connection of g, which is an invariant metric on the dual F-manifold $(M, *, E)$ and the coidentity $g(E)$ of $(M, *, E, g)$ is closed (because $g(E) = \tilde{g}(e)$, which is closed). From [7, Theorem 9.4 (a), (e)], $\widehat{\nabla}$ is related to $\tilde{\nabla}$ by

$$\widehat{\nabla}_X(Y) = E \circ \tilde{\nabla}_X(E^{-1} \circ Y) - \tilde{\nabla}_{E^{-1} \circ Y}(E) \circ X + \frac{1}{2}(D+1) X \circ Y \circ E^{-1}, \quad (4.1)$$

where D is the constant given by $L_E(\tilde{g}) = D\tilde{g}$.

4.2. The canonical family on the dual F-manifold

Motivated by the structure connections of a Frobenius manifold, we now consider an F-manifold $(M, \circ, e, \mathcal{E}, \tilde{\nabla})$ with an eventual identity $\mathcal{E}$ and a compatible, torsion-free connection $\tilde{\nabla}$. On the dual F-manifold $(M, *, \mathcal{E})$ we are looking for compatible, torsion-free connections, related to $\tilde{\nabla}$ in a similar way as the first and second structure connections of a Frobenius manifold are related (recall (4.1) above): that is, we consider connections of the form

$$\nabla^A_X Y = \mathcal{E} \circ \tilde{\nabla}_X(\mathcal{E}^{-1} \circ Y) + A(Y) \circ X, \quad \forall X, Y \in \mathcal{X}(M), \quad (4.2)$$

where A is a section of $\mathrm{End}(TM)$. In the case of a Frobenius manifold, the first and second structural connections are related as in (4.2) with

$$A(Y) = -\tilde{\nabla}_{E^{-1} \circ Y}(E) + \frac{D+1}{2} E^{-1} \circ Y.$$

Our main result in this section is the following.

Theorem 4.1. *Let $(M, \circ, e, \mathcal{E}, \tilde{\nabla})$ be an F-manifold with an eventual identity $\mathcal{E}$ and compatible, torsion-free connection $\tilde{\nabla}$ and a section A of* $\mathrm{End}(TM)$. *The connection ∇^A defined by* (4.2) *is torsion-free and compatible with the dual multiplication*

$$X * Y := X \circ Y \circ \mathcal{E}^{-1} \quad (4.3)$$

if and only if

$$A(Y) = \mathcal{E} \circ \tilde{\nabla}_Y(\mathcal{E}^{-1}) + V \circ Y, \quad \forall Y \in \mathcal{X}(M), \quad (4.4)$$

where V is an arbitrary vector field.

We divide the proof of Theorem 4.1 into two lemmas, as follows.

Lemma 4.2. *In the setting of Theorem* 4.1, *the connection* ∇^A *defined by* (4.2) *is compatible with* $*$ *if and only if*

$$\begin{aligned} &A(Y \circ Z) - A(Y) \circ Z - A(Z) \circ Y + A(e) \circ Y \circ Z \\ &= \mathcal{E} \circ \left(\tilde{\nabla}_Y(\circ)(\mathcal{E}^{-1}, Z) + \tilde{\nabla}_{\mathcal{E}^{-1}}(e) \circ Y \circ Z \right) \end{aligned} \tag{4.5}$$

for any vector fields $Y, Z \in \mathcal{X}(M)$.

Proof. Denote by $\tilde{\nabla}^c$ the connection conjugated to $\tilde{\nabla}$ using $\mathcal{E}$, *i.e*.

$$\tilde{\nabla}^c_X(Y) := \mathcal{E} \circ \tilde{\nabla}_X(\mathcal{E}^{-1} \circ Y), \quad \forall X, Y \in \mathcal{X}(M). \tag{4.6}$$

With this notation,

$$\nabla^A_X Y = \tilde{\nabla}^c_X Y + A(Y) \circ X\,. \tag{4.7}$$

From (4.3) and (4.6), for any $X, Y, Z \in \mathcal{X}(M)$,

$$\begin{aligned} \tilde{\nabla}^c_X(*)(Y, Z) &= \tilde{\nabla}^c_X(Y * Z) - \tilde{\nabla}^c_X(Y) * Z - Y * \tilde{\nabla}^c_X(Z) \\ &= \mathcal{E} \circ \tilde{\nabla}_X(\mathcal{E}^{-2} \circ Y \circ Z) - \tilde{\nabla}_X(\mathcal{E}^{-1} \circ Y) \circ Z - Y \circ \tilde{\nabla}_X(\mathcal{E}^{-1} \circ Z) \\ &= \mathcal{E} \circ \tilde{\nabla}_X(\circ)(\mathcal{E}^{-1} \circ Y, \mathcal{E}^{-1} \circ Z), \end{aligned}$$

where we used $\mathcal{E}^{-2} \circ Y \circ Z = (\mathcal{E}^{-1} \circ Y) \circ (\mathcal{E}^{-1} \circ Z)$ and

$$\begin{aligned} \tilde{\nabla}_X(\mathcal{E}^{-2} \circ Y \circ Z) = \tilde{\nabla}_X(\circ)(\mathcal{E}^{-1} \circ Y, \mathcal{E}^{-1} \circ Z) + \tilde{\nabla}_X(\mathcal{E}^{-1} \circ Y) \circ \mathcal{E}^{-1} \circ Z \\ + \mathcal{E}^{-1} \circ Y \circ \tilde{\nabla}_X(\mathcal{E}^{-1} \circ Z). \end{aligned}$$

Thus:

$$\tilde{\nabla}^c_X(*)(Y, Z) = \mathcal{E} \circ \tilde{\nabla}_X(\circ)(\mathcal{E}^{-1} \circ Y, \mathcal{E}^{-1} \circ Z), \quad \forall X, Y, Z \in \mathcal{X}(M). \tag{4.8}$$

Using (4.7) and (4.8), we get

$$\begin{aligned} \nabla^A_X(*)(Y, Z) &= \nabla^A_X(Y * Z) - \nabla^A_X(Y) * Z - Y * \nabla^A_X(Z) \\ &= \tilde{\nabla}^c_X(*)(Y, Z) + A(Y * Z) \circ X - (A(Y) \circ X) * Z - (A(Z) \circ X) * Y \\ &= \mathcal{E} \circ \tilde{\nabla}_X(\circ)(\mathcal{E}^{-1} \circ Y\,, \mathcal{E}^{-1} \circ Z) \\ &\quad + A(Y \circ Z \circ \mathcal{E}^{-1}) \circ X - A(Y) \circ X \circ Z \circ \mathcal{E}^{-1} \\ &\quad - A(Z) \circ X \circ Y \circ \mathcal{E}^{-1}. \end{aligned}$$

It follows that $\nabla^A(*)$ is totally symmetric if and only if

$$\begin{aligned} &\mathcal{E} \circ \tilde{\nabla}_X(\circ)(\mathcal{E}^{-1} \circ Y, \mathcal{E}^{-1} \circ Z) + A(Y \circ Z \circ \mathcal{E}^{-1}) \circ X - A(Y) \circ X \circ Z \circ \mathcal{E}^{-1} \\ &= \mathcal{E} \circ \tilde{\nabla}_Y(\circ)(\mathcal{E}^{-1} \circ X, \mathcal{E}^{-1} \circ Z) + A(X \circ Z \circ \mathcal{E}^{-1}) \circ Y - A(X) \circ Y \circ Z \circ \mathcal{E}^{-1}. \end{aligned}$$

Multiplying the above relation with $\mathcal{E}^{-1}$ and replacing Z by $Z\circ\mathcal{E}$ we see that $\nabla^A(*)$ is totally symmetric if and only if

$$\begin{aligned}&\tilde{\nabla}_X(\circ)(\mathcal{E}^{-1}\circ Y, Z) - \tilde{\nabla}_Y(\circ)(\mathcal{E}^{-1}\circ X, Z)\\ &= \mathcal{E}^{-1}\circ(A(X\circ Z) - A(X)\circ Z)\circ Y\\ &\quad - \mathcal{E}^{-1}\circ(A(Y\circ Z) - A(Y)\circ Z)\circ X.\end{aligned}\tag{4.9}$$

Letting in this expression $X := e$ we get

$$\begin{aligned}&A(Y\circ Z) - A(Y)\circ Z - A(Z)\circ Y + A(e)\circ Y\circ Z\\ &= \mathcal{E}\circ\Big(\tilde{\nabla}_Y(\circ)(\mathcal{E}^{-1}, Z) - \tilde{\nabla}_e(\circ)(\mathcal{E}^{-1}\circ Y, Z)\Big).\end{aligned}\tag{4.10}$$

On the other hand, for any vector field $Z\in\mathcal{X}(M)$,

$$\tilde{\nabla}_Z(e) = \tilde{\nabla}_e(e)\circ Z,\tag{4.11}$$

which is a rewriting of the equality

$$\tilde{\nabla}_Z(\circ)(e,e) = \tilde{\nabla}_e(\circ)(Z,e).$$

Using (4.11) and the total symmetry of $\tilde{\nabla}$ we get

$$\tilde{\nabla}_e(\circ)(\mathcal{E}^{-1}\circ Y, Z) = \tilde{\nabla}_Z(\circ)(\mathcal{E}^{-1}\circ Y, e) = -\mathcal{E}^{-1}\circ Y\circ\tilde{\nabla}_Z(e) = -\tilde{\nabla}_{\mathcal{E}^{-1}}(e)\circ Y\circ Z.$$

Combining this relation with (4.10) we get (4.5). We proved that if $\nabla^A(*)$ is totally symmetric, then (4.5) holds. Conversely, we now assume that (4.5) holds and we show that $\nabla^A(*)$ is totally symmetric. Using (4.5), relation (4.9) which characterizes the symmetry of $\nabla^A(*)$ is equivalent to

$$\begin{aligned}&\tilde{\nabla}_X(\circ)(\mathcal{E}^{-1}\circ Y, Z) - \tilde{\nabla}_Y(\circ)(\mathcal{E}^{-1}\circ X, Z)\\ &= \tilde{\nabla}_X(\circ)(\mathcal{E}^{-1}, Z)\circ Y - \tilde{\nabla}_Y(\circ)(\mathcal{E}^{-1}, Z)\circ X\end{aligned}$$

or to

$$\begin{aligned}&\tilde{\nabla}_Z(\circ)(\mathcal{E}^{-1}\circ Y, X) - \tilde{\nabla}_Z(\circ)(\mathcal{E}^{-1}\circ X, Y)\\ &= \tilde{\nabla}_Z(\circ)(\mathcal{E}^{-1}, X)\circ Y - \tilde{\nabla}_Z(\circ)(\mathcal{E}^{-1}, Y)\circ X,\end{aligned}\tag{4.12}$$

where we used the symmetry of $\tilde{\nabla}(\circ)$. Using the definition of $\tilde{\nabla}(\circ)$, it may be checked that (4.12) holds. Our claim follows. □

The next lemma concludes the proof of Theorem 4.1.

Lemma 4.3. *In the setting of Theorem* 4.1, *the connection* ∇^A *defined by* (4.2) *is torsion-free and compatible with* $*$ *if and only if*

$$A(Y) = \mathcal{E}\circ\tilde{\nabla}_Y(\mathcal{E}^{-1}) + V\circ Y,\quad \forall Y\in\mathcal{X}(M),\tag{4.13}$$

where V *is an arbitrary vector field.*

Proof. Using the definition of ∇^A, the torsion-free property of $\tilde{\nabla}$ and the total symmetry of $\tilde{\nabla}(\circ)$, it can be checked that ∇^A is torsion-free if and only if, for any vector fields $X, Y \in \mathcal{X}(M)$,

$$A(X) \circ Y - A(Y) \circ X = \mathcal{E} \circ \left(\tilde{\nabla}_X(\mathcal{E}^{-1}) \circ Y - \tilde{\nabla}_Y(\mathcal{E}^{-1}) \circ X\right), \tag{4.14}$$

or, equivalently,

$$A(Y) = \mathcal{E} \circ \tilde{\nabla}_Y(\mathcal{E}^{-1}) + \left(A(e) - \mathcal{E} \circ \tilde{\nabla}_e(\mathcal{E}^{-1})\right) \circ Y, \quad \forall Y \in \mathcal{X}(M). \tag{4.15}$$

In particular, A is of the form (4.13), with

$$V = A(e) - \mathcal{E} \circ \tilde{\nabla}_e(\mathcal{E}^{-1}).$$

We now check that the connection ∇^A, with A given by (4.13), is compatible with $*$. For this, we apply Lemma 4.2. Thus, we have to check that relation (4.5) holds, with A defined by (4.13). When A is given by (4.13), relation (4.5) becomes

$$\begin{aligned} &\tilde{\nabla}_{Y \circ Z}(\mathcal{E}^{-1}) - \tilde{\nabla}_Y(\mathcal{E}^{-1}) \circ Z - \tilde{\nabla}_Z(\mathcal{E}^{-1}) \circ Y + \tilde{\nabla}_e(\mathcal{E}^{-1}) \circ Y \circ Z \\ &= \tilde{\nabla}_{\mathcal{E}^{-1}}(\circ)(Y, Z) + \tilde{\nabla}_{\mathcal{E}^{-1}}(e) \circ Y \circ Z. \end{aligned} \tag{4.16}$$

From the definition of $\tilde{\nabla}(\circ)$, the second line of (4.16) is equal to

$$\tilde{\nabla}_{\mathcal{E}^{-1}}(Y \circ Z) - \tilde{\nabla}_{\mathcal{E}^{-1}}(Y) \circ Z - Y \circ \tilde{\nabla}_{\mathcal{E}^{-1}}(Z) + \tilde{\nabla}_{\mathcal{E}^{-1}}(e) \circ Y \circ Z.$$

With this remark it is easy to check that (4.16) holds (use that $\mathcal{E}^{-1}$ is an eventual identity and $\tilde{\nabla}$ is torsion-free). Our claim follows. □

The proof of Theorem 4.1 is now completed.

4.3. The second structure connection of an F-manifold

In analogy with Frobenius manifolds, we now define the notion of second structure connection in the larger setting of F-manifolds.

Definition 4.4. Let $(M, \circ, e, \mathcal{E}, \tilde{\nabla})$ be an F-manifold with an eventual identity $\mathcal{E}$ and a compatible, torsion-free connection $\tilde{\nabla}$. The connection

$$\nabla^{\mathcal{F}}_X(Y) := \mathcal{E} \circ \tilde{\nabla}_X(\mathcal{E}^{-1} \circ Y) - \tilde{\nabla}_{\mathcal{E}^{-1} \circ Y}(\mathcal{E}) \circ X \tag{4.17}$$

is called the second structure connection of $(M, \circ, e, \mathcal{E}, \tilde{\nabla})$.

Remark 4.5. When the F-manifold $(M, \circ, e)$ underlies a Frobenius manifold $(M, \circ, e, E, \tilde{g})$, $\mathcal{E} = E$ is the Euler field (assumed to be invertible) and $\tilde{\nabla}$ is the first structure connection, the Frobenius second structure connection $\widehat{\nabla}$ given by (4.1) and the $\mathcal{F}$-manifold second structure connection $\nabla^{\mathcal{F}}$ given by Definition 4.4 differ by a (constant) multiple of the Higgs field $\tilde{\mathcal{C}}_X(Y) = X \circ Y$ - hence, they belong to the same pencil of connections on $(M, \circ, e)$. Both are flat and compatible with the dual multiplication $X * Y = X \circ Y \circ E^{-1}$.

Our main result from this section shows that $\nabla^{\mathcal{F}}$ is compatible with $*$ in cases where $(M, \circ, e)$ does not necessarily come equipped with a Frobenius metric. More precisely, the following holds:

Proposition 4.6. *The second structure connection $\nabla^{\mathcal{F}}$ of an F-manifold $(M, \circ, e, \mathcal{E}, \tilde{\nabla})$ with an eventual identity $\mathcal{E}$ and a compatible, torsion-free connection $\tilde{\nabla}$, is torsion-free and compatible on the dual F-manifold $(M, *, \mathcal{E})$. It belongs to the family of connections $\{\nabla^{A}\}$ (given by* (4.2) *and* (4.4)*), determined in Theorem* 4.1.

Proposition 4.6 is a consequence of Theorem 4.1, the definition of $\nabla^{\mathcal{F}}$ and the following lemma.

Lemma 4.7. *Let $(M, \circ, e, \mathcal{E}, \tilde{\nabla})$ be an F-manifold with an eventual identity $\mathcal{E}$ and a compatible, torsion-free connection $\tilde{\nabla}$. Then, for any $X \in \mathcal{X}(M)$,*

$$\tilde{\nabla}_{\mathcal{E}^{-1}\circ X}(\mathcal{E}) = -\mathcal{E}\circ\tilde{\nabla}_X(\mathcal{E}^{-1}) + \left(\frac{1}{2}\left(\tilde{\nabla}_{\mathcal{E}^{-1}}(\mathcal{E}) + \tilde{\nabla}_{\mathcal{E}}(\mathcal{E}^{-1})\right) + \tilde{\nabla}_e(e)\right)\circ X. \tag{4.18}$$

Proof. Using that $\mathcal{E}$ is an eventual identity and $\tilde{\nabla}$ is torsion-free, we get:

$$\begin{aligned}\tilde{\nabla}_{\mathcal{E}^{-1}\circ X}(\mathcal{E}) &= \tilde{\nabla}_{\mathcal{E}}(\mathcal{E}^{-1}\circ X) + L_{\mathcal{E}^{-1}\circ X}(\mathcal{E})\\ &= \tilde{\nabla}_{\mathcal{E}}(\mathcal{E}^{-1}\circ X) - [\mathcal{E}, \mathcal{E}^{-1}]\circ X - \mathcal{E}^{-1}\circ[\mathcal{E}, X] - [e, \mathcal{E}]\circ\mathcal{E}^{-1}\circ X\\ &= \tilde{\nabla}_{\mathcal{E}}(\mathcal{E}^{-1})\circ X + \mathcal{E}^{-1}\circ\tilde{\nabla}_{\mathcal{E}}(X) + \tilde{\nabla}_{\mathcal{E}}(\circ)(\mathcal{E}^{-1}, X)\\ &\quad -[\mathcal{E}, \mathcal{E}^{-1}]\circ X - \mathcal{E}^{-1}\circ[\mathcal{E}, X] - [e, \mathcal{E}]\circ\mathcal{E}^{-1}\circ X.\end{aligned} \tag{4.19}$$

On the other hand, using the total symmetry of $\tilde{\nabla}(\circ)$ and (4.11),

$$\begin{aligned}\tilde{\nabla}_{\mathcal{E}}(\circ)(\mathcal{E}^{-1}, X) &= \tilde{\nabla}_X(\circ)(\mathcal{E}, \mathcal{E}^{-1})\\ &= \tilde{\nabla}_e(e)\circ X - \tilde{\nabla}_X(\mathcal{E})\circ\mathcal{E}^{-1} - \mathcal{E}\circ\tilde{\nabla}_X(\mathcal{E}^{-1}).\end{aligned} \tag{4.20}$$

From (4.19), (4.20), and the torsion-free property of $\tilde{\nabla}$, we get

$$\tilde{\nabla}_{\mathcal{E}^{-1}\circ X}(\mathcal{E}) = -\mathcal{E}\circ\tilde{\nabla}_X(\mathcal{E}^{-1}) + \left(\tilde{\nabla}_{\mathcal{E}^{-1}}(\mathcal{E}) - \mathcal{E}^{-1}\circ[e, \mathcal{E}] + \tilde{\nabla}_e(e)\right)\circ X. \tag{4.21}$$

On the other hand, $\mathcal{E}$ is an eventual identity and relation (2.2) with $n = -1$ and $m = 1$ gives

$$\mathcal{E}^{-1}\circ[e, \mathcal{E}] = \frac{1}{2}[\mathcal{E}^{-1}, \mathcal{E}]. \tag{4.22}$$

Relation (4.21), (4.22) and again the torsion-free property of $\tilde{\nabla}$ imply our claim. □

In the next sections, we shall use Theorem 4.1 in the following equivalent form:

Corollary 4.8. *Let* $(M, \circ, e, \mathcal{E}, \tilde{\nabla})$ *be an* F*-manifold with an eventual identity* $\mathcal{E}$ *and a compatible, torsion-free connection* $\tilde{\nabla}$. *Any torsion-free connection compatible with the dual multiplication* $*$ *and of the form* (4.2) *is given by*

$$\nabla_X^W(Y) := \mathcal{E} \circ \tilde{\nabla}_X(\mathcal{E}^{-1} \circ Y) - \tilde{\nabla}_{\mathcal{E}^{-1} \circ Y}(\mathcal{E}) \circ X + W * X * Y$$

where W *is an arbitrary vector field.*

Proof. Trivial, from Theorem 4.1, Lemma 4.7 and the definition of $*$. □

5. Duality for F-manifolds with special families of connections

We now interpret Theorem 4.1 as a duality for F-manifolds with eventual identities and so called special families of connections (see Section 5.1). Then we discuss a class of special families of connections and the dual families (see Section 5.2).

5.1. Special families of connections and duality

Definition 5.1. A family of connections $\tilde{\mathcal{S}}$ on an F-manifold $(M, \circ, e)$ is called special if it is of the form

$$\tilde{\mathcal{S}} = \{\tilde{\nabla}^V, V \in \mathcal{X}(M)\}$$

where

$$\tilde{\nabla}_X^V(Y) := \tilde{\nabla}_X(Y) + V \circ X \circ Y, \quad \forall X, Y \in \mathcal{X}(M), \tag{5.1}$$

and $\tilde{\nabla}$ is a fixed connection, torsion-free and compatible with $\circ$.

Remark 5.2. It is easy to check that if $\tilde{\nabla}^V$ and $\tilde{\nabla}$ are any two connections related by (5.1), then

$$\tilde{\nabla}_X^V(\circ)(Y, Z) = \tilde{\nabla}_X(\circ)(Y, Z) - V \circ X \circ Y \circ Z, \quad \forall X, Y, Z \in \mathcal{X}(M).$$

Thus, $\tilde{\nabla}^V$ is compatible with $\circ$ if and only if $\tilde{\nabla}$ is compatible with $\circ$. Moreover, $\tilde{\nabla}^V$ is torsion-free if and only if $\tilde{\nabla}$ is torsion-free. It follows that all connections from a special family are torsion-free and compatible with the multiplication of the F-manifold.

In the language of special families of connections, Theorem 4.1 can be reformulated as follows:

Theorem 5.3. *Let* $(M, \circ, e, \mathcal{E}, \tilde{\mathcal{S}})$ *be an* F*-manifold with an eventual identity* $\mathcal{E}$ *and a special family of connections* $\tilde{\mathcal{S}}$. *Choose any* $\tilde{\nabla} \in \tilde{\mathcal{S}}$ *and define the family of connections*

$$\mathcal{D}_{\mathcal{E}}(\tilde{\mathcal{S}}) = \mathcal{S} := \{\nabla^W, W \in \mathcal{X}(M)\}$$

where

$$\nabla^W_X(Y) = \mathcal{E}\circ\tilde{\nabla}_X(\mathcal{E}^{-1}\circ Y) - \tilde{\nabla}_{\mathcal{E}^{-1}\circ Y}(\mathcal{E})\circ X + W * X * Y, \quad \forall X, Y \in \mathcal{X}(M) \quad (5.2)$$

and $*$ is the dual multiplication

$$X * Y = X \circ Y \circ \mathcal{E}^{-1}.$$

Then $\mathcal{S}$ is special on the dual F-manifold $(M, *, \mathcal{E})$ and the map

$$(M, \circ, e, \mathcal{E}, \tilde{\mathcal{S}}) \to (M, *, \mathcal{E}, e, \mathcal{S}) \quad (5.3)$$

is an involution on the set of F-manifolds with eventual identities and special families of connections.

Proof. The family $\mathcal{S}$ is well defined, *i.e.* independent of the choice of connection $\tilde{\nabla}$ from $\tilde{\mathcal{S}}$, and, from Corollary 4.8, it is a special family on $(M, *, \mathcal{E})$. It remains to prove that the map (5.3) is an involution. Recall, from Theorem 2.2, that the map

$$(M, \circ, e, \mathcal{E}) \to (M, *, \mathcal{E}, e)$$

is an involution on the set of F-manifolds with eventual identities. Thus, we only need to prove the statement about the special families. This reduces to showing that for (any) $\tilde{\nabla} \in \tilde{\mathcal{S}}$,

$$\tilde{\nabla}_X(Y) = e * \nabla^W_X(e^{-1,*} * Y) - \nabla^W_{e^{-1,*}*Y}(e) * X + V \circ X \circ Y, \quad (5.4)$$

where ∇^W is given by (5.2), V is a vector field which needs to be determined and $e^{-1,*} = \mathcal{E}^2$ is the inverse of the eventual identity e on the F-manifold $(M, *, \mathcal{E})$. From definitions, it is straightforward to check that

$$e * \nabla^W_X(e^{-1,*} * Y) - \nabla^W_{e^{-1,*}*Y}(e) * X = \tilde{\nabla}_X(Y) - \left(\tilde{\nabla}_Y(\mathcal{E})\circ\mathcal{E}^{-1} + \tilde{\nabla}_{\mathcal{E}\circ Y}(\mathcal{E}^{-1})\right)\circ X + \tilde{\nabla}_{\mathcal{E}^{-1}}(\mathcal{E})\circ X\circ Y,$$

for any $X, Y \in \mathcal{X}(M)$. Moreover, from Lemma 4.7 with $\mathcal{E}$ replaced by $\mathcal{E}^{-1}$,

$$\tilde{\nabla}_Y(\mathcal{E})\circ\mathcal{E}^{-1} + \tilde{\nabla}_{\mathcal{E}\circ Y}(\mathcal{E}^{-1}) = \left(\tilde{\nabla}_e(e) + \frac{1}{2}\left(\tilde{\nabla}_{\mathcal{E}}(\mathcal{E}^{-1}) + \tilde{\nabla}_{\mathcal{E}^{-1}}(\mathcal{E})\right)\right)\circ Y.$$

We get

$$e*\nabla^W_X(e^{-1,*}*Y) - \nabla^W_{e^{-1,*}*Y}(e)*X = \tilde{\nabla}_X(Y) - \left(\frac{1}{2}[\mathcal{E}, \mathcal{E}^{-1}] + \tilde{\nabla}_e(e)\right)\circ X\circ Y. \quad (5.5)$$

We deduce that (5.4) holds, with

$$V := \frac{1}{2}[\mathcal{E}, \mathcal{E}^{-1}] + \tilde{\nabla}_e(e).$$

Our claim follows. □

5.2. A class of special families of connections

In the following example we discuss a class of special families of connections and how they behave under the duality.

Example 5.4. Let $(M, \circ, e)$ be an F-manifold of dimension n and $\tilde{X}_0$ a vector field such that the system $\{\tilde{X}_0, \tilde{X}_0^2, \cdots, \tilde{X}_0^n\}$ is a frame of TM. Assume that there is a torsion-free, compatible connection $\tilde{\nabla}$ on $(M, \circ, e)$ satisfying

$$\tilde{\nabla}_Y(\tilde{X}_0) \circ Z = \tilde{\nabla}_Z(\tilde{X}_0) \circ Y, \quad \forall Y, Z \in \mathcal{X}(M). \tag{5.6}$$

The following facts hold:

i) The set $\tilde{\mathcal{S}}$ of all torsion-free, compatible connections on $(M, \circ, e)$, satisfying (5.6), is special.
ii) The image $\mathcal{S} := \mathcal{D}_{\mathcal{E}}(\tilde{\mathcal{S}})$ of $\tilde{\mathcal{S}}$ through the duality of Theorem 5.3, defined by an eventual identity $\mathcal{E}$ on $(M, \circ, e)$, is the special family of compatible, torsion-free connections ∇ on the dual F-manifold $(M, *, \mathcal{E})$, satisfying

$$\nabla_Y(\mathcal{E} \circ \tilde{X}_0) * Z = \nabla_Z(\mathcal{E} \circ \tilde{X}_0) * Y, \quad \forall Y, Z \in \mathcal{X}(M). \tag{5.7}$$

Proof. Suppose that $\tilde{\nabla}$ is a torsion-free connection, compatible with $\circ$ and satisfying (5.6), and let

$$\tilde{\nabla}^B_X(Y) = \tilde{\nabla}_X(Y) + B_X(Y), \quad \forall X, Y \in \mathcal{X}(M)$$

be another connection with these properties, where $B \in \Omega^1(M, \mathrm{End}TM)$. Since $\tilde{\nabla}$ and $\tilde{\nabla}^B$ are torsion-free,

$$B_X(Y) = B_Y(X), \quad \forall X, Y \in \mathcal{X}(M). \tag{5.8}$$

Since $\tilde{\nabla}(\circ)$ is totally symmetric, the total symmetry of $\tilde{\nabla}^B(\circ)$ is equivalent to

$$B_X(Y \circ Z) - B_X(Z) \circ Y = B_Y(X \circ Z) - B_Y(Z) \circ X, \tag{5.9}$$

where we used (5.8). Letting $Z := e$ in the above relation we get

$$B_Y(e) = B_e(Y) = V \circ Y, \quad \forall Y \in \mathcal{X}(M), \tag{5.10}$$

where $V := B_e(e)$. On the other hand, since (5.6) is satisfied by both $\tilde{\nabla}$ and $\tilde{\nabla}^B$,

$$B_Y(\tilde{X}_0) \circ Z = B_Z(\tilde{X}_0) \circ Y, \quad \forall Y, Z \in \mathcal{X}(M). \tag{5.11}$$

We obtain:

$$B_{\tilde{X}_0}(Y) = B_Y(\tilde{X}_0) = B_e(\tilde{X}_0) \circ Y = V \circ \tilde{X}_0 \circ Y, \quad \forall Y \in \mathcal{X}(M), \tag{5.12}$$

where in the first equality we used (5.8), in the second equality we used (5.11) and in the third equality we used (5.10). Letting in (5.9) $X := \tilde{X}_0$ and using (5.12) we get

$$B_Y(\tilde{X}_0 \circ Z) = B_Y(Z) \circ \tilde{X}_0, \quad \forall Y, Z \in \mathcal{X}(M).$$

An induction argument now shows that

$$B_Y(\tilde{X}_0^k) = V \circ \tilde{X}_0^k \circ Y, \quad \forall Y \in \mathcal{X}(M), \quad \forall k \in \mathbb{N}.$$

Since $\{\tilde{X}_0, \tilde{X}_0^2, \cdots, \tilde{X}_0^n\}$ is a frame of TM,

$$B_Y(X) = V \circ X \circ Y, \quad \forall X, Y \in \mathcal{X}(M)$$

and thus $\tilde{\mathcal{S}}$ is a special family on $(M, \circ, e)$. Claim *i)* follows. For claim *ii)*, one checks, using (5.6), that any connection $\nabla \in \mathcal{S}$ satisfies (5.7). On the other hand, $(\mathcal{E} \circ \tilde{X}_0) * \cdots * (\mathcal{E} \circ \tilde{X}_0)$ (k-times) is equal to $\mathcal{E} \circ \tilde{X}_0^k$ and $\{\mathcal{E} \circ \tilde{X}_0, \cdots, \mathcal{E} \circ \tilde{X}_0^n\}$ is a frame of TM. Therefore, from claim *i)*, the set of compatible, torsion-free connections on $(M, *, \mathcal{E})$ satisfying (5.7) is special. It coincides with $\mathcal{S}$. □

Remark 5.5. i) Consider the setting of Example 5.4 and assume, moreover, that the F-manifold $(M, \circ, e)$ is semi-simple. We claim that under this additonal assumption, one can always find a compatible, torsion-free connection satisfying relation (5.6) (and hence the family of all such connections is special). Using the canonical coordinates $(u^1, \cdots, u^n)$ on the F-manifold $(M, \circ, e)$, we can give a direct proof for this claim, as follows. As proved in [12], the Christoffel symbols Γ^k_{ij} of a compatible, torsion-free connection $\tilde{\nabla}$ on $(M, \circ, e)$ satisfy, in the coordinate system $(u^1, \cdots, u^n)$,

$$\Gamma^s_{ij} = 0, \quad \forall i \neq j,\ s \notin \{i, j\} \tag{5.13}$$

and

$$\Gamma^i_{ij} = -\Gamma^i_{jj}, \quad \forall i \neq j, \tag{5.14}$$

while Γ^i_{ii} are arbitrary. Given a vector field $\tilde{X}_0 = \sum_{k=1}^n X^k \frac{\partial}{\partial u^k}$ relation (5.6) is equivalent to

$$\frac{\partial X^j}{\partial u^i} + (X^i - X^j)\Gamma^j_{ii} = 0, \quad i \neq j. \tag{5.15}$$

If, moreover, $\{\tilde{X}_0, \tilde{X}_0^2, \cdots, \tilde{X}_0^n\}$ is a frame of TM, $X^i(p) \neq X^j(p)$ at any $p \in M$, for any $i \neq j$, and $\Gamma^j_{ii}, i \neq j$, are determined by $\tilde{X}_0$ using (5.15). We proved that a connection $\tilde{\nabla}$ on $(M, \circ, e)$ is torsion-free, compatible, and satisfies (5.6) if and only if its Christoffel symbols Γ^i_{jk}, with i, j, k not all equal, are determined by (5.13), (5.14), (5.15), and the remaining Γ^i_{ii} are arbitrary. It follows that the family of all such connections is non-empty (and special).

ii) In the semi-simple setting, relation (5.6) was considered in [13], in connection with semi-Hamiltonian systems. It is worth to remark a difference between our conventions and those used in [12, 13]: while for us a compatible connection on an F-manifold $(M, \circ, e)$ is a connection $\tilde{\nabla}$ for which the vector valued $(3, 0)$-tensor field $\tilde{\nabla}(\circ)$ is totally symmetric, in the language of [12, 13] a compatible connection satisfies, besides this condition, another additional condition, involving the curvature, see (7.1) from the Section 7. Hopefully this will not generate confusion.

In the following sections we discuss several applications of Theorem 5.3.

6. Duality for F-manifolds with eventual identities and compatible, torsion-free connections

It is natural to ask if the duality of Theorem 5.3 induces a duality for F-manifolds with eventual identities and compatible, torsion-free connections, rather than special families. This amounts to choosing, in a way consistent with the duality, a preferred connection in a special family. We will show that this can be done by prescribing the covariant derivative of the unit field. We shall consider two cases of this construction: when the unit fields are parallel (see Section 6.1) and when the preferred connections are the second structure connections (see Section 6.2).

6.1. Duality and parallel unit fields

Lemma 6.1. *Let $\tilde{\mathcal{S}}$ be a special family on an F-manifold $(M, \circ, e)$ and U a vector field on M. There is a unique connection $\tilde{\nabla}$ in $\tilde{\mathcal{S}}$ such that*

$$\tilde{\nabla}_X(e) = U \circ X, \quad \forall X \in \mathcal{X}(M).$$

In particular, any special family on an F-manifold contains a unique connection for which the unit field is parallel.

Proof. Straightforward from (4.11). □

The following proposition with $U = 0$ gives a duality for F-manifolds with eventual identities and compatible, torsion-free connections preserving the unit fields.

Proposition 6.2. *Let M be a manifold and U a fixed vector field on M. The map*

$$(\circ, e, \mathcal{E}, \tilde{\nabla}) \to (*, \mathcal{E}, e, \nabla) \tag{6.1}$$

where $*$ is related to $\circ$ by (4.3) and ∇ is related to $\tilde{\nabla}$ by

$$\nabla_X(Y) = \mathcal{E} \circ \tilde{\nabla}_X(\mathcal{E}^{-1} \circ Y) - \tilde{\nabla}_{\mathcal{E}^{-1} \circ Y}(\mathcal{E}) \circ X + \frac{1}{2}[\mathcal{E}^{-1}, \mathcal{E}] \circ X \circ Y + U * X * Y, \tag{6.2}$$

is an involution on the set of quatruples $(\circ, e, \mathcal{E}, \tilde{\nabla})$, where $\circ$ is the multiplication of an F-manifold structure on M with unit field e, $\mathcal{E}$ is an eventual identity on $(M, \circ, e)$ and $\tilde{\nabla}$ is torsion-free connection, compatible with $\circ$, such that

$$\tilde{\nabla}_X(e) = U \circ X, \quad \forall X \in \mathcal{X}(M). \tag{6.3}$$

Proof. Assume that $(\circ, e, \mathcal{E}, \tilde{\nabla})$ is a quatruple like in the statement of the proposition. From Theorem 2.2, $*$ defines an F-manifold structure on M with unit field $\mathcal{E}$ and e is an eventual identity on $(M, *, \mathcal{E})$. From Corollary 4.8, the connection ∇, related to $\tilde{\nabla}$ by (6.2), is compatible with $*$ and is torsion-free. Moreover, it is easy to check, using (4.22), (6.3) and the torsion-free property of $\tilde{\nabla}$, that

$$\nabla_X(\mathcal{E}) = U * X, \quad \forall X \in \mathcal{X}(M).$$

It follows that the map (6.1) is well defined. It remains to prove that it is an involution. This amounts to showing that

$$\tilde{\nabla}_X(Y) = e*\nabla_X(e^{-1,*}*Y)-\nabla_{e^{-1,*}*Y}(e)*X+\frac{1}{2}[e^{-1,*}, e]*X*Y+U\circ X\circ Y, \quad (6.4)$$

for any vector fields $X, Y \in \mathcal{X}(M)$. To prove (6.4) we make the following computation: from (5.5) and the definition of $*$,

$$\begin{aligned}
&e * \nabla_X(e^{-1,*} * Y) - \nabla_{e^{-1,*}*Y}(e) * X + \frac{1}{2}[e^{-1,*}, e] * X * Y + U \circ X \circ Y\\
&= \tilde{\nabla}_X(Y) - \left(\frac{1}{2}[\mathcal{E}, \mathcal{E}^{-1}] + \tilde{\nabla}_e(e)\right) \circ X \circ Y + \frac{1}{2}[\mathcal{E}^2, e] \circ \mathcal{E}^{-2} \circ X \circ Y + U \circ X \circ Y\\
&= \tilde{\nabla}_X(Y) - \tilde{\nabla}_e(e) \circ X \circ Y + U \circ X \circ Y,
\end{aligned}$$

where in the last equality we used $[\mathcal{E}^2, e] = 2[\mathcal{E}, e] \circ \mathcal{E}$ and (4.22). From (6.3) $\tilde{\nabla}_e(e) = U$ and relation (6.4) follows. □

6.2. Duality and second structure connections

From Definition 4.4, the second structure connection $\nabla^{\mathcal{F}}$ of an F-manifold $(M, \circ, e, \mathcal{E}, \tilde{\nabla})$ with an eventual identity $\mathcal{E}$ and a compatible, torsion-free connection $\tilde{\nabla}$, is given by (6.2), with

$$U := \frac{1}{2}[\mathcal{E}, \mathcal{E}^{-1}] \circ \mathcal{E}^2, \quad (6.5)$$

but despite this it does not fit into the involution of Proposition 6.2. The reason is that U is a fixed vector field in Proposition 6.2, while in (6.5) U changes when $(\circ, e, \mathcal{E}, \tilde{\nabla})$ varies in the domain of the map (6.1). If we want to construct a duality for F-manifolds with eventual identities and second structure connections, we therefore need a different type of map, like in the following proposition.

Proposition 6.3. *The map*

$$(M, \circ, e, \mathcal{E}, \tilde{\nabla}) \to (M, *, \mathcal{E}, e, \nabla^{\mathcal{F}}) \quad (6.6)$$

where $*$ is related to $\circ$ by (4.3) *and $\nabla^{\mathcal{F}}$ is the second structure connection of $(M, \circ, e, \mathcal{E}, \tilde{\nabla})$, is an involution on the set of F-manifolds $(M, \circ, e, \mathcal{E}, \tilde{\nabla})$ with eventual identities and compatible, torsion-free connections satisfying*

$$\tilde{\nabla}_X(e) = \frac{1}{2}[\mathcal{E}^{-1}, \mathcal{E}] \circ X, \quad \forall X \in \mathcal{X}(M). \tag{6.7}$$

Proof. Choose $(M, \circ, e, \mathcal{E}, \tilde{\nabla})$ from the domain of the map (6.6). From the torsion-free property of $\tilde{\nabla}$ and (6.7)

$$\begin{aligned} \tilde{\nabla}_e(\mathcal{E}) &= [e, \mathcal{E}] + \tilde{\nabla}_{\mathcal{E}}(e)\,, \\ &= [e, \mathcal{E}] + \frac{1}{2}[\mathcal{E}^{-1}, \mathcal{E}] \circ \mathcal{E}\,. \end{aligned} \tag{6.8}$$

Let $\nabla^{\mathcal{F}}$ be the second structure connection of $(M, \circ, e, \mathcal{E}, \tilde{\nabla})$. From its definition and equations (6.7) and (6.8)

$$\nabla^{\mathcal{F}}_X(\mathcal{E}) = ([\mathcal{E}, e] \circ \mathcal{E}) * X = \frac{1}{2}[e^{-1,*}, e] * X, \quad \forall X \in \mathcal{X}(M). \tag{6.9}$$

So the map (6.6) is well defined. Moreover, from (5.5) and (6.7),

$$\tilde{\nabla}_X Y = e * \nabla^{\mathcal{F}}_X(e^{-1,*} * Y) - \nabla^{\mathcal{F}}_{e^{-1,*} * Y}(e) * X,$$

i.e. $\tilde{\nabla}$ is the second structure connection of $(M, *, \mathcal{E}, e, \nabla^{\mathcal{F}})$. It follows that the map (6.6) is an involution. □

7. Duality and curvature

Let $(M, \circ, e, \tilde{\nabla})$ be an F-manifold with a compatible, torsion-free connection $\tilde{\nabla}$, with curvature $R^{\tilde{\nabla}}$. We assume that for any $X, Y, Z, V \in \mathcal{X}(M)$,

$$V \circ R^{\tilde{\nabla}}_{Z,Y} X + Y \circ R^{\tilde{\nabla}}_{V,Z} X + Z \circ R^{\tilde{\nabla}}_{Y,V} X = 0. \tag{7.1}$$

In the following theorem we show that condition (7.1) (if true) is independent of the choice of connection in a special family and is preserved under the duality of Theorem 5.3.

Theorem 7.1.

i) *Let $\tilde{\nabla}$ be a compatible, torsion-free connection on an F-manifold $(M, \circ, e)$ and*

$$\tilde{\mathcal{S}} := \{\tilde{\nabla}^V_X(Y) = \tilde{\nabla}_X(Y) + V \circ X \circ Y, \quad V \in \mathcal{X}(M)\}$$

the associated special family. Assume that (7.1) *holds for $\tilde{\nabla}$. Then* (7.1) *holds for all connections $\tilde{\nabla}^V$ from $\tilde{\mathcal{S}}$.*

ii) *The involution*

$$(M, \circ, e, \mathcal{E}, \tilde{\mathcal{S}}) \to (M, *, \mathcal{E}, e, \mathcal{S}) \tag{7.2}$$

from Theorem 5.3 *preserves the class of special families of connections which satisfy condition* (7.1). *More precisely, if* $\tilde{\nabla} \in \tilde{\mathcal{S}}$ *and* $\nabla \in \mathcal{S}$ *are any two connections belonging, respectively, to a special family* $\tilde{\mathcal{S}}$ *and its dual* $\mathcal{S} = \mathcal{D}_{\mathcal{E}}(\tilde{\mathcal{S}})$, *then*

$$V \circ R^{\tilde{\nabla}}_{Z,Y} X + Y \circ R^{\tilde{\nabla}}_{V,Z} X + Z \circ R^{\tilde{\nabla}}_{Y,V} X = 0, \quad \forall X, Y, Z, V \in \mathcal{X}(M) \tag{7.3}$$

if and only if

$$V * R^{\nabla}_{Z,Y} X + Y * R^{\nabla}_{V,Z} X + Z * R^{\nabla}_{Y,V} X = 0, \quad \forall X, Y, Z, V \in \mathcal{X}(M). \tag{7.4}$$

Before proving Theorem 7.1 we make some preliminary remarks. Note that claim i) from the above theorem is a particular case of claim ii): if in (7.2) $\mathcal{E} = e$, then $\circ = *$, $\tilde{\mathcal{S}} = \mathcal{S}$ and claim ii) reduces to claim i). Therefore, it is enough to prove claim ii). For this, we use that any two connections, one from $\tilde{\mathcal{S}}$ and the other from the dual $\mathcal{S} = \mathcal{D}_{\mathcal{E}}(\tilde{\mathcal{S}})$, are related by

$$\nabla^A_X Y = \mathcal{E} \circ \tilde{\nabla}_X(\mathcal{E}^{-1} \circ Y) + A(Y) \circ X, \quad \forall X, Y \in \mathcal{X}(M), \tag{7.5}$$

where A is a section of $\mathrm{End}(TM)$. Another easy but useful fact is that (7.4) holds if and only if it holds with $*$ replaced by $\circ$. With these preliminary remarks, Theorem 7.1 is a consequence of the following general result:

Proposition 7.2. *Let* $(M, \circ, e, \tilde{\nabla})$ *be an* F*-manifold with a compatible, torsion-free connection* $\tilde{\nabla}$, *such that*

$$V \circ R^{\tilde{\nabla}}_{Z,Y} X + Y \circ R^{\tilde{\nabla}}_{V,Z} X + Z \circ R^{\tilde{\nabla}}_{Y,V} X = 0, \tag{7.6}$$

for any $X, Y, Z, V \in \mathcal{X}(M)$. *Let* A *be a section of* $\mathrm{End}(TM)$ *and* $\mathcal{E}$ *an invertible vector field* (*not necessarily an eventual identity*). *Let* ∇^A *be the connection given by* (7.5). *Then also*

$$V \circ R^{\nabla^A}_{Z,Y} X + Y \circ R^{\nabla^A}_{V,Z} X + Z \circ R^{\nabla^A}_{Y,V} X = 0, \tag{7.7}$$

for any $X, Y, Z, V \in \mathcal{X}(M)$.

Proof. A straightforward computation which uses (7.5), the symmetry of $\tilde{\nabla}(\circ)$ and the torsion-free property of $\tilde{\nabla}$, gives

$$R^{\nabla^A}_{Y,Z} X = \mathcal{E} \circ R^{\tilde{\nabla}}_{Y,Z}(\mathcal{E}^{-1} \circ X) - Q(Y, X) \circ Z + Q(Z, X) \circ Y \tag{7.8}$$

where

$$Q(Y, X) := A(\mathcal{E} \circ \tilde{\nabla}_Y(\mathcal{E}^{-1} \circ X)) + A(A(X) \circ Y) - \mathcal{E} \circ \tilde{\nabla}_Y(\mathcal{E}^{-1} \circ A(X)).$$

From (7.8) we readily obtain that

$$V \circ R^{\nabla^A}_{Z,Y} X + Y \circ R^{\nabla^A}_{V,Z} X + Z \circ R^{\nabla^A}_{Y,V} X$$
$$= \mathcal{E} \circ \Big(V \circ R^{\tilde{\nabla}}_{Z,Y} (\mathcal{E}^{-1} \circ X) + Y \circ R^{\tilde{\nabla}}_{V,Z} (\mathcal{E}^{-1} \circ X) + Z \circ R^{\tilde{\nabla}}_{Y,V} (\mathcal{E}^{-1} \circ X) \Big).$$

The claim follows. □

The following remark describes various classes of compatible torsion-free connections whose curvature satisfies condition (7.1).

Remark 7.3. In the setting of Frobenius manifolds, both the Saito metric and the intersection form are automatically flat, so condition (7.1) and its dual are trivially satisfied. However, by twisting by powers of the Euler field (which is an eventual identity) one may construct a hierarchy of connections each of which satisfies this condition (this is just a statement, in the semi-simple case, of the semi-Hamiltonian property of these hierarchies). An alternative way to construct examples (in the case of Coxeter group orbit spaces) is to introduce conformal curvature [5]. This results in a constant curvature/zero curvature pair of connections both of which satisfy (7.1) and one may again keep twisting with the Euler field to obtain hierarchies of such connections. It is easy to see that (7.1) holds for the Levi-Civita connection of any metric of constant sectional curvature.

8. Duality and flat connections

While condition (7.1) is well suited for the duality of F-manifolds with eventual identities and special families of connections, it is natural to ask if other curvature conditions are well suited, too. Along these lines it natural to consider the flatness condition in relation with this duality.

A first remark is that the flatness condition depends on the choice of connection in a special family. More precisely, the following lemma holds:

Lemma 8.1. *Let $\tilde{\nabla}$ and $\tilde{\nabla}^V$ be two torsion-free, compatible connections on an F-manifold $(M, \circ, e)$, related by*

$$\tilde{\nabla}^V_X(Y) = \tilde{\nabla}_X(Y) + V \circ X \circ Y, \quad \forall X, Y \in \mathcal{X}(M), \tag{8.1}$$

where V is a vector field. Assume that $\tilde{\nabla}$ is flat. Then $\tilde{\nabla}^V$ is also flat if and only if

$$\tilde{\nabla}_X(\tilde{\mathcal{C}}_V)(Y) \circ Z = \tilde{\nabla}_Z(\tilde{\mathcal{C}}_V)(Y) \circ X, \quad \forall X, Y, Z \in \mathcal{X}(M), \tag{8.2}$$

where $\tilde{\mathcal{C}}_V$ is the section of $\mathrm{End}(TM)$ *given by*

$$\tilde{\mathcal{C}}_V(X) = X \circ V, \quad \forall X \in \mathcal{X}(M).$$

Proof. With no flatness assumptions, one can show that the curvatures of two compatible, torsion-free connections $\tilde{\nabla}^V$ and $\tilde{\nabla}$ like in (8.1), are related by

$$\begin{aligned} R^{\tilde{\nabla}^V}_{X,Z} Y = R^{\tilde{\nabla}}_{X,Z} Y &+ \Big(\tilde{\nabla}_X(V \circ Y) - V \circ \tilde{\nabla}_X(Y)\Big) \circ Z \\ &- \Big(\tilde{\nabla}_Z(V \circ Y) - V \circ \tilde{\nabla}_Z(Y)\Big) \circ X, \end{aligned} \tag{8.3}$$

for any vector fields X, Y, Z. Thus, if $\tilde{\nabla}$ is flat then $\tilde{\nabla}^V$ is also flat if and only if (8.2) holds. □

Given a special family which contains a flat connection, it is natural to ask when the dual family has this property, too. The answer is given in the following theorem, which is our main result from this section.

Theorem 8.2. *Let $(M, \circ, e, \mathcal{E}, \tilde{\mathcal{S}})$ be an F-manifold with an eventual identity $\mathcal{E}$ and a special family of connections $\tilde{\mathcal{S}}$. Assume that there is $\tilde{\nabla} \in \tilde{\mathcal{S}}$ which is flat. Then the dual family $\mathcal{S} = \mathcal{D}_{\mathcal{E}}(\tilde{\mathcal{S}})$ on the dual F-manifold $(M, *, \mathcal{E})$ contains a flat connection if and only if there is a vector field $\tilde{W}$, such that, for any $X, Y, Z \in \mathcal{X}(M)$,*

$$\Big(\tilde{\nabla}^2_{X,Y}(\mathcal{E}) - \tilde{\nabla}_X(\tilde{\mathcal{C}}_{\tilde{W}})(Y)\Big) \circ Z = \Big(\tilde{\nabla}^2_{Z,Y}(\mathcal{E}) - \tilde{\nabla}_Z(\tilde{\mathcal{C}}_{\tilde{W}})(Y)\Big) \circ X. \tag{8.4}$$

If (8.4) *holds then the connection*

$$\nabla^W_X(Y) := \mathcal{E} \circ \tilde{\nabla}_X(\mathcal{E}^{-1} \circ Y) - \tilde{\nabla}_{\mathcal{E}^{-1} \circ Y}(\mathcal{E}) \circ X + W * X * Y, \tag{8.5}$$

with $W := \tilde{W} \circ \mathcal{E}$, is flat (and belongs to $\mathcal{D}_{\mathcal{E}}(\tilde{\mathcal{S}})$).

Proof. Let W be a vector field and ∇^W the connection defined by (8.5). We need to compute the curvature of ∇^W (knowing that $\tilde{\nabla}$ is flat). For this, note that

$$\nabla^W_X(Y) = \nabla^{\mathcal{F}}_X(Y) + W * X * Y$$

where $\nabla^{\mathcal{F}}$ is the second structure connection of $(M, \circ, e, \mathcal{E}, \tilde{\nabla})$. Since both ∇^W and $\nabla^{\mathcal{F}}$ are torsion-free and compatible with $*$, their curvatures are related by (8.3), with $\circ$ replaced by $*$ and V replaced by W. Thus,

$$\begin{aligned} R^{\nabla^W}_{Z,X} Y = R^{\nabla^{\mathcal{F}}}_{Z,X} Y \quad &+ \Big(\nabla^{\mathcal{F}}_Z(W * Y) - W * \nabla^{\mathcal{F}}_Z(Y)\Big) * X \\ &- \Big(\nabla^{\mathcal{F}}_X(W * Y) - W * \nabla^{\mathcal{F}}_X(Y)\Big) * Z \\ = R^{\nabla^{\mathcal{F}}}_{Z,X}(Y) &+ \Big(\nabla^{\mathcal{F}}_Z(\mathcal{E}^{-1} \circ W \circ Y) - \mathcal{E}^{-1} \circ W \circ \nabla^{\mathcal{F}}_Z(Y)\Big) \circ X \circ \mathcal{E}^{-1} \\ &- \Big(\nabla^{\mathcal{F}}_X(\mathcal{E}^{-1} \circ W \circ Y) - \mathcal{E}^{-1} \circ W \circ \nabla^{\mathcal{F}}_X(Y)\Big) \circ Z \circ \mathcal{E}^{-1}. \end{aligned}$$

Using the definition (4.17) of $\nabla^{\mathcal{F}}$ to compute the right-hand side of the above relation, we get

$$R^{\nabla^W}_{Z,X}Y = R^{\nabla^{\mathcal{F}}}_{Z,X}Y + \Big(\mathcal{E}\circ\tilde{\nabla}_Z(W\circ Y\circ\mathcal{E}^{-2}) - W\circ\tilde{\nabla}_Z(\mathcal{E}^{-1}\circ Y)\Big)\circ X\circ\mathcal{E}^{-1}$$
$$- \Big(\mathcal{E}\circ\tilde{\nabla}_X(W\circ Y\circ\mathcal{E}^{-2}) - W\circ\tilde{\nabla}_X(\mathcal{E}^{-1}\circ Y)\Big)\circ Z\circ\mathcal{E}^{-1},$$

or, by replacing Y with $\mathcal{E}\circ Y$,

$$R^{\nabla^W}_{Z,X}(\mathcal{E}\circ Y) = R^{\nabla^{\mathcal{F}}}_{Z,X}(\mathcal{E}\circ Y) + \tilde{\nabla}_Z(\tilde{\mathcal{C}}_{W\circ\mathcal{E}^{-1}})(Y)\circ X - \tilde{\nabla}_X(\tilde{\mathcal{C}}_{W\circ\mathcal{E}^{-1}})(Y)\circ Z, \quad (8.6)$$

for any vector fields $X, Y, Z \in \mathcal{X}(M)$. We now compute the curvature of $\nabla^{\mathcal{F}}$. For this, let Y be a local $\tilde{\nabla}$-flat vector field and $\tilde{Y} := \tilde{\nabla}_Y(\mathcal{E})$. Then, for any $X, Z \in \mathcal{X}(M)$,

$$\nabla^{\mathcal{F}}_X(\mathcal{E}\circ Y) = -\tilde{Y}\circ X$$

and

$$\nabla^{\mathcal{F}}_Z\nabla^{\mathcal{F}}_X(\mathcal{E}\circ Y) = -\mathcal{E}\circ\tilde{\nabla}_Z(\mathcal{E}^{-1}\circ\tilde{Y}\circ X) + \tilde{\nabla}_{\mathcal{E}^{-1}\circ\tilde{Y}\circ X}(\mathcal{E})\circ Z. \quad (8.7)$$

Since $\mathcal{E}$ is an eventual identity, we can apply Lemma 4.7 to compute the second term in the right-hand side of (8.7). We get:

$$\tilde{\nabla}_{\mathcal{E}^{-1}\circ\tilde{Y}\circ X}(\mathcal{E}) = -\mathcal{E}\circ\tilde{\nabla}_{\tilde{Y}\circ X}(\mathcal{E}^{-1}) + \frac{1}{2}\Big(\tilde{\nabla}_{\mathcal{E}^{-1}}(\mathcal{E}) + \tilde{\nabla}_{\mathcal{E}}(\mathcal{E}^{-1})\Big)\circ\tilde{Y}\circ X$$
$$+ \tilde{\nabla}_e(e)\circ\tilde{Y}\circ X$$

and therefore

$$\nabla^{\mathcal{F}}_Z\nabla^{\mathcal{F}}_X(\mathcal{E}\circ Y) = -\mathcal{E}\circ\tilde{\nabla}_Z(\mathcal{E}^{-1}\circ\tilde{Y}\circ X) - \mathcal{E}\circ\tilde{\nabla}_{\tilde{Y}\circ X}(\mathcal{E}^{-1})\circ Z$$
$$+ \frac{1}{2}\Big(\tilde{\nabla}_{\mathcal{E}^{-1}}(\mathcal{E}) + \tilde{\nabla}_{\mathcal{E}}(\mathcal{E}^{-1})\Big)\circ\tilde{Y}\circ X\circ Z$$
$$+ \tilde{\nabla}_e(e)\circ\tilde{Y}\circ X\circ Z.$$

Assume now that X and Z are both $\tilde{\nabla}$-flat. Since $\tilde{\nabla}$ is torsion-free, $[X, Z] = 0$ and the above relation gives, by skew-symmetrizing in Z and X,

$$R^{\nabla^{\mathcal{F}}}_{Z,X}(\mathcal{E}\circ Y) = \mathcal{E}\circ\Big(\tilde{\nabla}_X(\mathcal{E}^{-1}\circ\tilde{Y}\circ Z) - \tilde{\nabla}_{\tilde{Y}\circ X}(\mathcal{E}^{-1})\circ Z\Big)$$
$$- \mathcal{E}\circ\Big(\tilde{\nabla}_Z(\mathcal{E}^{-1}\circ\tilde{Y}\circ X) - \tilde{\nabla}_{\tilde{Y}\circ Z}(\mathcal{E}^{-1})\circ X\Big)$$

or, using the $\tilde{\nabla}$-flatness of X, Z and the total symmetry of $\tilde{\nabla}$,

$$R^{\nabla^{\mathcal{F}}}_{Z,X}(\mathcal{E}\circ Y) = \mathcal{E}\circ\Big(\tilde{\nabla}_X(\mathcal{E}^{-1}\circ\tilde{Y}) - \tilde{\nabla}_{\tilde{Y}\circ X}(\mathcal{E}^{-1})\Big)\circ Z$$
$$- \mathcal{E}\circ\Big(\tilde{\nabla}_Z(\mathcal{E}^{-1}\circ\tilde{Y}) - \tilde{\nabla}_{\tilde{Y}\circ Z}(\mathcal{E}^{-1})\Big)\circ X.$$

We now simplify the right-hand side of this expression. Define

$$E(X,\tilde{Y}) = \tilde{\nabla}_X(\mathcal{E}^{-1}\circ\tilde{Y}) - \tilde{\nabla}_{\tilde{Y}\circ X}(\mathcal{E}^{-1}).$$

Then

$$\begin{aligned}
E(X,\tilde{Y}) &= \tilde{\nabla}_X(\mathcal{E}^{-1})\circ\tilde{Y} + \mathcal{E}^{-1}\circ\tilde{\nabla}_X(\tilde{Y}) + \tilde{\nabla}_{\mathcal{E}^{-1}}(\circ)(X,\tilde{Y}) - \tilde{\nabla}_{\tilde{Y}\circ X}(\mathcal{E}^{-1})\\
&= \tilde{\nabla}_X(\mathcal{E}^{-1})\circ\tilde{Y} + \mathcal{E}^{-1}\circ\tilde{\nabla}_X(\tilde{Y})\\
&\quad + \tilde{\nabla}_{\mathcal{E}^{-1}}(\tilde{Y}\circ X) - X\circ\tilde{\nabla}_{\mathcal{E}^{-1}}(\tilde{Y}) - \tilde{\nabla}_{\tilde{Y}\circ X}(\mathcal{E}^{-1})\\
&= \tilde{\nabla}_X(\mathcal{E}^{-1})\circ\tilde{Y} + \mathcal{E}^{-1}\circ\tilde{\nabla}_X(\tilde{Y}) + L_{\mathcal{E}^{-1}}(\tilde{Y}\circ X) - X\circ\tilde{\nabla}_{\mathcal{E}^{-1}}(\tilde{Y})\\
&= \mathcal{E}^{-1}\circ\tilde{\nabla}_X(\tilde{Y}) + \Big([e,\mathcal{E}^{-1}]\circ\tilde{Y} - \tilde{\nabla}_{\tilde{Y}}(\mathcal{E}^{-1})\Big)\circ X,
\end{aligned}$$

where in the first equality we used the symmetry of $\tilde{\nabla}(\circ)$; in the third equality we used the torsion-free property of $\tilde{\nabla}$; in the fourth equality we used that $\mathcal{E}^{-1}$ is an eventual identity, the $\tilde{\nabla}$-flatness of X and again the torsion-free property of $\tilde{\nabla}$. Since $\tilde{\nabla}_X(\tilde{Y}) = \tilde{\nabla}^2_{X,Y}(\mathcal{E})$ (Y being $\tilde{\nabla}$-flat) and similarly $\tilde{\nabla}_Z(\tilde{Y}) = \tilde{\nabla}^2_{Z,Y}(\mathcal{E})$, we get

$$R^{\nabla^{\mathcal{F}}}_{Z,X}(\mathcal{E}\circ Y) = \tilde{\nabla}^2_{X,Y}(\mathcal{E})\circ Z - \tilde{\nabla}^2_{Z,Y}(\mathcal{E})\circ X. \tag{8.8}$$

Combining (8.6) with (8.8) we finally obtain

$$\begin{aligned}
R^{\nabla^W}_{Z,X}(\mathcal{E}\circ Y) &= \Big(\tilde{\nabla}^2_{X,Y}(\mathcal{E}) - \tilde{\nabla}_X(\tilde{\mathcal{C}}_{W\circ\mathcal{E}^{-1}})(Y)\Big)\circ Z\\
&\quad - \Big(\tilde{\nabla}^2_{Z,Y}(\mathcal{E}) - \tilde{\nabla}_Z(\tilde{\mathcal{C}}_{W\circ\mathcal{E}^{-1}})(Y)\Big)\circ X,
\end{aligned} \tag{8.9}$$

for any $\tilde{\nabla}$-flat vector fields X, Y, Z. Since $\tilde{\nabla}$ is flat, relation (8.9) holds for any $X, Y, Z \in \mathcal{X}(M)$ (not necessarily flat). Our claim follows. □

We end this section with several comments and remarks.

Remark 8.3. i) Condition (8.2) is clearly satisfied when V is the unit field or a (constant) multiple of the unit field of the F-manifold. Therefore, if $\tilde{\nabla}$ is a compatible, torsion-free connection on an F-manifold $(M,\circ,e)$ and the pencil

$$\tilde{\nabla}^z_X(Y) = \tilde{\nabla}_X(Y) + zX\circ Y$$

(z-constant) contains a flat connection, then all connections from this pencil are flat. Such pencils of flat torsion-free connections appear naturally on Frobenius manifolds. More generally, it can be checked that on a semi-simple F-manifold $(M,\circ,e)$ with canonical coordinates $(u^1,\cdots,u^n)$, a vector field $V = \sum_{k=1}^n V^k\frac{\partial}{\partial u^k}$ satisfies (8.2) if and only if

$$V^j = V^j(u^j),\quad \Gamma^j_{ii}(V^i - V^j) = 0,\quad \forall i,j$$

where V^j are functions depending on u^j only and Γ^k_{ij} are the Christoffel symbols of $\tilde{\nabla}$ in the coordinate system $(u^1, \cdots, u^n)$.

ii) We claim that condition (8.4) from Theorem 8.2 is independent of the choice of flat connection $\tilde{\nabla}$ from $\tilde{\mathcal{S}}$. To prove this claim, let $(M, \circ, e, \mathcal{E})$ be an F-manifold and $\tilde{\mathcal{C}}_X(Y) := X \circ Y$ the associated Higgs field. Let $\mathcal{E}$ be an eventual identity and $\tilde{\nabla}$, $\tilde{\nabla}^V$ any two compatible, torsion-free, flat connections on $(M, \circ, e)$, related by (5.1). Since both $\tilde{\nabla}$ and $\tilde{\nabla}^V$ are flat, (8.2) holds. A long but straightforward computation which uses (8.2) shows that

$$(\tilde{\nabla}^V)^2_{X,Y}(\mathcal{E}) \circ Z - (\tilde{\nabla}^V)^2_{Z,Y}(\mathcal{E}) \circ X = \tilde{\nabla}^2_{X,Y}(\mathcal{E}) \circ Z - \tilde{\nabla}^2_{Z,Y}(\mathcal{E}) \circ X, \qquad (8.10)$$

for any vector fields X, Y, Z. On the other hand, it is easy to see that

$$\tilde{\nabla}^V_X(\tilde{\mathcal{C}}_S)(Y) \circ Z - \tilde{\nabla}^V_Z(\tilde{\mathcal{C}}_S)(Y) \circ X = \tilde{\nabla}_X(\tilde{\mathcal{C}}_S)(Y) \circ Z - \tilde{\nabla}_Z(\tilde{\mathcal{C}}_S)(Y) \circ X, \quad (8.11)$$

for any vector fields X, Y, Z, S. Using (8.10) and (8.11) we deduce that (8.4) holds for $\tilde{\nabla}^V$ if and only if it holds for $\tilde{\nabla}$.

iii) In the setting of Theorem 8.2 assume that the initial F-manifold $(M, \circ, e)$ underlies a Frobenius manifold $(M, \circ, e, E, \tilde{g})$, $\mathcal{E} = E$ is the Euler field (assumed to be invertible) and $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{g}$. Since $\tilde{\nabla}^2 E = 0$, by applying Proposition 8.2 with $\tilde{W} = 0$, we recover the well-known fact that the second structure connection of $(M, \circ, e, E, \tilde{g})$ is flat.

9. Duality and Legendre transformations

We now adopt the abstract point of view of external bundles to construct F-manifolds with compatible, torsion-free connections (see Section 9.1). This construction is particularly suitable in the setting of special families of connections. Then we consider the particular case when the external bundle is the tangent bundle of an F-manifold with a special family of connections and we define and study the notions of Legendre (or primitive) field and Legendre transformation of a special family (see Section 9.2). Finally, we show that our duality for F-manifolds with eventual identities and special families of connections commutes with the Legendre transformations so defined (see Section 9.3).

9.1. External bundles and F-manifolds

Let $V \to M$ be a vector bundle (sometimes called an external bundle), D a connection on V and $A \in \Omega^1(M, \mathrm{End}V)$ such that

$$(d^D A)_{X,Y} := D_X(A_Y) - D_Y(A_X) - A_{[X,Y]} = 0 \qquad (9.1)$$

and

$$A_X A_Y = A_Y A_X \qquad (9.2)$$

for any vector fields $X, Y \in \mathcal{X}(M)$. Let u be a section of V such that

$$A_Y(D_Z u) = A_Z(D_Y u), \quad \forall Y, Z \in \mathcal{X}(M). \tag{9.3}$$

Assume that the map

$$F : TM \to V, \quad F(X) := A_X(u) \tag{9.4}$$

is a bundle isomorphism. In this setting, the following proposition holds (see [15] for the flat case).

Proposition 9.1.

i) *The multiplication*

$$X \circ Y = F^{-1}\left(A_X A_Y u\right), \quad X, Y \in \mathcal{X}(M) \tag{9.5}$$

is commutative, associative, with unit field $F^{-1}(u)$.

ii) *The pull-back connection* F^*D *is torsion-free and compatible with* $\circ$. *In particular,* $(M, \circ, F^{-1}(u))$ *is an* F*-manifold.*

Proof. It is easy to check, using (9.2) and the bijectivity of F, that

$$A_{X\circ Y} = A_X A_Y, \quad \forall X, Y \in \mathcal{X}(M), \tag{9.6}$$

which readily implies, from (9.2) and the bijectivity of F again, the commutativity and associativity of $\circ$. From (9.4),

$$A_{F^{-1}(v)}(u) = v, \quad \forall v \in V \tag{9.7}$$

and, for any $X \in \mathcal{X}(M)$,

$$X \circ F^{-1}(u) = F^{-1}\left(A_X A_{F^{-1}(u)} u\right) = F^{-1}\left(A_X(u)\right) = X,$$

i.e. $F^{-1}(u)$ is the unit field for $\circ$. Claim *i)* follows. For claim *ii)*, recall that the pull-back connection F^*D is defined by

$$(F^*D)_X Y := F^{-1} D_X(F(Y)), \quad \forall X, Y \in \mathcal{X}(M).$$

To prove that F^*D is torsion-free, let $X, Y \in \mathcal{X}(M)$. Then

$$\begin{aligned}(F^*D)_X Y - (F^*D)_Y X &= F^{-1}\left(D_X(F(Y)) - D_Y(F(X))\right)\\ &= F^{-1}\left(D_X(A_Y u) - D_Y(A_X u)\right)\\ &= F^{-1}(A_{[X,Y]} u) = [X, Y],\end{aligned}$$

where we used (9.1) and (9.3). It remains to show that F^*D is compatible with $\circ$. For this, let $\tilde{\mathcal{C}}$ be the $\mathrm{End}(TM)$-valued 1-form defined by

$$\tilde{\mathcal{C}}_X(Y) = X \circ Y, \quad \forall X, Y \in \mathcal{X}(M).$$

Using (9.1) and (9.4), we get

$$(d^{F^*D}\tilde{\mathcal{C}})_{X,Y}(Z) = F^{-1}(d^D A)_{X,Y} F(Z) = 0. \tag{9.8}$$

This relation and the torsion-free property of F^*D imply that F^*D is compatible with $\circ$ (see relations (2.3) and (2.5) from Section 2.2). Relation (9.8) also implies that $(M, \circ, F^{-1}(u))$ is an F-manifold (from [8, Lemma 4.3] already mentioned in Section 2.2). Our claim follows. □

It is worth to make some comments on Proposition 9.1.

Remark 9.2. i) The torsion-free property of F^*D relies on the condition (9.3) satisfied by u. Note that by dropping (9.3) from the setting of Proposition 9.1, $(M, \circ, F^{-1}(u))$ remains an F-manifold. The reason is that relation (9.8), which implies that $(M, \circ, F^{-1}(u))$ is an F-manifold, does not use (9.3), but only (9.1).

ii) The pull-back connections $F^*(D + zA)$ (z-constant) are given by

$$F^*(D + zA)_X(Y) = (F^*D)_X(Y) + zX \circ Y, \quad \forall X, Y \in \mathcal{X}(M)$$

because

$$(F^*A)_X(Y) := F^{-1}\left(A_X F(Y)\right) = F^{-1}\left(A_X A_Y u\right) = X \circ Y.$$

Thus, $F^*(D + zA)$ is a pencil of compatible, torsion-free connections on the F-manifold $(M, \circ, F^{-1}(u))$. When D is flat, all connections $D + zA$ are flat and we recover [15, Theorem 4.3] (which states that a pencil of flat connections on an external bundle $V \to M$ together with a primitive section – the section u in our notations – induces an F-manifold structure on M together with a pencil of flat, torsion-free, compatible connections on this F-manifold).

9.2. Legendre transformations and special families of connections

We now apply the results from the previous section to the particular case when V is the tangent bundle of an F-manifold. We begin with the following definition.

Definition 9.3.

i) A vector field u on an F-manifold $(M, \circ, e, \tilde{\mathcal{S}})$ with a special family of connections $\tilde{\mathcal{S}}$ is called a Legendre (or primitive) field if it is invertible and for one (equivalently, any) $\tilde{\nabla} \in \tilde{\mathcal{S}}$,

$$\tilde{\nabla}_X(u) \circ Y = \tilde{\nabla}_Y(u) \circ X, \quad \forall X, Y \in \mathcal{X}(M). \tag{9.9}$$

ii) The family of connections

$$\{\mathcal{L}_u(\tilde{\nabla}) := u^{-1} \circ \tilde{\nabla} \circ u,\ \tilde{\nabla} \in \tilde{\mathcal{S}}\} \tag{9.10}$$

is called the Legendre transformation of $\tilde{\mathcal{S}}$ by u and is denoted by $\mathcal{L}_u(\tilde{\mathcal{S}})$.

Remark 9.4. The notions of Legendre field and Legendre transformation on F-manifolds with special families of connections are closely related to the corresponding notions from the theory of Frobenius manifolds [2]. The reason is that if u is a Legendre field on an F-manifold $(M, \circ, e, \tilde{\mathcal{S}})$ with a special family $\tilde{\mathcal{S}}$, then there is a unique connection $\tilde{\nabla}$ in $\tilde{\mathcal{S}}$ for which u is parallel (this can be easily checked). On the other hand, recall that a Legendre field on a Frobenius manifold $(M, \circ, e, E, \tilde{g})$ is, by definition, a parallel, invertible vector field ∂. It defines a new invariant flat metric by

$$g(X, Y) = \tilde{g}(\partial \circ X, \partial \circ Y),$$

see [15]. The passage from $\tilde{g}$ to g is usually called a Legendre-type transformation. The Levi-Civita connections of $\tilde{g}$ and g are related by

$$\nabla_X(Y) = \partial^{-1} \circ \tilde{\nabla}_X(\partial \circ Y), \quad \forall X, Y \in \mathcal{X}(M),$$

like in (9.10).

In the next proposition we describe some basic properties of Legendre transformations.

Proposition 9.5. *Let $(M, \circ, e, \tilde{\mathcal{S}}, u)$ be an F-manifold with a special family of connections $\tilde{\mathcal{S}}$ and a Legendre field u. The following facts hold*:

i) *The Legendre transformation $\mathcal{L}_u(\tilde{\mathcal{S}})$ of $\tilde{\mathcal{S}}$ by u is also special on $(M, \circ, e)$.*
ii) *If (any) connection $\tilde{\nabla} \in \tilde{\mathcal{S}}$ satisfies the condition (7.1) from Section 7, i.e.*

$$V \circ R^{\tilde{\nabla}}_{Z,Y} X + Y \circ R^{\tilde{\nabla}}_{V,Z} X + Z \circ R^{\tilde{\nabla}}_{Y,V} X = 0, \tag{9.11}$$

then the same is true for any connection from $\mathcal{L}_u(\tilde{\mathcal{S}})$.
iii) *If $\tilde{\mathcal{S}}$ contains a flat connection, then so does $\mathcal{L}_u(\tilde{\mathcal{S}})$.*

Proof. For the first claim, we prove that for any $\tilde{\nabla} \in \tilde{\mathcal{S}}$, $\mathcal{L}_u(\tilde{\nabla})$ is torsion-free and compatible with $\circ$. To do this, we use Proposition 9.1, with $V := TM$, $D := \tilde{\nabla}$, $A := \tilde{\mathcal{C}}$ the Higgs field, given by

$$\tilde{\mathcal{C}}_X(Y) := X \circ Y, \quad \forall X, Y \in \mathcal{X}(M), \tag{9.12}$$

and u the Legendre field. It is easy to check that conditions (9.1)-(9.3) are satisfied. The map (9.4) is given by

$$F : TM \to TM, \quad F(X) = X \circ u$$

and is an isomorphism because u is invertible. The induced multiplication (9.5) from Proposition 9.1 coincides with $\circ$, because

$$F^{-1}(A_X A_Y u) = F^{-1}(X \circ Y \circ u) = X \circ Y, \quad \forall X, Y \in \mathcal{X}(M).$$

From Proposition 9.1, the connection

$$F^*(\tilde{\nabla})_X(Y) = u^{-1} \circ \tilde{\nabla}_X(u \circ Y) = \mathcal{L}_u(\tilde{\nabla})_X(Y), \quad \forall X, Y \in \mathcal{X}(M)$$

is torsion-free and compatible with $\circ$, as required. It also belongs to the Legendre transformation $\mathcal{L}_u(\tilde{\mathcal{S}})$ of $\tilde{\mathcal{S}}$. It follows that $\mathcal{L}_u(\tilde{\mathcal{S}})$ is a special family on $(M, \circ, e)$. This proves our first claim.

For the second and third claims, we notice that the curvatures of $\tilde{\nabla}$ and $\mathcal{L}_u(\tilde{\nabla})$ are related by

$$R^{\mathcal{L}_u(\tilde{\nabla})}_{X,Y} Z = u^{-1} \circ R^{\tilde{\nabla}}_{X,Y}(u \circ Z). \tag{9.13}$$

Thus, if $\tilde{\nabla}$ is flat then so is $\mathcal{L}_u(\tilde{\nabla})$. Similarly, if the curvature of $\tilde{\nabla}$ satisfies the condition (9.11), then, from (9.13), also

$$V \circ R^{\mathcal{L}_u(\tilde{\nabla})}_{Z,Y} X + Y \circ R^{\mathcal{L}_u(\tilde{\nabla})}_{V,Z} X + Z \circ R^{\mathcal{L}_u(\tilde{\nabla})}_{Y,V} X = 0.$$

Our second and third claims follow. □

9.3. Legendre transformations and eventual identities

In this section we prove a compatibility property between Legendre transformations, eventual identities and our duality for F-manifolds with eventual identities and special families of connections. It is stated as follows:

Theorem 9.6. *Let $(M, \circ, e, \tilde{\mathcal{S}}, u)$ be an F-manifold with a special family of connections $\tilde{\mathcal{S}}$ and a Legendre field u. Let $\mathcal{E}$ be an eventual identity on $(M, \circ, e)$ and $(M, *, \mathcal{E}, e, \mathcal{S})$ the dual of $(M, \circ, e, \mathcal{E}, \tilde{\mathcal{S}})$, as in Theorem* 5.3. *Then $\mathcal{E} \circ u$ is a Legendre field on $(M, *, \mathcal{E}, \mathcal{S})$ and*

$$\mathcal{L}_{\mathcal{E}\circ u}(\mathcal{S}) = (\mathcal{D}_{\mathcal{E}} \circ \mathcal{L}_u)(\tilde{\mathcal{S}}), \tag{9.14}$$

where $(\mathcal{D}_{\mathcal{E}} \circ \mathcal{L}_u)(\tilde{\mathcal{S}}) = \mathcal{D}_{\mathcal{E}}(\mathcal{L}_u(\tilde{\mathcal{S}}))$ is the image of the special family $\mathcal{L}_u(\tilde{\mathcal{S}})$ on $(M, \circ, e)$ through the duality defined by $\mathcal{E}$.

Proof. Recall from Theorem 5.3 and Lemma 4.7 that $\mathcal{S} = \mathcal{D}_{\mathcal{E}}(\tilde{\mathcal{S}})$ is the special family on $(M, *, \mathcal{E})$ which contains the connection

$$\nabla_X(Y) = \mathcal{E} \circ \tilde{\nabla}_X(\mathcal{E}^{-1} \circ Y) + \mathcal{E} \circ \tilde{\nabla}_Y(\mathcal{E}^{-1}) \circ X, \quad \forall X, Y \in \mathcal{X}(M) \tag{9.15}$$

where $\tilde{\nabla}$ is any connection from $\tilde{\mathcal{S}}$. Since u is a Legendre field on $(M, \circ, e, \tilde{\mathcal{S}})$, relation (9.15) implies that

$$\nabla_X(\mathcal{E} \circ u) * Y = \nabla_Y(\mathcal{E} \circ u) * X, \quad X, Y \in \mathcal{X}(M),$$

i.e. $\mathcal{E} \circ u$ is a Legendre field on $(M, *, \mathcal{E}, \mathcal{S})$. The connection

$$\mathcal{L}_{\mathcal{E}\circ u}(\nabla)_X(Y) := (\mathcal{E} \circ u)^{-1,*} * \nabla_X\left((\mathcal{E} \circ u) * Y\right)$$

$$= u^{-1} \circ \mathcal{E} \circ \left(\tilde{\nabla}_X(\mathcal{E}^{-1} \circ u \circ Y) + \tilde{\nabla}_{u\circ Y}(\mathcal{E}^{-1}) \circ X\right)$$

belongs to the special family $\mathcal{L}_{\mathcal{E}\circ u}(\mathcal{S})$, where $(\mathcal{E} \circ u)^{-1,*} = u^{-1} \circ \mathcal{E}$ is the inverse of $\mathcal{E} \circ u$ with respect to the dual multiplication $*$. On the other hand, $\mathcal{L}_u(\tilde{\mathcal{S}})$ contains the connection $\mathcal{L}_u(\tilde{\nabla}) = u^{-1} \circ \tilde{\nabla} \circ u$ and thus, from Theorem 5.3 and Lemma 4.7, $(\mathcal{D}_{\mathcal{E}} \circ \mathcal{L}_u)(\tilde{\mathcal{S}})$ contains the connection

$$(\mathcal{D}_{\mathcal{E}} \circ \mathcal{L}_u)(\tilde{\nabla})_X(Y) = \mathcal{E} \circ \left(u^{-1} \circ \tilde{\nabla} \circ u\right)_X (\mathcal{E}^{-1} \circ Y)$$

$$+ \mathcal{E} \circ \left(u^{-1} \circ \tilde{\nabla} \circ u\right)_Y (\mathcal{E}^{-1}) \circ X$$

$$= \mathcal{E} \circ u^{-1} \circ \left(\tilde{\nabla}_X(u \circ \mathcal{E}^{-1} \circ Y) + \tilde{\nabla}_Y(u \circ \mathcal{E}^{-1}) \circ X\right).$$

In order to prove our claim we need to show that $\mathcal{L}_{\mathcal{E}\circ u}(\nabla)$ and $(\mathcal{D}_{\mathcal{E}} \circ \mathcal{L}_u)(\tilde{\nabla})$ belong to the same special family of connections on $(M, *, \mathcal{E})$, *i.e.* that there is a vector field U, which needs to be determined, such that

$$\tilde{\nabla}_Y(u \circ \mathcal{E}^{-1}) = \tilde{\nabla}_{u\circ Y}(\mathcal{E}^{-1}) + U \circ Y, \quad \forall Y \in \mathcal{X}(M). \tag{9.16}$$

In order to determine U, we note that

$$\tilde{\nabla}_Y(u \circ \mathcal{E}^{-1}) = \tilde{\nabla}_Y(u) \circ \mathcal{E}^{-1} + \tilde{\nabla}_{\mathcal{E}^{-1}}(\circ)(u, Y) + u \circ \tilde{\nabla}_Y(\mathcal{E}^{-1})$$

$$= \tilde{\nabla}_Y(u) \circ \mathcal{E}^{-1} + \tilde{\nabla}_{\mathcal{E}^{-1}}(u \circ Y) - \tilde{\nabla}_{\mathcal{E}^{-1}}(u) \circ Y - u \circ \tilde{\nabla}_{\mathcal{E}^{-1}}(Y)$$

$$+ u \circ \tilde{\nabla}_Y(\mathcal{E}^{-1})$$

$$= L_{\mathcal{E}^{-1}}(u \circ Y) + \tilde{\nabla}_{u\circ Y}(\mathcal{E}^{-1}) - u \circ L_{\mathcal{E}^{-1}}(Y)$$

$$= \left([\mathcal{E}^{-1}, u] + [e, \mathcal{E}^{-1}] \circ u\right) \circ Y + \tilde{\nabla}_{u\circ Y}(\mathcal{E}^{-1}),$$

where in the first equality we used the total symmetry of $\tilde{\nabla}(\circ)$; in the third equality we used

$$\tilde{\nabla}_Y(u) \circ \mathcal{E}^{-1} = \tilde{\nabla}_{\mathcal{E}^{-1}}(u) \circ Y, \quad \forall Y \in \mathcal{X}(M),$$

(because u is a Legendre field) and the torsion-free property of $\tilde{\nabla}$; in the fourth equality we used that $\mathcal{E}^{-1}$ is an eventual identity. We proved that (9.16) holds, with

$$U := [\mathcal{E}^{-1}, u] + [e, \mathcal{E}^{-1}] \circ u. \tag{9.17}$$

Our claim follows. □

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