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# The Extended Future Tube Conjecture for SO(1, $n$)

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**Abstract.** Let $C$ be the open upper light cone in $\mathbb{R}^{1+n}$ with respect to the Lorentz product. The connected linear Lorentz group $\mathrm{SO}_{\mathbb{R}}(1,n)^0$ acts on $C$ and therefore diagonally on the $N$-fold product $T^N$ where $T = \mathbb{R}^{1+n} + iC \subset \mathbb{C}^{1+n}$. We prove that the extended future tube $\mathrm{SO}_{\mathbb{C}}(1,n) \cdot T^N$ is a domain of holomorphy.

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For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ let $\mathbb{K}^{1+n}$ denote the $(1+n)$-dimensional Minkowski space, i.e., on $\mathbb{K}^{1+n}$ we have given the bilinear form

$$(x, y) \mapsto x \bullet y := x_0 y_0 - x_1 y_1 - \cdots - x_n y_n$$

where $x_j$ respectively $y_j$ are the components of $x$ respectively $y$ in $\mathbb{K}^{1+n}$. The group $\mathrm{O}_{\mathbb{K}}(1,n) = \{g \in \mathrm{Gl}_{\mathbb{K}}(1+n); gx \bullet gy = x \bullet y \text{ for all } x, y \in \mathbb{K}^{1+n}\}$ is called the linear Lorentz group. For $n \geq 2$ the group $\mathrm{O}_{\mathbb{R}}(1,n)$ has four connected components and $\mathrm{O}_{\mathbb{C}}(1,n)$ has two connected components. The connected component of the identity $\mathrm{O}_{\mathbb{K}}(1,n)^0$ of $\mathrm{O}_{\mathbb{K}}(1,n)$ will be called the connected linear Lorentz group. Note that $\mathrm{SO}_{\mathbb{R}}(1,n) = \{g \in \mathrm{O}_{\mathbb{R}}(1,n); \det(g) = 1\}$ has two connected components and $\mathrm{O}_{\mathbb{R}}(1,n)^0 = \mathrm{SO}_{\mathbb{R}}(1,n)^0$. In the complex case we have $\mathrm{SO}_{\mathbb{C}}(1,n) = \mathrm{O}_{\mathbb{C}}(1,n)^0$.

The forward cone $C$ is by definition the set $C := \{y \in \mathbb{R}^{1+n}; y \bullet y > 0 \text{ and } y_0 > 0\}$ and the future tube $T$ is the tube domain over $C$ in $\mathbb{C}^{1+n}$, i.e., $T = \mathbb{R}^{1+n} + iC \subset \mathbb{C}^{1+n}$. Note that $T^N = T \times \cdots \times T$ is the tube domain in the space of complex $(1+n) \times N$-matrices $\mathbb{C}^{(1+n)\times N}$ over $C^N = C \times \cdots \times C \subset \mathbb{R}^{(1+n)\times N}$. The group $\mathrm{SO}_{\mathbb{C}}(1,n)$ acts by matrix multiplication on $\mathbb{C}^{(1+n)\times N}$ and the subgroup $\mathrm{SO}_{\mathbb{R}}(1,n)^0$ stabilizes $T^N$. In this note we prove the

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Extended future tube conjecture:

$$\mathrm{SO}_{\mathbb{C}}(1,n)\cdot T^N = \bigcup_{g\ \in\ \mathrm{SO}_{\mathbb{C}}(1,n)} g\cdot T^N \text{ is a domain of holomorphy.}$$

This conjecture arise in the theory of quantized fields for about 50 years. We refer the interested reader to the literature ([HW], [J], [SV], [StW], [W]). There is a proof of this conjecture in the case where $n = 3$ ([He2]), [Z]). The proof there uses essentially that $T$ can be realized as the set $\{Z \in \mathbb{C}^{2\times 2};\ \frac{1}{2i}(Z - {}^t\bar{Z})$ is positive definite$\}$. Moreover the proof for $n = 3$ is unsatisfactory. It does not give much information about $\mathrm{SO}_{\mathbb{C}}(1,n)\cdot T^N$ except for holomorphic convexity.

Here we prove that more is true. Roughly speaking, we show that the basic Geometric Invariant Theory results known for compact groups (see [He1]) also holds for $X := T^N$ and the non compact group $\mathrm{SO}_{\mathbb{R}}(1,n)^0$. More precisely this means $\mathrm{SO}_{\mathbb{C}}(1,n)\cdot X = Z$ is a universal complexification of the $G$-space $X$, $G = \mathrm{SO}_{\mathbb{R}}(1,n)^0$, in the sense of [He1]. There exists complex analytic quotients $X//G$ and $Z/\!/G^{\mathbb{C}}$, $G^{\mathbb{C}} = \mathrm{SO}_{\mathbb{C}}(1,n)$, given by the algebra of invariant holomorphic functions and there is a $G$-invariant strictly plurisubharmonic function $\rho : X \to \mathbb{R}$, which is an exhaustion on $X/G$. Let

$$\mu : X \to \mathfrak{g}^*, \quad \mu(z)(\xi) = \frac{d}{dt}\Big|_{t=0}(t \to \rho(\exp it\xi\cdot z)),$$

be the corresponding moment map. Then the diagram

$$\begin{array}{ccccc} \mu^{-1}(0) & \hookrightarrow & X & \hookrightarrow & Z \\ \downarrow & & \downarrow \pi & & \downarrow \pi^{\mathbb{C}} \\ \mu^{-1}(0)/G & \equiv & X/\!/G & \equiv & Z/\!/G^{\mathbb{C}} \end{array}$$

where all maps are induced by inclusion is commutative, $X/\!/G, X, Z$ and $Z/\!/G^{\mathbb{C}}$ are Stein spaces and $\rho|\mu^{-1}(0)$ induces a strictly plurisubharmonic exhaustion on $\mu^{-1}(0)/G = X/\!/G = Z/\!/G^{\mathbb{C}}$. Moreover the same statement holds if we replace $X = T^N$ with a closed $G$-stable analytic subset $A$ of $X$.

## 1. – Geometric Invariant Theory of Stein spaces

Let $Z$ be a Stein space and $G$ a real Lie group acting as a group of holomorphic transformations on $Z$. A complex space $Z/\!/G$ is said to be an analytic Hilbert quotient of $Z$ by the given $G$-action if there is a $G$-invariant surjective holomorphic map $\pi : Z \to Z/\!/G$, such that for every open Stein subspace $Q \subset Z/\!/G$

i. its inverse image $\pi^{-1}(Q)$ is an open Stein subspace of $Z$ and
ii. $\pi^*\mathcal{O}_{Z/\!/G}(Q) = \mathcal{O}(\pi^{-1}(Q))^G$, where $\mathcal{O}(\pi^{-1}(Q))^G$ denotes the algebra of $G$-invariant holomorphic functions on $\pi^{-1}(Q)$ and $\pi^*$ is the pull back map.

Now let $G^c$ be a linearly reductive complex Lie group. A complex space $Z$ endowed with a holomorphic action of $G^c$ is called a holomorphic $G^c$-space.

THEOREM 1.1. *Let $Z$ be a holomorphic $G^c$-space, where $G^c$ is a linearly reductive complex Lie group.*
i. *If $Z$ is a Stein space, then the analytic Hilbert quotient $Z/\!/G^c$ exists and is a Stein space.*
ii. *If $Z/\!/G^c$ exists and is a Stein space, then $Z$ is a Stein space.*

PROOF. Part i. is proven in [He1] and part ii. in [HeMP]. □

REMARK 1.1.
i. If the analytic Hilbert quotient $\pi : Z \to Z/\!/G^c$ exists, then every fiber $\pi^{-1}(q)$ of $\pi$ contains a unique $G^c$-orbit $E_q$ of minimal dimension. Moreover, $E_q$ is closed and $\pi^{-1}(q) = \{z \in Z;\, E_q \subset \overline{G^c.z}\}$. Here $\overline{\phantom{x}}$ denotes the topological closure.
ii. Let X be a subset of $Z$, such that $G^c \cdot X := \bigcup_{g \in G^c} g \cdot X = Z$ and assume that $Z/\!/G^c$ exists. Then $G^c \cdot X$ is a Stein space if and only if $Z/\!/G^c = \pi(X)$ is a Stein space.
iii. Let $V^c$ be a finite dimensional complex vector space with a holomorphic linear action of $G^c$. Then the algebra $\mathbb{C}[V^c]^{G^c}$ of invariant polynomials is finitely generated (see e.g. [Kr]).

In particular, the inclusion $\mathbb{C}[V^c]^{G^c} \hookrightarrow \mathbb{C}[V^c]$ defines an affine variety $V^c/\!/G^c$ and an affine morphism $\pi^c : V^c \to V^c/\!/G^c$. If we regard $V^c/\!/G^c$ as a complex space, then $\pi^c : V^c \to V^c/\!/G^c$ gives the analytic Hilbert quotient of $V^c$ (see e.g. [He1]).

REMARK 1.2. For a non-connected linearly reductive complex group $G$ let $G^0$ denote the connected component of the identity and let $Z$ be a holomorphic $G$-space. The analytic Hilbert quotient $Z/\!/G$ exists if and only if the quotient $Z/\!/G^0$ exists. Moreover, the quotient map $\pi_G : Z \to Z/\!/G$ induces a map $\pi_{G/G^0} : Z/\!/G^0 \to Z/\!/G$ which is finite. In fact the diagram

$$\begin{array}{ccccc} & & Z & & \\ & \pi_{G^0} \swarrow & & \searrow \pi_G & \\ Z/\!/G^0 & & \underset{\pi_{G/G^0}}{\longrightarrow} & & Z/\!/G \end{array}$$

commutes and $\pi_{G/G^0}$ is the quotient map for the induced action of the finite group $G/G^0$ on $Z/\!/G^0$.

## 2. – The geometry of the Minkowski space

Let $\mathbb{K}$ denote either the field $\mathbb{R}$ or $\mathbb{C}$ and $(e_0, \ldots, e_n)$ the standard orthonormal basis for $\mathbb{K}^{1+n}$. The space $\mathbb{K}^{1+n}$ together with the quadratic form $\eta(z) = z_0^2 - z_1^2 - \cdots - z_n^2$, where $z_j$ are the components of $z$, is called the $(1+n)$-dimensional linear Minkowski space. Let $<,>_L$ denote the symmetric non-degenerated bilinear form which corresponds to $\eta$, i.e., $z \bullet w := < z, w >_L = {}^t z J w$ where ${}^t z$ denotes the transpose of $z$ and $J = (e_0, -e_1, \ldots ., -e_n)$ or equivalently $z \bullet w = < z, Jw >_E$ where $<,>_E$ denotes the standard Euclidean product on $\mathbb{R}^{1+n}$, respectively its $\mathbb{C}$-linear extension to $\mathbb{C}^{1+n}$.

Let $\mathrm{O}_{\mathbb{K}}(1,n)$ denote the subgroup of $\mathrm{Gl}_{\mathbb{K}}(1+n)$ which leave $\eta$ fixed, i.e., $\mathrm{O}_{\mathbb{K}}(1,n) = \{g \in \mathrm{Gl}_{\mathbb{K}}(1+n); gz \bullet gw = z \bullet w$ for all $z, w \in \mathbb{K}^{1+n}\}$. Note that $\mathrm{SO}_{\mathbb{K}}(1,n) = \{g \in \mathrm{O}_{\mathbb{K}}(1,n); \det g = 1\}$ is an open subgroup of $\mathrm{O}_{\mathbb{K}}(1,n)$. For $\mathbb{K} = \mathbb{C}$, $\mathrm{SO}_{\mathbb{C}}(1,n)$ is connected. But in the real case $\mathrm{SO}_{\mathbb{R}}(1,n)$ consists of two connected components $(n \geq 2)$. The connected component $\mathrm{SO}_{\mathbb{R}}(1,n)^0 = \mathrm{O}_{\mathbb{R}}(1,n)^0$ of the identity is called the connected linear Lorentz group. Note that $\mathrm{SO}_{\mathbb{R}}(1,n)^0$ is not an algebraic subgroup of $\mathrm{SO}_{\mathbb{R}}(1,n)$ but is Zariski dense in $\mathrm{SO}_{\mathbb{R}}(1,n)$. We have $\mathbb{K}[\eta] = \mathbb{K}[\mathbb{K}^{1+n}]^{\mathrm{SO}_{\mathbb{K}}(1,n)} = \mathbb{K}[\mathbb{K}^{1+n}]^{\mathrm{O}_{\mathbb{K}}(1,n)}$.

Now let $\mathbb{C}^{(1+n)\times N} = \mathbb{C}^{1+n} \times \cdots \times \mathbb{C}^{1+n}$ be the $N$-fold product of $\mathbb{C}^{1+n}$, i.e., the space of complex $(1+n) \times N$- matrices. The group $\mathrm{O}_{\mathbb{C}}(1,n)$ acts on $\mathbb{C}^{(1+n)\times N}$ by left multiplication. A classical result in Invariant Theory says that $\mathbb{C}[\mathbb{C}^{(1+n)\times N}]^{\mathrm{O}_{\mathbb{C}}(1,n)}$ is generated by the polynomials $p_{kj}(z_1, \ldots, z_N) = z_k \bullet z_j$ where $z = (z_1, \ldots, z_N) \in \mathbb{C}^{(1+n)\times N}$.

REMARK 2.1. The (algebraic) Hilbert quotient $\mathbb{C}^{(1+n)\times N}/\!/\mathrm{O}_{\mathbb{C}}(1,n)$ can be identified with the space $\mathrm{Sym}_N(\min\{1+n, N\})$ of symmetric $N \times N$-matrices of rank smaller or equal $\min\{1+n, N\}$.

With this identification the quotient map $\pi_{\mathbb{C}} : \mathbb{C}^{(1+n)\times N} \to \mathbb{C}^{(1+n)\times N}/\!/\mathrm{O}_{\mathbb{C}}(1,n)$ is given by $\pi_{\mathbb{C}}(Z) = {}^t Z J Z$ where ${}^t Z$ denotes the transpose of $Z$ and $J$ is as above. For the group $\mathrm{SO}_{\mathbb{C}}(1,n)$ the situation is slightly more complicated. If $N \geq 1+n$ additional invariants appear, but they are not relevant for our considerations, since the induced map $\mathbb{C}^{(1+n)\times N}/\!/\mathrm{SO}_{\mathbb{C}}(1,n) \to \mathbb{C}^{(1+n)\times N}/\!/\mathrm{O}_{\mathbb{C}}(1,n)$ is finite.

There is a well known characterization of closed $\mathrm{O}_{\mathbb{C}}(1,n)$-orbits in $\mathbb{C}^{(1+n)\times N}$. In order to formulate this we need more notations. Let $z = (z_1, \ldots, z_N) \in \mathbb{C}^{(1+n)\times N}$ and $L(z) := \mathbb{C}z_1 + \cdots + \mathbb{C}z_N$ be the subspace of $\mathbb{C}^{1+n}$ spanned by $z_1, \ldots, z_N$. The Lorentz product $<,>_L$ restricted to $L(z)$ is in general degenerated. Thus let $L(z)^0 = \{w \in L(z); < w, v >_L = 0$ for all $v \in L(z)\}$. It follows that $\dim L(z)/L(z)^0 = \mathrm{rank}({}^t z J z) = \mathrm{rank}\ \pi_{\mathbb{C}}(z)$. Elementary consideration show the following.

LEMMA 2.1. *The orbit* $\mathrm{O}_{\mathbb{C}}(1,n) \cdot z$ *through* $z \in \mathbb{C}^{(1+n)\times N}$ *is closed if and only if the orbit* $\mathrm{SO}_{\mathbb{C}}(1,n) \cdot z$ *is closed and this is the case if and only if* $L(z)^0 = \{0\}$, *i.e.,* $\dim L(z) = \mathrm{rank}\ \pi_{\mathbb{C}}(z)$.

The light cone $N := \{y \in \mathbb{R}^{1+n}; \eta(y) = 0\}$ is of codimension one and its complement $\mathbb{R}^{1+n} \backslash N$ consists of three connected components (here of course we assume $n \geq 2$). By the forward cone $C$ we mean the connected component which contains $e_0$. It is easy to see that $C = \{y \in \mathbb{R}^{1+n}; y \bullet e_0 > 0 \text{ and } \eta(y) > 0\}$ $= \{y \in \mathbb{R}^{1+n}; y \bullet x > 0 \text{ for all } x \in N^+\}$ where $N^+ = \{x \in N; x \bullet e_0 > 0\}$. In particular, $C$ is an open convex cone in $\mathbb{R}^{1+n}$. Since $J$ has only one positive Eigenvalue, the following version of the Cauchy-Schwarz inequality holds.

LEMMA 2.2. *If $\eta(y) > 0$, then $\tilde{x} \bullet y \leq 0$ for $\tilde{x} := x - \frac{x \bullet y}{\eta(y)^2} y$ and all $x \in \mathbb{R}^{1+n}$. In particular*

$$\eta(x) \cdot \eta(y) \leq (x \bullet y)^2$$

*and equality holds if and only if $x$ and $y$ are linearly dependent.*

The elementary Lemma has several consequences which are used later on. For example,

- if $y_1, y_2 \in C^{\pm} := C \cup (-C) = \{y \in \mathbb{R}^{1+n}; \eta(y) > 0\}$, then $y_1 \bullet y_2 \neq 0$. Moreover,
- if $y_1, y_2 \in N = \{y \in \mathbb{R}^{1+n}; \eta(y) = 0\}$, and $y_1 \bullet y_2 = 0$, then $y_1$ and $y_2$ are linearly dependent.

The tube domain $T = \mathbb{R}^{1+n} + iC \subset \mathbb{C}^{1+n}$ over $C$ is called the future tube. Note that $\mathrm{SO}_{\mathbb{R}}(1,n)^0$ acts on $T$ by $g \cdot (x + iy) = gx + igy$ and therefore on the $N$-fold product $T^N = T \times \cdots \times T \subset \mathbb{C}^{(1+n)\times N}$ by matrix multiplication.

REMARK 2.2. It is easy to show that the $\mathrm{SO}_{\mathbb{R}}(1,n)^0$-action on $C$ and consequently also on $T^N$ is proper. In particular $T^N/\mathrm{SO}_{\mathbb{R}}(1,n)^0$ is a Hausdorff space.

The complexified group $\mathrm{SO}_{\mathbb{C}}(1,n)$ does not stabilize $T^N$. The domain

$$\mathrm{SO}_{\mathbb{C}}(1,n) \cdot T^N = \bigcup_{g \in \mathrm{SO}_{\mathbb{C}}(1,n)} g \cdot T^N$$

is called the extended future tube.

## 3. – Orbit connectedness of the future tube

Let $G$ be a Lie group acting on $Z$. A subset $X \subset Z$ is called orbit connected with respect to the $G$-action on $Z$ if $\Sigma(z) = \{g \in G; g \cdot z \in X\}$ is connected for all $z \in X$.

In this section we prove the following

THEOREM 3.1. *The $N$-fold product $T^N$ of the future tube is orbit connected with respect to the* $\mathrm{SO}_{\mathbb{C}}(1,n)$*-action on* $\mathbb{C}^{(1+n)\times N}$.

We first reduce the proof of this Theorem for the $\mathrm{SO}_{\mathbb{C}}(1,n)$-action to the proof of the related statement about the Cartan subgroups of $\mathrm{SO}_{\mathbb{C}}(1,n)$. For this we use the results of Bremigan in [B]. For the convenience of the reader we briefly recall those parts, which are relevant for the proof of Theorem 3.1.

Starting with a simply connected complex semisimple Lie group $G^{\mathbb{C}}$ with a given real form $G$ defined by an anti-holomorphic group involution, $g \mapsto \bar{g}$, there is a subset $S$ of $G^{\mathbb{C}}$ such that $GSG$ contains an open $G \times G$-invariant dense subset of $G^{\mathbb{C}}$. The set $S$ is given as follows.

Let $\mathrm{Car}(G^{\mathbb{C}}) = \{H_1, \ldots, H_\ell\}$ be a complete set of representatives of the Cartan subgroups of $G^{\mathbb{C}}$, which are defined over $\mathbb{R}$. Associated to each $H \in \mathrm{Car}(G^{\mathbb{C}})$ are the Weyl group $\mathcal{W}(H) := N_{G^{\mathbb{C}}}(H)/H$, the real Weyl group $\mathcal{W}_{\mathbb{R}}(H) := \{gH \in \mathcal{W}(H);\, \bar{g}H = gH\}$ and the totally real Weyl group $\mathcal{W}_{\mathbb{R}!}(H) := \{gH \in \mathcal{W}_{\mathbb{R}}(H);\, \bar{g} = g\}$. Here $N_{G^{\mathbb{C}}}(H)$ denotes the normalizer of $H$ in $G^{\mathbb{C}}$.

For $H \in \mathrm{Car}(G^{\mathbb{C}})$ let $R(H)$ be a complete set of representatives of the double coset space $\mathcal{W}_{\mathbb{R}!}(H)\backslash\mathcal{W}_{\mathbb{R}}(H)/\mathcal{W}_{\mathbb{R}!}(H)$ chosen in such a way that $\bar{\epsilon} = \epsilon^{-1}$ holds for all $\epsilon \in R(H)$. Then $S := \cup H\epsilon$ has the claimed properties.

Although $\mathrm{SO}_{\mathbb{C}}(1,n)$ is not simply connected, the results above remain true for $G := \mathrm{SO}_{\mathbb{R}}(1,n)^0$ and $G^{\mathbb{C}} := \mathrm{SO}_{\mathbb{C}}(1,n)$, as one can see by going over to the universal covering.

REMARK 3.1. Using the classification of the $\mathrm{SO}_{\mathbb{R}}(1,n)^0 \times \mathrm{SO}_{\mathbb{R}}(1,n)^0$-orbits in $\mathrm{SO}_{\mathbb{C}}(1,n)$ as presented in [J], the same result can be obtained for $G^{\mathbb{C}} = \mathrm{SO}_{\mathbb{C}}(1,n)$.

Since $T^N$ is $\mathrm{SO}_{\mathbb{R}}(1,n)^0$-stable, $\mathrm{SO}_{\mathbb{R}}(1,n)^0$ is connected and $\mathrm{SO}_{\mathbb{R}}(1,n)^0 \cdot S \cdot \mathrm{SO}_{\mathbb{R}}(1,n)^0$ is dense in $\mathrm{SO}_{\mathbb{C}}(1,n)$, Theorem 3.1 follows from

PROPOSITION 3.1. *The set* $\Sigma_S(w) := \{g \in S;\, g \cdot w \in T^N\}$ *is connected for all* $w \in T^N$.

In the case $n = 2m-1$ we may choose $\mathrm{Car}(\mathrm{SO}_{\mathbb{C}}(1,n)) = \{H_0\}$ where

$$H_0 = \left\{ \begin{pmatrix} \sigma & 0 & \cdots & 0 \\ 0 & \tau_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \tau_{m-1} \end{pmatrix} ; \sigma \in \mathrm{SO}_{\mathbb{C}}(1,1),\, \tau_j \in \mathrm{SO}_{\mathbb{C}}(2) \right\} \text{ and } R(H_0) = \{\mathrm{Id}\}.$$

In the even case $n = 2m$ we make the choice $\mathrm{Car}(\mathrm{SO}_{\mathbb{C}}(1,n)) = \{H_1, H_2\}$ where

$$H_1 = \left\{ \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} ; h \in H_0 \right\},\ H_2 = \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \tau_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \tau_m \end{pmatrix} ; \tau_j \in \mathrm{SO}_{\mathbb{C}}(2) \right\},$$

$$R(H_1) = \{\mathrm{Id}\} \text{ and } R(H_2) = \{\mathrm{Id}, \epsilon\} \text{ with } \epsilon = \begin{pmatrix} -1 & & & 0 \\ & 0 & 1 & \\ & 1 & 0 & \\ 0 & & & \mathrm{Id}_{2m-3} \end{pmatrix}.$$

Observe that in the case $H_2$, where $\epsilon$ is present, $S$ is not connected. But the "$\epsilon$-part" of $S$ is not relevant, since any $h \in H_2$ does not change the sign of the first component of the imaginary part of $z_j \in T$ and therefore $\Sigma_{H_2\epsilon}(z)$ is empty for all $z \in T^N$. Thus it is sufficient to prove the following

PROPOSITION 3.2. *For every possible $H \in \{H_0, H_1, H_2\}$ and every $w \in T^N$ the set $\Sigma_H(w) = \{h \in H; h \cdot w \in T^N\}$ is connected.*

PROOF. We will carry out the proof in the case where $n = 2m - 1$ and $H = H_0$. The proof in the other cases is analogous. Note that $H$ splits into its real and imaginary part, i.e., $H = H_{\mathbb{R}} \cdot H_I \cong H_{\mathbb{R}} \times H_I$ where $H_{\mathbb{R}}$ denotes the connected component of the identity of $\mathrm{SO}_{\mathbb{R}}(1,n)^0 \cap H = \{h \in H; \bar{h} = h\}$ and $H_I = \exp i\mathfrak{h}_{\mathbb{R}}$. Thus the $2 \times 2$ blocks appearing for $h \in H_I$ are given by

$$\sigma = \begin{pmatrix} a & ib \\ ib & a \end{pmatrix} \qquad \text{where } a^2 + b^2 = 1 \qquad \text{and}$$

$$\tau_j = \begin{pmatrix} c_j & -id_j \\ id_j & c_j \end{pmatrix} \text{ where } c_j^2 - d_j^2 = 1, c_j > 0.$$

Let $S^1 := \{(x, y) \in \mathbb{R}^2;\ x^2 + y^2 = 1\}$, $\mathcal{H} := \{(x, y) \in \mathbb{R}^2;\ x^2 - y^2 = 1 \text{ and } x > 0\}$, identify $H_I$ with $S^1 \times \mathcal{H} \times \cdots \times \mathcal{H} \subset \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 = \mathbb{R}^{2m}$ and let

$$\tilde{\psi} : \mathbb{R}^{2m} \to \mathbb{R}^{(1+n)\times(1+n)},\ \tilde{\psi}(a, b, c_1, d_1, ..., c_{m-1}, d_{m-1}) = \begin{pmatrix} \sigma & 0 & \cdots & 0 \\ 0 & \tau_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \tau_{m-1} \end{pmatrix}$$

where $\sigma = \begin{pmatrix} a & ib \\ ib & a \end{pmatrix}$ and $\tau_j = \begin{pmatrix} c_j & -id_j \\ id_j & c_j \end{pmatrix}$. The restriction $\psi$ of $\tilde{\psi}$ to $S^1 \times \mathcal{H} \times \cdots \times \mathcal{H}$ is a diffeomorphism onto its image $H_I$.

For every $w_k \in T$, $k = 1, \dots, N$ we get the linear map $\tilde{\varphi}_k : \mathbb{R}^{2m} \to \mathbb{R}^{1+n}$, $p \mapsto \mathrm{Im}(\tilde{\psi}(p) \cdot w_k)$. Note that

- If $p = (p_1, \dots, p_m) \in \tilde{\varphi}_k^{-1}(C)$, then $(p_1, \dots, rp_j, \dots, p_m) \in \tilde{\varphi}_k^{-1}(C)$ for all $0 < r \le 1$ and $j = 2, \dots, m$.
- If $p = (p_1, \dots, p_m)$, $p_j \in \tilde{\varphi}_k^{-1}(C)$, then $(s \cdot p_1, p_2, \dots, p_m) \in \tilde{\varphi}_k^{-1}(C)$ for all $s > 1$.

where $p_1 = (a, b)$, $p_j = (c_j, d_j) \in \mathbb{R}^2$, $j = 2, \dots, m$.

It remains to show that $\Sigma_{H_I}(w)$ is connected for all $w \in T^N$.

Let $e := ((1, 0), (1, 0), \dots., (1, 0)) = \psi^{-1}(\mathrm{Id}) \in \psi^{-1}(\Sigma_{H_I}(w))$ and $p = (p_1, \dots, p_m) := \psi^{-1}(h) \in \psi^{-1}(\Sigma_{H_I}(w))$. From the convexity of $C$ and the linearity of $\tilde{\varphi}_k$ it follows that $q(t) = (q_1(t), \dots, q_m(t)) = e + t(p - e)$ is contained in $\bigcap_{k=1}^N \varphi_k^{-1}(C)$ for $t \in [0, 1]$. Thus

$$\tilde{\gamma}_p(t) := \left( \frac{q_1(t)}{\|q_1(t)\|_E}, \frac{q_2(t)}{\sqrt{\eta(q_2(t))}}, \dots., \frac{q_m(t)}{\sqrt{\eta(q_m(t))}} \right) \in \psi^{-1}(\Sigma_H(w))$$

for $t \in [0, 1]$. Here $\| \cdot \|_E$ denotes the standard Euclidean norm. Thus $\gamma_h(t) := \psi(\tilde{\gamma}_p(t))$ gives a curve which connects Id with $h$. □

Since $\mathrm{SO}_{\mathbb{R}}(1,n)^0$ is a real form of $\mathrm{SO}_{\mathbb{C}}(1,n)$, orbit connectness implies the following (see [He1])

COROLLARY 3.1. *Let $Y$ be a complex space with a holomorphic $\mathrm{SO}_{\mathbb{C}}(1,n)$-action. Then every holomorphic $\mathrm{SO}_{\mathbb{R}}(1,n)^0$-equivariant map $\varphi : T^N \to Y$ extends to a holomorphic $\mathrm{SO}_{\mathbb{C}}(1,n)$-equivariant map $\Phi : \mathrm{SO}_{\mathbb{C}}(1,n) \cdot T^N \to Y$.*

In the terminology of [He1] Corollary 3.1 means that $\mathrm{SO}_{\mathbb{C}}(1,n) \cdot T^N$ is the universal complexification of the $\mathrm{SO}_{\mathbb{R}}(1,n)^0$-space $T^N$.

## 4. – The strictly plurisubharmonic exhaustion of the tube

Let $X, Q, P$ be topological spaces, $q : X \to Q$ and $p : X \to P$ continuous maps. A function $f : X \to \mathbb{R}$ is said to be an exhaustion of $X$ mod $p$ along $q$ if for every compact subset $K$ of $Q$ and $r \in \mathbb{R}$ the set $p(q^{-1}(K) \cap f^{-1}((-\infty, r]))$ is compact.

The characteristic function of the forward cone $C$ is up to a constant given by the function $\tilde{\rho} : C \to \mathbb{R}$, $\tilde{\rho}(y) = \eta(y)^{-\frac{n+1}{2}}$. It follows from the construction of the characteristic function, that $\log \tilde{\rho}$ is a $\mathrm{SO}_{\mathbb{R}}(1,n)^0$-invariant strictly convex function on $C$ (see [FK] for details). In particular

$$\rho : T^N \to \mathbb{R}, \quad (x_1 + iy_1, \dots, x_N + iy_N) \mapsto \frac{1}{\eta(y_1)} + \cdots + \frac{1}{\eta(y_N)}$$

is a $\mathrm{SO}_{\mathbb{R}}(1,n)^0$-invariant strictly plurisubharmonic function on $T^N$. Of course this may also be checked by direct computation.

Let $\pi_{\mathbb{C}} : \mathbb{C}^{(1+n)\times N} \to \mathbb{C}^{(1+n)\times N} /\!/ \mathrm{SO}_{\mathbb{C}}(1,n)$ be the analytic Hilbert quotient and $\pi_{\mathbb{R}} : T^N \to T^N / \mathrm{SO}_{\mathbb{R}}(1,n)^0$ the quotient by the $\mathrm{SO}_{\mathbb{R}}(1,n)^0$-action. In the following we always write $z = x + iy$, i.e., $z_j = x_j + iy_j$ where $x_j$ denote the real and $y_j$ the imaginary part of $z_j$. For example $z_j \bullet z_k = x_j \bullet x_k - y_j \bullet y_k + i(x_j \bullet y_k + x_k \bullet y_j)$.

The main result of this section is the following

THEOREM 4.1. *The function $\rho : T^N \to \mathbb{R}$, is an exhaustion of $T^N$ mod $\pi_{\mathbb{R}}$ along $\pi_{\mathbb{C}}$.*

We do the case of one copy first.

LEMMA 4.1. *Let $D_1 \subset T$ and assume that $\pi_{\mathbb{C}}(D_1) \subset \mathbb{C}$ is bounded. Then $\{(x \bullet y, \eta(x), \eta(y)) \in \mathbb{R}^3; z = x + iy \in D_1\}$ is bounded.*

PROOF. The condition on $D_1$ means, that there is a $M \geq 0$ such that

$$|\eta(x) - \eta(y)| \leq M \qquad \text{and} \qquad |x \bullet y| \leq M$$

for all $z = x + iy \in D_1$. Since $\eta(x)\eta(y) \leq (x \bullet y)^2$ and $\eta(y) \geq 0$, this implies that $\{(x \bullet y, \eta(x), \eta(y)) \in \mathbb{R}^3; z \in D_1\}$ is bounded. □

LEMMA 4.2. *Let $D_2 \subset T \times T$ be such that $\pi_{\mathbb{C}}(D_2)$ is bounded. Then $\{(\eta(x_1), \eta(y_1), \eta(x_2), \eta(y_2), x_1 \bullet x_2, y_1 \bullet y_2) \in \mathbb{R}^6; (z_1, z_2) \in D_2\}$ is bounded.*

PROOF. Lemma 4.1 implies that there is a $M_1 \geq 0$ such that $|\eta(x_j)| \leq M_1, |\eta(y_j)| \leq M_1$ and $|x_j \bullet y_j| \leq M_1$, $j = 1, 2$, for all $(z_1, z_2) \in D_2$. Now $\eta(z_1+z_2) = \eta(z_1)+\eta(z_2)+2\cdot z_1 \bullet z_2$ shows that $\{\eta(z_1+z_2) \in \mathbb{R}; (z_1, z_2) \in D_2\}$ is bounded. But $z_1 + z_2 \in T$, thus Lemma 4.1 implies $|\eta(x_1 + x_2)| \leq M_2$ and $|\eta(y_1 + y_2)| \leq M_2$ for some $M_2 \geq 0$ and all $(z_1, z_2) \in D_2$. This gives

$$|x_1 \bullet x_2| \leq \frac{3}{2} \max\{M_1, M_2\} \qquad \text{and} \qquad |y_1 \bullet y_2| \leq \frac{3}{2} \max\{M_1, M_2\}. \qquad \square$$

REMARK 4.1. Based on the following we only need, that the set $\{(\eta(y_1), \eta(y_2), y_1 \bullet y_2) \in \mathbb{R}^3; (z_1, z_2) \in D_2\}$ is bounded. We apply this to points $y_j + iy_1$ where $\pi_{\mathbb{C}}(y_j + iy_1) = \eta(y_j) - \eta(y_1) + 2iy_j \bullet y_1$.

REMARK 4.2. For every subset $X$ of $T$, we have

$$X \subset \mathrm{SO}_{\mathbb{R}}(1, n)^0 \cdot (X \cap (\mathbb{R}^{1+n} + i(\mathbb{R}^{>0} \cdot e_0))),$$

where $\mathbb{R}^{>0} \cdot e_0 = \{te_0; t > 0\} \subset \mathbb{R}^{1+n}$.

LEMMA 4.3. *For every compact sets $B \subset C$ and $K \subset \mathbb{C}$ the set*

$$M(B, K) := \{x \in \mathbb{R}^{1+n}; \pi_{\mathbb{C}}(x + iy) \in K \text{ for some } y \in B\}$$

*is compact.*

PROOF. Since $B$ and $K$ are compact, $M(B, K)$ is closed. We have to show that it is bounded. First note that $B_1 \subset B_2$ implies $M(B_1, K) \subset M(B_2, K)$. Using the properness of the $\mathrm{SO}_{\mathbb{R}}(1, n)^0$-action on $C$, we see, that there is an interval $I = \{t \cdot e_0; a \leq t \leq b\}$, $a > 0$ in $\mathbb{R} \cdot e_0$ and a compact subset $N$ in $\mathrm{SO}_{\mathbb{R}}(1, n)^0$, such that $N \cdot I := \bigcup_{g\in N} g \cdot I \supset B$. Thus $M(B, K) \subset M(N \cdot I, K) = N \cdot M(I, K) := \bigcup_{g\in N} g \cdot M(I, K)$.

It remains to show that $M(I, K)$ is bounded. For $x \in M(I, K)$, $x = \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}$, there exists a $M_1 \geq 0$ such that $|x \bullet (y_0 \cdot e_0)| = |x_0 \cdot y_0| \leq M_1$ for all $y_0 \cdot e_0 \in I$. Since $a \leq y_0 \leq b$ and $a > 0$, this implies $|x_0^2| \leq \frac{{M_1}^2}{|y_0^2|} \leq \frac{{M_1}^2}{a^2}$. There also exists a $M_2 \geq 0$ such that $|\eta(x)| = |x_0^2 - x_1^2 - \cdots x_n^2| \leq M_2$, so we get $x_1^2 + \cdots x_n^2 \leq \frac{M_1^2}{a^2} + M_2$. $\square$

COROLLARY 4.1. *For every $r > 0$ the set $M(B, K) \cap \{y \in \mathbb{R}^{1+n}; r \leq \eta(y)\}$ is compact.*

PROOF OF THEOREM 4.1. Using Remark 4.2 it is sufficient to prove that the set

$$S := (\pi_{\mathbb{C}}^{-1}(K) \cap \{\rho \leq r\}) \cap ((\mathbb{R}^{1+n} + i(\mathbb{R}^{>0} \cdot e_0)) \times T^{N-1})$$

is compact. For $z = (z_1, \ldots, z_N) \in S$ let $z_j = x_j + iy_j$, where $x_j$ denotes the real part and $y_j$ the imaginary part of $z_j$. By the definition of $S$ we have $y_1 = y_{10} \bullet e_0$ where $y_{10} = y_1 \cdot e_0$. Moreover, we get $\frac{1}{r} \leq \eta(y_1) = (y_{10})^2 \leq M$. Therefore the set $\{y_1 \in \mathbb{R}^{1+n}; (z_1, \ldots, z_N) \in S\} = \{t \cdot e_0; t^2 \in [\frac{1}{r}, M], t > 0\}$ is compact.

By Remark 4.1 we get that the sets $\{(\eta(y_1), \eta(y_j), y_1 \bullet y_j) \in \mathbb{R}^3; (z_1, ..., z_N) \in S\}$ are bounded for $j = 2, \ldots, N$. Therefore we get the boundedness of $\{\pi_{\mathbb{C}}(y_j + iy_1) \in \mathbb{C}; (z_1, \ldots, z_N) \in S\}$. Thus the $y_j$, $j = 2, \ldots, N$, with $(z_1, \ldots, z_N) \in S$ are lying in the sets $M(I, B_j) \cap \{y \in \mathbb{R}^{1+n}; r \leq \eta(y)\}$, where $I := \{t \cdot e_0; t^2 \in [\frac{1}{r}, M], t > 0\}$ and $B_j$ are compact subsets of $\mathbb{C}$, containing $\{\pi_{\mathbb{C}}(y_j + iy_1) \in \mathbb{C}; (z_1, \ldots, z_N) \in S\}$. By Corollary 4.1 these sets are compact, which implies that the set $\{(y_1, \ldots, y_N) \in \mathbb{R}^{(1+n)\times N}; (z_1, \ldots, z_N) \in S\}$ is compact. Hence using Lemma 4.3 it follows that $\{(x_1, \ldots, x_N) \in \mathbb{R}^{(1+n)\times N}; (z_1, \ldots, z_N) \in S\}$ is bounded. Thus $S$ is bounded and therefore compact. □

## 5. – Saturatedness of the extended future tube

We call $A \subset X$ saturated with respect to a map $p : X \to Y$ if $A$ is the inverse image of a subset of $Y$.

Let $\pi_{\mathbb{C}} : \mathbb{C}^{(1+n)\times N} \to \mathbb{C}^{(1+n)\times N}//\mathrm{SO}_{\mathbb{C}}(1, n)$ be the analytic Hilbert quotient, which is given by the algebra of $\mathrm{SO}_{\mathbb{C}}(1, n)$-invariant polynomials functions on $\mathbb{C}^{(1+n)\times N}$ (see Section 1) and let $U_r$ denote the set $\{z \in T^N; \rho(z) < r\}$ for some $r \in \mathbb{R} \cup \{+\infty\}$, where $\rho$ is the strictly plurisubharmonic exhaustion function, which we defined in Section 4.

THEOREM 5.1. *The set* $\mathrm{SO}_{\mathbb{C}}(1, n) \cdot U_r = \mathrm{SO}_{\mathbb{C}}(1, n) \cdot \{z \in T^N; \rho(z) < r\}$ *is saturated with respect to* $\pi_{\mathbb{C}}$.

It is well known, that each fiber of $\pi_{\mathbb{C}}$ contains exactly one closed orbit of $\mathrm{SO}_{\mathbb{C}}(1, n)$ (see Section 1). Moreover, every orbit contains a closed orbit in its closure. Therefore it is sufficient to prove

PROPOSITION 5.1. *If* $z \in U_r$ *and* $\mathrm{SO}_{\mathbb{C}}(1, n) \cdot u$ *is the closed orbit in* $\overline{\mathrm{SO}_{\mathbb{C}}(1, n) \cdot z}$, *then* $\mathrm{SO}_{\mathbb{C}}(1, n) \cdot u \cap U_r \neq \emptyset$.

The idea of proof is to construct a one-parameter group $\gamma$ of $\mathrm{SO}_{\mathbb{C}}(1, n)$, such that $\gamma(t)z \in U_r$ for $|t| \leq 1$ and $\lim_{t\to 0} \gamma(t)z \in \mathrm{SO}_{\mathbb{C}}(1, n) \cdot u$.

In the following, let $z = (z_1, \ldots, z_N) \in U_r$ and denote by $L(z) = \mathbb{C}z_1 + \cdots + \mathbb{C}z_N$ the $\mathbb{C}$-linear subspace of $\mathbb{C}^{1+n}$ spanned by $z_1, \ldots, z_N$. The subspace of isotropic vectors in $L(z)$ with respect to the Lorentz product is denoted by $L(z)^0$, i.e., $L(z)^0 = \{w \in L(z); w \bullet v = 0 \text{ for all } v \in L(z)\}$. Let $\overline{L(z)^0}$ be its conjugate, i.e., $\overline{L(z)^0} = \{\bar{v}; v \in L(z)^0\}$.

LEMMA 5.1. *For all $\omega \neq 0$, $\omega \in L(z)^0$ we have $\eta(\mathrm{Im}(\omega)) < 0$.*

PROOF. Let $\omega = \omega_1 + i\omega_2$ with $\omega_1 = \mathrm{Re}(\omega), \omega_2 = \mathrm{Im}(\omega)$. Assume that $\eta(\mathrm{Im}(\omega)) = \eta(\omega_2) \geq 0$. Since $\omega \in L(z)^0$, we have $0 = \eta(\omega) = \eta(\omega_1) - \eta(\omega_2) + 2i\omega_1 \bullet \omega_2$.

If $\eta(\omega_2) > 0$, i.e., $\omega_2 \in C$ or $\omega_2 \in -C$, then $\omega_1 \bullet \omega_2 = 0$ contradicts $\eta(\omega_1) = \eta(\omega_2) > 0$. Thus assume $\eta(\omega_1) = \eta(\omega_2) = 0$ and $\omega_1 \bullet \omega_2 = 0$. Hence $\omega_1$ and $\omega_2$ are $\mathbb{R}$-linearly dependent and therefore there is a $\lambda \in \mathbb{C}, \omega_3 \in \mathbb{R}^{1+n}$ such that $\omega = \lambda\omega_3$ and $\omega_3 \bullet e_0 \geq 0$. We have $\eta(\omega_3) = 0$ and, since $\omega_3 \in L(z)^0, e_0 \bullet \omega_3 \geq 0$ and $z_1 \in T$, we also have $0 = \omega_3 \bullet \mathrm{Im}(z_1)$. This implies by the definition of $C$ that $\omega_3 = 0$. □

COROLLARY 5.1. *For $\omega \in L(z)^0, \omega \neq 0$, we have $\omega \bullet \bar{\omega} < 0$. In particular, $L(z)^0 \cap \overline{L(z)^0} = \{0\}$ and the complex Lorentz product is non-degenerate on $L(z)^0 \oplus \overline{L(z)^0}$.*

COROLLARY 5.2. *Let $W := (L(z) \oplus \overline{L(z)})^{\perp} := \{v \in \mathbb{C}^{1+n}; v \bullet u = 0$ for all $u \in L(z)^0 \oplus \overline{L(z)^0}\}$. Then*

$$L(z) = L(z)^0 \oplus (L(z) \cap W).$$

PROOF OF PROPOSITION 5.1. Let $z \in U_r$. We use the notation of Corollary 5.2. Define

$$\gamma : \mathbb{C}^* \to \mathrm{SO}_{\mathbb{C}}(1, n) \quad \text{by} \quad \gamma(t)v = \begin{cases} tv & \text{for } v \in L(z)^0 \\ t^{-1}v & \text{for } v \in \overline{L(z)^0} \\ v & \text{for } v \in W \end{cases}.$$

Every component $z_j$ of $z$ is of the form $z_j = u_j + \omega_j$ where $u_j \in W$ and $\omega_j \in L(z)^0$ are uniquely determined by $z_j$. Recall that $W$ is the set $\{v \in \mathbb{C}^{1+n}; v \bullet u = 0 \text{ for all } u \in L(z)^0 \oplus \overline{L(z)^0}\}$. Since $\lim_{t \to 0} \gamma(t)z_j = u_j$ and $L(u)^0 = \{0\}$ for $u = (u_1, \ldots, u_N)$, $u$ lies in the unique closed orbit in $\overline{\mathrm{SO}_{\mathbb{C}}(1, n).z}$ (see Lemma 2.1). It remains to show that $u \in U_r$. For every $t \in \mathbb{C}$ we have

$$\eta(\mathrm{Im}(u_j + t\omega_j)) = \eta(\mathrm{Im}(u_j)) + |t|^2\eta(\mathrm{Im}(\omega_j)).$$

Since $\eta(\mathrm{Im}(u_j + \omega_j)) > 0$ and $\eta(\mathrm{Im}(\omega_j)) \leq 0$, this implies $\eta(\mathrm{Im}(u_j + t\omega_j)) \in C^{\pm}$ for all $t \in [0, 1]$. Moreover, $\eta(\mathrm{Im}(z_j)) < \eta(\mathrm{Im}(u_j))$, for every $j$. Thus $\rho(z) > \rho(u)$ and therefore $u \in U_r$. □

COROLLARY 5.3. *The extended future tube is saturated with respect to $\pi_{\mathbb{C}}$.*

REMARK 5.1. The function $f : \mathbb{R} \to \mathbb{R}, t \mapsto \eta(\mathrm{Im}(u_j + t\omega_j))$, is strictly concave if $\omega_j \neq 0$. The proof shows $u_j + t\omega_j \in T$ for all $t \in \mathbb{R}$.

## 6. – The Kählerian reduction of the extended future tube

If one is only interested in the statement of the future tube conjecture, one can simply apply the main result in [He2] (Theorem 1 in Section 2). Our goal here is to show that much more is true.

For $z \in \mathbb{C}^{(1+n)\times N}$ let $x = \frac{1}{2}(z + \bar{z})$ be the real and $y = \frac{1}{2i}(z - \bar{z})$ the imaginary part of $z$, i.e., $z = (z_1, \dots, z_N) = (x_1, \dots, x_N) + i(y_1, \dots, y_N)$ in the obvious sense. The strictly plurisubharmonic function $\rho : T^N \to \mathbb{R}$, $\rho(z) = \frac{1}{\eta(y_1)} + \cdots + \frac{1}{\eta(y_N)}$ defines for every $\xi \in \mathfrak{so}(1, n) = \mathfrak{o}(1, n)$ the function

$$\mu_\xi(z) = d\rho(z)(i\xi z) = \frac{d}{dt}\Big|_{t=0} \rho(\exp it\xi \cdot z).$$

Here of course $\mathfrak{so}(1, n) = \mathfrak{o}(1, n)$ denotes the Lie algebra of $\mathrm{O}_{\mathbb{R}}(1, n)$. The real group $\mathrm{SO}_{\mathbb{R}}(1, n)^0$ acts by conjugation on $\mathfrak{so}(1, n)$ and therefore by duality on the dual vector space $\mathfrak{so}(1, n)^*$. It is easy to check that the map $\xi \to \mu_\xi$ depends linearly on $\xi$. Thus

$$\mu : T^N \to \mathfrak{so}(1, n)^*, \quad \mu(z)(\xi) := \mu_\xi(z),$$

is a well defined $\mathrm{SO}_{\mathbb{R}}(1, n)^0$-equivariant map. In fact $\mu$ is a moment map with respect to the Kähler form $\omega = 2i\partial\bar{\partial}\rho$.

In order to emphasizes the general ideas, we set $G := \mathrm{SO}_{\mathbb{R}}(1, n)^0$, $G^{\mathbb{C}} := \mathrm{SO}_{\mathbb{C}}(1, n)$, $X := T^N$ and $Z := G^{\mathbb{C}} \cdot X$. The corresponding analytic Hilbert quotient, induced by $\pi_{\mathbb{C}} : \mathbb{C}^{(1+n)\times N} \to \mathbb{C}^{(1+n)\times N} /\!/ \mathrm{SO}_{\mathbb{C}}(1, n)$ are denoted by $\pi_X : X \to X/\!/G$, $\pi_Z : Z \to Z/\!/G^{\mathbb{C}}$. Note that, by what we proved, we have $X/\!/G = Z/\!/G^{\mathbb{C}}$.

Proposition 6.1.

i. *For every* $q \in Z/\!/G^{\mathbb{C}}$ *we have* $(\pi_{\mathbb{C}})^{-1}(q) \cap \mu^{-1}(0) = G \cdot x_0$ *for some* $x_0 \in \mu^{-1}(0)$ *and* $G^{\mathbb{C}} \cdot x_0$ *is a closed orbit in* $Z$.

ii. *The inclusion* $\mu^{-1}(0) \xrightarrow{\iota} X \subset Z$ *induces a homeomorphism* $\mu^{-1}(0)/G \xrightarrow{\bar{\iota}} Z/\!/G^{\mathbb{C}}$.

Proof. A simple calculation shows that the set of critical points of $\rho|G^{\mathbb{C}} \cdot x \cap X$, i.e., $\mu^{-1}(0) \cap G^{\mathbb{C}} \cdot x$, consists of a discrete set of $G$-orbits. Moreover, every critical point is a local minimum (see [He2], Proof of Lemma 2 in Section 2).

On the other hand Remark 5.1 of Section 5 says that if $\rho|G^{\mathbb{C}} \cdot x \cap X$ has a local minimum in $x_0 \in G^{\mathbb{C}} \cdot x \cap X$, then $G^{\mathbb{C}} \cdot x_0 = G^{\mathbb{C}} \cdot x$ is necessarily closed in $Z$. Moreover, $\rho|G^{\mathbb{C}} \cdot x \cap X$ is then an exhaustion and therefore $\mu^{-1}(0) \cap (G^{\mathbb{C}} \cdot x_0 \cap X) = G \cdot x_0$ (see [He2], Lemma 2 in Section 2). This proves the first part.

The statement i. implies that $\iota : \mu^{-1}(0) \hookrightarrow X \subset Z$ induces a bijective continuous map $\bar{\iota} : \mu^{-1}(0)/G \to Z/\!/G^{\mathbb{C}}$. Since the $G$-action on $X$ is proper and $\mu^{-1}(0)$ is closed, the action on $\mu^{-1}(0)$ is proper. In particular $\mu^{-1}(0)/G$ is a Hausdorff topological space.

Theorem 5.1 implies that $\bar{\iota}$ is a homeomorphism, since for every sequence $q_\alpha \to q_0$ in $Z/\!/G^{\mathbb{C}}$ we find a sequence $(x_\alpha)$ such that $x_\alpha$ are contained in a compact subset of $\mu^{-1}(0)$ and $\pi_{\mathbb{C}}(x_\alpha) = q_\alpha$. Thus every convergent subsequence of $(x_\alpha)$ has a limit point in $G \cdot x_0$ where $\pi_{\mathbb{C}}(x_0) = q_0$. □

PROPOSITION 6.2. *The restriction* $\rho|\mu^{-1}(0) : \mu^{-1}(0) \to \mathbb{R}$ *induces a strictly plurisubharmonic continuous exhaustion* $\bar{\rho} : Z/\!/G^{\mathbb{C}} \to \mathbb{R}$.

PROOF. The exhaustion property for $\bar{\rho}$ follows from Theorem 4.1. The argument that $\bar{\rho}$ is strictly plurisubharmonic is the same as in [HeHuL]. □

THEOREM 6.1. *The extended future tube $Z$ is a domain of holomorphy.*

PROOF. Proposition 6.2 implies that $Z/\!/G^{\mathbb{C}}$ is a Stein space (see [N] Theorem II). Hence $Z$ is a Stein space. □

In fact, much more has been proved here. We would like to comment on this. By definition, an analytic subset of a complex manifold is closed. For the following recall that orbit-connectedness is a condition on the $G^{\mathbb{C}}$-orbits.

PROPOSITION 6.3. *Every analytic $G$-invariant subset $A$ of $X$ is orbit connected in $Z$ and $G^{\mathbb{C}} \cdot A$ is an analytic subset of $Z$. In particular, $G^{\mathbb{C}} \cdot A$ is a Stein space. Moreover the restriction maps*

$$\mathcal{O}(Z)^{G^{\mathbb{C}}} \to \mathcal{O}(G^{\mathbb{C}} \cdot A)^{G^{\mathbb{C}}} \to \mathcal{O}(A)^G$$

*are surjective.*

PROOF. If $b \in G^{\mathbb{C}} \cdot A \cap X$, then $b = g \cdot a$ for some $g \in G^{\mathbb{C}}$ and $a \in A$. Hence $g \in \Sigma_{G^{\mathbb{C}}}(a) = \{g \in G^{\mathbb{C}}; g \cdot a \in X\}$. The identity principle for holomorphic functions shows that $\Sigma_{G^{\mathbb{C}}}(a) \cdot a \in A$. Thus $b \in A$ This shows $G^{\mathbb{C}} \cdot A \cap X = A$. But $\{g \cdot X; g \in G^{\mathbb{C}}\}$ is an open covering of $X$ such that $G^{\mathbb{C}} \cdot A \cap g \cdot X = g \cdot A$. This shows that $G^{\mathbb{C}} \cdot A$ is an analytic subset of $Z$. In particular, it is a Stein space. The last statement follows from orbit connectedness (see [He1]). □

PROPOSITION 6.4. *For every $G$-invariant analytic subset $A$, its saturation $\hat{A} = \pi_X^{-1}(\pi_X(A))$ is an analytic subset of $X$. Moreover, $\hat{A}/\!/G$ is canonically isomorphic to $A/\!/G$ and $\pi_{\hat{A}} : \hat{A} \to \hat{A}/\!/G \subset X/\!/G$ is the Hilbert quotient of $\hat{A}$ whose restriction to $A$ gives the analytic Hilbert quotient of $A$*

PROOF. We already know that $A^c = G^{\mathbb{C}} \cdot A$ is an analytic subset of $Z$. Its saturation $\hat{A}^c = \pi_Z^{-1}(\pi_Z(A^c)) = \pi_Z^{-1}(\pi_Z(A))$ is an analytic subset of $Z$ and it is easily checked that $\hat{A} = \hat{A}^c \cap X = \pi_X^{-1}(\pi_X(A))$ has the desired properties. □

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