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Bootstrap regularity for integro-differential operators and its application to nonlocal minimal surfaces

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Abstract. We prove that $C^{1,\alpha}$ s-minimal surfaces are of class C^∞. For this, we develop a new bootstrap regularity theory for solutions of integro-differential equations of very general type, which we believe is of independent interest.

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1. Introduction

Motivated by the structure of interphases arising in phase transition models with long range interactions, in [4] the authors introduced a nonlocal version of minimal surfaces. These objects are obtained by minimizing a "nonlocal perimeter" inside a fixed domain Ω: fix $s \in (0,1)$, and given two sets $A, B \subset \mathbb{R}^n$, let us define the interaction term

$$L(A,B) := \int_A \int_B \frac{dx\,dy}{|x-y|^{n+s}}.$$

The nonlocal perimeter of E inside Ω is defined as

$$\begin{aligned}\mathrm{Per}(E,\Omega,s) := &\, L\big(E\cap\Omega, (\mathbb{R}^n\setminus E)\cap\Omega\big)\\ &+ L\big(E\cap\Omega, (\mathbb{R}^n\setminus E)\cap(\mathbb{R}^n\setminus\Omega)\big)\\ &+ L\big((\mathbb{R}^n\setminus E)\cap\Omega, E\cap(\mathbb{R}^n\setminus\Omega)\big).\end{aligned}$$

Then nonlocal (s-)minimal surfaces correspond to minimizers of the above functional with the "boundary condition" that $E\cap(\mathbb{R}^n\setminus\Omega)$ is prescribed.

It is proved in [4] that "flat s-minimal surface" are $C^{1,\alpha}$ for all $\alpha < s$, and in [1,9,10] that, as $s \to 1^-$, the s-minimal surfaces approach the classical ones,

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both in a geometric sense and in a Γ-convergence framework, with uniform estimates as $s \to 1^-$. In particular, when s is sufficiently close to 1, they inherit some nice regularity properties from the classical minimal surfaces (see also [8,13,14] for the relation between s-minimal surfaces and the interfaces of some phase transition equations driven by the fractional Laplacian).

On the other hand, all the previous literature only focused on the $C^{1,\alpha}$ regularity, and higher regularity was left as an open problem. In this paper we address this issue, and we prove that $C^{1,\alpha}$ s-minimal surfaces are indeed C^∞, according to the following result[1]:

Theorem 1.1. *Let $s \in (0,1)$, and ∂E be an s-minimal surface in K_R for some $R > 0$. Assume that*

$$\partial E \cap K_R = \left\{(x', x_n) \,:\, x' \in B_R^{n-1} \text{ and } x_n = u(x')\right\} \tag{1.1}$$

for some $u : B_R^{n-1} \to \mathbb{R}$, with $u \in C^{1,\alpha}(B_R^{n-1})$ for any $\alpha < s$ and $u(0) = 0$. Then

$$u \in C^\infty(B_\rho^{n-1}) \quad \forall \rho \in (0, R).$$

The regularity result of Theorem 1.1 combined with [4, Theorem 6.1] and [10, Theorems 1, 3, 4, 5], implies also the following results (here and in the sequel, $\{e_1, e_2, \dots, e_n\}$ denotes the standard Euclidean basis):

Corollary 1.2. *Fix $s_o \in (0,1)$. Let $s \in (s_o, 1)$ and ∂E be an s-minimal surface in B_R for some $R > 0$. There exists $\epsilon_\star > 0$, possibly depending on n, s_o and α, but independent of s and R, such that if*

$$\partial E \cap B_R \subseteq \{|x \cdot e_n| \leqslant \epsilon_\star R\}$$

then $\partial E \cap B_{R/2}$ is a C^∞-graph in the e_n-direction.

Corollary 1.3. *There exists $\epsilon_o \in (0,1)$ such that if $s \in (1-\epsilon_o, 1)$, then:*

- *If $n \leqslant 7$, any s-minimal surface is of class C^∞;*
- *If $n = 8$, any s-minimal surface is of class C^∞ except, at most, at countably many isolated points.*

[1] Here and in the sequel, we write $x \in \mathbb{R}^n$ as $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Moreover, given $r > 0$ and $p \in \mathbb{R}^n$, we define

$$K_r(p) := \{x \in \mathbb{R}^n \,:\, |x' - p'| < r \text{ and } |x_n - p_n| < r\}.$$

As usual, $B_r(p)$ denotes the Euclidean ball of radius r centered at p. Given $p' \in \mathbb{R}^{n-1}$, we set

$$B_r^{n-1}(p') := \{x' \in \mathbb{R}^{n-1} \,:\, |x' - p'| < r\}.$$

We also use the notation $K_r := K_r(0)$, $B_r := B_r(0)$, $B_r^{n-1} := B_r^{n-1}(0)$.

More generally, in any dimension n there exists $\epsilon_n \in (0,1)$ *such that if* $s \in (1-\epsilon_n, 1)$ *then any s-minimal surface is of class* C^∞ *outside a closed set* Σ *of Hausdorff dimension* $n-8$.

Also, Theorem 1.1 here combined with [15, Corollary 1] gives the following regularity result in the plane:

Corollary 1.4. *Let* $n = 2$. *Then, for any* $s \in (0,1)$, *any s-minimal surface is a smooth embedded curve of class* C^∞.

In order to prove Theorem 1.1 we establish in fact a very general result about the regularity of integro-differential equations, which we believe is of independent interest.

For this, we consider a kernel $K = K(x,w) : \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \to (0,+\infty)$ satisfying some general structural assumptions. In the following, $\sigma \in (1,2)$.

First of all, we suppose that K is close to an autonomous kernel of fractional Laplacian type, namely

$$\begin{cases} \text{there exist } a_0, r_0 > 0 \text{ and } \eta \in (0, a_0/4) \text{ such that} \\ \left| \dfrac{|w|^{n+\sigma} K(x,w)}{2-\sigma} - a_0 \right| \leqslant \eta \qquad \forall\, x \in B_1,\ w \in B_{r_0} \setminus \{0\}. \end{cases} \tag{1.2}$$

Moreover, we assume that[2]

$$\begin{cases} \text{there exist } k \in \mathbb{N} \cup \{0\} \text{ and } C_k > 0 \text{ such that} \\ K \in C^{k+1}\big(B_1 \times (\mathbb{R}^n \setminus \{0\})\big), \\ \|\partial_x^\mu \partial_w^\theta K(\cdot, w)\|_{L^\infty(B_1)} \leqslant \dfrac{C_k}{|w|^{n+\sigma+|\theta|}} \\ \qquad\qquad \forall\, \mu, \theta \in \mathbb{N}^n,\ |\mu| + |\theta| \leqslant k+1,\ w \in \mathbb{R}^n \setminus \{0\}. \end{cases} \tag{1.3}$$

Our main result is a "Schauder regularity theory" for solutions[3] of an integro-differential equation. Here and in the sequel we use the notation

$$\delta u(x,w) := u(x+w) + u(x-w) - 2u(x). \tag{1.4}$$

Theorem 1.5. *Let* $\sigma \in (1,2)$, $k \in \mathbb{N} \cup \{0\}$, *and* $u \in L^\infty(\mathbb{R}^n)$ *be a viscosity solution of the equation*

$$\int_{\mathbb{R}^n} K(x,w)\, \delta u(x,w) dw = f(x, u(x)) \qquad \text{inside } B_1, \tag{1.5}$$

with $f \in C^{k+1}(B_1 \times \mathbb{R})$. *Assume that* $K : B_1 \times (\mathbb{R}^n \setminus \{0\}) \to (0,+\infty)$ *satisfies assumptions* (1.2) *and* (1.3) *for the same value of* k.

[2] Observe that we use $|\cdot|$ both to denote the Euclidean norm of a vector and, for a multi-index case $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, to denote $|\alpha| := \alpha_1 + \cdots + \alpha_n$. However, the meaning of $|\cdot|$ will always be clear from the context.

[3] We adopt the notion of viscosity solution used in [5–7].

Then, if η *in* (1.2) *is sufficiently small (the smallness being independent of* k*), we have* $u \in C^{k+\sigma+\alpha}(B_{1/2})$ *for any* $\alpha < 1$, *and*

$$\|u\|_{C^{k+\sigma+\alpha}(B_{1/2})} \leqslant C\left(1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1\times\mathbb{R})}\right), \tag{1.6}$$

where[4] $C > 0$ *depends only on* n, σ, k, C_k, *and* $\|f\|_{C^{k+1}(B_1\times\mathbb{R})}$.

Let us notice that, since the right-hand side in (1.5) depends on u, there is no uniqueness for such an equation. In particular it is not enough for us to prove a-priori estimates for smooth solutions and then argue by approximation, since we do not know if our solution can be obtained as a limit of smooth solution.

We also note that, if in (1.3) one replaces the C^{k+1}-regularity of K with the $C^{k,\beta}$-assumption

$$\|\partial_x^\mu \partial_w^\theta K(\cdot, w)\|_{C^{0,\beta}(B_1)} \leqslant \frac{C_k}{|w|^{n+\sigma+|\theta|}}, \tag{1.7}$$

for all $|\mu| + |\theta| \leqslant k$, then we obtain the following:

Theorem 1.6. *Let* $\sigma \in (1, 2)$, $k \in \mathbb{N}\cup\{0\}$, *and* $u \in L^\infty(\mathbb{R}^n)$ *be a viscosity solution of equation* (1.5) *with* $f \in C^{k,\beta}(B_1 \times \mathbb{R})$. *Assume that* $K : B_1 \times (\mathbb{R}^n \setminus \{0\}) \to (0, +\infty)$ *satisfies assumptions* (1.2) *and* (1.7) *for the same value of* k.

Then, if η *in* (1.2) *is sufficiently small (the smallness being independent of* k*), we have* $u \in C^{k+\sigma+\alpha}(B_{1/2})$ *for any* $\alpha < \beta$, *and*

$$\|u\|_{C^{k+\sigma+\alpha}(B_{1/2})} \leqslant C\left(1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1\times\mathbb{R})}\right),$$

where $C > 0$ *depends only on* n, σ, k, C_k, *and* $\|f\|_{C^{k,\beta}(B_1\times\mathbb{R})}$.

The proof of Theorem 1.6 is essentially the same as the one of Theorem 1.5, the only difference being that instead of differentiating the equations (see for instance the argument in Section 2.4) one should use incremental quotients. Although this does not introduce any major additional difficulties, it makes the proofs longer and more tedious. Hence, since the proof of Theorem 1.5 already contains all the main ideas to prove also Theorem 1.6, we will show the details of the proof only for Theorem 1.5.

The paper is organized as follows: in the next section we prove Theorem 1.5, and then in Section 3 we write the fractional minimal surface equation in a suitable form so that we can apply Theorems 1.5 and 1.6 to prove Theorem 1.1.

[4] As customary, when $\sigma + \alpha \in (1, 2)$ (respectively $\sigma + \alpha > 2$), by (1.6) we mean that $u \in C^{k+1,\sigma+\alpha-1}(B_{1/2})$ (respectively $u \in C^{k+2,\sigma+\alpha-1}(B_{1/2})$). (To avoid any issue, we will always implicitly assume that α is chosen different from $2 - \sigma$, so that $\sigma + \alpha \neq 2$.)

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2. Proof of Theorem 1.5

The core of the proof of Theorem 1.5 is the step $k = 0$, which will be proved in several steps.

2.1. Toolbox

We collect here some preliminary observations on scaled Hölder norms, covering arguments, and differentiation of integrals that will play an important role in the proof of Theorem 1.5. This material is mainly technical, and the expert reader may go directly to Section 2.2 at page 619.

2.1.1. *Scaled Hölder norms and coverings*

Given $m \in \mathbb{N}$, $\alpha \in (0,1)$, $x \in \mathbb{R}^n$, and $r > 0$, we define the $C^{m,\alpha}$-norm of a function u in $B_r(x)$ as

$$\|u\|_{C^{m,\alpha}(B_r(x))} := \sum_{|\gamma|\leqslant m} \|D^\gamma u\|_{L^\infty(B_r(x))} + \sum_{|\gamma|=m} \sup_{y\neq z\in B_r(x)} \frac{|D^\gamma u(y) - D^\gamma u(z)|}{|y-z|^\alpha}.$$

For our purposes it is also convenient to look at the following classical rescaled version of the norm:

$$\|u\|^*_{C^{m,\alpha}(B_r(x))} := \sum_{j=0}^{m} \sum_{|\gamma|=j} r^j \|D^\gamma u\|_{L^\infty(B_r(x))} + \sum_{|\gamma|=m} r^{m+\alpha} \sup_{y\neq z\in B_r(x)} \frac{|D^\gamma u(y) - D^\gamma u(z)|}{|y-z|^\alpha}.$$

This scaled norm behaves nicely under covering, as the next observation points out:

Lemma 2.1. *Let $m \in \mathbb{N}$, $\alpha \in (0,1)$, $\rho > 0$, and $x \in \mathbb{R}^n$. Fix $\lambda \in (0,1)$, and suppose that $B_\rho(x)$ is covered by finitely many balls $\{B_{\lambda\rho/2}(x_k)\}_{k=1}^N$. Then, there exists $C_o > 0$, depending only on λ and m, such that*

$$\|u\|^*_{C^{m,\alpha}(B_\rho(x))} \leqslant C_o \sum_{k=1}^{N} \|u\|^*_{C^{m,\alpha}(B_{\lambda\rho}(x_k))}.$$

Proof. We first observe that, if $j \in \{0, \dots, m\}$ and $|\gamma| = j$,

$$\begin{aligned}
\rho^j \|D^\gamma u\|_{L^\infty(B_\rho(x))} &\leqslant \lambda^{-j} (\lambda\rho)^j \max_{k=1,\dots,N} \|D^\gamma u\|_{L^\infty(B_{\lambda\rho}(x_k))} \\
&\leqslant \lambda^{-m} \sum_{k=1}^{N} (\lambda\rho)^j \|D^\gamma u\|_{L^\infty(B_{\lambda\rho}(x_k))} \\
&\leqslant \lambda^{-m} \sum_{k=1}^{N} \|u\|^*_{C^{m,\alpha}(B_{\lambda\rho}(x_k))}.
\end{aligned}$$

Now, let $|\gamma| = m$: we claim that

$$\rho^{m+\alpha} \sup_{y \neq z \in B_\rho(x)} \frac{|D^\gamma u(y) - D^\gamma u(z)|}{|y-z|^\alpha} \leqslant 2\lambda^{-(m+\alpha)} \sum_{k=1}^{N} \|u\|^*_{C^{m,\alpha}(B_{\lambda\rho}(x_k))}.$$

To check this, we take $y, z \in B_\rho(x)$ with $y \neq z$ and we distinguish two cases. If $|y - z| < \lambda\rho/2$ we choose $k_o \in \{1, \dots, N\}$ such that $y \in B_{\lambda\rho/2}(x_{k_o})$. Then $|z - x_{k_o}| \leqslant |z - y| + |y - x_{k_o}| < \lambda\rho$, which implies $y, z \in B_{\lambda\rho}(x_{k_o})$, therefore

$$\begin{aligned}
\rho^{m+\alpha} \frac{|D^\gamma u(y) - D^\gamma u(z)|}{|y-z|^\alpha} &\leqslant \rho^{m+\alpha} \sup_{\tilde{y} \neq \tilde{z} \in B_{\lambda\rho}(x_{k_o})} \frac{|D^\gamma u(\tilde{y}) - D^\gamma u(\tilde{z})|}{|\tilde{y} - \tilde{z}|^\alpha} \\
&\leqslant \lambda^{-(m+\alpha)} \|u\|^*_{C^{m,\alpha}(B_{\lambda\rho}(x_{k_o}))}.
\end{aligned}$$

Conversely, if $|y - z| \geqslant \lambda\rho/2$, recalling that $\alpha \in (0, 1)$ we have

$$\begin{aligned}
\rho^{m+\alpha} \frac{|D^\gamma u(y) - D^\gamma u(z)|}{|y-z|^\alpha} &\leqslant 2\lambda^{-\alpha} \rho^m \|D^\gamma u\|_{L^\infty(B_\rho(x))} \\
&\leqslant 2\lambda^{-\alpha} \rho^m \sum_{k=1}^{N} \|D^\gamma u\|_{L^\infty(B_{\lambda\rho}(x_k))} \\
&\leqslant 2\lambda^{-(m+\alpha)} \sum_{k=1}^{N} \|u\|^*_{C^{m,\alpha}(B_{\lambda\rho}(x_k))}.
\end{aligned}$$

This proves the claim and concludes the proof. □

Scaled norms behave also nicely in order to go from local to global bounds, as the next result shows:

Lemma 2.2. *Let $m \in \mathbb{N}$, $\alpha \in (0, 1)$, and $u \in C^{m,\alpha}(B_1)$. Suppose that there exist $\mu \in (0, 1/2)$ and $\nu \in (\mu, 1]$ for which the following holds: for any $\epsilon > 0$ there exists $\Lambda_\epsilon > 0$ such that, for any $x \in B_1$ and any $r \in (0, 1 - |x|]$, we have*

$$\|u\|^*_{C^{m,\alpha}(B_{\mu r}(x))} \leqslant \Lambda_\epsilon + \epsilon \|u\|^*_{C^{m,\alpha}(B_{\nu r}(x))}. \tag{2.1}$$

Then there exist constants ϵ_o, $C > 0$, depending only on n, m, μ, ν, and α, such that

$$\|u\|_{C^{m,\alpha}(B_\mu)} \leqslant C \Lambda_{\epsilon_o}.$$

Proof. First of all we observe that

$$\|u\|^*_{C^{m,\alpha}(B_{\mu r}(x))} \leqslant \|u\|_{C^{m,\alpha}(B_{\mu r}(x))} \leqslant \|u\|^*_{C^{m,\alpha}(B_1)}$$

because $r \in (0, 1)$, which implies that

$$Q := \sup_{\substack{x\in B_1 \\ r\in(0,1-|x|]}} \|u\|^*_{C^{m,\alpha}(B_{\mu r}(x))} < +\infty.$$

We now use a covering argument: pick $\lambda \in (0, 1/2]$ to be chosen later, and fixed any $x \in B_1$ and $r \in (0, 1 - |x|]$ we cover $B_{\mu r}(x)$ with finitely many balls $\{B_{\lambda\mu r/2}(x_k)\}_{k=1}^N$, with $x_k \in B_{\mu r}(x)$, for some N depending only on λ and the dimension n. We now observe that, since $\mu < 1/2$,

$$|x_k| + r/2 \leqslant |x_k - x| + |x| + r/2 \leqslant \mu r + |x| + r/2 < r + |x| \leqslant 1. \tag{2.2}$$

Hence, since $\lambda \leqslant 1/2$, we can use (2.1) (with $x = x_k$ and r scaled to λr) to obtain

$$\|u\|^*_{C^{m,\alpha}(B_{\lambda\mu r}(x_k))} \leqslant \Lambda_\epsilon + \epsilon\|u\|^*_{C^{m,\alpha}(B_{\lambda\nu r}(x_k))}.$$

Then, using Lemma 2.1 with $\rho := \mu r$ and $\lambda = \mu/(2\nu)$, and recalling (2.2) and the definition of Q, we get

$$\begin{aligned}
\|u\|^*_{C^{m,\alpha}(B_{\mu r}(x))} &\leqslant C_o \sum_{k=1}^N \|u\|^*_{C^{m,\alpha}(B_{\lambda\mu r}(x_k))} \\
&\leqslant C_o N\Lambda_\epsilon + C_o\epsilon \sum_{k=1}^N \|u\|^*_{C^{m,\alpha}(B_{\lambda\nu r}(x_k))} \\
&= C_o N\Lambda_\epsilon + C_o\epsilon \sum_{k=1}^N \|u\|^*_{C^{m,\alpha}(B_{\mu r/2}(x_k))} \\
&\leqslant C_o N\Lambda_\epsilon + \epsilon C_o N Q.
\end{aligned}$$

Using the definition of Q again, this implies

$$Q \leqslant C_o N\Lambda_\epsilon + \epsilon C_o N Q,$$

so that, by choosing $\epsilon_o := 1/(2C_o N)$,

$$Q \leqslant 2C_o N\Lambda_{\epsilon_o}.$$

Thus we have proved that

$$\|u\|^*_{C^{m,\alpha}(B_{\mu r}(x))} \leqslant 2C_o N\Lambda_{\epsilon_o} \qquad \forall x \in B_1,\ r \in (0, 1 - |x|],$$

and the desired result follows setting $x = 0$ and $r = 1$. $\square$

2.1.2. *Differentiating integral functions*

In the proof of Theorem 1.5 we will need to differentiate, under the integral sign, smooth functions that are either supported near the origin or far from it. This purpose will be accomplished in Lemmata 2.5 and 2.6, after some technical bounds that are needed to use the Dominated Convergence Theorem.

Recall the notation in (1.4).

Lemma 2.3. *Let $r > r' > 0$, $v \in C^3(B_r)$, $x \in B_{r'}$, $h \in \mathbb{R}$ with $|h| < (r-r')/2$. Then, for any $w \in \mathbb{R}^n$ with $|w| < (r-r')/2$, we have*

$$|\delta v(x+he_1,w) - \delta v(x,w)| \leqslant |h|\,|w|^2 \|v\|_{C^3(B_r)}.$$

Proof. Fixed $x \in B_{r'}$ and $|w| < (r-r')/2$, for any $h \in [(r'-r)/2, (r-r')/2]$ we set $g(h) := v(x+he_1+w) + v(x+he_1-w) - 2v(x+he_1)$. Then

$$\begin{aligned} |g(h)-g(0)| &\leqslant |h| \sup_{|\xi|\leqslant|h|} |g'(\xi)| \\ &\leqslant |h| \sup_{|\xi|\leqslant|h|} \Big|\partial_1 v(x+\xi e_1+w) + \partial_1 v(x+\xi e_1 - w) - 2\partial_1 v(x+\xi e_1)\Big|. \end{aligned}$$

Noticing that $|x+\xi e_1 \pm w| \leqslant r' + |h| + |w| < r$, a second order Taylor expansion of $\partial_1 v$ with respect to the variable w gives

$$\Big|\partial_1 v(x+\xi e_1+w) + \partial_1 v(x+\xi e_1-w) - 2\partial_1 v(x+\xi e_1)\Big| \leqslant |w|^2 \|\partial_1 v\|_{C^2(B_r)}. \tag{2.3}$$

Therefore

$$|\delta v(x+he_1,w) - \delta v(x,w)| = |g(h)-g(0)| \leqslant |h|\,|w|^2 \|v\|_{C^3(B_r)},$$

as desired. □

Lemma 2.4. *Let $r > r' > 0$, $v \in W^{1,\infty}(\mathbb{R}^n)$, $h \in \mathbb{R}$. Then, for any $w \in \mathbb{R}^n$,*

$$|\delta v(x+he_1,w) - \delta v(x,w)| \leqslant 4|h| \|\nabla v\|_{L^\infty(\mathbb{R}^n)}.$$

Proof. It sufficed to proceed as in the proof of Lemma 2.3, but replacing (2.3) with the following estimate:

$$\Big|\partial_1 v(x+\xi e_1+w) + \partial_1 v(x+\xi e_1-w) - 2\partial_1 v(x+\xi e_1)\Big| \leqslant 4\|\partial_1 v\|_{L^\infty(\mathbb{R}^n)}.$$ □

Lemma 2.5. *Let $\ell \in \mathbb{N}$, $r \in (0,2)$, K satisfy (1.3), and $U \in C_0^{\ell+2}(B_r)$. Let $\gamma = (\gamma_1,\dots,\gamma_n) \in \mathbb{N}^n$ with $|\gamma| \leqslant \ell \leqslant k+1$. Then*

$$\begin{aligned} \partial_x^\gamma \int_{\mathbb{R}^n} K(x,w)\,\delta U(x,w)\,dw &= \int_{\mathbb{R}^n} \partial_x^\gamma \Big(K(x,w)\,\delta U(x,w)\Big)\,dw \\ &= \sum_{\substack{1\leqslant i\leqslant n \\ 0\leqslant\lambda_i\leqslant\gamma_i \\ \lambda=(\lambda_1,\dots,\lambda_n)}} \binom{\gamma_1}{\lambda_1}) \dots \binom{\gamma_n}{\lambda_n}) \int_{\mathbb{R}^n} \partial_x^\lambda K(x,w)\,\delta(\partial_x^{\gamma-\lambda} U)(x,w)\,dw \end{aligned} \tag{2.4}$$

for any $x \in B_r$.

Proof. The latter equality follows from the standard product derivation formula, so we focus on the proof of the first identity. The proof is by induction over $|\gamma|$. If $|\gamma| = 0$ the result is trivially true, so we consider the inductive step. We take x with $r' := |x| < r$, we suppose that $|\gamma| \leqslant \ell - 1$ and, by inductive hypothesis, we know that

$$g_\gamma(x) := \partial_x^\gamma \int_{\mathbb{R}^n} K(x, w)\, \delta U(x, w)\, dw = \int_{\mathbb{R}^n} \theta(x, w)\, dw$$

with

$$\theta(x, w) := \sum_{\substack{1\leqslant i\leqslant n\\ 0\leqslant \lambda_i\leqslant \gamma_i\\ \lambda=(\lambda_1,\dots,\lambda_n)}} \binom{\gamma_1}{\lambda_1}) \dots \binom{\gamma_n}{\lambda_n})\partial_x^\lambda K(x, w)\, \delta(\partial_x^{\gamma-\lambda} U)(x, w)\, dw.$$

By (1.3), if $0 < |h| < (r - r')/2$ then

$$|\partial_x^\lambda K(x + he_1, w) - \partial_x^\lambda K(x, w)| \leqslant C_{|\lambda|+1}|h|\, |w|^{-n-\sigma}. \tag{2.5}$$

Moreover, if $|w| < (r - r')/2$, we can apply Lemma 2.3 with $v := \partial_x^{\gamma-\lambda} U$ and obtain

$$|\delta(\partial_x^{\gamma-\lambda} U)(x + he_1, w) - \delta(\partial_x^{\gamma-\lambda} U)(x, w)| \leqslant |h|\, |w|^2 \|U\|_{C^{|\gamma-\lambda|+3}(B_r)}. \tag{2.6}$$

On the other hand, by Lemma 2.4 we obtain

$$|\delta(\partial_x^{\gamma-\lambda} U)(x + he_1, w) - \delta(\partial_x^{\gamma-\lambda} U)(x, w)| \leqslant 4\, |h|\, \|\partial_x^{\gamma-\lambda} U\|_{C^1(\mathbb{R}^n)}.$$

All in all,

$$\begin{aligned} &|\delta(\partial_x^{\gamma-\lambda} U)(x + he_1, w) - \delta(\partial_x^{\gamma-\lambda} U)(x, w)| \\ &\leqslant |h|\, \|U\|_{C^{|\gamma-\lambda|+3}(\mathbb{R}^n)} \min\{4, |w|^2\}. \end{aligned} \tag{2.7}$$

Analogously, a simple Taylor expansion provides also the bound

$$|\delta(\partial_x^{\gamma-\lambda} U)(x, w)| \leqslant \|U\|_{C^{|\gamma-\lambda|+2}(\mathbb{R}^n)} \min\{4, |w|^2\}. \tag{2.8}$$

Hence, (1.3), (2.5), (2.7), and (2.8) give

$$\begin{aligned} &\left|\partial_x^\lambda K(x + he_1, w)\, \delta(\partial_x^{\gamma-\lambda} U)(x + he_1, w) - \partial_x^\lambda K(x, w)\, \delta(\partial_x^{\gamma-\lambda} U)(x, w)\right| \\ \leqslant& \left|\partial_x^\lambda K(x + he_1, w) \left[\delta(\partial_x^{\gamma-\lambda} U)(x + he_1, w) - \delta(\partial_x^{\gamma-\lambda} U)(x, w)\right]\right| \\ &+\left|\left[\partial_x^\lambda K(x + he_1, w) - \partial_x^\lambda K(x, w)\right]\delta(\partial_x^{\gamma-\lambda} U)(x, w)\right| \\ \leqslant& C_1|h| \min\{|w|^{-n-\sigma}, |w|^{2-n-\sigma}\}, \end{aligned}$$

with $C_1 > 0$ depending only on ℓ, C_ℓ and $\|U\|_{C^{\ell+2}(\mathbb{R}^n)}$. As a consequence,

$$|\theta(x+he_1,w)-\theta(x,w)| \leqslant C_2|h|\,\min\{|w|^{-n-\sigma},|w|^{2-n-\sigma}\},$$

and, by the Dominated Convergence Theorem, we get

$$\begin{aligned}\int_{\mathbb{R}^n}\partial_{x_1}\theta(x,w)\,dw &= \lim_{h\to 0}\int_{\mathbb{R}^n}\frac{\theta(x+he_1,w)-\theta(x,w)}{h}\,dw\\ &= \lim_{h\to 0}\frac{g_\gamma(x+he_1)-g_\gamma(x)}{h}\\ &= \partial_{x_1}g_\gamma(x),\end{aligned}$$

which proves (2.4) with γ replaced by $\gamma+e_1$. Analogously one could prove the same result with γ replaced by $\gamma+e_i$, concluding the inductive step. □

The differentiation under the integral sign in (2.4) may also be obtained under slightly different assumptions, as next result points out:

Lemma 2.6. *Let $\ell\in\mathbb{N}$, $R>r>0$. Let $U\in C^{\ell+1}(\mathbb{R}^n)$ with $U=0$ in B_R. Let $\gamma=(\gamma_1,\dots,\gamma_n)\in\mathbb{N}^n$ with $|\gamma|\leqslant\ell$. Then* (2.4) *holds true for any $x\in B_r$.*

Proof. If $x\in B_r$, $w\in B_{(R-r)/2}$ and $|h|\leqslant(R-r)/2$, we have that $|x+w+he_1|<R$ and so $\delta U(x+he_1,w)=0$. In particular

$$\delta U(x+he_1,w)-\delta U(x,w)=0$$

for small h when $w\in B_{(R-r)/2}$. This formula replaces (2.6), and the rest of the proof goes on as the one of Lemma 2.5. □

2.1.3. *Integral computations*

Here we collect some integral computations which will be used in the proof of Theorem 1.5.

Lemma 2.7. *Let $v:\mathbb{R}^n\to\mathbb{R}$ be smooth and with all its derivatives bounded. Let $x\in B_{1/4}$, and $\gamma,\lambda\in\mathbb{N}^n$, with $\gamma_i\geqslant\lambda_i$ for any $i\in\{1,\dots,n\}$. Then there exists a constant $C'>0$, depending only on n and σ, such that*

$$\left|\int_{\mathbb{R}^n}\partial_x^\lambda K(x,w)\,\delta(\partial_x^{\gamma-\lambda}v)(x,w)\,dw\right| \leqslant C'\,C_{|\gamma|}\,\|v\|_{C^{|\gamma-\lambda|+2}(\mathbb{R}^n)}. \tag{2.9}$$

Furthermore, if

$$v=0 \quad\text{in}\quad B_{1/2} \tag{2.10}$$

we have

$$\left|\int_{\mathbb{R}^n} \partial_x^\lambda K(x,w)\,\delta(\partial_x^{\gamma-\lambda} v)(x,w)\,dw\right| \leqslant C'\,C_{|\gamma|}\,\|v\|_{L^\infty(\mathbb{R}^n)}. \tag{2.11}$$

Proof. By (1.3) and (2.8) (with $U = v$),

$$\begin{aligned}&\int_{\mathbb{R}^n} \left|\partial_x^\lambda K(x,w)\right| \left|\delta(\partial_x^{\gamma-\lambda} v)(x,w)\right| dw\\ &\leqslant C_{|\lambda|}\left(\|v\|_{C^{|\gamma-\lambda|+2}(\mathbb{R}^n)}\int_{B_2}|w|^{-n-\sigma+2}\,dw + 4\|v\|_{C^{|\gamma-\lambda|}(\mathbb{R}^n)}\int_{\mathbb{R}^n\setminus B_2}|w|^{-n-\sigma}\,dw\right),\end{aligned}$$

which proves (2.9).

We now prove (2.11). For this we notice that, thanks to (2.10), $v(x+w)$ and $v(x-w)$ (and also their derivatives) are equal to zero if x and w lie in $B_{1/4}$. Hence, by an integration by parts we see that

$$\begin{aligned}&\int_{\mathbb{R}^n} \partial_x^\lambda K(x,w)\,\delta(\partial_x^{\gamma-\lambda} v)(x,w)\,dw\\ &= \int_{\mathbb{R}^n} \partial_x^\lambda K(x,w)\,\partial_w^{\gamma-\lambda}\big[v(x+w)-v(x-w)\big]\,dw\\ &= \int_{\mathbb{R}^n\setminus B_{1/4}} \partial_x^\lambda K(x,w)\,\partial_w^{\gamma-\lambda}\big[v(x+w)-v(x-w)\big]\,dw\\ &= (-1)^{|\gamma-\lambda|}\int_{\mathbb{R}^n\setminus B_{1/4}} \partial_x^\lambda\partial_w^{\gamma-\lambda} K(x,w)\,\big[v(x+w)-v(x-w)\big]\,dw.\end{aligned}$$

Consequently, by (1.3),

$$\begin{aligned}&\left|\int_{\mathbb{R}^n} \partial_x^\lambda K(x,w)\,\delta(\partial_x^{\gamma-\lambda} v)(x,w)\,dw\right|\\ &\leqslant 2C_{|\gamma|}\,\|v\|_{L^\infty(\mathbb{R}^n)}\int_{\mathbb{R}^n\setminus B_{1/4}}|w|^{-n-\sigma-|\gamma-\lambda|}\,dw,\end{aligned}$$

proving (2.11). □

2.2. Approximation by nicer kernels

In what follows, it will be convenient to approximate the solution u of (1.5) with smooth functions u_ε obtained by solving equations similar to (1.5), but with kernels K_ε which coincide with the fractional Laplacian in a neighborhood of the origin. Indeed, this will allow us to work with smooth functions, ensuring that in our computations all integrals converge. We will then prove uniform estimates on u_ε, which will give the desired $C^{\sigma+\alpha}$-bound on u by letting $\varepsilon \to 0$.

To simplify the notation, up to multiply both K and f by $1/a_0$, we assume without loss of generality that the constant a_0 in (1.2) is equal to 1.

Let $\eta \in C^\infty(\mathbb{R}^n)$ satisfy

$$\eta = \begin{cases} 1 & \text{in } B_{1/2}, \\ 0 & \text{in } \mathbb{R}^n \setminus B_{3/4}, \end{cases}$$

and for any $\varepsilon, \delta > 0$ set $\eta_\varepsilon(w) := \eta\big(\frac{w}{\varepsilon}\big)$ for any $\varepsilon > 0$, $\hat{\eta}_\delta(x) := \delta^{-n}\eta(x/\delta)$. Then we define

$$K_\varepsilon(x, w) := \eta_\varepsilon(w)\frac{2-\sigma}{|w|^{n+\sigma}} + (1 - \eta_\varepsilon(w))\hat{K}_\varepsilon(x, w), \tag{2.12}$$

where

$$\hat{K}_\varepsilon(x, w) := K(x, w) * \Big(\hat{\eta}_{\varepsilon^2}(x)\hat{\eta}_{\varepsilon^2}(w)\Big), \tag{2.13}$$

and

$$L_\varepsilon v(x) := \int_{\mathbb{R}^n} K_\varepsilon(x, w)\, \delta v(x, w) dw. \tag{2.14}$$

We also define

$$f_\varepsilon(x) := f(x, u(x)) * \hat{\eta}_\varepsilon(x). \tag{2.15}$$

Note that we get a family $f_\varepsilon \in C^\infty(B_1)$ such that

$$\lim_{\varepsilon \to 0^+} f_\varepsilon = f \text{ uniformly in } B_{3/4}.$$

Finally, we define $u_\varepsilon \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ as the unique solution to the following linear problem:

$$\begin{cases} L_\varepsilon u_\varepsilon = f_\varepsilon(x) & \text{in } B_{3/4} \\ u_\varepsilon = u & \text{in } \mathbb{R}^n \setminus B_{3/4}. \end{cases} \tag{2.16}$$

It is easy to check that the kernels K_ε satisfy (1.2) and (1.3) with constants independent of ε (recall that, to simplify the notation, we are assuming $a_0 = 1$). Also, since K satisfies assumption (1.3) with $k = 0$ and the convolution parameter ε^2 in (2.12) is much smaller than ε, the operators L_ε converge to the operator associated to K in the weak sense introduced in [6, Definition 22]. Indeed, let v a smooth function satisfying

$$|v| \leqslant M \quad \text{in } \mathbb{R}^n, \qquad |v(w) - v(x) - (w - x)\cdot\nabla v(x)| \leqslant M|x - w|^2 \quad \forall\, w \in B_1(x), \tag{2.17}$$

for some $M > 0$. Then, from (1.3) and (2.17), it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left| \eta_\varepsilon(w) \frac{2-\sigma}{|w|^{n+\sigma}} + (1-\eta_\varepsilon(\omega))\big(K(x,w) * \hat{\eta}_{\varepsilon^2}(x)\hat{\eta}_{\varepsilon^2}(w)\big) - K(x,w) \right| \\
&\quad \times |\delta v(x,w)|\, dw \\
&\leqslant \int_{\mathbb{R}^n} \left(\eta_\varepsilon(w) \left| \frac{2-\sigma}{|w|^{n+\sigma}} - K(x,w) \right| \right. \\
&\qquad\qquad \left. + (1-\eta_\varepsilon(\omega)) \big| K(x,w) * \hat{\eta}_{\varepsilon^2}(x)\hat{\eta}_{\varepsilon^2}(w) - K(x,w) \big| \right) |\delta v(x,w)|\, dw \\
&\leqslant \int_{B_\varepsilon} C|w|^{2-n-\sigma} + \int_{\mathbb{R}^n\setminus B_\varepsilon} \big| K(x,w) * \hat{\eta}_{\varepsilon^2}(x)\hat{\eta}_{\varepsilon^2}(w) - K(x,w) \big| |\delta v(x,w)|\, dw \\
&\leqslant C\varepsilon^{2-\sigma} + I,
\end{aligned} \tag{2.18}$$

with

$$I := \int_{\mathbb{R}^n\setminus B_\varepsilon} \big| K(x,w) * \hat{\eta}_{\varepsilon^2}(x)\hat{\eta}_{\varepsilon^2}(w) - K(x,w) \big| \, |\delta v(x,w)|\, dw.$$

By (1.3), (2.17), and the fact that $\sigma > 1$, we have

$$\begin{aligned}
I &= \int_{\mathbb{R}^n\setminus B_\varepsilon} \int_{B_1} \int_{B_1} \Big| K(x-\varepsilon^2 y, w-\varepsilon^2 \tilde{w}) \eta(y)\eta(\tilde{w}) - K(x,w) \Big| \, dy\, d\tilde{w}\, |\delta v(x,w)|\, dw \\
&\leqslant \int_{\mathbb{R}^n\setminus B_\varepsilon} \frac{C\varepsilon^2}{|w|^{n+1+\sigma}} |\delta v(x,w)|\, dw \\
&\leqslant C \int_{B_1\setminus B_\varepsilon} \frac{\varepsilon^2}{|w|^{n-1+\sigma}}\, dw + C \int_{\mathbb{R}^n\setminus B_1} \frac{\varepsilon^2}{|w|^{n+1+\sigma}}\, dw \\
&\leqslant C(\varepsilon^{3-\sigma} + \varepsilon^2).
\end{aligned}$$

Combining this estimate with (2.18), we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left| \eta_\varepsilon(w) \frac{2-\sigma}{|w|^{n+\sigma}} + (1-\eta_\varepsilon(\omega))(K(x,w) * \hat{\eta}_{\varepsilon^2}(x)\hat{\eta}_{\varepsilon^2}(w)) - K(x,w) \right| \delta v(x,w)\, dw \\
&\qquad \leqslant C\varepsilon^{2-\sigma},
\end{aligned}$$

where C depends of M and σ. Since $\sigma < 2$ we conclude that

$$\|L_\varepsilon - L\| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Thanks to this fact, we can repeat almost *verbatim*[5] the proof of [6, Lemma 7] to obtain the uniform convergence

$$u_\varepsilon \to u \quad \text{on } \mathbb{R}^n \qquad \text{as } \varepsilon \to 0. \tag{2.20}$$

2.3. Smoothness of the approximate solutions

We prove now that the functions u_ε defined in the previous section are of class C^∞ inside a small ball (whose size is uniform with respect to ε): namely, there exists $r \in (0, 1/4)$ such that, for any $m \in \mathbb{N}^n$

$$\|D^m u_\varepsilon\|_{L^\infty(B_r)} \leqslant C \tag{2.21}$$

for some positive constant $C = C(m, \sigma, \varepsilon, \|u\|_{L^\infty(\mathbb{R}^n)}, \|f\|_{L^\infty(B_1\times\mathbb{R})})$.

For this, we observe that by (2.12)

$$\frac{2-\sigma}{|w|^{n+\sigma}} = K_\varepsilon(x, w) - (1-\eta_\varepsilon(w))\hat{K}_\varepsilon(x, w) + (1-\eta_\varepsilon(w))\,\frac{2-\sigma}{|w|^{n+\sigma}},$$

[5] In order to repeat the argument in the proof of [6, Lemma 7] one needs to know that the functions u_ε are equicontinuous, which is a consequence of [6, Lemmata 2 and 3]. To be precise, to apply [6, Lemma 3] one would need the kernels to satisfy the bounds $\frac{(2-\sigma)\lambda}{|w|^{n+\sigma}} \leqslant K_*(x, w) \leqslant \frac{(2-\sigma)\Lambda}{|w|^{n+\sigma}}$ for all $w \neq 0$, while in our case the kernel K (and so also K_ε) satisfies

$$\frac{(2-\sigma)\lambda}{|w|^{n+\sigma}} \leqslant K(x, w) \leqslant \frac{(2-\sigma)\Lambda}{|w|^{n+\sigma}} \qquad \forall\, |w| \leqslant r_0 \tag{2.19}$$

with $\lambda := a_0 - \eta$, $\Lambda := a_0 + \eta$, and $r_0 > 0$ (observe that, by our assumptions in (1.2), $\lambda \geqslant 3a_0/4$). However this is not a big problem: if $v \in L^\infty(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} K_*(x, w)\,\delta v(x, w)\,dw = f(x) \qquad \text{in } B_{3/4}$$

for some kernel satisfying (1.3) and $\frac{(2-\sigma)\lambda}{|w|^{n+\sigma}} \leqslant K_*(x, w) \leqslant \frac{(2-\sigma)\Lambda}{|w|^{n+\sigma}}$ only for $|w| \leqslant r_0$, we define $K'(x, w) := \zeta(w)K_*(x, w) + (2-\sigma)\frac{1-\zeta(w)}{|w|^{n+\sigma}}$, with ζ a smooth cut-off function supported inside B_{r_0}, to get

$$\int_{\mathbb{R}^n} K'(x, w)\,\delta v(x, w)\,dw = f(x) + \int_{\mathbb{R}^n} [1-\zeta(w)]\left(-K_*(x, w) + \frac{2-\sigma}{|w|^{n+\sigma}}\right)\delta v(x, w)\,dw.$$

Since $1-\zeta(w) = 0$ near the origin, by assumption (1.3), the second integral is uniformly bounded as a function of x, so [6, Lemma 3] applied to K' gives the desired equicontinuity.
Finally, the uniqueness for the boundary problem

$$\begin{cases} \int_{\mathbb{R}^n} K(x, w)\,\delta v(x, w)\,dw = f(x, u(x)) & \text{in } B_{3/4}, \\ v = u & \text{in } \mathbb{R}^n \setminus B_{3/4}. \end{cases}$$

follows by a standard comparison principle argument (see for instance the argument used in the proof of [2, Theorem 3.2]).

so for any $x \in B_{1/4}$

$$\begin{aligned}\frac{2-\sigma}{2c_{n,\sigma}}(-\Delta)^{\sigma/2}u_\varepsilon(x) &= \int_{\mathbb{R}^n} \frac{2-\sigma}{|w|^{n+\sigma}}\delta u_\varepsilon(x,w)dw \\ &= f_\varepsilon(x) - \int_{\mathbb{R}^n}(1-\eta_\varepsilon(w))\hat{K}_\varepsilon(x,w)\,\delta u_\varepsilon(x,w)dw \\ &\quad + \int_{\mathbb{R}^n}(1-\eta_\varepsilon(w))\frac{2-\sigma}{|w|^{n+\sigma}}\,\delta u_\varepsilon(x,w)dw\end{aligned}$$

(here $c_{n,\sigma}$ is the positive constant that appears in the definition of the fractional Laplacian, see *e.g.* [11,16]). Then, for any $x \in B_{1/4}$ it follows that

$$\begin{aligned}&(-\Delta)^{\sigma/2}u_\varepsilon(x) \\ &= d_{n,\sigma}\left[f_\varepsilon(x) + \int_{\mathbb{R}^n}(1-\eta_\varepsilon(w))\left(\frac{2-\sigma}{|w|^{n+\sigma}} - \hat{K}_\varepsilon(x,w)\right)\delta u_\varepsilon(x,w)dw\right] \\ &=: d_{n,\sigma}[f_\varepsilon(x) + h_\varepsilon(x)] \\ &=: d_{n,\sigma}\, g_\varepsilon(x),\end{aligned} \tag{2.22}$$

with $d_{n,\sigma} := \dfrac{2c_{n,\sigma}}{2-\sigma}$.

Making some changes of variables we can rewrite h_ε as follows:

$$\begin{aligned}h_\varepsilon(x) =& \int_{\mathbb{R}^n}(1-\eta_\varepsilon(w-x))\left(\frac{2-\sigma}{|w-x|^{n+\sigma}} - \hat{K}_\varepsilon(x,w-x)\right)u_\varepsilon(w)dw \\ &+ \int_{\mathbb{R}^n}(1-\eta_\varepsilon(x-w))\left(\frac{2-\sigma}{|w-x|^{n+\sigma}} - \hat{K}_\varepsilon(x,x-w)\right)u_\varepsilon(w)dw \\ &- 2u_\varepsilon(x)\int_{\mathbb{R}^n}(1-\eta_\varepsilon(w))\left(\frac{2-\sigma}{|w|^{n+\sigma}} - \hat{K}_\varepsilon(x,w)\right)dw.\end{aligned} \tag{2.23}$$

We now notice that "the function h_ε is locally as smooth as u_ε", is the sense that for any $m \in \mathbb{N}$ and $U \subset B_{1/4}$ open we have

$$\|h_\varepsilon\|_{C^m(U)} \leqslant C_{\varepsilon,m}\left(1 + \|u_\varepsilon\|_{C^m(U)}\right) \tag{2.24}$$

for some constant $C_{\varepsilon,m} > 0$. To see this observe that, in the first two integrals, the variable x appears only inside η_ε and in the kernel $\hat{K}_\varepsilon$, and η_ε is equal to 1 near the origin. Hence the first two integrals are smooth functions of x (recall that $\hat{K}_\varepsilon$ is smooth, see (2.13)). The third term is clearly as regular as u_ε because the third integral is smooth by the same reason as before. This proves (2.24).

We are now going to prove that the functions u_ε belong to $C^\infty(B_{1/5})$, with

$$\|u_\varepsilon\|_{C^m(B_{1/4-r_m})} \leqslant C(r_1, m, \sigma, \varepsilon, \|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)}, \|f\|_{L^\infty(B_1\times\mathbb{R})}) \tag{2.25}$$

for any $m \in \mathbb{N}$, where $r_m := 1/20 - 25^{-m}$ (so that $1/4 - r_m > 1/5$ for any m).

To show this, we begin by observing that, since $\sigma \in (1,2)$, by (2.22), (2.24), and [6, Theorem 61], we have that $u_\varepsilon \in L^\infty(\mathbb{R}^n) \cap C^{1,\beta}(B_{1/4-r_1})$ for any $\beta < \sigma - 1$ ($r_1 = 1/100$), and

$$\|u_\varepsilon\|_{C^{1,\beta}(B_{1/4-r_1})} \leqslant C_\varepsilon\Big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1\times\mathbb{R})}\Big). \tag{2.26}$$

Now, to get a bound on higher derivatives, the idea would be to differentiate (2.22) and use again (2.24) and [6, Theorem 61]. However we do not have C^1 bounds on the function u_ε outside $B_{1/4-r_1}$, and therefore we can not apply directly this strategy to obtain the $C^{2,\alpha}$ regularity of the function u_ε.

To avoid this problem we follow the localization argument in [5, Theorem 13.1]: we take $\delta > 0$ small (to be chosen) and we consider a smooth cut-off function

$$\vartheta := \begin{cases} 1 & \text{in } B_{1/4-(1+\delta)r_1}, \\ 0 & \text{on } \mathbb{R}^n \setminus B_{1/4-r_1}, \end{cases}$$

and for fixed $e \in S^{n-1}$ and $|h| < \delta r_1$ we define

$$v(x) := \frac{u_\varepsilon(x+eh) - u_\varepsilon(x)}{|h|}. \tag{2.27}$$

The function $v(x)$ is uniformly bounded in $B_{1/4-(1+\delta)r_1}$ because $u \in C^1(B_{1/4-r_1})$. We now write $v(x) = v_1(x) + v_2(x)$, being

$$v_1(x) := \frac{\vartheta u_\varepsilon(x+eh) - \vartheta u_\varepsilon(x)}{|h|} \quad \text{and} \quad v_2(x) := \frac{(1-\vartheta)u_\varepsilon(x+eh) - (1-\vartheta)u_\varepsilon(x)}{|h|}.$$

By (2.26) it is clear that

$$v_1 \in L^\infty(\mathbb{R}^n)$$

and that (recall that $|h| < \delta r_1$)

$$v_1 = v \qquad \text{inside } B_{1/4-(1+2\delta)r_1}. \tag{2.28}$$

Moreover, for $x \in B_{1/4-(1+2\delta)r_1}$, using (2.22), (2.15), and (2.24) we get

$$\begin{aligned} \Big|(-\Delta)^{\sigma/2} v_1(x)\Big| &= \Big|(-\Delta)^{\sigma/2} v(x) - (-\Delta)^{\sigma/2} v_2(x)\Big| \\ &= \left|\frac{g_\varepsilon(x+eh) - g_\varepsilon(x)}{|h|} - (-\Delta)^{\sigma/2} v_2(x)\right| \\ &\leqslant C_\varepsilon\Big(1 + \|u_\varepsilon\|_{C^1(B_{1/4-r_1})}\Big) + \Big|(-\Delta)^{\sigma/2} v_2(x)\Big|. \end{aligned} \tag{2.29}$$

Now, let us denote by $K_o(y) := \frac{c_{n,\sigma}}{|y|^{n+\sigma}}$ the kernel of the fractional Laplacian. Since for $x \in B_{1/4-(1+2\delta)r_1}$ and $|y| < \delta r_1$ we have that $(1-\vartheta)u_\varepsilon(x \pm y) = 0$, it follows from a change of variable that

$$
\begin{aligned}
|(-\Delta)^{\sigma/2} v_2(x)| &\leqslant \left| \int_{\mathbb{R}^n} (v_2(x+y) + v_2(x-y) - 2v_2(x)) K_o(y) dy \right| \\
&\leqslant \left| \int_{\mathbb{R}^n} \frac{(1-\vartheta)u_\varepsilon(x+y+eh) - (1-\vartheta)u_\varepsilon(x+y)}{|h|} K_o(y) dy \right| \\
&\quad + \left| \int_{\mathbb{R}^n} \frac{(1-\vartheta)u_\varepsilon(x-y+eh) - (1-\vartheta)u_\varepsilon(x-y)}{|h|} K_o(y) dy \right| \\
&\leqslant \int_{\mathbb{R}^n} (1-\vartheta)|u_\varepsilon|(x+y) \left| \frac{K_o(y-eh) - K_o(y)}{|h|} \right| dy \\
&\quad + \int_{\mathbb{R}^n} (1-\vartheta)|u_\varepsilon|(x-y) \left| \frac{K_o(y-eh) - K_o(y)}{|h|} \right| dy \\
&\leqslant \|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)} \int_{\mathbb{R}^n \setminus B_{\delta r_1}} \frac{1}{|y|^{n+\sigma+1}} dy \\
&\leqslant C_\delta \|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)}.
\end{aligned}
$$

Therefore, by (2.29) we obtain

$$
\left|(-\Delta)^{\sigma/2} v_1(x)\right| \leqslant C_{\varepsilon,\delta}\Big(1 + \|u_\varepsilon\|_{C^1(B_{1/4-r_1})} + \|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\Big), \quad x \in B_{1/4-(1+2\delta)r_1},
$$

and we can apply [6, Theorem 61] to get that $v_1 \in C^{1,\beta}(B_{1/4-r_2})$ for any $\beta < \sigma - 1$, with

$$
\|v_1\|_{C^{1,\beta}(B_{1/4-r_2})} \leqslant C_\varepsilon\Big(1 + \|v_1\|_{L^\infty(\mathbb{R}^n)} + \|u_\varepsilon\|_{C^1(B_{1/4-r_1})} + \|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\Big),
$$

provided $\delta > 0$ was chosen sufficiently small so that $r_2 > (1+2\delta)r_1$. By (2.27), (2.28), and (2.26), this implies that $u_\varepsilon \in C^{2,\beta}(B_{1/4-r_2})$, with

$$
\begin{aligned}
\|u_\varepsilon\|_{C^{2,\beta}(B_{1/4-r_2})} &\leqslant C_\varepsilon\Big(1 + \|u_\varepsilon\|_{C^1(B_{1/4-r_1})} + \|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\Big) \\
&\leqslant C_\varepsilon\Big(1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1 \times \mathbb{R})}\Big).
\end{aligned}
$$

Iterating this argument we obtain (2.25), as desired.

2.4. Uniform estimates and conclusion of the proof for $k = 0$

Knowing now that the functions u_ε defined by (2.16) are smooth inside $B_{1/5}$ (see (2.25)), our goal is to obtain a-priori bounds independent of ε.

By [6, Theorem 61] applied[6] to u, we have that $u \in C^{1,\beta}(B_{1-R_1})$ for any $\beta < \sigma - 1$ and $R_1 > 0$, with

$$
\|u\|_{C^{1,\beta}(B_{1-R_1})} \leqslant C\left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1 \times \mathbb{R})}\right). \tag{2.30}
$$

[6] As already observed in the footnote on page 622, the fact that the kernel satisfies (2.19) only for small w is not a problem, and one can easily check that [6, Theorem 61] still holds in our setting.

Then, for any ε sufficiently small, $f_\varepsilon \in C^1(B_{1/2})$ with

$$\begin{aligned} \|f_\varepsilon\|_{C^1(B_{1/2})} &\leqslant C'\left(1+\|u\|_{C^1(B_{1-R_1})}\right) \\ &\leqslant C'C\left(1+\|u\|_{L^\infty(\mathbb{R}^n)}+\|f\|_{L^\infty(B_1\times\mathbb{R})}\right), \end{aligned} \tag{2.31}$$

where $C' > 0$ depends on $\|f\|_{C^1(B_1\times\mathbb{R})}$ only.

Consider a cut-off function $\tilde\eta$ which is 1 inside $B_{1/7}$ and 0 outside $B_{1/6}$.

Then, recalling (2.16), we write the equation satisfied by u_ε as

$$f_\varepsilon(x) = \int_{\mathbb{R}^n} K_\varepsilon(x,w)\,\delta(\tilde\eta u_\varepsilon)(x,w)dw + \int_{\mathbb{R}^n} K_\varepsilon(x,w)\,\delta((1-\tilde\eta)u_\varepsilon)(x,w)dw,$$

and by differentiating it, say in direction e_1, we obtain (recall Lemmata 2.5 and 2.6)

$$\begin{aligned} \partial_{x_1} f_\varepsilon(x) = &\int_{\mathbb{R}^n} K_\varepsilon(x,w)\delta(\partial_{x_1}(\tilde\eta u_\varepsilon))(x,w)dw \\ &+\int_{\mathbb{R}^n} \partial_{x_1}\big[K_\varepsilon(x,w)\delta((1-\tilde\eta)u_\varepsilon)(x,w)\big]dw \\ &+\int_{\mathbb{R}^n} \partial_{x_1}K_\varepsilon(x,w)\delta(\tilde\eta u_\varepsilon)(x,w)dw \end{aligned}$$

for any $x \in B_{1/5}$. It is convenient to rewrite this equation as

$$\int_{\mathbb{R}^n} K_\varepsilon(x,w)\delta(\partial_{x_1}(\tilde\eta u_\varepsilon))(x,w)dw = A_1 - A_2 - A_3,$$

with

$$\begin{aligned} A_1 &:= \partial_{x_1} f_\varepsilon(x), \\ A_2 &:= \int_{\mathbb{R}^n} \partial_{x_1}K_\varepsilon(x,w)\delta(\tilde\eta u_\varepsilon)(x,w)dw \\ A_3 &:= \int_{\mathbb{R}^n} \partial_{x_1}\big[K_\varepsilon(x,w)\delta((1-\tilde\eta)u_\varepsilon)(x,w)\big]dw. \end{aligned}$$

We claim that

$$\|A_1 - A_2 - A_3\|_{L^\infty(B_{1/14})} \leqslant C\left(1+\|u\|_{L^\infty(\mathbb{R}^n)}+\|u_\varepsilon\|_{C^2(B_{1/6})}\right) \tag{2.32}$$

with C depending only on $\|f\|_{C^1(B_1\times\mathbb{R})}$. To prove this, we first observe that by (2.31)

$$\|A_1\|_{L^\infty(B_{1/14})} \leqslant C\left(1+\|u\|_{L^\infty(\mathbb{R}^n)}\right).$$

Also, since $|\partial_{x_1}\hat{K}_\varepsilon(x,w)|\leqslant C|w|^{-(n+\sigma)}$,[7] by (2.9) (used with $\gamma=\lambda:=(1,0,\dots,0)$ and $v:=\tilde{\eta}u_\varepsilon$) we get

$$\|A_2\|_{L^\infty(B_{1/14})}\leqslant C\|\tilde{\eta}u_\varepsilon\|_{C^2(\mathbb{R}^n)}\leqslant C\|u_\varepsilon\|_{C^2(B_{1/6})},$$

where we used that $\tilde{\eta}$ is supported in $B_{1/6}$.

Moreover, since $(1-\tilde{\eta})u_\varepsilon=0$ inside $B_{1/7}$, we can use (2.11) with $v:=(1-\tilde{\eta})u_\varepsilon$ to obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}\partial_{x_1}K_\varepsilon(x,w)\,\delta((1-\tilde{\eta})u_\varepsilon)(x,w)\,dw\right|\\&\quad+\left|\int_{\mathbb{R}^n}K_\varepsilon(x,w)\,\partial_{x_1}\delta((1-\tilde{\eta})u_\varepsilon)(x,w)\,dw\right|\\&\leqslant C\,C_k\,\|(1-\tilde{\eta})u_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\end{aligned}$$

for any $x\in B_{1/14}$, which gives (note that, by an easy comparison principle, $\|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)}\leqslant C(1+\|u\|_{L^\infty(\mathbb{R}^n)})$)

$$\|A_3\|_{L^\infty(B_{1/14})}\leqslant C(1+\|u\|_{L^\infty(\mathbb{R}^n)}).$$

The above estimates imply (2.32).

Since $\partial_{x_1}(\tilde{\eta}u_\varepsilon)$ is bounded on the whole of $\mathbb{R}^n$, by (2.32) and [6, Theorem 61] we obtain that $\partial_{x_1}(\tilde{\eta}u_\varepsilon)\in C^{1,\beta}(B_{1/14-R_2})$ for any $R_2>0$, with

$$\|\partial_{x_1}(\tilde{\eta}u_\varepsilon)\|_{C^{1,\beta}(B_{1/14-R_2})}\leqslant C\left(1+\|u\|_{L^\infty(\mathbb{R}^n)}+\|u_\varepsilon\|_{C^2(B_{1/6})}\right),$$

which implies

$$\|u_\varepsilon\|_{C^{2,\beta}(B_{1/15})}\leqslant C\left(1+\|u\|_{L^\infty(\mathbb{R}^n)}+\|u_\varepsilon\|_{C^2(B_{1/6})}\right).\tag{2.33}$$

To end the proof we need to reabsorb the C^2-norm on the right hand side. To do this, we observe that by standard interpolation inequalities (see for instance [12, Lemma 6.35]), for any $\delta\in(0,1)$ there exists $C_\delta>0$ such that

$$\|u_\varepsilon\|_{C^2(B_{1/6})}\leqslant\delta\|u_\varepsilon\|_{C^{2,\beta}(B_{1/5})}+C_\delta\|u_\varepsilon\|_{L^\infty(\mathbb{R}^n)}.\tag{2.34}$$

Hence, by (2.33) and (2.34) we obtain

$$\|u_\varepsilon\|_{C^{2,\beta}(B_{1/15})}\leqslant C_\delta(1+\|u\|_{L^\infty(\mathbb{R}^n)})+C\delta\|u_\varepsilon\|_{C^{2,\beta}(B_{1/5})}.\tag{2.35}$$

[7] This can be easily checked using the definition of $\hat{K}_\varepsilon$ and (1.3). Indeed, because of the presence of the term $(1-\eta_\varepsilon(w))$ which vanishes for $|w|\leqslant\varepsilon/2$, one only needs to check that

$$\int_{\mathbb{R}^n}|w-z|^{-n-\sigma}\,\hat{\eta}_{\varepsilon^2}(z)\,dz\leqslant C|w|^{-n-\sigma}\qquad\text{for }|w|\geqslant\varepsilon/2,$$

which is easy to prove (we leave the details to the reader).

To conclude, one needs to apply the above estimates at every point inside $B_{1/5}$ at every scale: for any $x \in B_{1/5}$, let $r > 0$ be any radius such that $B_r(x) \subset B_{1/5}$. Then we consider

$$v^x_{\varepsilon,r}(y) := u_\varepsilon(x + ry), \tag{2.36}$$

and we observe that $v^x_{\varepsilon,r}$ solves an analogous equation as the one solved by u_ε with the kernel given by

$$K^x_{\varepsilon,r}(y, z) := r^{n+\sigma} K_\varepsilon(x + ry, rz)$$

and with right-hand side

$$F_{\varepsilon,r}(y) := r^\sigma \int_{\mathbb{R}^n} f(x + ry - \tilde{x}, u(x + ry - \tilde{x}))\hat{\eta}_\varepsilon(\tilde{x})d\tilde{x}.$$

We now observe that the kernels $K^x_{\varepsilon,r}$ satisfy assumptions (1.2) and (1.3) uniformly with respect to ε, r, and x. Moreover, for $|x| + r \leqslant 1/5$, and ε small, we have

$$\|F_{\varepsilon,r}\|_{C^1(B_{1/2})} \leqslant r^\sigma C(1 + \|u\|_{C^1(B_{3/4})}),$$

with $C > 0$ depending on $\|f\|_{C^1(B_1\times\mathbb{R})}$ only. Hence, by (2.30) this implies

$$\|F_{\varepsilon,r}\|_{C^1(B_{1/2})} \leqslant r^\sigma C \left(1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1\times\mathbb{R})}\right).$$

Thus, applying (2.35) to $v^x_{\varepsilon,r}$ (by the discussion we just made, the constants are all independent of ε, r, and x) and scaling back, we get

$$\|u_\varepsilon\|^*_{C^{2,\beta}(B_{r/15}(x))} \leqslant C_\delta \left(1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1\times\mathbb{R})}\right) + C\delta\|u_\varepsilon\|^*_{C^{2,\beta}(B_{r/5}(x))}.$$

Using now Lemma 2.2 inside $B_{1/5}$ with $\mu = 1/15$, $\nu = 1/5$, $m = 2$, and $\Lambda_\delta = C_\delta(1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1\times\mathbb{R})})$, we conclude (observe that $1/15 \cdot 1/5 = 1/75$)

$$\|u_\varepsilon\|_{C^{2,\beta}(B_{1/75})} \leqslant C \left(1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1\times\mathbb{R})}\right),$$

which implies

$$\|u\|_{C^{2,\beta}(B_{1/75})} \leqslant C \left(1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1\times\mathbb{R})}\right)$$

by letting $\varepsilon \to 0$ (see (2.20)). Since $\beta < \sigma - 1$, this is equivalent to

$$\|u\|_{C^{\sigma+\alpha}(B_{1/75})} \leqslant C \left(1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1\times\mathbb{R})}\right), \quad \text{for any } \alpha < 1.$$

A standard covering/rescaling argument completes the proof of Theorem 1.5 in the case $k = 0$.

2.5. The induction argument

We already proved Theorem 1.5 in the case $k = 0$.

We now show by induction that, for any $k \geqslant 1$,

$$\|u\|_{C^{k+\sigma+\alpha}(B_{1/2^{3k+4}})} \leqslant C_k \left(1 + \|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1\times\mathbb{R})}\right), \tag{2.37}$$

for some constant $C_k > 0$: by a standard covering/rescaling argument, this proves (1.6) and so Theorem 1.5. As we shall see, the argument is more or less identical to the case $k = 0$. To be fully rigorous, we should apply the regularization argument with the functions u_ε as done in the previous step. However, to simplify the notation and make the argument more transparent, we will skip the regularization.

Define $g(x) := f(x, u(x))$, and consider a cut-off function $\tilde{\eta}$ which is 1 inside $B_{1/2^{3k+5}}$ and 0 outside $B_{1/2^{3k+4}}$.

By Lemmata 2.5 and 2.6 we differentiate the equation $k+1$ times according to the following computation: first we observe that, since (2.37) is true for $k-1$ and we can choose $\alpha \in (2-\sigma, 1)$ so that $\sigma+\alpha > 2$, we deduce that $g \in C^{k+1}(B_{1/2^{3k+4}})$ with

$$\begin{aligned}\|g\|_{C^{k+1}(B_{1/2^{3k+4}})} &\leqslant C\left(1 + \|u\|_{C^{k+1}(B_{1/2^{3k+4}})}\right)\\ &\leqslant C\left(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(B_1\times\mathbb{R}^n)}\right),\end{aligned} \tag{2.38}$$

with $C > 0$ depending on $\|f\|_{C^{k+1}(B_1\times\mathbb{R})}$ only. Now we take $\gamma \in \mathbb{N}^n$ with $|\gamma| = k+1$ and we differentiate the equation to obtain

$$\begin{aligned}&\partial^\gamma g(x)\\ &= \sum_{\substack{1\leqslant i\leqslant n\\ 0\leqslant\lambda_i\leqslant\gamma_i\\ \lambda=(\lambda_1,\dots,\lambda_n)}} \binom{\gamma_1}{\lambda_1}\cdots\binom{\gamma_n}{\lambda_n}\int_{\mathbb{R}^n}\partial_x^\lambda K(x,w)\,\delta(\partial_x^{\gamma-\lambda}(\tilde{\eta}u))(x,w)\,dw\\ &\quad+\sum_{\substack{1\leqslant i\leqslant n\\ 0\leqslant\lambda_i\leqslant\gamma_i\\ \lambda=(\lambda_1,\dots,\lambda_n)}} \binom{\gamma_1}{\lambda_1}\cdots\binom{\gamma_n}{\lambda_n}\int_{\mathbb{R}^n}\partial_x^\lambda K(x,w)\,\delta(\partial_x^{\gamma-\lambda}(1-\tilde{\eta})u)(x,w)\,dw.\end{aligned}$$

Then, we isolate the term with $\lambda = 0$ in the first sum:

$$\int_{\mathbb{R}^n} K(x,w)\,\delta(\partial_x^\gamma(\tilde{\eta}u))(x,w)\,dw = A_1 - A_2 - A_3$$

with

$$
\begin{aligned}
A_1 &:= \partial^\gamma g(x),\\
A_2 &:= \sum_{\substack{1\leqslant i\leqslant n\\ 0\leqslant \lambda_i\leqslant \gamma_i\\ \lambda=(\lambda_1,\dots,\lambda_n)\neq 0}} \binom{\gamma_1}{\lambda_1}\cdots\binom{\gamma_n}{\lambda_n}\int_{\mathbb{R}^n}\partial_x^\lambda K(x,w)\,\delta(\partial_x^{\gamma-\lambda}(\tilde\eta u))(x,w)\,dw\\
A_3 &:= \sum_{\substack{1\leqslant i\leqslant n\\ 0\leqslant \lambda_i\leqslant \gamma_i\\ \lambda=(\lambda_1,\dots,\lambda_n)}} \binom{\gamma_1}{\lambda_1}\cdots\binom{\gamma_n}{\lambda_n}\int_{\mathbb{R}^n}\partial_x^\lambda K(x,w)\,\delta(\partial_x^{\gamma-\lambda}(1-\tilde\eta)u)(x,w)\,dw.
\end{aligned}
$$

We claim that

$$
\|A_1-A_2-A_3\|_{L^\infty(B_{1/2^{3k+6}})}\leqslant C\left(1+\|u\|_{L^\infty(\mathbb{R}^n)}+\|u\|_{C^{k+2}(B_{1/2^{3k+4}})}\right). \quad (2.39)
$$

Indeed, by the fact that $|\gamma-\lambda|\leqslant k$ we see that

$$
\begin{aligned}
\|A_2\|_{L^\infty(B_{1/2^{3k+6}})} &\leqslant C\,C_k\,\|\tilde\eta u\|_{C^{k+2}(\mathbb{R}^n)}\\
&\leqslant C\,C_k\,\|u\|_{C^{k+2}(B_{1/2^{3k+4}})}.
\end{aligned} \quad (2.40)
$$

Furthermore, since $(1-\tilde\eta)u=0$ inside $B_{1/2^{3k+5}}$, we can use (2.11) with $v:=(1-\tilde\eta)u$ to obtain

$$
\|A_3\|_{L^\infty(B_{1/2^{3k+6}})}\leqslant C\|u\|_{L^\infty(\mathbb{R}^n)}.
$$

This last estimate, (2.38), and (2.40) allow us to conclude the validity of (2.39).

Now, by [6, Theorem 61] applied to $\partial_x^\gamma(\tilde\eta u)$ we get

$$
\|u\|_{C^{\sigma+k+\alpha}(B_{1/2^{3k+7}})}\leqslant C\left(1+\|u\|_{C^{k+2}(B_{1/2^{3k+4}})}+\|u\|_{L^\infty(\mathbb{R}^n)}\right),
$$

which is the analogous of (2.33) with $\sigma+\alpha=2+\beta$. Hence, arguing as in the case $k=0$ (see the argument after (2.33)) we conclude that

$$
\|u\|_{C^{\sigma+k+\alpha}(B_{1/2^{3(2k+1)+5}})}\leqslant C\left(1+\|u\|_{L^\infty(\mathbb{R}^n)}+\|f\|_{L^\infty(B_1\times\mathbb{R})}\right).
$$

A covering argument shows the validity of (2.37), concluding the proof of Theorem 1.5.

3. Proof of Theorem 1.1

The idea of the proof is to write the fractional minimal surface equation in a suitable form so that we can apply Theorem 1.5.

3.1. Writing the operator on the graph of u

The first step of our proof consists in writing the s-minimal surface functional in terms of the function u which (locally) parameterizes the boundary of a set E. More precisely, we assume that u parameterizes $\partial E \cap K_R$ and that (without loss of generality) $E \cap K_R$ is contained in the ipograph of u. Moreover, since by assumption $u(0) = 0$ and u is of class $C^{1,\alpha}$, up to rotating the system of coordinates (so that $\nabla u(0) = 0$) and reducing the size of R, we can also assume that

$$\partial E \cap K_R \subset B_R^{n-1} \times [-R/8, R/8]. \tag{3.1}$$

Let $\varphi \in C^\infty(\mathbb{R})$ be an even function satisfying

$$\varphi(t) = \begin{cases} 1 & \text{if } |t| \leqslant 1/4, \\ 0 & \text{if } |t| \geqslant 1/2, \end{cases}$$

and define the smooth cut-off functions

$$\zeta_R(x') := \varphi(|x'|/R), \qquad \eta_R(x) := \varphi(|x'|/R)\varphi(|x_n|/R).$$

Observe that

$$\zeta_R = 1 \quad \text{in } B_{R/4}^{n-1}, \qquad \zeta_R = 0 \quad \text{outside } B_{R/2}^{n-1},$$

$$\eta_R = 1 \quad \text{in } K_{R/4}, \qquad \eta_R = 0 \quad \text{outside } K_{R/2}.$$

We claim that, for any $x \in \partial E \cap \left(B_{R/2}^{n-1} \times [-R/8, R/8]\right)$,

$$\begin{aligned} &\int_{\mathbb{R}^n} \eta_R(y-x) \frac{\chi_E(y) - \chi_{\mathbb{R}^n \setminus E}(y)}{|x-y|^{n+s}}\, dy \\ &= 2 \int_{\mathbb{R}^{n-1}} F\left(\frac{u(x'-w') - u(x')}{|w'|}\right) \frac{\zeta_R(w')}{|w'|^{n-1+s}}\, dw', \end{aligned} \tag{3.2}$$

where

$$F(t) := \int_0^t \frac{d\tau}{(1+\tau^2)^{(n+s)/2}}.$$

Indeed, writing $y = x - w$ we have (observe that η_R is even)

$$\begin{aligned} &\int_{\mathbb{R}^n} \eta_R(y-x) \frac{\chi_E(y) - \chi_{\mathbb{R}^n \setminus E}(y)}{|x-y|^{n+s}}\, dy \\ &= \int_{\mathbb{R}^n} \eta_R(w) \frac{\chi_E(x-w) - \chi_{\mathbb{R}^n \setminus E}(x-w)}{|w|^{n+s}}\, dw \\ &= \int_{\mathbb{R}^{n-1}} \zeta_R(w') \left[\int_{-R/4}^{R/4} \frac{\chi_E(x-w) - \chi_{\mathbb{R}^n \setminus E}(x-w)}{\left(1 + (w_n/|w'|)^2\right)^{(n+s)/2}}\, dw_n\right] \frac{dw'}{|w'|^{n+s}}, \end{aligned} \tag{3.3}$$

where the last equality follows from the fact that $\varphi(|w_n|/R) = 1$ for $|w_n| \leqslant R/4$, and that by (3.1) and by symmetry the contributions of $\chi_E(x-w)$ and $\chi_{\mathbb{R}^n\setminus E}(x-w)$ outside $\{|w_n| \leqslant R/4\}$ cancel each other.

We now compute the inner integral: using the change of variable $t := w_n/|w'|$ we have

$$\begin{aligned}
&\int_{-R/4}^{R/4} \frac{\chi_E(x-w)}{\big(1+(w_n/|w'|)^2\big)^{(n+s)/2}}\,dw_n \\
&= \int_{u(x')-u(x'-w')}^{R/4} \frac{1}{\big(1+(w_n/|w'|)^2\big)^{(n+s)/2}}\,dw_n \\
&= |w'| \int_{(u(x')-u(x'-w'))/|w'|}^{R/(4|w'|)} \frac{1}{\big(1+t^2\big)^{(n+s)/2}}\,dt \\
&= |w'| \left[F\left(\frac{R}{4|w|'}\right) - F\left(\frac{u(x')-u(x'-w')}{|w|'}\right)\right].
\end{aligned}$$

In the same way,

$$\begin{aligned}
&\int_{-R/4}^{R/4} \frac{\chi_{\mathbb{R}^n\setminus E}(x-w)}{\big(1+(w_n/|w'|)^2\big)^{(n+s)/2}}\,dw_n \\
&= |w'| \left[F\left(\frac{u(x')-u(x'-w')}{|w|'}\right) - F\left(-\frac{R}{4|w'|}\right)\right].
\end{aligned}$$

Therefore, since F is odd, we immediately get that

$$\int_{-R/4}^{R/4} \frac{\chi_E(x-w)-\chi_{\mathbb{R}^n\setminus E}(x-w)}{\big(1+(w_n/|w'|)^2\big)^{(n+s)/2}}\,dw_n = 2|w'| F\left(\frac{u(x'-w')-u(x')}{|w'|}\right),$$

which together with (3.3) proves (3.2).

Let us point out that to justify these computations in a pointwise fashion one would need $u \in C^{1,1}(x)$ (in the sense of [3, Definition 3.1]). However, by using the viscosity definition it is immediate to check that (3.2) holds in the viscosity sense (since one only needs to verify it at points where the graph of u can be touched with paraboloids).

3.2. The right-hand side of the equation

Let us define the function

$$\Psi_R(x) := \int_{\mathbb{R}^n} [1-\eta_R(y-x)]\, \frac{\chi_E(y)-\chi_{\mathbb{R}^n\setminus E}(y)}{|x-y|^{n+s}}\,dy. \tag{3.4}$$

Since $1 - \eta_R(y - x)$ vanishes on a neighborhood of $\{x = y\}$, it is immediate to check that the function $\psi_R(z) := \frac{1-\eta_R(z)}{|z|^{n+s}}$ is of class C^∞, with

$$|\partial^\alpha \psi_R(z)| \leqslant \frac{C_{|\alpha|}}{1 + |z|^{n+s}} \qquad \forall\, \alpha \in \mathbb{N}^n.$$

Hence, since $1/(1 + |z|^{n+s}) \in L^1(\mathbb{R}^n)$ we deduce that

$$\Psi_R \in C^\infty(\mathbb{R}^n), \text{ with all its derivatives uniformly bounded.} \tag{3.5}$$

3.3. An equation for u and conclusion

By [4, Theorem 5.1] we have that the equation

$$\int_{\mathbb{R}^n} \frac{\chi_E(y) - \chi_{\mathbb{R}^n \setminus E}(y)}{|x - y|^{n+s}}\, dy = 0$$

holds in the viscosity sense for any $x \in (\partial E) \cap K_R$. Consequently, by (3.2) and (3.4) we deduce that u is a viscosity solution of

$$\int_{\mathbb{R}^{n-1}} F\left(\frac{u(x' - w') - u(x')}{|w'|}\right) \frac{\zeta_R(w')}{|w'|^{n-1+s}}\, dw' = -\frac{\Psi_R(x', u(x'))}{2}$$

inside $B^{n-1}_{R/2}$. Since F is odd, we can add the term $F\big(-\nabla u(x') \cdot \frac{w'}{|w'|}\big)$ inside the integral in the left hand side (since it integrates to zero), so the equation actually becomes

$$\begin{aligned}&\int_{\mathbb{R}^{n-1}} \left[F\left(\frac{u(x' - w') - u(x')}{|w'|}\right) - F\left(-\nabla u(x') \cdot \frac{w'}{|w'|}\right)\right] \frac{\zeta_R(w')}{|w'|^{n-1+s}}\, dw' \\ &= -\frac{\Psi_R(x', u(x'))}{2}.\end{aligned} \tag{3.6}$$

We would like to apply the regularity result from Theorem 1.6, exploiting (3.5) to bound the right-hand side of (3.6). To this aim, using the Fundamental Theorem of Calculus, we rewrite the left hand side in (3.6) as

$$\int_{\mathbb{R}^{n-1}} \big(u(x' - w') - u(x') + \nabla u(x') \cdot w'\big) \frac{a(x', -w')\zeta_R(w')}{|w'|^{n+s}}\, dw', \tag{3.7}$$

where

$$a(x', -w') := \int_0^1 \left(1 + \left(t\frac{u(x' - w') - u(x')}{|w'|} - (1-t)\nabla u(x') \cdot \frac{w'}{|w'|}\right)^2\right)^{-(n+s)/2} dt.$$

Now, we claim that

$$\int_{\mathbb{R}^{n-1}} \delta u(x', w')\, K_R(x', w')\, dw' = -\Psi_R(x', u(x')) + A_R(x'), \tag{3.8}$$

where

$$K_R(x', w') := \frac{[a(x', w') + a(x', -w')]\zeta_R(w')}{2|w'|^{(n-1)+(1+s)}}$$

and

$$A_R(x') := \int_{\mathbb{R}^{n-1}} [u(x'-w') - u(x') + \nabla u(x') \cdot w'] \frac{[a(x', w') - a(x', -w')]\zeta_R(w')}{|w'|^{n+s}}\, dw'.$$

To prove (3.8) we introduce a short-hand notation: we define

$$u^{\pm}(x', w') := u(x' \pm w') - u(x') \mp \nabla u(x') \cdot w', \quad a^{\pm}(x', w') := a(x', \pm w') \frac{\zeta_R(w')}{|w'|^{n+s}},$$

while the integration over $\mathbb{R}^{n-1}$, possibly in the principal value sense, will be denoted by $I[\cdot]$. With this notation, and recalling (3.7), it follows that (3.6) can be written

$$-\frac{\Psi_R}{2} = I[u^- a^-]. \tag{3.9}$$

By changing w' with $-w'$ in the integral given by I, we see that

$$I[u^+ a^+] = I[u^- a^-],$$

consequently (3.9) can be rewritten as

$$-\frac{\Psi_R}{2} = I[u^+ a^+]. \tag{3.10}$$

Notice also that

$$u^+ + u^- = \delta u, \qquad I[u^+(a^+ - a^-)] = I[u^-(a^- - a^+)]. \tag{3.11}$$

Hence, adding (3.9) and (3.10), and using (3.11), we obtain

$$\begin{aligned} -\Psi_R &= I[u^+ a^+] + I[u^- a^-] \\ &= \frac{1}{2} I[(u^+ + u^-)(a^+ + a^-)] + \frac{1}{2} I[(u^+ - u^-)(a^+ - a^-)] \\ &= \frac{1}{2} I[\delta u\, (a^+ + a^-)] + \frac{1}{2} I[(u^+ - u^-)(a^+ - a^-)] \\ &= \frac{1}{2} I[\delta u\, (a^+ + a^-)] - I[u^-(a^+ - a^-)], \end{aligned}$$

which proves (3.8).

Now, to conclude the proof of Theorem 1.1 it suffices to apply Theorem 1.6 iteratively: more precisely, let us start by assuming that $u \in C^{1,\beta}(B^{n-1}_{2r})$ for some $r \leqslant R/2$ and any $\beta < s$. Then, by the discussion above we get that u solves

$$\int_{\mathbb{R}^{n-1}} \delta u(x', w')\, K_r(x', w')\, dw' = -\Psi_r(x', u(x')) + A_r(x) \qquad \text{in } B^{n-1}_r.$$

Moreover, one can easily check that the regularity of u implies that the assumptions of Theorem 1.6 with $k = 0$ are satisfied with $\sigma := 1 + s$ and $a_0 := 1/(1 - s)$. (Observe that (1.7) holds since $\|u\|_{C^{1,\beta}(B^{n-1}_{2r})}$.) Furthermore, it is not difficult to check that, for $|w'| \leqslant 1$,

$$\big|[u(x' - w') - u(x') + \nabla u(x') \cdot w']\,[a(x', w') - a(x', -w')]\big| \leqslant C|w'|^{2\beta+1},$$

which implies that the integral defining A_r is convergent by choosing $\beta > s/2$. Furthermore, a tedious computation (which we postpone to Subsection 3.4 below) shows that

$$A_r \in C^{2\beta-s}(B^{n-1}_r). \tag{3.12}$$

Hence, by Theorem 1.6 with $k = 0$ we deduce that $u \in C^{1,2\beta}(B^{n-1}_{r/2})$. But then this implies that $A_r \in C^{4\beta-s}(B^{n-1}_{r/4})$ and so by Theorem 1.6 again $u \in C^{1,4\beta}(B^{n-1}_{r/8})$ for all $\beta < s$. Iterating this argument infinitely many times[8] we get that $u \in C^m(B^{n-1}_{\lambda^m r})$ for some $\lambda > 0$ small, for any $m \in \mathbb{N}$. Then, by a simple covering argument we obtain that $u \in C^m(B^{n-1}_\rho)$ for any $\rho < R$ and $m \in \mathbb{N}$, that is, u is of class C^∞ inside B_ρ for any $\rho < R$. This completes the proof of Theorem 1.1.

3.4. Hölder regularity of A_R

We now prove (3.12), *i.e.*, if $u \in C^{1,\beta}(B^{n-1}_{2r})$ then $A_r \in C^{2\beta-s}(B^{n-1}_r)$ ($r \leqslant R/2$). For this we introduce the following notation:

$$U(x', w') := u(x' - w') - u(x') + \nabla u(x') \cdot w'$$

and

$$p(\tau) := \frac{1}{(1 + \tau^2)^{\frac{n+s}{2}}}.$$

[8] Note that, once we know that $\|u\|_{C^{k,\beta}(B^{n-1}_{2r})}$ is bounded for some $k \geqslant 2$ and $\beta \in (0, 1]$, for any $|\gamma| \leqslant k - 1$ we get

$$\partial_x^\gamma A_r(x) = \int_{\mathbb{R}^{n-1}} \partial_x^\gamma \big([u(x' - w') - u(x') + \nabla u(x') \cdot w']\,[a(x', w') - a(x', -w')]\big) \frac{\zeta_r(w')}{|w'|^{n+s}}\, dw',$$

and exactly as in the case $k = 0$ one shows that

$$\big|\partial_x^\gamma \big([u(x' - w') - u(x') + \nabla u(x') \cdot w']\,[a(x', w') - a(x', -w')]\big)\big| \leqslant C|w'|^{2\beta+1} \quad \forall\, |w'| \leqslant 1,$$

and that $A_r \in C^{k-1,2\beta-s}(B^{n-1}_r)$.

In this way we can write

$$a(x', -w') = \int_0^1 p\left(t \frac{u(x'-w') - u(x')}{|w'|} - (1-t)\nabla u(x') \cdot \frac{w'}{|w'|}\right) dt. \quad (3.13)$$

Let us define

$$\mathcal{A}(x', w') := a(x', w') - a(x', -w').$$

Then we have

$$A_r(x') = \int_{\mathbb{R}^{n-1}} U(x', w') \frac{\mathcal{A}(x', w')}{|w'|^{n+s}} \zeta_r(w')\, dw'.$$

To prove the desired Hölder condition for the function $A_r(x')$, we first note that

$$U(x', w') = \int_0^1 \big[\nabla u(x') - \nabla u(x' - tw')\big] dt \; \cdot w'.$$

Since $u \in C^{1,\beta}(B_R^{n-1})$ and $2r \leqslant R$, we get

$$|U(x', w') - U(y', w')| \leqslant C \, \min\{|x'-y'|^\beta |w'|, |w'|^{\beta+1}\}, \quad \text{for } y' \in B_r^{n-1} \quad (3.14)$$

and

$$|U(x', w')| \leqslant C|w'|^{\beta+1}. \quad (3.15)$$

Therefore, from (3.14) and (3.15) it follows that, for any $y' \in B_r^{n-1}$,

$$\begin{aligned} &|A_r(x') - A_r(y')| \\ &= \left| \int_{\mathbb{R}^{n-1}} \big(U(x', w')\mathcal{A}(x', w') - U(y', w')\mathcal{A}(y', w')\big) \frac{\zeta_r(w')}{|w'|^{n+s}}\, dw' \right| \\ &\leqslant C \int_{\mathbb{R}^{n-1}} \min\{|x'-y'|^\beta |w'|, |w'|^{\beta+1}\} \frac{|\mathcal{A}(x', w')|}{|w'|^{n+s}} \zeta_r(w')\, dw' \\ &\quad + C \int_{\mathbb{R}^{n-1}} |w'|^{\beta+1} \frac{|\mathcal{A}(x', w') - \mathcal{A}(y', w')|}{|w'|^{n+s}} \zeta_r(w')\, dw' \\ &=: I_1(x', y') + I_2(x', y'). \end{aligned} \quad (3.16)$$

To estimate the last two integrals we define

$$\mathcal{A}_*(x', w') := a(x', w') - p\left(\nabla u(x') \cdot \frac{w'}{|w'|}\right).$$

With this notation we have

$$\mathcal{A}(x', w') = \mathcal{A}_*(x', w') - \mathcal{A}_*(x', -w'). \quad (3.17)$$

By (3.13) and (3.15), since $|p'(t)| \leqslant C$ and p is even, it follows that

$$\begin{aligned}
&|\mathcal{A}_*(x', -w')| \\
&\leqslant \int_0^1 \int_0^1 \left| \frac{d}{d\lambda} p\Big(\lambda t \frac{u(x'-w') - u(x')}{|w'|} \right. \\
&\qquad\qquad \left. - [\lambda(1-t) + (1-\lambda)] \nabla u(x') \cdot \frac{w'}{|w'|}\Big) \right| d\lambda\, dt \qquad (3.18)\\
&\leqslant \int_0^1 t \frac{|U(x', w')|}{|w'|} \Big(\int_0^1 \Big| p'\Big(\lambda t \frac{U(x', w')}{|w'|} - \nabla u(x') \cdot \frac{w'}{|w'|}\Big)\Big| d\lambda \Big) dt \\
&\leqslant C|w'|^\beta
\end{aligned}$$

for all $|w'| \leqslant r$.

Estimating $\mathcal{A}_*(x', w')$ in the same way, by (3.17) and (3.18), we get, for any $\beta > s/2$,

$$\begin{aligned}
I_1(x', y') &\leqslant C \int_{\mathbb{R}^{n-1}} \min\{|x'-y'|^\beta |w'|, |w'|^{\beta+1}\} |w'|^{\beta-n-s} \zeta_r(w')\, dw' \\
&\leqslant C|x'-y'|^\beta \int_{|x'-y'|}^r t^{\beta-s-1} dt + \int_0^{|x'-y'|} t^{2\beta-s-1}\, dt \qquad (3.19)\\
&\leqslant C|x'-y'|^{2\beta-s}.
\end{aligned}$$

On the other hand, to estimate I_2 we note that

$$\begin{aligned}
|\mathcal{A}(x', w') - \mathcal{A}(y', w')| &\leqslant |\mathcal{A}_*(x', w') - \mathcal{A}_*(y', w')| \\
&\quad + |\mathcal{A}_*(y', -w') - \mathcal{A}_*(x', -w')|. \qquad (3.20)
\end{aligned}$$

Hence, arguing as in (3.18) we have

$$\begin{aligned}
&|\mathcal{A}_*(x', -w') - \mathcal{A}_*(y', -w')| \\
&\leqslant \int_0^1 t \frac{|U(x', w')|}{|w'|} \int_0^1 \Big| p'\Big(\lambda t \frac{U(x', w')}{|w'|} - \nabla u(x') \cdot \frac{w'}{|w'|}\Big) \\
&\qquad\qquad - p'\Big(\lambda t \frac{U(y', w')}{|w'|} - \nabla u(y') \cdot \frac{w'}{|w'|}\Big)\Big| d\lambda\, dt \qquad (3.21)\\
&\quad + \int_0^1 t \frac{|U(x',w') - U(y',w')|}{|w'|} \int_0^1 \Big| p'\Big(\lambda t \frac{U(y',w')}{|w'|} - \nabla u(y') \cdot \frac{w'}{|w'|}\Big)\Big| d\lambda\, dt \\
&=: I_{2,1}(x', y') + I_{2,2}(x', y').
\end{aligned}$$

We bound each of these integrals separately. First, since $|p'(t)| \leqslant C$, it follows immediately from (3.14) that

$$I_{2,2}(x', y') \leqslant C \min\{|x'-y'|^\beta, |w'|^\beta\}. \qquad (3.22)$$

On the other hand, by (3.15), (3.14), and the fact that $u \in C^{1,\beta}(B_R^{n-1})$ and p' is uniformly Lipschitz, we get

$$\begin{aligned} I_{2,1}(x', y') &\leqslant C|w'|^\beta\left(\frac{|U(x', w') - U(y', w')|}{|w'|} + |\nabla u(x') - \nabla u(y')|\right) \\ &\leqslant C|w'|^\beta\Big(\min\{|x'-y'|^\beta, |w'|^\beta\} + |x'-y'|^\beta\Big) \\ &\leqslant C|w'|^\beta|x'-y'|^\beta. \end{aligned} \tag{3.23}$$

Then, assuming without loss of generality $r \leqslant 1$ (so that also $|x'-y'| \leqslant 1$), by (3.21), (3.22), and (3.23) it follows that

$$\begin{aligned} |\mathcal{A}_*(x',-w')-\mathcal{A}_*(y',-w')| &\leqslant C\left(\min\{|x'-y'|^\beta,|w'|^\beta\}+|w'|^\beta|x'-y'|^\beta\right) \\ &\leqslant C\min\{|x'-y'|^\beta, |w'|^\beta\}. \end{aligned} \tag{3.24}$$

As $|\mathcal{A}_*(y', w') - \mathcal{A}_*(x', w')|$ is bounded in the same way, by (3.20), we have

$$|\mathcal{A}(x', w') - \mathcal{A}(y', w')| \leqslant C\min\{|x'-y'|^\beta, |w'|^\beta\}.$$

By arguing as in (3.19), we get that, for any $s/2 < \beta < s$,

$$\begin{aligned} I_2(x', y') &\leqslant C\int_{\mathbb{R}^{n-1}} |w'|^{\beta+1}\frac{\min\{|x'-y'|^\beta, |w'|^\beta\}}{|w'|^{n+s}}\zeta_r(w')dw' \\ &\leqslant C|x'-y'|^{2\beta-s}. \end{aligned} \tag{3.25}$$

Finally, by (3.16), (3.19) and (3.25), we conclude that

$$|A_r(x') - A_r(y')| \leqslant C|x'-y'|^{2\beta-s}, \quad y' \in B_r^{n-1},$$

as desired.

References

[1] L. AMBROSIO, G. DE PHILIPPIS and L. MARTINAZZI, *Gamma-convergence of nonlocal perimeter functionals*, Manuscripta Math. **134** (2011), 377–403.
[2] C. BJORLAND, L. CAFFARELLI and A. FIGALLI, *Non-local tug-of-war and the infinity fractional Laplacian*, Comm. Pure Applied Math. **65** (2012), 337–380.
[3] C. BJORLAND, L. CAFFARELLI and A. FIGALLI, *Non-local gradient dependent operators*, Adv. Math. **230** (2012), 1859–1894.
[4] L. CAFFARELLI, J.-M. ROQUEJOFFRE and O. SAVIN, *Nonlocal minimal surfaces*, Comm. Pure Appl. Math. **63** (2010), 1111–1144.
[5] L. CAFFARELLI and L. SILVESTRE, *Regularity theory for fully nonlinear integro-differential equations*, Comm. Pure Appl. Math. **62** (2009), 597–638.
[6] L. CAFFARELLI and L. SILVESTRE, *Regularity results for nonlocal equations by approximation*, Arch. Ration. Mech. Anal. **200** (2011), 59–88.

[7] L. CAFFARELLI and L. SILVESTRE, *The Evans-Krylov theorem for nonlocal fully nonlinear equations*, Ann. of Math., **174** (2011), 1163–1187.
[8] L. CAFFARELLI and P. E. SOUGANIDIS, *Convergence of nonlocal threshold dynamics approximations to front propagation*, Arch. Ration. Mech. Anal. **195** (2010), 1–23.
[9] L. CAFFARELLI and E. VALDINOCI, *Uniform estimates and limiting arguments for nonlocal minimal surfaces*, Calc. Var. Partial Differential Equations **41** (2011), 203–240.
[10] L. CAFFARELLI and E. VALDINOCI, *Regularity properties of nonlocal minimal surfaces via limiting arguments*, Adv. Math. **248** (2013), 843–871.
[11] E. DI NEZZA, G. PALATUCCI and E. VALDINOCI, *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math. **136** (2012), 521–573.
[12] D. GILBARG and N. S. TRUDINGER, "Elliptic Partial Differential Equations of Second Order", Classics in Mathematics, Springer, Berlin, 2001.
[13] O. SAVIN and E. VALDINOCI, *Density estimates for a variational model driven by the Gagliardo norm*, J. Math. Pures Appl. **101** (2014), 1–26.
[14] O. SAVIN and E. VALDINOCI, *Γ-convergence for nonlocal phase transitions*, Ann. Inst. H. Poincaré Anal. Non Linéaire **29** (2012), 479–500.
[15] O. SAVIN and E. VALDINOCI, *Regularity of nonlocal minimal cones in dimension* 2, Calc. Var. Partial Differential Equations **48** (2013), 33–39.
[16] L. SILVESTRE, "Regularity of the Obstacle Problem for a Fractional Power of the Laplace Operator", Ph.D. Thesis, Austin University, 2005, available at http://www.math.uchicago.edu/~luis/preprints/luisdissreadable.pdf.

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