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The corona theorem for weighted Hardy and Morrey spaces

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Abstract. The main goal of this paper is to give a unified proof of the corona theorem for weighted Hardy spaces and for Morrey spaces. We use a technique that reduces the problem to the weighted Hardy spaces $H^2(\theta)$.

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1. Introduction

The corona theorem for a space X of holomorphic functions on the unit ball $\mathbb{B}$ of $\mathbb{C}^n$ consists in proving that if $g_1, \ldots, g_m$ are pointwise multipliers of X satisfying the corona condition

$$\inf\{|g(z)|^2 = |g_1(z)|^2 + \cdots + |g_m(z)|^2 : z \in \mathbb{B}\} > 0, \tag{1.1}$$

then the map $\mathcal{M}_g : X \times \cdots \times X \to X$ defined by $(f_1, \ldots, f_m) \to g_1 f_1 + \ldots + g_m f_m$ is onto.

In this work we give a unified proof of this corona theorem for weighted Hardy and Morrey spaces of holomorphic functions on $\mathbb{B}$. The methods we use to prove our main result give a simplified proof of the well-known unweighted corona theorem for Hardy spaces.

Before we state our main results, we introduce some notation and definitions. Let $\mathbb{S}$ be the unit sphere in $\mathbb{C}^n$ and let $d\nu$ and $d\sigma$ denote the corresponding Lebesgue measures on $\mathbb{B}$ and $\mathbb{S}$ respectively. For $1 \le p < \infty$ we denote by $\mathcal{A}_p$ the Muckenhoupt class of weights on $\mathbb{S}$. For $1 < p < \infty$, and $\theta \in \mathcal{A}_p$, the Hardy space $H^p(\theta)$ consists of the holomorphic functions f on $\mathbb{B}$ satisfying

$$\|f\|_{H^p(\theta)} = \left(\sup_r \int_{\mathbb{S}} |f(r\zeta)|^p \theta(\zeta) d\sigma(\zeta)\right)^{1/p} < \infty. \tag{1.2}$$

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For $1 < p < \infty$ and $-1 < s \le n/p$, we also define the Morrey-Campanato space $M^{p,s}$ on $\mathbb{S}$ given by

$$M^{p,s} = \left\{ f \in L^p(\mathbb{S}) : \|f\|_{p,s} < \infty \right\},$$

where

$$\|f\|_{p,s} = \|f\|_p + \sup_{I_{\zeta,\varepsilon}} \left(\varepsilon^{sp-n} \int_{I_{\zeta,\varepsilon}} |f(\eta) - f(\zeta)|^p d\sigma(\eta) \right)^{1/p}, \tag{1.3}$$

$\|f\|_p$ denotes the $L^p(d\sigma)$-norm of f and $I_{\zeta,\varepsilon} = \{\eta \in \mathbb{S};\ |1 - \bar{\zeta}\eta| < \varepsilon\}$.

It is clear that if $s = n/p$, then the space $M^{p,n/p}$ coincides with $L^p(\mathbb{S})$ and that if $s = 0$, then $M^{p,0}$ coincides with the non isotropic BMO space. It is also well-known that if $-1 < s < 0$, then $M^{p,s}$ coincides with a non isotropic Lipschitz space.

When $0 < s \le n/p$, the norm (1.3) is equivalent to the so-called Morrey norm (see for instance [20])

$$\|f\|_{M^{p,s}} = \sup_{I_{\zeta,\varepsilon}} \left(\varepsilon^{sp-n} \int_{I_{\zeta,\varepsilon}} |f(\eta)|^p d\sigma(\eta) \right)^{1/p}. \tag{1.4}$$

We denote by $HM^{p,s} = M^{p,s} \cap H^p$ the corresponding holomorphic Morrey space.

The main result of this work is the following theorem.

Theorem 1.1. *Let $1 < p < \infty$ and $0 < s < n/p$. Let $g_1, \dots, g_m \in H^\infty$. Then the following assertions are equivalent*:

(i) *The functions g_k, $k = 1, \dots, m$ satisfy* $\inf\{|g(z)| : z \in \mathbb{B}\} > 0$.
(ii) *$\mathcal{M}_g$ maps $H^p(\theta) \times \cdots \times H^p(\theta)$ onto $H^p(\theta)$ for any $1 < p < \infty$ and any $\theta \in \mathcal{A}_p$.*
(iii) *$\mathcal{M}_g$ maps $H^p(\theta) \times \cdots \times H^p(\theta)$ onto $H^p(\theta)$ for some $1 < p < \infty$ and some $\theta \in \mathcal{A}_p$.*
(iv) *$\mathcal{M}_g$ maps $HM^{p,s} \times \cdots \times HM^{p,s}$ onto $HM^{p,s}$ for any $1 < p < \infty$ and any $0 < s < n/p$.*
(v) *$\mathcal{M}_g$ maps $HM^{p,s} \times \cdots \times HM^{p,s}$ onto $HM^{p,s}$ for some $1 < p < \infty$ and some $0 < s < n/p$.*

Moreover, there exists a linear operator $\mathcal{T}_g$ such that $\mathcal{M}_g(\mathcal{T}_g(f)) = f$ for all the functions f in one of the above spaces.

Note that for $n = 1$ the above results are a consequence of the celebrated Carleson corona theorem [11]. Therefore, in what follows we will only consider the case $n > 1$. We recall that the corona problem for H^p was proved in [2] and [4,5].

There is an extensive literature on corona problems in several spaces of holomorphic functions For instance, the corona problem for the Morrey-Campanato spaces in the scale $-1 < s < 0$, which corresponds to Lipschitz spaces, was considered in [19], the case $s = 0$, which corresponds to BMOA, in [24] and [8]. Related results involving, among others, Bergman, Besov and Bloch spaces can be found in [7,8,15,16,22,24–26,32]. See also the book [29] and the references therein.

The study of the corona problem for certain weighted Hardy spaces has been the object of interest of several authors. It is interesting to remark that in [3] it is shown that solving the corona theorem for H^∞ is equivalent to solving the corona problem for all the weighted Hardy spaces $H^2(\mu)$, for any probabilistic measure μ on $\mathbb{S}$. In [32] the authors prove that it is possible to constrain the class of measures to a certain family of weights.

We believe that the interest of this paper lies not only on the results but on the techniques, which are based on the use of some extrapolation theorems, that allow to reduce the proof of the corona theorem for weighted Hardy spaces and Morrey spaces to the particular case $H^2(\theta)$ for any weight $\theta \in \mathcal{A}_2$. Since $H^2(\theta)$ coincides with a weighted Besov space (see Proposition 2.6 below), some computations are easier to deal with.

We will finish the introduction giving a brief sketch of the proof of Theorem 1.1 and the distribution of the parts of its proof in the different sections of the paper.

In the proof, it will be convenient to add to the list of assertions in Theorem 1.1 the following one

(vi) $\mathcal{M}_g$ maps $H^2(\theta) \times \ldots \times H^2(\theta)$ onto $H^2(\theta)$ for any $\theta \in \mathcal{A}_2$.

The scheme of the proof of the corona theorem for the case of weighted Hardy spaces will be the following:

$$\text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(i)} \Rightarrow \text{(vi)} \Rightarrow \text{(ii)}.$$

Clearly (ii) $\Rightarrow$ (iii). The implication (iii) $\Rightarrow$ (i), which states that the necessity of the condition (1.1), is proved in Section 4. The proof of (i) $\Rightarrow$ (vi) is quite technical. In order to make the paper more readable, we prove first this result for the particular case of two generators in Section 5, and next in Section 6 we prove the general case. For the case of two generators we will use minimal solutions of the $\bar{\partial}\partial$-equation and Wolff type techniques that allow to estimate the solutions of the corona problem using Carleson measures for $H^2(\theta)$.

For the proof of the general case, we will consider as usual the Koszul complex with estimates of the involved operators which are suitable for the study of the required continuities.

Finally, in Section 7 we prove that (vi) $\Rightarrow$ (ii). This result will be a consequence of an extrapolation theorem due to J.L. Rubio de Francia.

The scheme of the proof of the Morrey case is similar and we will show in this case that

$$\text{(iv)} \Rightarrow \text{(v)} \Rightarrow \text{(i)} \Leftrightarrow \text{(ii)} \Rightarrow \text{(iv)}.$$

The first implication is obvious, and the proof of the second will be given in Section 4. The proof of (ii) $\Rightarrow$ (iv), given in Section 7, follows from a theorem proved by H. Arai and T. Mizuhara in [9].

In Section 2 we recall some well-known results about weighted Hardy spaces and Morrey spaces. In Section 3 we prove some results about pointwise multipliers and Carleson measures needed to prove the main results.

2. Preliminaries

2.1. Notation

In this subsection we include most of the definitions of operators, spaces of functions and measures that we will use throughout the paper and that have not already been introduced.

As usual, we will adopt the convention of using the same letter for various absolute constants whose values may change in each occurrence, and we will write $A \lesssim B$ if there exists an absolute constant M such that $A \leq MB$. We will say that two quantities A and B are equivalent if both $A \lesssim B$ and $B \lesssim A$, and, in this case, we will write $A \approx B$.

2.1.1. *Sets*

For $\zeta \in \mathbb{S}$ and $r > 0$, let $I_{\zeta,r} = \{\eta \in \mathbb{S} : |1 - \eta\bar{\zeta}| < r\}$. When $\zeta = z/|z|$ and $r = (1 - |z|^2)$, for $z \in \mathbb{B}$, $z \neq 0$, we write I_z instead of $I_{\zeta,r}$. If $\zeta \in \mathbb{S}$, and $\alpha \geq 1$, the admissible region $\Gamma_{\zeta,\alpha}$ is defined by $\Gamma_{\zeta,\alpha} = \{z \in \mathbb{B}\,;\ |1 - z\bar{\zeta}| < \alpha(1 - |z|^2)\}$, and if $A \subset \mathbb{S}$, then $T_\alpha(A) = \mathbb{B} \setminus (\cup_{\zeta \notin A}\Gamma_\alpha(\zeta))$ is the tent over A. When $\alpha = 1$, we will write $\Gamma_\zeta = \Gamma_{\zeta,1}$, and $T(A) = T_1(A)$. If $\zeta \in \mathbb{S}$, and $r > 0$, we will write $\hat{I}_{\zeta,r} = T(I_{\zeta,r})$ and if $\zeta = z/|z|$ and $r = (1 - |z|^2)$, we write $\hat{I}_z = I_{\zeta,r}$.

By $|A|$ we denote the Lebesgue measure of a measurable set $A \subset \mathbb{S}$.

2.1.2. *Differential operators*

For $1 \leq j \leq n$, let $D_j = \frac{\partial}{\partial z_j}$. If $1 \leq i < j \leq n$, let $D_{i,j}$ be the complex-tangential differential operator defined by $D_{i,j} = \bar{z}_j D_i - \bar{z}_i D_j$. The tangential operators $D_{i,j}$ appear when one computes the coefficients of the (2,0)-form

$$\partial\varphi(z) \wedge \partial|z|^2 = \sum_{1\leq i<j\leq n} D_{i,j}\varphi(z) dz_i \wedge dz_j.$$

The pointwise norms of $\partial\varphi$ and of the above differential form are

$$|\partial\varphi(z)| = \sum_{j=1}^{n} |D_j\varphi(z)|, \quad \text{and}$$

$$|\partial_T\varphi(z)| = |\partial\varphi(z) \wedge \partial|z|^2| = \sum_{1\leq i<j\leq n} |D_{i,j}\varphi(z)|.$$

Let $\mathcal{R}$ be the radial derivative $\mathcal{R} = \sum_{j=1}^{n} z_j D_j$. For $l > 0$ and k a positive integer, we define

$$\mathcal{R}_l^k = \frac{\Gamma(l)}{\Gamma(l+k)}((l+k-1)\mathcal{I} + \mathcal{R}) \dots (l\mathcal{I} + \mathcal{R}), \tag{2.1}$$

where $\mathcal{I}$ denotes the identity operator.

2.1.3. *Integral operators*

We will denote by $\mathcal{C}$ the Cauchy projection and by $\mathbb{P}$ the Poisson-Szëgo projection, that is

$$\mathcal{C}(\varphi)(z) = \int_{\mathbb{S}} \frac{\varphi(\zeta)}{(1 - z\bar{\zeta})^n} d\sigma(\zeta), \qquad \mathbb{P}(\varphi)(z) = \int_{\mathbb{S}} \varphi(\zeta) \frac{(1 - |z|^2)^n}{|1 - z\bar{\zeta}|^{2n}} d\sigma(\zeta).$$

We introduce the following kernels and their corresponding integral operators.

Definition 2.1. Let N, M, L be real numbers satisfying $N > 0$ and $L < n$. For $z, w \in \mathbb{B}$, let

$$\mathcal{L}_{M,L}^N(w, z) = \frac{(1 - |w|^2)^{N-1}}{|1 - z\bar{w}|^M \phi(w, z)^L},$$

where $\phi(w, z) = |1 - z\bar{w}|^2 - (1 - |w|^2)(1 - |z|^2)$.

$\mathcal{L}_{M,L}^N$ will also denote the corresponding integral operator given by

$$\psi(z) \to \mathcal{L}_{M,L}^N(\psi)(z) = \int_{\mathbb{B}} \psi(w) \mathcal{L}_{M,L}^N(w, z) d\nu(w).$$

The proof of the following result can be found in [14, Lemma I.1].

Lemma 2.2. *Let N, M, L be real numbers satisfying $N > 0$ and $L < n$. If $n + N - M - 2L \neq 0$ then*

$$\int_{\mathbb{B}} \mathcal{L}_{M,L}^N(w, z) d\nu(w) \lesssim 1 + (1 - |z|^2)^{n+N-M-2L}, \quad z \in \mathbb{B}.$$

Definition 2.3. We define the type of the kernel $\mathcal{L}_{M,L}^N$ by

$$type(\mathcal{L}_{M,L}^N) = n + N - M - 2L.$$

2.2. The Muckenhoupt class $\mathcal{A}_p$ on $\mathbb{S}$

Given a non-negative weight $\theta \in L^1(d\sigma)$ and a measurable set E in $\mathbb{S}$, let $\theta(E) = \int_E \theta d\sigma$. For $z = r\zeta$, with $\zeta \in \mathbb{S}$ and $0 < r < 1$, we consider the average function on $\mathbb{B}$ associated to θ defined by $\Theta(z) = \frac{\theta(I_z)}{|I_z|}$, where $I_z = I_{r\zeta} = \{\eta \in \mathbb{S} : |1 - \eta\bar{\zeta}| < 1 - r^2\}$.

The Muckenhoupt class $\mathcal{A}_p$ on $\mathbb{S}$, with $1 < p < \infty$, consists of the non-negative weights $\theta \in L^1(d\sigma)$ satisfying

$$\mathcal{A}_p(\theta) = \sup_{z \in \mathbb{B}} (\Theta(z))^{1/p} \left(\Theta'(z)\right)^{1/p'} < \infty \tag{2.2}$$

where $\theta' = \theta^{-p'/p}$ and $\Theta'(z) = \frac{\theta'(I_z)}{|I_z|}$. Observe that $\theta \in \mathcal{A}_p$, if and only if, $\theta' \in \mathcal{A}_{p'}$.

If $p = 1$, the class $\mathcal{A}_1$ on $\mathbb{S}$ is the set of non-negative weights $\theta \in L^1(d\sigma)$ satisfying

$$\mathcal{A}_1(\theta) = \left\| \frac{M_{H-L}(\theta)(\zeta)}{\theta(\zeta)} \right\|_{L^\infty} < \infty, \tag{2.3}$$

where $M_{H-L}(\theta)(\zeta) = \sup_r \Theta(r\zeta)$ is the Hardy-Littlewood maximal function of θ in ζ.

Some well-known properties on Muckenhoupt weights are summarized in the following proposition.

Proposition 2.4.

(i) *If* $1 < p$ *then* $\mathcal{A}_1 \subset \mathcal{A}_p$.
(ii) *If* $1 < p < \infty$ *and* $\varphi \in \mathcal{A}_p$ *then there exists* $1 \le q < p$ *such that* $\varphi \in \mathcal{A}_q$.
(iii) *For any* $\theta \in \mathcal{A}_p$*, the measure* $\theta d\sigma$ *is a doubling measure. In fact, there exist* $C > 0$ *and* $0 < \lambda < np$ *such that for any* $\zeta \in \mathbb{S}$ *and any* $r > 0$, $\theta(I_{\zeta,2r}) \le C 2^\lambda \theta(I_{\zeta,r})$.

The proof of (i) can be found in [31, page 197] and the proof of (ii) in [31, page 202]. The estimate in (iii) with $\lambda \le np$ is proved in [31, page 196]. This estimate together with (ii) gives $\lambda < np$.

2.3. Holomorphic weighted Hardy spaces

Let us start by recalling some facts on weighted Hardy-Sobolev spaces $H^p(\theta)$, with $\theta \in \mathcal{A}_p$, which can be found for instance in [23, Section 5].

Proposition 2.5. *Let* $1 < p < \infty$ *and let* θ *be an* $\mathcal{A}_p$ *weight.*

(i) *If* $M_\alpha(f)$ *is the admissible maximal function*

$$M_\alpha(f)(\zeta) = \sup\{|f(z)| : z \in \Gamma_{\zeta,\alpha}\}$$

then $\|M_\alpha(f)\|_{L^p(\theta)} \approx \|f\|_{H^p(\theta)}$.
(ii) *There exist* $1 < p_1 < p < p_2$ *such that* $H^{p_2} \subset H^p(\theta) \subset H^{p_1}$.
(iii) *If* $1 < p < \infty$ *and* $\theta \in \mathcal{A}_p$ *then the Cauchy projection* $\mathcal{C}$ *maps* $L^p(\theta)$ *to* $H^p(\theta)$.

(iv) *If* $1 < p < \infty$ *then the dual of* $H^p(\theta)$ *can be identified with* $H^{p'}(\theta')$ *with respect to the pairing*

$$\langle f, g\rangle_S = \lim_{r\to 1}\int_S f_r \bar{g}_r d\sigma, \tag{2.4}$$

where $f_r(\zeta) = f(r\zeta)$.

The following result (see [12]) implies that, as it happens in the unweighted case, the weighted space $H^2(\theta)$ can be considered as a weighted Besov space.

Proposition 2.6. *Let* $\theta \in \mathcal{A}_2$ *and let k be any positive integer. The following assertions are equivalent*

(i) $f \in H^2(\theta)$,
(ii) $(1-|z|^2)^{k-1/2}\Theta(z)^{1/2}|(\mathcal{I}+\mathcal{R})^k f(z)| \in L^2(\mathbb{B})$,

with equivalence of norms.

Other equivalent norms in $H^2(\theta)$ can be obtained replacing in (2.6) the operator $(\mathcal{I}+\mathcal{R})^k$ by the operator $\mathcal{R}_l^k$ defined in (2.1), or using the estimates in [1, Lemma 3.6], which state that if $\alpha < \beta$, then there exists $C > 0$ such that

$$\int_{\Gamma_{\zeta,\alpha}} |\partial_T h(z)|^2 (1-|z|^2)^{-n} d\nu(z) \le C\int_{\Gamma_{\zeta,\beta}} |\mathcal{R}h(z)|^2(1-|z|^2)^{1-n} d\nu(z),$$

for any $\zeta \in \mathbb{S}$ and $h \in H(\mathbb{B})$. Therefore, using Fubini's theorem to compute the $L^1(\theta)$-norms of the above terms, we obtain the following estimate:

$$\int_{\mathbb{B}} |D_{i,j}h(z)|^2\Theta(z)d\nu(z) \lesssim \int_{\mathbb{B}} |\mathcal{R}h(z)|^2(1-|z|^2)\Theta(z)d\nu(z). \tag{2.5}$$

In particular, we have from this observation and Proposition 2.6 the following:

Proposition 2.7. *If* $f \in H^2(\theta)$ *then*

$$\left\|\Theta(z)^{1/2}\left(|\partial_T f(z)| + (1-|z|^2)^{1/2}|\partial f(z)|\right)\right\|_{L^2(\mathbb{B})} \lesssim \|f\|_{H^2(\theta)}.$$

2.4. Holomorphic Morrey spaces $0 < s < n/p$

The following embedding is a consequence of Hölder's inequality:

Proposition 2.8. *If* $1 < p < \infty$ *and* $0 < s \le n/p$ *then* $H^{n/s} \subset HM^{p,s} \subset H^p$.

3. Pointwise multipliers and Carleson measures

In this section we check that the space of pointwise multipliers of the holomorphic weighted Hardy spaces $H^p(\theta)$ and of the holomorphic Morrey spaces $M^{p,s}$ coincide with H^∞. We also give examples of Carleson measures for $H^p(\theta)$, which will play an important role in the proof of the main theorem.

3.1. Pointwise multipliers and Carleson measures on weighted Hardy spaces

In the next propositions we state some results on weighted Hardy spaces. These results are analogous to those corresponding to the unweighted case, but we include sketches of the proofs for a sake of completeness.

Proposition 3.1. *A holomorphic function g on $\mathbb{B}$ is a pointwise multiplier of $H^p(\theta)$ if and only if $g \in H^\infty$.*

Proof. It is clear that if $g \in H^\infty$ then g is a pointwise multiplier of $H^p(\theta)$.

The other implication is a consequence of the inequality $\|g^m\|_{H^p(\theta)} \leq \|\mathcal{M}_g\|^m \|1\|_{H^p(\theta)}$, where $\|\mathcal{M}_g\|$ denotes the norm of the operator $\mathcal{M}_g(f) = gf$, and that $\|g\|_{H^\infty} \lesssim \sup_m \|g^m\|_{H^p(\theta)}^{1/m}$. □

Proposition 3.2. *Let $1 < p < \infty$, $\theta \in \mathcal{A}_p$ and $N > n$. If $f_z(w) = \frac{1}{(1-w\bar{z})^N}$, for $z \in \mathbb{B}$, we then have*

$$\|f_z\|_{H^p(\theta)}^p \lesssim \frac{\theta(I_z)}{(1-|z|^2)^{Np}}.$$

Proof. By Proposition 2.4, the measure $\theta d\sigma$ is a doubling measure and $\theta(2I_z) \lesssim 2^\lambda \theta(I_z)$ with $\lambda < np \leq Np$. If $z = |z|\zeta$, and $2I_z = I_{\zeta, 2(1-|z|^2)}$, we have

$$\left\| \frac{1}{(1-w\bar{z})^N} \right\|_{H^p(\theta)}^p = \int_{\mathbb{S}} \frac{1}{|1-\zeta\bar{z}|^{Np}} \theta(\zeta) d\sigma(\zeta) \lesssim \sum_{k\geq 0} \frac{\theta(2^k I_z)}{2^{kNp}(1-|z|^2)^{Np}}$$
$$\lesssim \sum_{k\geq 0} \frac{2^{k\lambda}\theta(I_z)}{2^{kNp}(1-|z|^2)^{Np}} \lesssim \frac{\theta(I_z)}{(1-|z|^2)^{Np}},$$

where in last estimate we have used the fact that $Np > \lambda$. □

We recall that a positive Borel measure μ on $\mathbb{B}$ is a Carleson measure for a space X^p of functions on $\mathbb{B}$, if there exists $C > 0$ such that, for any $f \in X^p$,

$$\int_{\mathbb{B}} |f(z)|^p d\mu(z) \leq C \|f\|_{X^p}^p. \tag{3.1}$$

As it happens in the unweighted case, if $1 < p < \infty$, θ is an $\mathcal{A}_p$ and X^p is either $H^p(\theta)$ or the space $\mathbb{P}[L^p(\theta)]$, then these measures can be characterized in terms of size conditions on the measure of tents over balls.

The space $\mathbb{P}[L^p(\theta)]$ is normed by $\|u\|_{\mathbb{P}[L^p(\theta)]} = \|f\|_{L^p(\theta)}$, where $u = \mathbb{P}[f]$, and we recall that $\|u\|_{\mathbb{P}[L^p(\theta)]} \approx \|M_\alpha[u]\|_{L^p(\theta)}$.

We then have:

Proposition 3.3. *Let $1 < p < \infty$, μ a positive Borel measure on $\mathbb{B}$ and θ be a weight in $\mathcal{A}_p$. Then the following assertions are equivalent:*

(i) *μ is a Carleson measure for $\mathbb{P}[L^p(\theta)]$;*
(ii) *μ is a Carleson measure for $H^p(\theta)$;*
(iii) *There is a constant C such that for all $z \in \mathbb{B}$, $\mu(\hat{I}_z) \leq C\theta(I_z)$.*

Proof. From [28, Theorem 5.6.8] and Proposition 2.5, it is immediate to deduce that for any function $f \in H^p(\theta)$, $f = \mathcal{C}[f^*] = \mathbb{P}[f^*]$, and $\|f\|_{H^p(\theta)} \approx \|f\|_{\mathbb{P}[L^p(\theta)]} \approx \|f^*\|_{L^p(\theta)}$, where f^* are the boundary values of the function f. Hence, any Carleson measure for $\mathbb{P}[L^p[\theta]]$ is also a Carleson measure for $H^p(\theta)$, and, as a consequence, (i) implies (ii).

Next, assume that (ii) holds, and for $N > 0$ big enough, let $f_z(w) = \frac{1}{(1-w\bar{z})^N}$. We then have that for any $w \in \hat{I}_z$, $|1 - w\bar{z}| \lesssim 1 - |z|^2$. Thus Proposition 3.2 gives

$$\frac{\mu(\hat{I}_z)}{(1-|z|^2)^{Np}} \lesssim \int_{\mathbb{B}} \frac{d\mu(w)}{|1-w\bar{z}|^{Np}} \lesssim \|f_z\|^p_{H^p(\theta)} \lesssim \frac{\theta(I_z)}{(1-|z|^2)^{Np}}.$$

So we are left to show that (iii) implies (i). We have

$$\int_{\mathbb{B}} |\mathbb{P}[f]|^p d\mu = p \int_0^\infty \mu(\{\mathbb{P}[f] > \lambda\})\lambda^{p-1} d\lambda.$$

But $\{\mathbb{P}[f] > \lambda\} \subset T(\{\zeta \,;\, M_\alpha \mathbb{P}[f] > \lambda\})$. Since $A_\lambda = \{\zeta \,:\, M_\alpha \mathbb{P}[f] > \lambda\}$ is an open set, $A_\lambda = \cup I_{\zeta, r_\zeta}$, and for any compact $K \subset A_\lambda$, there exists a finite subfamily of pairwise disjoint open balls $I_{\zeta_i, r_{\zeta_i}}$ such that $K \subset \cup_1^M I_{\zeta_i, 3r_{\zeta_i}}$. Consequently,

$$\begin{aligned}\mu(T(K)) &\le \sum_{i=1}^M \mu(T(I_{\zeta_i,3r_{\zeta_i}})) \lesssim \sum_{i=1}^M \theta(I_{\zeta_i,3r_{\zeta_i}}) \\ &\lesssim \sum_{i=1}^M \theta(I_{\zeta_i,r_{\zeta_i}}) = \theta(\cup_{i=1}^M I_{\zeta_i,r_{\zeta_i}}) \lesssim \theta(A_\lambda),\end{aligned}$$

and

$$\int_{\mathbb{B}} |\mathbb{P}[f]|^p d\mu \lesssim \int_0^\infty \theta(A_\lambda)\lambda^{p-1} d\lambda \approx \int_{\mathbb{S}} |M_\alpha \mathbb{P}[f]|^p d\theta \lesssim \int_{\mathbb{S}} |f|^p d\theta,$$

which ends the proof. □

Proposition 3.4. *If $g \in H^\infty$ then the measures*

$$|\partial g(z)|^2(1-|z|^2)\Theta(z)d\nu(z) \quad \textit{and} \quad |\partial_T g(z)|^2\Theta(z)d\nu(z)$$

are Carleson measures for $H^2(\theta)$.

Proof. Let $f \in H^2(\theta)$. Since $gf \in H^2(\theta)$, by Proposition 2.7 we have

$$\int_{\mathbb{B}} |\partial(gf)(z)|^2(1-|z|^2)\Theta(z)d\nu(z) \approx \|gf\|^2_{H^2(\theta)} \lesssim \|f\|^2_{H^2(\theta)}.$$

This observation, the fact that $f \in H^2(\theta)$, $g \in H^\infty$ and $f\partial g = \partial(gf) - g\partial f$, give

$$\begin{aligned}&\int_{\mathbb{B}} |f(z)|^2 |\partial g(z)|^2 (1-|z|^2)\Theta(z)d\nu(z)\\ &\le \int_{\mathbb{B}} |\partial(gf)(z)|^2(1-|z|^2)\Theta(z)d\nu(z)\\ &\quad + \int_{\mathbb{B}} |\partial f(z)|^2 |g(z)|^2 (1-|z|^2)\Theta(z)d\nu(z) \lesssim \|f\|^2_{H^2(\theta)}.\end{aligned}$$

We now deal with the second assertion. We have that $fD_{i,j}g = D_{i,j}(gf) - gD_{i,j}f$. Hence,

$$\begin{aligned}\int_{\mathbb{B}} |f(z)|^2 |\partial_T g(z)|^2 \Theta(z)d\nu(z) &\le \int_{\mathbb{B}} |\partial_T(gf)(z)|^2\Theta(z)d\nu(z)\\ &\quad + \int_{\mathbb{B}} |\partial_T f(z)|^2 |g(z)|^2 (1-|z|^2)\Theta(z)d\nu(z).\end{aligned}$$

Applying Proposition 2.8 to both f and gf in the preceding estimate with $g \in H^\infty$, we obtain that

$$\int_{\mathbb{B}} |f(z)|^2 |\partial_T g(z)|^2 \Theta(z)d\nu(z) \lesssim \|gf\|^2_{H^2(\theta)} + \|f\|^2_{H^2(\theta)} \lesssim \|f\|^2_{H^2(\theta)},$$

which ends the proof. □

As a consequence of Propositions 3.3 and 3.4 we have:

Proposition 3.5. *Let $\theta \in \mathcal{A}_2$ and $g \in H^\infty$. Let*

$$d\mu_{g,\theta}(z) = \Theta(z)\left((1-|z|^2)|\partial g(z)|^2 + |\partial_T g(z)|^2\right)d\nu(z).$$

If $\varphi \in L^2(\theta)$ then $\displaystyle\int_{\mathbb{B}} |\mathbb{P}(\varphi)(z)|^2\, d\mu_{g,\theta}(z) \lesssim \|\varphi\|^2_{L^2(\theta)}$.

3.2. Multipliers on Morrey spaces

The next result gives a characterization of the pointwise multipliers of $HM^{p,s}$, for $0 < s \le n/p$.

Proposition 3.6. *If $1 < p < \infty$ and $0 < s \le n/p$, a function g is a pointwise multiplier of $HM^{p,s}$ if and only if $g \in H^\infty$.*

Proof. It is clear that if $g \in H^\infty$, then $\|gf\|_{M^{p,s}} \le \|g\|_\infty \|f\|_{M^{p,s}}$.

In order to prove the necessity we recall that, for a positive integer m,

$$\|g^m\|_p \le \|g^m\|_{M^{p,s}} \le \|\mathcal{M}_g\|^m \|1\|_{M^{p,s}},$$

where $\|\mathcal{M}_g\|$ denotes the norm of the operator $f \to gf$. Therefore, $\|g\|_{pm} \lesssim \|\mathcal{M}_g\|$. Since $\|g\|_\infty = \lim_{m\to\infty} \|g\|_{pm}$ we obtain the result. □

4. Necessary conditions on the corona data

In this section we prove that if any of the assertions (ii), (iii), (iv) or (v) in Theorem 1.1 holds, then the corona data g satisfies (i).

4.1. Necessary conditions for weighted Hardy spaces

In order to obtain necessary conditions on the corona problem we recall the following lemma which has been proved in [12]:

Lemma 4.1. *Let $1 < p < \infty$ and θ be a weight in $\mathcal{A}_p$. There exists $C > 0$ such that for any holomorphic function f in $\mathbb{B}$, and any $z = |z|\zeta$,*

$$|f(z)| \le C\left(|f(0)| + \int_{1-|z|^2}^{1} \frac{dt}{\theta(I_{\zeta,t})^{1/p}\, t}\|f\|_{H^p(\theta)}\right).$$

As a corollary we obtain:

Corollary 4.2. *Let $1 < p < \infty$ and θ be a weight in $\mathcal{A}_p$. There exists $C > 0$ such that for any holomorphic function f in $\mathbb{B}$, and any $z \in \mathbb{B}$,*

$$|f(z)| \le C\frac{\left(\theta^{-p'/p}(I_z)\right)^{1/p'}}{(1-|z|^2)^n}\|f\|_{H^p(\theta)}.$$

Proof. Let $z = |z|\zeta \neq 0$. The fact that θ is in $\mathcal{A}_p$ gives that $\theta^{-p'/p}$ is in $\mathcal{A}_{p'}$, it satisfies a doubling condition of order $\lambda < np'$, and consequently

$$\begin{aligned}\int_{1-|z|^2}^{1} \frac{1}{\theta(I_{\zeta,t})^{1/p}}\frac{dt}{t} &\approx \int_{1-|z|^2}^{1}\left(\frac{\theta^{-p'/p}(I_{\zeta,t})}{t^n}\right)^{1/p'}\frac{1}{t^{n/p}}\frac{dt}{t}\\ &\lesssim \sum_{k\ge 0} 2^{k(\lambda/p'-n)}\frac{\left(\theta^{-p'/p}(I_z)\right)^{1/p'}}{(1-|z|^2)^n} \approx \frac{\left(\theta^{-p'/p}(I_z)\right)^{1/p'}}{(1-|z|^2)^n}.\end{aligned}$$

In order to finish we just have to show that $|f(0)| \lesssim \frac{\left(\theta^{-p'/p}(I_z)\right)^{1/p'}}{(1-|z|^2)^n}$. This is a consequence from the fact that

$$|f(0)| \le \int_{\mathbb{S}}|f|d\sigma = \int_{\mathbb{S}}|f|\theta^{1/p}\theta^{-1/p}d\sigma \lesssim \|f\|_{H^p(\theta)},$$

and

$$1 \lesssim \frac{(\theta(I_z))^{1/p}\left(\theta^{-p'/p}(I_z)\right)^{1/p'}}{(1-|z|^2)^n} \lesssim \frac{\left(\theta^{-p'/p}(I_z)\right)^{1/p'}}{(1-|z|^2)^n}.$$

□

Proposition 4.3. *Let $g_1, \dots, g_m \in H^\infty$ and $\theta \in \mathcal{A}_p$. If the operator $\mathcal{M}_g : H^p(\theta) \times \cdots \times H^p(\theta) \to H^p(\theta)$ is onto, then $g = (g_1, \dots, g_m)$ satisfies* $\inf_{z \in \mathbb{B}} |g(z)| > 0$.

Proof. We first observe that by the Open Mapping Theorem, for every $f \in H^p(\theta)$, there exist functions $f_i \in H^p(\theta)$, $i = 1, \dots, m$, such that:

(i) $f = \sum_{i=1}^m f_i g_i$;
(ii) $\|f_i\|_{H^p(\theta)} \lesssim \|f\|_{H^p(\theta)}$ for $i = 1, \dots, m$.

If $z \in \mathbb{B}$, let $f_z(w) = \frac{1}{(1-w\bar{z})^N}$, where $N > 0$ is to be chosen. We then have that there exist f_i, with $i = 1, \dots, m$, satisfying conditions (i) and (ii) above for $f = f_z$. Therefore, if N is big enough, Corollary 4.2 and Proposition 3.2 give that

$$\begin{aligned}\frac{1}{(1-|z|^2)^N} &= |f_z(z)| \le \sum_{i=1}^m |f_i(z)||g_i(z)| \\ &\lesssim \frac{\left(\theta^{-p'/p}(I_z)\right)^{1/p'}}{(1-|z|^2)^n} \|f_z\|_{H^p(\theta)} \sum_{i=1}^m |g_i(z)| \\ &\lesssim \frac{\left(\theta^{-p'/p}(I_z)\right)^{1/p'}}{(1-|z|^2)^n} \frac{\theta(I_z)^{1/p}}{(1-|z|^2)^N} \sum_{i=1}^m |g_i(z)|,\end{aligned}$$

and since θ is in $\mathcal{A}_p$, we obtain that $1 \lesssim \sum_{i=1}^m |g_i(z)|$. □

4.2. Necessary conditions for Morrey spaces

The next lemma gives a pointwise growth estimate for functions in $HM^{p,s}$.

Proposition 4.4. *Let $1 \le p < \infty$, $0 < s < n/p$, f in $HM^{p,s}$, and $z \in B$. Then* $|f(z)| \lesssim \|f\|_{M^{p,s}} (1-|z|^2)^{-s}$.

Proof. Assume $z \neq 0$, and let $\zeta = z/|z|$. For a positive integer j, let $I_j = \{\eta \in \mathbb{S} : |1 - \eta\bar{\zeta}| \le 2^{j+1}(1-|z|^2)\}$.

Therefore, by Cauchy's formula,

$$\begin{aligned}|f(z)| &\le \int_{I_1} \frac{|f(\eta)|}{|1-z\bar{\eta}|^n} d\sigma(\eta) + \sum_{j>1} \int_{I_{j+1}-I_j} \frac{|f(\eta)|}{|1-z\bar{\eta}|^n} d\sigma(\eta) \\ &\lesssim \sum_{j\ge 1} (2^j(1-|z|^2))^{-n} \int_{I_{j+1}} |f(\eta)| d\sigma(\eta).\end{aligned}$$

Next, Hölder's inequality gives

$$|f(z)| \lesssim \sum_{j\ge 1} (2^j(1-|z|^2))^{-s} \|f\|_{M^{p,s}} \lesssim (1-|z|^2)^{-s} \|f\|_{M^{p,s}},$$

which concludes the proof. □

Proposition 4.5. *Assume that* $g_1, \dots, g_m \in H^\infty$. *If for some* $1 < p < \infty$ *and* $0 < s < n/p$ *the map* $\mathcal{M}_g : HM^{p,s} \times \cdots \times HM^{p,s} \to HM^{p,s}$ *is surjective, then* $\inf\{|g(z)| : z \in \mathbb{B}\} > 0$.

Proof. By the Open Mapping Theorem, for every function f in $HM^{p,s}$ there exist functions $f_1, \dots, f_m$ in $HM^{p,s}$, such that

$$\sum_{k=1}^{m} g_k f_k = f \qquad \text{and} \qquad \|f_k\|_{M^{p,s}} \lesssim \|f\|_{M^{p,s}}.$$

By Proposition 4.4

$$|f(z)| \leq \sum_{i=1}^{m} |f_i(z)|\,|g_i(z)| \lesssim \|f\|_{M^{p,s}}(1-|z|^2)^{-s} \sum_{i=1}^{m} |g_i(z)|. \tag{4.1}$$

For $N > s$, we consider the function $f_z(w) = (1 - w\bar{z})^{-N}$. Since, by Proposition 2.8, $H^{n/s} \subset HM^{p,s}$, we have

$$\|f_z\|_{M^{p,s}} \lesssim \|f_z\|_{H^{n/s}} \approx (1-|z|^2)^{s-N}.$$

Therefore, by (4.1)

$$\begin{aligned}(1-|z|^2)^{-N} = f_z(z) &\lesssim \|f_z\|_{M^{p,s}}(1-|z|^2)^{-s} \sum_{i=1}^{m} |g_i(z)| \\ &\lesssim (1-|z|^2)^{-N} \sum_{i=1}^{m} |g_i(z)|,\end{aligned}$$

which proves the result. □

5. The $H^2(\theta)$-corona theorem for 2 generators

Throughout this section we will assume that the functions $g_1, g_2 \in H^\infty$ satisfy $\inf_{z\in\mathbb{B}} |g(z)| > 0$.

We want to prove that the operator $\mathcal{M}_g$ defined by $\mathcal{M}_g(f_1, f_2) = g_1 f_1 + g_2 f_2$ maps $H^2(\theta) \times H^2(\theta)$ onto $H^2(\theta)$ for any weight $\theta \in \mathcal{A}_2$.

Let $g = (g_1, g_2)$ and let $G = (G_1, G_2)$ where $G_j = \frac{\bar{g}_j}{|g|^2}$, for $j = 1, 2$. An easy computation proves that

$$\bar\partial G_1 = -g_2 \Omega, \quad \bar\partial G_2 = g_1 \Omega, \tag{5.1}$$

where

$$\Omega = \frac{\overline{g_1 \partial g_2 - g_2 \partial g_1}}{|g|^4} = G_1 \bar\partial G_2 - G_2 \bar\partial G_1. \tag{5.2}$$

Clearly $g_1 G_1 + g_2 G_2 = 1$, $\bar\partial \Omega = 0$ and $\|Gf\|_{L^2(\theta)} \lesssim \|f\|_{H^2(\theta)}$.

Since the functions $G_j f$ are not holomorphic on $\mathbb{B}$, we must correct them using a solution of a $\bar\partial$-problem. Since $\bar\partial(\Omega f) = 0$ for any $f \in H^2(\theta)$, we will choose a suitable integral operator $\mathcal{K}$ such that $\bar\partial\mathcal{K}(\Omega f) = \Omega f$ and such that the linear operator

$$\mathcal{T}_g(f) = Gf + g^{\perp}\mathcal{K}(\Omega f), \quad g^{\perp} = (g_2, -g_1) \tag{5.3}$$

maps $H^2(\theta)$ to $H^2(\theta) \times H^2(\theta)$.

It is clear by construction that the components of $\mathcal{T}_g(f)$ are holomorphic functions on $\mathbb{B}$ and that $\mathcal{M}_g(\mathcal{T}_g(f)) = f$.

In order to choose a suitable operator $\mathcal{K}$, let

$$\mathcal{K}_0^N(w,z) = \sum_{k=0}^{n-1} c_{k,N} \frac{(1-|w|^2)^{N+k}(s \wedge (\bar\partial_w s)^{n-1-k})(w,z)}{(1-z\bar w)^{n+N}(1-w\bar z)^{n-k}} \wedge (\gamma(w))^k$$

where $\bar\partial = \bar\partial_w$ (differential respect w), $\gamma(w) = \bar\partial \frac{\partial |w|^2}{1-|w|^2}$ and $s(w,z) = (1-w\bar z)\partial|w|^2 - (1-|w|^2)\partial_w(w\bar z)$.

It is well-known that the corresponding integral operators associated to these kernels, also denoted by $\mathcal{K}_0^N$, that is, if ϑ is a (0,1)-form,

$$\mathcal{K}_0^N(\vartheta)(z) = \int_{\mathbb{B}} \mathcal{K}_0^N(w,z) \wedge \vartheta(w),$$

solve the $\bar\partial$-equation or the $\bar\partial_b$-equation on the unit ball of $\mathbb{C}^n$ (see for instance [30] or [14]).

The following proposition summarizes the main properties of these operators. In particular it gives a decomposition of $\mathcal{K}_0^N(\vartheta)$ as a sum of two functions. The first one is an antiholomorphic function on $\mathbb{B}$, and the other term involves $\partial\vartheta$. The main advantatge of this last term is that if ϑ is the form Ω defined in (5.2), then, by Proposition 3.4, we obtain expressions like $\Theta(w)|\partial\Omega(w)|^2(1-|w|^2)d\nu(w)$ or $\Theta(w)|\partial\Omega(w) \wedge \partial|w|^2 \wedge \bar\partial|w|^2|^2 d\nu(w)$, which are Carleson measures for $H^2(\theta)$, and these facts will play an important role in the calculus of the estimates.

Proposition 5.1. *Let ϑ be a* (0, 1)*-form with coefficients in $\mathcal{C}^1(\bar{\mathbb{B}})$. Then, for each positive integer N, there exist integral operators $\mathcal{Q}_0^{N,1}$ and $\mathcal{Q}_0^{N,2}$ satisfying the following properties*:

(i) $\mathcal{K}_0^N(\vartheta) = \mathcal{Q}_0^{N,1}(\vartheta) + \mathcal{Q}_0^{N,2}(\partial\vartheta)$;

(ii) $\bar\partial\mathcal{K}_0^N(\vartheta) = \vartheta$ *if* $\bar\partial\vartheta = 0$;

(iii) *The function $\mathcal{Q}_0^{N,1}(\vartheta)(z)$ is antiholomorphic on $\mathbb{B}$ and for $\zeta \in \mathbb{S}$*

$$\mathcal{Q}_0^{N,1}(\vartheta)(\zeta) = c_{N,n} \sum_{k=1}^{n+N} \int_{\mathbb{B}} \frac{(1-|w|^2)^N \vartheta(w) \wedge \partial(w\bar\zeta) \wedge (\partial\bar\partial|w|^2)^{n-1}}{(1-w\bar\zeta)^k};$$

(iv) *For any* $\zeta \in \mathbb{S}$,

$$|\mathcal{Q}_0^{N,2}(\partial\vartheta)(\zeta)| \leq C_{N,n} \int_{\mathbb{B}} \frac{(1-|w|^2)^{N+1}|\partial\vartheta(w)|}{|1-w\bar{\zeta}|^{n+N}} d\nu(w) + \int_{\mathbb{B}} \frac{(1-|w|^2)^{N}|\partial\vartheta(w)\wedge\partial|w|^2\wedge\bar{\partial}|w|^2|}{|1-w\bar{\zeta}|^{n+N}} d\nu(w).$$

Assertion (ii) is proved in [30] and [14]. Assertions (i) and (iii) can be found in [6, page 46]. Finally, (iv) follows from [6, Theorem 2].

We want to prove that, for $N > 0$ big enough, $\mathcal{T}_g^N$ maps $H^2(\theta)$ to $H^2(\theta) \times H^2(\theta)$, that is $\|\mathcal{T}_g^N(f)\|_{L^2(\theta)} \lesssim \|f\|_{H^2(\theta)}$ with a constant depending of n, N, g and θ.

Since $|\mathcal{T}_g^N(f)| \leq |G||f| + |g^{\perp}||\mathcal{K}^N(\Omega f)| \leq \|G\|_\infty|f| + \|g\|_\infty|\mathcal{K}^N(\Omega f)|$, we only need to prove that for $N > 0$ large enough we have the estimate $\|\mathcal{K}^N(\Omega f)\|_{L^2(\theta)} \lesssim \|f\|_{H^2(\theta)}$.

Since $\mathcal{K}_0^N(\Omega f) = \mathcal{Q}_0^{N,1}(\Omega f) + \mathcal{Q}_0^{N,2}(\partial(\Omega f))$, we will need the following estimates of Ωf and of $\partial(\Omega f)$.

Lemma 5.2. *Let* Ω *as in* (5.2). *Then*

$$\begin{aligned} |(\Omega f)(w)| &\lesssim |\partial g(w)||f(w)| \\ |\partial(\Omega f)(w)| &\lesssim |\partial g(w)|^2|f(w)| + |\partial g(w)||\partial f(w)|. \\ |\partial(\Omega f)(w)\wedge\partial|w|^2\wedge\bar{\partial}|w|^2| &\lesssim |\partial_T g(w)|^2|f(w)| + |\partial_T g(w)||\partial_T f(w)|. \end{aligned}$$

Proof. All the above estimates follow from the definition of Ω and the formulas $|\overline{\partial g_k(w)}\wedge\bar{\partial}|w|^2| = |\partial_T g_k(w)|$ and $|\partial f(w)\wedge\partial|w|^2| = |\partial_T f(w)|$. ☐

The estimate of the $L^2(\theta)$-norm of $\mathcal{K}_0^N(\Omega f)$ will be obtained by duality. We first state the following lemma that will be used in the proof of these $L^2(\theta)$-estimates.

Lemma 5.3. *If* $N > 0$ *then*

$$|\mathcal{K}_0^N(\Omega f)(\zeta)| \lesssim |\mathcal{Q}_0^N(\Omega f)(\zeta) + \mathcal{L}_{n+N,0}^{N+1}(W)(\zeta),$$

where

$$W(w) = (1-|w|^2)\Big(|\partial g(w)|^2|f(w)| + |\partial g(w)||\partial f(w)|\Big) + |\partial_T g(w)|^2|f(w)| + |\partial_T g(w)||\partial_T f(w)|.$$

Proof. The proof is a consequence of Proposition 5.1 and Lemma 5.2. ☐

Lemma 5.4. *Let ψ be a continuous function on $\mathbb{S}$. If $N > n$ then*

$$\begin{aligned}
&\left|\int_{\mathbb{S}} \mathcal{K}_0^N(\Omega f)\,\psi\, d\sigma\right| \\
&\lesssim \sum_{k=1}^{n+N}\sum_{j=1}^{n}\int_{\mathbb{B}} |\partial g(w)||f(w)|(1-|w|^2)^N \left|\int_{\mathbb{S}} \frac{\bar{\zeta}_j \psi(\zeta)}{(1-w\bar{\zeta})^k} d\sigma(\zeta)\right| d\nu(w) \\
&\quad + \int_{\mathbb{B}} (1-|w|^2)\Big(|\partial g(w)|^2|f(w)| + |\partial g(w)||\partial f(w)|\Big)\mathbb{P}(|\psi|)(w)d\nu(w) \\
&\quad + \int_{\mathbb{B}} \Big(|\partial_T g(w)|^2|f(w)| + |\partial_T g(w)||\partial_T f(w)|\Big)\mathbb{P}(|\psi|)(w)d\nu(w).
\end{aligned}$$

Proof. The proof is a consequence of Lemma 5.3, Fubini's Theorem and the estimate $1-|w|^2 \le 2|1-\zeta\bar{w}|$. □

Lemma 5.5. *If $\theta \in \mathcal{A}_2$ then for any positive integer N we have*

$$\sum_{k=1}^{n+N}\int_{\mathbb{B}} \Theta(w)(1-|w|^2)^{2N-1}\left|\int_{\mathbb{S}} \frac{\varphi(\zeta)}{(1-w\bar{\zeta})^k} d\sigma(\zeta)\right|^2 d\nu(w) \lesssim \|\varphi\|^2_{L^2(\theta)}.$$

Proof. Since $\theta \in \mathcal{A}_2$, we have

$$\|\mathcal{C}(\varphi)\|_{H^2(\theta)} = \left\|\int_{\mathbb{S}} \frac{\varphi(\zeta)}{(1-w\bar{\zeta})^n} d\sigma(\zeta)\right\|_{H^2(\theta)} \lesssim \|\varphi\|_{L^2(\theta)}.$$

Next, if $n \le k \le n+N$, we have

$$\int_{\mathbb{S}} \frac{\varphi(\zeta)}{(1-w\bar{\zeta})^k} d\sigma(\zeta) = \mathcal{R}_n^{k-n}\mathcal{C}(\varphi)(z),$$

(see (2.1) for the definition of $\mathcal{R}_n^{k-n}$). Therefore, the desired result follows from Proposition 2.6.

In order to prove the case $1 \le k < n$, observe that

$$\mathcal{C}(\varphi)(z) = \mathcal{R}_k^{n-k}\int_{\mathbb{S}} \frac{\varphi(\zeta)}{(1-w\bar{\zeta})^k} d\sigma(\zeta).$$

By Proposition 2.6 and the fact that $\|\mathcal{C}(\varphi)\|_{H^2(\theta)} \lesssim \|\varphi\|_{L^2(\theta)}$ we conclude the proof. □

Proposition 5.6. *If $N > n$ and $\theta \in \mathcal{A}_2$ then*

$$\|\mathcal{K}_0^N(\Omega f)\|_{L^2(\theta)} \lesssim \|f\|_{H^2(\theta)}.$$

Proof. Let $\theta' = \theta^{-1} \in \mathcal{A}_2$ and let Θ and Θ' be the corresponding averages of θ and θ'. Let also

$$d\mu_{g,\theta}(z) = \Theta(z)\left((1-|z|^2)|\partial g(z)|^2 + |\partial_T g(z)|^2\right) d\nu(z).$$

By Proposition 3.5, $\mu_{g,\theta}$ is a Carleson measure for $H^2(\theta)$.

Let $\psi \in L^2(\theta')$ and let

$$\Psi(w) = \sum_{k=1}^{n+N}\sum_{j=1}^{n}(1-|w|^2)^{N-1/2}\left|\int_{\mathbb{S}} \frac{\bar{\zeta}_j \psi(\zeta)\theta(\zeta)}{(1-w\bar{\zeta})^k} d\sigma(\zeta)\right|.$$

By Lemma 5.4, Hölder's Inequality and the fact that $\Theta(z)^{\frac{1}{2}}\Theta'(z)^{\frac{1}{2}} \approx 1$, we have

$$\begin{aligned}
\int_{\mathbb{S}} \mathcal{K}_0^N(\Omega f)\psi\, d\sigma &= \left|\int_{\mathbb{S}} \left(\mathcal{Q}_0^{N,1}(\Omega f) + \mathcal{Q}_0^{N,2}(\partial(\Omega f))\right) \psi d\sigma\right| \\
&\lesssim \left(\int_{\mathbb{B}} |f|^2 d\mu_{g,\theta}\right)^{1/2} \left(\int_{\mathbb{B}} \Theta'|\Psi|^2 d\nu\right)^{1/2} \\
&\quad + \left(\int_{\mathbb{B}} |f|^2 d\mu_{g,\theta}\right)^{1/2} \left(\int_{\mathbb{B}} |\mathbb{P}(\psi)|^2 d\mu_{g,\theta'}\right)^{1/2} \\
&\quad + \left(\int_{\mathbb{B}} ((1-|w|^2)|\partial f|^2 + |\partial_T f|^2)\Theta d\nu\right)^{1/2} \left(\int_{\mathbb{B}} |\mathbb{P}(\psi)|^2 d\mu_{g,\theta'}\right)^{1/2}.
\end{aligned}$$

Therefore, Propositions 2.7 and 3.5, and Lemma 5.5 give

$$\left|\int_{\mathbb{S}} \mathcal{K}_0^N(\Omega f)\, \psi d\sigma\right| \lesssim \|f\|_{H^2(\theta)}\|\psi\|_{L^2(\theta')}.$$

This estimate combined with the fact that $(L^2(\theta))' \equiv L^2(\theta')$ concludes the proof. □

As a consequence of the above proposition we have:

Theorem 5.7. *Let* $g_1, g_2 \in H^\infty$ *satisfy* $\inf\{|g(z)| : z \in \mathbb{B}\} > 0\}$. *If* $N > n$ *then* $\mathcal{T}_g^N(f) = Gf - g^\perp \mathcal{K}_0^N(\Omega f)$ *is a bounded operator from* $H^2(\theta)$ *to* $H^2(\theta) \times H^2(\theta)$ *for any* $\theta \in \mathcal{A}_2$.

6. The $H^2(\theta)$-corona problem for m generators

It is a well-known fact that one way to prove the corona problem with m generators is based on the so-called Koszul complex. In order to make the reading easier, we briefly recall this method using the notation of [8, Theorem 3.1], which provides solutions of the corona problem for H^p.

6.1. The Koszul complex

We begin with some notation and definitions. Let $E = \{e_1, ..., e_m\}$ be a basis of $\mathbb{C}^m$ and let E^* be the corresponding dual basis. We denote by $\Lambda^l = \Lambda^l(E)$ the set of all elements $e_I = e_{i_1} \sqcap \ldots \sqcap e_{i_l}$, where $I = \{i_1, \ldots, i_l\}$, of degree l of the exterior algebra $\Lambda = \Lambda(E)$. In order to avoid any confusion in our notation, we use $\sqcap$ to denote the exterior multiplication in Λ and $\wedge$ to denote the exterior product of differential forms. If $v^* \in E^*$ then $\delta_{v^*} : \Lambda^{l+1} \to \Lambda^l$ denotes the anti-derivation defined by

$$\delta_{v^*}(e_{i_1} \sqcap \ldots \sqcap e_{i_l}) = \sum_{j=1}^{l} (-1)^{j-1} v_j e_{i_1} \sqcap \ldots \sqcap e_{i_{j-l}} \sqcap e_{i_{j+l}} \ldots \sqcap e_{i_l}.$$

Let $\mathcal{E}_q$ denote the space of $(0, q)$-forms with coefficients in $\mathcal{C}^\infty(\bar{\mathbb{B}})$ and $\mathcal{E} = \cup_{q=0}^n \mathcal{E}_q$.

We also consider the space $\mathcal{E}_q(\Lambda)$ of Λ-valued forms

$$\sum_I \eta_I e_I, \qquad \eta_I \in \mathcal{E}_q,$$

and the union $\mathcal{E}(\Lambda) = \cup_{q=0}^n \mathcal{E}_q(\Lambda)$ of these spaces.

We will use similar notation to consider other Λ-valued spaces of functions. For instance, $H^2(\theta, \Lambda)$ consists of sums of $h_I(z) e_I$ with $h_I \in H^2(\theta)$.

For $F = \sum_I \eta_I e_I$ and $H = \sum_J \vartheta_J e_J \in \mathcal{E}(\Lambda)$ we let

$$F \sqcap G = \sum_{I,J} \eta_I \wedge \vartheta_J \, e_I \sqcap e_J.$$

If $\mathcal{K} : \mathcal{E} \to \mathcal{E}$ is a linear operator, we will also use $\mathcal{K}$ to denote the operator defined on $\mathcal{E}(\Lambda)$ by $\mathcal{K}(\eta_I \, e_I) = \mathcal{K}(\eta_I) \, e_I$. If $h^* = \sum_{j=1}^m h_j(z) \sqcap e_j* \in \mathcal{E}_0((\Lambda^*)^1)$ (that is $h_j \in \mathcal{C}^\infty(\bar{B})$), let $\delta_{h^*}(\eta_I e_I) = \eta_I \sqcap \delta_{h^*}(e_I) = \sum_{j=1}^m h_j \eta_I \delta_{e_j^*} e_I$. We denote by $\delta_{h^*}\mathcal{K} : \mathcal{E}(\Lambda) \to \mathcal{E}(\Lambda)$ the composition of $\mathcal{K}$ and δ_{h^*}, that is $(\delta_{h^*}\mathcal{K})(\eta_I e_I) = \mathcal{K}(\eta) \sqcap \delta_{h^*}(e_I) = \sum_{j=1}^m h_j \mathcal{K}(\eta_I) \delta_{e_j^*} e_I$.

Now we can give an explicit formula that solves the corona problem. Observe that $\sum_{j=1}^m g_j F_j = f$ can be written as $\delta_{g^*} F = f$, where $g^* = \sum_{j=1}^m g_j(z) e_j^*$ and $F = \sum_{j=1}^m F_j(z) e_j$.

As usual, if $F = \sum_I \eta_I \, e_I$, then $|F|$ denotes the pointwise norm $|F| = \sum_I |\eta_I|$.

Let us introduce a family of kernels $\mathcal{K}^N$ and their corresponding integral operators $\mathcal{K}^N : \mathcal{E} \to \mathcal{E}$ which satisfy $\bar\partial \mathcal{K}^N(\eta) = \eta$ for any $(0, q+1)$-form η such that $\bar\partial \eta = 0$ (see for instance [14]).

For $N \geq 0$, we consider the kernel

$$\mathcal{K}^N(w, z) = \sum_{k=0}^{n-1} c_{k,N} \frac{(1 - |w|^2)^{N+k}}{(1 - \bar{w}z)^{N+k}} \frac{(s \wedge (\bar\partial \, s)^{n-1-k})(w, z)}{\phi^{n-k}(w, z)} \wedge (\gamma(w))^k,$$

where $\bar\partial = \bar\partial_w + \bar\partial_z$ (∂ in both the variables the w and z), and

$$\begin{aligned} \gamma(w) &= \bar\partial\,\frac{\partial|w|^2}{1-|w|^2} = \frac{\bar\partial\partial|w|^2}{1-|w|^2} + \frac{\bar\partial|w|^2\wedge\partial|w|^2}{(1-|w|^2)^2} \\ s(w,z) &= (1-w\bar z)\partial|w|^2 - (1-|w|^2)\partial(w\bar z) \\ &= (1-w\bar z)\partial(|w|^2 - w\bar z) + (|w|^2 - w\bar z)\partial(w\bar z) \\ \phi(w,z) &= |1-\bar w z|^2 - (1-|w|^2)(1-|z|^2) \\ &= |(w-z)\bar w|^2 + (1-|w|^2)|w-z|^2. \end{aligned} \tag{6.1}$$

Set $\mathcal{K}^N = \sum_{q=0}^{n-1}\mathcal{K}_q^N$, where $\mathcal{K}_q^N$ denotes the component in $\mathcal{K}^N$ of bidegree $(0,q)$ in z and $(n, n-q-1)$ in w. If $q=0$, then $\mathcal{K}_0^N(w,z)$ coincides with the kernel in Proposition 5.1.

Formulas (6.1) together with

$$\begin{aligned} \bar\partial_w s(w,z) &= (1-w\bar z)\bar\partial_w\partial_w|w|^2 - \partial_w(w\bar z)\wedge\bar\partial_w|w|^2 \\ \bar\partial_z s(w,z)) &= -\bar\partial_z(w\bar z)\wedge\partial_w|w|^2 - (1-|w|^2)\bar\partial_z\partial_w(w\bar z) \\ &= \bar\partial_z(|z|^2 - w\bar z)\wedge\partial_w|w|^2 - (1-|w|^2)\bar\partial_z\partial_w(w\bar z) - \bar\partial_z|z|^2\wedge\partial_w|w|^2, \\ \bar\partial_w(w\bar z)\wedge\partial_w|w|^2 &= \partial_w(w\bar z - |w|^2)\wedge\partial_w|w|^2 \end{aligned}$$

give (see [17, page 69]) the following decomposition of $\mathcal{K}_q^N(w,z)$:

Lemma 6.1.

$$\mathcal{K}_q^N(w,z) = \mathcal{K}_q^{N,1}(w,z) + \mathcal{K}_q^{N,2}(w,z)\wedge\bar\partial|w|^2 + \mathcal{K}_q^{N,3}(w,z)\wedge\bar\partial|z|^2, \tag{6.2}$$

with the following estimates:

$$\begin{aligned} |\mathcal{K}_q^{N,1}(w,z)| &\lesssim \mathcal{L}^{N+1}_{N+1-n,n-1/2}(w,z), \\ |\mathcal{K}_q^{N,2}(w,z)|,\ |\mathcal{K}_q^{N,3}(w,z)| &\lesssim \mathcal{L}^{N+1/2}_{N+1-n,n-1/2}(w,z). \end{aligned} \tag{6.3}$$

(*The kernels* $\mathcal{L}^N_{M,L}$ *are introduced in Definition* 2.1).

Note that, if $q=0$, then $\mathcal{K}_0^N$ does not contain the terms $d\bar z_j$, and therefore $\mathcal{K}_0^{N,3}=0$. Analogously, $\mathcal{K}_{n-1}^{N,2}=0$.

If $\zeta\in\mathbb{S}$ then $\phi(w,\zeta) = |1-\zeta\bar w|^2$ and

$$|\mathcal{K}_0^{N,1}(w,\zeta)| \lesssim \mathcal{L}^{N+1}_{n+N,0}(w,\zeta), \qquad |\mathcal{K}_0^{N,2}(w,\zeta)| \lesssim \mathcal{L}^{N+1/2}_{n+N,0}(w,\zeta). \tag{6.4}$$

Observe that, by (6.3), $|\mathcal{K}_q^{N,1}|$ is bounded by a kernel of type 1, and $|\mathcal{K}_q^{N,2}|$ and $|\mathcal{K}_q^{N,3}|$ are bounded by kernels of type 1/2. Therefore, $|\mathcal{K}_q^N|$ is globally bounded by a kernel of type 1/2.

Now, given $g = (g_1, \dots, g_m) \in H^\infty$ satisfying $\inf_{z\in\mathbb{B}} |g(z)| > 0$, let $G_j = \frac{\bar g_j}{|g|^2}$ and let $G = \sum_{j=1}^m G_j(z)e_j$. Clearly, $\delta_{g^*}(G) = g\,G = 1$.

Then, we will use the following formula which provides a solution of the corona problem on Hardy spaces.

Theorem 6.2 ([8]). *If* $1 \le p < \infty$ *and* $g = (g_1, \dots, g_m)$*, with* $g_j \in H^\infty$*, satisfies* $\inf_{z\in\mathbb{B}} |g(z)| > 0$*, then the linear operator*

$$\mathcal{T}_g^N(f) = \sum_{k=0}^{\min(n,m-1)} (-1)^k \left(\delta_{g^*}\mathcal{K}^N\right)^k \left(fG \sqcap (\bar\partial G)^k\right) \tag{6.5}$$

maps H^p *to* $H^p(\Lambda^1)$*, with* $1 \le p < \infty$*, and* $\delta_{g^*}(\mathcal{T}_g^N(f)) = f$.

In order to facilitate the reading of this paper, we will give the explicit computations of $\mathcal{T}_g^N(f)$ for $m = 2$ and $m = 3$, and $n \ge 3$.

If $m = 2$ formula (6.5) coincides with that of Section 5. In order to prove this observe that, by bidegree arguments, the term $k = 1$ in (6.5) is

$$\begin{aligned}&(\delta_{g^*}\mathcal{K}^N)\left(f(G_1e_1 + G_2e_2) \sqcap (\bar\partial G_1 e_1 + \bar\partial G_2 e_2)\right)\\ &= (\delta_{g^*}\mathcal{K}_0^N)\left(f(G_1e_1 + G_2e_2) \sqcap (\bar\partial G_1 e_1 + \bar\partial G_2 e_2)\right)\end{aligned}$$

and that

$$\begin{aligned}&(\delta_{g^*}\mathcal{K}_0^N)\left(f(G_1e_1 + G_2e_2) \sqcap (\bar\partial G_1 e_1 + \bar\partial G_2 e_2)\right)\\ &= (\delta_{g^*}\mathcal{K}_0^N)\left(f(G_1\bar\partial G_2 - G_2\bar\partial G_1)e_1 \sqcap e_2\right)\\ &= \mathcal{K}_0^N(fG_1\bar\partial G_2 - fG_2\bar\partial G_1)(g_1e_2 - g_2e_1).\end{aligned}$$

Employing the notation of Section 5, by (5.2) we have $fG_1\bar\partial G_2 - fG_2\bar\partial G_1 = f\Omega$ and, therefore,

$$\mathcal{T}_g^N(f) = (fG_1 + g_2\mathcal{K}_0^N(f\Omega))e_1 + (fG_2 - g_1\mathcal{K}_0^N(f\Omega))e_2, \tag{6.6}$$

which coincides with (5.3).

If $m = 3$ then similar computations prove that the term $k = 1$ in (6.5) is

$$\begin{aligned}(\delta_{g^*}\mathcal{K}_0^N)(G \sqcap \bar\partial G) &= \mathcal{K}_0^N(fG_1\bar\partial G_2 - fG_2\bar\partial G_1)(g_1e_2 - g_2e_1)\\ &\quad + \mathcal{K}_0^N(fG_2\bar\partial G_3 - fG_3\bar\partial G_2)(g_2e_3 - g_3e_2)\\ &\quad + \mathcal{K}_0^N(fG_3\bar\partial G_1 - fG_1\bar\partial G_3)(g_3e_1 - g_1e_3).\end{aligned}$$

Now, if $\Omega_{i,j} = \begin{vmatrix} G_i & G_j \\ \bar\partial G_i & \bar\partial G_j \end{vmatrix} = G_i\bar\partial G_j - G_j\bar\partial G_i$ then

$$\begin{aligned}(\delta_{g^*}\mathcal{K}_0^N)(G \sqcap \bar\partial G) &= (g_2\mathcal{K}_0^N)(f\Omega_{2,1}) + g_3\mathcal{K}_0^N(f\Omega_{3,1}))e_1\\ &\quad + (g_1\mathcal{K}_0^N(f\Omega_{1,2}) + g_3\mathcal{K}_0^N(f\Omega_{3,2}))e_2\\ &\quad + (g_1\mathcal{K}_0^N(f\Omega_{1,3}) + g_2\mathcal{K}_0^N(f\Omega_{2,3}))e_3.\end{aligned} \tag{6.7}$$

In order to calculate the term $k = 2$ in (6.5), set

$$\Omega_{123} = \begin{vmatrix} G_1 & G_2 & G_3 \\ \bar\partial G_1 & \bar\partial G_2 & \bar\partial G_3 \\ \bar\partial G_1 & \bar\partial G_2 & \bar\partial G_3 \end{vmatrix} = 2\left(G_1\bar\partial G_2 \wedge \bar\partial G_3 + G_2\bar\partial G_3 \wedge \bar\partial G_1 + G_3\bar\partial G_1 \wedge \bar\partial G_2\right).$$

It is easy to check that $G \sqcap \bar\partial G \sqcap \bar\partial G = \Omega_{123}\ e_1 \sqcap e_2 \sqcap e_3$. The use of the determinants of forms to formulate the Koszul complex can be found in [22]. Therefore,

$$(\delta_{g^*}\mathcal{K}_1^N)(\Omega_{123}\ e_1 \sqcap e_2 \sqcap e_3) = g_1\mathcal{K}_1^N(\Omega_{123})\, e_2 \sqcap e_3 + g_2\mathcal{K}_1^N(\Omega_{123})\, e_3 \sqcap e_1 + g_3\mathcal{K}_1^N(\Omega_{123})\, e_1 \sqcap e_2,$$

and

$$\begin{aligned}(\delta_{g^*}\mathcal{K}_0^N)(\delta_{g^*}\mathcal{K}_1^N)(\Omega_{123}\ e_1 \sqcap e_2 \sqcap e_3) = {} & \mathcal{K}_0^N(g_1\mathcal{K}_1^N(\Omega_{123}))\,(g_2e_3 - g_3e_2) \\ & + \mathcal{K}_0^N(g_2\mathcal{K}_1^N(\Omega_{123}))\,(g_3e_1 - g_1e_3) \\ & + \mathcal{K}_0^N(g_3\mathcal{K}_1^N(\Omega_{123}))\,(g_1e_2 - g_2e_1).\end{aligned}$$

Observe that in general we have the following:

Lemma 6.3. *The coefficients of* $\left(\delta_{g^*}\mathcal{K}\right)^k\left(fG \sqcap (\bar\partial G)^k\right)$, *for* $k \geq 1$, *are linear combinations of terms of type*

$$F_l e_l = g_{i_0}\mathcal{K}_0(g_{i_1}(\mathcal{K}_1(....(g_{i_{k-1}}\mathcal{K}_{k-1}(fG_{j_0}\bar\partial G_{j_1} \wedge \ldots \wedge \bar\partial G_{j_k})))))\, e_l. \qquad (6.8)$$

To conclude, for completeness, we recall the proof of the fact that $\mathcal{T}_g^N(f) \in H(\Lambda^1)$, which can be found in [8, Theorem 3.1].

Let $r = \min\{n, m-1\}$ and let $V_k = G \sqcap (\bar\partial G)^k$. We define by induction the forms $U_r = V_r$ and $U_k = V_k - (\delta_{g^*}\mathcal{K})(U_{k+1})$, $0 \leq k < r$. Observe that $U_0 = \mathcal{T}_g^N(f)$, $\delta_{g^*}(\bar\partial G) = (\bar\partial(\delta_{g^*}G)) = 0$ and $\delta_{g^*}(V_k) = (\bar\partial G)^k$.

We want to prove that $\bar\partial U_k = 0$ for all $0 \leq k \leq r$. If $r = n$, then by bidegree arguments, the form $V_r = G \sqcap (\bar\partial G)^r$ satisfies $\bar\partial V_r = 0$. If $r = m-1$, then using $\delta_{g^*}\bar\partial G = 0$ we also obtain $\bar\partial V_r = 0$. Assume that $\bar\partial U_{k+1} = 0$. Since $\delta_{g^*}^2 = 0$, we have $\delta_{g^*}U_{k+1} = \delta_{g^*}V_{k+1} = (\bar\partial G)^{k+1} = \bar\partial V_k$. Therefore, we have

$$\bar\partial((\delta_{g^*}\mathcal{K}_k)(U_{k+1})) = \delta_{g^*}(\bar\partial\mathcal{K}_k(U_{k+1})) = \delta_{g^*}U_{k+1} = \bar\partial V_k,$$

which proves that $\bar\partial U_k = 0$.

6.2. Estimates of the terms appearing in (6.5)

We want to obtain $L^2(\theta)$-norm estimates of

$$\left|(\delta_{g^*}\mathcal{K}^N)^k\left(fG\sqcap(\bar\partial G)^k\right)(\zeta)\right|,\qquad k=0,\cdots,\min\{n,m-1\}.$$

By (6.6), the estimates with $k=0,1$ have been obtained in Section 5. In order to obtain the corresponding estimates for $k\ge 2$ we need the next:

Proposition 6.4. *For N large enough and $k\ge 2$, we have*

$$\left|(\delta_{g^*}\mathcal{K}^N)^k\left(fG\sqcap(\bar\partial G)^k\right)(\zeta)\right|$$
$$\lesssim\int_{\mathbb{B}}\frac{|f(w)|((1-|w|^2)|\partial g(w)|^2+|\partial_T g(w)|^2)\,(1-|w|^2)^n}{|1-\zeta\bar w|^{2n}}d\nu(w).$$

Assuming this result, it is easy to prove the corona theorem for $p=2$.

Theorem 6.5. *Let $g=(g_1,\dots,g_m)$, with $g_j\in H^\infty$, satisfy* $\inf_{z\in\mathbb{B}}|g(z)|>0$. *If N is large enough, then the operator $\mathcal{T}_g^N$ in Theorem* 6.2 *maps $H^2(\theta)$ to $H^2(\theta,\Lambda^1)$ for all $\theta\in\mathcal{A}_2$.*

Proof. The estimate of $(\delta_{g^*}\mathcal{K}^N)^0(fG)=fG$, corresponding to the term $k=0$ in (6.5), is clear. The estimate of the term $k=1$, that is $(\delta_{g^*}\mathcal{K}^N)(fG\sqcap\bar\partial G)$, follows arguing as in the case of two generators. Observe that as it happens in (6.7) the coefficients of the terms that appear in the representation of $(\delta_{g^*}\mathcal{K}^N)(fG\sqcap\bar\partial G)$ are of the same type of the expressions $g_j\mathcal{K}_0^N(f\Omega)$ considered in Section 5.

Therefore, it remains to consider the terms $k\ge 2$. For any $\psi\in L^2(\theta^{-1})$, Proposition 6.4 gives

$$\left|\int_{\mathbb{S}}F_l(\zeta)\psi(\zeta)d\sigma(\zeta)\right|\lesssim\int_{\mathbb{B}}|f(w)||\partial_T g(w)|^2\mathbb{P}(|\psi|)(w)d\nu(w)$$
$$+\int_{\mathbb{B}}(1-|w|^2)|f(w)||\partial g(w)|^2\mathbb{P}(|\psi|)(w)d\nu(w).$$

Thus, arguing as in Proposition 5.6, Hölder's inequality, the fact that $\Theta^{\frac12}\Theta'^{\frac12}\approx 1$ and Proposition 3.5 give

$$\left|\int_{\mathbb{S}}F_l(\zeta)\psi(\zeta)d\sigma(\zeta)\right|\lesssim\left(\int_{\mathbb{S}}|f|^2d\mu_{g,\theta}\right)^{1/2}\left(\int_{\mathbb{S}}|\mathbb{P}(\psi)|^2d\mu_{g,\theta'}\right)^{1/2}$$
$$\lesssim\|f\|_{H^2(\theta)}\|\psi\|_{L^2(\theta')},$$

which proves that $F_l(\zeta)\in L^2(\theta)$ and $\|F_l\|_{L^2(\theta)}\lesssim\|f\|_{H^2(\theta)}$. □

Therefore, it remains to prove Proposition 6.4.

Lemma 6.6. *For $k \geq 2$,*

$$(G_{j_0}\bar{\partial}G_{j_1} \wedge \ldots \wedge \bar{\partial}G_{j_k})(w) = \tilde{G}_R(w) + \tilde{G}_T(w) \wedge \bar{\partial}|w|^2 \tag{6.9}$$

with

$$\begin{aligned} |\tilde{G}_R(w)| &\lesssim (1-|w|^2)^2|\partial g(w)|^2 + (1-|w|^2)^{1-k/2}|\partial_T g(w)|^2 \\ |\tilde{G}_T(w)| &\lesssim (1-|w|^2)^{1/2-k/2}\Big[(1-|w|^2)|\partial g(w)|^2 + |\partial_T g(w)|^2\Big], \end{aligned} \tag{6.10}$$

and consequently

$$|\tilde{G}(w)| \lesssim (1-|w|^2)^{1/2-k/2}\Big[(1-|w|^2)|\partial g(w)|^2 + |\partial_T g(w)|^2\Big].$$

Proof. The decomposition

$$\begin{aligned} \bar{\partial}G_l(w) &= (1-|w|^2)\bar{\partial}G_l(w) + \sum_{j=1}^n\sum_{i=1}^n \bar{w}_i w_j \bar{D}_i G_l(w) d\bar{w}_j \\ &\quad + \sum_{j=1}^n\sum_{i=1}^n \bar{w}_i(w_i\bar{D}_j G_l(w) - w_j\bar{D}_i G_l(w))d\bar{w}_j \\ &= (1-|w|^2)\bar{\partial}G_l(w) + \bar{\mathcal{R}}G_l(w)\bar{\partial}|w|^2 + \sum_{i,j}\bar{w}_i\bar{D}_{i,j}G_l(w)d\bar{w}_j, \end{aligned}$$

and $(1-|w|^2)^{1/2}|\partial_T g(w)| + (1-|w|^2)|\partial g(w)| \lesssim 1$, prove (6.9) with

$$\begin{aligned} |\tilde{G}_R(w)| &\lesssim \Big[(1-|w|^2)|\partial g(w)| + |\partial_T g(w)|\Big]^k \\ &\lesssim (1-|w|^2)^2|\partial g(w)|^2 + (1-|w|^2)^{1-k/2}|\partial_T g(w)|^2 \\ |\tilde{G}_T(w)| &\lesssim \Big[(1-|w|^2)|\partial g(w)| + |\partial_T g(w)|\Big]^{k-1}|\partial g(w)|. \end{aligned}$$

Since $k \geq 2$, $(1-|w|^2)^{k-1}|\partial g(w)|^k \lesssim (1-|w|^2)|\partial g(w)|^2$ and

$$\begin{aligned} |\partial_T g(w)|^{k-1}|\partial g(w)| &\lesssim (1-|w|^2)^{1-k/2}|\partial_T g(w)||\partial g(w)| \\ &\lesssim (1-|w|^2)^{1/2-k/2}\Big(|\partial_T g(w)|^2 + (1-|w|^2)|\partial g(w)|^2\Big) \end{aligned}$$

which concludes the proof. □

The next lemma is well-known (see for instance [24, Lemma 2.5]).

Lemma 6.7. *If* $0 \le A, B < N < n + A + B$ *and* $z, w \in \mathbb{B}$*, then*

$$\int_{\mathbb{B}} \frac{(1-|u|^2)^{N-1}}{|1-z\bar{u}|^{n+A}|1-u\bar{w}|^{n+B}}\, d\nu(u) \lesssim \frac{1}{|1-z\bar{w}|^{n+A+B-N}}.$$

Lemma 6.8. *If the kernel* $\mathcal{L}^N_{M,L}$ *has type* $\kappa = n + N - M - 2L > 0$*, then for* $\kappa - n < A \le N$ *and* $B \ge 0$*,*

$$\int_{\mathbb{B}} \mathcal{L}^N_{n+A,0}(z,\zeta)\mathcal{L}^{N+B}_{M+B,L}(w,z)d\nu(z) \lesssim \mathcal{L}^N_{n+A-\kappa,0}(w,\zeta).$$

Observe that the type of $\mathcal{L}^N_{n+A-\kappa,0}$ *is the sum of the types of* $\mathcal{L}^N_{n+A,0}$ *and* $\mathcal{L}^N_{M,L}$.

Proof. Since $\mathcal{L}^{N+B}_{M+B,L}(w,z) \lesssim \mathcal{L}^N_{M,L}(w,z)$ we can consider $B = 0$.

The left-hand side term in the above inequality is

$$I(w,\zeta) = (1-|w|^2)^{N-1}\int_{\mathbb{B}} \frac{(1-|z|^2)^{N-1}}{|1-\zeta\bar{z}|^{n+A}|1-z\bar{w}|^M\,\phi(w,z)^L} d\nu(z).$$

Let $\varphi_w(z)$ denotes the automorphism of the unit ball which maps w to 0. We will use the change of variables $u = \varphi_w(z)$ and the formulas in [28, Section 2.2] to reduce the estimate to that of Lemma 6.7.

Since

$$1 - \varphi_w(z)\overline{\varphi_w(u)} = \frac{(1-|w|^2)(1-z\bar{u})}{(1-z\bar{w})(1-w\bar{u})}, \tag{6.11}$$

we have

$$1-|\varphi_w(u)|^2 = \frac{(1-|w|^2)(1-|u|^2)}{|1-u\bar{w}|^2}, \qquad \text{and} \qquad \phi(w,z) = |1-z\bar{w}|^2|\varphi_w(z)|^2.$$

Therefore, the change of variables $u = \varphi_w(z)$ gives

$$I(w,\zeta) = c\int_{\mathbb{B}} \frac{(1-|w|^2)^{N-1}(1-|\varphi_w(u)|^2)^{N-1}}{|1-\zeta\overline{\varphi_w(u)}|^{n+A}|1-\varphi_w(u)\bar{w}|^{M+2L}|u|^{2L}} \frac{(1-|w|^2)^{n+1}}{|1-u\bar{w}|^{2n+2}}\, d\nu(u).$$

By (6.11),

$$\begin{aligned} 1-\varphi_w(u)\bar{w} &= 1-\varphi_w(u)\overline{\varphi_w(0)} = \frac{1-|w|^2}{1-u\bar{w}} \\ 1-\zeta\overline{\varphi_w(u)} &= 1-\varphi_w(\varphi_w(\zeta))\overline{\varphi_w(u)} = \frac{(1-\zeta\bar{w})(1-\varphi_w(\zeta)\bar{u})}{1-u\bar{w}}, \end{aligned}$$

therefore,

$$I(w,\zeta)=c\frac{(1-|w|^2)^{N-1+\kappa}}{|1-\zeta\bar{w}|^{N+A}}\int_{\mathbb{B}}\frac{(1-|u|^2)^{N-1}}{|1-\varphi_w(\zeta)\bar{u}|^{n+A}|1-u\bar{w}|^{N+\kappa-A}|u|^{2L}}\,d\nu(u).$$

We decompose the above integral in the sum of the integral in the ball $\frac{1}{2}\mathbb{B}=\{u\in\mathbb{B};\,|u|\le 1/2\}$ and of the integral in its complementary set. Since $L<2n$ and $|1-\bar{u}z|\approx 1$ on $\frac{1}{2}\mathbb{B}$, the integral in this set is bounded. By Lemma 6.7, the integral over the complementary of $\frac{1}{2}\mathbb{B}$ is bounded by $\frac{1}{|1-\varphi_w(\zeta)\bar{w}|^{\kappa}}=\frac{|1-\zeta\bar{w}|^{\kappa}}{(1-|w|^2)^{\kappa}}$, which concludes the estimate. □

Proof of Proposition 6.4. Observe that by the decomposition obtained in (6.2), and the facts that $\bar\partial|w|^2\wedge\bar\partial|w|^2=0$,

$$\begin{aligned}&\mathcal{K}_q^{N,2}(z,u)\wedge\mathcal{K}_{q+1}^{N,3}(w,z)=0,\quad\text{for all } 0\le q\le n-1,\ \text{and}\\&\mathcal{K}_0^{N,3}=0,\end{aligned}\tag{6.12}$$

the term in (6.8) is a sum of terms of type

$$F_1=g_{i_0}\mathcal{K}_0^{N,j_0}(g_{i_1}(\mathcal{K}_1^{N,j_1}(\dots(g_{i_{k-1}}\mathcal{K}_{k-1}^{N,j_{k-1}}(f\tilde{G}))\dots))),$$

with $j_l=1,2,3$ and at least one of them equal to 1, and one term of type

$$F_2=g_{i_0}\mathcal{K}_0^{N,2}(g_{i_1}(\mathcal{K}_1^{N,2}(\dots(g_{i_{k-1}}\mathcal{K}_{k-1}^{N,2}(f\tilde{G}_T))\dots))).$$

Observe that, by (6.12), all terms including $\mathcal{K}_q^{N,3}(f\tilde{G})$ are considered in the first type F_1.

Since $|\mathcal{K}_q^{N,1}|$ is bounded by a kernel of type 1 and $|\mathcal{K}_q^{N,2}|$, $|\mathcal{K}_q^{N,3}|$ are bounded by a kernel of type 1/2, the kernels in F_1 are bounded by a product of kernels of type 1 or 1/2 whose sum of types is greater than or equal to $(k-1)/2+1=(k+1)/2$.

Analogously, the kernels in F_2 are bounded by a product of kernels of type 1/2 and whose sum of types is equal to $k/2$.

Therefore, if N is large enough, the pointwise estimate of $\tilde{G}$ in Lemma 6.6, together Lemma 6.8 give

$$\begin{aligned}|F_1(\zeta)|&\lesssim\int_{\mathbb{B}}\frac{(1-|w|^2)^{N-1/2}}{|1-\zeta\bar{w}|^{n+N-k/2}}(1-|w|^2)^{1/2-k/2}d\mu_g(w)\\&=\int_{\mathbb{B}}\frac{(1-|w|^2)^{N-k/2}}{|1-\zeta\bar{w}|^{n+N-k/2}}d\mu_g(w),\end{aligned}$$

where $d\mu_g(w)=\big[(1-|w|^2)|\partial g(w)|^2+|\partial_T g(w)^2|\big]\,d\nu(w)$.

Analogously,

$$|F_2(\zeta)| \lesssim \int_{\mathbb{B}} \frac{(1-|w|^2)^{N-1/2}}{|1-\zeta\bar{w}|^{n+N+1/2-k/2}}(1-|w|^2)^{1-k/2}d\mu_g(w)$$
$$= \int_{\mathbb{B}} \frac{(1-|w|^2)^{N+1/2-k/2}}{|1-\zeta\bar{w}|^{n+N+1/2-k/2}}d\mu_g(w).$$

Therefore, taking $N \geq n + k/2$ we obtain the estimate in Proposition 6.4. □

7. End of the proof of Theorem 1.1

7.1. Corona theorem for weighted Hardy spaces

The following extrapolation theorem was proved in [27] (see also [31, page 233]).

Theorem 7.1. *Let* $1 < r < \infty$, *and* $\mathcal{T}$ *be a sublinear operator which is bounded on* $L^r(\theta)$ *for any* $\theta \in \mathcal{A}_r$, *with constant depending only on the constant* $\mathcal{A}_r(\theta)$ *of the condition* $\mathcal{A}_r$. *Then* $\mathcal{T}$ *is bounded on* $L^p(\theta)$ *for any* $1 < p < \infty$ *and any* $\theta \in \mathcal{A}_p$, *with constant depending only on* $\mathcal{A}_p(\theta)$.

Next we will prove the corona theorem for $H^p(\theta)$.

Theorem 7.2. *Let* $1 < p < \infty$ *and* $0 < s < n/p$. *Let* $g_1, \dots, g_m \in H^\infty$. *Then the following assertions are equivalent*:

(i) *The functions* g_k, $k = 1, \dots, m$ *satisfy* $\inf\{|g(z)| : z \in \mathbb{B}\} > 0$*;*
(ii) $\mathcal{M}_g$ *maps* $H^p(\theta) \times \cdots \times H^p(\theta)$ *onto* $H^p(\theta)$ *for any* $1 < p < \infty$ *and any* $\theta \in \mathcal{A}_p$*;*
(iii) $\mathcal{M}_g$ *maps* $H^p(\theta) \times \cdots \times H^p(\theta)$ *onto* $H^p(\theta)$ *for some* $1 < p < \infty$ *and some* $\theta \in \mathcal{A}_p$*;*
(iv) $\mathcal{M}_g$ *maps* $H^2(\theta) \times \cdots \times H^2(\theta)$ *onto* $H^2(\theta)$ *for any* $\theta \in \mathcal{A}_2$.

Proof. We will follow the scheme (ii) ⇒ (iii) ⇒ (i) ⇒ (iii) ⇒ (ii).

Clearly (ii) ⇒ (iii). The implication (iii) ⇒ (i) is proved in Proposition 4.3. The proof of (i) ⇒ (iv) is given in Theorem 6.5 using the linear operator $\mathcal{T}_g^N$. The proof of (iv) ⇒ (ii) follows from Theorem 7.1 applied to $r = 2$ and to each one of the operators $\mathcal{T} = \mathcal{T}_{g,i}^N \circ \mathcal{C}$, $i = 1, \dots, m$. Here $\mathcal{T}_{g,i}^N$ are the components of the operator $\mathcal{T}_g^N$ and $\mathcal{C}$ is the Cauchy kernel. □

7.2. Corona theorem for Morrey spaces

The following result was proved in [9, Theorem 3.1].

Theorem 7.3. *Let φ and ψ be non-negative Borel measurable functions on $\mathbb{S}$. Suppose that for each $\alpha \geq 1$ and every bounded weight $\theta \in \mathcal{A}_1$, such that $\mathcal{A}_1(\theta) \leq \alpha$, there exists $c(\alpha)$ such that*

$$\int_{\mathbb{S}} \varphi\theta d\sigma \leq c(\alpha) \int_{\mathbb{S}} \psi\theta d\sigma.$$

Then, for $0 < t < n$, there exists a constant C depending on n and t, such that $\|\varphi\|_{M^{1,t}} \leq C\|\psi\|_{M^{1,t}}$ for any $\varphi, \psi \in M^{1,t}$.

Theorem 7.4. *Let $1 < p < \infty$ and $0 < s < n/p$. Let $g_1, \ldots, g_m \in H^\infty$. Then the following assertions are equivalent*:

(i) *The functions g_k, $k = 1, \ldots, m$ satisfy $\inf\{|g(z)| : z \in \mathbb{B}\} > 0$;*
(ii) *$\mathcal{M}_g$ maps $HM^{p,s} \times \cdots \times HM^{p,s}$ onto $HM^{p,s}$ for any $1 < p < \infty$ and any $0 < s < n/p$;*
(iii) *$\mathcal{M}_g$ maps $HM^{p,s} \times \cdots \times HM^{p,s}$ onto $HM^{p,s}$ for some $1 < p < \infty$ and some $0 < s < n/p$.*

Proof. The scheme of the proof of the Morrey case is similar and we will show in this case that

$$\text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(i)} \Leftrightarrow \text{(ii)[Theorem 7.2]} \Rightarrow \text{(ii)}.$$

The first implication is obvious, and the proof of the second one is given in Proposition 4.5. The proof of (ii)[Theorem 7.2] $\Rightarrow$ (ii) follows from Theorem 7.3. Observe that, if $1 < p < \infty$, $\varphi = |\mathcal{T}_g(f)|^p$, $\psi = |f|^p$ and $t = sp < n$, then the fact that $\mathcal{A}_1 \subset \mathcal{A}_p$, (ii)[Theorem 7.2] and Theorem 7.3 give

$$\||\mathcal{T}_g(f)|\|^p_{M^{p,s}} = \||\mathcal{T}_g(f)|^p\|_{M^{1,sp}} \leq C\||f|^p\|_{M^{1,sp}} = \|f\|^p_{M^{p,s}},$$

which proves the result. □

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