Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)
Vol. XIII (2014), 551-577

Monodromy of Lauricella's hypergeometric F_A-system

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Abstract. We give a monodromy representation of Lauricella's system of differential equations annihilating the hypergeometric series $F_A(a, (b), (c); x)$ of k-variables; its rank is 2^k. Under some non-integral conditions for parameters a, $(b) = (b_1, \dots, b_k)$, $(c) = (c_1, \dots, c_k)$, we find circuit matrices with respect to solutions represented by integrals. We make use of the intersection numbers of the domains of the integrals.

Mathematics Subject Classification (2010): 32S40 (primary); 33C65 (secondary).

1. Introduction

We give a monodromy representation of Lauricella's system of differential equations annihilating the hypergeometric series $F_A(a, (b), (c); x)$ of k-variables; its rank is 2^k. Under some non-integral conditions for parameters a, $(b) = (b_1, \dots, b_k)$, $(c) = (c_1, \dots, c_k)$, we find circuit matrices with respect to solutions represented by integrals. We make use of the intersection numbers of the domains of the integrals regarded as bases of a twisted homology group.

In general, we have the following principle: Suppose that a local solution space of a system of hypergeometric differential equations can be identified with a twisted homology group with intersection form $\mathcal{I}$. If the Jordan normal form of the circuit transformation m_ρ along a loop ρ is diagonal with two eigenvalues, say α and β, and either the eigenspace belonging to the eigenvalue α or that to β is specified then m_ρ is uniquely determined by the specified eigenspace and the intersection form $\mathcal{I}$.

We apply this principle in this paper to Lauricella's system of type A, and find a set of generators of the monodromy group. When the number of variables is two, this system is called Appell's F_2, of which monodromy group is studied by several authors; refer to [4] and the references therein.

This principle is applied to finding generators of the monodromy group of Lauricella's system of type D in [8].

Received October 26, 2011; accepted in revised form August 1, 2012.

2. Lauricella's F_A-system of hypergeometric differential equations

In this section, we collect some facts about Lauricella's hypergeometric F_A-system of differential equations, for which we refer to [7] and [1]. The hypergeometric series F_A of complex variables $x = (x_1, \dots, x_k)$ is defined by

$$F_A(a,(b),(c);x) = \sum_{(n)\in\mathbb{N}^k} \frac{\left(a, \sum_{i=1}^{k} n_i\right) \prod_{i=1}^{k}(b_i, n_i)}{\prod_{i=1}^{k}(c_i, n_i) \prod_{i=1}^{k}(1, n_i)} \prod_{i=1}^{k} x_i^{n_i},$$

where $\mathbb{N} = \{0, 1, 2, \dots,\}$, a, $(b) = (b_1, \dots, b_k)$ and $(c) = (c_1, \dots, c_k)$ are complex parameters satisfying $c_1, \dots, c_k \notin -\mathbb{N} = \{0, -1, -2, \dots,\}$, and $(a, m) = a(a+1)\cdots(a+m-1) = \Gamma(a+m)/\Gamma(a)$. This series converges in the domain

$$\mathbb{D} = \left\{ x \in \mathbb{C}^k \middle| \sum_{i=1}^{k} |x_i| < 1 \right\}$$

and admits the integral representation

$$\left[\prod_{i=1}^{k} \frac{\Gamma(c_i)}{\Gamma(b_i)\Gamma(c_i-b_i)}\right] \int_{(0,1)^k} u(a,(b),(c);x,t)dt, \tag{2.1}$$

where $dt = dt_1 \wedge \cdots \wedge dt_k$,

$$u(x,t) = u(a,(b),(c);x,t) = \left[\prod_{i=1}^{k} t_i^{b_i-1}(1-t_i)^{c_i-b_i-1}\right](1-\Sigma_{i=1}^{k} x_i t_i)^{-a}, \tag{2.2}$$

and parameters (b) and (c) satisfy $\mathrm{Re}(c_i) > \mathrm{Re}(b_i) > 0$ $(i = 1, \dots, k)$.

Differential operators

$$x_i(1-x_i)\partial_i^2 - x_i \sum_{1\le j\le k}^{j\ne i} x_j \partial_i \partial_j + [c_i - (a+b_i+1)x_i]\partial_i - b_i \sum_{1\le j\le k}^{j\ne i} x_j \partial_j - ab_i \tag{2.3}$$

for $i = 1, \dots, k$ annihilate the series $F_A(a,(b),(c);x)$. We define Lauricella's hypergeometric F_A-system $E_A(a,(b),(c))$ by differential equations corresponding to these operators.

We define the local solution space $Sol(U)$ of the system $E_A(a,(b),(c))$ on a domain U in $\mathbb{C}^k$ by the $\mathbb{C}$-vector space

$$\{F(x) \in \mathcal{O}(U) \mid P(x,\partial)\cdot F(x) = 0 \text{ for } {}^{\forall}P(x,\partial) \in E_A(a,(b),(c))\},$$

where $\mathcal{O}(U)$ is the $\mathbb{C}$-algebra of single valued holomorphic functions on U. The rank of $E_A(a,(b),(c))$ is defined by $\sup_U \dim(Sol(U))$. If the rank of $E_A(a,(b),(c))$

is greater than $\dim(Sol(U_x))$ for any neighborhood U_x of $x \in \mathbb{C}^m$ then x is called a singular point of $E_A(a,(b),(c))$. The singular locus S of $E_A(a,(b),(c))$ is defined as the set of such points.

We show that the rank of $E_A(a,(b),(c))$ is 2^k. Denote by F the unknown, and by $F_{ij\dots}$ the derivatives $(\partial_i\partial_j\cdots)F$. Let L_1 be the linear span of $\{F,\ F_i\ (i=1,2,\dots)\}$ over the ring $R_1=\mathbb{C}[x_i, 1/x_i (i=1,...,k)]$, and L_2 the linear span of

$$\{F,\ F_i,\ F_{ij}\ (i<j)\}$$

over the ring

$$R_2 = R_1[(x_i-1)^{-1}\ (i=1,2,\dots)],$$

and L_3 the linear span of

$$\{F,\ F_i,\ F_{ij},\ F_{ij\ell}\quad (i<j<\ell)\}$$

over the ring

$$R_3 = R_2[(x_i+x_j-1)^{-1}\quad (i<j)],$$

and L_4 the linear span of

$$\{F,\ F_i,\ F_{ij},\ F_{ij\ell},\ F_{ij\ell n}\quad (i<j<\ell<n)\}$$

over the ring

$$R_4 = R_3[(x_i+x_j+x_\ell-1)^{-1}\quad (i<j<\ell)],$$

and so on. Note that this procedure becomes stable after k: $R_{k+1}=R_{k+2}=\cdots$, $L_{k+1}=L_{k+2}=\cdots$.

The operators (2.3) lead to the linear expressions

$$[ii]:\ (x_i-1)F_{ii}+\sum_{j\neq i}x_jF_{ij}\in L_1,$$

which shows $F_{ii}\in L_2$.

Differentiating the expression $[ii]$ by x_ℓ $(\ell\neq i)$, we have

$$[ii\ell]:\ (x_i-1)F_{ii\ell}+x_\ell F_{i\ell\ell}+\sum_{j\neq i,\ell}x_jF_{ij\ell}\in L_2.$$

Since we have

$$[i\ell\ell]-[ii\ell]:\ F_{ii\ell}-F_{i\ell\ell}\in L_2,$$

the expression $[ii\ell]$ above can be written as

$$(x_i+x_\ell-1)F_{ii\ell}+\sum_{j\neq i,\ell}x_jF_{ij\ell}\in L_2,$$

which implies $F_{ii\ell} \in L_3$. The expression

$$[iii]: \ (x_i - 1)F_{iii} + \sum_{j \neq i} x_j F_{iij} \in L_2$$

leads to $F_{iii} \in L_3$.

Differentiating the expression $[ii\ell]$ by x_n ($n \neq i, \ell$), we have

$$[ii\ell n]: \ (x_i - 1)F_{ii\ell n} + x_\ell F_{i\ell\ell n} + x_n F_{in\ell n} + \sum_{j \neq i, \ell, n} x_j F_{ij\ell n} \in L_3.$$

Since we have

$$[ii\ell n] - [i\ell\ell n]: \ F_{ii\ell n} - F_{i\ell\ell n} \in L_3 \quad \text{and} \quad [ii\ell n] - [i\ell nn]: \ F_{ii\ell n} - F_{i\ell nn} \in L_3,$$

the expression $[ii\ell n]$ above can be written as

$$(x_i + x_\ell + x_n - 1)F_{ii\ell n} + \sum_{j \neq i, \ell n} x_j F_{ij\ell n} \in L_3,$$

which implies $F_{ii\ell n} \in L_4$.

Differentiating the expression $[ii\ell]$ by x_i and x_ℓ, we have

$$[ii\ell i]: \ (x_i - 1)F_{iii\ell} + x_\ell F_{ii\ell\ell} + \sum_{j \neq i, \ell} x_j F_{iij\ell} \in L_3$$

and

$$[ii\ell\ell]: \ (x_i - 1)F_{ii\ell\ell} + x_\ell F_{i\ell\ell\ell} + \sum_{j \neq i, \ell} x_j F_{ij\ell\ell} \in L_3.$$

Since we have

$$[\ell\ell i\ell] - [ii\ell\ell]: \ F_{ii\ell\ell} - F_{i\ell\ell\ell} \in L_3,$$

the expression $[\ell\ell i\ell]$ above can be written as

$$(x_i + x_\ell - 1)F_{i\ell\ell\ell} + \sum_{j \neq i, \ell} x_j F_{ij\ell\ell} \in L_3,$$

which implies $F_{i\ell\ell\ell}, F_{ii\ell\ell} \in L_4$. The expression

$$[iiii]: \ (x_i - 1)F_{iiii} + \sum_{j \neq i} x_j F_{iiji} \in L_3$$

leads to $F_{iiii} \in L_4$.

In this way, we can show that all the derivatives of F belongs to L_{k+1}. In particular, all the derivatives of F can be linearly expressed in terms of the derivatives $F_{ij\ldots}$, with distinct indices $i, j, \ldots$; cardinality of these derivatives is 2^k. Thus the rank of the system $E_A(a, (b), (c))$ is not greater than 2^k. Moreover the argument above shows that the singular locus of the system is included in the variety defined by

$$\prod_i x_i(x_i-1)\prod_{i<j}(x_i+x_j-1)\prod_{i<j<\ell}(x_i+x_j+x_\ell-1)\cdots.$$

An expression of the singular locus more suitable for this paper is given below in Section 4.

We give two fundamental systems of solutions to $E_A(a,(b),(c))$ in a small neighborhood $\dot{U}$ of the reference point

$$\dot{x}=(\dot{x}_1,\dot{x}_2,\dots,\dot{x}_k)=(2^{-1},2^{-2},\dots,2^{-k})\in\mathbb{D}. \tag{2.4}$$

Since each system consists of 2^k lineally independent solutions, we conclude that the rank of the system is 2^k. From now on, we assume that

$$a,\ b_1,\dots,b_k,\ c_1-b_1,\dots,c_k-b_k,\ a-\sum_{i\in I}c_i\notin\mathbb{Z}, \tag{2.5}$$

where I runs over the subsets of $\{1,\dots,k\}$. This condition (2.5) coincides with the condition that the intersection matrix H in Section 6 is well-defined and non-degenerate (Proposition 6.2). Moreover this is equivalent also to the condition of irreducibility of the system $E_A(a,(b),(c))$, refer to [3]. (The authors thank to N. Takayama for pointing out this fact.)

Fact 2.1 ([7]). Under the condition

$$c_1,\dots,c_k\notin\mathbb{Z},$$

the following 2^k functions are linearly independent solutions of $E_A(a,(b),(c))$ in $\dot{U}$:

1	$F_A(a,(b),(c);x)$
k	$x_1^{\lambda_1}F_A(a+\lambda_1,(b)+\lambda_1e_1,(c)+2\lambda_1e_1;x)$ $\vdots$ $x_k^{\lambda_k}F_A(a+\lambda_k,(b)+\lambda_ke_k,(c)+2\lambda_ke_k;x)$
$\vdots$	$\vdots$
$\binom{k}{r}$	$\vdots$ $\left[\prod_{i\in I_r}x_i^{\lambda_i}\right]F_A\left(a+\sum_{i\in I_r}\lambda_i,(b)+\sum_{i\in I_r}\lambda_ie_i,(c)+2\sum_{i\in I_r}\lambda_ie_i;x\right)$ $\vdots$
$\vdots$	$\vdots$
1	$\left[\prod_{i=1}^k x_i^{\lambda_i}\right]F_A\left(a+\sum_{i=1}^k\lambda_i,(b)+\sum_{i=1}^k\lambda_ie_i,(c)+2\sum_{i=1}^k\lambda_ie_i;x\right)$

where $I_r=\{i_1,\dots,i_r\}$ $(1\le i_1<\cdots<i_r\le k)$, $\lambda_i=1-c_i$ and e_i is the i-th unit row vector.

We fix $x \in \dot{U} \cap \mathbb{R}^k$ for a while and consider $(2k+1)$ hyperplanes in the t-space $\mathbb{R}^k$ defined by

$$t_1 = 0,\ t_1 = 1,\ \ldots,\ t_k = 0,\ t_k = 1,\ x_1t_1 + \cdots + x_kt_k = 1;$$

the complement of these hyperplanes in $\mathbb{R}^k$ will be denoted by $T_{\mathbb{R}}(x)$. There are 2^k bounded chambers in $T_{\mathbb{R}}(x)$. Note that if $t = (t_1, \ldots, t_k)$ belongs to a bounded chamber then we necessarily have

$$t_1, \ldots, t_k > 0, \quad x_1t_1 + \cdots + x_kt_k < 1.$$

Let $\mathbb{Z}_2$ be the set $\{0, 1\} \subset \mathbb{Z}$. Each element $v = (v_1, \ldots, v_k) \in \mathbb{Z}_2^k$ determines a bounded chamber $D_v = D_v(x)$

$$D_v(x) : x_1t_1 + \cdots + x_kt_k < 1, \quad \begin{cases} 0 < t_i < 1 & \text{if } v_i = 0, \\ 1 < t_i & \text{if } v_i = 1. \end{cases} \tag{2.6}$$

For example, if $v = (0, \ldots, 0)$, D_v is the k-dimensional cube $[0, 1]^k$, if $v = (1, \ldots, 1)$, D_v is the k-dimensional simplex given by

$$t_1 > 1, \ldots, t_k > 1, \quad x_1t_1 + \cdots + x_kt_k < 1.$$

In general, for $v = (v_1, \ldots, v_k) \in \mathbb{Z}_2^k$ with $|v| = \sum_{i=1}^k v_i = r$, D_v is a polytope isomorphic to the direct product of the $(k - r)$-dimensional cube $[0, 1]^{k-r}$ and the r-dimensional standard simplex

$$\Delta^r = \{s = (s_1, \ldots, s_r) \in \mathbb{R}^r \mid s_1 > 0, \ldots, s_r > 0,\ s_1 + \cdots + s_r < 1\}.$$

Fact 2.2 ([5]). Under the conditions (2.5) and

$$\mathrm{Re}(c_i) > \mathrm{Re}(b_i) > 0\ (i = 1, \ldots, k), \quad \mathrm{Re}(a) < 1, \tag{2.7}$$

the integrals

$$\int_{D_v} u(a, (b), (c); x, t) dt_1 \wedge \cdots \wedge dt_k, \quad (v \in \mathbb{Z}_2^k)$$

are solutions of $E_A(a, (b), (c))$ in $\dot{U} \cap \mathbb{R}^k$.

Remark 2.1. These can be extended to linearly independent solutions of $E_A(a, (b), (c))$ in $\dot{U}$ by Fact 3.1 and Proposition 6.2.

We define a partial order $\succ$ on $\mathbb{Z}_2^k$.

Definition 2.2. For $v = (v_1, \ldots, v_k)$, $w = (w_1, \ldots, w_k) \in \mathbb{Z}_2^k$,

(1) $v \succeq w$ if and only if $w_i = 1 \Rightarrow v_i = 1$.
(2) $v \succ w$ if and only if $w \succeq v$ and $w \neq v$.

Note that the elements $(0, \dots, 0)$ and $(1, \dots, 1)$ are the minimum and the maximum, respectively.

Lemma 2.3.

(i) *The cardinality of the set* $\{v \in \mathbb{Z}_2^k \mid v \succeq w\}$ *is* $2^{k-|w|}$, *where* $|w| = \sum_{i=1}^k w_i$.
(ii) *If* $v \succ w$ *then the intersection* $\overline{D_v} \cap \overline{D_w}$ *is contained in the hyperplane* $t_i = 1$ *for any index* i *satisfying* $v_i > w_i$, *where* $\overline{D_w}$ *and* $\overline{D_v}$ *are the closures of* D_v *and* D_w, *respectively.*
(iii) *For* $x \in \mathbb{R}^k - S$, *the interior of the union*

$$\cup_{v \succeq w} \overline{D_v}$$

is the simplex $\Delta_w = \Delta_w(x)$:

$$\begin{aligned} \Delta_w &= \{t \in \mathbb{R}^k \mid t_1 > w_1, \dots, t_k > w_k,\ x_1 t_1 + \cdots + x_k t_k < 1\} \\ &= \Big\{w + (1 - \sum_{i=1}^k w_i x_i) s/x \Big| s = (s_1, \dots, s_k) \in \Delta^k\Big\}, \end{aligned} \tag{2.8}$$

where $s/x = (s_1/x_1, \dots, s_k/x_k)$.

Proof.

(i) If $v \succeq w$ then $v_i = 1$ for an index i with $w_i = 1$ and $v_i = 0, 1$ for an index i with $w_i = 0$. Thus there are $2^{k-|w|}$ v's such that $v \succeq w$.
(ii) If $v \succ w$ and $v_i > w_i$ then $v_i = 1$ and $w_i = 0$. By (2.6), the intersection of the boundaries of D_v and D_w is contained in the hyperplane $t_i = 1$.
(iii) For any $v \in \mathbb{Z}_2^k$, if t belongs to D_v then $\Sigma_{i=1}^k x_i t_i < 1$ and $t_i > v_i$ for $i = 1, \dots, k$. Thus if $v \succeq w$ then $D_v \subset \Delta_w$. If $v \not\succeq w$ then there exists an index i such that $v_i = 0$ and $w_i = 1$. Since the point $t = v \in \overline{D_v}$ is not in $\overline{\Delta_w}$, D_v is not contained in Δ_w for $v \not\succeq w$. We have only to note that Δ_w can be expressed as the interior of the union of some $\overline{D_v}$'s. □

3. Twisted homology group

Set

$$\mu_a = \exp(-\pi\sqrt{-1}a),\ \mu_{0i} = \exp(\pi\sqrt{-1}b_i),\ \mu_{1i} = \exp(\pi\sqrt{-1}(c_i - b_i)),$$
$$\mu = (\mu_a, \mu_{01}, \dots, \mu_{0k}, \mu_{11}, \dots, \mu_{1k}).$$

We consider the parameters a, b, c and μ as indeterminates. When we assign complex values to them, we assume the condition (2.5), or equivalently

$$\mu_a^2,\ \mu_{01}^2, \dots, \mu_{0k}^2,\ \mu_{11}^2, \dots, \mu_{1k}^2,\ \mu_a^2 \prod_{i \in I} (\mu_{0i}^2 \mu_{1i}^2) \neq 1,$$

where I runs over the subsets of $\{1, \dots, k\}$.

Let $\mathbb{Q}(\mu)$ be the rational function field over $\mathbb{Q}$ generated by the entries of μ. We fix x in the neighborhood $\dot{U}$ of $\dot{x}$. The multi-valued holomorphic function $u = u(t) = u(a, (b), (c); x, t)$ on

$$T(x) = \{t \in \mathbb{C}^k \mid t_1(1-t_1)\cdots t_k(1-t_k)(1-x_1t_1-\cdots-x_kt_k) \neq 0\}$$

defines the twisted homology groups $H_i(T(x), u)$ and the locally finite ones $H_i^{\mathrm{lf}}(T(x), u)$, where we regard the complexes of twisted chains as defined over the field $\mathbb{Q}(\mu)$. Elements of these homology groups are called twisted cycles or loaded cycles. It is known [2] that they are purely k-dimensional, and the natural map (regularization)

$$\mathrm{reg} : H_k^{\mathrm{lf}}(T(x), u) \longrightarrow H_k(T(x), u)$$

is an isomorphism between 2^k-dimensional vector spaces over $\mathbb{Q}(\mu)$.

Now fix $x \in \dot{U} \cap \mathbb{R}^k$, and *load* on D_v a (constant multiple of) branch of u:

$$u_v = \left[\prod_{i=1}^{k} t_i^{b_i-1}\{(-1)^{v_i}(1-t_i)\}^{c_i-b_i-1}\right](1-\Sigma_{i=1}^k x_i t_i)^{-a}.$$

Note that each linear form in u_v is positive on D_v. Its argument is assigned to be zero. This chamber D_v loaded with the branch of u_v defines an element D_v^u of $H_k^{\mathrm{lf}}(T(x), u)$. This loading is called the *standard loading*. The loaded cycles $D_v^u (v \in \mathbb{Z}_2^k)$ form a basis of $H_k^{\mathrm{lf}}(T(x), u)$.

Thanks to the local triviality of the bundle

$$\bigcup_{x \in \mathbb{C}^k - S} H_k^{\mathrm{lf}}(T(x), u),$$

these D_v^u are defined as elements of $H_k^{\mathrm{lf}}(T(x), u)$ for $x \in \mathbb{C}^k - S$. By this extension and Fact 2.2, we have the following identification.

Fact 3.1. For $x \in \mathbb{C}^k - S$, the germ of the local solution space $Sol(U_x)$ at x can be identified with $H_k^{\mathrm{lf}}(T(x), u)$ and $H_k(T(x), u)$ as vector spaces over $\mathbb{Q}(\mu)$.

4. Singular locus

Set

$$S_w = \left\{x \in \mathbb{C}^k \mid w \cdot x := \sum_{i=1}^{k} w_i x_i = 1\right\}, \quad w \in \check{\mathbb{Z}}_2^k,$$
$$S_0^i = \{x \in \mathbb{C}^k \mid x_i = 0\}, \qquad i = 1, \ldots, k,$$

where $\check{\mathbb{Z}}_2^k = \mathbb{Z}_2^k - \{(0, \ldots, 0)\}$.

By the expression (2.8), we have the following:

Lemma 4.1. *The simplex $\Delta_w(x)$ vanishes when x is in the set S_w.*

Proposition 4.2. *Under the assumption* (2.5)*, the singular locus S of $E_A(a,(b),(c))$ consists of the hyperplanes S_w $(w \in \check{\mathbb{Z}}_2^k)$ and S_0^i $(i = 1, \dots, k)$.*

Proof. A point $x \in \mathbb{C}^k$ satisfying

$$\dim H_k(T(x), u) < \text{rank of } E_A(a, (b), (c)) = 2^k$$

is a singular point of $E_A(a, (b), (c))$ by Fact 3.1. If x does not belong to

$$(\cup_{w\in\check{\mathbb{Z}}_2^k} S_w) \cup (\cup_{i=1}^k S_0^i),$$

then there is a homotopy equivalence between $T(x)$ and $T(\dot{x})$. Thus $H_k(T(x), u)$ is isomorphic to $H_k(T(\dot{x}), u)$, which is of rank 2^k.

Recall that $D_v^u(v \in \mathbb{Z}_2^k)$ form a basis of $H_k(T(x), u)$. By Lemma 4.1, if x belongs to S_w $(w \in \check{\mathbb{Z}}_2^k)$, then Δ_w^u with suitable loading of u degenerates. Thus $\dim H_k(T(x), u)$ for $x \in S_w$ is less than 2^k. The expression of local solutions (Fact 2.1) tells that any element x of S_0^i is a singular point. □

For an element $w \in \check{\mathbb{Z}}_2^k$ with $|w| = r$, we define $\dot{x}_w \in \mathbb{D}$ as follows: read the array w from the left; at the first 1 we put 2^{-1}, at the second 1 we put $2^{-2}, \dots,$ and at the last 1 we put 2^{-r}, go back to the left-end and re-start: at the first 0 we put $2^{-(r+1)}$, and at the second 0 we put $2^{-(r+2)}$ and so on. For example,

$$\dot{x}_w = \left(2^{-2}, 2^{-3}, 2^{-1}\right), \quad \text{when} \quad w = (0, 0, 1).$$

Define a line $\mathbb{C}_w$ in $\mathbb{C}^k$ as the image of a map

$$\eta_w : \mathbb{C} \ni y \mapsto \dot{x}_w + yw \in \mathbb{C}^k.$$

We study the intersection $S_v \cap \mathbb{C}_w$ for $v \in \check{\mathbb{Z}}_2^k$. If $v \cdot w = 0$, then $S_v \cap \mathbb{C}_w = \phi$. If $v \cdot w \neq 0$, then by solving

$$(\dot{x}_w + yw) \cdot v = 1,$$

we find the intersection point $S_v \cap \mathbb{C}_w$ as $\eta_w(y_v)$, where

$$y_v = (1 - \dot{x}_w \cdot v)/(v \cdot w) \in \mathbb{R} \subset \mathbb{C}.$$

In particular, $S_w \cap \mathbb{C}_w$ is given by $\eta_w(y_w)$, where

$$y_w = (1 - \dot{x}_w \cdot w)/|w| \in \mathbb{R} \subset \mathbb{C}.$$

For example, when $w = (0, 0, 1)$, we show the intersection points $S_v \cap \mathbb{C}_w$ on the complex y-plane $\mathbb{C}$ for

$$v = (1, 1, 1), \quad (1, 0, 1), \quad (0, 1, 1), \quad (0, 0, 1)$$

in Figure 4.1. Here note that $\dot{x}_{001} = (2^{-2}, 2^{-3}, 2^{-1})$, which corresponds to the origin of the complex y-plane. The line $\mathbb{C}_w$ is parameterized as

$$\eta_w : y \longmapsto (x_1, x_2, x_3) = \left(2^{-2}, 2^{-3}, 2^{-1}\right) + (0, 0, y) \in \mathbb{C}^3,$$

and the intersections of the line $\mathbb{C}_w$ with the lines

$$x_1 + x_2 + x_3 = 1, \quad x_1 + x_3 = 1, \quad x_2 + x_3 = 1, \quad x_3 = 1$$

are given by

$$y_{111} = 1/8, \quad y_{101} = 2/8, \quad y_{011} = 3/8, \quad y_{001} = 4/8,$$

respectively.

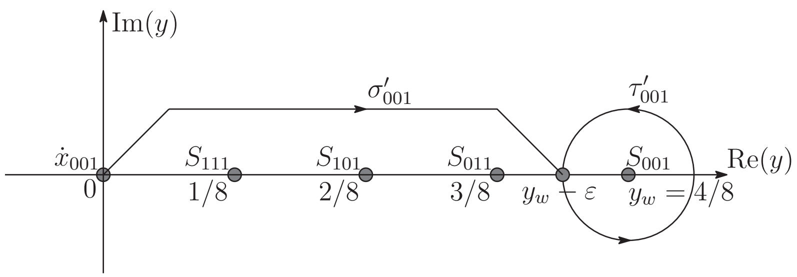

Figure 4.1. The loop τ'_{001} and the path σ'_{001}.

Lemma 4.3. *Suppose that* $v \cdot w \neq 0$. *If* $w \preceq v$ *then* $0 < y_v \leq y_w$, *otherwise* $y_w < y_v$.

Proof. Recall that

$$y_w = (1 - \dot{x}_w \cdot w)/|w|, \quad y_v = (1 - \dot{x}_w \cdot v)/(v \cdot w).$$

Since $1 - \dot{x}_w \cdot v > 0$, we have $y_v > 0$ for any $v \in \check{\mathbb{Z}}_2^k$ with $v \cdot w \neq 0$. If $w \preceq v$ then

$$\dot{x}_w \cdot v \geq \dot{x}_w \cdot w, \quad v \cdot w = w \cdot w = r.$$

Thus $0 < y_v \leq y_w$. If $w \not\preceq v$ then $v \cdot w < r$ and

$$\dot{x}_w \cdot v \leq (2^{-1} + \cdots + 2^{1-r}) + (2^{-1-r} + \cdots + 2^{-k}) < 2^{-1} + \cdots + 2^{-r} = \dot{x}_w \cdot w.$$

Thus we have $y_w < y_v$. □

Let τ'_w be a positively oriented circle with center y_w and terminal $y_w - \varepsilon$ in $\mathbb{C}$, and let σ'_w be a path in $\mathbb{C}$ starting from 0, traveling in the upper half space, and ending at $y_w - \varepsilon$, where ε is a small positive number; see Figure 4.1. Define a loop τ_w and a path σ_w in $\mathbb{C}_w(\subset \mathbb{C}^k)$ as the images of τ'_w and σ'_w by the map η_w, respectively. We define a loop ρ_w in $X = \mathbb{C}^k - S$ by connecting the segment from $\dot{x}$ to $\dot{x}_w$, the path σ_w, the loop τ_w, the path σ_w^{-1}, and the segment from $\dot{x}_w$ to $\dot{x}$.

On the other hand, we define a loop ρ_0^i in X with base point $\dot{x}$ by

$$(2^{-1}, \dots, 2^{-i+1}, \sigma_i(s), 2^{-i-1}, \dots, 2^{-k}),$$

where $\sigma_i(s)$ is a path starting from $s = 2^{-i}$, turning around the point $s = 0$ counterclockwise, and coming back. The Lefschetz hyperplane theorem and the van Kampen theorem imply the following.

Proposition 4.4. *The fundamental group $\pi_1(X, \dot{x})$ is generated by ρ_w for $w \in \check{\mathbb{Z}}_2^k$ and $\rho_0^1, \dots, \rho_0^k$.*

5. Local monodromy

A loop ρ with base point $\dot{x}$ induces a linear transformation m_ρ of $H_k(T(\dot{x}), u)$, which is called the circuit transform (or monodromy) with respect to ρ. By Fact 3.1, this transformation can be regarded as that of the local solution space $Sol(\dot{U})$.

Proposition 5.1. *Suppose that*

$$\alpha_w = \mu_a^2 \cdot \mu_{w_1 1}^2 \cdots \mu_{w_k k}^2 \neq 1.$$

Then the Jordan normal form of the circuit transform $m_w = m_{\rho_w}$ with respect to the loop ρ_w ($w \in \check{\mathbb{Z}}_2^k$) is given by

$$\mathrm{diag}(\alpha_w, 1, \dots, 1).$$

Proof. Take the end point $x_{\sigma_w} = \eta_w(y_w - \varepsilon)$ of the path σ_w for $w \in \check{\mathbb{Z}}_2^k$, where $y_w = (1 - \dot{x}_w \cdot w)/|w|$. Note that the simplex $\Delta_w = \Delta_w(x_{\sigma_w})$ is contained in a small neighborhood of the vertex w of the cube $[0, 1]^k$. We deform Δ_w along the loop

$$\tau_w : [-\pi, \pi] \ni \theta \mapsto x_\theta = \dot{x}_w + (\varepsilon e^{\sqrt{-1}\theta} + y_w)w \in X.$$

Note that if $w_i = 0$ then x_i does not move, and that

$$1 - x_\theta \cdot w = -\varepsilon e^{\sqrt{-1}\theta}|w|.$$

By using the expression (2.8) of Δ_w, we express the deformation of Δ_w along the loop τ_w as

$$\Delta_w(x_\theta) = \{w - \varepsilon e^{\sqrt{-1}\theta}|w|(s_1/x_1, \dots, s_k/x_k) \mid s \in \Delta^k,\ -\pi \le \theta \le \pi\},$$

where $x_\theta = (x_1, \dots, x_k)$.

We trace the value of the function $u = u(x, t)$ while x travels along the loop τ_w. The argument of

$$t_i - w_i = -\varepsilon e^{\sqrt{-1}\theta}|w|s_i/x_i$$

increases by 2π by the continuation along the loop τ_w. Since we have

$$1 - \sum_{i=1}^{k} x_i t_i = -\varepsilon e^{\sqrt{-1}\theta}|w| \left(1 - \sum_{i=1}^{k} s_i\right),$$

its argument also increases by 2π by the continuation. Hence the loaded cycle $\Delta_w^u(x_{\sigma_w})$ supported by $\Delta_w(x_{\sigma_w})$ loaded with $u = u(x, t)$ is multiplied by α_w by the continuation.

We have 2^k chambers around the vertex $(1, \dots, 1) - w$ of the cube $[0, 1]^k$. We give a basis of $H_k^{\mathrm{lf}}(T(x_{\sigma_w}), u)$ as the simplex Δ_w and the $2^k - 1$ chambers outside of the cube $[0, 1]^k$ loaded with u. It is geometrically clear that the move τ_w does not affect the other $2^k - 1$ chambers. See Figure 5.1 for the case $k = 2$ and $w = (1, 1)$. Hence the circuit matrix is diagonal as stated. □

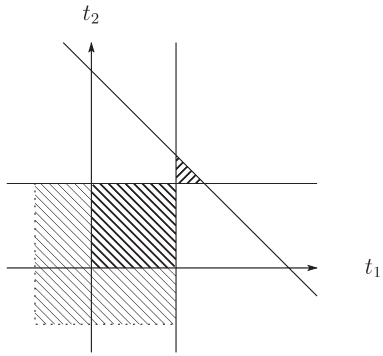

Figure 5.1. Vanishing and invariant chambers

Proposition 5.2. *Suppose that $c_i \notin \mathbb{Z}$. Then the Jordan normal form of the circuit transformation $m_0^i = m_{\rho_0^i}$ with respect to the loop ρ_0^i is given by*

$$\mathrm{diag}(\overbrace{1, \dots, 1}^{2^{k-1}}, \overbrace{\alpha_0^i, \dots, \alpha_0^i}^{2^{k-1}}),$$

where

$$\alpha_0^i = \mu_{0i}^{-2}\mu_{1i}^{-2} = \exp(-2\pi\sqrt{-1}c_i) \neq 1.$$

Proof. We make use of the local solutions given in Fact 2.1. The analytic continuation of the these solutions along the loop ρ_0^i is quite obvious: we have 2^{k-1} invariant solutions and 2^{k-1} solutions multiplied by $\exp(2\pi\sqrt{-1}\lambda_i)$. □

6. Intersection form

Let $z \mapsto z^\vee$ be the isomorphism of $\mathbb{Q}(\mu)$ over $\mathbb{Q}$ induced by

$$\mu_a \mapsto \mu_a^{-1}, \quad \mu_{0j} \mapsto \mu_{0j}^{-1}, \quad \mu_{1j} \mapsto \mu_{1j}^{-1} \qquad j = 1, \dots, k.$$

Note that if we assign real numbers to the entries of a, (b) and (c), then $z^\vee$ is the complex conjugate $\overline{z}$ of $z \in \mathbb{Q}(\mu) \subset \mathbb{C}$.

We define the intersection form $\mathcal{I}$ on $H_k(T(x), u) \times H_k(T(x), u)$ as follows. Let D^u and $\acute{D}^u$ be elements of $H_k(T(x), u)$ given by

$$D^u = \sum_{i \in I} d_i D_i^{u_i}, \quad \acute{D}^u = \sum_{j \in J} \acute{d}_j \acute{D}_j^{u_j}, \qquad d_i, \acute{d}_j \in \mathbb{Q}(\mu),$$

where $D_i^{u_i}$ denotes a singular k-simplex D_i loaded with a branch $u_i = u_i(t)$ of u. The intersection number $\mathcal{I}(D^u, \acute{D}^u)$ is given, by definition, as

$$\mathcal{I}(D^u, \acute{D}^u) = \sum_{i \in I, j \in J} \sum_{p \in D_i \cap \acute{D}_j} d_i \acute{d}_j^\vee (D_i \cdot \acute{D}_j)_p \frac{u_i(p)}{u_j(p)},$$

where $(D_i \cdot \acute{D}_j)_p$ is the topological intersection number of k-chains D_i and $\acute{D}_j$ at p. We have

$$\mathcal{I}(\acute{D}^u, D^u) = (-1)^k \mathcal{I}(D^u, \acute{D}^u)^\vee,$$
$$\mathcal{I}(zD^u, \acute{D}^u) = z\mathcal{I}(D^u, \acute{D}^u), \quad \mathcal{I}(D^u, z\acute{D}^u) = z^\vee \mathcal{I}(D^u, \acute{D}^u),$$

for $z \in \mathbb{Q}(\mu)$.

Proposition 6.1. *For $v \in \mathbb{Z}_2^k$, let $D_v^u \in H_k^{\mathrm{lf}}(T(\dot{x}), u)$ be the chamber D_v standardly loaded with u. We have*

$$\mathcal{I}(D_v^u, D_{v'}^u) = \left[\prod_{1 \le i \le k}^{v_i \neq v_i'} \frac{\mu_{1i}}{\mu_{1i}^2 - 1} \right] \cdot \left[\prod_{1 \le i \le k}^{v_i = v_i' = 0} (-1) \frac{\mu_{0i}^2 \mu_{1i}^2 - 1}{(\mu_{0i}^2 - 1)(\mu_{1i}^2 - 1)} \right]$$
$$\cdot (-1)^{\sum_i \min(v_i, v_i')} \left[\frac{\mu_a^2 \prod_{1 \le i \le k}^{v_i = v_i' = 1} \mu_{1i}^2 - 1}{(\mu_a^2 - 1) \prod_{1 \le i \le k}^{v_i = v_i' = 1} (\mu_{1i}^2 - 1)} \right],$$

where

$$v = (v_1, \dots, v_k), \quad v' = (v_1', \dots, v_k') \in \mathbb{Z}_2^k.$$

Proof. The intersection of the (closure of the) chambers D_v and $D_{v'}$ is the direct product of

- the point 1 on the t_i-line if $v_i \neq v'_i$, let I_1 be the set of such indices i,
- the interval [0, 1] on the t_i-line if $v_i = v'_i = 0$, let I_2 be the set of such indices i,
- the simplex in the remaining coordinate space $(t_j)_{j\in J}$, where $J = \{1, \dots, k\} - I_1 - I_2$, bounded by the hyperplanes $t_j = 1$ and

$$\sum_{i\in I_1} x_i + \sum_{i\in I_2} x_i + \sum_{j\in J} x_j t_j = 1.$$

Note that $j \in J$ if and only if $v_j = v'_j = 1$, and the cardinality of J is given by $\sum_i \min(v_i, v'_i)$. The intersection number of D^u_v and $D^u_{v'}$ is the product of the three kinds of factors:

- the intersection number of the two intervals at the point 1 with exponent μ^2_{1i} for $i \in I_1$,
- the self-intersection numbers of the 1-dimensional cycles supported by the interval [0, 1] with exponents μ^2_{0i} at 0 and μ^2_{1i} at 1 for $i \in I_2$,
- the self-intersection number of the cycle supported by the simplex with exponents $\mu^2_{j1} (j \in J)$ and μ^2_a.

These self-intersection numbers can be found in [6]. Since we load u standardly, the intersection number in the first factor is $\frac{\mu_{1i}}{\mu^2_{1i}-1}$. □

Note that the intersection number $\mathcal{I}(D^u_v, D^u_{v'})$ is complex valued whenever we assign values to μ under the condition (2.5).

We array the basis $\{D^u_v\}_{v\in\mathbb{Z}^k_2}$ in a total order on $v \in \mathbb{Z}^k_2$, say the total-lexicographic order: $w = (w_1, \dots, w_k) < v = (v_1, \dots, v_k)$ if either (i) or (ii) is satisfied:

(i) $|w| < |v|$
(ii) $|w| = |v|$ and $w_j < v_j$,
where $j = \min\{i \in \{1, \dots k\} \mid w_i \neq v_i\}$.

Note that if $w \prec v$ then $w < v$.

We define the intersection matrix with respect to this basis as

$$H = (\mathcal{I}(D^u_v, D^u_{v'}))_{v,v'\in\mathbb{Z}^k_2}, \tag{6.1}$$

where v and v' are arranged in the total-lexicographic order. The determinant of the intersection matrix H is given as

$$\frac{d_a(1)}{d_a d_1} \quad (k = 1), \qquad \frac{d_a(12)d_a(1)d_(2)}{d_a^3 d_1^2 d_2^2} \quad (k = 2),$$

$$\frac{d_a(123)d_a(12)d_a(23)d_a(31)d_a(1)d_a(2)d_a(3)}{d_a^7 d_1^4 d_2^4 d_3^4} \quad (k = 3),$$

where

$$\nu_a = \mu_a^2, \quad \nu_{0i} = \mu_{0i}^2, \quad \mu_{1i} = \mu_{1i}^2,$$
$$d_a = \nu_a - 1, \quad d_i = (\nu_{0i} - 1)(\nu_{1i} - 1),$$
$$d_a(i \cdots j) = \nu_a(\nu_{0i}\nu_{1i}) \cdots (\nu_{0j}\nu_{1j}) - 1.$$

In general, we have the following expression, wich will be proved in the appendix:

Proposition 6.2. *We have*

$$\det(H) = \frac{\prod_{p=1}^{k} \prod_{1 \le i_1 < \cdots < i_p \le k} d_a(i_1 \cdots i_p)}{d_a^{2^k - 1} \prod_{p=1}^{k} d_i^{2^{k-1}}}.$$

In particular, the intersection form $\mathcal{I}$ is non-degenerate under the condition (2.5).

Lemma 6.3. *Let m_ρ be the circuit transformation of $H_k(T(\dot{x}), u)$ with respect to a loop ρ in X.*

(i)
$$\mathcal{I}(m_\rho(D^u), m_\rho(\acute{D}^u)) = \mathcal{I}(D^u, \acute{D}^u), \quad D^u, \acute{D}^u \in H_k(T(\dot{x}), u).$$

(ii)
$$M_\rho H \, {}^t M_\rho^\vee = H,$$

where H is the intersection matrix in (6.1) *and M_ρ is the matrix representation (circuit matrix) of m_ρ with respect to the basis $\{D_v^u\}_{v \in \mathbb{Z}_2^k}$ of $H_k(T(\dot{x}), u)$.*

(iii) *Let D^u be an eigenvector of m_ρ with eigenvalue $\alpha \in \mathbb{Q}(\mu)$ and let $\acute{D}^u$ be that with eigenvalue $\alpha' \in \mathbb{Q}(\mu)$. Then*

$$\mathcal{I}(D^u, D^u) \neq 0 \Rightarrow \alpha \cdot \alpha^\vee = 1,$$
$$\alpha^\vee \cdot \alpha' \neq 1 \Rightarrow \mathcal{I}(D^u, \acute{D}^u) = 0.$$

Proof. Since the intersection form is stable under deformation of x as far as the topology of $T(x)$ does not change, we have (i). The statement (ii) is a matrix representation of (i) for the basis $\{D_v^u\}_{v \in \mathbb{Z}_2^k}$ of $H_k(T(\dot{x}), u)$. Let us show (iii). Note that

$$\begin{aligned}\mathcal{I}(D^u, \acute{D}^u) &= \mathcal{I}(m_\rho(D^u), m_\rho(\acute{D}^u)) = \mathcal{I}(\alpha D^u, \alpha' \acute{D}^u) \\ &= \alpha \cdot (\alpha')^\vee \mathcal{I}(D^u, \acute{D}^u).\end{aligned}$$

Thus if $\alpha' \cdot \alpha^\vee \neq 1$ then $\mathcal{I}(D^u, \acute{D}^u) = 0$. By putting $\acute{D}^u = D^u$, we have $\mathcal{I}(D^u, D^u) \neq 0 \Rightarrow \alpha \cdot \alpha^\vee = 1$. □

For $i = 1, \dots, k$, we set $\mathbb{Z}_2^k(i) = \{w \in \mathbb{Z}_2^k \mid w_i = 0\}$ and

$$\begin{aligned} W_i &= \langle D_w^u \mid w \in \mathbb{Z}_2^k(i)\rangle \subset H_k(T(\dot{x}), u), \\ W_i^\perp &= \{\acute{D}^u \in H_k(T(\dot{x}), u) \mid \mathcal{I}(\acute{D}^u, D^u) = 0 \text{ for any } D^u \in W_i\}. \end{aligned}$$

Lemma 6.4. *Suppose that $c_i \notin \mathbb{Z}$. Then the eigenspace of the circuit transform m_0^i with eigenvalue* 1 *is W_i and that with eigenvalue α_0^i is $W_i^\perp$, and*

$$W_i \oplus W_i^\perp = H_k(T(\dot{x}), u) \quad (1 \le i \le k).$$

Proof. Consider the circuit transformation m_0^i. By Proposition 5.2, the space $H_k(T(\dot{x}), u)$ is decomposed into 2^{k-1}-dimensional eigenspaces with eigenvalues 1 and $\alpha_0^i \neq 1$. Note that any cycle D_w ($w \in \mathbb{Z}_2^k(i)$) is invariant under the continuation along the loop ρ_0^i. Thus it belongs to W_i. Lemma 6.3 implies that any α_0^i-eigenvector belongs to $W_i^\perp$. Hence W_i is the eigenspace of the circuit transform m_0^i with eigenvalue 1 and $W_i^\perp$ includes that with eigenvalue α_0^i. Since $\dim(W_i^\perp) = 2^{k-1}$ by Proposition 6.2, $W_i^\perp$ coincides with the eigenspace of the circuit transform m_0^i with eigenvalue α_0^i, and $W_i \oplus W_i^\perp = H_k(T(\dot{x}), u)$. □

7. Monodromy representation

For $w = (w_1, \dots, w_k) \in \mathbb{Z}_2^k$, we set

$$\Delta_w^u = \sum_{v \succeq w} \left(\prod_{i=1}^k \mu_{1i}^{v_i - w_i} \right) D_v^u \in H_k(T(\dot{x}), u).$$

Theorem 7.1.

(i) *For each $w \in \check{\mathbb{Z}}_2^k$, the circuit transform m_w for the loop ρ_w is*

$$\begin{aligned} m_w : D^u \mapsto D^u &- (1 - \alpha_w)\mathcal{I}(D^u, \Delta_w^u)\mathcal{I}(\Delta_w^u, \Delta_w^u)^{-1}\Delta_w^u \\ &= D^u - (1 - \mu_a^2)\left[\prod_{i=1}^k (1 - \mu_{w_i i}^2)\right]\mathcal{I}(D^u, \Delta_w^u)\Delta_w^u. \end{aligned}$$

If we assign complex values to μ with condition

$$\alpha_w = \mu_a^2 \cdot \mu_{w_1 1}^2 \cdots \mu_{w_k k}^2 \neq 1$$

then it is the reflection of root Δ_w^u and eigenvalue α_w with respect to the intersection form $\mathcal{I}$.

(ii) *For $i = 1, \dots, k$, the circuit transform m_0^i for the loop ρ_0^i is given by*

$$m_0^i : D^u \mapsto \alpha_0^i D^u - (\alpha_0^i - 1)\mathrm{pr}_i(D^u),$$

where $\alpha_0^i = \mu_{0i}^{-2}\mu_{1i}^{-2}$ and pr_i is the projection from $H_k(T(\dot{x}), u)$ to W_i:

$$\mathrm{pr}_i : D^u = \acute{D}^u + \grave{D}^u \mapsto \acute{D}^u, \quad \acute{D}^u \in W_i,\ \grave{D}^u \in W_i^{\perp}.$$

We array the basis $\{D_v^u\}_{v\in\mathbb{Z}_2^k}$ as a column vector in the total-lexicographic order on $v \in \mathbb{Z}_2^k$. Let e_w be the unit row vector such that

$$e_w\ {}^t(\cdots, D_v^u, \dots) = D_w^u.$$

We define row vectors

$$\varepsilon_w = \sum_{v\succeq w}\left(\prod_{i=1}^{k}\mu_{1i}^{v_i-w_i}\right)e_v \in \mathbb{Q}(\mu)^{2^k}, \qquad w \in \check{\mathbb{Z}}_2^k$$

and arrange them in the total-lexicographic order, and define $(2^{k-1}, 2^k)$-matrices as

$$E_i = \begin{pmatrix} \vdots \\ e_w \\ \vdots \end{pmatrix}_{w\in\mathbb{Z}_2^k(i)}, \qquad i = 1, \dots, k.$$

If a 2^k-row-vector f is identified with $f\ {}^t(\cdots, D_v^u, \dots)$, then m_w and m_0^i are expressed as $2^k \times 2^k$-matrices by the intersection matrix H in (6.1).

Corollary 7.2.

(i) *The circuit transform m_w is expressed by the matrix*

$$\begin{aligned} M_w &= I_{2^k} - (1-\alpha_w)H\ {}^t\varepsilon_w^{\vee}(\varepsilon_w H\ {}^t\varepsilon_w^{\vee})^{-1}\varepsilon_w \\ &= I_{2^k} - (1-\mu_a^2)\left[\prod_{i=1}^{k}(1-\mu_{w_i i}^2)\right]H\ {}^t\varepsilon_w^{\vee}\varepsilon_w. \end{aligned}$$

(ii) *The circuit transform m_0^i is expressed by the matrix*

$$M_0^i = \alpha_0^i I_{2^k} - (\alpha_0^i - 1)H\ {}^tE_i^{\vee}(E_i H\ {}^tE_i^{\vee})^{-1}E_i.$$

These matrices act on 2^k-row-vectors from the right.

Proof.
(i) Suppose that $\alpha_w \neq 1$ for $w \in \check{\mathbb{Z}}_2^k$. We show that Δ_w^u is the eigenvector of m_w with eigenvalue α_w for any $w \in \check{\mathbb{Z}}_2^k$. It is shown in the proof of Proposition 5.1 that the loaded cycle $\Delta_w^{u'}(x_{\sigma_w})$ is an eigenvector belonging to the eigenvalue α_w of the transformation caused by the continuation along the loop τ_w. Here $x_{\sigma_w} = \eta_w(y_w - \varepsilon)$ is the end point of the path σ_w and we load u' on the small simplex $\Delta_w(x_{\sigma_w})$ by the assignments $\arg(t_i) = \arg(1 - \sum_{i=1}^k x_i t_i) = 0$ and

$$\arg(1 - t_i) = \begin{cases} 0 & \text{if } w_i = 0, \\ \pi & \text{if } w_i = 1. \end{cases}$$

We deform the simplex $\Delta_w(x_{\sigma_w})$ along the path σ_w^{-1} from x_{σ_w} to $\dot{x}_w$. Lemma 2.3 tells that the resulting simplex $\Delta_w(\dot{x}_w)$ is (the closure of) the union of the chambers D_v $(v \succeq w)$. At the same time, we trace the change of the function $u'(x)$ along the path σ_w^{-1} from x_{σ_w} to $\dot{x}_w$; the resulting loaded cycle $\Delta_w^{u'}(\dot{x}_w)$ would be a linear combination

$$\sum_{v \succeq w} d_v D_v^u.$$

We determine the coefficients. The key is the expression (2.8) of Δ_w. For any $v \succeq w$, there exits $s_v \in \Delta^k$ such that

$$w + (1 - w \cdot \dot{x}_w) s_v / \dot{x}_w = t_v \in D_v.$$

By comparing the value of $u(\dot{x}_w, t_w)$ with that of loaded function on D_w^u, we have

$$d_w = \prod_{1 \le i \le k}^{w_i = 1} \mu_{1i}.$$

For $v \succ w$, we follow the deformation of the i-th coordinates t_i of

$$t = w + (1 - w \cdot x) s_v / x$$

along the path $\sigma_w^{-1} : x = \dot{x} + wy$ for $y \in (\sigma_v)^{-1}$. If the index i satisfies $v_i = w_i$ then $\mathrm{Re}(1 - t_i) > 0$, otherwise $1 - t_i$ changes from positive to negative via the upper half space. Thus $\arg(t_i) = \arg(1 - \sum_{i=1}^k x_i t_i) = 0$ and

$$\arg(1 - t_i) = \begin{cases} 0 & \text{if } v_i = 0, \\ \pi & \text{if } v_i = 1, \end{cases}$$

on D_v. Hence we have

$$d_v = \prod_{1 \le i \le k}^{v_i = 1} \mu_{1i}, \quad \text{and so} \quad \sum_{v \succeq w} d_v D_v^u = d_w \Delta_w^u.$$

By Lemma 6.3, the eigenspace with eigenvalue 1 of m_w is the orthogonal complement of $\varDelta^u_w$. Therefore we have the first expression of m_w. By following the proof of Proposition 6.1, we have

$$\mathcal{I}(\varDelta^u_w, \varDelta^u_w) = \frac{1-\alpha_w}{(1-\mu_a^2)\prod_{i=1}^{k}(1-\mu_{w_i i}^2)},$$

which implies the second expression of m_w.

We consider the case $\alpha_w = 1$. Under our assumption (2.5), the intersection form $\mathcal{I}$ on $H_k(T(\dot{x}), u) \times H_k(T(\dot{x}), u)$ does not degenerate and $\{D^u_v\}_{v\in\mathbb{Z}_2^k}$ is a basis even in this case. Since we can regard the second expression of m_w as a limit of parameters, it is valid as the circuit transform.

(ii) Suppose that $c_i \notin \mathbb{Z}$. Under the linear map

$$D^u \mapsto \alpha_0^i D^u - (\alpha_0^i - 1)\mathrm{pr}_i(D^u),$$

$\acute{D}^u \in W_i$ is invariant and $\grave{D}^u \in W_i^\perp$ is transformed into $\alpha_0^i \grave{D}^u$. By Lemma 6.4, this map coincides with m_0^i. It is easy to see that m_0^i is represented by the matrix M_0^i for the basis $\{D^u_v\}_{v\in\mathbb{Z}_2^k}$.

We consider the case $c_i \in \mathbb{Z}$. Under our assumption (2.5), the intersection form $\mathcal{I}$ on $H_k(T(\dot{x}), u) \times H_k(T(\dot{x}), u)$ does not degenerate and $\{D^u_v\}_{v\in\mathbb{Z}_2^k}$ is a basis even in this case. Note that the map $(\alpha_0^i - 1)\mathrm{pr}_i$ is represented by $2^k \times 2^k$-matrix

$$(\alpha_0^i - 1)H\,{}^tE_i^\vee(E_iH\,{}^tE_i^\vee)^{-1}E_i \tag{7.1}$$

for the basis $\{D^u_v\}_{v\in\mathbb{Z}_2^k}$. The $2^{k-1} \times 2^{k-1}$-matrix $E_iH\,{}^tE_i^\vee$ has the factor $(\alpha_0^i - 1)$ by Propositions 6.1 and 6.2. Thus this factor in the expression (7.1) is canceled. If we regard this case as a limit of parameters then $(\alpha_0^i - 1)\mathrm{pr}_i$ converges to a linear transformation satisfying

$$\ker((\alpha_0^i - 1)\mathrm{pr}_i) = \mathrm{Im}((\alpha_0^i - 1)\mathrm{pr}_i) = W_i,$$

and the expression of m_0^i is valid as the circuit transform. □

Remark 7.3.

(i) The eigenspace of the circuit transform m_w with eigenvalue 1 is the orthogonal complement

$$(\varDelta^u_w)^\perp = \{D^u \in H_k(T(\dot{x}), u) \mid \mathcal{I}(D^u, \varDelta^u_w) = 0\}$$

of Δ^u_w. If $\alpha_w = 1$ then Δ^u_w belongs to $(\Delta^u_w)^\perp$, otherwise $H_k(T(\dot{x}), u)$ is spanned by Δ^u_w and $(\Delta^u_w)^\perp$. If $\alpha_w = 1$ then the Jordan normal form of m_w is given by

$$\begin{pmatrix} J_{1,2} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad J_{1,2} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(ii) If $c_i \in \mathbb{Z}$ then the Jordan normal form of m_0^i is the direct sum of 2^{k-1} copies of $J_{1,2}$:

$$\begin{pmatrix} J_{1,2} & & \\ & \ddots & \\ & & J_{1,2} \end{pmatrix}.$$

A. Sketch of a proof of Proposition 6.2

A.1. Determinant formula

Set

$$\alpha = \mu_a^2, \quad \beta_i = \mu_{0i}^2, \quad \gamma_i = \mu_{1i}^2, \quad \sqrt{\gamma_i} = \mu_{1i}.$$

Then Proposition 6.2 reads that $\det(H)$ equals

$$\frac{\alpha\beta_1\gamma_1 - 1}{(\alpha-1)(\beta_1-1)(\gamma_1-1)} \qquad (k=1),$$

$$\frac{(\alpha\beta_1\gamma_1\beta_2\gamma_2 - 1)(\alpha\beta_1\gamma_1 - 1)(\alpha\beta_2\gamma_2 - 1)}{(\alpha-1)^3(\beta_1-1)^2(\gamma_1-1)^2(\beta_2-1)^2(\gamma_2-1)^2} \qquad (k=2),$$

$$\frac{(\alpha\beta_1\gamma_1\beta_2\gamma_2\beta_3\gamma_3 - 1)\prod_{1\le i<j\le 3}(\alpha\beta_i\gamma_i\beta_j\gamma_j - 1)\prod_{i=1}^{3}(\alpha\beta_i\gamma_i - 1)}{(\alpha-1)^7\prod_{i=1}^{3}(\beta_i-1)^4(\gamma_i-1)^4} \qquad (k=3),$$

and in general,

$$\det(H) = \frac{\prod_{v\in\mathbb{Z}_2^k}\left[\alpha\prod_{j=1}^{k}(\beta_j\gamma_j)^{v_j} - 1\right]}{(\alpha-1)^{2^k}\prod_{j=1}^{k}[(\beta_j-1)(\gamma_j-1)]^{2^{k-1}}},$$

where $v = (v_1, \ldots, v_k) \in \mathbb{Z}_2^k$.

A.2. Outline of the proof

We index the rows and the columns of the intersection matrix H by elements $v \in \mathbb{Z}_2^k$, which are arranged in the lexicographic order:

$$(0,\dots,0) < (0,\dots,0,1) < (0,\dots,1,0) < (0,\dots,1,1) < \cdots < (1,\dots,1).$$

We apply the Laplace expansion to the determinant $\det(H)$ with respect to the 2^{k-1} rows:

$$(0,\check{v}), \quad \check{v} = (v_2,\dots,v_k) \in \mathbb{Z}_2^{k-1}.$$

We choose 2^{k-1} columns with indices $v^{(1)}, v^{(2)}, \dots, v^{(2^{k-1})} \in \mathbb{Z}_2^k$ and make the minor. Let us write their entries as

$$v^{(i)} = (\epsilon_i, \check{v}^{(i)}), \quad \check{v}^{(i)} = (v_2^{(i)},\dots,v_k^{(i)}) \in \mathbb{Z}_2^{k-1} \quad (1 \le i \le 2^{k-1}).$$

Lemma A.1. *The minor is zero unless $\check{v}^{(i)}$ are distinct, that is,*

$$\{\check{v}^{(i)} \mid 1 \le i \le 2^{k-1}\} = \mathbb{Z}_2^{k-1}.$$

Proof. Note that if $\check{v}^{(i)} = \check{v}^{(j)}$ for $i \neq j$ then the two columns of the minor are proportional. □

Let the chosen columns be indexed as

$$(1,\dots,1) = v^{(1)} > v^{(2)} > \cdots > v^{(2^{k-1})} = (0,\dots,0).$$

We make the product of this minor and the complementary minor, the minor made by complementary rows and the complementary columns, and denote this product as

$$m(\epsilon_{2^{k-1}} \ \cdots \ \epsilon_1).$$

Set

$$f_{0,\check{v}^{(i)}} = \frac{\beta_1\gamma_1 - 1}{\gamma_1 - 1}\left(\alpha\gamma_1 \prod_{j=2}^{k} (\beta_j\gamma_j)^{v_j^{(i)}} - 1\right),$$

$$f_{1,\check{v}^{(i)}} = \frac{(\beta_1 - 1)\gamma_1}{\gamma_1 - 1}\left(\alpha \prod_{j=2}^{k} (\beta_j\gamma_j)^{v_j^{(i)}} - 1\right).$$

For example, we have

$$
\begin{aligned}
f_{0,0}&=\frac{(\alpha\gamma_1-1)(\beta_1\gamma_1-1)}{\gamma_1-1}, & f_{1,0}&=\frac{(\alpha-1)(\beta_1-1)\gamma_1}{\gamma_1-1},\\
f_{0,1}&=\frac{(\alpha\gamma_1\beta_2\gamma_2-1)(\beta_1\gamma_1-1)}{\gamma_1-1}, & f_{1,1}&=\frac{(\alpha\beta_2\gamma_2-1)(\beta_1-1)\gamma_1}{\gamma_1-1},\\
f_{0,00}&=\frac{(\alpha\gamma_1-1)(\beta_1\gamma_1-1)}{\gamma_1-1}, & f_{1,00}&=\frac{(\alpha-1)(\beta_1-1)\gamma_1}{\gamma_1-1},\\
f_{0,10}&=\frac{(\alpha\gamma_1\beta_2\gamma_2-1)(\beta_1\gamma_1-1)}{\gamma_1-1}, & f_{1,10}&=\frac{(\alpha\beta_2\gamma_2-1)(\beta_1-1)\gamma_1}{\gamma_1-1},\\
f_{0,01}&=\frac{(\alpha\gamma_1\beta_3\gamma_3-1)(\beta_1\gamma_1-1)}{\gamma_1-1}, & f_{1,01}&=\frac{(\alpha\beta_3\gamma_3-1)(\beta_1-1)\gamma_1}{\gamma_1-1},\\
f_{0,11}&=\frac{(\alpha\gamma_1\beta_2\gamma_2\beta_3\gamma_3-1)(\beta_1\gamma_1-1)}{\gamma_1-1}, & f_{1,11}&=\frac{(\alpha\beta_2\gamma_2\beta_3\gamma_3-1)(\beta_1-1)\gamma_1}{\gamma_1-1},
\end{aligned}
\tag{A.1}
$$

for $k=2$ and $k=3$. Note that

$$
\begin{aligned}
&f_{0,\check{v}}=f_{0,(\check{v},0,\dots,0)}, \quad f_{1,\check{v}}=f_{1,(\check{v},0,\dots,0)},\\
&f_{0,\check{v}^{(i)}}-f_{1,\check{v}^{(i)}}=\alpha\prod_{j=1}^{k}(\beta_j\gamma_j)^{v_j^{(i)}}-1.
\end{aligned}
\tag{A.2}
$$

The products of the minors can be expressed in terms of these f's as follows:

Lemma A.2.

$$
m(\epsilon_{2^{k-1}}\ \cdots\ \epsilon_1)=\frac{\prod_{i=1}^{2^{k-1}}\left[(-1)^{\epsilon_i}\left(\alpha\prod_{j=2}^{k}(\beta_j\gamma_j)^{v_j^{(i)}}-1\right)f_{\epsilon_i,\check{v}^{(i)}}\right]}{(\alpha-1)^{2^{k-1}}\prod_{i=2}^{k}[(\beta_i-1)(\gamma_i-1)]^{2^{k-1}}}.
$$

Since we have the relations (A.2), we sum up $m(\epsilon_{2^{k-1}}\ \cdots\ \epsilon_1)$ two by two: set

$$
m(\epsilon_{2^{k-1}}\ \cdots\ \epsilon_2):=m(\epsilon_{2^{k-1}}\ \cdots\ \epsilon_2\,0)+m(\epsilon_{2^{k-1}}\ \cdots\ \epsilon_2\,1),
$$

and then set

$$
m(\epsilon_{2^{k-1}}\ \cdots\ \epsilon_3):=m(\epsilon_{2^{k-1}}\ \cdots\ \epsilon_3\,0)+m(\epsilon_{2^{k-1}}\ \cdots\ \epsilon_3\,1),
$$

and so on. We end up with

$$\begin{aligned}\det(H) &= \sum_{\text{all}} m(\epsilon_{2^{k-1}} \ \cdots \ \epsilon_1) = \cdots \\ &= m(00) + m(01) + m(10) + m(11) = m(0) + m(1) \\ &= \frac{\prod_{p=1}^{k} (\alpha \prod_{1\le i_1 < \cdots < i_p \le k} \beta_{i_1}\gamma_{i_1} \ldots \beta_{i_p}\gamma_{i_p} - 1)}{(\alpha-1)^{2^k-1} \prod_{i=1}^{k} (\beta_i - 1)^{2^{k-1}} (\gamma_i - 1)^{2^{k-1}}} \\ &= \frac{\prod_{v \in \mathbb{Z}_2^k} \left[\alpha \prod_{j=1}^{k} (\beta_j\gamma_j)^{v_j} - 1\right]}{(\alpha-1)^{2^k} \prod_{i=1}^{k} [(\beta_i - 1)(\gamma_i - 1)]^{2^{k-1}}}.\end{aligned}$$

Instead of giving proofs to the statements and Lemma A.2, we show the procedure when $k = 1, 2, 3$. These will light up the way in general cases.

A.3. $k = 1$

$$\begin{vmatrix} D_0 \cdot D_0 & D_0 \cdot D_1 \\ D_1 \cdot D_0 & D_1 \cdot D_1 \end{vmatrix} = \begin{vmatrix} -\dfrac{\beta_1\gamma_1 - 1}{(\beta_1 - 1)(\gamma_1 - 1)} & \dfrac{\sqrt{\gamma_1}}{\gamma_1 - 1} \\ \dfrac{\sqrt{\gamma_1}}{\gamma_1 - 1} & -\dfrac{\alpha\gamma_1 - 1}{(\alpha - 1)(\gamma_1 - 1)} \end{vmatrix}$$

$$= \frac{1}{(\beta_1-1)(\gamma_1-1)(\alpha-1)} \left(\frac{(\alpha\gamma_1-1)(\beta_1\gamma_1-1)}{\gamma_1-1} - \frac{(\alpha-1)(\beta_1-1)\gamma_1}{\gamma_1-1} \right)$$

$$= \frac{\alpha\beta_1\gamma_1 - 1}{(\beta_1 - 1)(\gamma_1 - 1)(\alpha - 1)}.$$

We use the identity

$$\frac{(CB-1)(AB-1)}{B-1} - \frac{(C-1)(A-1)B}{B-1} = CAB - 1$$

often.

A.4. $k = 2$

Set

$$F_2 := \frac{\alpha\beta_2\gamma_2 - 1}{(\alpha-1)^3(\beta_1-1)^2(\gamma_1-1)^2(\beta_2-1)^2(\gamma_2-1)^2};$$

this is the factor with the expected denominator and the numerator which does not contain β_1 nor γ_1. Then we have

$$m(00) = F_2 \cdot f_{0,1} \cdot f_{0,0}, \ -m(01) = F_2 \cdot f_{0,1} \cdot f_{1,0},$$

$$-m(10) = F_2 \cdot f_{1,1} \cdot f_{0,0}, \quad m(11) = F_2 \cdot f_{1,1} \cdot f_{1,0}.$$

Write down the identities (A.2) as

$$f_{0,0} - f_{1,0} = \alpha\beta_1\gamma_1 - 1, \quad f_{0,1} - f_{1,1} = \alpha\beta_1\gamma_1\beta_2\gamma_2 - 1,$$

which imply

$$\begin{aligned} m(0) &:= m(00) + m(01) = F_2 \cdot f_{0,1} \cdot (\alpha\beta_1\gamma_1 - 1), \\ -m(1) &:= m(10) + m(11) = F_2 \cdot f_{1,1} \cdot (\alpha\beta_1\gamma_1 - 1); \end{aligned}$$

$$\begin{aligned} m(0) + m(1) &= F_2 \cdot (\alpha\beta_1\gamma_1\beta_2\gamma_2 - 1)(\alpha\beta_1\gamma_1 - 1) \\ &= \frac{(\alpha\beta_1\gamma_1\beta_2\gamma_2 - 1)(\alpha\beta_1\gamma_1 - 1)(\alpha\beta_2\gamma_2 - 1)}{(\alpha - 1)^3(\beta_1 - 1)^2(\gamma_1 - 1)^2(\beta_2 - 1)^2(\gamma_2 - 1)^2}. \end{aligned}$$

Every 2×2-minor can be computed by the determinant formula when $k = 1$ established in the previous subsection. For example, we compute $m(00)$:

$$\begin{aligned} &\begin{vmatrix} D_{00} \cdot D_{00} & D_{00} \cdot D_{01} \\ D_{01} \cdot D_{00} & D_{01} \cdot D_{01} \end{vmatrix} \\ &= \begin{vmatrix} \frac{\beta_1\gamma_1 - 1}{(\beta_1 - 1)(\gamma_1 - 1)} \frac{\beta_2\gamma_2 - 1}{(\beta_2 - 1)(\gamma_2 - 1)} & \frac{\beta_1\gamma_1 - 1}{(\beta_1 - 1)(\gamma_1 - 1)} \frac{\sqrt{\gamma_2}}{\gamma_2 - 1} \\ \frac{\beta_1\gamma_1 - 1}{(\beta_1 - 1)(\gamma_1 - 1)} \frac{\sqrt{\gamma_2}}{\gamma_2 - 1} & \frac{\beta_1\gamma_1 - 1}{(\beta_1 - 1)(\gamma_1 - 1)} \frac{\alpha\gamma_2 - 1}{(\alpha - 1)(\gamma_2 - 1)} \end{vmatrix} \\ &= \left(\frac{\beta_1\gamma_1 - 1}{(\beta_1 - 1)(\gamma_1 - 1)} \right)^2 \begin{vmatrix} \frac{\beta_2\gamma_2 - 1}{(\beta_2 - 1)(\gamma_2 - 1)} & \frac{\sqrt{\gamma_2}}{\gamma_2 - 1} \\ \frac{\sqrt{\gamma_2}}{\gamma_2 - 1} & \frac{\alpha\gamma_2 - 1}{(\alpha - 1)(\gamma_2 - 1)} \end{vmatrix}. \end{aligned}$$

The second term of the last line is the intersection determinant appeared in the last subsection with the substitution:

$$\beta_1 \to \beta_2, \quad \gamma_1 \to \gamma_2.$$

$$\begin{vmatrix} D_{10} \cdot D_{10} & D_{10} \cdot D_{11} \\ D_{11} \cdot D_{10} & D_{11} \cdot D_{11} \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{\alpha\gamma_1 - 1}{(\gamma_1 - 1)(\alpha - 1)} \dfrac{\beta_2\gamma_2 - 1}{(\beta_2 - 1)(\gamma_2 - 1)} & \dfrac{\alpha\gamma_1 - 1}{(\gamma_1 - 1)(\alpha - 1)} \dfrac{\sqrt{\gamma_2}}{\gamma_2 - 1} \\ \dfrac{\alpha\gamma_1 - 1}{(\gamma_1 - 1)(\alpha - 1)} \dfrac{\sqrt{\gamma_2}}{\gamma_2 - 1} & \dfrac{\alpha\gamma_1\gamma_2 - 1}{(\alpha - 1)(\gamma_1 - 1)(\gamma_2 - 1)} \end{vmatrix}$$

$$= \left(\frac{\alpha\gamma_1 - 1}{(\gamma_1 - 1)(\alpha - 1)}\right)^2 \begin{vmatrix} \dfrac{\beta_2\gamma_2 - 1}{(\beta_2 - 1)(\gamma_2 - 1)} & \dfrac{\sqrt{\gamma_2}}{\gamma_2 - 1} \\ \dfrac{\sqrt{\gamma_2}}{\gamma_2 - 1} & \dfrac{\alpha\gamma_1\gamma_2 - 1}{(\alpha\gamma_1 - 1)(\gamma_2 - 1)} \end{vmatrix}.$$

The second term of the last line is the intersection determinant appeared in the last subsection with the substitution:

$$\beta_1 \to \beta_2, \quad \gamma_1 \to \gamma_2, \quad \text{and} \quad \alpha \to \alpha\gamma_1.$$

(Geometrically, the last substitution corresponds to the blow up at the intersection point of the lines labeled γ_1 and α; the exceptional curve corresponds to $\alpha\gamma_1$.) At any rate, the product of the two minors above give $m(00)$.

A.5. $k = 3$

Set

$$F_3 := \frac{(\alpha\beta_2\gamma_2 - 1)(\alpha\beta_3\gamma_3 - 1)(\alpha\beta_2\gamma_2\beta_3\gamma_3 - 1)}{(\alpha - 1)^7(\beta_1 - 1)^4(\gamma_1 - 1)^4(\beta_3 - 1)^4(\gamma_3 - 1)^4(\beta_3 - 1)^4(\gamma_3 - 1)^4};$$

this is the factor with the expected denominator and the numerator which does not contain β_1 nor γ_1. See (A.1) for $f_{0,\check{v}}$ and $f_{1,\check{v}}$ ($\check{v} \in \mathbb{Z}_2^2$). The products of two minors are given as

$$(-1)^{\epsilon_4+\epsilon_3+\epsilon_2+\epsilon_1} m(\epsilon_4, \epsilon_3, \epsilon_2, \epsilon_1) = F_3 \cdot f_{\epsilon_4,11} \cdot f_{\epsilon_3,01} \cdot f_{\epsilon_2,10} \cdot f_{\epsilon_1,00}.$$

By (A.2), we have

$$f_{0,00} - f_{1,00} = \alpha\beta_1\gamma_1 - 1, \qquad f_{0,10} - f_{1,10} = \alpha\beta_1\gamma_1\beta_2\gamma_2 - 1,$$
$$f_{0,01} - f_{1,01} = \alpha\beta_1\gamma_1\beta_3\gamma_3 - 1, \quad f_{0,11} - f_{1,11} = \alpha\beta_1\gamma_1\beta_2\gamma_2\beta_3\gamma_3 - 1.$$

These identities imply

$$(-1)^{\epsilon_4+\epsilon_3+\epsilon_2} m(\epsilon_4,\epsilon_3,\epsilon_2) = F_3 \cdot f_{\epsilon_4,11} \cdot f_{\epsilon_3,01} \cdot f_{\epsilon_2,10} \cdot (\alpha\beta_1\gamma_1 - 1),$$

$$(-1)^{\epsilon_4+\epsilon_3} m(\epsilon_4,\epsilon_3) = F_3 \cdot f_{\epsilon_4,11} \cdot f_{\epsilon_3,01} \cdot (\alpha\beta_1\gamma_1\beta_2\gamma_2 - 1)(\alpha\beta_1\gamma_1 - 1),$$

$$(-1)^{\epsilon_4} m(\epsilon_4) = F_3 \cdot f_{\epsilon_4,11} \cdot (\alpha\beta_1\gamma_1\beta_3\gamma_3 - 1)(\alpha\beta_1\gamma_1\beta_2\gamma_2 - 1)(\alpha\beta_1\gamma_1 - 1),$$

$$\begin{aligned} &m(0) + m(1) \\ &= F_3 \cdot (\alpha\beta_1\gamma_1\beta_2\gamma_2\beta_3\gamma_3 - 1)(\alpha\beta_1\gamma_1\beta_3\gamma_3 - 1)(\alpha\beta_1\gamma_1\beta_2\gamma_2 - 1)(\alpha\beta_1\gamma_1 - 1). \end{aligned}$$

The last identity is the expected expression.

Let us have a look at a typical minor:

$$\det((D_I \cdot D_J)_{I,J=\{000,001,010,011\}});$$

this turns out to be the product of

$$\left(\frac{\beta_1\gamma_1 - 1}{(\beta_1 - 1)(\gamma_1 - 1)}\right)^4$$

and the determinant of the intersection matrix when $k = 2$ with the substitution:

$$\beta_1 \to \beta_2,\ \gamma_1 \to \gamma_2,\ \beta_2 \to \beta_3,\ \gamma_2 \to \gamma_3;$$

and the complementary one

$$\det((D_I \cdot D_J)_{I,J=\{100,101,110,111\}});$$

this turns out to be the product of

$$\left(\frac{\alpha\gamma_1 - 1}{(\alpha - 1)(\gamma_1 - 1)}\right)^4$$

and the determinant of the intersection matrix when $k = 2$ with the substitution:

$$\beta_1 \to \beta_2,\ \gamma_1 \to \gamma_2,\ \beta_2 \to \beta_3,\ \gamma_2 \to \gamma_3, \quad \text{and} \quad \alpha \to \alpha\gamma_1.$$

Geometrically, the last substitution corresponds to the blow up along the intersection line of the planes labeled by γ_1 and α; the exceptional surface corresponds to $\alpha\gamma_1$.

References

[1] P. APPELL and J. KAMPÉ DE FÉRIET, "Fonctions Hypergéométriques et Hypersphériques: Polynomes d'Hermite", Gauthier-Villars, Paris, 1926.

[2] K. AOMOTO and M. KITA, "Theory of Hypergeometric Functions", translated by K. Iohara, Springer Monographs in Mathematics, Springer-Verlag, Tokyo, 2011.

[3] F. BEUKERS, *Algebraic A-hypergeometric functions*, Invent. Math. **180** (2010), 589–610.

[4] M. KATO, *Connection formulas and irreducibility conditions for Appell's* F_2, Kyushu J. Math. **66** (2012), 325–363.

[5] M. KITA and M. NOUMI, *On the structure of cohomology groups attached to the integral of certain many-valued analytic functions*, Japan. J. Math. **9** (1983), 113–157.

[6] M. KITA and M. YOSHIDA, *Intersection theory for twisted cycles*, Math. Nachr. **166** (1994), 287–304.

[7] G. LAURICELLA, *Sulle funzioni ipergeometriche a più variabili*, Rend. Circ. Mat. Palermo **7** (1893), 111–158.

[8] K. MATSUMOTO, *Monodromy and Pfaffian of Lauricella's* F_D *in terms of the intersection forms of twisted (co)homology groups*, Kyushu J. Math. **67** (2013), 367–387.

[9] K. MATSUMOTO, T. SASAKI, N. TAKAYAMA and M. YOSHIDA, *Monodromy of the hypergeometric differential equation of type* (3, 6) *I*, Duke Math. J. **71** (1993), 403–426.

[10] M. YOSHIDA, "Hypergeometric functions, my love, –Modular interpretations of configuration spaces–", Aspects of Mathematics E32., Vieweg & Sohn, Braunschweig, 1997.

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