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Factoring threefold divisorial contractions to points

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Abstract. We show that any terminal 3-fold divisorial contraction to a point of index > 1 with non-minimal discrepancy may be factored into a sequence of flips, flops and divisorial contractions to points with minimal discrepancies.

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1. Introduction

In minimal model program, the elementary birational maps consist of flips, flops and divisorial contractions. Birational maps in dimension three are reasonably well understood. The milestone work of Mori (*cf.* [14]) can be regarded as the starting point of explicit geometry. The detailed geometry of flips and flops in dimension three can be found in the seminal papers of Kollár and Mori (*cf.* [11, 12, 15]). Divisorial contractions to curves were studies by Cutkosky and intensively by Tziolas (*cf.* [2, 16–18]). Divisorial contractions to points are most well-understood. By results of Hayakawa, Kawakita, and Kawamata (*cf.* [3, 4, 6–8, 10]), it is now known that divisorial contractions to higher index points in dimension three are weighted blowups (under suitable embedding) and completely classified. It is expected that all divisorial contractions to points can be realized as weighted blowups.

Let $f : Y \to X$ be a divisorial contraction to a point $P \in X$ of index $n > 1$ in dimension three. We say that f has minimal discrepancy if the discrepancy of f is the minimal possible $1/n$ (which is the "w-morphism" in [1]). Divisorial contractions to higher index points with minimal discrepancies play a very interesting role at least for the following two reasons.

(1) For any terminal singularities $P \in X$ of index $n > 1$, there exists a partial resolution $X_k \to \ldots \to X_0 := X$ such that X_k has only terminal Gorenstein singularities, *i.e.* terminal singularity of index 1, and each map $X_{i+i} \to X_i$

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is a divisorial contraction to a higher index point with minimal discrepancy (*cf.* [4]).

(2) For any flipping contraction or divisorial contraction to a curve $Y \to X$, by taking a divisorial contraction over the highest index point with minimal discrepancy in Y, one gets a factorization into "simpler" birational maps (*cf.* [1]).

On the other hand, divisorial contractions to points with non-minimal discrepancies are rather special. For example, if $P \in X$ is of type $cAx/2$, $cAx/4$ or $cD/3$, then there is no divisorial contraction with non-minimal discrepancy (*cf.* [7]).

The purpose of this note is to show that divisorial contractions to a higher index points with non-minimal discrepancies can be factored into a sequence of divisorial contractions of minimal discrepancies, flips and flops (*cf.* [1]).

Let $f : Y \to X$ be a divisorial contraction to a point $P \in X$ of index $n > 1$ with discrepancy $a/n > 1/n$. By the classification of [2, 14], Y contains some higher index points. Let $Q \in Y$ be the highest index point of index $p > 1$. Thanks to Kawamata's result that there exists a divisorial contraction over any higher index point with minimal discrepancy (*cf.* [9]), we may pick $g : Z \to Y$ a divisorial contraction to $Q \in Y$ with discrepancy $1/p$, which is a weighted blowup.

Theorem 1.1. *Let $f : Y \to X$ be a divisorial contraction to a point $P \in X$ of index $n > 1$ with discrepancy $a/n > 1/n$. Let $Q \in Y$ be a point of highest index p in Y and $g : Z \to Y$ be an divisorial contraction to the point Q with discrepancy $1/p$. Then either $-K_{Z/X}$ is ample or $-K_{Z/X}$ is nef and that there is a member $S_Y \in |-K_Y|$ such that $S_Y \cap E$ contains an irreducible curve l with its proper transform $l_Z \cdot K_Z < 0$.*

Notice that the relative Picard number $\rho(Z/X) = 2$. Therefore, we are able to play the so called 2-ray game. By contracting the other extremal ray, we have $Z \dashrightarrow Z^\sharp \xrightarrow{g^\sharp} Y^\sharp \xrightarrow{f^\sharp} X$, where $Z \dashrightarrow Z^\sharp$ consists of a sequence of flips and flops (or identity map), $Z^\sharp \to Y^\sharp$ is a divisorial contraction. Let E, F be the exceptional divisor of f and g respectively. Let $F_{Y^\sharp}$ (respectively, $F_{Z^\sharp}$, $E_{Z^\sharp}$) be the proper transform of F (respectively F, E) in $Y^\sharp$ (respectively $Z^\sharp$). We have the following more precise description.

Theorem 1.2. *Keep the notation as above. We have the following diagram*

$$\begin{tikzcd} Z \arrow[rr, dashed] \arrow[d, "g"'] & & Z^\sharp \arrow[d, "g^\sharp"] \\ Y \arrow[dr, "f"'] & & Y^\sharp \arrow[dl, "f^\sharp"] \\ & X & \end{tikzcd}$$

where $f^\sharp$ is a divisorial contraction to a point $P \in X$ with discrepancy $\frac{a'}{n} < \frac{a}{n}$ and $g^\sharp$ is a divisorial contraction to a singular point $Q' \in F_{Y^\sharp}$ of index p' with discrepancy $\frac{q'}{p'}$. We may rite $g^{\sharp *}F_{Y^\sharp} = F_{Z^\sharp} + \frac{\mathfrak{q}}{p'}E_{Z^\sharp}$, then

$$\frac{a}{n} = \frac{a'}{n} \cdot \frac{\mathfrak{q}}{p'} + \frac{q'}{p'}.$$

More specifically, exactly one of the following holds.

(1) *If $P \in X$ is of type other than $cE/2$, then Q' is a point of index n, $\frac{\mathfrak{q}}{p'} = 1$, and $g^\sharp$ has discrepancy $\frac{a''}{n}$ with $a' + a'' = a$.*
(2) *If $P \in X$ is of type $cE/2$, then $a = 2$, $a' = 1$, Q' is a point of index $p' = 3$, $\frac{\mathfrak{q}}{p'} = \frac{4}{3}$, and $g^\sharp$ has minimal discrepancy $\frac{1}{3}$.*

As an immediate corollary by induction on discrepancy a, we have:

Corollary 1.3. *For any divisorial contraction $Y \to X$ to a point $P \in X$ of index $n > 1$ with discrepancy $\frac{a}{n} > \frac{1}{n}$. There exists a sequence of birational maps*

$$Y = X_k \dashrightarrow \dots \dashrightarrow X_0 = X$$

such that each map $X_{i+1} \dashrightarrow X_i$ is one of the following:

(1) *a divisorial contraction to a point of index $r_i > 1$ with minimal discrepancy $\frac{1}{r_i}$ or its inverse;*
(2) *a flip or a flop.*

We now briefly explain the main idea. According the 2-ray game, we have the following diagram of birational maps.

$$\begin{tikzcd}
Z \arrow[rr, dashed] \arrow[d, "g"'] & & Z^\sharp \arrow[d, "g^\sharp"] \\
Y \arrow[dr, "f"'] & & Y^\sharp \arrow[dl, "f^\sharp"] \\
& X &
\end{tikzcd}$$

Notice that in this diagram the order of exceptional divisors of the tower $Z^\sharp \to Y^\sharp \to X$ and $Z \to Y \to X$ are reversed. A priori, $g^\sharp$ could be a divisorial contraction to a point or a curve. To understand the diagram explicitly, the usual difficulty is that we need to determine the center of $E_{Z^\sharp}$ in $Y^\sharp$.

On the other hand, since $f : Y \to X$ is a weighted blowup, one can embed X into a toric variety $\mathcal{X}_0$ and understand $f : Y \to X$ as the proper transform of a toric weighted blowup $\mathcal{X}_1 \to \mathcal{X}_0$, which is nothing but a subdivision of a cone along a vector v_1. If $Z \to Y$ can be realized as the proper transform of a toric weighted blowup $\mathcal{X}_2 \to \mathcal{X}_1$ over the origin of the standard coordinate charts, then we can view $\mathcal{X}_2 \to \mathcal{X}_1$ as a toric weighted blowup along a vector v_2. Therefore, the tower $\mathcal{X}_2 \to \mathcal{X}_1 \to \mathcal{X}_0$ is obtained by subdivision along v_1 and then v_2.

We may consider a tower $\mathcal{X}'_2 \to \mathcal{X}'_1 \to \mathcal{X}_0$ of toric weighted blowup by subdivision along v_2 and then v_1 instead. The proper transforms of X in this tower then gives $Z' \to Y' \to X$. Clearly, this is a tower reversing the order of exceptional divisors of $Z \to Y \to X$ by construction. In fact, one can verify that those two induced tower $Z' \to Y' \to X$ and $Z^\sharp \to Y^\sharp \to X$ matches isomorphically. That is, $Y' \cong Y^\sharp$, $Z' \cong Z^\sharp$ and both $f^\sharp, g^\sharp$ are weighted blowups.

In Section 2, we recall and generalize the construction of weighted blowup. We also derive a criterion for $-K_{Z/X}$ being nef or ample. Moreover, we show that if the tower $Z \to Y \to X$ can be embedded into a tower of weighted blowup $\mathcal{X}_2 \to \mathcal{X}_1 \to \mathcal{X}_0$ and $-K_{Z/X}$ is ample, then the output of 2-ray game coincides with the output by "reversing order of vectors" of the tower of weighted blowups.

In Section 3, we study divisorial contractions to higher index points with non-minimal discrepancies case by case. By [5,7,8], any divisorial contraction to a higher index point with non-minimal discrepancy is described as a weighted blowup explicitly. We verify Theorem 1.1 and 1.2 for each case.

We always work over complex number field $\mathbb{C}$ and in dimension three. We assume that threefold X, Y are $\mathbb{Q}$-factorial. We freely use the standard notions in minimal model program such as terminal singularities, divisorial contractions, flips, and flops. For the precise definition, we refer to [13].

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2. Preliminaries

2.1. Weighted blowups

We recall the construction of weighted blowups by using the toric language.

Let $N = \mathbb{Z}^d$ be a free abelian group of rank d. Let $v = \frac{1}{n}(a_1, \dots, a_d) \in \mathbb{Q}^d$ be a vector. We may assume that $gcd(a_1, \dots, a_d) = 1$. We consider $\overline{N} := N + \mathbb{Z}v$. Let M (respectively $\overline{M}$) be the dual lattice of N (respectively $\overline{N}$). Let σ be the first orthant and Σ be the fan consists of σ and all the subcones of σ. We consider

$$\mathcal{X}_0 := \mathcal{X}_{\overline{N},\Sigma} := \mathrm{Spec}\mathbb{C}[\sigma^\vee \cap \overline{M}],$$

which is a quotient of $\mathbb{C}^n$ by the cyclic group $\mathbb{Z}/n\mathbb{Z}$ with weights $(a_1, \dots, a_d)$, which we denote it as $\mathbb{C}^n/\frac{1}{n}(a_1, \dots, a_d)$ or simply $\mathbb{C}^n/v$.

Let $v_1 = \frac{1}{r_1}(b_1, \ldots, b_d)$ be a primitive vector in $\overline{N}$. We assume that $b_i \in \mathbb{Z}_{>0}$ and $gcd(b_1, \ldots, b_d) = 1$. We are interested in the weighted blowup over $o \in \mathbb{C}^d/\frac{1}{n}(a_1, \ldots, a_d) = \mathcal{X}_{\overline{N},\Sigma} =: \mathcal{X}_0$ with weights $v_1 = \frac{1}{r_1}(b_1, \ldots, b_d)$ which we describe now.

Let $\overline{\Sigma}$ be the fan obtained by subdivision of Σ along v_1. One thus have a toric variety $\mathcal{X}_{\overline{N},\overline{\Sigma}}$ together with the natural map $\mathcal{X}_{\overline{N},\overline{\Sigma}} \to \mathcal{X}_{\overline{N},\Sigma}$. More concretely, let σ_i be the cone generated by $\{e_1, \ldots, e_{i-1}, v_1, e_{i+1}, \ldots, e_d\}$, then

$$\mathcal{X}_1 := \mathcal{X}_{\overline{N},\overline{\Sigma}} = \cup_{i=1}^d \mathcal{U}_i,$$

where $\mathcal{U}_i = \mathcal{X}_{\overline{N},\sigma_i} = \mathrm{Spec}\mathbb{C}[\sigma_i^\vee \cap \overline{M}]$. We always denote the origin of $\mathcal{U}_i$ as Q_i. In each affine chart $\mathcal{U}_i$, the natural map $\mathcal{U}_i \to \mathcal{X}_0$ is given by

$$\begin{cases} x_j \mapsto \bar{x}_j \bar{x}_i^{b_j/r_1}, & \text{if } j \neq i; \\ x_i \mapsto \bar{x}_i^{b_i/r_1}. \end{cases}$$

We denote the exceptional divisor $\mathcal{E} \cong \mathbb{P}((b_1, b_2, \ldots, b_d))$ by $\mathbb{P}(v_1)$.

Let $X \in \mathcal{X}_0$ be a complete intersection defined by semi-invariants $\varphi_1 = \ldots = \varphi_c = 0$. Let Y be the proper transform of X in $\mathcal{X}_1$. By abuse the notation, we also call the induced map $f : Y \to X$ the weighted blowups of X of weights v_1. Recall that for a semi-invariant $\varphi = \sum \alpha_{i_1,\ldots,i_d} x_1^{i_1} \ldots x_d^{i_d}$ on the quotient variety $\mathcal{X}_0$ and a vector $\nu = \frac{1}{r}(\beta_1, \beta_2, \ldots, \beta_n) \in \overline{N}$, we define

$$wt_\nu(\varphi) := \min\left\{\sum_{j=1}^d \frac{\beta_j i_j}{r} | \alpha_{i_1,\ldots,i_d} \neq 0\right\}.$$

Now the local chart $\mathcal{U}_i \cap Y \subset \mathcal{U}_i$ is defined by $\bar{\varphi}_1 = \ldots = \bar{\varphi}_c = 0$ with

$$\bar{\varphi}_j := \varphi_j(\bar{x}_1 \bar{x}_i^{b_1/r_1}, \ldots, \bar{x}_{i-1} \bar{x}_i^{b_{i-1}/r_1}, \bar{x}_i^{b_i/r_1}, \bar{x}_{i+1} \bar{x}_i^{b_{i+1}/r_1}, \ldots, \bar{x}_n \bar{x}_i^{b_d/r_1}) \bar{x}_i^{-wt_{v_1}(\varphi_j)},$$

for all i, j. Let $E := \mathcal{E} \cap Y \subset \mathbb{P}(v)$ denotes the exceptional divisor. If E is irreducible and reduced, then the adjunction formula yields that

$$K_Y = f^* K_X + a(v_1, X)E,$$

where $a(v_1, X) = \sum_i wt_{v_1}(x_i) - \sum_j wt_{v_1}(\varphi_j) - 1$.

Quite often, we need to compare weighted blowups of a given variety X embedded in different ambient spaces. Let $X \subset \mathcal{X}_0 = \mathbb{C}^d/v = \mathbb{C}^d/\frac{1}{n}(a_1, \ldots, a_d)$ be a complete intersection defined by semi-invariants $(\varphi_1 = \ldots = \varphi_c = 0)$. Suppose that $\varphi_c = f_0 + f_1 f_2$ with f_0, f_1, f_2 being semi-invariants. We set $v' := (\frac{a_1}{n}, \ldots, \frac{a_d}{n}, wt_v(f_1))$ and consider $X' \subset \mathcal{X}'_0 := \mathbb{C}^{d+1}/v'$ defined by semi-invariants $(\varphi_1 = \ldots = \varphi_{c-1} = \varphi'_c = \varphi'_{c+1} = 0)$, where

$$\begin{cases} \varphi'_c := f_0 + x_{d+1} f_2, \\ \varphi'_{c+1} := x_{d+1} - f_1. \end{cases}$$

It is straightforward to check that $X \cong X'$.

Definition 2.1. Let $X \subset \mathcal{X}_0 = \mathbb{C}^d/v$ and $X' \subset \mathcal{X}_0' = \mathbb{C}^{d+1}/v'$ be complete intersections defined as above. Let $f : Y \to X \subset \mathcal{X}_0$ be the weighted blowup with weights $v_1 = \frac{1}{r_1}(b_1, \dots, b_d)$ and let $f' : Y' \to X' \subset \mathcal{X}_0'$ be the weighted blowup with weights $v_1' = (\frac{b_1}{r_1}, \dots, \frac{b_d}{r_1}, wt_{v_1}(f_1))$.

Then one has the following commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\cong"] \arrow[d, "f"'] & Y' \arrow[d, "f'"] \\ X \arrow[r, "\cong"] & X' \end{tikzcd}$$

We say that the weighted blowup f and f' are *compatible* in this situation.

2.2. Tower of toric weighted blowups

Suppose that there is a primitive vector $v_2 = \frac{1}{r_2}(c_1, \dots, c_d) \in \overline{N}$ such that v_2 is in the interior of σ_i. We can write

$$v_2 = \frac{1}{p}(q_1e_1 + \ldots + q_{i-1}e_{i-1} + q_iv_1 + q_{i+1}e_{i+1} + \ldots + q_de_d),$$

for some $p, q_i \in \mathbb{Z}_{>0}$. We denote $w_2 = \frac{1}{p}(q_1, \dots, q_d)$ to be the weight of v_2 in the cone σ_i, or simply the *corresponding weight of* v_2 if no confusion is likely.

Observation. If $\overline{N}$ is generated by $\{v_1, v_2, e_1, \dots, \hat{e}_i, \dots, e_d\}$, then $\mathcal{X}_{\overline{N}, \sigma_i} \cong \mathbb{C}^d/w_2 = \mathbb{C}^d/\frac{1}{p}(q_1, q_2, \dots, q_d)$ has only quotient singularity at Q_i.

We can consider the second weighted blowup with vector v_2. Let $\overline{\overline{\Sigma}}$ be the fan obtained by subdivision of σ_i along v_2. One thus have a toric variety $\mathcal{X}_2 := \mathcal{X}_{\overline{N}, \overline{\overline{\Sigma}}}$. More explicitly, let τ_{ij} be the cone generated by

$$\begin{cases} \{e_1, \dots, e_{j-1}, v_2, e_{j+1}, \dots, e_{i-1}, v_1, e_{i+1}, \dots, e_d\}, & \text{if } j \neq i; \\ \{e_1, \dots, e_{i-1}, v_2, e_{i+1}, \dots, e_d\}, & \text{if } j = i. \end{cases}$$

Then $\mathcal{X}_{\overline{N}, \overline{\overline{\Sigma}}} = (\cup_{k \neq i} \mathcal{U}_k) \bigcup (\cup_{j=1}^d \mathcal{V}_{ij})$, where $\mathcal{V}_{ij} = \operatorname{Spec}\mathbb{C}[\tau_{ij}^\vee \cap \overline{M}]$.

Definition 2.2. We say that $\mathcal{X}_1 \to \mathcal{X}_0$ is the weighted blowup with with vector v_1 or with weights w_1. We say that $\mathcal{X}_2 \to \mathcal{X}_1$ is the weighted blowup with vector v_2 or with weights $w_2 = \frac{1}{p}(q_1, \dots, q_d)$.

We can consider $\mathcal{X}_1' \to \mathcal{X}_0$ the weighted blowup with vector v_2, then $\mathcal{X}_1' = \cup \mathcal{U}_j' = \cup \operatorname{Spec}\mathbb{C}[\sigma_j'^\vee \cap \overline{M}]$, where σ_j' is the cone generated by $\{e_1, \dots, e_{j-1}, v_2, e_{j+1}, \dots, e_d\}$. Clearly,

$$\mathcal{U}_i' = \operatorname{Spec}\mathbb{C}[\sigma_i'^\vee \cap \overline{M}] = \operatorname{Spec}\mathbb{C}[\tau_{ii}^\vee \cap \overline{M}] = \mathcal{V}_{ii}.$$

Notice also that the exceptional divisor $\mathcal{F}$ of $\mathcal{X}_2 \to \mathcal{X}_1$ and the exceptional divisor $\mathcal{F}'$ of $\mathcal{X}'_1 \to \mathcal{X}_0$ defines the same valuations, which is given by the ray generated by v_2.

Suppose furthermore that v_1 is in the interior of σ'_k for some k. Then we can consider a weighted blowup $\mathcal{X}'_2 \to \mathcal{X}'_1$ with vector v_1. Notice also that the exceptional divisor $\mathcal{E}$ of $\mathcal{X}_2 \to \mathcal{X}_1$ and the exceptional divisor $\mathcal{E}'$ of $\mathcal{X}'_1 \to \mathcal{X}_0$ defines the same valuations, which is given by the ray generated by v_1.

Definition 2.3. Keep the notation as above. We say that v_1 and v_2 are *interchangeable* if v_2 is in the interior of σ_i for some i and v_1 is in the interior of σ'_j for some j.

In the situation that v_1, v_2 are interchangeable, we have the following diagram

$$\begin{tikzcd}
\mathcal{X}_2 \arrow[r, dashed] \arrow[d, "v_2"'] & \mathcal{X}'_2 \arrow[d, "v_1"] \\
\mathcal{X}_1 \arrow[d, "v_1"'] & \mathcal{X}'_1 \arrow[d, "v_2"] \\
\mathcal{X}_0 \arrow[r, "="] & \mathcal{X}_0
\end{tikzcd}$$

such that $\mathcal{X}_2 \dashrightarrow \mathcal{X}'_2$ is isomorphic in codimension one and all the vertical maps are weighted blowups over points.

2.3. 2-ray game

Let $f : Y \to X$ be a divisorial contraction to a point $P \in X$ of index n with discrepancy $\frac{a}{n} \geq \frac{1}{n}$ and E be the exceptional divisor of f. Let $g : Z \to Y$ be a divisorial contraction to a point $Q \in E$ of index p with discrepancy $\frac{b}{p} \geq \frac{1}{p}$. Now the relative Picard number $\rho(Z/X) = 2$. If $-K_{Z/X}$ is nef, then there exists the so-called "2-ray game" (see [1] for example). We examine conditions under which $-K_{Z/X}$ is nef.

We fix some notations. Let $D \neq E$ be a divisor on Y passing through Q such that $D \cap E$ is irreducible (possibly non-reduced). Let $D_X = f_*D$, $D_Z = g_*^{-1}D$ be the proper transform of D on X, Z respectively and E_Z be the proper transform of E on Z. We have

$$f^*D_X = D + \frac{c_0}{n}E, \quad g^*D = D_Z + \frac{q_0}{p}F, \quad g^*E = E_Z + \frac{\mathfrak{q}}{p}F$$

for some $c_0, q_0, \mathfrak{q} \in \mathbb{Z}_{>0}$.

Proposition 2.4. *Let $f : Y \to X$ be a divisorial contraction to a point $P \in X$ of index n with discrepancy $\frac{a}{n}$ and E be the exceptional divisor of f. Let $g : Z \to Y$ be a divisorial contraction to a point $Q \in E$ of index p with discrepancy $\frac{b}{p}$. Suppose* *that there is a divisor D on Y such that $D \cap E$ is irreducible. Then $-K_{Z/X}$ is nef if the following inequalities holds*:

$$\begin{cases} T(f, g, D) := \dfrac{-ac_0}{n^2} E^3 + \dfrac{q_0 \mathfrak{q} b}{p^3} F^3 \leq 0; \\ bc_0 - aq_0 \leq 0. \end{cases}$$

Proof. To show that $-K_{Z/X}$ is nef, we need to show that for any curve $\gamma \subset E$ its proper transform γ_Z is non-positive, *i.e*. the inequality $\gamma_Z \cdot K_Z \leq 0$ holds.

Let $[l]$ (respectively $[l_Z]$, $[l_F]$) be the associated 1-cycle of $[D \cap E]$ (respectively $[D_Z \cap E_Z], [E_Z \cap F]$). We compute the intersection number.

$$l_Z \cdot K_Z = D_Z \cdot E_Z \cdot K_Z = D \cdot E \cdot K_Y + \frac{q_0 \mathfrak{q} b}{p^3} F^3 = \frac{-ac_0}{n^2} E^3 + \frac{q_0 \mathfrak{q} b}{p^3} F^3. \quad (2.1)$$

For any curve $\gamma \subset E$ not passing through Q, one has $\gamma_Z \cdot K_Z = \gamma \cdot K_Y < 0$ immediately. Now for any irreducible curve $l \subset E$ passing through Q. Since $\rho(Y/X) = 1$, we have $[\gamma] \equiv \frac{c}{c_0}[l]$ numerically for some $c \in \mathbb{R}_{>0}$ as 1-cycles. It follows that $\gamma \cdot K_Y = \frac{-ac}{n^2} E^3$. Since $\rho(Z/X) = 2$ and the relative Picard group $N_1(Z/X) \otimes \mathbb{R}$ is generated by l_F and γ_Z. We can write

$$g^*[\gamma] = [\gamma_Z] + \frac{q}{p}[l_F]$$

for some $q \in \mathbb{R}$. Intersecting with F, one finds that $q > 0$ by the facts that $\gamma_Z \cdot F > 0$ and $l_F \cdot F = \frac{-\mathfrak{q}}{p} F^3 < 0$.

Computation shows that

$$\gamma_Z \cdot K_Z = \gamma \cdot K_Y + \frac{q \mathfrak{q} b}{p^3} F^3 = \frac{-ac}{n^2} E^3 + \frac{q \mathfrak{q} b}{p^3} F^3. \quad (2.2)$$

Notice that $\gamma \cdot_Y D = \frac{c}{c_0} l \cdot D = \frac{c}{c_0} D^2 \cdot E = \frac{cc_0}{n^2} E^3$. On the other hand,

$$\begin{aligned} \gamma \cdot_Y D &= \gamma_Z \cdot_Z g^* D = \gamma_Z \cdot D_Z + \frac{q_0}{p} \gamma_Z \cdot F \\ &= \gamma_Z \cdot D_Z + \frac{q \mathfrak{q} q_0}{p^3} F^3 = \gamma_Z \cdot_{E_Z} l_Z + \frac{q \mathfrak{q} q_0}{p^3} F^3. \end{aligned}$$

If $\gamma \neq l$, then $\gamma_Z \cdot_{E_Z} l_Z \geq 0$. So we have

$$\frac{cc_0}{n^2} E^3 \geq \frac{q \mathfrak{q} q_0}{p^3} F^3.$$

Compare with (2.2), we have that for $\gamma \neq l$,

$$\gamma_Z \cdot K_Z \leq \frac{c}{q_0 n^2} (bc_0 - aq_0) E^3. \quad (2.3)$$

It follows that $-K_{Z/X}$ is nef provided the quantities in (2.1), (2.3) are ≤ 0. This completes the proof. $\square$

In [8, Theorem 1.5], Kawakita gives an affirmative answer to the General Elephant Conjecture. In particular, let $f : Y \to X$ be a divisorial contraction, then a general element $S_Y \in |-K_Y|$ is normal and has only Du Val singularities. The existence of a good member in $|-K_Y|$ indeed provides a very useful tool.

Proposition 2.5. *Let $f : Y \to X$ be a divisorial contraction to a point with exceptional divisors E and $g : Z \to Y$ be a divisorial contraction to a point $Q \in E \subset Y$ of index p with discrepancy $\frac{1}{p}$. Let F be the exceptional divisor of g. Suppose that $-K_{Z/X}$ is nef and there is an irreducible curve $l \subset S_Y \cap E$ such that $l_Z \cdot K_Z < 0$, then we have the following diagram of birational maps.*

$$\begin{tikzcd}
Z \arrow[rr, dashed, "\phi"] \arrow[d, "g"'] & & Z^\sharp \arrow[d, "g^\sharp"] \\
Y \arrow[dr, "f"'] & & Y^\sharp \arrow[dl, "f^\sharp"] \\
& X &
\end{tikzcd}$$

where ϕ is a sequence of flips and flops (or just the identity map), $g^\sharp$ is a divisorial contraction contracting $E_{Z^\sharp}$ and $f^\sharp$ is a divisorial contraction contracting $F_{Y^\sharp}$ to the point $P \in X$.

Proof. Suppose that $-K_{Z/X}$ is nef, then one can play the so-called "2-ray game" as in [1]. Let $\overline{NE}(Z/X)$ be the relative Mori cone, generated by $[l_F]$ and $[l_R]$, where l_F is a curve in the g-exceptional divisor F and l_R is the other generator. Let $h : Z \to W$ be the extremal contraction corresponding to the extremal ray $[l_R]$.

Claim. h cannot be a crepant divisorial contraction if there is an irreducible curve $l \subset S_Y \cap E$ such that $l_Z \cdot K_Z < 0$.[1]

Proof of the Claim. Suppose on the contrary that h is a crepant divisorial contraction. Then h contracts the divisor E_Z to a curve Γ. To see this, note that if either h contracts F or h contracts E_Z to a point, then there is a curve $\gamma \subset F$ or $\gamma \subset F \cap E_Z$ contracted by h. This is impossible because $[\gamma] \in \mathbb{R}_{>0}[l_F]$ cannot be contracted by h.

Let S_Y be an Du Val element in $|-K_Y|$ and $S_Z := g_*^{-1} S_Y$. By [1, Lemma 2.7], $S_Z \in |-K_Z|$. Let C be an h-exceptional curve. Then $C \cdot S_Z = 0$ and therefore either $C \cap S_Z = \emptyset$ or $C \subset S_Z$.

If there exists a curve $l \subset S_Y \cap E$ such that $l_Z \cdot K_Z < 0$, then l_Z is not contracted by h. It follows that l_Z maps onto the image Γ of E_Z and then l_Z meets

[1] The proof of this Claim was provided by the anonymous referee. We are grateful to the referee for pointing out a gap in our first version and kindly providing the proof.

all fibers of $E_Z \to \Gamma$ and hence if C is any h-exceptional curve, $C \cdot S_Z > 0$, which is impossible. $\square$

If $h : Z \to W$ is a divisorial contraction, then we set $Z^\sharp := Z, Y^\sharp := W, g^\sharp = h$, and ϕ the identity map. If $h : Z \to W$ is a flipping or a flopping contraction, then there exists Z^+ the flip or the flop of Z. By running the relative minimal model program of Z^+/X, one ends up with $Z^+ \dashrightarrow Z^\sharp \to Y^\sharp$ so that $Z^+ \dashrightarrow Z^\sharp$ consists of a sequence of flips and flops, and $Z^\sharp \to Y^\sharp$ is a divisorial contraction.

We now show that $K_{Y^\sharp/X}$ is $-f^\sharp$-ample. Since $\rho(Y^\sharp/X) = 1$, $Y^\sharp$ is terminal $\mathbb{Q}$-factorial and the $f^\sharp$-exceptional set support on a divisor, say D. Let $\gamma \subset D$ be any curve. Pick any very ample divisor H on $Y^\sharp$, then we have $f^{\sharp *} H_X = H + \mu D$ for some $\mu > 0$. Intersect with γ, we have

$$0 = \gamma \cdot f^{\sharp *} H_X = \gamma \cdot H + \mu\gamma \cdot D.$$

Hence $\gamma \cdot D < 0$. Now $\gamma \cdot K_{Y^\sharp} = \gamma \cdot aD < 0$, for X is terminal. Therefore the discrepancy $a = a(D, X) > 0$.

It remains to show that $f^\sharp$ contracts $F_{Y^\sharp}$. Suppose on the contrary that $f^\sharp$ contracts $E_{Y^\sharp}$. By [6, Lemma 3.4], we thus have $Y^\sharp \cong Y$ for E and $E_{Y^\sharp}$ clearly defines the same valuation. By the same argument, we also have $Z \cong Z^\sharp$, which is a contradiction. $\square$

Corollary 2.6. *Keep the notation as in Proposition* 2.5. *Suppose that there exist a normal Du Val element $S_Y \in |-K_Y|$ such that $S_Y \cap E$ is irreducible and*

$$T(f, g) := \frac{-a^2}{n^2} E^3 + \frac{\mathfrak{q}}{p^3} F^3 < 0.$$

Then $-K_{Z/X}$ is nef and there is a diagram as in the Proposition 2.5

Proof. In this situation $c_0 = a, q_0 = b = 1$. Hence $-K_{Z/X}$ is nef if $T(f, g) \leq 0$ by Proposition 2.4. Let $l = S_Y \cap E$, then $l_Z \cdot K_Z = T(f, g)$. There is a diagram as in Proposition 2.5 if $T(f, g) < 0$. $\square$

2.4. Weighted blowups and 2-ray game

We fix an embedding $P \in X \hookrightarrow \mathcal{X}_0$ such that the divisorial contraction $f : Y \to X$ is given by the weighted blowup $\mathcal{X}_1 \to \mathcal{X}_0$ with weights v_1. That is, Y is the proper transform of X in $\mathcal{X}_1$. Let $g : Z \to Y$ be a divisorial contraction with minimal discrepancy over a point Q_i of index $p > 1$.

Suppose that, under such embedding, the following hypotheses holds. **Hypothesis $\flat$.**

(1) The divisorial extraction $g : Z \to Y$ is given by a weighted blowup $\mathcal{X}_2 \to \mathcal{X}_1$ over a point Q_i with vector v_2.
(2) The vectors v_1, v_2 are interchangeable (*cf.* Remark 2.3).
(3) $-K_{Z/X}$ is nef and there is an irreducible curve $l \subset S_Y \cap E$ such that $l_Z \cdot K_Z < 0$.

Then we have the following diagram.

$$\begin{tikzcd}
Z^{\sharp} & Z \arrow[l, "\dashleftarrow"'] \arrow[r, "\hookrightarrow"] \arrow[d, "g"'] & \mathcal{X}_2 \arrow[r, "\dashrightarrow"] \arrow[d, "v_2"'] & \mathcal{X}'_2 \arrow[d, "v_1"] & Z' \arrow[l, "\hookleftarrow"'] \arrow[d, "g'"] \\
Y^{\sharp} \arrow[u, leftarrow, "g^{\sharp}"] \arrow[d, "f^{\sharp}"'] & Y \arrow[r, "\hookrightarrow"] \arrow[d, "f"'] & \mathcal{X}_1 \arrow[d, "v_1"'] & \mathcal{X}'_1 \arrow[d, "\eta"', "v_2"] & Y' \arrow[l, "\hookleftarrow"'] \arrow[d, "f'"] \\
X & X \arrow[l, "="'] \arrow[r, "\hookrightarrow"] & \mathcal{X}_0 \arrow[r, "="] & \mathcal{X}_0 & X, \arrow[l, "\hookleftarrow"']
\end{tikzcd}$$

where $Z^{\sharp} \to Y^{\sharp} \to X$ is the output of the 2-ray game and Z', Y' are proper transform of X in $\mathcal{X}'_2, \mathcal{X}'_1$ respectively.

Theorem 2.7. *Keep the notation as above and suppose that Hypothesis $\flat$ holds. Then $Y^{\sharp} \cong Y'$ and $Z^{\sharp} \cong Z'$. Moreover, both $f^{\sharp} \cong f'$ and $g^{\sharp} \cong g'$ are weighted blowups and divisorial contractions to a points.*

Proof. Let $\mathcal{F}$ be the exceptional divisor of $\mathcal{X}_2 \to \mathcal{X}_1$. It is the exceptional divisor induced by the vector v_2. Hence its proper transform $\mathcal{F}'$ in $\mathcal{X}'_1$ is the exceptional divisor of $\eta : \mathcal{X}'_1 \to \mathcal{X}_0$. Recall that by the construction in Subsection 2.2, there is a canonical isomorphism $\mathcal{V}_{ii} \cong \mathcal{U}'_i$ for some i, where $\mathcal{V}_{ii} \subset \mathcal{X}_2$ and $\mathcal{U}'_i \subset \mathcal{X}'_1$ are coordinate charts. Surely, we have an induced isomorphism $Z \cap \mathcal{V}_{ii} \cong Y' \cap \mathcal{U}'_i$. Since F is irreducible and

$$F \cap \mathcal{V}_{ii} = (\mathcal{F} \cdot Z) \cap \mathcal{V}_{ii} \cong (\mathcal{F}' \cdot Y') \cap \mathcal{U}'_i.$$

It follows that $F_{Y'} := \mathcal{F}' \cdot Y'$ is irreducible, which coincides with the exceptional set. On the other hand, the proper transform of F in $Y^{\sharp}$ is $F_{Y^{\sharp}}$, which is the exceptional divisor of $f^{\sharp}$. One sees immediately that $F_{Y'}$ and $F_{Y^{\sharp}}$ define the same valuation in the function field.

Note that $-\mathcal{F}'$ is clearly η-ample. It follows that $-F_{Y'}$ is f'-ample. Hene we have

$$Y^{\sharp} = \mathrm{Proj}(\oplus_{m \geq 0} f^{\sharp}_{*}\mathcal{O}(-mF_{Y^{\sharp}})) \cong \mathrm{Proj}(\oplus_{m \geq 0} f'_{*}\mathcal{O}(-mF_{Y'})) = Y'.$$

The proof for $Z^{\sharp} \cong Z'$ is similar. $\square$

3. Case studies

In this section we study divisorial contractions to a higher index point with non-minimal discrepancy case by case. For each case, we consider the extraction over a higher index point. We shall show that the Hypothesis $\flat$ holds by extracting over the highest index point. In fact, Hypothesis $\flat$ also holds for some other extractions over higher index point. In any event, Theorem 1.1 follows from our explicit case by case analysis.

Moreover, the output of 2-ray game and interchanging vectors of weighted blowups coincide. Hence we end up with a diagram for each case, where vertical maps are weighted blowups. Theorem 1.2 then follows by checking the diagram for each case.

3.1. Discrpancy $= 4/2$ over a $cD/2$ point

Let $f : Y \to X$ be a divisorial contraction to a $cD/2$ point $P \in X$ with discrepancy 2. By Kawakita's work (*cf.* [8]), it is known that there exists an embedding

$$(P \in X) \cong o \in \begin{pmatrix} \varphi_1 : x_1^2 + x_4x_5 + p(x_2, x_3, x_4) = 0 \\ \varphi_2 : x_2^2 + q(x_1, x_3, x_4) + x_5 = 0 \end{pmatrix} \subset \mathbb{C}^5/v$$

with $v = \frac{1}{2}(1, 1, 1, 0, 0)$ and f is the weighted blowup with weights $v_1 = (4l + 1, 4l, 2, 1, 8l + 1)$ or $(4l, 4l - 1, 2, 1, 8l - 1)$.

We treat this case in greater detail. The remaining cases can be treated similarly. Note that we can write $p(x_2, x_3, x_4) = x_4 p_1(x_2, x_3, x_4) + p_0(x_2, x_3)$. Therefore, replacing x_5 by $x_5 + p_1(x_2, x_3, x_4)$, we may assume that $\varphi_1 = x_1^2 + x_4x_5 + p(x_2, x_3)$.

Case 1. $v_1 = (4l + 1, 4l, 2, 1, 8l + 1)$.

Note that $wt_{v_1}(p(x_2, x_3)) \geq 8l + 1$, $wt_{v_1}(q(x_1, x_3, x_4)) \geq 8l$.

1. We consider higher index points in Y. We first look at Q_3. By computation of local charts, one sees that $Q_3 \in \mathcal{X}_1$ is a quotient singularity of type $\frac{1}{4}(1, 2, 1, 3, 3)$.

Claim 1. $Q_3 \notin Y$ and $x_3^{4l} \in \varphi_2$.

To see this, according to Kawakita's description, there is only one non-hidden non-Gorenstein singularity and also the hidden singularities has index at most 2. Hence $Q_3 \notin Y$. In other words, one must have either $x_3^{4l+1} \in \varphi_1$ or $x_3^{4l} \in \varphi_2$. Since φ_1 is a semi-invariant, it follows that $x_3^{4l} \in \varphi_2$.

We can check that $Q_5 \in \mathcal{X}_1$ is a quotient singularity of type $\frac{1}{2(8l+1)}(6l + 1, 10l + 1, 1, 12l + 2, 4l)$ with index $2(8l + 1)$. We set $w_2 = \frac{1}{2(8l+1)}(6l + 1, 10l + 1, 1, 12l + 2, 4l)$ so that $v_2 = \frac{1}{2}(2l + 1, 2l + 1, 1, 2, 4l)$.

Remark 3.1. The point $Q_4 \in Y$ is a "hidden" $cD/2$ point (see [7, page 68]). By the classification of Hayakawa (*cf.* [4]), any divisorial contraction $g : Z \to Y$ over Q_4 with discrepancy $\frac{1}{2}$ has the property that $g^*E = E_Z + \frac{t}{2}F$ with $t > 0$ and even. Therefore, $a(F, X) = \frac{t}{2}2 + \frac{1}{2} > 2$. Hence our theorem does not hold for arbitrary extraction over a point Q of index $p > 1$.

2. The weighted blowup $\mathcal{X}_2 \to \mathcal{X}_1$ with weights w_2 gives a divisorial contraction $g : Z \to Y$ of discrepancy $\frac{1}{2(8l+1)}$.

To see this, note that the local equation of Q_5 is given by

$$\begin{pmatrix} \overline{\varphi}_1 : \overline{x}_1^2 + \overline{x}_4 + \overline{p}(\overline{x}_2, \overline{x}_3) = 0 \\ \overline{\varphi}_2 : \overline{x}_2^2 + \overline{q}(\overline{x}_1, \overline{x}_3, \overline{x}_4) + \overline{x}_5 = 0 \end{pmatrix} \subset \mathbb{C}^5/w_2.$$

We have natural isomorphism between $(Q_5 \in Y) \subset \mathbb{C}^5/w_2$ and $(o \in \tilde{Y} := \mathbb{C}^3/\frac{1}{2(8l+1)}(6l+1, 10l+1, 1))$. The only extremal extraction over o with discrepancy $\frac{1}{2(8l+1)}$ is the Kawamata blowup $\tilde{Z} \to \tilde{Y}$, which is the weighted blowup with weights $\overline{w}_2 = \frac{1}{2(8l+1)}(6l+1, 10l+1, 1)$. Since $\overline{x}_3^{4l} \in \overline{\varphi}_2$, one sees that

$$\begin{cases} wt_{w_2}(\overline{x}_4) = wt_{\overline{w}_2}(\overline{x}_1^2) = wt_{\overline{w}_2}(\overline{x}_1^2 + \overline{p}(\overline{x}_2, \overline{x}_3)), \\ wt_{w_2}(\overline{x}_5) = wt_{\overline{w}_2}(\overline{x}_3^{4l}) = wt_{\overline{w}_2}(\overline{x}_2^2 + \overline{q}(\overline{x}_1, \overline{x}_3, \overline{x}_4)). \end{cases}$$

Therefore, the weighted blowup $Z \to Y$ with weights w_2 and the weighted blowup $\tilde{Z} \to \tilde{Y}$ with weights $\overline{w}_2$ are compatible (*cf.* Subsection 2.3). This verifies Hypothesis $\flat(1)$. The hypothesis $\flat(2)$ can be verified trivially.

3. We now checked the numerical conditions for 2-ray game. By Kawakita's Table (*cf.* [7, Table 1,2,3]), we have

$$E^3 = \frac{2}{2(8l+1)}, \quad F^3 = \frac{(2(8l+1))^2}{(6l+1)(10l+1)}.$$

Note that the exceptional divisor E can be realized as a $\mathbb{Z}_2$-quotient of complete intersection

$$\tilde{E} := (\varphi_{1,8l+2} = \varphi_{2,8l} = 0) \subset \mathbb{P}(4l+1, 4l, 2, 1, 8l+1),$$

where $\varphi_{i,k}$ denotes the homogeneous part of φ_i of v_1-weight $k/2$. Indeed, if we pick $S_X = \mathrm{div}(x_3)$ and set $S_Y = f_*^{-1} S_X = \mathrm{div}(\overline{x}_3)$, then S_Y is a normal Du Val element in $|-K_Y|$. We have that $E \cap S_Y$ is defined by $\mathbb{Z}_2$-quotient of the complete intersection

$$\begin{cases} x_3 = 0; \\ \varphi_{1,8l+2|x_3=0} = x_1^2 + x_4 x_5 = 0; \\ \varphi_{2,8l|x_3=0} = x_2^2 + q_{8l}(x_1, 0, x_4) = 0. \end{cases}$$

If $q_{8l}(x_1, 0, x_4)$ is not a perfect square, then $S_Y \cap E$ is clearly irreducible. If $q_{8l}(x_1, 0, x_4)$ is a perfect square, then this is reducible on $\tilde{E}$ but irreducible on E after the $\mathbb{Z}_2$-quotient. Therefore

$$T(f, g) = \frac{1}{2(8l+1)} \left(-8 + \frac{4l}{(6l+1)(10l+1)} \right) < 0$$

implies Hypothesis $\flat(3)$.

4. The weighted blowup $\mathcal{X}' \to \mathcal{X}_0$ with vector v_2 gives a divisorial contraction $f' : Y' \to X$ of discrepancy $\frac{1}{2}$. This follows from Theorem 2.7. In fact, we can check this directly as well by considering an embedding $(o \in \tilde{X}) \subset \mathbb{C}^4/\frac{1}{2}(1,1,1,0)$ defined by

$$(\tilde{\varphi} : x_1^2 + x_2^2 x_4 + q(x_1, x_3, x_4)x_4 + p(x_2, x_3, x_4) = 0).$$

Note that $x_3^{4l} x_4 \in \tilde{\varphi}$. Let $\overline{v}_2 = \frac{1}{2}(2l+1, 2l+1, 1, 2)$, then the weighted blowup $Y' \to X$ with weight v_2 is compatible with weighted blowup of $\tilde{Y} \to \tilde{X}$ with weight $\overline{v}_2$. It is easy to see that $wt_{v_1}(p) \geq 8l+1$ implies that $wt_{\overline{v}_2}(p) > 2l$ and $wt_{v_1}(q) \geq 8l$ implies that $wt_{\overline{v}_2}(qx_4) \geq 2l+1$. Therefore, the weighted blowup $\tilde{Y} \to \tilde{X}$ with weight $\overline{v}_2$ is indeed the weighted blowup given in Proposition 5.8 of [4], which is a divisorial contraction over a $cD/2$ with minimal discrepancy $\frac{1}{2}$. Hence so is $Y' \to X$.

5. One sees that $v_1 = \frac{6l+1}{2}e_1 + \frac{6l-1}{2}e_2 + \frac{3}{2}e_3 + v_2 + \frac{12l+2}{2}e_5$. Therefore, we consider the weighted blowup $\mathcal{X}_2' \to \mathcal{X}_1'$ with weights $w_2' = \frac{1}{2}(6l+1, 6l-1, 3, 2, 12l+2)$ over $Q_4' \in \mathcal{X}_1'$. Let Z' be the proper transform in $\mathcal{X}_2'$. Notice that $Z' \to Y'$ is a divisorial contraction over Q_4' with discrepancy $\frac{3}{2}$. This is indeed the map in Case 1 of Subsection 3.2 (after changing embedding into $\mathbb{C}^4/v$ as in Step 4.)

We summarize this case into the following diagram

$$\begin{array}{ccc}
Z & \overset{-\,-\rightarrow}{\longrightarrow} & Z' \\
{\scriptstyle \frac{1}{2(8l+1)}}\Big\downarrow {\scriptstyle wt=w_2} & & {\scriptstyle \frac{3}{2}}\Big\downarrow {\scriptstyle wt=w_2'} \\
Q_5 \in Y & & Y' \ni Q_4' \\
{\scriptstyle \frac{4}{2}}\Big\downarrow {\scriptstyle wt=w_1} & & {\scriptstyle \frac{1}{2}}\Big\downarrow {\scriptstyle wt=w_1'} \\
X & \overset{=}{\longrightarrow} & X
\end{array}$$

where

$$\begin{aligned}
w_1 &= v_1 = (4l+1, 4l, 2, 1, 8l+1),\\
w_1' &= v_2 = \frac{1}{2}(2l+1, 2l+1, 1, 2, 4l),\\
w_2 &= \frac{1}{2(8l+1)}(6l+1, 10l+1, 1, 12l+2, 4l),\\
w_2' &= \frac{1}{2}(6l+1, 6l-1, 3, 2, 12l+2).
\end{aligned}$$

Case 2. $v_1 = (4l, 4l-1, 2, 1, 8l-1)$.

We first look at Q_3, which is a quotient singularity of type $\frac{1}{4}(2,3,1,3,1)$ in $\mathcal{X}_1$.

Claim. $Q_3 \notin Y$ and $x_3^{4l} \in \varphi_1$.

To see this, according to Kawakita's description, there is only one non-hidden non-Gorenstein singularity and also the hidden singularities have indices at most 2. Hence $Q_3 \notin Y$. In other words, one must have either $x_3^{4l} \in \varphi_1$ or $x_3^{4l-1} \in \varphi_2$. Note that $x_3^{4l-1} \notin \varphi_2$ since φ_2 is a semi-invariant. We thus conclude that $x_3^{4l} \in \varphi_1$.

Next notice that $Q_5 \in \mathcal{X}_1$ is a quotient singularity of type $\frac{1}{2(8l-1)}(10l-1, 6l-1, 1, 4l, 12l-2)$. We set $w_2 = \frac{1}{2(8l-1)}(10l-1, 6l-1, 1, 4l, 12l-2)$ so that $v_2 = \frac{1}{2}(6l+1, 6l-1, 3, 2, 12l-2)$.

As before, the weighted blowup $\mathcal{X}_2 \to \mathcal{X}_1$ with vector v_2 gives a divisorial contraction $g : Z \to Y$ of discrepancy $\frac{1}{2(8l-1)}$, which is compatible with the Kawamata blowup. This can be seen by examining the local equation at Q_5 and the weight as in Case 1.

$$\begin{pmatrix} \overline{x}_1^2 + \overline{x}_4 + \overline{p}(\overline{x}_2, \overline{x}_3, \overline{x}_4) = 0 \\ \overline{x}_2^2 + \overline{q}(\overline{x}_1, \overline{x}_3, \overline{x}_4) + \overline{x}_5 = 0 \end{pmatrix} \subset \mathbb{C}^5/w_2.$$

We now checked the numerical conditions for 2-ray game. We have

$$E^3 = \frac{2}{2(8l-1)}, \quad F^3 = \frac{(2(8l-1))^2}{(6l-1)(10l-1)},$$

and

$$T(f, g) = \frac{1}{2(8l-1)}\left(-8 + \frac{2}{10l-1}\right) < 0.$$

We pick $S_X = \operatorname{div}(x_3)$ and set $S_Y = f_*^{-1} S_X = \operatorname{div}(\overline{x}_3)$, then S_Y is a normal Du Val element in $|-K_Y|$. One sees that $E \cap S_Y$ is defined by the $\mathbb{Z}_2$-quotient of the complete intersection

$$\begin{cases} x_3 = 0; \\ \varphi_{1,8l}{}_{|x_3=0} = x_1^2 + x_4 x_5 = 0; \\ \varphi_{2,8l-2}{}_{|x_3=0} = x_2^2 + q_{8l-2}(x_1, 0, x_4) = 0, \end{cases}$$

which is irreducible. Therefore, $T(f, g) < 0$ implies Hypothesis $\flat(3)$.

The weighted blowup $\mathcal{X}_1' \to \mathcal{X}_0$ with vector v_2 gives a divisorial contraction $f' : Y' \to X$ of discrepancy $\frac{3}{2}$. This can be seen to be a compatible embedding of Kawakita's description by eliminating x_5.

One sees that $v_1 = \frac{2l-1}{2} e_1 + \frac{2l-1}{2} e_2 + \frac{1}{2} e_3 + v_2 + \frac{4l}{2} e_5$. Therefore, we consider the weighted blowup $\mathcal{X}_2' \to \mathcal{X}_1'$ with weights $w_2' = \frac{1}{2}(2l-1, 2l-1, 1, 2, 4l)$ over $Q_4' \in \mathcal{X}_1'$. Let Z' be the proper transform in $\mathcal{X}_2'$, then one can easily check that $Z' \to Y'$ is a divisorial contraction over Q_4' with discrepancy $\frac{1}{2}$.

We summarize this case into the following diagram:

$$\begin{tikzcd} Z \arrow[r, dashed] \arrow[d, "\frac{1}{2(8l-1)}"', "wt=w_2"] & Z' \arrow[d, "\frac{1}{2}"', "wt=w'_2"] \\ Q_5 \in Y \arrow[d, "\frac{4}{2}"', "wt=w_1"] & Y' \ni Q'_4 \arrow[d, "\frac{3}{2}"', "wt=w'_1"] \\ X \arrow[r, "="] & X \end{tikzcd}$$

where

$$\begin{aligned} w_1 &= v_1 = (4l, 4l-1, 2, 1, 8l-1),\\ w_2 &= \frac{1}{2(8l-1)}(10l-1, 6l-1, 1, 4l, 12l-2),\\ w'_1 &= v_2 = \frac{1}{2}(6l+1, 6l-1, 3, 2, 12l-2),\\ w'_2 &= \frac{1}{2}(2l-1, 2l-1, 1, 2, 4l). \end{aligned}$$

3.2. Discrepancy $= a/2$ over a $cD/2$ point

Let $Y \to X$ be a divisorial contraction to a $cD/2$ point $P \in X$ with discrepancy $\frac{a}{2}$. This was classified by Kawakita into two cases (*cf.* [7, Theorem 1.2.ii]).

Case 1. In the case (1.2.ii.a), the local equation of $P \in X$ is given by

$$(\varphi : x_1^2 + x_2^2 x_4 + x_1 x_3 q(x_3^2, x_4) + \lambda x_2 x_3^{2\alpha-1} + p(x_3^2, x_4) = 0) \subset \mathbb{C}^4/v$$

with $v = \frac{1}{2}(1,1,1,0)$, f is the weighted blowup with weights $v_1 = \frac{1}{2}(r+2, r, a, 2)$, $r+1 = 2ad$ and both a, r are odd. Notice that $wt_{v_1}(\varphi) = r+1$ and as observed in [1], we have that $x_3^{4d} \in p(x_3^2, x_4)$.

There are two quotient singularities Q_1, Q_2 of index $r+2, r$ respectively.

Subcase 1. We take $g : Z \to Y$ the weighted blowup with weights $w_2 = \frac{1}{r+2}(4d, 4d, 1, r+2-4d)$ over Q_1. One sees that this is compatible with the Kawamata blowup over a point of type $\frac{1}{r+2}(4d, 1, r+2-4d)$. We have

$$E^3 = \frac{4(r+1)}{ar(r+2)}, \quad F^3 = \frac{(r+2)^2}{4d(r+2-4d)}.$$

In this case, we pick of $S_Y = \mathrm{div}(\overline{x}_3) \in |-K_Y|$, then $S_Y \cap E$ is irreducible. Now

$$T(f,g) = \frac{1}{r+2}\left(-\frac{a(r+1)}{r} + \frac{1}{r+2-4d}\right) < 0.$$

Therefore $-K_{Z/X}$ is nef and Hypothesis $\flat(3)$ holds.

We summarize this case into the following diagram.

$$\begin{tikzcd} Z \arrow[r, dashed] \arrow[d, "\frac{1}{r+2}"', "wt=w_2"] & Z' \arrow[d, "\frac{a-2}{2}"', "wt=w_2'"] \\ Q_1 \in Y \arrow[d, "\frac{a}{2}"', "wt=w_1"] & Y' \ni Q_4' \arrow[d, "\frac{2}{2}"', "wt=w_1'"] \\ X \arrow[r, "="] & X \end{tikzcd}$$

where

$$w_1 = v_1 = \frac{1}{2}(r+2, r, a, 2), \qquad w_1' = v_2 = (2d, 2d, 1, 1),$$
$$w_2 = \frac{1}{r+2}(4d, 4d, 1, r+2-4d), \ w_2' = \frac{1}{2}(r+2-4d, r-4d, a-2, 2).$$

Notice also that g' is a divisorial contraction of the same type over a $cD/2$ point with smaller discrepancy $\frac{a-2}{2}$, where $r+1-4d = 2d(a-2)$. The map f' is a contraction with discrepancy $2/2$ which is in Case 1 of Subsection 3.4.

Subcase 2. We take $g : Z \to Y$ the weighted blowup with weight $w_2 = \frac{1}{r}(4d, r-4d, 1, 4d)$ over Q_2. One sees that this is compatible with the Kawamata blowup over a point of type $\frac{1}{r}(4d, r-4d, 1)$.

One has

$$E^3 = \frac{4(r+1)}{ar(r+2)}, \quad F^3 = \frac{r^2}{4d(r-4d)}.$$

We pick S_Y as in Subcase 1, then we have

$$T(f,g) = \frac{1}{r}\left(-\frac{a(r+1)}{r+2} + \frac{1}{4d}\right) < 0.$$

Therefore $-K_{Z/X}$ is nef and hence Hypothesis $\flat(3)$ holds.

We summarize this case into the following diagram:

$$\begin{tikzcd} Z \arrow[r, dashed] \arrow[d, "\frac{1}{r}"', "wt=w_2"] & Z' \arrow[d, "\frac{2}{2}"', "wt=w_2'"] \\ Q_2 \in Y \arrow[d, "\frac{a}{2}"', "wt=w_1"] & Y' \ni Q_4' \arrow[d, "\frac{a-2}{2}"', "wt=w_1'"] \\ X \arrow[r, "="] & X \end{tikzcd}$$

where

$$w_1 = v_1 = \frac{1}{2}(r+2, r, a, 2), \ w_1' = v_2 = \frac{1}{2}(r+2-4d, r-4d, a-2, 2),$$
$$w_2 = \frac{1}{r}(4d, r-4d, 1, 4d), \quad w_2' = (2d, 2d, 1, 1).$$

Notice also that f' is a divisorial contraction of the same type over a $cD/2$ point with smaller discrepancy $\frac{a-2}{2}$, where $r+1-4d = 2d(a-2)$. The map g' is a contraction with discrepancy 1 which is in Case 1 of Subsection 3.4.

Case 2. In the case (1.2.ii.b), the local equation of $P \in X$ is given by

$$(P \in X) \cong o \in \begin{pmatrix} \varphi_1 : x_4^2 + x_2x_5 + p(x_1, x_3) = 0 \\ \varphi_2 : x_2x_3 + x_1^{2d+1} + q(x_1, x_3)x_1x_3 + x_5 = 0 \end{pmatrix} \subset \mathbb{C}^5/v,$$

with $v = \frac{1}{2}(1, 1, 0, 1, 1)$, f is a weighted blowup with weights $v_1 = \frac{1}{2}(a, r, 2, r+2, r+4)$, and $r+2 = (2d+1)a$. Notice that a is allowed to be any positive integer in this case.

There are quotient singularities Q_2, Q_5 of index $r, r+4$ respectively.

Subcase 1. We take $g : Z \to Y$ the weighted blowup with weights $w_2 = \frac{1}{r+4}(1, 4d+2, r-2d+3, 2d+1, 2d+1)$ over Q_5. One sees that this is compatible with the Kawamata blowup over a point of type $\frac{1}{r+4}(1, r-2d+3, 2d+1)$.

We check that

$$E^3 = \frac{4(r+2)}{ar(r+4)}, \quad F^3 = \frac{(r+4)^2}{(2d+1)(r-2d+3)}.$$

In this case, we pick of $S_X = \operatorname{div}(x_1)$ and $S_Y = f_*^{-1}S_X \in |-K_Y|$, then $S_Y \cap E = [l_1] + [l_2]$ cycle-theoretically, where

$$\begin{cases} l_1 := (x_1 = x_3 = \varphi_{1,2r+4} = 0); \\ l_2 := (x_1 = x_2 = \varphi_{1,2r+4} = 0). \end{cases}$$

We therefore pick $D = f_*^{-1}\operatorname{div}(x_3)$ instead. It is elementary to check that $D \cap E = (2d+1)[l_1]$. Hence $E \cap D$ is irreducible but non-reduced. We have $c_0 = 2, q_0 = r-2d+3$, hence $c_0 - aq_0 < 0$ and moreover

$$(2d+1)l_{1,Z} \cdot K_Z = T(f, g, D) = \frac{1}{r+4}\left(-\frac{2(r+2)}{r} + 1\right) < 0.$$

Therefore, Hypothesis $\flat(3)$ holds.

We summarize this case into the following diagram.

$$\begin{array}{ccc} Z & \stackrel{--\to}{\longrightarrow} & Z' \\ {\scriptstyle \frac{1}{r+4}}\Big\downarrow{\scriptstyle wt=w_2} & & {\scriptstyle \frac{a-1}{2}}\Big\downarrow{\scriptstyle wt=w_2'} \\ Q_5 \in Y & & Y' \ni Q_3' \\ {\scriptstyle \frac{a}{2}}\Big\downarrow{\scriptstyle wt=w_1} & & {\scriptstyle \frac{1}{2}}\Big\downarrow{\scriptstyle wt=w_1'} \\ X & \stackrel{=}{\longrightarrow} & X \end{array}$$

where

$$\begin{aligned}
w_1 &= v_1 = \frac{1}{2}(a, r, 2, r+2, r+4),\\
w_2 &= \frac{1}{r+4}(1, 4d+2, r-2d+3, 2d+1, 2d+1),\\
w_1' &= v_2 = \frac{1}{2}(1, 2d+1, 2, 2d+1, 2d+1),\\
w_2' &= \frac{1}{2}(a-1, r-2d-1, 2, r-2d+1, r-2d+3).
\end{aligned}$$

Notice also that g' is a divisorial contraction of the same type over a $cD/2$ point Q_3' with smaller discrepancy $\frac{a-1}{2}$, where $r-2d+1=(2d+1)(a-1)$. The map f' is a contraction with discrepancy $\frac{1}{2}$ which is compatible with the weighted blowup of [4, Proposition 5.8] by eliminating x_5.

Subcase 2. We take $g : Z \to Y$ the weighted blowup with weight $w_2 = \frac{1}{r}(1, r-2d-1, 2d+1, 2d+1, 4d+2)$ over Q_2. This is compatible with the Kawamata blowup over a point of type $\frac{1}{r}(1, r-2d-1, 2d+1)$.

We check that

$$E^3 = \frac{4(r+2)}{ar(r+4)}, \quad F^3 = \frac{r^2}{(2d+1)(r-2d-1)}.$$

We pick S_Y, D and l_1 as in the Subcase 1. We have $c_0 = 2$, $q_0 = 2d+1$ and hence $c_0 - aq_0 < 0$ and moreover

$$(2d+1)l_{1,Z} \cdot K_Z = T(f, g, D) = \frac{1}{r}\left(-\frac{2(r+2)}{r+4} + 1\right) < 0.$$

Therefore, Hypothesis $\flat(3)$ holds.

We summarize this case into the following diagram.

$$\begin{array}{ccc}
Z & \overset{-->}{\longrightarrow} & Z' \\
\frac{1}{r}\Big\downarrow wt=w_2 & & \frac{1}{2}\Big\downarrow wt=w_2' \\
Q_2 \in Y & & Y' \ni Q_3' \\
\frac{a}{2}\Big\downarrow wt=w_1 & & \frac{a-1}{2}\Big\downarrow wt=w_1' \\
X & \overset{=}{\longrightarrow} & X
\end{array}$$

where

$$\begin{aligned}
w_1 &= v_1 = \frac{1}{2}(a, r, 2, r+2, r+4),\\
w_2 &= \frac{1}{r}(1, r-2d-1, 2d+1, 2d+1, 4d+2),\\
w_1' &= v_2 = \frac{1}{2}(a-1, r-2d-1, 2, r-2d+1, r-2d+3),\\
w_2' &= \frac{1}{2}(1, 2d+1, 2, 2d+1, 2d+1).
\end{aligned}$$

3.3. Discrepancy $2/2$ to a $cE/2$ point

In this case, by [5, Theorem 1.2],

$$(P \in X) \cong 0 \in (\varphi : x_4^2 + x_1^3 + x_2^4 + x_3^8 + \text{others} = 0) \subset \mathbb{C}^4/v,$$

with $v = \frac{1}{2}(0,1,1,1)$ and $f : Y \to X$ is the weighted blowup with weights $v_1 = (3,2,1,4)$. There is a quotient singularity Q_1 of index 6.

Remark 3.2. There is another quotient singularity R_3 of index 2 in the fixed locus of $\mathbb{Z}_2$ action on U_3, which is not Q_3.

We can take $w_2 = \frac{1}{6}(2,5,1,1)$, then $v_2 = \frac{1}{2}(2,3,1,3)$. We pick $S_Y = f_*^{-1}\mathrm{div}(x_3) \in |-K_Y|$. One sees that $S_Y \cap E$ is $\mathbb{Z}_2$-quotient of $(x_4^2 + x_2^4 = 0) \subset \mathbb{P}(3,2,1,4)$, which is irreducible. We checked that

$$E^3 = \frac{1}{6}, \quad F^3 = \frac{36}{5}, \quad T(f,g) = \frac{-1}{10} < 0.$$

We summarize this case into the following diagram.

$$\begin{array}{ccc}
Z & \overset{-\,-\to}{\longrightarrow} & Z' \\
\tfrac{1}{6}\Big\downarrow wt=w_2 & & \tfrac{1}{3}\Big\downarrow wt=w_2' \\
Q_1 \in Y & & Y' \ni Q_2' \\
\tfrac{2}{2}\Big\downarrow wt=w_1 & & \tfrac{1}{2}\Big\downarrow wt=w_1' \\
X & \overset{=}{\longrightarrow} & X
\end{array}$$

where

$$\begin{aligned} w_1 = v_1 = (3,2,1,4),\ & w_1' = v_2 = \frac{1}{2}(2,3,1,3), \\ w_2 = \frac{1}{6}(2,5,1,1), \qquad & w_2' = \frac{1}{3}(5,4,1,6). \end{aligned}$$

Notice that $f' : Y' \to X$ is the weighted blowup with vector v_2 with discrepancy $\frac{1}{2}$ as in [5, Theorem 10.41]. The point $Q_2' \in Y'$ is a $cD/3$ point with local equation

$$\bar{x}_4^2 + \bar{x}_1^3 + \bar{x}_2^3 + \bar{x}_3^8\bar{x}_2 + \text{others} = 0 \subset \mathbb{C}^4/v,$$

with $v = \frac{1}{3}(2,1,1,0)$. Hence $Z' \to Y'$ is the weighted blowup with weights w_2' with discrepancy $\frac{1}{3}$ as in [5, Theorem 9.25].

3.4. Discrepancy $2/2$ to a $cD/2$ point

There are three cases to consider according Hayakawa's classification [5, Theorem 1.1]. Note that the case of Theorem 1.1.(iii) was treated in Subsection 3.2 already.

Case 1. The case of Theorem 1.1.(i) in [5].

In this case, we have

$$(P \in X) \cong o \in (x_1^2 + x_2^2 x_4 + s(x_3, x_4)x_2x_3x_4 + r(x_3)x_2 + p(x_3, x_4) = 0) \subset \mathbb{C}^4/v,$$

with $v = \frac{1}{2}(1, 1, 1, 0)$. The map $f : Y \to X$ is given by weighted blowup with weights $v_1 = (2l, 2l, 1, 1)$. Moreover, $wt_{v_1}(\varphi) = 2l$ and $x_3^{4l} \in p(x_3, x_4)$.

There is a singularity Q_2 of type $cA/4l$ with $aw = 2$. The local equation of $Q_2 \in U_2$ is given by

$$(\overline{\varphi} : \overline{x}_1^2 + \overline{x}_2\overline{x}_4 + \overline{x}_3^{4l} + \text{others} = 0) \subset \mathbb{C}^4/\frac{1}{4l}(0, 2l-1, 1, 2l+1).$$

Since $\overline{x}_3^{4l} \in \overline{\varphi}$, in terms of the terminology as in [3, Section 6], one has $\tau\text{-}wt(\overline{x}_3^{4l}) = 1$. This implies that there is only one divisorial contraction $Z \to Y$ with minimal discrepancy $\frac{1}{4l}$ which is a weighted blowup with weights $w_2 = \frac{1}{4l}(4l, 2l–1, 1, 2l+1)$.

We pick $S_Y = f_*^{-1}\mathrm{div}(x_3) \in |-K_Y|$ and check that $S_Y \cap E$ is $\mathbb{Z}_2$-quotient of $(x_1^2 + a_{0,4l}x_4^{4l} = 0) \subset \mathbb{P}(3, 2, 1, 4)$ for some $a_{0,4l}$. In any event, this is irreducible.

We checked that

$$E^3 = \frac{2}{4l}, \quad F^3 = \frac{(4l)^2}{(2l+1)(2l-1)}, \quad T(f, g) = \frac{1}{4l}\left(-2 + \frac{1}{2l+1}\right) < 0.$$

Hence Hypothesis $\flat(3)$ holds.

Hence we can summarize this case into the following diagram.

$$\begin{tikzcd}
Z \arrow[r, dashed] \arrow[d, "\frac{1}{4l}"', "wt=w_2"] & Z' \arrow[d, "\frac{1}{2}"', "wt=w_2'"] \\
Q_2 \in Y \arrow[d, "\frac{2}{2}"', "wt=w_1"] & Y' \ni Q_4' \arrow[d, "\frac{1}{2}"', "wt=w_1'"] \\
X \arrow[r, "="] & X
\end{tikzcd}$$

where

$$w_1 = v_1 = (2l, 2l, 1, 1), \qquad w_1' = v_2 = \frac{1}{2}(2l+1, 2l-1, 1, 2),$$
$$w_2 = \frac{1}{4l}(4l, 2l-1, 1, 2l+1), \quad w_2' = \frac{1}{2}(2l-1, 2l+1, 1, 2).$$

In this case, both f' and g' are divisorial contractions to a $cD/2$ point as in [4, Proposition 5.8].

Case 2. The case of [5, Theorem 1.1.(i')].

In this case, we have

$$(P \in X) \cong o \in (\varphi : x_1^2 + x_2x_3x_4 + x_2^4 + x_3^{2b} + x_4^c = 0) \subset \mathbb{C}^4/v,$$

with $b \geq 2$, $c \geq 4$ and $v = \frac{1}{2}(1, 1, 1, 0)$. The map $f : Y \to X$ is given by weighted blowup with weights $v_1 = (2, 2, 1, 1)$. Moreover, $wt_{v_1}(\varphi) = 4$.

There is a singularity Q_2 of type $cA/4$ with local equation of $Q_2 \in U_2$ is given by

$$(\overline{\varphi} : \overline{x}_1^2 + \overline{x}_3\overline{x}_4 + \overline{x}_2^4 + \overline{x}_3^{2b}\overline{x}_2^{2b-4} + \overline{x}_4^c\overline{x}_2^{c-4} = 0) \subset \mathbb{C}^4/\frac{1}{4}(0, 1, 1, 3).$$

Since $\overline{x}_2^4 \in \overline{\varphi}$, one has $\tau\text{-}wt = 1$. This implies that there is only one divisorial contraction $Z \to Y$ with minimal discrepancy $\frac{1}{4}$ which is the weighted blowup with weights $w_2 = \frac{1}{4}(4, 1, 1, 3)$.

We pick $S_Y = f_*^{-1}\mathrm{div}(x_3) \in |-K_Y|$ again and it is easy to see that $S_Y \cap E$ is $\mathbb{Z}_2$-quotient of $(x_1^2 + \delta_{4,c}x_4^c = 0) \subset \mathbb{P}(3, 2, 1, 4)$, where $\delta_{4,c}$ is the Kronecker's delta symbol. In any event, this is irreducible.

Then the invariant and diagram is exactly the same as the $l = 1$ in Case 1. For reference, we have

$$E^3 = \frac{2}{4}, \quad F^3 = \frac{4^2}{3l}, \quad T(f, g) = \frac{1}{4}\left(-2 + \frac{1}{3}\right) < 0.$$

We summarize the result into the following diagram.

$$\begin{array}{ccc}
Z & \overset{-- \to}{\longrightarrow} & Z' \\
\frac{1}{4}\Big\downarrow{\scriptstyle wt=w_2} & & \frac{1}{2}\Big\downarrow{\scriptstyle wt=w_2'} \\
Q_2 \in Y & & Y' \ni Q_4' \\
\frac{2}{2}\Big\downarrow{\scriptstyle wt=w_1} & & \frac{1}{2}\Big\downarrow{\scriptstyle wt=w_1'} \\
X & \overset{=}{\longrightarrow} & X
\end{array}$$

where

$$w_1 = (2, 2, 1, 1), \quad w_1' = \frac{1}{2}(3, 1, 1, 2),$$
$$w_2 = \frac{1}{4}(4, 1, 1, 3), \; w_2' = \frac{1}{2}(1, 3, 1, 2).$$

Note that f', g' are the weighted blowup of type v_1 as in [4, Section 4].

Case 3. The case of [5, Theorem 1.1.(ii)].

We have

$$(P \in X) \cong o \in \begin{pmatrix} \varphi_1 : x_1^2 + x_4x_5 + r(x_3)x_2 + p(x_3, x_4) = 0 \\ \varphi_2 : x_2^2 + s(x_3, x_4)x_1x_3 + q(x_3, x_4) - x_5 = 0 \end{pmatrix} \subset \mathbb{C}^5/v,$$

with $v = \frac{1}{2}(1, 1, 1, 0, 0)$. The map $f : Y \to X$ is the weighted blowup with vector $v_1 = (l + 1, l, 1, 1, 2l + 1)$. We can write $p(x_3, x_4) = p_0(x_3) + x_4 p_1(x_3, x_4)$. By replacing x_5 with $x_5 - p_1(x_3, x_4)$, we may and do assume that $p = p_0(x_3)$.

The point $Q_5 \in Y$ is a quotient singularity of type $\frac{1}{4l+2}(3l + 2, l, 1)$. We will need to take a weighted blowup $Z \to Y \ni Q_5$ which is compatible with the Kawamata blowup. However, according to the parity of l, we need to distinguish two subcases.

Subcase 3.1 l is odd.

In this situation, by the fact that either $x_3^{2l+2} \in \varphi_1$ or $x_2x_3^{l+2} \in \varphi_1$ (*cf.* [5, Theorem 1.1.ii.b,c]), one sees that the compatible weighted blowup is given by $w_2 = \frac{2l}{4l+2}(3l + 2, l, 1, 2l + 2, 2l)$.

We now pick $S_Y = \operatorname{div}(\overline{x}_3) \in |-K_Y|$ again and it is easy to see that $S_Y \cap E$ is $\mathbb{Z}_2$-quotient of $(\varphi_{1,2l+2} = \varphi_{2,2l} = x_3 = 0) \subset \mathbb{P}(l + 1, l, 1, 1, 2l + 1)$, which is irreducible.

We have

$$E^3 = \frac{4}{4l+2}, \quad F^3 = \frac{(4l+2)^2}{l(3l+2)},$$
$$T(f, g) = \frac{1}{4l+2}\left(-4 + \frac{2l}{l(3l+2)}\right) < 0.$$

Hence Hypothesis $\flat(3)$ holds.

We summarize the result into the following diagram.

$$\begin{array}{ccc} Z & \overset{--\to}{\longrightarrow} & Z' \\ {\scriptstyle \frac{1}{4l+2}}\big\downarrow {\scriptstyle wt=w_2} & & {\scriptstyle \frac{1}{2}}\big\downarrow {\scriptstyle wt=w_2'} \\ Q_5 \in Y & & Y' \ni Q_4' \\ {\scriptstyle \frac{2}{2}}\big\downarrow {\scriptstyle wt=w_1} & & {\scriptstyle \frac{1}{2}}\big\downarrow {\scriptstyle wt=w_1'} \\ X & \overset{=}{\longrightarrow} & X \end{array}$$

where

$$w_1 = (l + 1, l, 1, 1, 2l + 1), \qquad w_1' = \frac{1}{2}(l + 2, l, 1, 2, 2l),$$
$$w_2 = \frac{1}{4l+2}(3l + 2, l, 1, 2l + 2, 2l), \quad w_2' = \frac{1}{2}(l, l, 1, 2, 2l - 1).$$

Subcase 3.2 l is even.

In this situation, we need to use the fact that either $x_3^{2l} \in \varphi_2$ or $x_1 x_3^{l-1} \in \varphi_2$ (*cf*. [5, Theorem 1.1.ii.a]). Then the compatible weighted blowup is given by $w_2 = \frac{1}{4l+2}(l+1, 3l+1, 1, 2l+2, 2l)$.

We pick $S_Y = \operatorname{div}(\overline{x}_3) \in |-K_Y|$ again, then $S_Y \cap E$ is irreducible similarly. We have

$$E^3 = \frac{4}{4l+2}, \quad F^3 = \frac{(4l+2)^2}{(l+1)(3l+1)},$$
$$T(f,g) = \frac{1}{4l+2}\left(-4 + \frac{2l}{(l+1)(3l+1)}\right) < 0.$$

Hence Hypothesis $\flat(3)$ holds.

We summarize the result into the following diagram

$$\begin{array}{ccc}
Z & \xrightarrow{--\to} & Z' \\
{\scriptstyle \frac{1}{4l+2}}\big\downarrow {\scriptstyle wt=w_2} & & {\scriptstyle \frac{1}{2}}\big\downarrow {\scriptstyle wt=w_2'} \\
Q_5 \in Y & & Y' \ni Q_4' \\
{\scriptstyle \frac{2}{2}}\big\downarrow {\scriptstyle wt=w_1} & & {\scriptstyle \frac{1}{2}}\big\downarrow {\scriptstyle wt=w_1'} \\
X & \xrightarrow{=} & X
\end{array}$$

where

$$w_1 = (l+1, l, 1, 1, 2l+1), \qquad w_1' = \frac{1}{2}(l+1, l+1, 1, 2, 2l),$$
$$w_2 = \frac{1}{4l+2}(l+1, 3l+1, 1, 2l+2, 2l), \quad w_2' = \frac{1}{2}(l+1, l-1, 1, 2, 2l+2).$$

3.5. Discrepancy a/n to a cA/n point

This case is described in [7, Theorem 1.1.i]. We have

$$(P \in X) \cong o \in (\varphi : x_1x_2 + g(x_3^r, x_4) = 0) \subset \mathbb{C}^4/v,$$

where $v = \frac{1}{n}(1, -1, b, 0)$.

The map f is given by weighted blowup with weight $v_1 = \frac{1}{n}(r_1, r_2, a, n)$. We may write $r_1 + r_2 = dan$ for some $d > 0$ with the term $x_3^{dn} \in \varphi$. We also have that $s_1 := \frac{a-br_1}{n}$ is relatively prime to r_1 and $s_2 := \frac{a+br_2}{n}$ is relatively prime to r_2 (*cf*. [7, Lemma6.6]). We thus have the following:

$$\begin{cases} a = br_1 + ns_1, \\ 1 = q_1r_1 + s_1^*s_1, \\ a = -br_2 + ns_2, \\ 1 = q_2r_2 + s_2^*s_2, \end{cases}$$

for some $0 \le s_i^* < r_i$ and some q_i.

We set

$$\delta_1 := -nq_1 + bs_1^*, \quad \delta_2 := -nq_2 - bs_2^*.$$

One sees easily that

$$\begin{cases} \delta_1 r_1 + n = as_1^*, \\ \delta_2 r_2 + n = as_2^*. \end{cases}$$

Claim 1. $a > \delta_i \neq 0$ for $i = 1, 2$.

To see this, first notice that if $\delta_1 = 0$, then $s_1^* = tn, q_1 = tb$ for some integer t. It follows that $1 = ta$, which contradicts to $a > 1$. Hence $\delta_1 \neq 0$ and similarly $\delta_2 \neq 0$.

Note that $\delta_i r_i = as_i^* - n < as_i^* < ar_i$. Hence we have $\delta_i < a$ for $i = 1, 2$. This completes the proof of the Claim 1.

Moreover, we need the following:

Claim 2. $\delta_i > 0$ for some i.

If $\delta_i < 0$, then $n = -\delta_i r_i + as_i^* \geq r_i$. In fact, the equality holds only when $s_i^* = 0$, which implies in particular that $r_i = 1$. We can not have the equalities simultaneously for $i = 1, 2$ otherwise, $r_1 = r_2 = 1$ yields $2 = r_1 + r_2 = adn \geq 2n \geq 4$, which is absurd. This completes the proof of the Claim.

Remark 3.3. Suppose that both $\delta_1, \delta_2 > 0$ and $(a, r_1) = 1$, then we have $\delta_1 + \delta_2 = a$. To see this, note that $as_2^* = n + \delta_2 r_2 = n + \delta_2(adn - r_1)$. Therefore,

$$a(s_2^* - \delta_2 dn) = n + (-\delta_2) r_1.$$

By $(a, r_1) = 1$ and comparing it with $as_1^* = n + (\delta_1) r_1$, we have $\delta_1 = -\delta_2 + ta$ for some $t \in \mathbb{Z}$. Since $0 < \delta_1 + \delta_2 < 2a$, it follows that $\delta_1 + \delta_2 = a$.

Subcase 1. Suppose that $\delta_1 > 0$.

Notice that $r_1 = 1$ implies that $s_1^* = 1, q_1 = 1$ and hence $\delta_1 = -n$. Therefore, we must have $r_1 > 1$. Let $g : Z \to Y$ be Kawamata blowup over Q_1, which is a quotient singularity of type $\frac{1}{r_1}(r_1 - s_1^*, 1, s_1^*)$. We take $w_2 = \frac{1}{r_1}(r_1 - s_1^*, dr, 1, s_1^*)$ which is a compatible weighted blowup.

We pick $S_Y = f_*^{-1}\mathrm{div}(x_3) \in |-K_Y|$, then $S_Y \cap E = (x_1x_2 + a_{0,d}x_4^{da} = 0) \subset \mathbb{P}(v_1)$ for some $a_{0,d}$. If $a_{0,d} \neq 0$, then $S_Y \cap E$ is irreducible. If $a_{0,d} = 0$, then $S_Y \cap E = [l_1] + [l_2]$, where $l_i = (x_i = x_3 = 0) \subset \mathbb{P}(v_1)$.

We also pick $D = f_*^{-1}(x_4 = 0)$ then $E \cap D = (x_1x_2 + x_3^{dn} = 0)$ which is clearly irreducible. We have $c_0 = n, q_0 = s_1^*$, hence $c_0 - aq_0 = -\delta_1 r_1 < 0$ and moreover

$$E^3 = \frac{dn^2}{r_1 r_2}, \quad F^3 = \frac{(r_1)^2}{s_1^*(r_1 - s_1^*)},$$

$$T(f, g, D) = \frac{1}{r_1}\left(-\frac{adn}{r_2} + 1\right) < 0.$$

Hence $-K_{X/Z}$ is nef. Note also that $T(f,g) = \frac{1}{r_1}(-\frac{a^2d}{r_2} + \frac{1}{s_1^*}) < 0$. Therefore, Hypothesis $\flat(3)$ holds if $S_Y \cap E$ is irreducible. If $S_Y \cap E = [l_1] + [l_2]$, then one has $l_{i,Z} \cdot K_Z < 0$ for some i thanks to $l_{1,Z} \cdot K_Z + l_{2,Z} \cdot K_Z = T(f,g) < 0$. Therefore, Hypothesis $\flat(3)$ also holds even when $S_Y \cap E$ is not irreducible.

We summarize the result into the following diagram

$$\begin{array}{ccc} Z & \overset{\dashrightarrow}{\longrightarrow} & Z' \\ {\scriptstyle \frac{1}{r_1}}\Big\downarrow {\scriptstyle wt=w_2} & & {\scriptstyle \frac{\delta_1}{n}}\Big\downarrow {\scriptstyle wt=w_2'} \\ Q_1 \in Y & & Y' \ni Q_4' \\ {\scriptstyle \frac{a}{n}}\Big\downarrow {\scriptstyle wt=w_1} & & {\scriptstyle \frac{a-\delta_1}{n}}\Big\downarrow {\scriptstyle wt=w_1'} \\ X & \overset{=}{\longrightarrow} & X \end{array}$$

where

$$w_1 = \frac{1}{n}(r_1, r_2, a, n), \qquad w_1' = \frac{1}{n}(r_1 - s_1^*, r_2 - \delta_1 dn + s_1^*, a - \delta_1, n),$$
$$w_2 = \frac{1}{r_1}(r_1 - s_1^*, dn, 1, s_1^*), \ w_2' = \frac{1}{n}(s_1^*, \delta_1 dn - s_1^*, \delta_1, n).$$

Note that $0 < a' := a - \delta_1 < a$ and both f', g' are extremal contractions with discrepancies $< \frac{a}{r}$.

Subcase 2. Suppose that $\delta_2 > 0$.

Again, $r_2 > 1$ under this assumption. Let $g : Z \to Y$ be Kawamata blowup over Q_2, which is a quotient singularity of type $\frac{1}{r_2}(r_2 - s_2^*, 1, s_2^*)$. We take $w_2 = \frac{1}{r_2}(dr, r_2 - s_2^*, 1, s_2^*)$ which is a compatible weighted blowup.

We pick $D = f_*^{-1}\mathrm{div}(x_4)$ and $S_Y = f_*^{-1}\mathrm{div}(x_3)$ as in Subcase 1. We have that $D \cap E$ is irreducible and $c_0 = n$, $q_0 = s_2^*$, hence $c_0 - aq_0 = -\delta_2 r_2 < 0$ and moreover

$$E^3 = \frac{dr^2}{r_1 r_2}, \quad F^3 = \frac{(r_2)^2}{s_2^*(r_2 - s_2^*)},$$

$$T(f,g,D) = \frac{1}{r_2}\left(-\frac{adn}{r_1} + 1\right) < 0,$$

$$T(f,g) = \frac{1}{r_2}\left(-\frac{a^2d}{r_1} + \frac{1}{s_2^*}\right) < 0.$$

Hence by the same argument as in Subcase 1, Hypothesis $\flat(3)$ holds.

We summarize the result into the following diagram

$$
\begin{array}{ccc}
Z & \overset{-\,-\to}{\longrightarrow} & Z' \\
\tfrac{1}{r_1}\Big\downarrow wt=w_2 & & \tfrac{\delta_2}{n}\Big\downarrow wt=w_2' \\
Q_2 \in Y & & Y' \ni Q_4' \\
\tfrac{a}{n}\Big\downarrow wt=w_1 & & \tfrac{a-\delta_2}{n}\Big\downarrow wt=w_1' \\
X & \overset{=}{\longrightarrow} & X
\end{array}
$$

where

$$
\begin{aligned}
w_1 &= \frac{1}{n}(r_1, r_2, a, n), & w_1' &= \frac{1}{n}(r_1 + s_2^* - \delta_2 dn, r_2 - s_2^*, a - \delta_2, n),\\
w_2 &= \frac{1}{r_2}(dn, r_2 - s_2^*, 1, s_2^*), & w_2' &= \frac{1}{n}(\delta_2 dn - s_2^*, s_2^*, \delta_2, n).
\end{aligned}
$$

It is easy to see that if $r_1 \geq r_2$, then $\delta_1 > 0$. Hence extracting over Q_1 provides the desired factorization. Similar argument holds if $r_2 \geq r_1$. Therefore, one can conclude that Theorems holds by extracting over the point of highest index.

4. Further remarks

It is easy to see that our method also work for any divisorial contraction to a point of index 1 which is a weighted blowup. Let us take $f : Y \to X \ni P$ the weighted blowup with weight $(1, a, b)$ for example, where $a < b$ are relatively prime. One can write $ap = bq + 1$ for some $p, q > 0$ and $p < b, q < a$. Then by our method, we can take $g : Z \to Y$ the weighted blowup with weights $\frac{1}{b}(p, 1, b - p)$ over Q_3. After 2-ray game, we have that g' is the weighted blowup with weight $(1, q, p)$ over Q_1' and f' is the weighted blowup with weight $(1, a - q, b - p)$ over P. All the other known examples fit into our framework nicely.

We would like to raise the following:

Problem 4.1. Can every 3-fold divisorial contraction to a point be realized as a weighted blowup?

Assuming the affirmative answer, then by the method we provided in this article, it is reasonable to expect, as in Corollary 1.3, that for any 3-fold divisorial contraction $Y \to X$ to a singular point $P \in X$ of index $r = 1$ with discrepancy $a > 1$, there exists a sequence of birational maps

$$Y = X_n \dashrightarrow \ldots \dashrightarrow X_0 = X$$

such that each map $X_{i+1} \dashrightarrow X_i$ is one of the following:

(1) a divisorial contraction to a singular point of index $r_i \geq 1$ with discrepancy $\frac{1}{r_i}$ or its inverse;
(2) a flip or flop.

Together with the factorization result of [1], we have the following:

Conjecture 4.2. Let $Y \dashrightarrow X$ be a birational map which is flip, a divisorial contraction to a point, or a divisorial contraction to a curve. There exists a sequence of birational maps

$$Y = X_n \dashrightarrow \ldots \dashrightarrow X_0 = X$$

such that each map $X_{i+1} \dashrightarrow X_i$ is one of the following:

(1) a divisorial contraction to a point with minimal discrepancy or its inverse;
(2) a blowup of a lci curve;
(3) a flop.

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