Semigroups generated by elliptic operators in non-divergence form on $C_0(\Omega)$

WOLFGANG ARENDT AND REINER MICHAEL SCHÄTZLE

Abstract. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set satisfying the uniform exterior cone condition. Let $\mathcal{A}$ be a uniformly elliptic operator given by

$$\mathcal{A}u = \sum_{i,j=1}^{n} a_{ij}\partial_{ij}u + \sum_{j=1}^{n} b_j\partial_j u + cu$$

where

$$a_{ji} = a_{ij} \in C(\bar{\Omega}) \text{ and } b_j, c \in L^\infty(\Omega), c \leq 0 .$$

We show that the realization A_0 of $\mathcal{A}$ in

$$C_0(\Omega) := \{u \in C(\bar{\Omega}) : u_{|\partial\Omega} = 0\}$$

given by

$$D(A_0) := \{u \in C_0(\Omega) \cap W^{2,n}_{\text{loc}}(\Omega) : \mathcal{A}u \in C_0(\Omega)\}$$
$$A_0u := \mathcal{A}u$$

generates a bounded holomorphic C_0-semigroup on $C_0(\Omega)$. The result is in particular true if Ω is a Lipschitz domain. So far the best known result seems to be the case where Ω has C^2-boundary [12, Section 3.1.5]. We also study the elliptic problem

$$-\mathcal{A}u = f$$
$$u_{|\partial\Omega} = g .$$

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1. Introduction

The aim of this paper is to study elliptic and parabolic problems for operators in non-divergence form with continuous second order coefficients and to prove existence (and uniqueness) of solutions which are continuous up to the boundary of the

domain. Such parabolic equations are a classical subject in pde. The consideration of operators in non-divergence form occurs naturally in many situations, for example in the context of stochastic differential equations, where the corresponding parabolic equation is frequently called Kolmogorov's equation [8, page 4], or in the non-linear theory, where regularity results play an important role. Here we are interested in continuity up to the boundary. More precisely, we want to investigate when an elliptic operator generates an analytic semigroup on

$$C_0(\Omega) := \{u \in C(\bar{\Omega}) : u_{|\partial\Omega} = 0\}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded, open set. We refer to Lunardi's monograph [12] for a systematic theory of holomorphic semigroups generated by elliptic operators. Also in this monograph such semigroups are studied on spaces of continuous functions.

Throughout this paper Ω is a bounded open set in $\mathbb{R}^n$, $n \geq 2$, with boundary $\partial\Omega$. We consider the operator $\mathcal{A}$ given by

$$\mathcal{A}u := \sum_{i,j=1}^{n} a_{ij}\partial_{ij}u + \sum_{j=1}^{n} b_j\partial_j u + cu$$

with real-valued coefficients a_{ij}, b_j, c satisfying

$$
\begin{aligned}
&b_j \in L^\infty(\Omega) \,, \ j = 1, \ldots, n \,, \ c \in L^\infty(\Omega) \,, \ c \leq 0 \\
&a_{ij} \in C(\Omega) \cap L^\infty(\Omega) \,, \ a_{ij} = a_{ji} \,, \\
&\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \Lambda|\xi|^2 \qquad (x \in \Omega, \xi \in \mathbb{R}^n)
\end{aligned}
$$

where $\Lambda > 0$ is a fixed constant.

Only for part of the results we need that the a_{ij} are continuous up to the boundary (as is assumed in the Abstract). Of course, the assumption that $a_{ij} = a_{ji}$ is no restriction of generality (replacing a_{ij} by $\frac{a_{ij}+a_{ji}}{2}$ otherwise).

Our best results are obtained under the hypothesis that Ω satisfies the uniform exterior cone condition (and thus in particular if Ω has Lipschitz boundary). Then we show that for each $f \in L^n(\Omega)$, $g \in C(\partial\Omega)$ there exists a unique $u \in C(\bar{\Omega}) \cap W^{2,n}_{\text{loc}}(\Omega)$ such that

$$(E) \begin{cases} -\mathcal{A}u = f \\ u_{|\partial\Omega} = g \,. \end{cases}$$

(Corollary 3.4). This result is proved with the help of Alexandrov's maximum principle (which is responsible for the choice of $p = n$) and other standard results for elliptic second order differential operators (put together in the appendix). Our main concern is the parabolic problem

$$(P) \begin{cases} u_t = \mathcal{A}u \\ u(0, \cdot) = u_0 \\ u(t, x) = 0 \qquad x \in \partial\Omega \,, \ t > 0 \,. \end{cases}$$

with Dirichlet boundary conditions. Under the uniform exterior cone condition, we show that the realization A_0 of $\mathcal{A}$ in $C_0(\Omega)$ given by

$$D(A_0) := \{v \in C_0(\Omega) \cap W^{2,n}_{\text{loc}}(\Omega) : \mathcal{A}v \in C_0(\Omega)\}$$
$$A_0 v := \mathcal{A}v$$

generates a bounded, C_0-semigroup on $C_0(\Omega)$, which is holomorphic, if in addition $a_{ij} \in C(\bar{\Omega})$. This improves the known results, which are presented in the mono-graphe of Lunardi [12, Corollary 3.1.21] for Ω of class C^2 (and b_j, c uniformly continuous).

We would like to mention one particular reason why continuity at the boundary is important. If one considers semilinear equations the semilinear term $f(u)$ is usually treated by a fixed point argument. For this it is useful that the composition mapping $u \mapsto f(u)$ is locally Lipschitz continuous on the underlying function space X, where $f : \mathbb{R} \to \mathbb{R}$ defines the semilinear term. If $X = L^p$, then this is merely the case if f is already globally Lipschitz continuous, a condition which is too strong for many applications. However, on $C_0(\Omega)$ local Lipschitz continuity of f implies the same property of the composition operator. We refer to the mono-graph [7] of Cazenave-Haraux where the space $C_0(\Omega)$ is used for this purpose.

For applications, the improvement from C^2 to Lipschitz boundary is certainly the most important case of our results. But it is natural to ask for the minimal hy-potheses on coefficients and boundary. It is interesting that one may relax the condi-tion at the boundary if more regularity of the coefficients is assumed and vice versa. Three regularity properties at the boundary are considered in this paper: Lipschitz boundary (as the strongest concerning our results), the uniform exterior cone con-dition and Wiener regularity as the weakest property. Concerning the coefficients a_{ij} we assume global Lipschitz-continuity as the strongest, Dini continuity, Hölder continuity, continuity up to the boundary, and just continuity in the interior as the weakest possible assumption, formulated as our standing hypothesis above. The interplay among these properties seems interesting to us and we devote some par-ticular attention to it. For example, our weakest condition at the boundary, namely Wiener regularity, suffices for the generation of a holomorphic semigroup on $C_0(\Omega)$ and for well-posedness of (E) if the second order coefficients are globally Lipschitz continuous. On the other hand if merely $a_{ij} \in C(\bar{\Omega})$ then we need that Ω fulfills the exterior cone condition.

In order to show the existence of a semigroup, we have to study the elliptic problem (E) before (to verify the range condition in the Hille-Yosida theorem). At first we consider the Poisson equation with homogeneous boundary data $g = 0$ (Section 1). This is done merely on the basis of the standard elliptic theory, as presented in the monograph of Gilbarg-Trudinger [9], and leads in relatively ele-mentary way to two important cases (a_{ij} Lipschitz, Ω Wiener-regular in Theorem 1.1 and uniform exterior cone condition for Ω and a_{ij} merely bounded and contin-uous in the interior in Theorem 1.4).

In Section 2 we show that the Poisson problem is equivalent to the Dirichlet problem (*i.e.*, $f = 0$). This is of independent interest. But it also allows us to

use results of Krylov [10] to obtain a further case for the Generation Theorem 3.1 (namely a_{ij} Dini-continuous and Ω merely Wiener regular). The main results on the parabolic equations are obtained in Section 3.

We conclude the introduction by mentioning which results are known if an elliptic operator in divergence form with measurable bounded coefficients is considered (instead of our operator in non-divergence form). Such operator defines a mapping $\mathcal{A} : H^1_{\mathrm{loc}}(\Omega) \to \mathcal{D}(\Omega)'$. If Ω is Wiener regular, then for each $f \in L^p(\Omega)$, $p > \frac{n}{2}$, and each $g \in C(\partial\Omega)$ there exists a unique $u \in C(\bar{\Omega}) \cap H^1_{\mathrm{loc}}(\Omega)$ such that $\mathcal{A}u = f$, $u_{|\partial\Omega} = g$ (see [9, Theorem 8.31]). However, in general the solution is no longer in $W^{2,n}_{\mathrm{loc}}(\Omega)$ as in the case considered in the present paper. Concerning the generation theorem, one knows that the part A_0 of $\mathcal{A}$ in $C_0(\Omega)$, defined on the domain $D(A_0) := \{u \in H^1_{\mathrm{loc}}(\Omega) \cap C_0(\Omega) : \mathcal{A}u \in C_0(\Omega)\}$ generates holomorphic C_0-semigroup on $C_0(\Omega)$, see [1], again on the sole hypothesis that Ω is Wiener regular. The same remains true for operators in non-divergence form of the type $m \cdot \Delta$; *i.e.* in the very special case of isotropic coefficients ($a_{ij}(x) = m(x)$, $i, j = 1, \dots, n$). This remains even true in the case where $m(x)$ degenerates at the boundary (see [3]).

2. The Poisson problem

We consider the bounded open set $\Omega \subset \mathbb{R}^n$ and the elliptic operator $\mathcal{A}$ from the introduction. At first we consider the case where the second order coefficients are Lipschitz continuous. Then we merely need a very mild regularity condition on Ω. We say that Ω is *Wiener regular* if for each $g \in C(\partial\Omega)$ there exists a solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ of the Dirichlet problem

$$\Delta u = 0$$
$$u_{|\partial\Omega} = g \ .$$

The name comes from Wiener's characterization via capacity [9, (2.37)]. If Ω satisfies the exterior cone condition, then Ω is Wiener regular.

Theorem 2.1. *Assume that the second order coefficients a_{ij} are globally Lipschitz continuous. If Ω is Wiener regular, then for each $f \in L^n(\Omega)$, there exists a unique $u \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap C_0(\Omega)$ such that*

$$-\mathcal{A}u = f \ .$$

The point is that for Lipschitz continuous a_{ij} the operator $\mathcal{A}$ may be written in divergence form. This is due to the following lemma.

Lemma 2.2. *Let $h : \Omega \to \mathbb{R}$ be Lipschitz continuous. Then $h \in W^{1,\infty}(\Omega)$. In particular, $hu \in W^{1,2}(\Omega)$ for all $u \in W^{1,2}(\Omega)$ and $\partial_j(hu) = (\partial_j h)u + h\partial_j u$.*

Proof. One can extend h to a Lipschitz function on $\mathbb{R}^n$ (without increasing the Lipschitz constant, see [15]). Now the result follows from [8, 5.8 Theorem 4]. $\square$

Proof of Theorem 2.1. We assume that Ω is Dirichlet regular. Uniqueness follows from Aleksandrov's maximum principle Theorem A.1. In order to solve the problem we replace $\mathcal{A}$ by an operator in divergence form in the following way. Let $\tilde{b}_j := b_j - \sum_{i=1}^{n} \partial_i a_{ij}, j = 1, \ldots, n$. Then $\tilde{b}_j \in L^\infty(\Omega)$. Consider the elliptic operator $\mathcal{A}_d$ in divergence form given by

$$\mathcal{A}_d u = \sum_{i,j=1}^{n} \partial_i (a_{ij} \partial_j u) + \sum_{j=1}^{n} \tilde{b}_j \partial_j u + cu .$$

Let $f \in L^n(\Omega)$. By [9, Theorem 8.31] or [1, Corollary 4.6] there exists a unique $u \in C_0(\Omega) \cap W^{1,2}_{\text{loc}}(\Omega)$ such that $-\mathcal{A}_d u = f$ weakly, *i.e.*,

$$\sum_{i,j=1}^{d} \int_\Omega a_{ij} \partial_j u \partial_i v - \sum_{j=1}^{d} \int_\Omega \tilde{b}_j \partial_j uv - \int_\Omega cuv = \int_\Omega fv$$

for all $v \in \mathcal{D}(\Omega)$ (the space of all test functions). We mention in passing that $u \in W^{1,2}_0(\Omega)$ by [1, Lemma 4.2]. For our purposes, it is important that $u \in W^{2,2}_{\text{loc}}(\Omega)$ by Friedrich's theorem [9, Theorem 8.8]. Here we use again that the a_{ij} are uniformly Lipschitz continuous but do not need any further hypothesis on b_j and c. It follows from Lemma 2.2 that $a_{ij} \partial_j u \in W^{1,2}_{\text{loc}}(\Omega)$ and $\partial_i (a_{ij} \partial_j u) = (\partial_i a_{ij}) \partial_j u + a_{ij} \partial_{ij} u$. Thus $\mathcal{A}_d u = \mathcal{A}u$. Now it follows from [9, Lemma 9.16] that $u \in W^{2,n}_{\text{loc}}(\Omega)$. $\square$

Remark 2.3. Theorem 1.1 is formulated as it is needed for the main result in Section 4. Actually, more is true. Let $f \in L^p(\Omega)$ where $p > \frac{n}{2}$. Then there exists a unique $u \in W^{2,p}_{\text{loc}}(\Omega) \cap C_0(\Omega)$ such that $-\mathcal{A}u = f$. Uniqueness follows from [9, Lemma 9.16]: If $\mathcal{A}u = 0$ then $u \in W^{2,q}_{\text{loc}}(\Omega)$ for all $q \geq p$. So we can apply Alexandrov's maximum principle. In order to prove existence, we choose $f_k \in L^n(\Omega)$ converging to f in $L^p(\Omega)$ as $k \to \infty$. By the proof of Theorem 2.1 we find $u_k \in C_0(\Omega) \cap W^{2,n}_{\text{loc}}(\Omega)$ such that $-\mathcal{A}_d u_k = -\mathcal{A}u_k = f_k$. Choose $\Omega_m \subset\subset \Omega$ such that $\overline{\Omega_m} \subset \Omega_{m+1}$, $\bigcup_{m \in \mathbb{N}} \Omega_m = \Omega$. Then clearly $u_k \in W^{2,2}(\Omega_m) \subset W^{1,2}(\Omega_m)$. Thus [9, Theorem 8.16] shows that

$$\|u_k - u_\ell\|_{L^\infty(\Omega_m)} \leq \|u_k - u_\ell\|_{C(\partial \Omega_m)} + C\|f_k - f_\ell\|_{L^p(\Omega_m)}$$

for all $m \in \mathbb{N}$. Letting $m \to \infty$ it follows that

$$\|u_k - u_\ell\|_{L^\infty(\Omega)} \leq C\|f_k - f_\ell\|_{L^p(\Omega)} .$$

Thus $u := \lim_{k \to \infty} u_k$ converges in $C_0(\Omega)$. By the Calderon-Zygmund estimate Theorem A.2

$$\|u_k - u_\ell\|_{W^{2,p}(B_\varrho)} \leq c_\varrho(p)(\|u_k - u_\ell\|_{L^p(B_{2\varrho})} + \|f_k - f_\ell\|_{L^p(B_{2\varrho})})$$

whenever $\bar{B}_{2\varrho} \subset \Omega$. Thus $u \in W^{2,p}_{\text{loc}}(\Omega)$ and $\lim_{k \to \infty} u_k = u$ in $W^{2,p}_{\text{loc}}(\Omega)$. It follows that $-\mathcal{A}u = \lim_{k \to \infty} -\mathcal{A}u_k = f$.

Now we return to the general assumption $a_{ij} \in C(\Omega) \cap L^\infty(\Omega)$ and do no longer assume that the a_{ij} are Lipschitz continuous. The following result is stated in [9, Theorem 9.30] with a sketchy proof referring to the Perron method. We include a direct proof based on our Theorem 2.1, for the convenience of the reader.

Theorem 2.4. *Assume that Ω satisfies the uniform exterior cone condition. Then for all $f \in L^n(\Omega)$ there exists a unique $u \in C_0(\Omega) \cap W^{2,n}_{\text{loc}}(\Omega)$ such that $-\mathcal{A}u = f$.*

Proof. As for Theorem 2.1 we merely have to prove existence of a solution. We extend a_{ij} by δ_{ij} outside Ω and mollify to obtain $a_{ij}^k \in C^\infty(\mathbb{R}^n) \subset Lip(\Omega)$ uniformly elliptic and bounded, converging to a_{ij} locally uniformly in Ω. Let $\mathcal{A}_k$ be the elliptic operator with the second order coefficients a_{ij} of $\mathcal{A}$ replaced by a_{ij}^k. Let $f \in L^n(\Omega)$. By Theorem 2.1, for each $k \in \mathbb{N}$ there exists a unique $u_k \in W^{2,n}_{\text{loc}}(\Omega) \cap C_0(\Omega)$ such that $-\mathcal{A}_k u_k = f$. By Hölder regularity (Theorem A.3) there exists a constant c which does not depend on $k \in \mathbb{N}$ such that

$$\|u_k\|_{C^\alpha(\Omega)} \le c(\|f\|_{L^n(\Omega)} + \|u_k\|_{L^n(\Omega)}).$$

By Aleksandrov's maximum principle $\|u_k\|_{L^\infty(\Omega)} \le c_1\|f\|_{L^n(\Omega)}$ for all $k \in \mathbb{N}$ and some constant c_1. Thus $(u_k)_{k\in\mathbb{N}}$ is bounded in $C^\alpha(\Omega)$. By the Arzela-Ascoli theorem we may assume that u_k converges uniformly to $u \in C_0(\Omega)$ as $k \to \infty$ (passing to a subsequence if necessary). Let $\overline{B_{2\varrho}} \subset \Omega$ where $B_{2\varrho}$ is a ball of radius 2ϱ. Since the modulus of continuity of the a_{ij}^k is locally bounded in Ω, by the interior Calderon-Zygmund estimate Theorem A.2

$$\|u_k\|_{W^{2,n}(B_\varrho)} \le c_2(\|u_k\|_{L^n(B_{2\varrho})} + \|f\|_{L^n(B_{2\varrho})})$$

for all $k \in \mathbb{N}$ and some constant c_2, which depends on ϱ and the position of B_ϱ. It follows from reflexivity that $u \in W^{2,n}(B_\varrho)$ and $u_k \rightharpoonup u$ in $W^{2,n}(B_\varrho)$ as $k \to \infty$ after extraction of a subsequence. Since $-\mathcal{A}_k u_k = f$, it follows that $-\mathcal{A}u = f$. In fact, since $u_k \rightharpoonup u$ weakly in $W^{2,n}(B_\varrho)$, it follows that $\partial_{ij}u_k \rightharpoonup \partial_{ij}u$ in $L^n(B_\varrho)$ as $k \to \infty$. Thus $\sup_k \|\partial_{ij}u_k\|_{L^n(B_\varrho)} < \infty$. It follows that

$$(a_{ij}^k - a_{ij})\partial_{ij}u_k \to 0 \quad \text{in } L^n(B_\varrho) \text{ as } k \to \infty$$

and consequently $a_{ij}^k \partial_{ij}u_k \rightharpoonup a_{ij}\partial_{ij}u$ in $L^n(B_\varrho)$. $\qquad \square$

3. The Dirichlet problem

In this section we show the equivalence between well-posedness of the *Poisson problem*

$$(P) \qquad \begin{aligned} -\mathcal{A}u &= f \\ u_{|\partial\Omega} &= 0 \end{aligned}$$

and the *Dirichlet problem*

$$(D) \qquad\qquad \begin{aligned} \mathcal{A}u &= 0 \\ u_{|\partial\Omega} &= g \end{aligned}$$

where $f \in L^n(\Omega)$ and $g \in C(\partial\Omega)$ are given. We consider the operator $\mathcal{A}$ defined in the previous section and define its realization A in $L^n(\Omega)$ (recall that $\Omega \subset \mathbb{R}^n$) by

$$\begin{aligned} D(A) &:= \{u \in C_0(\Omega) \cap W^{2,n}_{\text{loc}}(\Omega) : \mathcal{A}u \in L^n(\Omega)\} \\ Au &:= \mathcal{A}u \, . \end{aligned}$$

Thus the Poisson problem can be formulated in a more precise way by asking under which conditions A is invertible (*i.e.* bijective from $D(A)$ to $L^n(\Omega)$ with bounded inverse $A^{-1} : L^n(\Omega) \to L^n(\Omega)$). By Aleksandrov's maximum principle, A is always injective. Note that for $\mu > 0$, the operator $A - \mu := A - \mu I$ has the same form as A (the order-0-coefficient c being just replaced by $c - \mu$).

Proposition 3.1. *The operator A is closed, injective and has closed range. If A is invertible, then the operator $\tilde{A}$ obtained by replacing the coefficient of order 0 by another coefficient $0 \geq \tilde{c} \in L^\infty(\Omega)$ is also invertible. In particular, if $A - \mu_0$ has dense range in $L^n(\Omega)$ for some $\mu_0 \geq 0$, then $A - \mu$ is invertible for all $\mu \geq 0$,*

Proof. By the Aleksandrov maximum principle (Theorem A.1) there exists a constant $c_1 > 0$ such that

$$\|u\|_\infty \leq c_1 \|\mu u - Au\|_{L^n(\Omega)} \tag{3.1}$$

for all $u \in D(A)$, $\mu \geq 0$. In order to show that A is closed, let $u_k \in D(A)$ such that $u_k \to u$ in $L^n(\Omega)$ and $Au_k \to f$ in $L^n(\Omega)$. It follows from (3.1) that $u \in C_0(\Omega)$ and $\lim_{k\to\infty} u_k = u$ in $C_0(\Omega)$. Let $B_{2\varrho}$ be a ball of radius 2ϱ such that $\overline{B_{2\varrho}} \subset \Omega$. By the Calderon-Zygmund estimate (Theorem A.2)

$$\|u_k - u_\ell\|_{W^{2,n}(B_\varrho)} \leq c_\varrho (\|u_k - u_\ell\|_{L^n(B_{2\varrho})} + \|A(u_k - u_\ell)\|_{L^n(B_{2\varrho})}) \, .$$

It follows that $(u_k)_{k\in\mathbb{N}}$ converges to u in $W^{2,n}(B_\varrho)$. Consequently $\mathcal{A}u_k \to \mathcal{A}u$ in $L^n(B_\varrho)$. Thus $\mathcal{A}u = f$ on B_ϱ. Since the ball is arbitrary, it follows that $u \in D(A)$ and $Au = f$. We have thus shown that A is closed. Now it follows from (3.1) that the range of A is closed. Next we show that invertibility is independent of the choice of the coefficient of order 0. Assume that A is invertible. Consider $A(t) = A + t(\tilde{c} - c)$, which corresponds to replacing c by $c + t(\tilde{c} - c)$, which is still ≤ 0 for $t \in [0, 1]$. Moreover, $D(A(t)) = D(A)$. We want to show that $A(1)$ is invertible. The set

$$M := \{t \in [0, 1] : A(t) \text{ is invertible}\}$$

contains $t = 0$ by hypothesis. Since $u \mapsto (\tilde{c} - c)u$ is a bounded operator on $L^n(\Omega)$, the set M is open. In fact, assume that $A(t_0)$ is invertible. If

$$|t - t_0| \|c - \tilde{c}\|_{L^\infty(\Omega)} \|A(t_0)^{-1}\|_{\mathcal{L}(L^n(\Omega))} < 1 \, ,$$

then $S := I - (t - t_0)(c - \tilde{c})A(t_0)^{-1}$ is invertible in $\mathcal{L}(L^n(\Omega))$ by the Neumann series. It follows that $A(t) = S A(t_0)$ is invertible.

It remains to show that M is closed. Let $t_k \in M$ such that $\lim_{k \to \infty} t_k = t_0$. Let $f \in L^n(\Omega)$. There exist $u_k \in D(A)$ such that $A(t_k)u_k = f$. By Aleksandrov's maximum principle we have

$$\|u_k\|_\infty \leq c_1 \|f\|_{L^n(\Omega)} \quad (k \in \mathbb{N}) \, .$$

Moreover,

$$\begin{aligned} A(t_k)(u_k - u_\ell) &= (A(t_\ell) - A((t_k))u_\ell \\ &= (t_\ell - t_k)(\tilde{c} - c)u_\ell \, . \end{aligned}$$

From Aleksandrov's maximum principle we conclude now that

$$\|u_k - u_\ell\|_\infty \leq c_1^2 |t_\ell - t_k| \, \|\tilde{c} - c\|_\infty \|f\|_{L^n(\Omega)} \, .$$

It follows that $\lim_{k \to \infty} u_k = u$ exists in $C_0(\Omega)$. Observe that

$$\begin{aligned} Au_k &= A(t_k)u_k - t_k(\tilde{c} - c)u_k \\ &= f - t_k(\tilde{c} - c)u_k \end{aligned}$$

converges to $f - t_0(\tilde{c} - c)u$ in $C_0(\Omega)$ and hence in $L^n(\Omega)$. Since A is closed, it follows that $u \in D(A)$ and $Au = f - t_0(\tilde{c} - c)u$; *i.e.*, $A(t_0)u = f$. We have shown that $A(t_0)$ is surjective. As noted before, $A(t_0)$ is injective. Hence $t_0 \in M$. □

We call a function u on Ω $\mathcal{A}$-*harmonic* if $u \in W^{2,p}_{loc}(\Omega)$ for some $p > 1$ and $\mathcal{A}u = 0$. By [9, Theorem 9.16] each $\mathcal{A}$-harmonic function u is in $\bigcap_{q>1} W^{2,q}_{loc}(\Omega)$. Given $g \in C(\partial\Omega)$, the *Dirichlet problem* consists in finding an $\mathcal{A}$-harmonic function $u \in C(\bar{\Omega})$ such that $u_{|\partial\Omega} = g$. We say that Ω is $\mathcal{A}$-*regular* if for each $g \in C(\partial\Omega)$ there is a solution of the Dirichlet problem. Uniqueness follows from the maximum principle [9, Theorem 9.6]

$$-\|u^-_{|\partial\Omega}\|_{C(\partial\Omega)} \leq u(x) \leq \|u^+_{|\partial\Omega}\|_{C(\partial\Omega)} \tag{3.2}$$

for all $x \in \bar{\Omega}$, which holds for each $\mathcal{A}$-harmonic function $u \in C(\bar{\Omega})$. In particular,

$$\|u\|_{C(\bar{\Omega})} \leq \|u\|_{C(\partial\Omega)} \, . \tag{3.3}$$

Theorem 3.2. *If the operator A is invertible then Ω is $\mathcal{A}$-regular. Conversely, if $a_{ij} \in C(\bar{\Omega})$ and Ω is $\mathcal{A}$-regular, then A is invertible.*

Thus for $a_{ij} \in C(\bar{\Omega})$ well-posedness of the problems (P) and (D) are equivalent.

For the proof we need the following lemma which we learnt from H. Amann. We are grateful to him for the permission to include it. Recall from the Introduction that we assume throughout that $a_{ij} = a_{ji}$.

Lemma 3.3. a) *There exist $\tilde{a}_{ij} \in C^b(\mathbb{R}^n)$ such that $\tilde{a}_{ij} = \tilde{a}_{ji}, \tilde{a}_{ij}(x) = a_{ij}(x)$ if $x \in \Omega$ and*

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \frac{\Lambda}{2}|\xi|^2$$

for all $\xi \in \mathbb{R}^n, x \in \Omega$.

b) *There exist $a_{ij}^k \in C^\infty(\bar{\Omega})$ such that $a_{ij}^k = a_{ji}^k, \sum_{i,j=1}^n a_{ij}^k(x)\xi_i\xi_j \geq \frac{\Lambda}{2}|\xi|^2$ and $\lim_{k\to\infty} a_{ij}^k(x) = a_{ij}(x)$ uniformly on $\bar{\Omega}$.*

Proof. a) Let $b_{ij} : \mathbb{R}^n \to \mathbb{R}$ be a bounded, continuous extension of a_{ij} to $\mathbb{R}^n$. Replacing b_{ij} by $\frac{b_{ij}+b_{ji}}{2}$, we may assume that $b_{ij} = b_{ji}$. Since the function φ : $\mathbb{R}^n \times S^1 \to \mathbb{R}$ given by $\varphi(x,\xi) := \sum_{i,j=1}^n b_{ij}(x)\xi_i\xi_j$ is continuous and $S^1 := \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ is compact, the set $\Omega_1 := \{x \in \mathbb{R}^n : \varphi(x,\xi) > \frac{\Lambda}{2}$ for all $\xi \in S^1\}$ is open and contains $\bar{\Omega}$. Let $0 \leq \varphi_1, \varphi_2 \in C(\mathbb{R}^n)$ such that $\varphi_1(x) + \varphi_2(x) = 1$ for all $x \in \mathbb{R}^n$ and $\varphi_2(x) = 1$ for $x \in \mathbb{R}^n \setminus \Omega_1, \varphi_1(x) = 1$ for $x \in \bar{\Omega}$. Then $\tilde{a}_{ij} := \varphi_1 b_{ij} + \frac{\Lambda}{2}\varphi_2\delta_{ij}$ fulfills the requirements.

b) Let $(\varrho_k)_{k\in\mathbb{N}}$ be a mollifier satisfying $\text{supp}\,\varrho_k \subset B_{1/k}(0)$. Then $a_{ij}^k = \tilde{a}_{ij} * \varrho_k \in C^\infty(\mathbb{R}^n)$ and $\lim_{k\to\infty} a_{ij}^k(x) = \tilde{a}_{ij}(x) = a_{ij}(x)$ uniformly in $x \in \bar{\Omega}$. If $\frac{1}{k} < \text{dist}(\partial\Omega_1, \Omega)$, then for $x \in \Omega, \xi \in \mathbb{R}^n$

$$\sum_{i,j=1}^n a_{ij}^k(x)\xi_i\xi_j = \int_{|y|<1/k} \sum_{i,j=1}^n \tilde{a}_{ij}(x-y)\xi_i\xi_j\varrho_k(y)\,dy$$

$$\geq \frac{\Lambda}{2} \int_{|y|<1/k} \varrho_k(y)\,dy = \frac{\Lambda}{2}\,. \qquad \square$$

Proof of Theorem 3.2. a) Assume that A is invertible.

First step. Let $g \in C(\partial\Omega)$ be of the form $g = G_{|\partial\Omega}$ where $G \in C^2(\bar{\Omega})$. Then $\mathcal{A}G \in L^n(\Omega)$. Let $v = A^{-1}(\mathcal{A}G)$, then $u := G - v$ solves the Dirichlet problem for g.

Second step. Let $g \in C(\partial\Omega)$ be arbitrary. Extending g continuously and mollifying we find $g_k \in C(\partial\Omega)$ of the kind considered in the first step such that $g = \lim_{k\to\infty} g_k$ in $C(\partial\Omega)$. Let $u_k \in C(\bar\Omega)$ be $\mathcal{A}$-harmonic satisfying $u_k|_{\partial\Omega} = g_k$. By (3.3) $u := \lim_{k\to\infty} u_k$ exists in $C(\bar\Omega)$. In particular, $u|_{\partial\Omega} = g$. Let $\overline{B_{2\varrho}} \subset \Omega$. Then by the Calderon-Zygmund estimate Theorem A.2

$$\|u_k\|_{W^{2,p}(B_\varrho)} \le c_\varrho \|u_k\|_{L^p(B_{2\varrho})} \le c_\varrho c \|u_k\|_{C(\bar\Omega)}$$

(remember that $\mathcal{A}u_k = 0$). Thus $(u_k)_{k\in\mathbb{N}}$ is bounded in $W^{2,p}(B_\varrho)$. Passing to a subsequence, we can assume that $u_k \rightharpoonup u$ in $W^{2,p}(B_\varrho)$. This implies that $\mathcal{A}u = 0$ in B_ϱ. Since the ball is arbitrary, it follows that u is $\mathcal{A}$-harmonic. Thus u is a solution of the Dirichlet problem (D).

b) Conversely, assume that Ω is $\mathcal{A}$-regular. Let $f \in L^n(\Omega)$. We want to find $u \in D(A)$ such that $Au = f$. Let B be a ball containig $\bar\Omega$ and extend f by 0 to B. Then by Theorem 2.4 we find $v \in C_0(B) \cap W^{2,n}_{\mathrm{loc}}(B)$ such that $\tilde{\mathcal{A}}v = f$. Here $\tilde{\mathcal{A}}$ is an extension of $\mathcal{A}$ to the ball B according to Lemma 3.3a). Let $g = v_{|\partial\Omega}$. Then by our assumption there exists an $\mathcal{A}$-harmonic function $w \in C(\bar\Omega)$ such that $w_{|\partial\Omega} = g$. Let $u = v - w$. Then $u \in C_0(\Omega) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ and $\mathcal{A}u = \mathcal{A}v = f$; i.e. $u \in D(A)$ and $Au = f$. We have shown that A is surjective, which implies invertibility by Proposition 3.1. □

Corollary 3.4. *Assume that one of the following two conditions is satisfied:*

a) Ω *is Wiener regular and the coefficients* a_{ij} *are globally Lipschitz continuous, or*

b) *the* a_{ij} *are in* $C(\bar\Omega)$ *and* Ω *satisfies the exterior cone condition.*

Then Ω *is* $\mathcal{A}$-regular. *More generally, for all* $f \in L^n(\Omega)$, $g \in C(\partial\Omega)$ *there exists a unique* $u \in C(\bar\Omega) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ *satisfying*

$$-\mathcal{A}u = f$$
$$u_{|\partial\Omega} = g \,.$$

Proof. Since A is closed by Proposition 3.1 it follows from Theorem 2.1 (in the case a)) and from Theorem 2.4 (in the case b)) that A is invertible. Thus Ω is $\mathcal{A}$ regular by Theorem 3.2. Let $f \in L^n(\Omega)$, $g \in C(\partial\Omega)$. Since Ω is $\mathcal{A}$-regular, there exists an $\mathcal{A}$-harmonic function $u_1 \in C(\bar\Omega)$ such that $u_{1_{|\partial\Omega}} = g$. Since A is invertible, there exists a function $u_0 \in C_0(\Omega) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ such that $-\mathcal{A}u_0 = f$. Let $u := u_0 + u_1$. Then $u \in C(\bar\Omega) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$, $u_{|\partial\Omega} = g$ and $-\mathcal{A}u = f$. Uniqueness follows from Theorem A.1. □

For the Laplacian $\mathcal{A} = \Delta$, Δ-regularity is Wiener regularity as defined in the beginning of Section 1. It is a most interesting question how $\mathcal{A}$-regularity and Δ-regularity are related. In general it is not true that $\mathcal{A}$-regularity implies Wiener

regularity. In fact, K. Miller [14] gives an example of an elliptic operator $\mathcal{A}$ with $a_{ij} \in C(\bar{\Omega})$ and $b_j = c = 0$ such that the pointed unit disc $\{x \in \mathbb{R}^2 : 0 < |x| < 1\}$ is $\mathcal{A}$-regular even though it is not Δ-regular. The other implication seems to be open (it is known to be false for non-continuous measurable coefficients, see [13]). The fact that the uniform exterior cone property (which is much stronger than Δ-regularity) implies $\mathcal{A}$-regularity (Corollary 3.4) had been proved before by Krylov [10, Theorem 5] with the help of partially probabilistic methods. If Ω is merely Δ-regular, then it seems not to be known whether Ω is $\mathcal{A}$-regular. Known results concerning this question are based on further restrictive conditions on the coefficients a_{ij}. By Theorem 2.1 in connection with Theorem 2.2 we obtain a positive answer for globally Lipschitz continuous a_{ij}. The best result seems to be [10, Theorem 4] which goes in both directions: If the a_{ij} are Dini-continuous (in particular, if they are Hölder-continuous), then Ω is Δ-regular if and only if Ω is $\mathcal{A}$-regular.

Things are different in the isotropic case (*i.e.* if a_{ij} is independent of $i, j \in 1, \ldots, n$): In that case, for merely measurable a_{ij} the operator A is invertible if and only if Ω is Wiener regular, see [3].

4. Generation results

An operator B on a complex Banach space X is said to *generate a bounded holomorphic semigroup* if $(\lambda - B)$ is invertible for $\operatorname{Re} \lambda > 0$ and

$$\sup_{\operatorname{Re} \lambda > 0} \|\lambda (\lambda - B)^{-1}\| < \infty .$$

Then there exist $\theta \in (0, \pi/2)$ and a holomorphic bounded function $T : \Sigma_\theta \to \mathcal{L}(X)$ satisfying $T(z_1 + z_2) = T(z_1)T(z_2)$ such that

$$\lim_{n \to \infty} e^{tB_n} = T(t) \text{ in } \mathcal{L}(X) \tag{4.1}$$

for all $t > 0$, where $B_n = nB(n - B)^{-1} \in \mathcal{L}(X)$. Here Σ_θ is the sector $\Sigma_\theta := \{re^{i\alpha} : r > 0, |\alpha| < \theta\}$.

If B is an operator on a real Banach space X we say that B *generates a bounded holomorphic semigroup* if its linear extension $B_\mathbb{C}$ to the complexification $X_\mathbb{C}$ of X generates a bounded holomorphic semigroup $T_\mathbb{C}$ on $X_\mathbb{C}$. In that case $T_\mathbb{C}(t)X \subseteq X$ (see [12, Corollary 2.1.3]); in particular $T(t) := T_\mathbb{C}(t)_{|X} \in \mathcal{L}(X)$. We call $T = (T(t))_{t>0}$ the semigroup generated by B. It satisfies $\lim_{t \downarrow 0} T(t)x = x$ for all $x \in X$ (*i.e.*, it is a C_0-semigroup) if and only if $\overline{D(B)} = X$. We refer to [12, Chapter 2] and [2, Sec. 3.7] for these facts and further information.

In this section we consider the parts A_c and A_0 of $\mathcal{A}$ in $C(\bar{\Omega})$ and $C_0(\Omega)$ as follows:

$$D(A_c) := \{u \in C_0(\Omega) \cap W^{2,n}_{\text{loc}}(\Omega) : \mathcal{A}u \in C(\bar{\Omega})\}$$
$$A_c u := \mathcal{A}u \quad \text{and}$$
$$D(A_0) := \{u \in C_0(\Omega) \cap W^{2,n}_{\text{loc}}(\Omega) : \mathcal{A}u \in C_0(\Omega)\}$$
$$A_0 u := \mathcal{A}u .$$

Thus A_c is the part of A in $C(\bar{\Omega})$ and A_0 the part of A_c in $C_0(\Omega)$. Note that $D(A_0) \subseteq D(A_c) \subseteq \bigcap_{q>1} W^{2,q}_{\text{loc}}$ by [9, Lemma 9.16]. The main result of this section is the following.

Theorem 4.1. *Assume that the a_{ij} are in $C(\bar{\Omega})$ and that Ω is $\mathcal{A}$-regular. Then A_c generates a bounded holomorphic semigroup T on $C(\bar{\Omega})$. The operator A_0 generates a bounded holomorphic C_0-semigroup T_0 on $C_0(\Omega)$. Moreover, $T(t)C_0(\Omega) \subseteq C_0(\Omega)$ and*

$$T_0(t) = T(t)_{|C_0(\Omega)} \ .$$

Recall that Ω is $\mathcal{A}$-regular if one of the following conditions is satisfied:

(a) Ω satisfies the uniform exterior cone condition or
(b) Ω is Wiener regular and the coefficients a_{ij} are Dini-continuous.
    In particular, Ω is $\mathcal{A}$-regular if
(a') Ω is a Lipschitz-domain or
(b') Ω is Wiener-regular and the a_{ij} are Hölder continuous.

In the following complex maximum principle (Proposition 4.3) we extend $\mathcal{A}$ to the complex space $W^{2,p}_{\text{loc}}(\Omega)$ without changing the notation. We first prove a lemma.

Lemma 4.2. *Let $B \subseteq \Omega$ be a ball of center x_0 and let $u \in W^{2,p}(B)$, $p > n$, be a complex-valued function such that $\mathcal{A}u \in C(B)$. If $|u(x_0)| \geq |u(x)|$ for all $x \in B$, then*

$$\text{Re} \left[\overline{u(x_0)}(\mathcal{A}u)(x_0) \right] \leq 0 \ .$$

Proof. We may assume that $x_0 = 0$. If the claim is wrong, then there exist $\varepsilon > 0$ and a ball $B_\varrho \subset B$ such that $\text{Re} \left[\overline{u(x)}(\mathcal{A}u)(x) \right] \geq \varepsilon$ on B_ϱ.

Since $\partial_j |u|^2 = (\partial_j u)\bar{u} + u\overline{\partial_j u} = 2\text{Re} \left[\partial_j u\bar{u} \right]$, and $\partial_{ij}(u\bar{u}) = (\partial_{ij}u)\bar{u} + \partial_i u\overline{\partial_j u} + \partial_j u\overline{\partial_i u} + u\overline{\partial_{ij}u}$, and since by ellipticity

$$\text{Re} \sum_{i,j} a_{ij}\partial_i u\overline{\partial_j u} \geq 0 \ , \quad \text{Re} \sum_{i,j} a_{ij}\partial_j u\overline{\partial_i u} \geq 0 \ ,$$

it follows that

$$\mathcal{A}|u|^2 \geq \text{Re} \sum_{i,j} a_{ij}(\partial_{ij}u)\bar{u} + \text{Re} \sum_{i,j} a_{ij}u\overline{\partial_{ij}u}$$
$$+ \sum_j b_j 2\text{Re} \left[\partial_j u\bar{u} \right] + cu\bar{u}$$
$$\geq 2\text{Re} \ (\mathcal{A}u\bar{u}) \geq 2\varepsilon \ \text{on} \ B_\varrho \ .$$

Let $\psi(x) = |u|^2 - \tau|x|^2$, $\tau > 0$. Choosing $\tau > 0$ small enough, we have $\mathcal{A}|\psi| \geq \varepsilon$ on B_ϱ.

Since $\psi \in W^{2,p}(B_\varrho) \cap C(\overline{B_\varrho})$, and since $-\mathcal{A}|\psi| \leq -\varepsilon$ on B_ϱ, by Aleksandrov's maximum principle [9, Theorem 9.1], see Theorem A.1, it follows that

$$|u(0)|^2 = |\psi(0)| \leq \sup_{\partial B_\varrho(0)} \psi$$

$$= \sup_{\partial B_\varrho(0)} |u|^2 - \tau \varrho^2$$

$$\leq |u(0)|^2 - \tau \varrho^2 < |u(0)|^2 \,,$$

a contradiction. For the last inequality we used the hypothesis on $x_0 = 0$. $\qquad \square$

Proposition 4.3 (complex maximum principle). *Let* $u \in C(\bar{\Omega}) \cap W^{2,n}_{\text{loc}}(\Omega)$ *such that* $\lambda u - \mathcal{A}u = 0$ *where* $\text{Re }\lambda > 0$. *If there exists* $x_0 \in \Omega$ *such that* $|u(x)| \leq |u(x_0)|$ *for all* $x \in \Omega$, *then* $u \equiv 0$. *Consequently,*

$$\max_{\bar{\Omega}} |u(x)| = \max_{\partial \Omega} |u(x)| \,.$$

Proof. If $|u(x)| \leq |u(x_0)|$ for all $x \in \Omega$, then by Lemma 4.2, $\text{Re}\left[\overline{u(x_0)}(\mathcal{A}u)(x_0)\right] \leq 0$. Since $\lambda u = \mathcal{A}u$, it follows that

$$\text{Re }\lambda |u(x_0)|^2 = \text{Re}\left[\overline{u(x_0)}(\mathcal{A}u)(x_0)\right] \leq 0 \,.$$

Hence $u(x_0) = 0$. $\qquad \square$

Next, recall that an operator B on a real Banach space X is called *m-dissipative* if $\lambda - B$ is invertible and

$$\lambda \|(\lambda - B)^{-1}\| \leq 1 \ \text{ for all } \ \lambda > 0 \,.$$

Now we show that the operator A_c is *m*-dissipative and that the resolvent is positive (*i.e.*, maps non-negative functions to non-negative functions).

Proposition 4.4. *Assume that A is invertible. Then A_c is m-dissipative and* $(\lambda - A_c)^{-1} \geq 0$ *for* $\lambda > 0$.

Proof. Let $\lambda > 0$. Since by Theorem 3.2 the operator $(\lambda - A)$ is bijective, also $(\lambda - A_c)$ is bijective.

a) We show that $(\lambda - A_c)^{-1} \geq 0$. Let $f \in C(\bar{\Omega}), f \leq 0, u := (\lambda - A_c)^{-1}f$. Assume that $u^+ \neq 0$. Since $u \in C_0(\Omega)$, there exists $x_0 \in \Omega$ such that $u(x_0) = \max_{\Omega} u > 0$. Then by Lemma 4.2, $\mathcal{A}u(x_0) \leq 0$. Since $\lambda u - \mathcal{A}u = f$, it follows that $\lambda u(x_0) \leq f(x_0) \leq 0$ a contradiction.

b) Let $f \in C(\bar{\Omega}), u = (\lambda - A_c)^{-1}f$. We show that $\|\lambda u\|_{C(\bar{\Omega})} \leq \|f\|_{C(\bar{\Omega})}$. Assume first that $f \geq 0, f \neq 0$. Then $u \geq 0$ by a) and $u \neq 0$. Let $x_0 \in \Omega$ such that $u(x_0) = \|u\|_{C(\bar{\Omega})}$. Then $(A_c u)(x_0) \leq 0$ by Lemma 4.2. Hence $\lambda u(x_0) \leq \lambda u(x_0) - (A_c u)(x_0) = f(x_0) \leq \|f\|_{C(\bar{\Omega})}$.

If $f \in C(\bar{\Omega})$ is arbitrary, then by a) $|(\lambda - A_c)^{-1}f| \leq (\lambda - A_c)^{-1}|f|$ and so $\|\lambda(\lambda - A_c)^{-1}f\|_{C(\bar{\Omega})} \leq \|f\|_{C(\bar{\Omega})}$. $\qquad \square$

Now we consider the complex extension of A_c (still denoted by A_c) to the space of all complex-valued functions on $\bar\Omega$ which we still denote by $C(\bar\Omega)$. Our aim is to prove that for $\mathrm{Re}\,\lambda > 0$ the operator $(\lambda - A_c)^{-1}$ is invertible and

$$\|(\lambda - A_c)^{-1}\| \le \frac{M}{|\lambda|}\,,$$

where M is a constant. For that, we extend the coefficents a_{ij} to uniformly continuous bounded real-valued functions on $\mathbb{R}^n$ satisfying the strict ellipticity condition

$$\mathrm{Re}\ \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\bar\xi_j \ge \frac{\Lambda}{2}|\xi|^2$$

($\xi \in \mathbb{R}^n$, $x \in \mathbb{R}^n$), keeping the same notation, see Lemma 3.3a). We extend b_j, c to bounded measurable functions on $\mathbb{R}^n$ such that $c \le 0$ (keeping the same notation). Now we define the operator B_∞ on $L^\infty(\mathbb{R}^n)$ by

$$D(B_\infty) := \{u \in \bigcap_{p>1} W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n) : u, \mathcal{B}u \in L^\infty(\mathbb{R}^n)\}$$

where

$$B_\infty u := \mathcal{B}u\,,$$

$$\mathcal{B}u := \sum_{i,j=1}^{d} a_{ij}\partial_{ij}u + \sum_{j=1}^{d} b_j\partial_j u + cu \text{ for } u \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)\,.$$

We will show that the operator B_∞ is sectorial (Theorem 4.6 below). This is proved in [12, Theorem 3.1.7] under the assumption that the coefficients b_j, c are uniformly continuous. We give a perturbation argument to deduce the general case from the case $b_j = c = 0$. The following lemma shows in particular that the domain of B_∞ is independent of b_j and c.

Lemma 4.5. *One has $D(B_\infty) \subset W^{1,\infty}(\mathbb{R}^n)$. Moreover, for each $\varepsilon > 0$ there exists $c_\varepsilon \ge 0$ such that*

$$\|u\|_{W^{1,\infty}(\mathbb{R}^n)} \le \varepsilon\|B_\infty u\|_{L^\infty(\mathbb{R}^n)} + c_\varepsilon\|u\|_{L^\infty(\mathbb{R}^n)}$$

for all $u \in D(B_\infty)$.

Proof. Consider an arbitrary ball B_1 in $\mathbb{R}^n$ of radius 1 and the corresponding ball B_2 of radius 2. Let $p > n$. Since the injection of $W^{2,p}(B_1)$ into $C^1(\bar B_1)$ is compact, for each $\varepsilon > 0$ there exists $c'_\varepsilon > 0$ such that

$$\|u\|_{C^1(\bar B_1)} \le \varepsilon\|u\|_{W^{2,p}(B_1)} + c'_\varepsilon\|u\|_{L^\infty(B_1)}\,.$$

By the Calderon-Zygmund estimate this implies that

$$\begin{aligned}
\|u\|_{C^1(\bar B_1)} &\le \varepsilon c_1(\|B_\infty u\|_{L^\infty(B_2)} + \|u\|_{L^\infty(B_2)})\\
&\quad + c'_\varepsilon\|u\|_{L^\infty(B_1)}\\
&\le \varepsilon c_1\|B_\infty u\|_{L^\infty(\mathbb{R}^n)} + (\varepsilon c_1 + c'_\varepsilon)\cdot\|u\|_{L^\infty(\mathbb{R}^n)}\,.
\end{aligned}$$

Since $\|u\|_{W^{1,\infty}(\mathbb{R}^n)} = \sup_{B_1} \|u\|_{C^1(\overline{B_1})}$, where the supremum is taken over all balls of radius 1 in $\mathbb{R}^n$, the claim follows. $\qquad\square$

Theorem 4.6. *There exist $M \geq 0$, $\omega \in \mathbb{R}$ such that $(\lambda - B_\infty)$ is invertible and*

$$\|\lambda(\lambda - B_\infty)^{-1}\| \leq M \quad (\mathrm{Re}\,\lambda > \omega) .$$

Proof. Denote by B_∞^0 the operator with the coefficients b_j, c replaced by 0. Lemma 4.5 implies that $D(B_\infty^0) = D(B_\infty)$ and (applied to B_∞^0) that

$$\|(B_\infty - B_\infty^0)u\|_{L^\infty(\mathbb{R}^n)} \leq \varepsilon\|B_\infty^0 u\|_{L^\infty(\mathbb{R}^n)} + c'_\varepsilon\|u\|_{L^\infty(\mathbb{R}^n)}$$

for all $u \in D(B_\infty^0)$, $\varepsilon > 0$ and some $c'_\varepsilon \geq 0$. Since B_∞^0 is sectorial by [12, Theorem 3.1.7] the claim follows from the usual holomorphic perturbation result [2, Theorem 3.7.23]. $\qquad\square$

Now we use the maximum principle, Lemma 4.2, to carry over the sectorial estimate from $\mathbb{R}^n$ to Ω. This is done in a very abstract framework by Lumer-Paquet [11], see [6, Section 2.5] for the Laplacian.

Proof of Theorem 4.1. Let ω be the constant from Theorem 4.6 and let $\mathrm{Re}\,\lambda > \omega$, $f \in C(\bar\Omega)$, $u = (\lambda - A_c)^{-1}f$. Then

$$u \in C_0(\Omega) \cap \bigcap_{p>1} W^{2,p}_{\mathrm{loc}}(\Omega) \text{ and } \lambda u - \mathcal{A}u = f .$$

Extend f by 0 to $\mathbb{R}^n$ and let $v = (\lambda - B_\infty)^{-1}f$. Then $\lambda v - \mathcal{A}v = f$ on Ω and $\|\lambda v\|_{L^\infty(\Omega)} \leq M\|f\|_{C(\bar\Omega)}$ by Theorem 4.6. Moreover, $w := v - u \in C(\bar\Omega) \cap \bigcap_{p\geq 1} W^{2,p}_{\mathrm{loc}}(\Omega)$, $\lambda w - \mathcal{A}w = 0$ on Ω and $w(z) = v(z)$ for all $z \in \partial\Omega$. Then by the complex maximum principle Proposition 4.3,

$$\|w\|_{C(\bar\Omega)} = \max_{z\in\partial\Omega} |v(z)| \leq \frac{M}{|\lambda|}\|f\|_{C(\bar\Omega)} .$$

Consequently,

$$\begin{aligned}\|u\|_{C(\bar\Omega)} &= \|u - v + v\|_{C(\bar\Omega)}\\ &\leq \|w\|_{C(\bar\Omega)} + \|v\|_{C(\bar\Omega)}\\ &\leq \frac{2M}{|\lambda|}\|f\|_{C(\bar\Omega)} .\end{aligned}$$

This is the desired estimate which shows that A_c is sectorial. By [12, Proposition 2.1.11] there exist a sector $\Sigma_\theta + \omega := \{\omega + re^{i\alpha} : r > 0, |\alpha| < \theta\}$ with $\theta \in (\frac{\pi}{2}, \pi)$, $\omega \geq 0$, and a constant $M_1 > 0$ such that

$$(\lambda - A_c)^{-1} \text{ exists for } \lambda \in \Sigma_\theta + \omega \text{ and } \|\lambda(\lambda - A_c)^{-1}\| \leq M_1 .$$

Thus there exists $r > 0$ such that $(\lambda - A_c)$ is invertible and $\|\lambda(\lambda - A_c)^{-1}\| \leq M$ whenever $\mathrm{Re}\,\lambda > 0$ and $|\lambda| > r$. Since A is invertible by Theorem 3.2, it follows that A_c is bijective. Since the resolvent set of A_c is nonempty, A_c is closed. Thus A_c is invertible. Since by Proposition 4.4 A_c is resolvent positive, it follows from [2, Proposition 3.11.2] that there exists $\varepsilon > 0$ such that $(\lambda - A_c)$ is invertible whenever $\mathrm{Re}\,\lambda > -\varepsilon$. As a consequence,

$$\sup_{\substack{|\lambda| \leq r \\ \mathrm{Re}\,\lambda > 0}} \|\lambda(\lambda - A_c)^{-1}\| < \infty\,.$$

Together with the previous estimates this implies that

$$\|\lambda(\lambda - A_c)^{-1}\| \leq M_2$$

whenever $\mathrm{Re}\,\lambda > 0$ for some constant M_2. Thus A_c generates a bounded holomorphic semigroup T on $C(\bar{\Omega})$. Since $D(A_c) \subset C_0(\Omega)$ and $\mathcal{D}(\Omega) \subset D(A_c)$ it follows that $\overline{D(A_c)} = C_0(\Omega)$. The part of A_c in $C_0(\Omega)$ is A_0. So it follows from [12, Remark 2.1.5, Proposition 2.1.4] that A_0 generates a bounded, holomorphic C_0-semigroup T_0 on $C_0(\Omega)$ and $T_0(t) = T(t)_{|C_0(\Omega)}$ on $C_0(\Omega)$. $\qquad\square$

If the coefficients a_{ij} are merely in $C(\Omega) \cap L^\infty(\Omega)$ as in our standing hypothesis we cannot extend the operator to $\mathbb{R}^n$ and our proof of analyticity of the semigroup breaks down. But we still obtain a C_0-semigroup on $C_0(\Omega)$

More precisely, we have the following result.

Proposition 4.7. *Assume that Ω satisfies the uniform exterior cone condition. Then A_0 generates a positive, contractive, irreducible C_0-semigroup on $C_0(\Omega)$. Moreover,*

$$\|T(t)\| \leq M e^{-\varepsilon t} \quad (t \geq 0)$$

for some $M > 0$, $\varepsilon > 0$. The resolvent $(\lambda - A_0)^{-1}$ is compact for all $\lambda \in \varrho(A_0)$.

Proof. By Theorem 2.4 the operator A is invertible. It follows from Proposition 4.3 that A_0 is m-dissipative. Since $D(A_0)$ is dense, by the Lumer-Phillips Theorem the operator A_0 generates a positive, contractive C_0-semigroup $(T_0(t))_{t \geq 0}$ on $C_0(\Omega)$. It follows from Theorem A.3 that $D(A_0) \subset C^\alpha(\Omega)$. Since the embedding of $C^\alpha(\Omega)$ into $C(\bar{\Omega})$ is compact, it follows that the resolvent of A_0 is compact. Since the resolvent is positive by Proposition 4.3, it follows from [2, 3.11] that the semigroup is positive.

Next show that $u := (\lambda - A_0)^{-1} f$ is strictly positive if $0 \leq f \in C_0(\Omega)$, $f \neq 0$. Assume that $u(x) \leq 0$ for some $x \in \Omega$. Let $v = -u$. Then $\mathcal{A}v - \lambda v = f \geq 0$. It follows from the maximum principle [9, Theorem 9.6] that v is constant. Since $v \in C_0(\Omega)$, it follows that $v \equiv 0$. Hence also $f \equiv 0$. This proves the claim. It follows from [16, B-III.3.1] that the semigroup $(T_0(t))_{t \geq 0}$ is irreducible. [2, Theorem 5.3.8] implies that the semigroup is exponentially stable. $\qquad\square$

A. Results on elliptic partial differential equations

In this section, we collect some results on elliptic partial differential equations, which can be found in text books, for example [9]. We consider the elliptic operator $\mathcal{A}$ from the Introduction.

Theorem A.1 (Aleksandrov's maximum principle, [9, Theorem 9.1]). *Let* $f \in L^n(\Omega)$, $u \in C(\bar{\Omega}) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ *such that*

$$-\mathcal{A}u \leq f \ .$$

Then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + c_1 \|f^+\|_{L^n(\Omega)}$$

where the constant c_1 *depends merely on* n, $\mathrm{diam}\,\Omega$, *the ellipticity constant* Λ *and* $\|b_j\|_{L^n(\Omega)}$, $j = 1\ldots, n$. *Consequently, if* $u \in C_0(\Omega)$ *and* $-\mathcal{A}u = f$, *then*

$$\|u\|_{L^\infty(\Omega)} \leq c_1 \|f\|_{L^n(\Omega))}$$

and $u \leq 0$ *if* $f \leq 0$.

For the next two results we assume that the ellipticity constant $\Lambda > 0$ is so small that $\|a_{ij}\|_{L^\infty}$, $\|b_j\|_{L^\infty}$, $\|c\|_{L^\infty} \leq \frac{1}{\Lambda}$.

Theorem A.2 (Interior Calderon-Zygmund estimate, [9, Theorem 9.11]).
Let $B_{2\varrho}$ *be a ball of radius* 2ϱ *such that* $\overline{B_{2\varrho}} \subset \Omega$, *and let* $u \in W^{2,p}(B_{2\varrho})$, *where* $1 < p < \infty$. *Then*

$$\|u\|_{W^{2,p}(B_\varrho)} \leq c_\varrho (\|\mathcal{A}u\|_{L^p(B_{2\varrho})} + \|u\|_{L^p(B_{2\varrho})})$$

where B_ϱ *is the ball of radius* ϱ *concentric with* $B_{2\varrho}$. *The constant* ϱ *merely depends on* Λ, n, ϱ, p *and the continuity moduli of the* a_{ij}.

Theorem A.3 (Hölder regularity, [9, Corollary 9.29]). *Assume that* Ω *satisfies the uniform exterior cone condition. Let* $u \in C_0(\Omega) \cap W^{2,n}_{\mathrm{loc}}(\Omega)$ *and* $f \in L^n(\Omega)$ *such that* $-\mathcal{A}u = f$. *Then* $u \in C^\alpha(\Omega)$ *and*

$$\|u\|_{C^\alpha(\Omega)} \leq C(\|f\|_{L^n(\Omega)} + \|u\|_{L^2(\Omega)})$$

where $\alpha > 0$ *and* $c > 0$ *depend merely on* Ω, Λ *and* n.

In [9, Corollary 9.29] it is supposed that $u \in W^{2,n}(\Omega)$. But an inspection of the proof and of the results preceding [9, Corollary 9.29] shows that $u \in W^{2,n}_{\mathrm{loc}}(\Omega)$ suffices. The above Hölder regularity also holds for solutions of equations in divergence form when the right-hand side f is in $L^q(\Omega)$ for some $q > \frac{n}{2}$, see [9, Theorem 8.29].

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Institute of Applied Analysis
University of Ulm
D - 89069 Ulm, Germany
wolfgang.arendt@uni-ulm.de

Department of Mathematics
Eberhard-Karls-University of Tübingen
D-72076 Tübingen, Germany
schaetz@everest.mathematik.uni-tuebingen.de