

Deformation openness and closedness of various classes of compact complex manifolds; examples

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Abstract. We review the relations between compact complex manifolds carrying various types of Hermitian metrics (Kähler, balanced, or *strongly Gauduchon*) and those satisfying topological properties such as the $\partial\bar{\partial}$ -lemma, or the degeneration at E_1 of the Frölicher spectral sequence. We also review the behaviour of these properties under holomorphic deformations. The emphasis will be placed on the notion of *strongly Gauduchon* (sG) manifold that we introduced recently in the study of deformation limits of projective and Moishezon manifolds. Besides its expository aspect, the paper presents new results such as exhibiting various examples of sG and non-sG manifolds. These are then used to review and reinterpret from this new standpoint a range of constructions already known in the literature.

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1. Introduction

The main theme of this text is the interaction between *metric* and *topological* properties satisfied by compact complex manifolds.

1.1. Metric properties: smooth metrics

A Hermitian metric on a given compact complex manifold X ($\dim_{\mathbb{C}} X := n$) will be identified with the associated positive-definite C^∞ $(1, 1)$ -form $\omega > 0$ on X . The metric ω is said to be:

- *Kähler* if $d\omega = 0$;
- *Balanced*¹ (or *semi-Kähler*² or *co-Kähler*) if $d\omega^{n-1} = 0$;
- *Strongly Gauduchon*³ (or *sG*) if $\partial\omega^{n-1}$ is $\bar{\partial}$ -exact;
- *Gauduchon* if $\partial\omega^{n-1}$ is $\bar{\partial}$ -closed or, equivalently, if $\partial\bar{\partial}\omega^{n-1} = 0$.

¹ Terminology of [35].

² Terminology of [19], unrelated to the balanced condition used by Donaldson, Luo and others in the context of cscK metrics.

³ Notion introduced in [41].

Although Hermitian metrics always exist, the stronger Kähler, balanced and sG metrics need not exist on an arbitrary X . A compact complex manifold carrying one of the first three types of metrics described above is said to be a compact *Kähler*, *balanced* or, respectively, *strongly Gauduchon* (or *sG*) manifold. By contrast, Gauduchon metrics exist on any compact complex manifold (*cf.* [18]). Actually, Gauduchon's main result in [18] asserts far more: there exists a Gauduchon metric (unique up to normalisation) in the conformal class of *any* Hermitian metric on X . At the level of Hermitian metrics ω on X , the following obvious implications hold:

$$\omega \text{ is Kähler} \implies \omega \text{ is balanced} \implies \omega \text{ is sG} \implies \omega \text{ is Gauduchon.}$$

In the case of complex surfaces X (*i.e.* $n = 2$), it is obvious that ω is a Kähler metric if and only if ω is a balanced metric. An sG metric ω need not be Kähler even when $n = 2$, but compact complex surfaces X carrying sG metrics also carry Kähler metrics (see [41, Section 3]). Thus, at the level of compact complex manifolds, we have the following equivalences:

$$X \text{ is a Kähler surface} \iff X \text{ is a balanced surface} \iff X \text{ is an sG surface.}$$

However, if compact complex surfaces are replaced by compact complex manifolds of complex dimension $n \geq 3$, the above equivalences break down, and only strict implications from left to right hold.

1.2. Metric properties: currents and birational transformations

Recall that the so-called *Kähler currents* provide a singular (thus weaker) substitute for Kähler metrics. A d -closed positive $(1, 1)$ -current T on X is said to be a *Kähler current*⁴ if it satisfies the strong positivity condition

$$T \geq \varepsilon \omega \quad \text{on } X,$$

for some constant $\varepsilon > 0$ and some Hermitian metric $\omega > 0$. Kähler currents need not exist on an arbitrary X , but they may exist when Kähler metrics do not. Recall that X is said to be a Fujiki *class- \mathcal{C}* manifold if X is bimeromorphic to a compact Kähler manifold, or equivalently, if there exists a proper holomorphic bimeromorphic map (*i.e.* a *modification*)

$$\mu : \tilde{X} \rightarrow X$$

from a compact Kähler manifold \tilde{X} . Fujiki introduced *class- \mathcal{C}* manifolds X as meromorphic images of compact Kähler manifolds in [17], while Varouchas gave them the above nice characterisation in [47]. It is a result of Demailly and Paun that *class- \mathcal{C}* manifolds are characterised by the existence of a Kähler current.

⁴ Terminology introduced in [27].

Theorem 1.1 (Demailly-Paun [15]). *A compact complex manifold X is of class \mathcal{C} if and only if there exists a Kähler current T on X .*

Recall that a *Moishezon manifold* is, by definition, a compact complex manifold that is bimeromorphic to a projective manifold. Equivalently, X is Moishezon if and only if there exists a modification

$$\mu : \tilde{X} \rightarrow X$$

from a projective manifold \tilde{X} . Thus Moishezon manifolds are to projective manifolds what *class- \mathcal{C}* manifolds are to compact Kähler manifolds.

The special case of integral cohomology classes is relevant in characterisations of some of the above classes of manifolds. Recall that the De Rham 2-cohomology class $\{\omega\} \in H^2_{DR}(X, \mathbb{R})$ (respectively, $\{T\} \in H^2_{DR}(X, \mathbb{R})$) defined by a C^∞ d -closed real $(1, 1)$ -form ω (respectively, by a d -closed real $(1, 1)$ -current T) is said to be *integral* if it is the first Chern class of a holomorphic line bundle $L \rightarrow X$ or, equivalently, if ω (respectively, T) is the curvature form (respectively, curvature current) $\frac{i}{\pi} \Theta_h(L)$ of a holomorphic line bundle $(L, h) \rightarrow X$ endowed with a C^∞ (respectively, singular) Hermitian fibre metric h .

There are neat characterisations of projective and Moishezon manifolds mirroring the general case of arbitrary (*i.e.* possibly transcendental) classes that occur on Kähler and *class- \mathcal{C}* manifolds.

Theorem 1.2 (Kodaira’s embedding theorem). *A compact complex manifold X is projective if and only if there exists a Kähler metric ω on X whose De Rham cohomology class $\{\omega\} \in H^2_{DR}(X, \mathbb{R})$ is integral.*

Thus projective manifolds are *integral-class* special cases of compact Kähler manifolds. Likewise, Moishezon manifolds are *integral-class* special cases of *class- \mathcal{C}* manifolds, as the following characterisation shows:

Theorem 1.3 (Ji-Shiffman [27]). *A compact complex manifold X is Moishezon if and only if there exists a Kähler current T on X whose De Rham cohomology class $\{T\} \in H^2_{DR}(X, \mathbb{R})$ is integral.*

An interesting result of Alessandrini and Bassanelli (see [4–6]) asserts that every *class- \mathcal{C}* manifold is *balanced* (*i.e.* carries a *balanced metric*). They actually accomplished rather more proving that *balanced manifolds* are stable under modifications:

Theorem 1.4 (Alessandrini-Bassanelli [6]). *Let $\mu : \tilde{X} \rightarrow X$ be a modification of compact complex manifolds. Then X is balanced if and only if \tilde{X} is balanced.*

Now if X is a *class- \mathcal{C}* manifold, then by [47] there exists a modification $\mu : \tilde{X} \rightarrow X$ where \tilde{X} is a compact Kähler manifold. Then \tilde{X} is also balanced and, by Theorem 1.4, X must be balanced as well.

It is worth mentioning that the sG condition enjoys the same modification stability property.

Theorem 1.5 (Theorem 1.3. in [43]). *Let $\mu : \tilde{X} \rightarrow X$ be a modification of compact complex manifolds. Then X is strongly Gauduchon if and only if \tilde{X} is strongly Gauduchon.*

Once again, the picture is known to be much simpler in the case of compact complex surfaces (*i.e.* when $n = 2$): all *class-C* surfaces are actually Kähler, while all Moishezon surfaces are actually projective. However, in dimension $n \geq 3$, there exist Moishezon manifolds that are not projective and *class-C* manifolds that are not Kähler (see *e.g.* the examples of Hironaka and Moishezon described in [21, Appendix B, Example 3.4.2.], from which the only known class of *class-C* manifolds that are neither Kähler nor Moishezon can be obtained). A host of other examples are provided by Lebrun’s twistor spaces in [33]: these are all Moishezon (hence *class-C*) 3-dimensional compact complex manifolds that are not Kähler.

1.3. Topological properties

First recall the following standard piece of terminology.

Definition 1.6. A compact complex manifold X is said to satisfy the $\partial\bar{\partial}$ -lemma if, for any d -closed *pure-type* form u on X , the following exactness properties are equivalent:

$$u \text{ is } d\text{-exact} \iff u \text{ is } \partial\text{-exact} \iff u \text{ is } \bar{\partial}\text{-exact} \iff u \text{ is } \partial\bar{\partial}\text{-exact}$$

If the *pure-type* assumption on u is dropped, then u must be assumed to be both d -closed and d^c -closed (or, equivalently, both ∂ -closed and $\bar{\partial}$ -closed) before the above exactness properties are required to be equivalent (*cf.* [14]). For a *pure-type* form u , the sole d -closedness is equivalent to u being both ∂ -closed and $\bar{\partial}$ -closed. However, if u is not of pure-type, $du = 0$ does not imply $\partial u = 0$ and $\bar{\partial}u = 0$.

It is a standard fact in Hodge theory that any compact Kähler manifold satisfies the $\partial\bar{\partial}$ -lemma. Moreover, if $\mu : \tilde{X} \rightarrow X$ is a modification between compact complex manifolds and if the $\partial\bar{\partial}$ -lemma holds for \tilde{X} , then the $\partial\bar{\partial}$ -lemma also holds for X (see *e.g.* [14, Theorem 5.22.]). In particular, *class-C* manifolds (hence also Moishezon manifolds) satisfy the $\partial\bar{\partial}$ -lemma.

Now recall that on any compact complex manifold X , we always have

$$b_k(X) \leq \sum_{p+q=k} h^{p,q}(X), \quad k = 0, 1, \dots, 2n = \dim_{\mathbb{R}} X, \tag{1.1}$$

where $b_k(X) := \dim_{\mathbb{C}} H_{DR}^k(X, \mathbb{C})$ is the k -th Betti number of X and $h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}(X, \mathbb{C})$ is the Hodge number of type (p, q) of X . The Dolbeault cohomology group associated with (p, q) -forms on X is denoted, as usual, by $H^{p,q}(X, \mathbb{C})$. Recall that the Frölicher spectral sequence of X degenerates at E_1 (a property that will be denoted by $E_1(X) = E_{\infty}(X)$) if and only if equality is achieved in all of the above inequalities (1.1). Indeed, we always have $H_{DR}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} E_{\infty}^{p,q}(X)$ for all k and $E_1^{p,q}(X) = H^{p,q}(X, \mathbb{C})$, as well as

$\dim E_{\infty}^{p,q}(X) \leq \dim E_1^{p,q}(X)$, for all p, q . Frölicher degeneration at E_1 means that $E_1^{p,q}(X) = E_{\infty}^{p,q}(X)$ for all p, q . Hence we have:

Proposition 1.7 (Standard). *On any compact complex manifold X ($\dim_{\mathbb{C}} X =: n$), the property $E_1(X) = E_{\infty}(X)$ is equivalent to the existence of a Hodge decomposition on X , possibly without Hodge symmetry, in the form:*

$$H^k_{DR}(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C}), \quad k = 0, 1, \dots, 2n,$$

where the isomorphism need not be canonical.

On the other hand, recall that the $\partial\bar{\partial}$ -lemma implies both the Hodge decomposition and the Hodge symmetry. In particular, the Frölicher spectral sequence of any compact complex manifold X that satisfies the $\partial\bar{\partial}$ -lemma degenerates at E_1 .

Indeed, it is easily verified that the $\partial\bar{\partial}$ -lemma implies that every Dolbeault cohomology class of type (p, q) can be represented by a d -closed (p, q) -form. This allows one to define, for every (p, q) , a linear map $H^{p,q}(X, \mathbb{C}) \rightarrow H^{p+q}_{DR}(X, \mathbb{C})$ by sending every Dolbeault class to the De Rham class of any of its d -closed representatives. This De Rham class is independent of the choice of the d -closed representative of the initial Dolbeault class. Furthermore, the $\partial\bar{\partial}$ -lemma also implies that the map defined in this way for every (p, q) is injective. Consequently, when the $\partial\bar{\partial}$ -lemma holds on X , there is, for every k , a canonical injection

$$\bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C}) \hookrightarrow H^k_{DR}(X, \mathbb{C})$$

which must be an isomorphism thanks to the reverse-way dimension inequality (1.1) which always holds on X .

The $\partial\bar{\partial}$ -lemma is also known to imply the Hodge symmetry because:

- On any X , there are well-defined linear maps from the Bott-Chern cohomology to the Dolbeault cohomology

$$H^{p,q}_{BC}(X, \mathbb{C}) \rightarrow H^{p,q}(X, \mathbb{C}), \quad p, q = 0, 1, \dots, n = \dim_{\mathbb{C}} X, \quad (1.2)$$

mapping the Bott-Chern cohomology class $[u]_{BC}$ of any d -closed (p, q) -form u to the Dolbeault cohomology class $[u]$ of the same form u ;

- On any X , the map $H^{p,q}_{BC}(X, \mathbb{C}) \ni [u]_{BC} \mapsto [\bar{u}]_{BC} \in H^{q,p}_{BC}(X, \mathbb{C})$ is well-defined, \mathbb{C} -anti-linear and bijective, so the Hodge symmetry analogue for the Bott-Chern cohomology always holds in a trivial way;

- If the $\partial\bar{\partial}$ -lemma holds on X , the maps (1.2) are easily seen to be isomorphisms.

The conclusion is summed up in the following:

Proposition 1.8 (Standard). *If the $\partial\bar{\partial}$ -lemma holds on a compact complex manifold X ($\dim_{\mathbb{C}} X := n$), then there are canonical isomorphisms:*

$$H^k_{DR}(X, \mathbb{C}) \simeq \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C}) \quad \text{and} \quad H^{p,q}(X, \mathbb{C}) \simeq \overline{H^{q,p}(X, \mathbb{C})}$$

for all $k = 0, 1, \dots, 2n$ and all $p, q = 0, 1, \dots, n$.

As usual, the situation is much simpler on surfaces than in higher dimension: every compact complex *surface* X satisfies the property $E_1(X) = E_\infty(X)$ (see *e.g.* [8]), while the only compact complex surfaces satisfying the $\partial\bar{\partial}$ -lemma are the Kähler ones. The latter statement can either be verified directly or seen using sG surfaces: the $\partial\bar{\partial}$ -lemma always implies the sG property (see next subsection), while for compact complex *surfaces* the sG property and the Kähler property are equivalent (see Subsection 1.1).

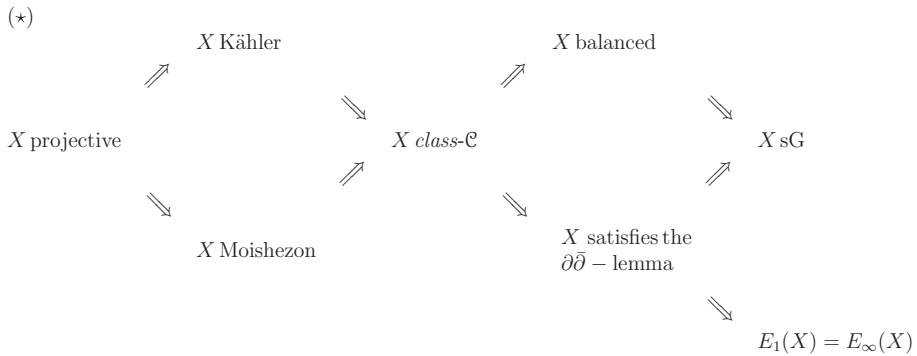
1.4. Metric-topological interplay

The $\partial\bar{\partial}$ -lemma property and the balanced property are unrelated (see *e.g.* examples below), but they are both implied by the *class-C* property and, in turn, they both imply the sG property.

Let us recall the argument given in [41] proving that on every compact complex manifold X which satisfies the $\partial\bar{\partial}$ -lemma, the notions of Gauduchon and sG metrics are equivalent, hence every such X is an sG manifold. If ω is a Gauduchon metric on X , then the pure-type $(n, n - 1)$ -form $\partial\omega^{n-1}$ is $\bar{\partial}$ -closed by definition. Since $\partial\omega^{n-1}$ is also ∂ -closed, it must be d -closed. Thus, if the $\partial\bar{\partial}$ -lemma holds on X , the d -closed and ∂ -exact pure-type form $\partial\omega^{n-1}$ must also be $\bar{\partial}$ -exact. Hence ω is an sG metric on X .

While the sG property of X and the degeneration at E_1 of the spectral sequence of X are unrelated (see Theorem 1.11 below), we have seen above that they are both implied by the $\partial\bar{\partial}$ -lemma.

The relations among these properties for a compact complex manifold X are summed up in the following diagram:



The above diagram simplifies dramatically, with many implications becoming equivalences, when $\dim_{\mathbb{C}} X = 2$:

$$X \text{ projective} \iff X \text{ Moiseszon} \implies X \text{ Kähler} \iff X \text{ class-C} \iff X \text{ balanced} \\ \iff X \text{ satisfies the } \partial\bar{\partial}\text{-lemma} \iff X \text{ sG};$$

and we always have $E_1(X) = E_\infty(X)$ for surfaces.

1.5. Examples

One of the purposes of this work is to explain how the new notion of *strongly Gauduchon manifold* fits into the context of the well-known earlier other notions featuring in diagram (★) above. On one hand, we shall provide examples of compact complex manifolds that are not sG (cf. Section 2) in order to stress that the sG condition is not automatically satisfied:

Theorem 1.9. *The Calabi-Eckmann manifolds [12], the Hopf manifolds [26] and Tsuji’s manifolds [46] are not strongly Gauduchon.*

On the other hand, we shall provide examples of compact complex manifolds that drive home the peculiarity of the sG condition distinguishing it from the two stronger notions that immediately precede it in diagram (★) (cf. also Theorem 3.15 in Section 3).

Theorem 1.10. *There exist compact complex manifolds that are strongly Gauduchon but are not balanced and on which the $\partial\bar{\partial}$ -lemma does not hold.*

We also point out that the last two notions in diagram (★) are unrelated:

Theorem 1.11.

(a) *There exist strongly Gauduchon compact complex manifolds whose Frölicher spectral sequence does not degenerate at E_1 (e.g. the Iwasawa manifold, which is even balanced).*

(b) *There exist compact complex manifolds whose Frölicher spectral sequence degenerates at E_1 but which are not strongly Gauduchon (e.g. any compact non-Kähler complex surface).*

This observation seems to indicate the existence of two disjoint realms in diagram (★): the *metric* notions (i.e. *balanced* and *sG*) branching off upwards from the *class- \mathcal{C}* notion; the *topological* notions (i.e. $\partial\bar{\partial}$ -lemma and $E_1 = E_\infty$) branching off downwards. The overall idea underlying the proof of our main result in [41] was based on switching from the latter to the former realm after observing that the *topological* condition of non-jumping at $t = 0$ of the Hodge number $h^{0,1}(t)$ is sufficient for our purposes but next to impossible to guarantee on an *a priori* basis, we replaced it with the *metric* sG condition on the central fibre that we managed to guarantee.

1.6. Holomorphic deformation behaviour

Another purpose of this work is to review known results and open questions about the behaviour of these properties under holomorphic deformations, with a special emphasis on the new notion of *strongly Gauduchon* (or sG) manifolds. We adopt the Kodaira-Spencer terminology (cf. e.g. [28]):

A complex analytic (or holomorphic) family of compact complex manifolds is a proper holomorphic submersion $\pi : \mathcal{X} \rightarrow \Delta$ from an arbitrary complex analytic manifold \mathcal{X} to some open ball $\Delta \subset \mathbb{C}^m$ centered at the origin $0 \in \mathbb{C}^m$.

Thus the fibres $X_t := \pi^{-1}(t)$ for $t \in \Delta$, are all (smooth) compact complex manifolds of the same dimension varying holomorphically with the parameter $t \in \Delta$. It is well-known that any such family is differentiably trivial, *i.e.* there exists a C^∞ manifold X independent of $t \in \Delta$ such that the fibre X_t is C^∞ -diffeomorphic to X for all $t \in \Delta$. Only the complex structure J_t of X_t varies with $t \in \Delta$. Thus the holomorphic family $(X_t)_{t \in \Delta}$ can be identified with a fixed C^∞ manifold X endowed with a holomorphic family of complex structures $(J_t)_{t \in \Delta}$ (*cf.* [28]).

It will be sufficient to restrict our attention to the case where the base Δ is an open disc centered at the origin in \mathbb{C} , *i.e.* $m = 1$. We shall be concerned with stability properties of the notions featuring in diagram (\star) when they appear in holomorphic families as above. Two points of view will be adopted:

Definition 1.12. (i) A given property (P) of a compact complex manifold is said to be *open* under holomorphic deformations if for every holomorphic family of compact complex manifolds $(X_t)_{t \in \Delta}$ and for every $t_0 \in \Delta$, the following implication holds:

X_{t_0} has property $(P) \implies X_t$ has property (P) for all $t \in \Delta$ sufficiently close to t_0 .

(ii) A given property (P) of a compact complex manifold is said to be *closed* under holomorphic deformations if for every holomorphic family of compact complex manifolds $(X_t)_{t \in \Delta}$ and for every $t_0 \in \Delta$, the following implication holds:

X_t has property (P) for all $t \in \Delta \setminus \{t_0\} \implies X_{t_0}$ has property (P) .

It is obvious that if a property (P) is both open and closed, then all the fibres of a family satisfy (P) whenever one of them satisfies (P) .

The interest in deformation stability questions was sparked by the following celebrated result of Kodaira and Spencer:

Theorem 1.13 (Kodaira-Spencer [29]). *The Kähler property of compact complex manifolds is open under holomorphic deformations.*

However, when Hironaka constructed an example in 1962, the following fact came as a bit of a surprise.

Theorem 1.14 (Hironaka [23]). *The Kähler property of compact complex manifolds of complex dimension ≥ 3 is not closed under holomorphic deformations.*

Our main result of [41] states that the degeneration of the projective property of compact complex manifolds in the deformation limit is relatively mild, *i.e.* projective manifolds degenerate to Moishezon manifolds. In view of Hironaka's example that proved Theorem 1.14, the following result is optimal:

Theorem 1.15 (Theorem 1.1. in [41]). *Let $\pi : \mathcal{X} \rightarrow \Delta$ be a complex analytic family of compact complex manifolds. If the fibre X_t is projective for every $t \in \Delta^*$, then X_0 is Moishezon.*

Hironaka's example does not cover the case of holomorphic families of compact complex surfaces, which turned out to behave very differently from manifolds of higher dimensions. Indeed, Kodaira's classification of surfaces, Miyaoka's result [36] asserting that an elliptic surface is Kähler if and only if its first Betti number is even, and Siu's result [45] asserting that every K3 surface is Kähler showed that the Kählerness of compact complex surfaces is a topological property:

Theorem 1.16 (Kodaira, Miyaoka, Siu). *A compact complex surface X is Kähler if and only if its first Betti number $b_1(X)$ is even.*

Direct proofs of this theorem that do not invoke Kodaira's classification of compact complex surfaces were subsequently given by Buchdahl [9] and Lamari [31], independently. The reader will find further details on the history of the result in these references.

Since all the fibres X_t in a holomorphic family of compact complex manifolds $(X_t)_{t \in \Delta}$ are differentiably diffeomorphic, they have the same Betti numbers. In particular, we have the following:

Corollary 1.17. *The Kähler property of compact complex surfaces is both open and closed under holomorphic deformations. In particular, if in a holomorphic family of compact complex surfaces some fibre is Kähler, then all the fibres are Kähler.*

Let us also recall that the deformation behaviour of the *class- \mathcal{C}* property is the opposite of that of the Kähler property:

Theorem 1.18 (Campana [10], Lebrun-Poon [34]). *The class \mathcal{C} property of compact complex manifolds is not open under holomorphic deformations.*

The examples proving this statement use families of twistor spaces and will be alluded to in section 4. Since it is known by a result of Campana [11] that the Moishezon and *class- \mathcal{C}* properties are equivalent for twistor spaces, the Moishezon property is also seen not to be open under deformations. This latter fact is hardly surprising since a property associated with integral classes is not naturally expected to be deformation open. However, the following long-standing conjecture is still unsettled:

Conjecture 1.19. *The class \mathcal{C} property of compact complex manifolds is closed under holomorphic deformations.*

Our main result in [42] amounts to a confirmation of this conjecture in the *integral-class* case, since Moishezon manifolds can be seen as integral-class versions of *class- \mathcal{C}* manifolds (thanks to Theorems 1.1 and 1.3):

Theorem 1.20 ([42, Theorem 1.1.]). *The Moishezon property of compact complex manifolds is closed under holomorphic deformations.*

We have shown in [42] (see also Theorem 3.1 below) that the *sG* property is deformation open. We hope that the following also holds:

Conjecture 1.21. ⁵ The sG property of compact complex manifolds is *closed* under holomorphic deformations.

The proof is still elusive, but if this turns out to be the case, the sG property would be the only known property of compact complex manifolds to be stable under all known operations (*i.e.* both open and closed under deformations, as well as stable under modifications as shown in *cf.* [43, Theorem 1.5 above]). In particular, it would suffice for just one fibre X_{t_0} to be sG in order to guarantee that all the fibres X_t are sG. We have proved the conclusion of Conjecture 1.21 under a stronger assumption on the generic fibres and this has played a major part in our proofs of the main results of [41] and [42]:

Proposition 1.22 ([41, Proposition 4.1.]). *Let $\pi : \mathcal{X} \rightarrow \Delta$ be a complex analytic family of compact complex manifolds. If the $\partial\bar{\partial}$ -lemma holds on X_t for every $t \in \Delta^*$, then X_0 is a strongly Gauduchon manifold.*

When it comes to balanced manifolds, Alessandrini and Bassanelli showed in [2] (see Theorem 3.3 below) that the *balanced* property of compact complex manifolds is *not* deformation *open*. This difference in deformation behaviour between sG and balanced manifolds will be exploited in Section 3. However, we hope that the balanced analogue of Conjecture 1.21 holds:

Conjecture 1.23. The balanced property of compact complex manifolds is closed under holomorphic deformations.

If this turns out to be the case, Conjecture 1.23 might be used to tackle the standard Conjecture 1.19. Indeed, by the Alessandrini-Bassanelli Theorem 1.4, proving that the limit fibre X_0 is balanced when X_t has been supposed to be *class C* (hence also balanced) for all $t \neq 0$, is necessary to proving the stronger *class C* property of X_0 .

As for the property of Frölicher degeneration at E_1 , we have the following:

Theorem 1.24. (a) [29, Kodaira-Spencer] *For compact complex manifolds, the property of the Frölicher spectral sequence degenerating at E_1 is open under holomorphic deformations.*

(b) (Eastwood-Singer [16, Theorem 5.4.]) *For compact complex manifolds, the property of the Frölicher spectral sequence degenerating at E_1 is not closed under holomorphic deformations.*

Part (a) is by now a classical statement that follows immediately from the inequality (1.1) satisfied by every fibre X_t (in which equality is equivalent to $E_1(X_t) = E_\infty(X_t)$), from the Betti numbers $b_k(X_t)$ of the fibres being independent of t (thanks to the C^∞ triviality of the family) and from the upper-semicontinuity of every Hodge number $h^{p,q}(X_t)$ with respect to $t \in \Delta$ (which is another classical

⁵ This conjecture was *a posteriori* suggested to the author by Jean-Pierre Demailly as a reinforcement of our earlier Proposition 1.22.

result of Kodaira and Spencer). An overview of the Eastwood-Singer proof of part (b) will be given in Section 4.

Turning now to the $\partial\bar{\partial}$ -lemma, we have the following:

Theorem 1.25 ([49, C.-C. Wu]). ⁶ *For compact complex manifolds, the $\partial\bar{\partial}$ -lemma property is open under holomorphic deformations.*

We give a brief account of Wu's strategy of proof in Subsection 4.3 of the last section.

We are at a loss to know anything about the deformation limits of compact complex manifolds satisfying the $\partial\bar{\partial}$ -lemma. It might well be the case that this property is not deformation-closed although the current evidence is tenuous. We will outline a possible approach to this question via twistor spaces in Subsection 4.4 of the last Section 4.

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2. Examples of non-sG compact complex manifolds

In this section we prove Theorem 1.9 by pointing out three well-known classes of compact complex manifolds as not being sG: the Calabi-Eckmann manifolds, the Hopf manifolds and Tsuji's manifolds constructed in [46]. The underlying space of all these manifolds is a product $X := S^{2p+1} \times S^{2q+1}$ of two real odd-dimensional spheres, so they all share the property $H_{DR}^2(X, \mathbb{R}) = 0$ for the second De Rham cohomology group. This implies that any d -closed positive current T of type $(1, 1)$ on X , should it exist, must be d -exact since the associated De Rham cohomology 2-class $\{T\} \in H_{DR}^2(X, \mathbb{R})$ must vanish. The existence of a non-trivial $(1, 1)$ -current T on X that is both positive and d -exact amounts to X being non-sG as the following intrinsic characterisation of sG manifolds obtained in [41] shows:

Proposition 2.1 ([41, Proposition 3.3.]). *Let X be a compact complex manifold. Then X carries a strongly Gauduchon metric if and only if there exists no non-zero current T of type $(1, 1)$ such that $T \geq 0$ and T is d -exact on X .*

We shall briefly review the three classes of compact complex manifolds mentioned above and notice that every such manifold X possesses complex hypersurfaces $Y \subset X$. Thus, since $H_{DR}^2(X, \mathbb{R}) = 0$, the current of integration on any of

⁶ The author is grateful to Professor Jixiang Fu for pointing out to him this work of Wu of which he was unfortunately unaware when writing the first version of the present paper.

these complex hypersurfaces Y is a current as in Proposition 2.1, ruling out the possibility that any manifold X in one of these classes be sG.

(a) **Calabi-Eckmann manifolds.** For all $p, q \in \mathbb{N}$, Calabi and Eckmann [12] constructed a complex structure on the Cartesian product $S^{2p+1} \times S^{2q+1}$ of odd-dimensional spheres. The case $p = q = 0$ being equivalent to a closed Riemann surface of genus 1 and periods $1, \tau$, they assume $p > 0$. In the case $q = 0$, the Calabi-Eckmann complex structure on $S^{2p+1} \times S^1$, although constructed by a different method, coincides with the complex structure constructed earlier by Hopf in [26] starting from the universal covering space of $S^{2p+1} \times S^1$ equipped with the complex structure of $\mathbb{C}^{p+1} \setminus \{0\}$. The simply connected manifolds $S^{2p+1} \times S^{2q+1}$ ($p, q > 0$) are given in [12] complex structures making them into compact, simply connected, non-Kähler complex manifolds $M^{p,q}$ of complex dimension $p + q + 1$ enjoying, among other things, the following properties (for all p, q , including $q = 0$):

- (i) There exists a complex analytic fibring $\sigma : M^{p,q} \rightarrow \mathbb{P}^p \times \mathbb{P}^q$ over the product of complex projective spaces \mathbb{P}^p and \mathbb{P}^q whose fibres are tori of real dimension 2 (or algebraic curves of genus 1) (cf. [12, Theorem II]);
- (ii) Every compact complex subvariety of $M^{p,q}$ is the set of all points that are mapped by σ onto an algebraic subvariety of $\mathbb{P}^p \times \mathbb{P}^q$; it is therefore also fibred by tori (cf. [12, Theorem IV]).

It is clear that the inverse image under σ of any complex hypersurface of $\mathbb{P}^p \times \mathbb{P}^q$ defines a complex hypersurface of the Calabi-Eckmann manifold $M^{p,q}$. Thus no Calabi-Eckmann manifold $M^{p,q}$ ($p > 0$) can be an sG manifold.⁷

(b) **Hopf manifolds.** As mentioned above (and proved in [12, Section 3]), the Hopf manifolds $S^{2p+1} \times S^1$ ($p > 0$) endowed with the complex structure constructed in [Hop48] can be seen in retrospect as special cases for $q = 0$ of Calabi-Eckmann manifolds. Thus they contain complex hypersurfaces and are not sG manifolds by the above arguments.

(c) **Tsuji’s manifolds.** Generalising the Calabi-Eckmann complex structures, Tsuji constructed in [46] complex structures on $S^3 \times S^3$ in the following way. Starting from an arbitrary $(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{C}^3$ satisfying

$$0 < |\alpha_1| \leq |\alpha_2| < 1 \quad \text{and} \quad 0 < |\alpha_3| < 1,$$

the author of [46] considers the primary Hopf manifold

$$H(\alpha) := \mathbb{C}^3 \setminus \{0\} / \langle h \rangle$$

⁷ This same argument was invoked in [35, page 263], to show that Calabi-Eckmann manifolds are not balanced.

of complex dimension 3, where the automorphism $h : \mathbb{C}^3 \rightarrow \mathbb{C}^3$ is defined by $h(z_1, z_2, z_3) := (\alpha_1 z_1, \alpha_2 z_2, \alpha_3 z_3)$ for all $(z_1, z_2, z_3) \in \mathbb{C}^3$ and $\langle h \rangle \subset \text{Aut}(\mathbb{C}^3)$ denotes the automorphism group generated by h . He then goes on to consider

$$C := \{[z_1, z_2, z_3] \in H(\alpha) ; z_1 = z_2 = 0\} \subset H(\alpha),$$

an elliptic curve contained in $H(\alpha)$ and

$$S_0 := \{[z_1, z_2, z_3] \in H(\alpha) ; z_3 = 0\} \subset H(\alpha),$$

a primary Hopf surface which is a complex hypersurface of $H(\alpha)$. For every

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \quad \text{and} \quad m = (m_1, m_2) \in \mathbb{Z}^2, m_1, m_2 \gg 1,$$

he shows the existence of $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{C}^3$ defining biholomorphisms

$$L^*(\beta) \stackrel{\Phi^\pm}{\simeq} L^*(\alpha),$$

where $L^*(\alpha)$ and $L^*(\beta)$ are obtained from $L(\alpha)$ and $L(\beta)$ by removing the respective zero section, while $L(\alpha)$ and $L(\beta)$ are holomorphic line bundles over the respective primary Hopf surfaces

$$S_{\alpha_1, \alpha_2, 0} := \mathbb{C}^2 \setminus \{0\} / \langle g_\alpha \rangle \quad \text{and} \quad S_{\beta_1, \beta_2, 0} := \mathbb{C}^2 \setminus \{0\} / \langle g_\beta \rangle$$

associated with automorphisms of \mathbb{C}^2

$$g_\alpha(z_1, z_2) := (\alpha_1 z_1, \alpha_2 z_2) \quad \text{and} \quad g_\beta(z_1, z_2) := (\beta_1 z_1, \beta_2 z_2)$$

defined by

$$L(\alpha) := \mathbb{C}^2 \setminus \{0\} \times \mathbb{C} / \langle h_\alpha \rangle$$

$$L(\beta) := \mathbb{C}^2 \setminus \{0\} \times \mathbb{C} / \langle h_\beta \rangle,$$

where the automorphisms h_α and h_β of \mathbb{C}^3 are defined by

$$h_\alpha(z_1, z_2, z_3) := (\alpha_1 z_1, \alpha_2 z_2, \alpha_3 z_3)$$

$$h_\beta(z_1, z_2, z_3) := (\beta_1 z_1, \beta_2 z_2, \beta_3 z_3).$$

Considering a compactification of $L(\beta)$ as a \mathbb{P}^1 -bundle $\mathbb{P}(\beta) \rightarrow S_{\beta_1, \beta_2, 0}$, the infinity section of $\mathbb{P}(\beta)$ is denoted S_∞ , while $U(S_\infty)$ denotes a tubular neighbourhood of S_∞ in $\mathbb{P}(\beta)$. The author defines compact complex manifolds

$$M^\pm(\alpha, A, m)$$

by identifying

$$L^*(\beta) \subset \mathbb{P}(\beta) \setminus (\text{zero section})$$

with

$$L^*(\alpha) \simeq H(\alpha) \setminus (S_0 \cup C) \subset H(\alpha)$$

using Φ^\pm . These compact complex manifolds are seen to arise as

$$M^\pm(\alpha, A, m) = (H(\alpha) \setminus C) \cup U(S_\infty), \tag{2.1}$$

or equivalently, $M^\pm(\alpha, A, m)$ are obtained from $H(\alpha)$ by a surgery which replaces C with $U(S_\infty)$.

Theorem 2.2 ([46, Theorem 1.13]). *$M^\pm(\alpha, A, m)$ is diffeomorphic to $S^3 \times S^3$ if and only if A is of the form $A = \begin{pmatrix} a & b \\ \pm 1 & d \end{pmatrix}$.*

Consequently, if A has the above shape, $M^\pm(\alpha, A, m)$ is diffeomorphic to an S^3 -bundle over a lens space, hence $M^\pm(\alpha, A, m)$ has a complex structure.

With this outline of Tsuji’s construction understood, we see that the complex hypersurface $S_0 \subset H(\alpha)$ satisfies $S_0 \cap C = \emptyset$. Thus, in view of the description (2.1) of $M^\pm(\alpha, A, m)$, we get a complex hypersurface

$$S_0 \subset M^\pm(\alpha, A, m)$$

whose existence, along with the property $H_{DR}^2(M^\pm(\alpha, A, m), \mathbb{R}) = 0$, shows that Tsuji’s compact complex manifolds $M^\pm(\alpha, A, m)$ are not sG for any $\alpha \in \mathbb{C}^3$, $A \in SL(2, \mathbb{Z})$, $m = (m_1, m_2) \in \mathbb{Z}^2$ as above.

3. Examples of sG manifolds

As pointed out in the Introduction, all balanced manifolds and all compact complex manifolds on which the $\partial\bar{\partial}$ -lemma holds provide examples of sG manifolds (cf. implication diagram (\star)). In this section we shall exhibit compact complex manifolds that are sG but neither are they balanced nor do they satisfy the $\partial\bar{\partial}$ -lemma. These examples are thus meant to emphasise the difference between sG manifolds and stronger earlier types of possibly non-Kähler compact complex manifolds.

3.1. Deformation stability

The examples we shall exhibit will be provided by holomorphic families of compact complex manifolds. The starting point is the following stability property of sG manifolds under small deformations:

Theorem 3.1 ([42, Conclusion 2.4.]). *The sG property of compact complex manifolds is open under holomorphic deformations.*

We now recall the proof of this *small deformation* stability result. The main argument is provided by the following characterisation of sG manifolds:

Lemma 3.2 ([41, Lemma 3.2]). *Let X be a compact complex manifold of complex dimension n . Then X carries an sG metric if and only if there exists a C^∞ $(2n - 2)$ -form Ω on X satisfying the following three conditions:*

- (a) $\Omega = \bar{\Omega}$ (i.e. Ω is real);
- (b) $d\Omega = 0$;

- (c) $\Omega^{n-1,n-1} > 0$ on X (i.e. the component of type $(n - 1, n - 1)$ of Ω with respect to the complex structure of X is positive-definite).

Note that conditions (a) and (b) are independent of the complex structure of X , while a change of complex structure changes the $(n - 1, n - 1)$ -component of a given $(2n - 2)$ -form Ω . Thus condition (c) is the only one to be dependent on the complex structure of X .

Proof of Lemma 3.2 (cf. [41]). The vanishing of the $(2n - 1)$ -form $d\Omega$ (cf. (b)) amounts to the simultaneous vanishing of its components $\partial\Omega^{n-1,n-1} + \bar{\partial}\Omega^{n,n-2}$ (of type $(n, n - 1)$) and $\partial\Omega^{n-2,n} + \bar{\partial}\Omega^{n-1,n-1}$ (of type $(n - 1, n)$). These two components are conjugate to each other if Ω satisfies (a). Thus, if (a) holds, (b) is equivalent to $\partial\Omega^{n-1,n-1} + \bar{\partial}\Omega^{n,n-2} = 0$.

Suppose there exists an sG metric ω on X . This means that ω is a C^∞ positive-definite $(1, 1)$ -form on X such that the $(n, n - 1)$ -form $\partial\omega^{n-1}$ is $\bar{\partial}$ -exact. Then the $(n - 1, n - 1)$ -form $\Omega^{n-1,n-1} := \omega^{n-1}$ is positive-definite on X and there exists a C^∞ $(n, n - 2)$ -form $\Omega^{n,n-2}$ on X satisfying $\partial\Omega^{n-1,n-1} = -\bar{\partial}\Omega^{n,n-2}$. Considering the $(n - 2, n)$ -form $\Omega^{n-2,n} := \overline{\Omega^{n,n-2}}$, we see that the C^∞ $(2n - 2)$ -form

$$\Omega := \Omega^{n,n-2} + \Omega^{n-1,n-1} + \Omega^{n-2,n}$$

satisfies conditions (a), (b), (c).

Conversely, suppose there exists a C^∞ $(2n - 2)$ -form Ω on X satisfying conditions (a), (b), (c). According to an observation in linear algebra due to Michelsohn [35], every C^∞ positive-definite $(n - 1, n - 1)$ -form admits a unique $(n - 1)^{st}$ root. Applying this to $\Omega^{n-1,n-1} > 0$, we get a unique C^∞ positive-definite $(1, 1)$ -form $\omega > 0$ on X such that

$$\omega^{n-1} = \Omega^{n-1,n-1}.$$

By condition (b) satisfied by Ω , we see that $\partial\omega^{n-1}$ is $\bar{\partial}$ -exact, which means that the Hermitian metric ω of X is *strongly Gauduchon*. □

Proof of Theorem 3.1. If we are given a holomorphic family of compact complex manifolds $(X_t)_{t \in \Delta}$ with $\dim_{\mathbb{C}} X_t = n$, we denote by X the C^∞ manifold underlying the fibres X_t and by J_t the complex structure of X_t for all $t \in \Delta$. The family $(X_t)_{t \in \Delta}$ is thus equivalent to the holomorphic family of complex structures $(J_t)_{t \in \Delta}$ on X . If we have a C^∞ $(2n - 2)$ -form Ω on X , its components $\Omega_t^{n-1,n-1}$ of type $(n - 1, n - 1)$ with respect to the complex structures J_t vary in a C^∞ way with $t \in \Delta$. Consequently, if $\Omega_0^{n-1,n-1} > 0$ then $\Omega_t^{n-1,n-1} > 0$ for $t \in \Delta$ sufficiently close to $0 \in \Delta$. Thus condition (c) of Lemma 3.2 is preserved under small deformations by mere continuity. Since conditions (a) and (b) of Lemma 3.2 are independent of the complex structure of X , it follows that any C^∞ $(2n - 2)$ -form Ω on X satisfying conditions (a), (b) and (c) of Lemma 3.2 with respect to J_0 also satisfies these conditions with respect to J_t for all t sufficiently near 0. The proof of Theorem 3.1 is complete. □

One fundamental difference between balanced and sG manifolds that we shall exploit is that, unlike sG manifolds, balanced manifolds are not stable under small deformations. This result was first observed by Alessandrini and Bassanelli [2] and refutes Michelsohn’s claim of the contrary made in the introduction to [35]:

Theorem 3.3 (Alessandrini-Bassanelli [2]). *The balanced property of compact complex manifolds is not open under holomorphic deformations.*

3.2. The Iwasawa manifold

Alessandrini and Bassanelli use the explicit description of the Kuranishi family of the Iwasawa manifold (known to be balanced) calculated by Nakamura in [37] and observe that one particular direction among the six dimensions of the base space yields the example that proves Theorem 3.3. We now review a few basic facts about compact *complex parallelisable* manifolds to the class of which the Iwasawa manifold belongs, before surveying the arguments and results of [2,37] and [3] that are necessary to the understanding of the observation of Alessandrini and Bassanelli (*cf.* Proposition 3.13) which proves Theorem 3.3.

Let X be a compact complex manifold, $\dim_{\mathbb{C}}X = n$. Recall the following:

Fact. *If X is Kähler, then for every $p = 0, 1, \dots, n$ and for every form $u \in C^\infty(X, \Lambda^{p,0}T^*X)$ such that $\bar{\partial}u = 0$, we have $du = 0$. In other words, every holomorphic p -form is d -closed on a compact Kähler manifold.*

To see this, fix any Kähler metric ω on X and recall that the associated d -Laplacian $\Delta_\omega := dd^* + d^*d$, ∂ -Laplacian $\Delta'_\omega := \partial\bar{\partial}^* + \bar{\partial}^*\partial$ and $\bar{\partial}$ -Laplacian $\Delta''_\omega := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ (where the formal adjoints d^* , ∂^* , $\bar{\partial}^*$ are calculated with respect to ω) are related by

$$\Delta'_\omega = \Delta''_\omega = \frac{1}{2}\Delta_\omega. \tag{3.1}$$

This property is peculiar to Kähler metrics; it fails dramatically for arbitrary, non-Kähler metrics. Now, given any smooth $(p, 0)$ -form u on X , we clearly have $\bar{\partial}^*u = 0$ for trivial bidegree reasons. If u is holomorphic (*i.e.* $\bar{\partial}u = 0$), then $\Delta''_\omega u = 0$ because $\ker \Delta''_\omega = \ker \bar{\partial} \cap \ker \bar{\partial}^*$. (This last identity of kernels is valid for any Hermitian metric ω on a compact manifold.) It then follows from (3.1) that $\Delta_\omega u = 0$. Since $\ker \Delta_\omega = \ker d \cap \ker d^*$, we see that $du = 0$. \square

Let us now recall that this standard fact enables one to see that the Iwasawa manifold is not Kähler. The *Iwasawa manifold* is, by definition, the compact complex manifold of complex dimension 3 defined as the quotient

$$X := G/\Gamma$$

of the simply connected, connected *complex* Lie group (the Heisenberg group)

$$G := \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} ; z_1, z_2, z_3 \in \mathbb{C} \right\} \subset \text{Gl}_3(\mathbb{C})$$

by the discrete subgroup $\Gamma \subset G$ of matrices with entries $z_1, z_2, z_3 \in \mathbb{Z}[i]$. The complex manifold structure on G is defined by the complex structure of \mathbb{C}^3 via the obvious diffeomorphism $G \simeq \mathbb{C}^3$, while the group structure on G is defined by the multiplication of matrices

$$\begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & w_1 & w_3 \\ 0 & 1 & w_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z_1 + w_1 & z_3 + w_3 \\ 0 & 1 & z_2 + w_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since the holomorphic 1-form on G

$$G \ni M \mapsto M^{-1} dM$$

is invariant under the action of Γ , it descends to a holomorphic 1-form on X . An elementary calculation shows that

$$\text{if } M = \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} \text{ then } M^{-1} dM = \begin{pmatrix} 0 & dz_1 & dz_3 - z_1 dz_2 \\ 0 & 0 & dz_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus we get holomorphic 1-forms on the Iwasawa manifold X induced by the following forms on \mathbb{C}^3 :

$$\varphi_1 := dz_1, \quad \varphi_2 := dz_2, \quad \varphi_3 := dz_3 - z_1 dz_2. \tag{3.2}$$

Denoting the induced forms by the same symbols $\varphi_1, \varphi_2, \varphi_3$, it is obvious that

$$d\varphi_1 = d\varphi_2 = 0 \quad \text{but} \quad d\varphi_3 = -\varphi_1 \wedge \varphi_2 \neq 0 \quad \text{on } X. \tag{3.3}$$

Since the holomorphic 1-form φ_3 on X is not d -closed, X is not Kähler.

Finally recall that the forms defined in (3.2), which are linearly independent at every point of X , can be used to completely calculate the De Rham, Dolbeault and Bott-Chern cohomologies of the Iwasawa manifold (see *e.g.* [44, page 4-6] for details). For example, $\varphi_1, \varphi_2, \overline{\varphi_1}, \overline{\varphi_2}$ are all d -closed, but not d -exact, 1-forms on X and any two of them are not d -cohomologous. For instance, if we had $\varphi_1 = df = \partial f + \bar{\partial} f$ for a smooth function f on X , then $\bar{\partial} f = 0$ since φ_1 is of type $(1, 0)$. Thus f would be holomorphic, hence constant, on the compact X , which is impossible since φ_1 is not zero. It is readily seen that $H^1_{DR}(X, \mathbb{C})$ is the \mathbb{C} -vector space generated as follows:

$$H^1_{DR}(X, \mathbb{C}) = \langle \{\varphi_1\}, \{\varphi_2\}, \{\overline{\varphi_1}\}, \{\overline{\varphi_2}\} \rangle, \quad \text{hence } b_1(X) = 4, \tag{3.4}$$

where $\{ \}$ denotes a De Rham cohomology class. Using (3.3), we can easily see that $\varphi_1 \wedge \varphi_3$ and $\varphi_2 \wedge \varphi_3$ (both of type $(2, 0)$) are d -closed but not d -exact on X , hence they and their conjugates (the latter being of type $(0, 2)$) induce non-zero elements in $H^2_{DR}(X, \mathbb{C})$, while $\varphi_1 \wedge \varphi_2$ is d -exact on X hence it induces the zero

class. On the other hand, the $(1, 1)$ -forms $\varphi_1 \wedge \bar{\varphi}_1, \varphi_2 \wedge \bar{\varphi}_2, \varphi_1 \wedge \bar{\varphi}_2$ and $\varphi_2 \wedge \bar{\varphi}_1$ are all d -closed but not d -exact. We easily get that $H^2_{DR}(X, \mathbb{C})$ is the \mathbb{C} -vector space generated as follows:

$$\begin{aligned}
 H^2_{DR}(X, \mathbb{C}) = & \langle \{\varphi_1 \wedge \varphi_3\}, \{\varphi_2 \wedge \varphi_3\} \rangle \\
 & \oplus \langle \{\varphi_1 \wedge \bar{\varphi}_1\}, \{\varphi_2 \wedge \bar{\varphi}_2\}, \{\varphi_1 \wedge \bar{\varphi}_2\}, \{\varphi_2 \wedge \bar{\varphi}_1\} \rangle \\
 & \oplus \langle \{\bar{\varphi}_1 \wedge \bar{\varphi}_3\}, \{\bar{\varphi}_2 \wedge \bar{\varphi}_3\} \rangle, \quad \text{hence } b_2(X) = 8.
 \end{aligned}
 \tag{3.5}$$

We have recalled (3.4) and (3.5) since they will be used further on.

We now review the argument showing that the Iwasawa manifold is balanced. Actually every compact *complex parallelisable* manifold will be seen to be balanced. The point of view presented here is that of [3].

Observation 3.4 (cf. e.g. [37, Lemma 1.2.] or [3, Remark 3.1.]). Let X be any compact complex (possibly non-Kähler) manifold, $\dim_{\mathbb{C}} X = n$. Then for every form $u \in C^\infty(X, \Lambda^{n-1,0} T^* X)$ such that $\bar{\partial}u = 0$, we have $du = 0$. In other words, every holomorphic $(n - 1)$ -form is d -closed on any compact complex manifold of dimension n .

Proof. Let $u \in C^\infty(X, \Lambda^{n-1,0} T^* X)$ such that $\bar{\partial}u = 0$. Then $du = \partial u$ is of type $(n, 0)$ and $d\bar{u} = \bar{\partial}u = 0$ is of type $(0, n)$. We get

$$i^{n^2} du \wedge d\bar{u} \geq 0 \quad \text{as an } (n, n) \text{ - form on } X \tag{3.6}$$

and

$$\int_X i^{n^2} du \wedge d\bar{u} = i^{n^2} \int_X d(u \wedge d\bar{u}) = 0 \quad \text{by Stokes.} \tag{3.7}$$

Thus (3.6) and (3.7) yield $i^{n^2} du \wedge d\bar{u} = 0$ everywhere on X , hence $du = 0$ everywhere on X . This proves the contention. To justify (3.6), write in local holomorphic coordinates z_1, \dots, z_n :

$$du = f dz_1 \wedge \dots \wedge dz_n, \quad \text{hence } i^{n^2} du \wedge d\bar{u} = |f|^2 i dz_1 \wedge d\bar{z}_1 \dots \wedge i dz_n \wedge d\bar{z}_n \geq 0.$$

It is clear that $i^{n^2} du \wedge d\bar{u} = 0$ iff $f = 0$ iff $du = 0$. □

Corollary 3.5. Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = n$. Suppose we have a form $u \in C^\infty(X, \Lambda^{n-1,0} T^* X)$ such that $\bar{\partial}u = 0$.

Then the $(n - 1, n - 1)$ -form $i^{(n-1)^2} u \wedge \bar{u}$ satisfies

$$i^{(n-1)^2} u \wedge \bar{u} \geq 0 \quad \text{and} \quad d\left(i^{(n-1)^2} u \wedge \bar{u}\right) = 0 \quad \text{on } X.$$

Proof. The first inequality is checked to hold for any $(n - 1, 0)$ -form u by a trivial calculation. If $\bar{\partial}u = 0$, then $du = 0$ by Observation 3.4. Then we also have $d\bar{u} = 0$ and the second part follows. □

Now recall the following standard notion introduced by Wang [Wan54]. A compact complex manifold X is said to be *complex parallelisable* if its holomorphic tangent bundle $T^{1,0}X$ is trivial. This condition is, of course, equivalent to the sheaf of germs of holomorphic 1-forms Ω_X^1 being trivial. If $n = \dim_{\mathbb{C}}X$, the *complex parallelisable* condition is equivalent to the existence of n holomorphic vector fields $\theta_1, \dots, \theta_n \in H^0(X, T^{1,0}X)$ that are linearly independent at every point of X . It is again equivalent to the existence of n holomorphic 1-forms $\varphi_1, \dots, \varphi_n \in H^0(X, \Omega_X^1)$ that are linearly independent at every point of X .

By a result of Wang [48], every compact *complex parallelisable* manifold is the compact quotient $X = G/\Gamma$ of a simply connected, connected *complex* Lie group G by a discrete subgroup $\Gamma \subset G$. Conversely, it is obvious that any such quotient is *complex parallelisable*. In particular, for any compact *complex parallelisable* manifold X , $H^0(X, T^{1,0}X) \simeq \mathfrak{g}$ where \mathfrak{g} is the Lie algebra of G . A compact *complex parallelisable* manifold X is said to be *nilpotent* (respectively *solvable*)⁸ if the corresponding complex Lie group G is *nilpotent* (respectively *solvable*). The Heisenberg group defining the Iwasawa manifold being nilpotent, the Iwasawa manifold is a nilpotent compact *complex parallelisable* manifold.

Corollary 3.6 (cf. e.g. [3, Remark 3.1.]). *Every compact complex parallelisable manifold is balanced.*

In particular, the Iwasawa manifold is balanced.

Proof. Let X be an arbitrary compact *complex parallelisable* manifold, $\dim_{\mathbb{C}}X = n$. Let $\varphi_1, \dots, \varphi_n \in H^0(X, \Omega_X^1)$ be n holomorphic 1-forms that are linearly independent at every point of X . Consider the $(n - 1, n - 1)$ -form on X :

$$\Omega := i^{(n-1)^2} \sum_{i=1}^n \varphi_1 \wedge \dots \wedge \widehat{\varphi}_i \wedge \dots \wedge \varphi_n \wedge \bar{\varphi}_1 \wedge \dots \wedge \widehat{\bar{\varphi}}_i \wedge \dots \wedge \bar{\varphi}_n = \sum_{i=1}^n i^{(n-1)^2} u_i \wedge \bar{u}_i,$$

where $u_i := \varphi_1 \wedge \dots \wedge \widehat{\varphi}_i \wedge \dots \wedge \varphi_n \in C^\infty(X, \Lambda^{n-1,0}T^*X)$ and $\widehat{}$ indicates a missing factor. Since $\bar{\partial}\varphi_k = 0$ for all $k = 1, \dots, n$, we see that $\bar{\partial}u_i = 0$ for all $i = 1, \dots, n$. Then Observation 3.4 gives $du_i = 0$ for all $i = 1, \dots, n$, while Corollary 3.5 gives

$$\Omega \geq 0 \quad \text{and} \quad d\Omega = 0 \quad \text{on } X.$$

Furthermore, since $\varphi_1, \dots, \varphi_n$ are linearly independent at every point of X , we must even have $\Omega > 0$. Thus Ω is a C^∞ $(n - 1, n - 1)$ -form on X satisfying

$$\Omega > 0 \quad \text{and} \quad d\Omega = 0 \quad \text{on } X.$$

Applying Michelsohn’s procedure [35] for extracting the $(n - 1)^{st}$ root of a smooth positive-definite $(n - 1, n - 1)$ -form, there exists a unique C^∞ positive-definite $(1, 1)$ -form $\omega > 0$ on X such that $\omega^{n-1} = \Omega$. Since $d(\omega^{n-1}) = d\Omega = 0$, we see that ω is a *balanced* metric on X . The proof is complete. \square

⁸ Or to be a compact *complex parallelisable nilmanifold* (respectively *solvmanifold*).

Note, however, that very few compact *complex parallelisable* manifolds are Kähler, thanks to a result of Wang:

Remark 3.7 ([48, Corollary 2, page 776]). Let $X = G/\Gamma$ be a compact complex parallelisable manifold. Then

$$X \text{ is Kähler} \iff G \text{ is Abelian} \iff X \text{ is a complex torus.}$$

3.3. The Kuranishi family of the Iwasawa manifold (after Nakamura [37])

(a) Let X be a compact complex manifold, with $\dim_{\mathbb{C}} X = n$. Since there are no non-zero $\bar{\partial}$ -exact $(1, 0)$ -forms on X (for obvious bidegree reasons), we have

$$H^{1,0}(X, \mathbb{C}) = \{u \in C^\infty(X, \Lambda^{1,0} T^* X); \bar{\partial} u = 0\},$$

i.e. $H^{1,0}(X, \mathbb{C})$ consists of holomorphic 1-forms on X . Denoting $h^{1,0}(X) := \dim_{\mathbb{C}} H^{1,0}(X, \mathbb{C})$ we have the following trivial:

Observation 3.8. If X is complex parallelisable, then $h^{1,0}(X) = n$.

Proof. By the *complex parallelisability* hypothesis on X , the rank- n analytic sheaf Ω_X^1 is trivial, hence it is generated by n holomorphic 1-forms $\varphi_1, \dots, \varphi_n \in H^{1,0}(X, \mathbb{C})$ that are linearly independent at every point of X . In particular, $\{\varphi_1, \dots, \varphi_n\}$ is a basis of $H^{1,0}(X, \mathbb{C}) \simeq H^0(X, \Omega_X^1)$. □

Suppose now that X is compact *complex parallelisable*. Let $\theta_1, \dots, \theta_n \in H^0(X, T^{1,0} X)$ be n holomorphic vector fields that are linearly independent at every point of X , chosen to be dual to the holomorphic $(1, 0)$ -forms $\varphi_1, \dots, \varphi_n \in H^{1,0}(X, \mathbb{C})$ considered in the above proof. For every smooth function $g : X \rightarrow \mathbb{C}$, we have

$$\partial g = \sum_{\lambda=1}^n (\theta_\lambda g) \varphi_\lambda, \quad \bar{\partial} g = \sum_{\lambda=1}^n (\bar{\theta}_\lambda g) \bar{\varphi}_\lambda, \tag{3.8}$$

i.e. the familiar formalism induced by local holomorphic coordinates finds a global analogue on a compact *complex parallelisable* manifold in a formalism where θ_λ replaces $\partial/\partial z_\lambda$ and φ_λ replaces dz_λ . Thus any $(0, 1)$ -form φ on X has a unique decomposition

$$\varphi = \sum_{\lambda=1}^n f_\lambda \bar{\varphi}_\lambda$$

with $f_1, \dots, f_n : X \rightarrow \mathbb{C}$ functions on X . Thus there is an implicit L^2 inner product on $C^\infty(X, \Lambda^{0,1} T^* X)$ defined as follows (no Hermitian metric is needed on X): for any $\varphi = \sum_{\lambda=1}^n f_\lambda \bar{\varphi}_\lambda, \psi = \sum_{\lambda=1}^n g_\lambda \bar{\varphi}_\lambda \in C^\infty(X, \Lambda^{0,1} T^* X)$, set

$$\langle\langle \varphi, \psi \rangle\rangle := \int_X \left(\sum_{\lambda=1}^n f_\lambda \bar{g}_\lambda \right) i^{n^2} \varphi_1 \wedge \dots \wedge \varphi_n \wedge \bar{\varphi}_1 \wedge \dots \wedge \bar{\varphi}_n. \tag{3.9}$$

It is clear that $dV := i^{n^2} \varphi_1 \wedge \cdots \wedge \varphi_n \wedge \bar{\varphi}_1 \wedge \cdots \wedge \bar{\varphi}_n > 0$ is a global volume form on X and that the above L^2 inner product is independent of the choices made. We can define the formal adjoint $\bar{\partial}^*$ of $\bar{\partial}$ with respect to this L^2 inner product in the usual way: for any smooth $(0, 1)$ -form φ , define $\bar{\partial}^* \varphi$ to be the unique smooth function on X satisfying

$$\langle \bar{\partial}^* \varphi, g \rangle = \langle \varphi, \bar{\partial} g \rangle$$

for any smooth function g on X . A trivial calculation using Stokes' theorem gives

$$\bar{\partial}^* \varphi = - \sum_{\lambda=1}^n \theta_\lambda f_\lambda \tag{3.10}$$

for any smooth $(0, 1)$ -form $\varphi = \sum_{\lambda=1}^n f_\lambda \bar{\varphi}_\lambda$ on X . Thus we see that

$$\bar{\partial}^* \bar{\varphi}_\nu = 0, \quad \nu = 1, \dots, n, \tag{3.11}$$

because $\bar{\varphi}_\nu = \sum_{\lambda=1}^n \delta_{\nu\lambda} \bar{\varphi}_\lambda$ and $\theta_\lambda \delta_{\nu\lambda} = 0$ (since the $\delta_{\nu\lambda}$ are constants).

Now denote by $r \in \{0, 1, \dots, n\}$ the number of d -closed forms among $\varphi_1, \dots, \varphi_n$. After a possible reordering, we can suppose that $\varphi_1, \dots, \varphi_r$ are d -closed and $\varphi_{r+1}, \dots, \varphi_n$ are not d -closed. Then we have

$$\partial \varphi_1 = \cdots = \partial \varphi_r = 0 \quad \text{or equivalently} \quad \bar{\partial} \bar{\varphi}_1 = \cdots = \bar{\partial} \bar{\varphi}_r = 0. \tag{3.12}$$

Thus the $\bar{\partial}$ -closed $(0, 1)$ -forms $\bar{\varphi}_1, \dots, \bar{\varphi}_r$ define Dolbeault $(0, 1)$ -cohomology classes in $H^{0,1}(X, \mathbb{C})$.

We can define the $\bar{\partial}$ -Laplacian on forms of X in the usual way:

$$\Delta'' := \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}.$$

The corresponding harmonic space of $(0, 1)$ -forms $\mathcal{H}_{\Delta''}^{0,1}(X, \mathbb{C}) := \ker \Delta'' = \ker \bar{\partial} \cap \ker \bar{\partial}^*$ satisfies the Hodge isomorphism $\mathcal{H}_{\Delta''}^{0,1}(X, \mathbb{C}) \simeq H^{0,1}(X, \mathbb{C})$. Notice that (3.11) and (3.12) give

$$\Delta'' \bar{\varphi}_\nu = 0, \quad \nu = 1, \dots, r, \tag{3.13}$$

i.e. the forms $\bar{\varphi}_1, \dots, \bar{\varphi}_r$ are Δ'' -harmonic. On the other hand, $\bar{\varphi}_{r+1}, \dots, \bar{\varphi}_n$ are not Δ'' -harmonic. Thus the number r of linearly independent d -closed holomorphic 1-forms of X (independent of the choice of $\varphi_1, \dots, \varphi_n$) satisfies:

$$r \leq h^{0,1}(X). \tag{3.14}$$

Suppose now that the compact complex parallelisable X is nilpotent.

Fact 3.9 (see e.g. [37] or [13, page 5405-5406]). If X is a compact complex parallelisable nilmanifold, then the holomorphic 1-forms $\varphi_1, \dots, \varphi_n$ that form a basis of $H^{1,0}(X, \mathbb{C})$ can be chosen such that

$$d\varphi_\mu = \sum_{1 \leq \lambda < \nu \leq n} c_{\mu\lambda\nu} \varphi_\lambda \wedge \varphi_\nu, \quad 1 \leq \mu \leq n, \tag{3.15}$$

with constant coefficients $c_{\mu\lambda\nu} \in \mathbb{C}$ satisfying

$$c_{\mu\lambda\nu} = 0 \quad \text{whenever} \quad \mu \leq \lambda \quad \text{or} \quad \mu \leq \nu.$$

Taking this standard fact (which in [37] follows from the existence of a Chevalley decomposition of the nilpotent Lie algebra \mathfrak{g}) for granted, we now spell out the details of the proof of the following result of Kodaira along the lines of [37, Theorem 3, page 100]:

Theorem 3.10 (Kodaira). *If X is a compact complex parallelisable nilmanifold then $h^{0,1}(X) = r$. Moreover, the Δ'' -harmonic $(0, 1)$ -forms $\bar{\varphi}_1, \dots, \bar{\varphi}_r$ form a basis of the harmonic space $\mathcal{H}_{\Delta''}^{0,1}(X, \mathbb{C})$. Equivalently, the Dolbeault $(0, 1)$ -cohomology classes $\{\bar{\varphi}_1\}, \dots, \{\bar{\varphi}_r\}$ form a basis of $H^{0,1}(X, \mathbb{C})$.*

Proof. The only thing that needs proving is that the linearly independent forms $\bar{\varphi}_1, \dots, \bar{\varphi}_r \in \mathcal{H}_{\Delta''}^{0,1}(X, \mathbb{C})$ generate $\mathcal{H}_{\Delta''}^{0,1}(X, \mathbb{C})$. Pick an arbitrary C^∞ $(0, 1)$ -form φ on X and write

$$\varphi = \sum_{\lambda=1}^n f_\lambda \bar{\varphi}_\lambda$$

with C^∞ functions f_1, \dots, f_n on X . Using formula (3.8) for $\bar{\partial}$ and the obvious identities $\bar{\partial} \bar{\varphi}_\lambda = d \bar{\varphi}_\lambda$, $\lambda = 1, \dots, n$, due to φ_λ being holomorphic, we get:

$$\begin{aligned} \bar{\partial} \varphi &= \sum_{\lambda, \nu=1}^n (\bar{\theta}_\nu f_\lambda) \bar{\varphi}_\nu \wedge \bar{\varphi}_\lambda + \sum_{\mu=1}^n f_\mu d \bar{\varphi}_\mu \\ &= \sum_{\lambda, \nu=1}^n (\bar{\theta}_\nu f_\lambda) \bar{\varphi}_\nu \wedge \bar{\varphi}_\lambda + \sum_{\mu=1}^n f_\mu \sum_{1 \leq \nu < \lambda \leq n} \overline{c_{\mu\nu\lambda}} \bar{\varphi}_\nu \wedge \bar{\varphi}_\lambda \tag{3.16} \\ &= \sum_{1 \leq \nu < \lambda \leq n} \left(\bar{\theta}_\nu f_\lambda - \bar{\theta}_\lambda f_\nu + \sum_{\mu=1}^n \overline{c_{\mu\nu\lambda}} f_\mu \right) \bar{\varphi}_\nu \wedge \bar{\varphi}_\lambda, \end{aligned}$$

where the second line above follows from the conjugate of (3.15).

Now φ is Δ'' -harmonic if and only if

$$(i) \quad \bar{\partial}\varphi = 0 \iff \bar{\theta}_\nu f_\lambda - \bar{\theta}_\lambda f_\nu + \sum_{\mu=1}^n \overline{c_{\mu\nu\lambda}} f_\mu = 0 \quad \text{for } 1 \leq \nu < \lambda \leq n \text{ (cf. (3.16));}$$

and

$$(ii) \quad \bar{\partial}^*\varphi = 0 \iff \sum_{\lambda=1}^n \theta_\lambda f_\lambda = 0 \quad \text{(cf. (3.10)).}$$

Suppose that φ is Δ'' -harmonic. Then the above (i) reads:

$$\bar{\theta}_\lambda f_\nu = \sum_{\mu=1}^n \overline{c_{\mu\nu\lambda}} f_\mu + \bar{\theta}_\nu f_\lambda, \quad 1 \leq \nu < \lambda \leq n.$$

Summing over $\lambda = 1, \dots, n$ and using formula (3.8) for $\bar{\partial}$, we get

$$\bar{\partial} f_\nu = \sum_{\lambda=1}^n (\bar{\theta}_\lambda f_\nu) \bar{\varphi}_\lambda = \sum_{\lambda, \mu=1}^n \overline{c_{\mu\nu\lambda}} f_\mu \bar{\varphi}_\lambda + \sum_{\lambda=1}^n (\bar{\theta}_\nu f_\lambda) \bar{\varphi}_\lambda, \quad \nu = 1, \dots, n,$$

with the understanding that $c_{\mu\nu\lambda} = 0$ if $\nu \geq \lambda$. Now $\Delta'' f_\nu = \bar{\partial}^* \bar{\partial} f_\nu$ since f_ν is a function. Taking $\bar{\partial}^*$ on either side above and using formula (3.10) for $\bar{\partial}^*$, we get

$$\Delta'' f_\nu = - \sum_{\lambda, \mu=1}^n \theta_\lambda (\overline{c_{\mu\nu\lambda}} f_\mu) - \sum_{\lambda=1}^n \theta_\lambda (\bar{\theta}_\nu f_\lambda) \tag{3.17}$$

$$= - \sum_{\lambda, \mu=1}^n \overline{c_{\mu\nu\lambda}} \theta_\lambda f_\mu, \quad \text{for all } \nu = 1, \dots, n,$$

because $\theta_\lambda (\overline{c_{\mu\nu\lambda}} f_\mu) = \overline{c_{\mu\nu\lambda}} \theta_\lambda f_\mu$ due to $\overline{c_{\mu\nu\lambda}}$ being constant, while $\sum_{\lambda=1}^n \theta_\lambda f_\lambda = 0$ due to φ being Δ'' -harmonic (cf. (ii) or (3.10)).

Taking now $\nu = n$ in (3.18), we get $\Delta'' f_n = 0$ since $c_{\mu n \lambda} = 0$ for all μ, λ by Fact 3.9 and the obvious inequality $\mu \leq \nu = n$. Thus the compactness of X and the ellipticity of Δ'' yield

$$f_n \text{ is constant on } X \text{ if } \Delta''\varphi = 0. \tag{3.18}$$

Taking now $\nu = n - 1$ in (3.18) and using the fact that $\theta_\lambda f_n = 0$ for all λ (since f_n is constant by (3.18)), we get

$$\Delta'' f_{n-1} = - \sum_{\lambda=1}^n \left(\sum_{\mu=1}^{n-1} \overline{c_{\mu n-1 \lambda}} \theta_\lambda f_\mu \right) = 0 \quad \text{on } X,$$

since $c_{\mu n-1 \lambda} = 0$ for all $\mu = 1, \dots, n - 1$ and $\lambda = 1, \dots, n$ by Fact 3.9 and the obvious inequality $\mu \leq \nu = n - 1$. Hence we get

$$f_{n-1} \text{ is constant on } X \text{ if } \Delta''\varphi = 0. \tag{3.19}$$

We can now continue by decreasing induction on ν . Taking $\nu = n - 2$ in (3.18) and using the fact that $\theta_\lambda f_n = \theta_\lambda f_{n-1} = 0$ for all λ (since f_n is constant by (3.18) and f_{n-1} is constant by (3.19)), we get

$$\Delta'' f_{n-2} = - \sum_{\lambda=1}^n \left(\sum_{\mu=1}^{n-2} \overline{c_{\mu n-2\lambda}} \theta_\lambda f_\mu \right) = 0 \quad \text{on } X,$$

since $c_{\mu n-2\lambda} = 0$ for all $\mu = 1, \dots, n - 2$ and $\lambda = 1, \dots, n$ by Fact 3.9 and the obvious inequality $\mu \leq \nu = n - 2$. Hence we get

$$f_{n-2} \text{ is constant on } X \text{ if } \Delta''\varphi = 0. \tag{3.20}$$

Running a decreasing induction on ν , we get

$$f_\nu := C_\nu \text{ is constant on } X \text{ for all } \nu = 1, \dots, n \text{ if } \Delta''\varphi = 0. \tag{3.21}$$

We conclude that whenever $\Delta''\varphi = 0$ we have

$$\varphi = \sum_{\nu=1}^n C_\nu \bar{\varphi}_\nu \quad \text{with } C_\nu \text{ constant for all } \nu = 1, \dots, n.$$

On the other hand, since $\Delta''\varphi = 0$, we must have $\bar{\partial}\varphi = 0$ which amounts to $\sum_{\nu=1}^n C_\nu \bar{\partial}\bar{\varphi}_\nu = 0$. However, we know that $\bar{\partial}\bar{\varphi}_\nu = 0$ for all $\nu \in \{1, \dots, r\}$ (cf.

(3.12)), hence $\sum_{\nu=r+1}^n C_\nu \bar{\partial}\bar{\varphi}_\nu = 0$. Now the forms

$$\bar{\partial}\bar{\varphi}_\nu = d\bar{\varphi}_\nu = \sum_{\lambda,\mu} \overline{c_{\nu\lambda\mu}} \bar{\varphi}_\lambda \wedge \bar{\varphi}_\mu, \quad \nu = 1, \dots, n,$$

are linearly independent because $\bar{\varphi}_1, \dots, \bar{\varphi}_n$ are linearly independent at every point of X . Hence $C_\nu = 0$ for all $\nu = r + 1, \dots, n$. We get

$$\varphi = \sum_{\nu=1}^r C_\nu \bar{\varphi}_\nu \quad \text{with } C_\nu \text{ constant for all } \nu = 1, \dots, r.$$

Since φ has been chosen arbitrary in $\mathcal{H}_{\Delta''}^{0,1}(X, \mathbb{C})$, we have proved that the linearly independent forms $\bar{\varphi}_1, \dots, \bar{\varphi}_r \in \mathcal{H}_{\Delta''}^{0,1}(X, \mathbb{C})$ generate $\mathcal{H}_{\Delta''}^{0,1}(X, \mathbb{C})$. The proof of Kodaira's theorem 3.10 is complete. \square

When applying Observation 3.8 and Kodaira's Theorem 3.10 to the Iwasawa manifold (a compact *complex parallelisable nilmanifold* of dimension $n = 3$ having $r = 2$), we get the following classical fact:

Observation 3.11. For the Iwasawa manifold, we have

$$h^{1,0} = 3 \quad \text{and} \quad h^{0,1} = 2.$$

Since, on the other hand, the first Betti number is $b_1 = 4$ (see e.g. (3.4)), we see that $b_1 < h^{1,0} + h^{0,1}$. Thus the Frölicher spectral sequence of the Iwasawa manifold does not degenerate at E_1^9 . In particular, the $\partial\bar{\partial}$ -lemma does not hold on the Iwasawa manifold.

On the other hand, we have seen in Corollary 3.6 that the Iwasawa manifold is balanced, hence sG. Combined with Observation 3.11, this fact proves part (a) of Theorem 1.11. The same statement will be proved again in section 4 (see comments after Gauduchon’s Theorem 4.9) by a twistor space featuring as the central fibre in the Eastwood-Singer family [16].

(b) Now suppose that X is the *Iwasawa manifold*. In particular, X is a compact complex parallelisable nilmanifold of complex dimension 3. Let $\varphi_1 = dz_1, \varphi_2 = dz_2, \varphi_3 = dz_3 - z_1 dz_2$ be the holomorphic 1-forms on X defined in (3.2); they are linearly independent at every point of X . Since φ_1 and φ_2 are d -closed while φ_3 is not d -closed, $r = 2$ for the Iwasawa manifold. By Kodaira’s Theorem 3.10, the \mathbb{C} -vector space $H^{0,1}(X, \mathbb{C})$ has complex dimension 2 and is spanned by the Dolbeault cohomology classes $\{\bar{\varphi}_1\}$ and $\{\bar{\varphi}_2\}$. Let $\theta_1, \theta_2, \theta_3 \in H^0(X, \Omega_X^1)$ be the holomorphic vector fields dual to $\varphi_1, \varphi_2, \varphi_3$. They are given by

$$\theta_1 = \frac{\partial}{\partial z_1}, \quad \theta_2 = \frac{\partial}{\partial z_2} + z_1 \frac{\partial}{\partial z_3}, \quad \theta_3 = \frac{\partial}{\partial z_3} \tag{3.22}$$

and satisfy the relations

$$[\theta_1, \theta_2] = -[\theta_2, \theta_1] = \theta_3, \quad [\theta_1, \theta_3] = [\theta_2, \theta_3] = 0, \tag{3.23}$$

i.e. $[\theta_i, \theta_j] = 0$ whenever $\{i, j\} \neq \{1, 2\}$.

Since the holomorphic tangent bundle $T^{1,0}X$ is trivial and spanned by $\theta_1, \theta_2, \theta_3$, the cohomology group $H^{0,1}(X, T^{1,0}X)$ of $T^{1,0}X$ -valued $(0, 1)$ -forms on X is a \mathbb{C} -vector space of dimension 6 spanned by the classes of $\theta_i \bar{\varphi}_\lambda$:

$$H^{0,1}(X, T^{1,0}X) = \bigoplus_{1 \leq i \leq 3, 1 \leq \lambda \leq 2} \mathbb{C}\{\theta_i \bar{\varphi}_\lambda\}, \quad \dim_{\mathbb{C}} H^{0,1}(X, T^{1,0}X) = 6. \tag{3.24}$$

This will be seen to imply that the Kuranishi family of the Iwasawa manifold is a 6-parameter family.

(c) We now briefly recall a few basic facts in the deformation theory of compact complex manifolds in order to fix the notation. The standard reference is Kodaira’s

⁹ Historically, the Iwasawa manifold was the first example exhibited of a compact complex manifold with this property. It is even known to have $E_1 \neq E_2 = E_\infty$ in its Frölicher spectral sequence.

book [28]. Given a compact complex manifold X of complex dimension n , let $\bar{\partial}$ be the Cauchy-Riemann operator representing the complex structure $J = J_0$ of X and let z_1, \dots, z_n be local holomorphic coordinates about an arbitrary point on X . A deformation X_t of the complex structure of X is represented by a vector $(0, 1)$ -form

$$\psi(t) \in C^\infty(X, \Lambda^{0,1} T^* X \otimes T^{1,0} X)$$

in the following sense: the Cauchy-Riemann operator $\bar{\partial}_t$ representing the (almost) complex structure J_t of X_t is defined by

$$\bar{\partial}_t := \bar{\partial} - \psi(t).$$

Equivalently, a locally defined C^∞ function f on X satisfies the equivalence:

$$f \text{ is } J_t\text{-holomorphic} \quad \text{if and only if} \quad (\bar{\partial} - \psi(t))f = 0.$$

The integrability condition reads:

$$J_t \text{ is integrable} \quad \text{if and only if} \quad \bar{\partial}\psi(t) = \frac{1}{2}[\psi(t), \psi(t)] \tag{3.25}$$

$$\begin{aligned} &\text{if and only if the system of n PDE's } \bar{\partial}_t f = 0 \text{ has} \\ &\quad n \text{ linearly independent } C^\infty \text{ solutions in} \\ &\quad \text{a neighbourhood of any point of } X. \end{aligned} \tag{3.26}$$

Recall furthermore that the *Kodaira-Spencer map* of a holomorphic family $\pi : \mathcal{X} \rightarrow B \subset \mathbb{C}^m$ at $t = 0$ (having supposed that $0 \in B$) is

$$\rho_0 : T_0 B \rightarrow H^{0,1}(X, T^{1,0} X), \quad \frac{\partial}{\partial t} \mapsto -\frac{\partial \psi(t)}{\partial t} \Big|_{t=0} := \frac{\partial X_t}{\partial t} \Big|_{t=0}.$$

Thus $\frac{\partial X_t}{\partial t} \Big|_{t=0}$ denotes the infinitesimal deformation of X at $t = 0$. Recall now the fundamental Kuranishi Theorem of Existence:

Theorem 3.12 (Kuranishi [30]). *Given any compact complex manifold X , there exists a complete holomorphic family*

$$\pi : \mathcal{X} \rightarrow B \subset \Delta_\varepsilon := \{t \in \mathbb{C}^m ; |t| < \varepsilon\}$$

such that $X_0 = X$, for some small $\varepsilon > 0$ and some analytic subset B of the ball Δ_ε , where $m := \dim_{\mathbb{C}} H^{0,1}(X, T^{1,0} X)$.

Complete means that the Kodaira-Spencer map $\rho_0 : T_0 B \rightarrow H^{0,1}(X, T^{1,0} X)$ is surjective. This family is called the *Kuranishi family* of X . The base B of the family may have singularities and arises as

$$B = \{t \in \Delta_\varepsilon ; f_1(t) = \dots = f_l(t) = 0\},$$

where $l := \dim_{\mathbb{C}} H^{0,2}(X, T^{1,0}X)$ and $f_1, \dots, f_l : \Delta_\varepsilon \rightarrow \mathbb{C}$ are holomorphic functions. In the special case of a manifold X satisfying $H^{0,2}(X, T^{1,0}X) = 0$, the base B is smooth and Kuranishi's theorem reduces to the earlier Kodaira-Nirenberg-Spencer theorem of existence.

The construction of the Kuranishi family of a given X amounts to the construction of a family of vector $(0, 1)$ -forms $\psi(t) \in C^\infty(X, \Lambda^{0,1}T^*X \otimes T^{1,0}X)$ satisfying the integrability condition (3.25) for $t = (t_1, \dots, t_m)$ in the largest possible subset of some $\Delta_\varepsilon \subset \mathbb{C}^m$. Given an arbitrary basis $\{\beta_1, \dots, \beta_m\}$ of $H^{0,1}(X, T^{1,0}X)$, set

$$\psi_1(t) := t_1\beta_1 + \dots + t_m\beta_m \in H^{0,1}(X, T^{1,0}X). \tag{3.27}$$

Identifying $H^{0,1}(X, T^{1,0}X)$ with $\mathcal{H}_{\Delta''}^{0,1}(X, T^{1,0}X)$ by the Hodge isomorphism, we see that $\Delta''\psi_1(t) = 0$ (i.e. $\psi_1(t)$ is Δ'' -harmonic).

Since ψ_1 need not satisfy the integrability condition (3.25), we search for a power series

$$\psi(t) = \psi_1(t) + \sum_{v=2}^{+\infty} \psi_v(t), \tag{3.28}$$

where $\psi_v(t) = \sum_{v_1+\dots+v_m=v} \psi_{v_1\dots v_m} t_1^{v_1} \dots t_m^{v_m}$ is a homogeneous polynomial of degree v in t_1, \dots, t_m whose coefficients $\psi_{v_1\dots v_m} \in C^\infty(X, \Lambda^{0,1}T^*X \otimes T^{1,0}X)$ will be determined such that the following two conditions are fulfilled:

- The power series defining $\psi(t)$ converges and its sum is C^∞ on $X \times \Delta_\varepsilon$ for some small $\varepsilon > 0$;

Kuranishi's proof achieves convergence in a Hölder norm $\|\cdot\|_{k,\alpha}$ for all $k \geq 2$ and all $t \in \Delta_\varepsilon$ provided that $\varepsilon > 0$ is small enough.

- The integrability condition $\bar{\partial}\psi(t) = \frac{1}{2}[\psi(t), \psi(t)]$ holds (cf. (3.25)).

To fulfill the integrability condition (3.25), it suffices to ensure that the vector $(0, 2)$ -form $[\psi(t), \psi(t)] \in C^\infty(X, \Lambda^{0,2}T^*X \otimes T^{1,0}X)$ is $\bar{\partial}$ -exact and that $\psi(t) - \psi_1(t)$ is the minimal L^2 -norm solution of equation $\bar{\partial}u = \frac{1}{2}[\psi(t), \psi(t)]$ (recall that $\bar{\partial}\psi_1(t) = 0$). Minimality of the solution's L^2 -norm translates to

$$\psi(t) = \psi_1(t) + \frac{1}{2}\bar{\partial}^*\Delta''^{-1}[\psi(t), \psi(t)],$$

a formula that is easily seen to be equivalent to

$$\psi_v(t) = \frac{1}{2} \sum_{\mu=1}^{v-1} \bar{\partial}^*\Delta''^{-1}[\psi_\mu(t), \psi_{v-\mu}(t)] \quad \text{for all } v \geq 2. \tag{3.29}$$

This means that $\psi_v(t)$ is the minimal L^2 -norm solution of the equation $\bar{\partial}u = v_v$, where v_v is the projection of $1/2 \sum_{1 \leq \mu \leq v-1} [\psi_\mu(t), \psi_{v-\mu}(t)]$ onto $\text{Im } \bar{\partial}$. In particular, $\psi_v(t) \in \text{Im } \bar{\partial}^*$ for all $v \geq 2$.

Identities (3.29) allow one to construct $\psi_\nu(t)$, $\nu \geq 2$, inductively from $\psi_1(t)$ defined in (3.27). Convergence in Hölder norm $\|\cdot\|_{k,\alpha}$ of the resulting series (3.28) follows from *a priori estimates* on the Laplacian Δ'' , while the integrability condition (3.25) for the sum $\psi(t)$ of this series is seen to be equivalent to

$$H[\psi(t), \psi(t)] = 0, \tag{3.30}$$

where $H : C^\infty(X, \Lambda^{0,2}T^*X \otimes T^{1,0}X) \rightarrow \mathcal{H}_{\Delta''}^{0,2}(X, T^{1,0}X)$ is the harmonic projector. Condition (3.30) requires $[\psi(t), \psi(t)]$ to have no harmonic component which, for a $\bar{\partial}$ -closed form, is equivalent to $\bar{\partial}$ -exactness (precisely what is needed in view of (3.25)). If $\{\gamma_1, \dots, \gamma_l\}$ is any orthonormal basis of $\mathcal{H}_{\Delta''}^{0,2}(X, T^{1,0}X)$, then

$$H[\psi(t), \psi(t)] = \sum_{k=1}^l \langle [\psi(t), \psi(t)], \gamma_k \rangle \gamma_k, \quad t \in \Delta_\varepsilon,$$

and we see that the vanishing condition (3.30) is equivalent to $f_1(t) = \dots = f_l(t) = 0$, where $f_k(t) := \langle [\psi(t), \psi(t)], \gamma_k \rangle$ for all $k = 1, \dots, l$ and $t \in \Delta_\varepsilon$. Thus the integrability condition (3.25) is satisfied for $t \in B$, where

$$B := \{t \in \Delta_\varepsilon ; f_1(t) = \dots = f_l(t) = 0\} \subset \Delta_\varepsilon \tag{3.31}$$

is analytic.

(d) *Nakamura’s calculation of the Kuranishi family of the Iwasawa manifold*

Given a compact complex manifold X of dimension n , for any vector $(0, 1)$ -forms $\psi, \tau \in C^\infty(X, \Lambda^{0,1}T^*X \otimes T^{1,0}X)$ written locally as

$$\psi = \sum_{\alpha=1}^n \psi^\alpha \frac{\partial}{\partial z_\alpha}, \quad \tau = \sum_{\beta=1}^n \tau^\beta \frac{\partial}{\partial z_\beta}, \quad \psi^\alpha, \tau^\beta \in C^\infty(X, T^{1,0}X),$$

Kuranishi defines in general

$$[\psi, \tau] := \sum_{\alpha, \beta=1}^n \left(\psi^\alpha \wedge \frac{\partial \tau^\beta}{\partial z_\alpha} + \tau^\alpha \wedge \frac{\partial \psi^\beta}{\partial z_\alpha} \right) \frac{\partial}{\partial z_\beta}.$$

Now fix $X = \mathbb{C}^3 / \Gamma$ to be the Iwasawa manifold. Then $n = 3$ and we get:

$$[\theta_i \bar{\varphi}_\lambda, \theta_k \bar{\varphi}_\nu] = [\theta_i, \theta_k] \bar{\varphi}_\lambda \wedge \bar{\varphi}_\nu, \quad i, k, \lambda, \nu = 1, 2, 3, \tag{3.32}$$

with $[\theta_i, \theta_k]$ given in (3.23).

We have seen in (3.24) that the classes $\{\theta_i \bar{\varphi}_\lambda\}$, with $1 \leq i \leq 3, 1 \leq \lambda \leq 2$, form a basis of $H^{0,1}(X, T^{1,0}X)$. Consequently the Kuranishi family of X can be described by 6 parameters $t = (t_{i\lambda})_{1 \leq i \leq 3, 1 \leq \lambda \leq 2}$. By (3.13), the $T^{1,0}X$ -valued $(0, 1)$ -forms $\theta_i \bar{\varphi}_\lambda$ are Δ'' -harmonic when $1 \leq \lambda \leq 2$. In order to construct the

vector $(0, 1)$ -forms $\psi(t) \in C^\infty(X, \Lambda^{0,1} T^* X \otimes T^{1,0} X)$ that describe the Kuranishi family of $X = \mathbb{C}^3 / \Gamma$, formula (3.27) prescribes to start off by setting

$$\psi_1(t) := \sum_{i=1}^3 \sum_{\lambda=1}^2 t_{i\lambda} \theta_i \bar{\varphi}_\lambda, \quad t = (t_{i\lambda})_{1 \leq i \leq 3, 1 \leq \lambda \leq 2}, \tag{3.33}$$

for which we see that

$$\frac{1}{2} [\psi_1(t), \psi_1(t)] = \frac{1}{2} \sum_{i,j=1,2,3} \sum_{\lambda,\mu=1,2} t_{i\lambda} t_{j\mu} [\theta_i, \theta_j] \bar{\varphi}_\lambda \wedge \bar{\varphi}_\mu.$$

By (3.23), this translates to

$$\begin{aligned} \frac{1}{2} [\psi_1(t), \psi_1(t)] &= \frac{1}{2} (t_{11} t_{22} \theta_3 \bar{\varphi}_1 \wedge \bar{\varphi}_2 + t_{12} t_{21} \theta_3 \bar{\varphi}_2 \wedge \bar{\varphi}_1 \\ &\quad - t_{21} t_{12} \theta_3 \bar{\varphi}_1 \wedge \bar{\varphi}_2 - t_{22} t_{11} \theta_3 \bar{\varphi}_2 \wedge \bar{\varphi}_1). \end{aligned}$$

Since $\bar{\varphi}_1 \wedge \bar{\varphi}_2 = -\bar{\varphi}_2 \wedge \bar{\varphi}_1$, we get

$$\frac{1}{2} [\psi_1(t), \psi_1(t)] = (t_{11} t_{22} - t_{12} t_{21}) \theta_3 \bar{\varphi}_1 \wedge \bar{\varphi}_2. \tag{3.34}$$

On the other hand, for the choice (3.33) we see that

$$\bar{\partial} \psi_1(t) = d\psi_1(t) = \sum_{i=1}^3 \sum_{\lambda=1}^2 t_{i\lambda} \theta_i d\bar{\varphi}_\lambda = 0 \tag{3.35}$$

since $d\bar{\varphi}_1 = d\bar{\varphi}_2 = 0$. Now setting

$$\psi_2(t) := -(t_{11} t_{22} - t_{12} t_{21}) \theta_3 \bar{\varphi}_3, \tag{3.36}$$

and using (3.3) and (3.34), we find

$$\begin{aligned} \bar{\partial} \psi_2(t) &= d\psi_2(t) = (t_{11} t_{22} - t_{12} t_{21}) \theta_3 (-d\bar{\varphi}_3) \\ &= (t_{11} t_{22} - t_{12} t_{21}) \theta_3 \bar{\varphi}_1 \wedge \bar{\varphi}_2 = \frac{1}{2} [\psi_1(t), \psi_1(t)]. \end{aligned} \tag{3.37}$$

In particular, $[\psi_1(t), \psi_1(t)]$ is seen to be $\bar{\partial}$ -exact here (although it need not be so in the case of an arbitrary manifold, see comments after (3.29)), but the solution $\psi_2(t)$ of equation (3.37) need not be of minimal L^2 -norm (unlike the $\psi_2(t)$ defined in the case of a general manifold by formula (3.29) for $\nu = 2$). In other words, in the special case of the Iwasawa manifold, a solution $\psi_2(t)$ of (3.37) is easily observed and we are spared the application of the general formulae (3.29). This readily yields the desired $\psi(t)$ by setting

$$\psi(t) := \psi_1(t) + \psi_2(t) = \sum_{i=1}^3 \sum_{\lambda=1}^2 t_{i\lambda} \theta_i \bar{\varphi}_\lambda - (t_{11} t_{22} - t_{12} t_{21}) \theta_3 \bar{\varphi}_3, \tag{3.38}$$

for which we find

$$\frac{1}{2}[\psi(t), \psi(t)] = \sum_{j,k=1}^2 \frac{1}{2}[\psi_j(t), \psi_k(t)] = \frac{1}{2}[\psi_1(t), \psi_1(t)]. \tag{3.39}$$

Indeed, $[\psi_j(t), \psi_k(t)] = 0$ for all $(i, j) \neq (1, 1)$ since these terms involve only brackets of the shape $[\theta_3, \theta_i] = 0$ and $[\theta_i, \theta_3] = 0$ which vanish by (3.23).

On the other hand, combining (3.35) and (3.37), we get

$$\bar{\partial}\psi(t) = \bar{\partial}\psi_1(t) + \bar{\partial}\psi_2(t) = \bar{\partial}\psi_2(t) = \frac{1}{2}[\psi_1(t), \psi_1(t)]. \tag{3.40}$$

Then (3.39) and (3.40) yield

$$\bar{\partial}\psi(t) = \frac{1}{2}[\psi(t), \psi(t)], \tag{3.41}$$

showing that $\psi(t)$ defined in (3.38) satisfies the integrability condition (3.25).

By Kuranishi’s Theorem 3.12, this $T^{1,0}X$ -valued $(0, 1)$ -form $\psi(t)$ defines a *locally complete* complex analytic family of deformations X_t of X depending on 6 *effective* parameters $t = (t_{i\lambda})_{1 \leq i \leq 3, 1 \leq \lambda \leq 2}$ such that the complex structure of each fibre X_t is defined by $\bar{\partial}_t := \bar{\partial} - \psi(t)$ and $X_0 = X = \mathbb{C}^3 / \Gamma$ is the Iwasawa manifold. It is noteworthy that in the special case of the Iwasawa manifold, the power series (3.28) can be built with only two terms ($\psi_1(t)$ and $\psi_2(t)$) and the above simple calculations show $\psi(t) = \psi_1(t) + \psi_2(t)$ to satisfy the integrability condition (3.25) for all $t = (t_{i\lambda})_{1 \leq i \leq 3, 1 \leq \lambda \leq 2} \in \Delta_\varepsilon \subset \mathbb{C}^6$ if $\varepsilon > 0$ is small. With the notation of (3.31), this means that $B = \Delta_\varepsilon$.

Nakamura goes on to calculate holomorphic coordinates $\zeta_1 = \zeta_1(t)$, $\zeta_2 = \zeta_2(t)$, $\zeta_3 = \zeta_3(t)$ on X_t such that $\zeta_\nu(0) = z_\nu$ for $\nu = 1, 2, 3$ starting from arbitrary holomorphic coordinates z_1, z_2, z_3 given beforehand on the Iwasawa manifold $X_0 = X = \mathbb{C}^3 / \Gamma$. Here is the way he proceeds.

We are looking for C^∞ functions $\zeta_\nu(t)$, $\nu = 1, 2, 3$, on X satisfying the holomorphicity condition

$$\bar{\partial}_t \zeta_\nu(t) = 0 \iff \bar{\partial} \zeta_\nu(t) - \psi(t) \zeta_\nu(t) = 0, \quad \nu = 1, 2, 3. \tag{3.42}$$

Given the definition (3.38) of $\psi(t)$ and the formulae (3.22) for $\theta_1, \theta_2, \theta_3$, condition (3.42) reads for $\nu = 1, 2, 3$:

$$\begin{aligned} \bar{\partial} \zeta_\nu - \sum_{\lambda=1}^2 t_{1\lambda} \frac{\partial \zeta_\nu}{\partial z_1} d\bar{z}_\lambda - \sum_{\lambda=1}^2 t_{2\lambda} \left(\frac{\partial \zeta_\nu}{\partial z_2} + z_1 \frac{\partial \zeta_\nu}{\partial z_3} \right) d\bar{z}_\lambda \\ - \sum_{\lambda=1}^2 t_{3\lambda} \frac{\partial \zeta_\nu}{\partial z_3} d\bar{z}_\lambda + (t_{11}t_{22} - t_{12}t_{21}) \frac{\partial \zeta_\nu}{\partial z_3} (d\bar{z}_3 - \bar{z}_1 d\bar{z}_2) = 0. \end{aligned} \tag{3.43}$$

For $\nu = 1$, we arrange to have $\frac{\partial \zeta_1}{\partial z_1} = 1$ (in order to get $\zeta_1(t) = z_1 + (\text{terms depending only on } \bar{z}_\lambda)$) and $\frac{\partial \zeta_1}{\partial z_2} = \frac{\partial \zeta_1}{\partial z_3} = 0$. With these choices, condition (3.43) for $\nu = 1$ becomes:

$$\bar{\partial} \zeta_1 - \sum_{\lambda=1}^2 t_{1\lambda} \bar{\partial} \bar{z}_\lambda = 0 \iff \bar{\partial} \zeta_1(t) = \bar{\partial} \left(\sum_{\lambda=1}^2 t_{1\lambda} \bar{z}_\lambda \right).$$

Thus we can take

$$\zeta_1(t) = z_1 + \sum_{\lambda=1}^2 t_{1\lambda} \bar{z}_\lambda. \tag{3.44}$$

For $\nu = 2$, we similarly require $\frac{\partial \zeta_2}{\partial z_2} = 1$ and $\frac{\partial \zeta_2}{\partial z_1} = \frac{\partial \zeta_2}{\partial z_3} = 0$ and condition (3.43) for $\nu = 2$ similarly yields:

$$\zeta_2(t) = z_2 + \sum_{\lambda=1}^2 t_{2\lambda} \bar{z}_\lambda. \tag{3.45}$$

For $\nu = 3$, we require $\frac{\partial \zeta_3}{\partial z_3} = 1$, $\frac{\partial \zeta_3}{\partial z_2} = 0$ and $\frac{\partial \zeta_3}{\partial z_1} = \sum_{\lambda=1}^2 t_{2\lambda} \bar{z}_\lambda$. With these choices, (3.43) for $\nu = 3$ reads

$$\begin{aligned} \bar{\partial} \zeta_3 - \left(\sum_{\lambda=1}^2 t_{1\lambda} d\bar{z}_\lambda \right) \left(\sum_{\lambda=1}^2 t_{2\lambda} \bar{z}_\lambda \right) - z_1 \sum_{\lambda=1}^2 t_{2\lambda} d\bar{z}_\lambda \\ - \sum_{\lambda=1}^2 t_{3\lambda} d\bar{z}_\lambda + (t_{11}t_{22} - t_{12}t_{21})(d\bar{z}_3 - \bar{z}_1 d\bar{z}_2) = 0. \end{aligned}$$

We thus get

$$\zeta_3(t) = z_3 + \sum_{\lambda=1}^2 (t_{3\lambda} + t_{2\lambda} z_1) \bar{z}_\lambda + A(t, \bar{z}) - D(t) \bar{z}_3, \tag{3.46}$$

where we have denoted $A(t, \bar{z}) := \frac{1}{2}(t_{11}t_{21}\bar{z}_1^2 + 2t_{11}t_{22}\bar{z}_1\bar{z}_2 + t_{12}t_{22}\bar{z}_2^2)$ and $D(t) := (t_{11}t_{22} - t_{12}t_{21})$. We clearly have

$$d\zeta_1(t) \wedge d\zeta_2(t) \wedge d\zeta_3(t) = C(t) dz_1 \wedge dz_2 \wedge dz_3$$

for a constant $C(t)$ depending in a C^∞ way on t such that $C(0) = 1$. Hence $\zeta_1(t), \zeta_2(t), \zeta_3(t)$ define holomorphic coordinates on X_t for all $t = (t_{i\lambda})_{1 \leq i \leq 3, 1 \leq \lambda \leq 2}$ such that $\sum_{i=1,2,3; \lambda=1,2} |t_{i\lambda}| < \varepsilon$ if $\varepsilon > 0$ is small enough.

3.4. The example of Alessandrini and Bassanelli proving Theorem 3.3

In the 6-parameter Kuranishi family $(X_t)_{t \in B}$, with $t = (t_{i\lambda})_{1 \leq i \leq 3, 1 \leq \lambda \leq 2}$, of the Iwasawa manifold $X_0 = X = \mathbb{C}^3 / \Gamma$, Alessandrini and Bassanelli [2] single out the direction corresponding to parameters t such that

$$t_{12} \neq 0, \quad t_{ij} = 0 \quad \text{for all } (i, j) \neq (1, 2). \tag{3.47}$$

With this choice of t , they have

$$A(t, \bar{z}) = 0 \quad \text{and} \quad D(t) = 0.$$

Thus, setting $t := t_{12}$, the holomorphic coordinates of X_t calculated in (3.44), (3.45) and (3.46) reduce to

$$\zeta_1(t) = z_1 + t\bar{z}_2, \quad \zeta_2(t) = z_2, \quad \zeta_3(t) = z_3. \tag{3.48}$$

Implicitly $z_1 = \zeta_1(t) - t\bar{\zeta}_2(t)$, which yields

$$\varphi_3(t) := dz_3 - z_1 dz_2 = d\zeta_3(t) + (t\bar{\zeta}_2(t) - \zeta_1(t)) d\zeta_2(t), \tag{3.49}$$

$$\varphi_2(t) := dz_2 = d\zeta_2(t), \quad \tilde{\varphi}_1(t) := dz_1 = d\zeta_1(t) - t d\bar{\zeta}_2(t).$$

Set

$$\varphi_1(t) := d\zeta_1(t). \tag{3.50}$$

The above 1-forms $\varphi_1(t), \varphi_2(t), \varphi_3(t)$ are all of J_t -type $(1, 0)$ since $\zeta_1(t), \zeta_2(t), \zeta_3(t)$ are holomorphic coordinates for the complex structure J_t of X_t .

Proposition 3.13 (Alessandrini-Bassanelli [2, page 1062]). *Let $(X_t)_t$ be the Kuranishi family of the Iwasawa manifold $X = X_0$, $t = (t_{i\lambda})_{1 \leq i \leq 3, 1 \leq \lambda \leq 2}$. Then, for parameters such that $t_{i\lambda} = 0$ for all $(i, \lambda) \neq (1, 2)$, X_t is not balanced for any $t := t_{12} \neq 0$ satisfying $|t_{12}| < \varepsilon$ if $\varepsilon > 0$ is small enough.*

Proof. For the forms defined in (3.49) and (3.50), an immediate calculation shows

$$d\varphi_3(t) = (t d\bar{\zeta}_2(t) - d\zeta_1(t)) \wedge d\zeta_2(t) = -t \varphi_2(t) \wedge \bar{\varphi}_2(t) - \varphi_1(t) \wedge \varphi_2(t). \tag{3.51}$$

Thus the 2-form $d\varphi_3(t)$ has two components: $-t \varphi_2(t) \wedge \bar{\varphi}_2(t)$ is of J_t -type $(1, 1)$, while $-\varphi_1(t) \wedge \varphi_2(t)$ is of J_t -type $(2, 0)$. Recall that $\dim_{\mathbb{C}} X_t = 3$ for all t . Suppose that X_t were balanced for some $t = t_{12} \neq 0$ satisfying $|t_{12}| < \varepsilon$ with $\varepsilon > 0$ small. Then there would exist a balanced metric $\omega_t > 0$ on X_t . Thus $\Omega_t := \omega_t^2$ would be a C^∞ $(2, 2)$ -form on X_t satisfying

$$\Omega_t > 0, \quad d\Omega_t = 0. \tag{3.52}$$

In this case we would have:

$$0 = \int_{X_t} d\Omega_t \wedge i\bar{t}\varphi_3(t) = - \int_{X_t} \Omega_t \wedge i\bar{t} d\varphi_3(t) = |t|^2 \int_{X_t} \Omega_t \wedge i\varphi_2(t) \wedge \overline{\varphi_2(t)}. \tag{3.53}$$

Indeed, the first identity above follows from $d\Omega_t = 0$ (cf. (3.52)), the second one follows from Stokes's theorem, while the third one follows from formula (3.51) for $d\varphi_3(t)$ and the fact that the $(2, 0)$ -component $-\varphi_1(t) \wedge \varphi_2(t)$ is annihilated when wedged with the $(2, 2)$ -form Ω_t .

Now $\Omega_t > 0$ and $i\varphi_2(t) \wedge \overline{\varphi_2(t)} \geq 0$, hence $\Omega_t \wedge i\varphi_2(t) \wedge \overline{\varphi_2(t)} \geq 0$ at every point of X_t . It follows that the right-hand term in (3.53) is non-negative. However, since it must vanish by the first identity in (3.53), the $(3, 3)$ -form $\Omega_t \wedge i\varphi_2(t) \wedge \overline{\varphi_2(t)}$ must vanish identically on X_t , hence so must the $(1, 1)$ -form $i\varphi_2(t) \wedge \overline{\varphi_2(t)}$. This can only happen if $\varphi_2(t)$ vanishes identically on X_t , which is impossible since $\varphi_2(t) = d\zeta_2(t)$ and $\zeta_2(t)$ is a holomorphic coordinate on X_t if ε is small enough. This provides the desired contradiction.

Therefore X_t cannot be balanced for any $t = t_{12} \neq 0$ if $t_{i\lambda} = 0$ for all $(i, \lambda) \neq (1, 2)$ and $\varepsilon > 0$ is small. The proof is complete. \square

It is by means of this Proposition 3.13 that Alessandrini and Bassanelli proved Theorem 3.3: they observed that the fibres along one particular direction among the 6 directions available in the base space of the Kuranishi family of the Iwasawa manifold prove the non-openness of the balanced property under holomorphic deformations.

We now make the following:

Observation 3.14. (Implicit in [37]) In the Kuranishi family of the Iwasawa manifold, the Frölicher spectral sequence does not degenerate at E_1 (hence the $\partial\bar{\partial}$ -lemma does not hold) on any fibre X_t corresponding to parameters such that $t_{i\lambda} = 0$ for all $(i, \lambda) \neq (1, 2)$ and $t := t_{12}$ satisfies $|t_{12}| < \varepsilon$ with $\varepsilon > 0$ small enough.

Proof. We have seen in (3.5) that the second Betti number of the Iwasawa manifold is $b_2 = 8$. By the C^∞ -triviality of the family, all the fibres have the same Betti numbers. On the other hand, Nakamura concludes from his calculations reproduced above (via standard reasoning like that exemplified above between formulae (3.3) and (3.5)) that the Hodge numbers of weight 2 of any fibre X_t corresponding to parameters such that $t_{i\lambda} = 0$ for all $(i, \lambda) \neq (1, 2)$ and $t := t_{12} \neq 0$ are (see [37, table on page 96] for the case (ii) when $D(t) = 0$ and $(t_{11}, t_{12}, t_{21}, t_{22}) \neq (0, 0, 0, 0)$)

$$h^{2,0}(t) = h^{0,2}(t) = 2, \quad h^{1,1}(t) = 5.$$

Thus we see that for any such fibre X_t with $t := t_{12} \neq 0$, we have:

$$b_2 = 8 < h^{2,0}(t) + h^{1,1}(t) + h^{0,2}(t) = 9.$$

Hence the conclusion follows. Notice that for fibres as above with $t := t_{12} \neq 0$, Nakamura's table gives $b_1 = 4 = 2 + 2 = h^{1,0}(t) + h^{0,1}(t)$, while we have seen in Observation 3.11 that for X_0 we have $b_1 = 4 < 3 + 2 = h^{1,0}(0) + h^{0,1}(0)$. \square

3.5. The desired examples of sG manifolds

We can now conclude this section by exhibiting the desired examples of sG manifolds showing the difference between, on one hand, the sG property and, on the other hand, the (combined) balanced and $\partial\bar{\partial}$ -lemma properties. It suffices to bring together Theorem 3.1, Theorem 3.3 and Observation 3.14:

Theorem 3.15. *Let $(X_t)_t$ be the Kuranishi family of the Iwasawa manifold $X = X_0$, $t = (t_{i\lambda})_{1 \leq i \leq 3, 1 \leq \lambda \leq 2}$. Then, for parameters such that $t_{i\lambda} = 0$ for all $(i, \lambda) \neq (1, 2)$, X_t is a strongly Gauduchon manifold that is not balanced and whose Frölicher spectral sequence does not degenerate at E_1 (hence the $\partial\bar{\partial}$ -lemma does not hold) for any $t = t_{12} \neq 0$ satisfying $|t_{12}| < \varepsilon$ if $\varepsilon > 0$ is small enough.*

Proof. Since the Iwasawa manifold is balanced (cf. Corollary 3.6), it is also an sG manifold. Since the sG property is open under holomorphic deformations (cf. Theorem 3.1), all sufficiently nearby fibres X_t in the Kuranishi family of the Iwasawa manifold X_0 are again sG manifolds. However, by the observation of Alessandrini and Bassanelli (cf. Proposition 3.13), the fibres X_t corresponding to parameters for which $t_{i\lambda} = 0$ for all $(i, \lambda) \neq (1, 2)$ are not balanced if $t := t_{12} \neq 0$ is sufficiently close to 0. By Observation 3.14, the Frölicher spectral sequence does not degenerate at E_1 , hence the $\partial\bar{\partial}$ -lemma does not hold, on any of these fibres. \square

4. The Eastwood-Singer construction

In Subsection 4.2 of this section we give an outline of the Eastwood-Singer proof [16] of part (b) of Theorem 1.24 asserting the non-closedness of the Frölicher degeneration at E_1 under holomorphic deformations. All the fibres of the holomorphic family they construct in their example are twistor spaces (hence in particular compact complex 3-folds). We also outline in 4.3 the proof given in [49, Theorem 1.25]. We conclude by raising a few natural questions in 4.4 about other deformational stability properties of manifolds satisfying the $\partial\bar{\partial}$ -lemma to which twistor spaces might hold the key.

4.1. Background on twistor spaces

We briefly recall the barest essentials of Penrose's *twistor space* theory [40] for which the standard mathematical reference is [1]. Given a compact oriented Riemannian manifold (M, g) of real dimension 4, for every point $x \in M$ one defines L_x to be the set of all complex structures J_x on the tangent space $T_x M$ such that J_x is orthogonal with respect to g_x and J_x induces the negative (*i.e.* opposite to the given one) orientation on $T_x M$. Orthogonality with respect to g_x for a complex structure $J_x : T_x M \rightarrow T_x M$ means, as usual, that

$$g_x(J_x u, J_x v) = g_x(u, v), \quad \text{for all } u, v \in T_x M,$$

i.e. J_x is required to be a g_x -isometry of $T_x M$. When x varies in M , the union Z (which depends only on the conformal class $[g]$ of Riemannian metrics on M but not on the actual representative g) of all twistor lines L_x has a natural structure as a C^∞ manifold of real dimension 6 and a natural almost complex structure. The almost complex structure is *integrable* if and only if the conformal structure $[g]$ of M is *self-dual*. Recall that for an arbitrary-dimensional Riemannian manifold (M, g) , the curvature tensor R decomposes as

$$R = W + \rho,$$

where W is the Weyl tensor depending only on the conformal class $[g]$. Peculiar to the case when M has real dimension 4 is a further decomposition of the Weyl tensor as

$$W = W_+ + W_-,$$

where W_+ is the *self-dual* component and W_- is the *anti-self-dual* component. Reversing the orientation of M permutes W_+ and W_- . The conformal structure $[g]$ of M is said to be *self-dual* if the associated Weyl tensor reduces to its *self-dual* component W_+ (*i.e.* $W_- = 0$).

One thus gets the Penrose correspondence between *self-dual* compact connected oriented C^∞ real 4-manifolds M and the associated twistor spaces Z (which are compact complex 3-manifolds). The natural projection $\nu : Z \rightarrow M$ can be identified with the unit sphere subbundle of the rank-three real vector bundle of anti-self-dual 2-forms on M . Every twistor line $\nu^{-1}(x) = L_x$ is isomorphic to the complex projective line \mathbb{P}^1 .

For every r , denote by $\Lambda^r := C_r^\infty(M, \mathbb{C})$ the space of C^∞ complex-valued r -forms on M . Recall that the Hodge star operator of the Riemannian metric g of M acting on 2-forms

$$\star : \Lambda^2 \rightarrow \Lambda^2$$

satisfies $\star^2 = 1$. Hence it induces a direct-sum splitting

$$\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$$

into its ± 1 -eigenspaces. The 2-forms $u \in \Lambda_+^2$ (*i.e.* $\star u = u$) are termed *self-dual*, while the 2-forms $u \in \Lambda_-^2$ (*i.e.* $\star u = -u$) are termed *anti-self-dual*. If the Weyl curvature tensor W is viewed as a bundle-valued 2-form, its components satisfy $\star W_+ = W_+$ and respectively $\star W_- = -W_-$. One gets a corresponding splitting of the differential operator d acting on 1-forms of M :

$$\Lambda^1 \xrightarrow{d=d_++d_-} \Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$$

that induces cohomology groups $H_+^2(M, \mathbb{C})$ and $H_-^2(M, \mathbb{C})$.

4.2. The results of Eastwood and Singer

For a fixed compact connected oriented C^∞ manifold M of real dimension 4 endowed with a *self-dual* Riemannian metric g , Eastwood and Singer establish the following general facts about the associated twistor space Z ; the details may be found in [16, Section 2, 3, 4.]:

Fact 1. Given that the sheaves Ω^1, Ω^2 and Ω^3 of germs of holomorphic 1, 2 and respectively 3-forms on Z are explicitly given, on every twistor line $L = L_x = \mathbb{P}^1$, by the formulae

$$\begin{aligned} \Omega^1|_L &\simeq \mathcal{O}(-2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1), \\ \Omega^2|_L &\simeq \mathcal{O}(-3) \oplus \mathcal{O}(-3) \oplus \mathcal{O}(-2), \\ \Omega^3|_L &\simeq \mathcal{O}(-4), \end{aligned}$$

we see that the restricted bundles $\Omega^1|_L, \Omega^2|_L, \Omega^3|_L$ have no non-trivial sections over L . Since Z is fibred by projectives lines $L = L_x = \mathbb{P}^1$, it follows that the vector bundles $\Omega^1, \Omega^2, \Omega^3$ have no non-trivial global holomorphic sections over Z , hence

$$H^{1,0}(Z, \mathbb{C}) = 0, \quad H^{2,0}(Z, \mathbb{C}) = 0, \quad H^{3,0}(Z, \mathbb{C}) = 0. \tag{4.1}$$

By Serre’s duality, one also gets

$$H^{2,3}(Z, \mathbb{C}) = 0, \quad H^{1,3}(Z, \mathbb{C}) = 0, \quad H^{0,3}(Z, \mathbb{C}) = 0. \tag{4.2}$$

Since Z and M are compact, one infers that

$$H^{0,0}(Z, \mathbb{C}) = H^0(M, \mathbb{C}) = \mathbb{C} \quad \text{and} \quad H^{3,3}(Z, \mathbb{C}) = H^4(M, \mathbb{C}) = \mathbb{C}, \tag{4.3}$$

where the latter set of identities follows from the former by Serre duality on Z and Poincaré duality on M .

Fact 2. The Penrose transform relating analytic cohomology on the twistor space Z to solutions of differential equations on the base manifold M is used to see the following:

- For all $q = 0, 1, 2, 3$, $H^{0,q}(Z, \mathbb{C})$ is canonically isomorphic to the q^{th} cohomology of the complex

$$\Lambda^0 \longrightarrow \Lambda^1 \xrightarrow{d_-} \Lambda^2_- \longrightarrow 0;$$

- For all $q = 0, 1, 2, 3$, $H^{3,q}(Z, \mathbb{C})$ is canonically isomorphic to the q^{th} cohomology of the complex

$$0 \longrightarrow \Lambda^2_- \xrightarrow{d} \Lambda^3 \longrightarrow \Lambda^4.$$

It follows as a corollary that

$$\begin{aligned} H^{0,1}(Z, \mathbb{C}) &= H^1(M, \mathbb{C}), & H^{0,2}(Z, \mathbb{C}) &= H^2_-(M, \mathbb{C}) \\ H^{3,1}(Z, \mathbb{C}) &= H^2_-(M, \mathbb{C}), & H^{3,2}(Z, \mathbb{C}) &= H^3(M, \mathbb{C}). \end{aligned} \tag{4.4}$$

Fact 3. The Penrose transform of the vector bundles associated with Ω^1 and Ω^2 yields a commutative diagram with exact rows (cf. [16, Proposition 3.3]) from which it follows that:

- $H^{1,1}(Z, \mathbb{C})$ can be identified with the set of pairs $(f, \rho) \in \Lambda^0 \oplus \Lambda_+^2$ satisfying the equation

$$\mathcal{D}df - d\rho = 0 \quad \text{on } M, \tag{4.5}$$

where the operator $\mathcal{D} : \Lambda^1 \rightarrow \Lambda^3$ is defined by

$$\omega_b \mapsto \left(\nabla^a \nabla^b + 2R^{ab} - \frac{2}{3}R g^{ab} \right) \omega_b, \tag{4.6}$$

while ∇ denotes the Levi-Civita connection, R_{ab} denotes the Ricci curvature and R denotes the scalar curvature of (M, g) ;

- For $E_2^{1,1}$ featuring in the Frölicher spectral sequence of Z at E_2 level, we always have the canonical isomorphism of \mathbb{C} -vector spaces:

$$E_2^{1,1} := \ker \left(d_1 = \partial : H^{1,1}(Z, \mathbb{C}) \rightarrow H^{2,1}(Z, \mathbb{C}) \right) \simeq H^0(M, \mathbb{C}) \oplus H_+^2(M, \mathbb{C}), \tag{4.7}$$

hence $\dim_{\mathbb{C}} E_2^{1,1} = 1 + b_+(M)$, where $b_+(M) := \dim_{\mathbb{C}} H_+^2(M, \mathbb{C})$.

(Clearly $\dim_{\mathbb{C}} H^0(M, \mathbb{C}) = 1$ by compactness and connectedness of M .)

Fact 4. By [24], there exists a $(1, 1)$ -form h on Z such that $dh = 0$ and $h|_L$ generates $H^2(L, \mathbb{C})$ for every twistor line $L = L_x = \nu^{-1}(x) \subset Z$. Using the Leray-Hirsch theorem, we get isomorphisms

$$H^{r-2}(M, \mathbb{C}) \oplus H^r(M, \mathbb{C}) \simeq H^r(Z, \mathbb{C}), \quad 0 \leq r \leq 6,$$

induced by $\Lambda^{r-2} \oplus \Lambda^r \ni (\alpha, \beta) \mapsto h \wedge \nu^* \alpha + \nu^* \beta \in C_r^\infty(Z, \mathbb{C})$.

The conclusion of these four facts is that for any twistor space Z we always have $E_2(Z) = E_\infty(Z)$ (i.e. the Frölicher spectral sequence degenerates at the latest at E_2 level), while the degeneration at E_1 level depends exclusively on one arrow. Recall that for any complex manifold, the E_1 level of the Frölicher spectral sequence is given by the Dolbeault cohomology groups (i.e. $E_1^{p,q} = H^{p,q}$ for all p, q) and the d_1 arrows are induced by ∂ :

$$E_1^{p,q} \xrightarrow{d_1 = \partial} E_1^{p+1,q} \quad \text{for all } p, q.$$

When Z is a twistor space, it follows from the above facts (1) – (4) that all arrows $d_1 = \partial : E_1^{p,0} \rightarrow E_1^{p+1,0}$ are zero when $q = 0$ (hence also when $q = 3$ by duality),

while the part of the E_1 level of the Frölicher spectral sequence corresponding to $q = 1$ reads:

$$\begin{array}{ccccccc}
 0 \rightarrow & H^{0,1}(Z, \mathbb{C}) & \xrightarrow{d_1=0} & H^{1,1}(Z, \mathbb{C}) & \xrightarrow{d_1} & H^{2,1}(Z, \mathbb{C}) & \xrightarrow{d_1=0} & H^{3,1}(Z, \mathbb{C}) \rightarrow 0 \\
 & \parallel & & & & & & \parallel \\
 & H^1(M, \mathbb{C}) & & & & & & H_-^2(M, \mathbb{C}) .
 \end{array}$$

Since the part corresponding to $q = 2$ at E_1 level is dual to that for $q = 1$, we see that $E_1(Z) = E_\infty(Z)$ if and only if the middle arrow above, *i.e.*

$$d_1 = \partial : H^{1,1}(Z, \mathbb{C}) \rightarrow H^{2,1}(Z, \mathbb{C}), \tag{4.8}$$

vanishes. It is clear that the kernel of the arrow (4.8) equals $E_2^{1,1}$ (*cf.* (4.7)) since the arrow preceding it in the last complex vanishes. We get the following:

Conclusion 4.1 (*cf.* [16, page 653-662]). For any twistor space $\nu : Z \rightarrow M$, $H^0(M, \mathbb{C}) \oplus H_+^2(M, \mathbb{C})$ injects canonically into $H^{1,1}(Z, \mathbb{C})$.

Moreover, the Frölicher spectral sequence of Z degenerates at E_1 if and only if $H^{1,1}(Z, \mathbb{C})$ is isomorphic to $H^0(M, \mathbb{C}) \oplus H_+^2(M, \mathbb{C})$ iff $h^{1,1}(Z) = 1 + b_+(M)$.

To construct the actual examples that prove part (b) of Theorem 1.24, Eastwood and Singer [16, Section 5] go on to choose M to be a compact complex surface endowed with a Kähler metric ω of zero scalar curvature. By [32], any such M that has been given the conjugate orientation is self-dual, hence M possesses a twistor space Z .

On the other hand, for any compact Kähler manifold (X, ω) , the Lichnerowicz operator (*cf. e.g.* [7]) is defined on functions by

$$\mathcal{L} : C^\infty(X, \mathbb{C}) \rightarrow C^\infty(X, \mathbb{C}), \quad \mathcal{L}(f) := \Delta^2 f + \langle\langle dd^c f, \text{Ric } \omega \rangle\rangle, \tag{4.9}$$

where Δ is the Laplacian and $\text{Ric } \omega$ is the Ricci form of ω . A result of Lichnerowicz (*cf. e.g.* [7, Proposition 2.151]) guarantees that when the scalar curvature of (X, ω) is constant, there is an isomorphism

$$u(X)/u_0(X) \simeq (\ker \mathcal{L})/\mathbb{C},$$

where $u(X)$ denotes the complex Lie algebra of holomorphic vector fields on X , while $u_0(X)$ denotes the Abelian Lie algebra of parallel such vector fields.

It turns out that when $(X, \omega) = (M, \omega)$ is a compact Kähler complex surface of zero scalar curvature, the operator \mathcal{D} of (4.6) (defining equation (4.5) on M which characterises the Dolbeault cohomology group $H^{1,1}(Z, \mathbb{C})$) relates to the Lichnerowicz operator as follows (*cf.* [16, page 662]):

$$d\mathcal{D}d = \mathcal{L},$$

after identification of 4-forms with functions on M via the volume form.

This leads to the following consequence of the main theorem of [16]:

Theorem 4.2 ([16, Theorem 5.3.]). *Let (M, ω) be a compact Kähler complex surface of zero scalar curvature. If Z is the twistor space of M , then*

$$H^{1,1}(Z, \mathbb{C}) = H^0(M, \mathbb{C}) \oplus H_+^2(M, \mathbb{C}) \oplus \frac{\{\text{holomorphic vector fields on } M\}}{\{\text{parallel vector fields on } M\}}.$$

Since we always have $E_2^{1,1} \simeq H^0(M, \mathbb{C}) \oplus H_+^2(M, \mathbb{C}) = \mathbb{C} \oplus H_+^2(M, \mathbb{C})$ (cf. (4.7)), we see by Conclusion 4.1 that the Frölicher spectral sequence of Z degenerates at E_1 if and only if all holomorphic vector fields on M are parallel.

The simplest case to which Eastwood and Singer apply their results is that of $M := \Sigma_g \times \mathbb{P}^1$, where Σ_g is any compact complex curve of genus $g \geq 2$ endowed with the Poincaré metric ω_P (which is of constant curvature -1) and the complex projective line \mathbb{P}^1 is endowed with the metric $\omega_{\mathbb{P}^1} := 2\omega_{FS}$ (which is of constant curvature $+1$ if ω_{FS} denotes the Fubini-Study metric). The metric ω induced on the product compact complex surface $M = \Sigma_g \times \mathbb{P}^1$ is a Kähler metric of zero scalar curvature.

Proposition 4.3 ([16, page 663]). *If $M = \Sigma_g \times \mathbb{P}^1$ has been given the Kähler metric of zero scalar curvature $\omega = \omega_P \oplus \omega_{\mathbb{P}^1}$, the Frölicher spectral sequence of the twistor space Z of M does not degenerate at E_1 .*

Proof. We have seen in (4.5) that

$$E_1^{1,1} = H^{1,1}(Z, \mathbb{C}) \simeq \{(f, \rho) \in \Lambda^0 \oplus \Lambda_+^2; Ddf = d\rho \text{ on } M\}. \tag{4.10}$$

On $M = \Sigma_g \times \mathbb{P}^1$, we have

$$Ddf = (\nabla^a \nabla^b + 2R^{ab})\nabla_b f$$

because the scalar curvature $R \equiv 0$. If a C^∞ function $f : \Sigma_g \times \mathbb{P}^1 \rightarrow \mathbb{C}$ depends only on the \mathbb{P}^1 variable, we get:

$$\begin{aligned} Ddf &= (\nabla^i \nabla^j + 2\omega^{ij})\nabla_j f = \nabla^i (\nabla^j \nabla_j f) + 2\omega^{ij}\nabla_j f \\ &= \nabla^i (-\Delta + 2)f, \end{aligned}$$

where ∇_j is the Chern connection and $\Delta = -\nabla^j \nabla_j$ is the Laplacian on \mathbb{P}^1 , while $\omega = (\omega_{ij})_{i,j}$ locally on \mathbb{P}^1 and the identity $R^{ij} = \omega^{ij}$ holds because $\text{Ric } \omega = \omega$ on \mathbb{P}^1 . Using now the well-known fact that 2 is an eigenvalue of the Laplacian on \mathbb{P}^1 with eigenspace $E_\Delta(2)$ of complex dimension 3, we get

$$\{C^\infty \text{ functions } f : M \rightarrow \mathbb{C}; Ddf = 0\} \supset \{\text{Constants}\} \oplus E_\Delta(2) = \mathbb{C} \oplus E_\Delta(2),$$

hence $\dim_{\mathbb{C}}\{C^\infty \text{ functions } f : M \rightarrow \mathbb{C}; Ddf = 0\} \geq 4$. Thus (4.10) yields

$$h^{1,1}(Z) = \dim_{\mathbb{C}} H^{1,1}(Z, \mathbb{C}) \geq 4 + \dim_{\mathbb{C}}\{\rho \in \Lambda_+^2; d\rho = 0\} = 4 + b_+(M).$$

As $\dim_{\mathbb{C}} E_2^{1,1} = 1 + b_+(M)$ by (4.7), the claim follows from Conclusion 4.1. \square

It is obvious that $M = \Sigma_g \times \mathbb{P}^1$ is the ruled surface $\mathbb{P}(E_0)$ associated with the trivial rank-two vector bundle $E_0 := \Sigma_g \times \mathbb{C}^2 \rightarrow \Sigma_g$. This suggests a natural way of constructing a holomorphic family of compact complex surfaces $(M_t)_{t \in \Delta}$ such that $M_0 = M = \Sigma_g \times \mathbb{P}^1$ for which we can hope to ensure that $E_1(Z_t) = E_\infty(Z_t)$ for all $t \in \Delta^*$ in the associated family of twistor spaces $(Z_t)_{t \in \Delta}$: take $M_t := \mathbb{P}(E_t)$ when we have found a *suitable* family of rank-two holomorphic vector bundles $(E_t)_{t \in \Delta}$ over Σ_g with the trivial bundle $E_0 := \Sigma_g \times \mathbb{C}^2$ corresponding to $t = 0$. The authors make clear the meaning of *suitable* in the following form:

Proposition 4.4 ([16, page 663-664]). *If $E \rightarrow \Sigma_g$ is a stable rank-two holomorphic vector bundle with trivial determinant over a compact complex curve of genus $g \geq 2$, then the corresponding ruled surface $\mathbb{P}(E)$ satisfies:*

- (i) $\mathbb{P}(E)$ admits a Kähler metric of zero scalar curvature;
- (ii) $\mathbb{P}(E)$ has no non-zero holomorphic vector fields.

Proof. Eastwood and Singer deduce the above statements from classical results of Narasimhan-Seshadri [39] and Narasimhan-Ramanan [38].

By [39], any stable rank-two holomorphic vector bundle on Σ_g (with $g \geq 2$) arises from a representation of $\pi_1(\Sigma_g)$ into SU_2 , hence the corresponding ruled surface is a quotient $\mathbb{P}(E) = (\mathbb{P}^1 \times H)/\pi_1$ where H is the upper half-plane and π_1 acts by isometries of the natural metric on $\mathbb{P}^1 \times H$. This natural metric, obtained as the Riemannian product of the metric of curvature $+1$ on \mathbb{P}^1 with that of curvature -1 on H , is Kähler and of zero scalar curvature, hence (i) follows.

It follows from [39] that any vector bundle as in the statement satisfies $H^0(\Sigma_g, S^2 E^*) = 0$. Indeed, for any rank-two vector bundle E there is an isomorphism of bundles (see e.g. the proof in [25, Proposition 3.3.]):

$$\text{End}_0 E \simeq S^2 E^* \otimes \Lambda^2 E = S^2 E^* \otimes \det E \tag{4.11}$$

induced by the map T which associates with every $A \in \text{End } E$ the quadratic map $E \ni v \mapsto Av \wedge v \in \Lambda^2 E$. The kernel of T consists of the scalar endomorphisms (identified with \mathbb{C}), while $\text{End}_0 E$ denotes the traceless endomorphisms. It obviously satisfies $\text{End } E = \text{End}_0 E \oplus \mathbb{C}$. On the other hand, if E is stable then E is *simple* (cf. [39, Corollary to Proposition 4.3.]), i.e. $\dim_{\mathbb{C}} H^0(\Sigma_g, \text{End } E) = 1$, which means that the only endomorphisms of E are the scalar ones. Hence $H^0(\Sigma_g, \text{End}_0 E) = 0$ when E is stable. Thus, if E is stable and $\det E$ is trivial, we see by (4.11) that $H^0(\Sigma_g, S^2 E^*) = 0$.

Now, an easy argument explained in [16] shows that $H^0(\mathbb{P}(E), T^{1,0}\mathbb{P}(E)) \simeq H^0(\Sigma_g, S^2 E^*)$ for any rank-two holomorphic vector bundle $E \rightarrow \Sigma_g$ when $g \geq 2$. This is because any holomorphic vector field $\xi \in H^0(\mathbb{P}(E), T^{1,0}\mathbb{P}(E))$ must be *vertical*. Indeed, the normal bundle of any fibre of $\mathbb{P}(E) \rightarrow \Sigma_g$ being trivial, the component of ξ normal to any such fibre is constant along the fibre, hence it projects to a holomorphic vector field on Σ_g . Now any holomorphic vector field on Σ_g must vanish because, since $g \geq 2$, Σ_g embeds into a g -dimensional complex

torus \mathbb{C}^g/Λ whose flat metric induces a metric of negative curvature on $T^{1,0}\Sigma_g$ viewed as a holomorphic line subbundle of the tangent bundle of \mathbb{C}^g/Λ . Thus any $\xi \in H^0(\mathbb{P}(E), T^{1,0}\mathbb{P}(E))$ is indeed vertical, hence the restriction of ξ to any fibre of $\mathbb{P}(E) \rightarrow \Sigma_g$ defines a holomorphic vector field on the fibre $\mathbb{P}(E_t) \simeq \mathbb{P}^1$. Meanwhile, the holomorphic vector fields of \mathbb{P}^1 are the holomorphic sections of $-K_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2)$. Thus (ii) follows. \square

We now get an immediate corollary of Theorem 4.2 and Proposition 4.4. In view of Theorem 4.2, a weaker version of conclusion (ii) of Proposition 4.4 (with *non-parallel* in place of *non-zero*) would have sufficed.

Corollary 4.5 ([16, page 664]). *Let $E \rightarrow \Sigma_g$ be a stable rank-two holomorphic vector bundle with trivial determinant over a compact complex curve of genus $g \geq 2$. Let Z be the twistor space associated with the ruled surface $\mathbb{P}(E)$. Then the Frölicher spectral sequence of Z degenerates at E_1 .*

Putting together Proposition 4.3 and Corollary 4.5, we see that it suffices to show that the trivial rank-two vector bundle $E_0 = \Sigma_g \times \mathbb{C}^2 \rightarrow \Sigma_g$ deforms to *stable* rank-two holomorphic vector bundles $E_t \rightarrow \Sigma_g$ with trivial determinant, $t \in \Delta^*$. As explained before the statement of Proposition 4.4, associated with the family of bundles $(E_t)_{t \in \Delta}$ will be the family of ruled surfaces $(M_t := \mathbb{P}(E_t))_{t \in \Delta}$ whose corresponding family of twistor spaces $(Z_t)_{t \in \Delta}$ will provide the example proving the Eastwood-Singer part (b) of Theorem 1.24: $E_1(Z_t) = E_\infty(Z_t)$ for all $t \neq 0$ by Corollary 4.5, but $E_1(Z_0) \neq E_\infty(Z_0)$ by Proposition 4.3.

Theorem 4.6 ([16, page 664-665]) *Given any compact complex curve Σ_g of genus $g \geq 2$, the trivial rank-two vector bundle $E_0 = \Sigma_g \times \mathbb{C}^2 \rightarrow \Sigma_g$ deforms to stable rank-two holomorphic vector bundles with trivial determinant on Σ_g .*

Proof. We give an outline of the proof found in [16] which, as mentioned there, is a modification of an argument from [38]. Fix an arbitrary point $x \in \Sigma_g$ and denote by L_x the holomorphic line bundle on Σ_g defined by x (viewed as a divisor on the curve Σ_g). If L_x^{-1} denotes the line bundle dual to L_x , consider the rank-two holomorphic vector bundles $E \rightarrow \Sigma_g$ that are non-trivial extensions of L_x by L_x^{-1} . Equivalently, one considers short exact sequences of vector bundles on Σ_g :

$$0 \rightarrow L_x^{-1} \rightarrow E \rightarrow L_x \rightarrow 0 \tag{4.12}$$

that do not split holomorphically. There is a one-to-one correspondence between the equivalence classes of such non-trivial extensions and the non-zero classes $\{\beta^*\} \in H^{0,1}(\Sigma_g, \text{Hom}(L_x, L_x^{-1})) \simeq H^1(\Sigma_g, \mathcal{O}(L_x^{-2}))$, where β^* denotes the second fundamental form of extension (4.12). Any such E has trivial determinant (since $\det E = L_x^{-1} \otimes L_x$) and degree zero since L_x has degree $+1$, L_x^{-1} has degree -1 and the degree is additive in exact sequences. By [38, Lemma 5.1], any such E can have no holomorphic line subbundles of positive degree, hence any such E is at least *semi-stable*. To find *stable* vector bundles E arising as extensions (4.12),

it remains to rule out the existence of holomorphic line subbundles of degree zero in E .

Suppose that a rank-two holomorphic vector bundle E given by an extension (4.12) contains a degree-zero holomorphic line subbundle $L \subset E$. By the proof of [38, Lemma 5.2], any such L must be of the form $L = L_x \otimes L_y^{-1}$ for some point $y \in \Sigma_g$ and [38, Lemma 5.2] asserts that the class $\{\beta^*\}$ of extension (4.12) maps to the zero class in the cohomology group $H^{0,1}(\Sigma_g, \text{Hom}(L, L_x^{-1})) = H^1(\Sigma_g, \mathcal{O}(L_x^{-2} \otimes L_y))$ under the map

$$H^1(\Sigma_g, \mathcal{O}(L_x^{-2})) \rightarrow H^1(\Sigma_g, \mathcal{O}(L_x^{-2} \otimes L_y)). \tag{4.13}$$

Since (4.13) is a surjective linear mapping of complex vector spaces of dimensions $g + 1$ and respectively g thanks to Riemann-Roch, the kernel of (4.13) is a complex line l_y in the $(g + 1)$ -dimensional vector space $H^1(\Sigma_g, \mathcal{O}(L_x^{-2}))$. We conclude that $L = L_x \otimes L_y^{-1}$ is a holomorphic subbundle of E iff

$$\{\beta^*\} \in \ker \left(H^1(\Sigma_g, \mathcal{O}(L_x^{-2})) \rightarrow H^1(\Sigma_g, \mathcal{O}(L_x^{-2} \otimes L_y)) \right) \setminus \{0\} = l_y \setminus \{0\}.$$

By [38, Lemma 3.3], proportional extension classes $\{\beta^*\}$ and $\lambda\{\beta^*\}$, with $\lambda \in \mathbb{C}^*$, give rise to isomorphic bundles E and E_λ . Now the punctured line $l_y \setminus \{0\}$ defines a point in the complex projective space $\mathbb{P}H^1(\Sigma_g, \mathcal{O}(L_x^{-2})) \simeq \mathbb{P}^g$ and, when y varies in Σ_g , we get an analytic mapping

$$\Sigma_g \ni y \mapsto l_y \setminus \{0\} \in \mathbb{P}H^1(\Sigma_g, \mathcal{O}(L_x^{-2})) \simeq \mathbb{P}^g. \tag{4.14}$$

The conclusion is that an extension class $\{\beta^*\} \in H^1(\Sigma_g, \mathcal{O}(L_x^{-2})) \setminus \{0\}$ defines a holomorphic rank-two vector bundle $E \rightarrow \Sigma_g$ with trivial determinant that contains no degree-zero holomorphic line subbundles $L \subset E$ (hence E is *stable*) if and only if the image of $\{\beta^*\}$ in the projective space $\mathbb{P}H^1(\Sigma_g, \mathcal{O}(L_x^{-2})) \simeq \mathbb{P}^g$ under the natural projection $H^1(\Sigma_g, \mathcal{O}(L_x^{-2})) \setminus \{0\} \rightarrow \mathbb{P}H^1(\Sigma_g, \mathcal{O}(L_x^{-2}))$ lies in the complement of the curve which is the image of the map (4.14). Since $g \geq 2$, the complement of a complex curve in the g -dimensional complex projective space \mathbb{P}^g provides plenty of room for choice of deformations $E = E_t, t \neq 0$, of the trivial rank-two vector bundle E_0 . Of course, E_0 corresponds to the trivial extension (4.12) or, equivalently, to the zero class $\{\beta^*\} \in H^1(\Sigma_g, \mathcal{O}(L_x^{-2}))$. \square

As explained before the statement of Theorem 4.6, this result provides the final argument to the proof of the Eastwood-Singer part (b) of Theorem 1.24. We have chosen to reproduce the approach of [16] in some detail because it throws up new stimulating questions of which we now mention just a few.

4.3. Wu’s deformation openness of the $\partial\bar{\partial}$ -lemma property

In this subsection we outline the main ideas in the proof given by C.-C. Wu in [49] to the fact that the $\partial\bar{\partial}$ -lemma for compact complex manifolds is a deformation-open property (cf. Theorem 1.25).

The proof in [49] makes crucial use of a characterisation of the $\partial\bar{\partial}$ -lemma that was given in [14] (cf. Proposition 4.8 below). It is cast in terms of the relations between the Bott-Chern and the Aeppli cohomology groups of a compact complex manifold X ($\dim_{\mathbb{C}} X = n$) whose definitions (for all $p, q = 0, \dots, n$) we now recall:

$$H_{BC}^{p,q}(X, \mathbb{C}) := \frac{\ker \partial \cap \ker \bar{\partial}}{\text{Im}(\partial\bar{\partial})} \quad \text{and} \quad H_A^{p,q}(X, \mathbb{C}) := \frac{\ker(\partial\bar{\partial})}{\text{Im} \partial + \text{Im} \bar{\partial}}. \quad (4.15)$$

(The operators $\partial, \bar{\partial}$ and $\partial\bar{\partial}$ featuring in the numerators act on (p, q) -forms, while those in the denominators arrive in the subspace of (p, q) -forms.)

In a similar way, one also standardly defines

$$H_{\partial}^{p,q}(X, \mathbb{C}) := \frac{\ker \partial}{\text{Im} \partial} \quad \text{and} \quad H_{\bar{\partial}}^{p,q}(X, \mathbb{C}) := \frac{\ker \bar{\partial}}{\text{Im} \bar{\partial}}. \quad (4.16)$$

(Of course, $H_{\partial}^{p,q}(X, \mathbb{C})$ is the Dolbeault cohomology group of type (p, q) that was earlier denoted by $H^{p,q}(X, \mathbb{C})$.)

For all $p, q = 0, \dots, n$, there is always a natural (canonical) linear map

$$\begin{aligned} H_{BC}^{p,q}(X, \mathbb{C}) &\longrightarrow H_A^{p,q}(X, \mathbb{C}) \\ [u]_{BC} &\longmapsto [u]_A \end{aligned}$$

mapping the Bott-Chern class of any d -closed (p, q) -form u to the Aeppli class of u (which makes sense since $\partial\bar{\partial}u = 0$ whenever $\partial u = 0$ and $\bar{\partial}u = 0$). This map is always well defined but it need not be either injective or surjective on an arbitrary X . However, it is an isomorphism when X satisfies the $\partial\bar{\partial}$ -lemma. The converse also holds as Proposition 4.8 will implicitly show.

Before stating the $\partial\bar{\partial}$ -lemma characterisation of [14], we introduce some ad hoc terminology:

Definition 4.7. For a given $k = 0, 1, \dots, 2n$, a given compact complex manifold X (with $\dim_{\mathbb{C}} X = n$) is said to satisfy property:

(A_k) if the natural map $H_{BC}^{p,q}(X, \mathbb{C}) \longrightarrow H_A^{p,q}(X, \mathbb{C})$ is *injective* for all p, q such that $p + q = k$.

This property is clearly equivalent to property

(A'_k) $\ker \partial \cap \ker \bar{\partial} \cap (\text{Im} \partial + \text{Im} \bar{\partial}) = \text{Im}(\partial\bar{\partial})$, for all p, q s.t. $p + q = k$.

(B_k) if the natural map $H_{BC}^{p,q}(X, \mathbb{C}) \longrightarrow H_A^{p,q}(X, \mathbb{C})$ is *surjective* for all p, q such that $p + q = k$.

This property is clearly equivalent to property

(B'_k) $\text{Im} \partial + \text{Im} \bar{\partial} + (\ker \partial \cap \ker \bar{\partial}) = \ker(\partial\bar{\partial})$, for all p, q s.t. $p + q = k$.

(C_k) if the natural maps $H_{BC}^{p,q}(X, \mathbb{C}) \longrightarrow H_{\partial}^{p,q}(X, \mathbb{C})$ and

$H_{BC}^{p,q}(X, \mathbb{C}) \longrightarrow H_{\bar{\partial}}^{p,q}(X, \mathbb{C})$ are *injective* for all p, q such that $p + q = k$.

This property is clearly equivalent to the simultaneous occurrence of

- (C'_k) (i) $\text{Im } \partial \cap \ker \bar{\partial} = \text{Im } (\partial \bar{\partial})$ and (C'_k) (ii) $\text{Im } \bar{\partial} \cap \ker \partial = \text{Im } (\partial \bar{\partial})$,
for all p, q such that $p + q = k$.
- (D'_k) if (i) $\text{Im } \bar{\partial} + \ker \partial = \ker(\partial \bar{\partial})$ and (ii) $\text{Im } \partial + \ker \bar{\partial} = \ker(\partial \bar{\partial})$
for all p, q such that $p + q = k$.
- (L_k) if the $\partial \bar{\partial}$ -lemma holds on every space of forms $C_{p,q}^\infty(X, \mathbb{C})$ with $p + q = k$.

It is obvious that (L_k) implies each of the other properties listed above and that (L_k) is implied by the simultaneous occurrence of these other properties. As already pointed out, the following equivalences are obvious

$$(A_k) \iff (A'_k), \quad (B_k) \iff (B'_k), \quad (C_k) \iff (C'_k).$$

The inclusions \supset in (A'_k) , \subset in (B'_k) , \supset in (C'_k) (i), (ii) and \subset in (D'_k) (i), (ii) are obvious. The following statement is implicit in [14] and provides a key ingredient for the proof of Theorem 1.25 obtained in [49].

Proposition 4.8 (Contained in [14, Lemma 5.15]). *Let X be a compact complex manifold, $\dim_{\mathbb{C}} X = n$. For every $k = 1, \dots, 2n$, the following equivalences hold:*

$$(L_k) \iff (A_k) \iff (C_k) \iff (D'_{k-1}) \iff (B_{k-1}).$$

Proof. Fix an arbitrary $k \in \{1, \dots, 2n\}$. In view of what we have already noticed, it suffices to prove the equivalences

$$(A'_k) \iff (C'_k) \iff (D'_{k-1}) \iff (B'_{k-1}).$$

Proof of $(A'_k) \implies (C'_k)$. Let $u \in C_{p,q}^\infty(X, \mathbb{C})$ (where $p + q = k$) such that $\bar{\partial}u = 0$ and $u = \partial v$ for some $(p - 1, q)$ -form v . Then $u = \partial v + 0 \in \ker \partial \cap \ker \bar{\partial} \cap (\text{Im } \partial + \text{Im } \bar{\partial})$. Then (A'_k) forces $u \in \text{Im } (\partial \bar{\partial})$. This proves (i) of (C'_k) . The proof of (ii) of (C'_k) is similar with ∂ and $\bar{\partial}$ reversed.

Proof of $(C'_k) \implies (A'_k)$. Let $u \in C_{p,q}^\infty(X, \mathbb{C})$ (where $p + q = k$) such that $\partial u = 0$, $\bar{\partial}u = 0$ and $u = \partial v + \bar{\partial}w$ for some $(p - 1, q)$ -form v and some $(p, q - 1)$ -form w . Then we have:

- $\text{Im } \partial \ni \partial v = u - \bar{\partial}w \in \ker \bar{\partial}$, hence $\partial v \in \text{Im } \partial \cap \ker \bar{\partial} = \text{Im } (\partial \bar{\partial})$, the last identity of subspaces being given by the hypothesis (C'_k) (i). Thus $\partial v \in \text{Im } (\partial \bar{\partial})$.
- $\text{Im } \bar{\partial} \ni \bar{\partial}w = u - \partial v \in \ker \partial$, hence $\bar{\partial}w \in \text{Im } \bar{\partial} \cap \ker \partial = \text{Im } (\partial \bar{\partial})$, the last identity of subspaces being given by the hypothesis (C'_k) (ii). Thus $\bar{\partial}w \in \text{Im } (\partial \bar{\partial})$.

It is now clear that $u = \partial v + \bar{\partial}w \in \text{Im } (\partial \bar{\partial})$. This proves (A'_k) .

Proof of $(C'_k) \implies (D'_{k-1})$. Let $u \in C_{r,s}^\infty(X, \mathbb{C})$ (where $r + s = k - 1$) such that $\partial \bar{\partial}u = 0$. Then:

- ∂u is a k -form of type $(r + 1, s)$ and $\partial u \in \ker \bar{\partial} \cap \text{Im } \partial = \text{Im } (\partial \bar{\partial})$, the last identity of subspaces being given by the hypothesis (C'_k) (i). Hence $\partial u = \partial \bar{\partial} \zeta$ for some $(r, s - 1)$ -form ζ . This amounts to $\partial(u - \bar{\partial} \zeta) = 0$ or again to $u - \bar{\partial} \zeta \in \ker \partial$.

We get $u = \bar{\partial} \zeta + (u - \bar{\partial} \zeta) \in \text{Im } \bar{\partial} + \ker \partial$. This proves (D'_{k-1}) (i).

- $\bar{\partial} u$ is a k -form of type $(r, s + 1)$ and $\bar{\partial} u \in \ker \partial \cap \text{Im } \bar{\partial} = \text{Im } (\partial \bar{\partial})$, the last identity of subspaces being given by the hypothesis (C'_k) (ii). Hence $\bar{\partial} u = \partial \bar{\partial} w$ for some $(r - 1, s)$ -form w . This amounts to $\bar{\partial}(u + \partial w) = 0$ or again to $u + \partial w \in \ker \bar{\partial}$.

We get $u = -\partial w + (u + \partial w) \in \text{Im } \partial + \ker \bar{\partial}$. This proves (D'_{k-1}) (ii).

Proof of $(D'_{k-1}) \implies (C'_k)$. Let $u \in C_{p,q}^\infty(X, \mathbb{C})$ (where $p+q = k$) such that $\bar{\partial} u = 0$ and $u = \partial v$ for some $(k - 1)$ -form v of type $(p - 1, q)$. Then $0 = \bar{\partial} u = -\partial \bar{\partial} v$, hence $v \in \ker(\partial \bar{\partial})$. Since $\ker(\partial \bar{\partial}) = \text{Im } \bar{\partial} + \ker \partial$ for $(k - 1)$ -forms by (D'_{k-1}) (i), we can find a $(p - 1, q - 1)$ -form w and a $(p - 1, q)$ -form ζ such that

$$v = \bar{\partial} w + \zeta \quad \text{and} \quad \partial \zeta = 0.$$

Applying ∂ , we get: $u = \partial v = \partial \bar{\partial} w$. Thus $u \in \text{Im } (\partial \bar{\partial})$. This proves (C'_k) (i).

Reversing the roles of ∂ and $\bar{\partial}$, we get (C'_k) (ii) in a similar way from (D'_{k-1}) (ii).

Proof of $(D'_{k-1}) \implies (B'_{k-1})$. Let $u \in C_{r,s}^\infty(X, \mathbb{C})$ (where $r + s = k - 1$) such that $\partial \bar{\partial} u = 0$. Thanks to (D'_{k-1}) (ii), we can find an $(r - 1, s)$ -form v and an (r, s) -form w such that

$$u = \partial v + w \quad \text{and} \quad w \in \ker \bar{\partial}.$$

Now since $\bar{\partial} w = 0$, we also have $\partial \bar{\partial} w = 0$. Hence by (D'_{k-1}) (i) we can write

$$w = \bar{\partial} \zeta + \rho \quad \text{with} \quad \rho \in \ker \partial$$

for some $(r, s - 1)$ -form ζ and some (r, s) -form ρ . We get: $\rho = w - \bar{\partial} \zeta$ and since $w, \bar{\partial} \zeta \in \ker \bar{\partial}$, we finally get $\rho \in \ker \bar{\partial}$. Given the choice of ρ , this implies that $\rho \in \ker \partial \cap \ker \bar{\partial}$.

Putting the bits together, we have

$$u = \partial v + \bar{\partial} \zeta + \rho \in \text{Im } \partial + \text{Im } \bar{\partial} + (\ker \partial \cap \ker \bar{\partial}).$$

This proves (B'_{k-1}) .

Proof of $(B'_{k-1}) \implies (D'_{k-1})$. This implication is trivial because $\text{Im } \partial + (\ker \partial \cap \ker \bar{\partial}) \subset \ker \bar{\partial}$ and $\text{Im } \bar{\partial} + (\ker \partial \cap \ker \bar{\partial}) \subset \ker \bar{\partial}$.

The Proposition is proved. □

Outline of proof of Wu's Theorem 1.25 (according to [49]). The main idea is to exploit the one-unit discrepancy between k and $(k - 1)$ in the Deligne-Griffiths-Morgan-Sullivan characterisations given to the $\partial\bar{\partial}$ -lemma in Proposition 4.8, namely the equivalences

$$(L_k) \iff (A_k) \iff (B_{k-1}).$$

This enables an argument by induction on the degree k of forms. Note that the $\partial\bar{\partial}$ -lemma always holds in a trivial way for $k = 0$ (*i.e.* on functions). Fix an arbitrary holomorphic family of compact complex n -dimensional manifolds $(X_t)_{t \in \Delta}$. The proof falls into two steps.

Step 1. Assuming the $\partial\bar{\partial}$ -lemma on X_0 , Wu shows that the dimensions of the Bott-Chern and Aeppli cohomology spaces remain constant in a small neighbourhood of $0 \in \Delta$, *i.e.*

$$\begin{aligned} \dim_{\mathbb{C}} H_{BC}^{p,q}(X_t, \mathbb{C}) &= \dim_{\mathbb{C}} H_{BC}^{p,q}(X_0, \mathbb{C}) \quad \text{for all } t \in \Delta \text{ sufficiently close to } 0 \\ \dim_{\mathbb{C}} H_A^{p,q}(X_t, \mathbb{C}) &= \dim_{\mathbb{C}} H_A^{p,q}(X_0, \mathbb{C}) \quad \text{for all } t \in \Delta \text{ sufficiently close to } 0. \end{aligned}$$

This parallels the analogous results for the Dolbeault cohomology that are classical and known to hold under the weaker assumption that the Frölicher spectral sequence of X_0 degenerates at E_1 .

Step 2. Suppose the $\partial\bar{\partial}$ -lemma holds on X_0 . Wu shows that the same thing holds on every sufficiently close fibre X_t by induction on $k = 0, 1, \dots, 2n$. Suppose the $\partial\bar{\partial}$ -lemma holds for the k -forms of X_t . We have to show that it then also holds for the $(k + 1)$ -forms of X_t .

The induction hypothesis is that (L_k) holds on X_t which, by Proposition 4.8, is equivalent to (A_k) holding on X_t . This means that the natural map $H_{BC}^{p,q}(X_t, \mathbb{C}) \rightarrow H_A^{p,q}(X_t, \mathbb{C})$ is injective for all p, q such that $p + q = k$.

On the other hand, $\dim_{\mathbb{C}} H_{BC}^{p,q}(X_0, \mathbb{C}) = \dim_{\mathbb{C}} H_A^{p,q}(X_0, \mathbb{C})$ for all p, q thanks to the $\partial\bar{\partial}$ -lemma assumption on X_0 (see explanation above Definition 4.7). Since, by *Step 1*, the dimensions of these spaces do not change when t remains close to 0, we still have

$$\dim_{\mathbb{C}} H_{BC}^{p,q}(X_t, \mathbb{C}) = \dim_{\mathbb{C}} H_A^{p,q}(X_t, \mathbb{C}), \quad \text{for all } p, q \text{ and all } t \text{ near } 0.$$

Thus, for all t near 0, $H_{BC}^{p,q}(X_t, \mathbb{C})$ and $H_A^{p,q}(X_t, \mathbb{C})$ are finite-dimensional vector spaces of *equal* dimensions and the natural linear map between them is *injective* whenever $p + q = k$ by the induction hypothesis (A_k) . Then it must also be *surjective*, which means that property (B_k) holds on X_t for all t close to 0. By Proposition 4.8, this is equivalent to property (L_{k+1}) , *i.e.* to the $\partial\bar{\partial}$ -lemma for the $(k + 1)$ -forms of X_t .

4.4. Twistor spaces and the $\partial\bar{\partial}$ -lemma

The limiting behaviour under holomorphic deformations of compact complex manifolds satisfying the $\partial\bar{\partial}$ -lemma seems to be shrouded in mystery. A likely guess is that this property may not be deformation-closed. Our expectation is that deformations of twistor spaces may hold the key. We take this opportunity to raise a few natural questions.

We start by recalling the following result of Gauduchon.

Theorem 4.9 ([20, Proposition 11.b, page 618]). *Every twistor space is balanced¹⁰. Moreover, this is the case for any self-dual metric on the corresponding four-manifold (i.e. a balanced metric is obtained on the twistor space from any self-dual metric in a given conformal class of the base four-manifold).*

Combined with [ES93], this readily (re-)proves part (a) of Theorem 1.11. Indeed, by Gauduchon's Theorem 4.9, the central fibre Z_0 in the Eastwood-Singer family is an sG manifold since, as a twistor space, it even has the stronger *balanced* property. However, $E_1(Z_0) \neq E_\infty(Z_0)$ as has been seen in Proposition 4.3. Recall that the Iwasawa manifold had provided another example proving (a) of Theorem 1.11 (cf. comments after Observation 3.11).

On the other hand, Hitchin's main result in [24] states that there exist only two Kähler twistor spaces: \mathbb{P}^3 and the space of flags in \mathbb{C}^3 . Together with Gauduchon's Theorem 4.9 and Campana's result in [11] stating that a twistor space is Moishezon if and only if it is of *class-C*, this fact will point to a relative lack of variation in the properties of twistor spaces.

However, combining Wu's Theorem 1.25 with the examples of Campana [10] and Lebrun-Poon [34] that proved the deformation non-openness of the *class-C* property (cf. Theorem 1.18), we notice the first piece of evidence against the above mock "principle".

Observation 4.10. There exist twistor spaces satisfying the $\partial\bar{\partial}$ -lemma that are *not* of *class C*.

Indeed, in [10] and [34] families of twistor spaces $(Z_t)_{t \in \Delta}$ are constructed in which the central fibre Z_0 is a Moishezon (= *class-C*) twistor space (hence its algebraic dimension is maximal: $a(Z_0) = \dim_{\mathbb{C}} Z_0 = 3$), while all the nearby fibres can be chosen to be decidedly non-Moishezon (= non-*class-C*) twistor spaces, i.e. to have minimal algebraic dimension $a(Z_t) = 0$ for $t \neq 0$. Now, the Moishezon central fibre Z_0 must satisfy the $\partial\bar{\partial}$ -lemma, hence by Wu's Theorem 1.25 all the nearby fibres Z_t also satisfy the $\partial\bar{\partial}$ -lemma if $t \in \Delta^*$ is sufficiently close to 0. Thus any of these nearby fibres is an example of a twistor space that is not in the *class-C* but satisfies the $\partial\bar{\partial}$ -lemma.

In a similar vein, one may ask whether there exist twistor spaces whose Frölicher spectral sequence degenerates at E_1 but on which the $\partial\bar{\partial}$ -lemma does

¹⁰ The term *semi-Kähler* is used in [20] to mean *balanced*.

not hold. As pointed out in the introduction (cf. Proposition 1.8), the $\partial\bar{\partial}$ -lemma implies the Hodge symmetry. On the other hand, the Dolbeault cohomology of any twistor space Z can be calculated in terms of the De Rham cohomology of the base four-manifold M (see (4.1), (4.2), (4.4) taken from [16]). Hence the following are necessary conditions on the Betti numbers of M for its twistor space Z to satisfy the $\partial\bar{\partial}$ -lemma:

$$b_1(M) = b_3(M) = 0 \quad \text{and} \quad b_-(M) = 0, \tag{4.17}$$

where $b_-(M) := \dim_{\mathbb{C}} H_-^2(M, \mathbb{C})$. Now the twistor spaces $(Z_t)_{t \in \Delta}$ in the Eastwood-Singer family have $(M_t := \mathbb{P}(E_t))_{t \in \Delta}$ as respective underlying four-manifolds, where each $E_t \rightarrow \Sigma_g$ is a rank-two holomorphic vector bundle over a complex curve of genus $g \geq 2$. Thus each M_t is a non-rational ruled surface having $b_1(M_t) = 2g \geq 4$. In particular, $b_1(M_t) \neq 0$, showing that M_t does not satisfy conditions (4.17). Hence no fibre in the Eastwood-Singer family of twistor spaces $(Z_t)_{t \in \Delta}$ can satisfy the $\partial\bar{\partial}$ -lemma. Since $E_1(Z_t) = E_\infty(Z_t)$ for all $t \in \Delta \setminus \{0\}$, we obtain the following:

Observation 4.11. There exist twistor spaces whose Frölicher spectral sequence degenerates at E_1 but which do not satisfy the $\partial\bar{\partial}$ -lemma. Any fibre Z_t with $t \neq 0$ in the Eastwood-Singer family is such an example.

Another consequence of the Eastwood-Singer main theorem was that the Frölicher spectral sequence of the twistor space of any $K3$ surface degenerates at E_1 (cf. Corollary 5.2. in [16]). Now it is well-known that any $K3$ surface M satisfies

$$b_1(M) = b_3(M) = 0 \quad \text{and} \quad b_-(M) = 19,$$

meaning that the last of the conditions (4.17) is not satisfied by $K3$ surfaces. Hence we have yet another class of examples illustrating Observation 4.11.

Observation 4.12. On the twistor space of any $K3$ surface, the $\partial\bar{\partial}$ -lemma does not hold although $E_1 = E_\infty$ in the Frölicher spectral sequence.

It is thus natural to ask the following:

Question 4.13. Can one characterise the twistor spaces on which the $\partial\bar{\partial}$ -lemma holds? Is there a $\partial\bar{\partial}$ -lemma analogue of the Eastwood-Singer description of the Frölicher degeneration property for twistor spaces?

In other words, it would be interesting to know whether there exists an equation similar to (4.5) or any other condition on M that can be an indicator of the $\partial\bar{\partial}$ -lemma property on Z .

We conclude by briefly mentioning another class of compact complex manifolds that usually provide a host of examples of various sorts: *nilmanifolds*. Recall that a connected, simply connected *real* Lie group G possessing a discrete co-compact subgroup Γ and a left-invariant complex structure J defines a quotient compact complex manifold $X := \Gamma \backslash G$ (inheriting its complex structure from J by passing to the quotient) that is said to be a *nilmanifold* if the group G is *nilpotent*. In the special case of a *complex* Lie group G , the quotient X is a compact *complex parallelisable* nilmanifold (e.g. the Iwasawa manifold) of the class introduced

by Wang in [48], but G need not be a *complex* Lie group in general. Putting the following two pieces of information together:

- By [14], any compact complex manifold on which the $\partial\bar{\partial}$ -lemma holds is *formal*.
- By [22], the only *formal* nilmanifolds are the complex tori (*i.e.* those defined by an *Abelian* Lie group G),

and since complex tori are clearly Kähler, we see that Kählerness and the $\partial\bar{\partial}$ -lemma property are equivalent conditions on nilmanifolds. So we can ask whether one can construct a holomorphic family of *nilmanifolds* in which all the fibres, except one, are Kähler.

Added in proof. Conjectures 1.21 and 1.23 have been disproved in the paper *Classification of complex structure on 6-dimensional nilpotent Lie algebras* (arXiv e-print math.DG/1111.5873v4) by M. Ceballos, A. Otal, L. Ugarte and R. Villacampa.

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