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Quadratic Tilt-Excess Decay and Strong Maximum Principle for Varifolds

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Abstract. In this paper, we prove that integral n-varifolds μ in codimension 1 with $H_\mu \in L^p_{\text{loc}}(\mu)$, $p > n$, $p \geq 2$ have quadratic tilt-excess decay

$$\text{tiltex}_\mu(x, \varrho, T_x\mu) = O_x(\varrho^2)$$

for μ-almost all x, and a strong maximum principle which states that these varifolds cannot be touched by smooth manifolds whose mean curvature is given by the weak mean curvature H_μ, unless the smooth manifold is locally contained in the support of μ.

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1. – Introduction

The tilt-excess and height-excess of a rectifiable n-varifold μ measures the local deviation of the tangent plane to a given plane

$$\text{tiltex}_\mu(x, \varrho, T) := \varrho^{-n} \int_{B_\varrho(x)} \| T_\xi\mu - T \|^2 \, \mathrm{d}\mu(\xi) \tag{1.1}$$

and the distance of the support to a given plane

$$\text{heightex}_\mu(x, \varrho, T) := \varrho^{-n-2} \int_{B_\varrho(x)} \text{dist}(\xi - x, T)^2 \mathrm{d}\mu(\xi), \tag{1.2}$$

respectively. For notions in geometric measure theory, we refer to [F] and [Sim].

Tilt-excess decay estimates for rectifiable varifolds were established by Allard in [All72, Theorem 8.16] for the proof of his Regularity Theorem for unit-density.

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Brakke extended these estimates for integral varifolds to the higher multiplicity case by using a blow-up technic in [Bra78, Theorem 5.6]. The statement in [Bra78, p. 157] combined with the estimate in [Bra78, Theorem 5.5] reads

$$\operatorname{tiltex}_\mu(x, \varrho, T_x\mu) = o_x(\varrho^{2-\varepsilon})$$

for any $\varepsilon > 0$ and for μ-almost all x if $H_\mu \in L^2_{\text{loc}}(\mu)$.

Now quadratic tilt-excess decay estimates

$$\operatorname{tiltex}_\mu(x, \varrho, T_x\mu) = O_x(\varrho^2) \tag{1.3}$$

were obtained by the author in [Sch01, Lemma 5.4] almost everywhere on certain varifolds in codimension 1 which in particular were limits of smooth hypersurfaces. There, instead of a blow-up technic, a theorem in fully non-linear elliptic equations due to Caffarelli in [Caf89] and Trudinger in [T89], see also [CafCab, Lemma 7.8] and [CafCK96], was used. This theorem states that subsolutions of uniformly elliptic equations with right hand side in L^n are touched from above by paraboloids or equivalently have second order superdifferentials almost everywhere. Invoking a maximum principle for stationary varifolds in codimension 1 proved by Solomon and White in [SW89], one can establish (1.3) for any stationary varifold in codimension 1 without assuming the varifold to be a limit of smooth manifolds. Let us define the height functions of μ.

Definition 1.1. Let μ be an integral n-varifold in $\Omega \subseteq \mathbb{R}^{n+1}$. The upper and lower height functions $\varphi_\pm : \mathbb{R}^n \to [-\infty, \infty]$ of μ are defined by

$$\begin{aligned} \varphi_+(y) &:= \sup\{t \mid (y, t) \in \operatorname{spt}\mu\}, \\ \varphi_-(y) &:= \inf\{t \mid (y, t) \in \operatorname{spt}\mu\} \end{aligned} \tag{1.4}$$

for $y \in \mathbb{R}^n$, where we set $\varphi_+(y) = -\infty$ and $\varphi_-(y) = +\infty$ if $\operatorname{spt}\mu \cap (\{y\} \times \mathbb{R}) = \emptyset$.

The maximum principle in [SW89] states that φ_+ is a C^2-viscosity subsolution of the minimal surface equation

$$-\nabla\left(\frac{\nabla\varphi_+}{\sqrt{1 + |\nabla\varphi_+|^2}}\right) \leq 0.$$

For the notion of viscosity solutions, we refer to [CIL], [CafCab] and [CafCK96].

Then, adapting Caffarelli's and Trudinger's result to the non-uniformly elliptic minimal surface equation, we obtain that φ_+ is touched from above by paraboloids almost everywhere. Likewise considering the lower height function φ_-, we see that the distance to the tangent plane decays quadratically close to $x = (y, \varphi_\pm(y))$ for almost all $y \in [\varphi_+ = \varphi_-]$. Combining with the standard estimate of [Bra78, Theorem 5.5] or [Sim, Lemma 22.2] and a covering argument, we arrive at (1.3).

Now in trying to generalize (1.3) to integral n-varifolds in codimension 1 with $H_\mu \in L^p_{\text{loc}}(\mu)$, $p > n$, $p \geq 2$, we did not directly succeed to prove an extension of the maximum principle in [SW89] for these more general varifolds. Instead in the first run, we are only able to prove the following lemma. This will be done by thoroughly examining the blow-up technic already used by Brakke in [Bra78].

LEMMA 3.1. *Let μ be an integral n-varifold in $\Omega \subseteq \mathbb{R}^{n+1}$ with $H_\mu \in L^p_{\mathrm{loc}}(\mu)$, $p > n$, $p \geq 2$, $\Omega := U \times \mathbb{R}$, $U \subseteq \mathbb{R}^n$ open,* $\operatorname{spt}\mu \subseteq U \times]-1, 1[$ *and $\varphi_+ : U \to [-\infty, \infty[$ be the upper height function of μ.*

Then for any $n < q < p$, there exists $u \in L^q_{\mathrm{loc}}(U)$ such that φ_+ is a $W^{2,q}$-viscosity subsolution of

$$-F(\nabla\varphi_+, D^2\varphi_+) \leq u \quad \text{in } U,$$

where F is a continuous, fully non-linear elliptic operator which is uniformly elliptic for bounded gradients and is universal in the sense that F is independent of μ, n, p, q.

Nevertheless, this lemma is the key lemma which opens the path to quadratic tilt-excess decay and the maximum principle.

The assumption $\operatorname{spt}\mu \subseteq U \times]-1, 1[$ implies that the upper height function is upper semicontinuous, and we thereby avoid considering upper semicontinuous envelopes when dealing with viscosity subsolutions, see our Definition A.1. This assumption is always satisfied locally near points who have a tangent plane which is not vertical or when the varifold is touched from above in case of the maximum principle.

The quadratic tilt-excess decay readily follows when this Lemma is combined with Caffarelli's and Trudinger's theorem and a covering argument.

THEOREM 5.1 (Quadratic tilt-excess decay). *Let μ be an integral n-varifold in $\Omega \subseteq \mathbb{R}^{n+1}$ with $H_\mu \in L^p_{\mathrm{loc}}(\mu)$, $p > n$, $p \geq 2$. Then for μ-almost all $x \in \operatorname{spt}\mu$, the tilt-excess and the height-excess decay quadratically that is*

$$\operatorname{tiltex}_\mu(x, \varrho, T_x\mu), \operatorname{heightex}_\mu(x, \varrho, T_x\mu) = O_x(\varrho^2).$$

Secondly by standard PDE-technics, see [CafCK96, Propositions 3.4, 3.5] and [Wa92, Theorem 4.20], Lemma 3.1 implies that φ_+ is twice approximately differentiable almost everywhere. Combining this with the quadratic tilt-excess and height-excess decays in Theorem 5.1, we establish that φ_+ satisfies the minimal surface equation with right hand side given by the weak mean curvature of μ pointwise almost everywhere on the set where φ_+ is finite.

Next, we recall that by ABP-estimate, see [Caf89, Lemma 1], [CafCab, Theorem 3.2] and [CafCK96, Proposition 3.3], see also Alexandroff's Maximum Principle for strong solutions [GT, Theorem 9.1], supersolutions of uniformly elliptic equations with right hand side in L^n which have a strict minimum coincide with their convex envelope on a set of positive measure, hence have subgradients on a set of positive measure. Adapting this to the non-uniformly elliptic minimal surface equation, this yields that φ_+ is actually a viscosity subsolution of minimal surface equation.

THEOREM 6.1. *Let μ be an integral n-varifold in $\Omega \subseteq \mathbb{R}^{n+1}$ with $H_\mu \in L^p_{\mathrm{loc}}(\mu)$, $p > n$, $p \geq 2$, $\Omega := U \times \mathbb{R}$, $U \subseteq \mathbb{R}^n$ open,* $\operatorname{spt}\mu \subseteq U \times]-1, 1[$ *and $\varphi_+ : U \to [-\infty, \infty[$ be the upper height function of μ.*

Then φ_+ *is twice approximately differentiable* $\mathcal{L}^n$*-almost everywhere on* $[\varphi_\pm \in \mathbb{R}]$ *and the approximate differentials satisfy*

$$\vec{\mathbf{H}}_\mu(y, \varphi_+(y)) = \nabla \left(\frac{\nabla \varphi_+}{\sqrt{1 + |\nabla \varphi_+|^2}} \right)(y) \frac{(-\nabla \varphi_+(y), 1)}{\sqrt{1 + |\nabla \varphi_+(y)|^2}}$$

for $\mathcal{L}^n$*-almost all* $y \in [\varphi_+ \in \mathbb{R}]$*. Moreover* φ_+ *is a* $W^{2,p}$*-viscosity subsolution of*

$$-\nabla \left(\frac{\nabla \varphi_+}{\sqrt{1 + |\nabla \varphi_+|^2}} \right) \le \vec{\mathbf{H}}_\mu(., \varphi_+) \frac{(\nabla \varphi_+, -1)}{\sqrt{1 + |\nabla \varphi_+|^2}} \quad \text{in } U,$$

where the right hand side is extended arbitrarily on $U - [\varphi_+ \in \mathbb{R}]$ *to a function still in* $L^p_{\mathrm{loc}}(U)$*.*

The last statement yields that $\operatorname{spt} \mu$ cannot be touched from above by a regular manifold which is locally the graph of a function $\psi \in W^{2,p}$ satisfying

$$-\nabla \left(\frac{\nabla \psi}{\sqrt{1 + |\nabla \psi|^2}} \right)(y) \ge \vec{\mathbf{H}}_\mu(y, \varphi_+(y)) \frac{(-\nabla \varphi_+(y), 1)}{\sqrt{1 + |\nabla \varphi_+(y)|^2}} + \tau$$

for $\mathcal{L}^n$-almost all $y \in [\varphi_+ \in \mathbb{R}]$ and some $\tau > 0$. This statement extends [SW89] to a weak maximum principle for varifolds with $H_\mu \in L^p_{\mathrm{loc}}(\mu)$, $p > n$, $p \ge 2$.

For two smooth manifolds, this statement would immediately follow from Hopf's Maximum Principle even for $\tau = 0$, if the manifolds do not coincide. For area minimizing hypersurfaces, a strong maximum principle was proved in [Mo77] and [Sim87] independent of the singular structure of the hypersurfaces. In case of stationary varifolds, this was proved in [SW89] if one of the varifolds is smooth and extended in [Il96] when the singular set of the stationary varifolds are small. For comparison principles for hypersurfaces with prescribed bounded mean curvature in the area minimizing case or for graphs at the boundary, see [DuSt94].

Performing a perturbation argument on the minimal surface equation, we obtain the strong maximum principle from Theorem 6.1.

THEOREM 6.2 (Strong maximum principle). *Let* μ *be an integral n-varifold in* $\Omega \subseteq \mathbb{R}^{n+1}$ *with* $H_\mu \in L^p_{\mathrm{loc}}(\mu)$, $p > n$, $p \ge 2$, $\Omega := U \times \mathbb{R}, U \subseteq \mathbb{R}^n$ *open,* $\operatorname{spt} \mu \subseteq U \times]-1, 1[$ *and* $\varphi_+ : U \to [-\infty, \infty[$ *be the height upper function of* μ*.*

Then $\operatorname{spt} \mu$ *cannot be touched from above by the graph of a function* $\psi \in W^{2,p}(U')$, $U' \Subset U$*, open and connected, which satisfies*

$$-\nabla \left(\frac{\nabla \psi}{\sqrt{1 + |\nabla \psi|^2}} \right)(y) \ge \vec{\mathbf{H}}_\mu(y, \varphi_+(y)) \frac{(-\nabla \varphi_+(y), 1)}{\sqrt{1 + |\nabla \varphi_+(y)|^2}}$$

for $\mathcal{L}^n$*-almost all* $y \in U' \cap [\varphi_+ \in \mathbb{R}]$*, unless* $\operatorname{graph} \psi \subseteq \operatorname{spt} \mu$*.*

We fix some notations.

$G(n+m,n)$ denotes the set of unoriented n-planes in $\mathbb{R}^{n+m}$. In particular, we fix $P := \mathbb{R}^n \times \{0\}$ and $\pi : \mathbb{R}^{n+m} \to P$ the orthogonal projection onto P. Usually, we will not distinguish between the plane, its orthogonal projection π_T and the corresponding matrix. For $T \in G(n+1,n)$, we denote by $\nu(T)$ a normal of T. We adopt the convention that we identify a rectifiable varifold with its Radon measure.

Open balls in dimension n and $n+1$ will be denoted by $B^n_\varrho(x) \subseteq \mathbb{R}^n$ and by $B^{n+1}_\varrho(x) \subseteq \mathbb{R}^{n+1}$.

$\mathcal{L}^n$ is the Lebesgue-measures in dimension n. $\mathcal{H}^n$ is the n-dimensional Hausdorff-measure in any metric space. The volume of the n-dimensional unit-ball is abbreviated by $\omega_n := \mathcal{L}^n(B^n_1(0))$.

We define the n-dimensional density of a set Q in $x \in \mathbb{R}^{n+1}$ of a Radon measure μ on $\mathbb{R}^{n+1}$ by

$$\theta^n(\mu, Q, x) := \lim_{\varrho \to 0} \frac{\mu(B^{n+1}_\varrho(x) \cap Q)}{\omega_n \varrho^n}$$

if this limit exists.

$S(n)$ denotes the set of all symmetric $n \times n$-matrices. $X \in S(n)$ can uniquely by decomposed into a positive and negative part as $X = X^+ - X^-$, where $X^+, X^- \geq 0$ and $X^+ X^- = 0$. We recall the definition of the Pucci-extremal operators, see [CafCab, Section 2.2],

$$\mathcal{M}^-_\lambda(X) := \lambda \sum_{\varsigma_i > 0} \varsigma_i + \sum_{\varsigma_i < 0} \varsigma_i \qquad \mathcal{M}^+_\lambda(X) := \sum_{\varsigma_i > 0} \varsigma_i + \lambda \sum_{\varsigma_i < 0} \varsigma_i$$

for $0 < \lambda \leq 1$ and $X \in S(n)$ with eigenvalues ς_i counted according to their multiplicity.

We call a function $\varphi : U \to [-\infty, \infty]$, with $U \subseteq \mathbb{R}^n$ open, twice approximately differentiable at $y \in U$ if $\varphi(y) \in \mathbb{R}$ and there exist $b \in \mathbb{R}^n, X \in S(n)$ satisfying

$$\mathrm{ap} - \lim_{z \to y} \frac{\varphi(z) - \varphi(y) - b(z-y) - \frac{1}{2}(z-y)^T X (z-y)}{|z-y|^2} = 0.$$

In this case, we set the approximate differentials to be

$$\nabla\varphi(y) := b \quad \text{and} \quad D^2\varphi(y) := X.$$

We write $\overline{\theta}(\varphi, Q)(y)$ for the infimum of all positive constants M for which there is a convex paraboloid P of opening M that touches φ at y from above in $Q \subseteq U$, see [CafCab, Section 1.2]. Likewise, we define $\underline{\theta}(\varphi, Q)$ and $\theta(\varphi, Q) = \max(\overline{\theta}(\varphi, Q), \underline{\theta}(\varphi, Q))$.

For two real-valued functions, we put $[\varphi > \psi] := \{y | \varphi(y) > \psi(y)\}$ and similarly for analogous expressions.

We will denote any module of continuity by $\omega(\varrho)$ that means $\omega(\varrho) \to 0$ for $\varrho \to 0$.

For the reader's convenience, we just want to mention that by our definitions of the height functions for varifolds μ satisfying $\operatorname{spt}\mu \subseteq U \times]-1,1[$, we have

$$\varphi_- \leq \varphi_+ \iff \varphi_\pm \in \mathbb{R} \iff -\infty < \varphi_+ \text{ or } \varphi_- < \infty.$$

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2. – Blow-up

In this section, we reexamine the blow-up procedure used by Brakke in [Bra78, Theorem 5.6]. Unfortunately, this section is quite technical. Therefore we explain briefly our modifications to Brakke's blow-up procedure.

We consider a sequence of varifolds μ_j which are touched from above in $0 \in \operatorname{spt}\mu_j$ by regular graphs of functions ψ_j, that is

$$\varphi_{j,+} \leq \psi_j, \quad \varphi_{j,+}(0) = 0 = \psi_j(0),$$

where $\varphi_{j,+}$ is the upper height function of μ_j. We assume that ψ_j satisfy a certain uniformly elliptic equation

$$-F(D^2\psi_j) = u_j$$

which we specify below. We think of the varifolds μ_j as being rescaled of a given varifold that is $\mu_j := \zeta_{x_0,\varrho_j,\#}\mu$, where $\zeta_{x_0,\varrho}(x) := \varrho^{-1}(x - x_0)$, $\varrho_j \to 0$.

We will do the blow-up in $x_0 \in \operatorname{spt}\mu$. As this point of touching is apriori given, we cannot impose assumptions on x_0 which are satisfied only μ-almost everywhere. In particular, we cannot assume that $\theta^n(\mu)$ is approximately continuous at x_0 with respect to μ, see [Bra78, 5.6(3)]. On the other hand, the touching from above of μ in x_0 implies that the tangent plane $T_{x_0}\mu$ exists.

Proposition 2.1. *Let μ be an integral n-varifold in $\Omega \subseteq \mathbb{R}^{n+1}$ with $\vec{\mathbf{H}}_\mu \in L^p_{\text{loc}}(\mu)$, $p > n$ which is touched from above by a C^1-manifold M in $x_0 \in \operatorname{spt}\mu$. Then μ has a tangent plane at x_0, and*

$$T_{x_0}\mu = \theta^n(\mu, x_0) T_{x_0} M \quad \text{with } \theta^n(\mu, x_0) \in \mathbb{N}. \tag{2.1}$$

PROOF. We assume $T_{x_0}M = P$ and take any sequence $\varrho_j \to 0$ such that $\zeta_{x_0,\varrho_j,\#}\mu \to \mu_C$ weakly as varifolds. Since $p > n$, we know that μ_C is a stationary integral n-varifold which is a cone and $\theta^n(\mu_C, 0) = \theta^n(\mu, x_0) \in [1, \infty[$, see [Sim, Corollary 17.8, Theorem 19.3 and Section 42]. Since M touches $\operatorname{spt}\mu$ from above, we see that

$$\operatorname{spt}\mu_C \subseteq \{x_{n+1} \le 0\}. \tag{2.2}$$

We choose $\xi \in C_0^1([0, \infty[)$ satisfying $\xi' \le 0$,

$$\xi'(r) = 0 \text{ for } 0 \le r < 1/R, \qquad \xi'(r) < 0 \text{ for } 1/R < r < R$$

for some $R > 0$, and consider $\eta \in C_0^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ defined by $\eta(x) := \xi(|x|)e_{n+1}$. Since μ_C is stationary, we see

$$0 = \delta\mu_C(\eta) = \int \operatorname{div}_{T_x\mu_C} \eta(x) \mathrm{d}\mu_C(x). \tag{2.3}$$

Since μ_C is a cone, we have $x \in T_x\mu_C$ for μ_C-almost all $x \in \operatorname{spt}\mu_C - \{0\}$, hence $\nabla^{\mu_C}\xi(|x|) = \xi'(|x|)x/|x|$. We calculate

$$\operatorname{div}_{T_x\mu_C} \eta(x) = e_{n+1}\nabla^{\mu_C}\xi(|x|) = \frac{\xi'(|x|)x_{n+1}}{|x|} \ge 0,$$

as $\xi' \le 0$ and $\operatorname{spt}\mu_C \subseteq \{x_{n+1} \le 0\}$. Since $\xi' < 0$ on $]1/R, R[$, we see further

$$\operatorname{div}_{T\mu_C} \eta > 0 \quad \mu_C\text{-almost everywhere on } \{x_{n+1} < 0\} \cap \{1/R < |x| < R\}.$$

Therefore (2.3) yields

$$\mu_C(\{x_{n+1} < 0\} \cap \{1/R < |x| < R\}) = 0 \quad \forall R > 0,$$

hence $\operatorname{spt}\mu_C \subseteq \{x_{n+1} = 0\} = P$ and $\mu_C = \theta^n(\mu, x_0)\mathcal{H}^n \llcorner P$ by Constancy Theorem, see [Sim, Theorem 41.1]. This is the first part of (2.1), as $\varrho_j \to 0$ was arbitrary.

Finally by Allard's Integral Compactness Theorem, see [All72, Theorem 6.4] or [Sim, Remark 42.8], we have $\theta^n(\mu, x_0) = \theta^n(\mu_C, y) \in \mathbb{N}$ for any $y \in P$. □

We put $\theta_0 := \theta^n(\mu, x_0) \in \mathbb{N}$. Although $T_{x_0}\mu$ may not be horizontal, we can represent μ_j on a large set as a union of lipschitz graphs of functions $f_{ji}, i = 1, \dots, \theta_0$, over P by a tilted Lipschitz-Approximation, see Appendix D. With the assumption of approximate continuity, one would know that $f_{ji}, i = 1, \dots, \theta_0$, coincide on a set approaching full measure when $j \to \infty$. Therefore, the limit procedure $\delta_j^{-1} f_{ji}$ for any $i = 1, \dots, \theta_0$ was considered in [Bra78].

Here we have to modify and first average by putting $f_j := \frac{1}{\theta_0} \sum_{i=1}^{\theta_0} f_{ji}$. Writing $T_{x_0}\mu = \operatorname{graph} L$ for some linear function, we see $f_{ji} \to L$ and consider

$$\delta_j^{-1}(f_j - L) \to \bar{f}, \quad \delta_j^{-1}(\psi_j - L) \to \bar{\psi}.$$

As in [Bra78], we establish that $\bar{f}$ satisfies a linear elliptic equation with constant coefficients, more precisely

$$-\partial_{kl} A(\nabla L) \partial_{kl} \bar{f} = \frac{1}{\theta_0} \bar{v},$$

where $A(a) := \sqrt{1 + |a|^2}$ and $\bar{v}$ is related to $\vec{\mathbf{H}}_{\mu_j}$, see Proposition 2.2 below. We recall that in [Bra78] this was actually Laplace's equation.

Choosing F homogeneous of degree one, that is $F(\varrho X) = \varrho F(X)$, and convex, $\bar{\psi}$ is a supersolution of a uniformly elliptic equation $-F(D^2\bar{\psi}) \geq \bar{u}$, where $\delta_j^{-1} u_j \to \bar{u}$ weakly. If F were linear, we would get that $\bar{\psi}$ is actually a solution.

From the touching, we clearly know $\bar{f} \leq \bar{\psi}$. Since the convergence of $\delta_j^{-1}(\psi_j - L)$ will be strong in C^1, we get $\bar{\psi}(0) = 0, \quad \nabla\bar{\psi}(0) = 0$. To apply the strong maximum principle to $\bar{f}$ and $\bar{\psi}$, we have to establish $\bar{f}(0) = 0$. This will be achieved in Proposition 2.3 when δ_j controls the height-excess of μ_j on all scales, that is

$$\sup_{0<\varrho\leq 8} \operatorname{heightex}_{\mu_j}(0, \varrho, T_{x_0}\mu) \leq \delta_j$$

and not only $\operatorname{heightex}_{\mu_j}(0, 8, T_{x_0}\mu) \leq \delta_j$.

We also have to specify the choice of F. We choose it to be a Pucci-extremal operator $F = M_\lambda^+$ where $\lambda = \lambda(L)$ is such that

$$\partial_{kl} A(\nabla L) X_{kl} \leq M_\lambda^+(X) \quad \text{for all } X \in S(n).$$

This yields

$$-\partial_{kl} A(\nabla L) \partial_{kl}(\bar{\psi} - \bar{f}) \geq \bar{u} - \frac{1}{\theta_0}\bar{v} \geq \frac{1}{2}\bar{u} \geq 0$$

by choosing u_j appropriately such that $\frac{1}{2}\bar{u} \geq \frac{1}{\theta_0}\bar{v}, 0$. Then the maximum principle implies

$$\bar{f} = \bar{\psi}, \qquad \bar{v} = \bar{u} = 0,$$

and

$$M_\lambda^+(D^2\bar{\psi}) = \partial_{kl} A(\nabla L) \partial_{kl} \bar{\psi} = 0.$$

By our choice of F and λ, this implies

$$D^2\bar{\psi} = 0, \tag{2.4}$$

exploiting the fully non-linear structure of F. Therefore $\bar{\psi}$ is linear and $\bar{f} = \bar{\psi} = 0$, as $\bar{\psi}(0) = 0, \nabla\bar{\psi}(0) = 0$. The vanishing of the blow-up limits then implies a strong decay of the height-excess of μ. Here we have to observe that since f_j is an average in our definition, it cannot control the height-excess alone. Instead, we use

$$\sum_{i=1}^{\theta_0} |f_{ji}(y)| \le C(\theta_0)(|f_j(y)| + |\psi_j(y)|)$$

when $f_{ji}(y) \le \psi_j(y)$, see Proposition 2.5.

As still $\alpha_j := (\int_{B_8^{n+1}(0)} |\vec{\mathbf{H}}_{\mu_j}|^p \mathrm{d}\mu_j)^{1/p} \le \delta_j$ has to be satisfied, we have to more or less decouple α_j and $\mathrm{heightex}_{\mu_j}$ by choosing u_j in a balanced way, see Appendix B and the conclusion in (3.31). This balanced choice is one of the main reasons why we did not succeed in proving the maximum principle Theorem 6.1 or 6.2 directly, but had to go through the intermediate step of Lemma 3.1.

We start now with our general blow-up procedure and fix $n, \theta_0 \in \mathbb{N}, n < p < \infty, p \ge 2, \iota := 1 - n/p \in]0, 1[$ and $0 < \lambda_0 \le 1$. We consider a sequence of integral n-varifolds μ_j in $B_8^n(0) \times \mathbb{R}$ and $T_j \in G(n+1, n)$ satisfying

$$0 \in \operatorname{spt} \mu_j, \tag{2.5}$$

$$J_{T_j}\pi \ge \lambda_0, \tag{2.6}$$

$$|(\omega_n \varrho^n)^{-1} \mu_j(B_\varrho^{n+1}(0)) - \theta_0| \le \varepsilon_j \to 0 \quad \text{for } 0 < \varrho \le 8, \tag{2.7}$$

$$\operatorname{spt} \mu_j \subseteq \{(y,t) | |t| \le C(\lambda_0)|y|\}, \tag{2.8}$$

$$\alpha_j^p := \int_{B_8^n(0)\times\mathbb{R}} |\vec{\mathbf{H}}_{\mu_j}|^p \mathrm{d}\mu_j, \tag{2.9}$$

$$\gamma_{j,\varrho}^2 := \varrho^{-n-2} \int_{B_\varrho^{n+1}(0)} |\pi_{T_j}^{\perp}(x)|^2 \mathrm{d}\mu_j(x) = \mathrm{heightex}_{\mu_j}(0, \varrho, T_j), \quad \gamma_j := \gamma_{j,8}, \tag{2.10}$$

and

$$\max(\alpha_j, \sup_{0<\varrho\le 8} \gamma_{j,\varrho}) \le \delta_j \to 0, \quad \delta_j \ne 0. \tag{2.11}$$

We also put

$$\beta_j^2 := \int_{B_7^{n+1}(0)} \| T_x\mu_j - T_j \|^2 \mathrm{d}\mu_j(x) = 7^n \,\mathrm{tiltex}_{\mu_j}(0, 7, T_j) \tag{2.12}$$

and get from [Bra78, Theorem 5.5] or [Sim, Lemma 22.2] that

$$\beta_j^2 \le C_{n,p}(\alpha_j \gamma_j + \gamma_j^2) \le C\delta_j^2. \tag{2.13}$$

Next, we define $v_j : B_8^n(0) \to \mathbb{R}$ by putting

$$(2.14)\quad v_j(y) := \begin{cases} \sum\limits_{x\in\pi^{-1}(y)} -H_{\mu_j}(x)\theta^n(\mu_j, x) & \text{if} \quad \sum\limits_{x\in\pi^{-1}(y)} \theta^n(\mu_j, x) \leq \theta_0, \\ 0 & \text{if} \quad \sum\limits_{x\in\pi^{-1}(y)} \theta^n(\mu_j, x) > \theta_0, \end{cases}$$

where $H_{\mu_j} := \vec{\mathbf{H}}_{\mu_j}\nu(T\mu_j)$ for $\nu(T\mu_j)e_{n+1} \geq 0$. By Co-Area formula

$$(2.15)\quad \| v_j \|_{L^p(B_8^n(0))} \leq \theta_0{}^{1-1/p}\left(\int_{B_8^n(0)\times\mathbb{R}} |\vec{\mathbf{H}}_{\mu_j}|^p \mathrm{d}\mu_j\right)^{1/p} \leq C(\theta_0)\alpha_j \leq C\delta_j.$$

Choosing $C(\lambda_0)\delta < 1/2$, we get from Theorem D.1 a tilted Lipschitz-Approximation of μ_j over P, that is: There exists $\varrho_0(\lambda_0) > 0$, $B := B^n_{\varrho_0(\lambda_0)}(0)$, and lipschitz maps

$$f_{j1} \leq \ldots \leq f_{j\theta} : B \subseteq P \to P^\perp, \quad i = 1, \ldots, \theta_0,$$
$$F_{ji} : B \subseteq P \to \mathbb{R}^{n+1}, \qquad F_{ji}(y) = (y, f_{ji}(y)),$$

satisfying

$$(2.16)\qquad \operatorname{Lip} f_{ji} \leq C(\lambda_0), \qquad \| f_{ji} \|_{L^\infty(B)} \leq 1/4,$$

and putting $\tilde{f}_{ji} := f_{ji} - L_j$, where $L_j : P \to P^\perp$ is linear, $L_j := (\pi|T_j)^{-1} - id$

$$(2.17)\qquad \operatorname{Lip} \tilde{f}_{ji} \leq C(\lambda_0)\delta < 1/2, \| \tilde{f}_{ji} \|_{L^\infty(B)} \leq C(\lambda_0, n)\gamma_j^{\frac{2}{n+2}},$$

and there exists $Y_j \subseteq B$ such that

$$(2.18)\quad \theta^n(\mu_j, (y, t)) = \#\{i | f_{ji}(y) = t\} \text{ for all } y \in Y_j \subseteq P, t \in]-1/2, 1/2[\subseteq P^\perp,$$

and

$$(2.19)\qquad X_j := \operatorname{spt}\mu \cap (Y_j \times] - 1/2, 1/2[) = \cup_{i=1}^{\theta_0} F_{ji}(Y_j),$$

and satisfying the estimates

$$(2.20)\qquad \mu_j((B\times] - 1/2, 1/2[) - X_j) + \mathcal{L}^n(B - Y_j) \leq C\delta_j^2,$$

if $\delta_j \leq c_0(n, \theta_0, p, \delta_0, \delta)$ and where $C = C(\lambda_0, n, \theta_0, p, \delta_0, \delta) < \infty$.

By (2.8), we can choose $\varrho_0(\lambda_0)$ small enough such that

$$(2.21)\qquad \operatorname{spt}\mu_j \cap (B \times \mathbb{R}) \subseteq B\times] - 1/4, 1/4[\subseteq B_1^{n+1}(0).$$

From (2.6), we see that $|\nu(T_j)e_{n+1}| = J_{T_j}\mu_j \geq \lambda_0$, hence

$$\nu(T_j) = \frac{(-a_j, 1)}{\sqrt{1+|a_j|^2}}, \nabla L_j = a_j \quad \text{with } |a_j| \leq C(\lambda_0). \tag{2.22}$$

Since $(y, L_j y) \in T_j$, we see

$$|\pi_{T_j}^{\perp}(F_{ji}(y))| \leq |\tilde{f}_{ji}(y)|. \tag{2.23}$$

On the other hand, we obtain from (D.2) that

$$|\tilde{f}_{ji}(y)| \leq C(\lambda_0)|\pi_{T_j}^{\perp}(F_{ji}(y))|. \tag{2.24}$$

From (2.18), (2.19) and the Co-Area formula, we see for $\Phi \in (C^0 \cap L^\infty)(B\times]-1/2, 1/2[\times G(n+1, n)$ that

$$\int_{X_j} \Phi(x, T_x\mu_j) J_{\mu_j}\pi(x) \mathrm{d}\mu_j(x) = \int_{Y_j} \sum_{i=1}^{\theta_0} \Phi(F_{ji}(y), \mathrm{im}(DF_{ji}(y)))\mathrm{d}y$$

and

$$\int_{X_j} \Phi(x, T_x\mu_j)\mathrm{d}\mu_j(x) = \int_{Y_j} \sum_{i=1}^{\theta_0} \Phi(F_{ji}(y), \mathrm{im}(DF_{ji}(y)))\sqrt{Gr_n(DF_{ji}(y))}\mathrm{d}y \tag{2.25}$$

where $Gr_n(DF_{ji}(y))$ denotes the Gram-Determinant of the columns of $DF_{ji}(y) \in \mathbb{R}^{n,n+1}$.

First, we establish a $W^{1,2}(B)$-bound on f_{ji}. We get from (2.24) and (2.25) that

$$\begin{aligned}\int_{Y_j} \sum_{i=1}^{\theta_0} |\tilde{f}_{ji}(y)|^2 \mathrm{d}y &\leq \int_{Y_j} \sum_{i=1}^{\theta_0} |\tilde{f}_{ji}(y)|^2 \sqrt{Gr_n(DF_{ji}(y))}\mathrm{d}y \\ &\leq C(\lambda_0) \int_{X_j} |\pi_{T_j}^{\perp}(x)|^2 \mathrm{d}\mu_j(x) \leq C(\lambda_0, n)\gamma_j^2 \leq C(\lambda_0, n)\delta_j^2.\end{aligned}$$

Next (2.17) and (2.20) yield

$$\int_{B-Y_j} \sum_{i=1}^{\theta_0} |\tilde{f}_{ji}|^2 \leq C\delta_j^2 \theta_0 C(\lambda_0, n)\gamma_j^{\frac{4}{n+2}} \leq C\delta_j^{2+\frac{4}{n+2}}.$$

Combining the two estimates, we obtain

$$\limsup_{j\to\infty} \delta_j^{-2} \int_B \sum_{i=1}^{\theta_0} |\tilde{f}_{ji}|^2 \leq C(\lambda_0, n). \tag{2.26}$$

Next, we consider $y \in Y_j$ and $x := F_{ji}(y) = (y, f_{ji}(y))$. Clearly,

$$\nu(T_x\mu_j) = \frac{(-\nabla f_{ji}(y), 1)}{\sqrt{1 + |\nabla f_{ji}(y)|^2}}.$$

Recalling $\nu(T_j) = \frac{(-a_j, 1)}{\sqrt{1+|a_j|^2}}$, $\nabla L_j = a_j$ and $|\nabla f_{ji}|, |a_j| \leq C(\lambda_0)$ by (2.16) and (2.22), we get

$$|\nabla \tilde{f}_{ji}(y)| = |\nabla f_{ji}(y) - a_j| \leq C(\lambda_0)|\nu(T_x\mu_j) - \nu(T_j)| = C(\lambda_0) \parallel T_x\mu_j - T_j \parallel .$$

Plugging into (2.25), we obtain

$$\begin{aligned}\int_{Y_j} \sum_{i=1}^{\theta_0} |\nabla \tilde{f}_{ji}(y)|^2 \mathrm{d}y &\leq C(\lambda_0) \int_{Y_j} \sum_{i=1}^{\theta_0} |\operatorname{im} DF_{ji}(y) - T_j|^2 \mathrm{d}y \\ &\leq C(\lambda_0) \int_{X_j} \parallel T_x\mu_j - T_j \parallel^2 \mathrm{d}\mu_j(x) \\ &\leq C(\lambda_0, n)\beta_j^2 \leq C(\lambda_0, n, p)\delta_j^2.\end{aligned}$$

From (2.17) and (2.20), we see

$$\int_{B-Y_j} \sum_{i=1}^{\theta_0} |\nabla \tilde{f}_{ji}|^2 \leq C\delta_j^2 C(\lambda_0)\delta.$$

Combining the two estimates, we obtain

$$\limsup_{j\to\infty} \delta_j^{-2} \int_B \sum_{i=1}^{\theta_0} |\nabla \tilde{f}_{ji}|^2 < \infty. \tag{2.27}$$

We define

$$f_j := \frac{1}{\theta_0} \sum_{i=1}^{\theta_0} f_{ji}, \qquad \tilde{f}_j := \frac{1}{\theta_0} \sum_{i=1}^{\theta_0} \tilde{f}_{ji} = f_j - L_j$$

and see from (2.26) and (2.27) that

$$\limsup_{j\to\infty} \delta_j^{-2} \int_B |\tilde{f}_j|^2 \leq C(\lambda_0, n), \qquad \limsup_{j\to\infty} \delta_j^{-2} \int_B |\nabla \tilde{f}_j|^2 < \infty. \tag{2.28}$$

Selecting an appropriate subsequence, we obtain

$$\begin{aligned}&\delta_j^{-1} \tilde{f}_j \to \bar{f} \quad \text{weakly in } W^{1,2}(B) \text{ and strongly in } L^2(B), \\ &\parallel \bar{f} \parallel_{L^2(B)} \leq C(\lambda_0, n),\end{aligned} \tag{2.29}$$

(2.30) $$\tilde{f}_{ji} \to 0 \quad \text{strongly in } W^{1,2}(B),$$

using (2.15)

(2.31) $$\bar{v}_j := \delta_j^{-1} v_j \to \bar{v} \quad \text{weakly in } L^p(B),$$

and

(2.32) $$\begin{gathered} T_j \to T_0, a_j \to a_0, \\ |a_0| \leq C(\lambda_0). \end{gathered}$$

Then (2.30) and (2.32) yield

(2.33) $$\nabla f_{ji} \to a_0 \quad \text{strongly in } L^2(B).$$

$\bar{f}$ is a solution of a linear elliptic equation with constant coefficients. When $a_0 = 0$, this is Laplace's equation, compare [Bra78, Theorem 5.6].

PROPOSITION 2.2.

(2.34) $$-\partial_{kl} A(a_0) \partial_{kl} \bar{f} = \frac{1}{\theta_0} \bar{v} \quad \textit{weakly locally in } B$$

where $A(a) := \sqrt{1 + |a|^2}$.

PROOF. For $\eta \in C_0^1(B)$, we get

$$\begin{aligned} J_j &:= \int_B \partial_{kl} A(a_j) \partial_l \tilde{f}_j \partial_k \eta \\ &= \int_{B - Y_j} \frac{1}{\theta_0} \sum_{i=1}^{\theta_0} \left\langle \operatorname{im} DF_{ji}, \begin{pmatrix} 0 & 0 \\ \nabla \eta & 0 \end{pmatrix} \circ \pi \right\rangle \sqrt{Gr_n(DF_{ji})} \\ &\quad + \int_B \frac{1}{\theta_0} \sum_{i=1}^{\theta_0} \left(\partial_{kl} A(a_j) \partial_l \tilde{f}_{ji} \partial_k \eta - \left\langle \operatorname{im} DF_{ji}, \begin{pmatrix} 0 & 0 \\ \nabla \eta & 0 \end{pmatrix} \circ \pi \right\rangle \sqrt{Gr_n(DF_{ji})} \right) \\ &\quad + \frac{1}{\theta_0} \left(\int_{Y_j} \sum_{i=1}^{\theta_0} \left\langle \operatorname{im} DF_{ji}, \begin{pmatrix} 0 & 0 \\ \nabla \eta & 0 \end{pmatrix} \circ \pi \right\rangle \sqrt{Gr_n(DF_{ji})} - \delta \mu_j \left(\begin{pmatrix} 0 \\ \eta \end{pmatrix} \circ \pi \right) \right) \\ &\quad + \frac{1}{\theta_0} \delta \mu_j \left(\begin{pmatrix} 0 \\ \eta \end{pmatrix} \circ \pi \right) = J_{j,1} + \ldots J_{j,4}, \end{aligned}$$

where we consider $\binom{0}{\eta} \circ \pi \in C_0^1(\operatorname{spt} \mu \cap (B \times]-1/2, 1/2[))$ by (2.21).

From (2.16) and (2.20), we see

(2.35) $$|J_{j,1}| \leq C \delta_j^2 C(\lambda_0) \parallel \nabla \eta \parallel_{L^\infty} .$$

From (2.20) and (2.25), we get

$$(2.36)\quad |J_{j,3}| \le \frac{1}{\theta_0}\left|\int_{B\times]-1/2,1/2[-X_j} \left\langle T\mu_j, \begin{pmatrix} 0 & 0 \\ \nabla\eta & 0 \end{pmatrix} \circ \pi \right\rangle \mathrm{d}\mu_j\right| \le C\delta_j^2 \parallel \nabla\eta \parallel_{L^\infty} .$$

Next

$$\theta_0\delta_j^{-1} J_{j,4} = -\delta_j^{-1} \int_{B\times]-1/2,1/2[} \left\langle \vec{\mathbf{H}}_{\mu_j}(x), \eta(\pi(x)) e_{n+1} \right\rangle \mathrm{d}\mu_j(x)$$

$$(2.37)\quad = \int_{Y_j} \sum_{i=1}^{\theta_0} \frac{-\delta_j^{-1}\left\langle \vec{\mathbf{H}}_{\mu_j}(F_{ji}(y)), e_{n+1}\right\rangle}{J_{\mu_j}\pi(F_{ji}(y))} \eta(y)\mathrm{d}y$$

$$- \delta_j^{-1} \int_{B\times]-1/2,1/2[-X_j} \left\langle \vec{\mathbf{H}}_{\mu_j}(x), \eta(\pi(x)) e_{n+1} \right\rangle \mathrm{d}\mu_j(x).$$

Now

$$\vec{\mathbf{H}}_{\mu_j}(x) = H_{\mu_j}(x)\nu(T_x\mu_j), \qquad J_{\mu_j}\pi(x) = \nu(T_x\mu_j)e_{n+1}$$

for $\nu(T_x\mu_j)e_{n+1} \ge 0$, and

$$\sum_{x\in\pi^{-1}(y)} \theta^n(\mu_j, x) = \theta_0 \quad \text{for } y \in Y_j,$$

hence

$$v_j(y) = \sum_{i=1}^{\theta_0} -H_{\mu_j}(F_{ji}(y)) \quad \text{for } y \in Y_j.$$

Therefore

$$(2.38)\quad \theta_0\delta_j^{-1} J_{j,4} = \int_B \bar{\delta}_j^{-1} v_j \eta + R_j \quad \text{with} \quad R_j \to 0.$$

Indeed, we see from (2.37) that

$$|R_j| \le \delta_j^{-1} \parallel \eta \parallel_{L^\infty} \left(\parallel v_j \parallel_{L^p(B)} + \left(\int_{B_1^{n+1}(0)} |\vec{\mathbf{H}}_{\mu_j}|^p \mathrm{d}\mu_j \right)^{1/p} \right) C\delta_j^{2(1-1/p)}$$

$$\le C \parallel \eta \parallel_{L^\infty} \delta_j^{2(1-1/p)} \to 0,$$

since $p > 1$.

We turn to $J_{j,2}$. We put $S = \operatorname{im} DF_{ji} \in G(n+1, n)$. Then

$$\nu(S) := \frac{(-\nabla f_{ji}, 1)}{\sqrt{1 + |\nabla f_{ji}|^2}} \text{ and } Gr_n(DF_{ji}) = 1 + |\nabla f_{ji}|^2.$$

We calculate

$$
\begin{aligned}
&\left\langle \operatorname{im} DF_{ji}, \begin{pmatrix} 0 & 0 \\ \nabla\eta & 0 \end{pmatrix} \circ \pi \right\rangle \sqrt{Gr_n(DF_{ji})} \\
&\quad = \operatorname{tr}\left(\left(I_{n+1} - \nu(S)\nu(S)^T \right) \begin{pmatrix} 0 & 0 \\ \nabla\eta & 0 \end{pmatrix} \right) \sqrt{1 + |\nabla f_{ji}|^2} \\
&\quad = -\nu(S) \begin{pmatrix} 0 & 0 \\ \nabla\eta & 0 \end{pmatrix} \nu(S) \sqrt{1 + |\nabla f_{ji}|^2} \\
&\quad = -\frac{\left\langle (-\nabla f_{ji}, 1), (0, -\nabla\eta \nabla f_{ji}) \right\rangle}{\sqrt{1 + |\nabla f_{ji}|^2}} \\
&\quad = \frac{\nabla f_{ji} \nabla\eta}{\sqrt{1 + |\nabla f_{ji}|^2}}.
\end{aligned}
$$

Therefore

$$
\theta_0 J_{j,2} = \int_B \sum_{i=1}^{\theta_0} (\partial_{kl} A(a_j) \partial_l f_{ji} - \frac{\partial_k f_{ji}}{\sqrt{1 + |\nabla f_{ji}|^2}}) \partial_k \eta, \tag{2.39}
$$

since $\int_B \nabla\eta D^2 A(a_j) a_j = 0$. A short calculation yields for $a, b \in \mathbb{R}^n$

$$
\begin{aligned}
\partial_{kl} A(a) b_l - \frac{b_k}{\sqrt{1 + |b|^2}} &= b_k \left(\frac{1}{\sqrt{1 + |a|^2}} - \frac{1}{\sqrt{1 + |b|^2}} \right) - a_k \frac{ab}{(1 + |a|^2)^{1.5}} \\
&= \frac{b_k}{\sqrt{1 + |b|^2}\sqrt{1 + |a|^2}} \left(\sqrt{1 + |b|^2} - \sqrt{1 + |a|^2} \right) - a_k \frac{ab}{(1 + |a|^2)^{1.5}} \\
&= \frac{b_k}{\sqrt{1 + |b|^2}\sqrt{1 + |a|^2}} \frac{(b - a)(b + a)}{\sqrt{1 + |b|^2} + \sqrt{1 + |a|^2}} - a_k \frac{ab}{(1 + |a|^2)^{1.5}}.
\end{aligned}
$$

Together with (2.39), we obtain

$$
\begin{aligned}
&\theta_0 J_{j,2} = \\
&= \int_B \sum_{i=1}^{\theta_0} \Bigg(\frac{\nabla f_{ji} \nabla\eta}{\sqrt{1+|\nabla f_{ji}|^2}\sqrt{1+|a_j|^2}} \frac{\nabla \tilde{f}_{ji} (\nabla f_{ji} + a_j)}{\sqrt{1+|\nabla f_{ji}|^2} + \sqrt{1+|a_j|^2}} - a_j \nabla\eta \frac{a_j \nabla \tilde{f}_{ji}}{(1+|a_j|^2)^{1.5}} \Bigg),
\end{aligned}
$$

since $\int_B a_j \nabla\eta \frac{|a_j|^2}{(1+|a_j|^2)^{1.5}} = 0$. From (2.29), (2.32) and (2.33), we get

$$
\delta_j^{-1} J_{j,2} \to \int_B \left(\frac{a_0 \nabla\eta}{1 + |a_0|^2} \frac{\nabla \bar{f} 2 a_0}{2\sqrt{1 + |a_0|^2}} - a_0 \nabla\eta \frac{a_0 \nabla \bar{f}}{(1 + |a_0|^2)^{1.5}} \right) = 0. \tag{2.40}
$$

Then (2.35), (2.36), (2.38), (2.40) yield

$$\delta_j^{-1} J_j \to \int_B \frac{1}{\theta_0} \bar{v} \eta. \tag{2.41}$$

On the other hand,

$$\delta_j^{-1} J_j = \int_B \partial_{kl} A(a_j) \delta_j^{-1} \partial_l \tilde{f}_j \partial_k \eta \to \int_B \partial_{kl} A(a_0) \partial_l \bar{f} \partial_k \eta,$$

which yields (2.34) by (2.41). □

Since $\bar{v} \in L^p(B)$, we get by elliptic regularity theory that $\bar{f} \in W^{2,p}_{\mathrm{loc}}(B) \hookrightarrow C^{1,\iota}_{\mathrm{loc}}(B)$.

So far, we have only used that $\gamma_j = \gamma_{j,8} \leq \delta_j$. Using that δ_j controls $\mathrm{heightex}_{\mu_j}$ on all scales, we get the following proposition.

PROPOSITION 2.3.

$$\bar{f}(0) = 0. \tag{2.42}$$

PROOF. From (2.27) with $\varepsilon_j \leq 1/2$, (2.9) and (2.11), we get for $0 < \varrho \leq 1$

$$\begin{aligned} \| \delta\mu_j \| (B^{n+1}_\varrho(0)) &\leq \left(\int_{B^{n+1}_\varrho(0)} |\vec{\mathbf{H}}_{\mu_j}|^p \mathrm{d}\mu_j \right)^{1/p} \mu_j(B^{n+1}_\varrho(0))^{1-1/p} \\ &\leq \alpha_j (\omega_n \varrho^n/2)^{-1/p} \mu_j(B^{n+1}_\varrho(0)) \\ &\leq C(n,p) \alpha_j \varrho^{-n/p} \mu_j(B^{n+1}_\varrho(0)) \\ &\leq C(n,p) \delta_j \varrho^{\iota-1} \mu_j(B^{n+1}_\varrho(0)). \end{aligned} \tag{2.43}$$

Next from [Bra78, Theorem 5.5] or [Sim, Lemma 22.2], we obtain for $0 < \varrho \leq 1$ and (2.27) with $\varepsilon_j \leq 1/2$ that

$$\begin{aligned} \varrho^{-n} \int_{B^{n+1}_\varrho(0)} \| T_x\mu_j - T_j \| \mathrm{d}\mu_j(x) &\leq \mathrm{tiltex}_{\mu_j}(0,\varrho,T_j)^{1/2} (\varrho^{-n} \mu_j(B^{n+1}_\varrho(0)))^{1/2} \\ &\leq C(n,\theta_0) \Bigg(\bigg((2\varrho)^{2-n} \int_{B^{n+1}_{2\varrho}(0)} |\vec{\mathbf{H}}_{\mu_j}|^2 \mathrm{d}\mu_j \bigg)^{1/2} \\ &\qquad + \mathrm{heightex}_{\mu_j}(0,2\varrho,T_j)^{1/2} \Bigg) \\ &\leq C(n,\theta_0) \Bigg(\varrho \bigg(\varrho^{-n} \int_{B^{n+1}_{2\varrho}(0)} |\vec{\mathbf{H}}_{\mu_j}|^2 \mathrm{d}\mu_j \bigg)^{1/2} \\ &\qquad + \gamma_{j,2\varrho} \Bigg) \\ &\leq C(n,\theta_0)(\varrho^\iota \alpha_j + \gamma_{j,2\varrho}) \\ &\leq C(n,\theta_0) \delta_j, \end{aligned}$$

and

$$\int_{B_\varrho^{n+1}(0)} \| T_x\mu_j - T_j \| \,\mathrm{d}\mu_j(x) \le C(n,\theta_0)\delta_j \mu_j(B_\varrho^{n+1}(0)). \tag{2.44}$$

We put

$$B_\varrho^{j,\tau}(0) := \{x \in B_\varrho^{n+1}(0) \,|\, |\pi_{T_j}^\perp(x)| < \tau\},$$

and get for $0 < \tau, \delta_j \le 1$ by (2.43), (2.44), when adapting [Sim, 42.10(7)] to the case $0 < \iota < 1$,

$$\theta_0 = \theta^n(\mu_j, 0) \le \frac{\mu_j(B_\varrho^{j,\tau}(0))}{\omega_n \varrho^n} + C(n,p,\theta_0)\delta_j(\varrho/\tau + \varrho^\iota) \quad \text{for } 0 < \varrho \le 1. \tag{2.45}$$

We fix $\epsilon > 0$ small, set

$$\tau_j := \epsilon \delta_j$$

and consider small, but fixed ϱ such that

$$C(n,p,\theta_0)(\epsilon^{-1}\varrho + \varrho^\iota) < \epsilon. \tag{2.46}$$

Then (2.27) and (2.45) yields

$$\frac{\mu_j(B_\varrho^{n+1}(0) - B_\varrho^{j,\tau_j}(0))}{\omega_n \varrho^n} \le \varepsilon_j + \epsilon \le 2\epsilon$$

for large j, hence

$$\frac{\mathcal{L}^n(\pi(\operatorname{spt}\mu_j \cap (B_\varrho^{n+1}(0) - B_\varrho^{j,\tau_j}(0))))}{\omega_n \varrho^n} \le 2\epsilon. \tag{2.47}$$

We put

$$B_j := (B^n_{c_0(\lambda_0)\varrho}(0) \cap Y_j) - \pi(\operatorname{spt}\mu_j \cap (B_\varrho^{n+1}(0) - B_\varrho^{j,\tau_j}(0))),$$

where $c_0(\lambda_0)$ is small chosen below, and get from (2.20) and (2.47) that

$$\mathcal{L}^n(B_j) \ge \omega_n c_0(\lambda_0)^n \varrho^n - C\delta_j^2 - 2\epsilon\omega_n\varrho^n \ge (1 - \sqrt{\epsilon}) c_0(\lambda_0)^n \varrho^n \tag{2.48}$$

for large j and if

$$\sqrt{\epsilon} < c_0(\lambda_0)^n/4. \tag{2.49}$$

Now for $y \in B_j$, we get from (2.18) that $F_{ji}(y) = (y, f_{ji}(y)) \in \operatorname{spt}\mu_j$ and by (2.17) that

$$|f_{ji}(y)| \le |L_j y| + |\tilde{f}_{ji}(y)| \le C(\lambda_0)|y| + C(\lambda_0,n)\delta_j^{2/(n+2)} \le \varrho/2$$

if $c_0(\lambda_0)$ is small enough and j large, hence $F_{ji}(y) \in B^{n+1}_\varrho(0)$. By definition of B_j and (2.24), this yields

$$|\tilde{f}_{ji}(y)| \leq C(\lambda_0)|\pi^\perp_{T_j}(F_{ji}(y))| \leq C(\lambda_0)\tau_j,$$

hence

$$|\delta_j^{-1}\tilde{f}_{ji}(y)| \leq C(\lambda_0)\delta_j^{-1}\tau_j \leq C(\lambda_0)\epsilon,$$

and by (2.48)

$$\frac{\mathcal{L}^n([|\delta_j^{-1}\tilde{f}_j| \leq C(\lambda_0)\epsilon] \cap B^n_{c_0(\lambda_0)\varrho}(0))}{\mathcal{L}^n(B^n_{c_0(\lambda_0)\varrho}(0))} \geq 1 - \sqrt{\epsilon}.$$

Letting $j \to \infty$, we get by (2.29)

$$\frac{\mathcal{L}^n([|\bar{f}| > C(\lambda_0)\epsilon] \cap B^n_{c_0(\lambda_0)\varrho}(0))}{\mathcal{L}^n(B^n_{c_0(\lambda_0)\varrho}(0))} \leq \sqrt{\epsilon}$$

for all ϱ, ϵ satisfying (2.46) and (2.49). Letting first $\varrho \to 0$ then $\epsilon \to 0$, (2.42) follows as $\bar{f}$ is continuous. $\square$

Now, we add to our assumptions (2.5) - (2.11) that $\operatorname{spt}\mu_j$ are touched from above by regular graphs. More precisely we assume that there are $\psi_j \in W^{2,q}(B^n_1(0)) \hookrightarrow C^{1,\iota'}(B^n_1(0)), n < q < \infty, \iota' := 1 - n/q \in]0,1[$ satisfying

(2.50) $$\varphi_{j,+} \leq \psi_j \quad \text{in} \quad B^n_1(0), \qquad \psi_j(0) = 0,$$

where $\varphi_{j,+}$ is the upper height function of μ_j,

(2.51) $$\nabla\psi_j(0) = a_j, \qquad \frac{1}{\sqrt{1 + |\nabla\psi_j(0)|^2}} \geq \lambda_0,$$

where we have used (2.6),

(2.52) $$\| \psi_j - L_j \|_{C^1(B^n_1(0))} \leq \varepsilon_j \to 0,$$

and ψ_j satisfies a Pucci-equation almost everywhere

(2.53) $$-\mathcal{M}^+_\lambda(D^2\psi_j) = u_j \quad \mathcal{L}^n\text{-almost everywhere in } B^n_1(0),$$

where $0 < \lambda \leq 1$ and $u_j \in L^q(B^n_1(0))$ such that

(2.54) $$\| u_j \|_{L^q(B^n_1(0))} \leq \delta_j \to 0.$$

For a suitable subsequence

(2.55) $$\bar{u}_j := \delta_j^{-1}u_j \to \bar{u} \quad \text{weakly in } L^q(B^n_1(0)).$$

Clearly, we see from Proposition 2.1 and (2.5), (2.22), (2.50) and (2.51) that $T_0\mu_j = \theta_0 T_j$.

From (2.18) and (2.50), we get

(2.56) $$f_{ji} \leq \psi_j \quad \text{on } Y_j \text{ for } i = 1, \ldots, \theta_0.$$

In the next proposition, we derive by elliptic theory $W^{2,q}_{\text{loc}}$-estimates for ψ_j and get converging subsequences.

PROPOSITION 2.4. *Putting* $\tilde{\psi}_j := (\psi_j - L_j)$, *we get*

$$\limsup_{j\to\infty} \| \delta_j^{-1}\tilde{\psi}_j \|_{W^{2,q}(B^n_\varrho(0))} \leq C(\lambda_0, \lambda, n, q, \varrho) \quad \textit{for } 0 < \varrho < \varrho_0(\lambda_0), \tag{2.57}$$

hence for a subsequence

$$\delta_j^{-1}\tilde{\psi}_j \to \bar{\psi} \quad \textit{weakly in } W^{2,q}_{\mathrm{loc}}(B) \textit{ and strongly in } C^{1,\iota''}_{\mathrm{loc}}(B) \textit{ for } 0 < \iota'' < \iota', \tag{2.58}$$

$$\bar{\psi}(0) = 0, \nabla\bar{\psi}(0) = 0, \tag{2.59}$$

$$\bar{f} \leq \bar{\psi} \quad \textit{in } B, \tag{2.60}$$

$\bar{\psi}$ *satisfies*

$$-\mathcal{M}^+_\lambda(D^2\bar{\psi}) \geq \bar{u} \quad \mathcal{L}^n\textit{-almost everywhere in } B. \tag{2.61}$$

PROOF. In this proof, we abbreviate $\bar{\psi}_j := \delta_j^{-1}\tilde{\psi}_j = \delta_j^{-1}(\psi_j - L_j)$ and $B_\sigma := \sigma B = B_{\sigma\varrho_0(\lambda_0)}(0)$ for $0 < \sigma < 1$. Clearly $-\mathcal{M}^+_\lambda(D^2\bar{\psi}_j) = \bar{u}_j$ $\mathcal{L}^n$-almost everywhere in $B^n_1(0)$ by (2.53). We rewrite this to a linear equation

$$-a^{(j)}_{kl}\partial_{kl}\bar{\psi}_j = \bar{u}_j \quad \mathcal{L}^n\text{-almost everywhere in } B^n_1(0), \tag{2.62}$$

with measurable coefficients and bounded ellipticity $\lambda I_n \leq (a^{(j)}_{kl})_{kl} \leq I_n$. Next, (2.54) gives

$$\| \bar{u}_j \|_{L^q(B^n_1(0))} \leq 1. \tag{2.63}$$

First, we observe from (2.28) and (2.56) that

$$\limsup_{j\to\infty} \| (\bar{\psi}_j)_- \|_{L^2(Y_j)} \leq C(\lambda_0, n).$$

From (2.20) and (2.52), we see

$$\| \bar{\psi}_j \|_{L^2(B-Y_j)} \leq C\delta_j \| \delta_j^{-1}(\psi_j - L_j) \|_{L^\infty(B^n_1(0))} \leq C\varepsilon_j,$$

hence

$$\limsup_{j\to\infty} \| (\bar{\psi}_j)_- \|_{L^2(B)} \leq C(\lambda_0, n).$$

By local maximum estimates, see [GT, Theorem 9.20], we get from (2.62) and (2.63) that

$$\begin{aligned} \| (\bar{\psi}_j)_- \|_{L^\infty(B_{3/4})} &\leq C(\lambda_0, \lambda, n)(\| (\bar{\psi}_j)_- \|_{L^2(B)} + \| \bar{u}_j \|_{L^n(B)}) \\ &\leq C(\lambda_0, \lambda, n). \end{aligned} \tag{2.64}$$

for large j. Next, we put

$$\xi := \bar{\psi}_j + \| (\bar{\psi}_j)_- \|_{L^\infty(B_{3/4})} \geq 0 \quad \text{in } B_{3/4}$$

and get from Harnack-inequality, see [GT, Theorems 9.20 and 9.22], that

$$\begin{aligned} \| \xi \|_{L^\infty(B_{5/8})} &\leq C(\lambda_0, \lambda, n)(\inf_{B_{5/8}} \xi + \| \bar{u}_j \|_{L^n(B)}) \\ &\leq C(\lambda_0, \lambda, n)(\| (\bar{\psi}_j)_- \|_{L^\infty(B_{3/4})} + 1) \leq C(\lambda_0, \lambda, n) \end{aligned}$$

since $\bar{\psi}_j(0) = 0$ by (2.50) and using (2.64). Therefore

$$\limsup_{j\to\infty} \| \bar{\psi}_j \|_{L^\infty(B_{5/8})} \leq C(\lambda_0, \lambda, n). \tag{2.65}$$

Then (2.62), (2.63), (2.65) and interior $W^{2,q}$-estimates as $\mathcal{M}_\lambda^+$ is convex, see [Caf89, Theorem 1], [CafCab, Theorem 7.1] and [CafCK96, Theorem B.1], imply

$$\begin{aligned} \limsup_{j\to\infty} \| \bar{\psi}_j \|_{W^{2,q}(B_{1/2})} &\leq \limsup_{j\to\infty} C(\lambda_0, \lambda, n, q)(\| \bar{\psi}_j \|_{L^\infty(B_{5/8})} + \| \bar{u}_j \|_{L^q(B)}) \\ &\leq C(\lambda_0, \lambda, n, q) \end{aligned}$$

which is (2.57) for $\varrho = \varrho_0(\lambda_0)/2$. Of course this estimate is true for any $0 < \varrho < \varrho_0(\lambda_0)$ when $C(\lambda_0, \lambda, n, q)$ is replaced by $C(\lambda_0, \lambda, n, q, \varrho)$.

(2.57) implies the convergence of a subsequence as in (2.58). As this convergence is strong in C^1, (2.50) and (2.51) imply $\bar{\psi}(0) = 0$ and $\nabla\bar{\psi}(0) = 0$which is (2.59).

(2.60) follows from (2.20) and (2.56).

Since $(D^2\bar{\psi}_j, \bar{u}_j) \to (D^2\bar{\psi}, \bar{u})$ weakly in $L^q_{\text{loc}}(B)$, there exist convex combinations such that

$$\sum_j c_{kj}(D^2\bar{\psi}_j, \bar{u}_j) \to (D^2\bar{\psi}, \bar{u}) \quad \text{pointwise almost everywhere on } B \text{ as } k \to \infty.$$

Since $\mathcal{M}_\lambda^+$ is convex, we see from (2.62) that

$$-\mathcal{M}_\lambda^+(D^2\bar{\psi}) \leftarrow -\mathcal{M}_\lambda^+\left(\sum_j c_{kj} D^2\bar{\psi}_j\right) \geq \sum_j -c_{kj}\mathcal{M}_\lambda^+(D^2\bar{\psi}_j) = \sum_j c_{kj}\bar{u}_j \to \bar{u}$$

pointwise almost everywhere on B which is (2.61). □

We estimate the height-excess on balls $B_\sigma^{n+1}(0)$ with $0 < \sigma \leq \varrho_0(\lambda_0)$.

PROPOSITION 2.5. *There exists $C = C(\lambda_0, n, \theta_0) < \infty$ such that for any* $0 < \sigma \leq \varrho_0(\lambda_0)$

$$\limsup_{j\to\infty} \delta_j^{-1}\gamma_{j,\sigma} \leq C(\lambda_0, n, \theta_0)\sigma^{-\frac{n}{2}-1}\Big(\| \bar{f} \|_{L^2(B^n_\sigma(0))} + \| \bar{\psi} \|_{L^2(B^n_\sigma(0))} \Big). \tag{2.66}$$

PROOF. First, we get from (2.16), (2.23) and (2.25) that

$$\begin{aligned}\delta_j^{-2} \int_{B^{n+1}_\sigma(0)\cap X_j} |\pi^\perp_{T_j}(x)|^2 \mathrm{d}\mu_j(x) &\leq \int_{B^n_\sigma(0)\cap Y_j} \sum_{i=1}^{\theta_0} |\delta_j^{-1}\tilde{f}_{ji}(y)|^2 \sqrt{Gr_n(DF_{ji}(y))}\mathrm{d}y \\ &\leq C(\lambda_0)\int_{B^n_\sigma(0)\cap Y_j}\Big(\sum_{i=1}^{\theta_0}|\delta_j^{-1}(\tilde{f}_{ji}(y)-\tilde{\psi}_j(y))|^2 + \theta_0|\delta_j^{-1}\tilde{\psi}_j(y)|^2\Big)\mathrm{d}y.\end{aligned}$$

Recalling (2.56), we see for $y \in Y_j$ that

$$\begin{aligned}\sum_{i=1}^{\theta_0} |\tilde{f}_{ji}(y) - \tilde{\psi}_j(y)|^2 &\leq \sum_{i,k=1}^{\theta_0} (\tilde{f}_{ji}(y) - \tilde{\psi}_j(y))(\tilde{f}_{jk}(y) - \tilde{\psi}_j(y)) \\ &= \Big|\sum_{i=1}^{\theta_0}(\tilde{f}_{ji}(y) - \tilde{\psi}_j(y))\Big|^2 = \theta_0{}^2|\tilde{f}_j - \tilde{\psi}_j|^2,\end{aligned}$$

hence

$$\begin{aligned}\limsup_{j\to\infty} \delta_j^{-2} &\int_{B^{n+1}_\sigma(0)\cap X_j} |\pi^\perp_{T_j}(x)|^2 \mathrm{d}\mu_j(x) \\ &\leq C(\lambda_0, \theta_0)\Big(\| \bar{f} \|^2_{L^2(B^n_\sigma(0))} + \| \bar{\psi} \|^2_{L^2(B^n_\sigma(0))} \Big).\end{aligned} \tag{2.67}$$

For $x_0 \in B^{n+1}_\sigma(0) \cap \operatorname{spt}\mu_j$, we get by monotonicity formula, see [Sim, Theorem 17.7], that

$$\begin{aligned}\omega_n^{1/p} &\leq (\varrho^{-n}\mu_j(B^{n+1}_\varrho(x_0)))^{1/p} + \Big(\int_{B^{n+1}_2(0)} |\vec{\mathbf{H}}_{\mu_j}|^p \mathrm{d}\mu_j\Big)^{1/p}\frac{\varrho^\iota}{p-n} \\ &\leq (\varrho^{-n}\mu_j(B^{n+1}_\varrho(x_0)))^{1/p} + C_{n,p}\alpha_j \quad \text{for } 0 < \varrho \leq 1.\end{aligned}$$

From (2.11), we know $\alpha_j \leq \delta_j \to 0$, hence $(\varrho^{-n}\mu_j(B^{n+1}_\varrho(x_0))) \geq \omega_n/2$ for $0 < \varrho \leq 1$ and large j. Choosing $\varrho := \frac{1}{2}|\pi^\perp_{T_j}(x_0)| \leq \sigma/2 \leq \varrho_0(\lambda_0)/2 < 1/4$, we get

$$\begin{aligned}\delta_j^2 \geq \gamma_{j,8}^2 \geq 8^{-n-2}\int_{B^{n+1}_\varrho(x_0)} |\pi^\perp_{T_j}(x)|^2\mathrm{d}\mu_j(x) &\geq\geq 8^{-n-2}\varrho^2\mu_j(B^{n+1}_\varrho(x_0)) \\ \geq c_0(n)\varrho^{n+2} \geq c_0(n)|\pi^\perp_{T_j}(x_0)|^{n+2},\end{aligned}$$

hence $|\pi^{\perp}_{T_j}(x_0)| \leq C_n \delta_j^{\frac{2}{n+2}}$. Using (2.20), this yields

$$\limsup_{j\to\infty} \delta_j^{-2} \int_{B^{n+1}_\sigma(0)-X_j} |\pi^{\perp}_{T_j}(x)|^2 \mathrm{d}\mu_j(x) \leq \limsup_{j\to\infty} \delta_j^{-2} \mu_j(B^{n+1}_\sigma(0)-X_j) C_n \delta_j^{\frac{2}{n+2}} = 0.$$

Together with (2.67), we obtain

$$\begin{aligned}\limsup_{j\to\infty} \delta_j^{-2}\gamma^2_{j,\sigma} &\leq \limsup_{j\to\infty} \delta_j^{-2}\sigma^{-n-2} \int_{B^{n+1}_\sigma(0)} |\pi^{\perp}_{T_j}(x)|^2 \mathrm{d}\mu_j(x) \\ &\leq C(\lambda_0,\theta_0)\sigma^{-n-2}\Big(\parallel \bar f \parallel^2_{L^2(B^n_\sigma(0))} + \parallel \bar\psi \parallel^2_{L^2(B^n_\sigma(0))} \Big)\end{aligned}$$

which is (2.66). □

The following corollary will not be needed in the text. We state it for possible further applications.

COROLLARY 2.6. *There exists* $C(\lambda_0, n, \theta_0, p), C(\lambda_0, n, \theta_0, q) < \infty$ *such that for any* $0 < \sigma \leq \varrho_0(\lambda_0)/2$

$$\limsup_{j\to\infty} \delta_j^{-1}\gamma_{j,\sigma} \leq C(\lambda_0, n, \theta_0, p)\sigma^{\iota} + C(\lambda_0, n, \theta_0, q)\sigma^{\iota'}. \tag{2.68}$$

PROOF. From (2.15), (2.29), (2.34), and (2.57), we estimate

$$\begin{aligned}\parallel \bar f \parallel_{W^{2,p}(B^n_{\varrho_0(\lambda_0)/2}(0))} &\leq C(\lambda_0, n, p)\Big(\parallel \bar f \parallel_{L^2(B)} + \parallel \bar u \parallel_{L^p(B)} \Big) \\ &\leq C(\lambda_0, n, p), \parallel \bar\psi \parallel_{W^{2,q}(B^n_{\varrho_0(\lambda_0)/2}(0))} \\ &\leq C(\lambda_0, \lambda, n, q).\end{aligned} \tag{2.69}$$

Since $\bar\psi(0) = 0$, $\nabla\bar\psi(0) = 0$, $\bar f \leq \bar\psi$ and $\bar f(0) = 0$ by (2.42), (2.59) and (2.60), we get $\nabla \bar f(0) = 0$. Therefore by (2.69)

$$\begin{aligned}&\parallel \bar f \parallel_{L^\infty(B^n_\sigma(0))} \leq C(\lambda_0, n, p)\sigma^{1+\iota}, \\ &\parallel \bar\psi \parallel_{L^\infty(B^n_\sigma(0))} \leq C(\lambda_0, \lambda, n, q)\sigma^{1+\iota'},\end{aligned}$$

and (2.68) follows from (2.66). □

We compare the elliptic operators in (2.34) and (2.61). The eigenvalues of $(\partial_{kl}A(a))_{kl}$ are $\frac{1}{\sqrt{1+|a|^2}}$ counted $(n-1)$-times and $\frac{1}{(1+|a|^2)^{3/2}}$ counted once. We put

$$\lambda(a) := \frac{1}{(1+|a|^2)^{3/2}} \in]0,1] \tag{2.70}$$

and see

$$\partial_{kl} A(a) X_{kl} \leq \mathcal{M}^+_{\lambda(a)}(X) \quad \text{for any } X \in S(n). \tag{2.71}$$

We add to our assumptions (2.5) - (2.11) and (2.50) - (2.54) that

$$0 < \lambda < \lambda(a_0) \leq 1 \tag{2.72}$$

and

$$\frac{1}{\theta_0} v_j \leq \frac{1}{2} u_j, \quad 0 \leq u_j, \tag{2.73}$$

in particular

$$\bar{u} - \frac{1}{\theta_0}\bar{v} \geq \frac{1}{2}\bar{u} \geq 0. \tag{2.74}$$

We know that $\bar{f} \in W^{2,p}_{\text{loc}}(B)$ and $\bar{\psi} \in W^{2,q}_{\text{loc}}(B)$ and conclude from (2.61), (2.71) and (2.72) that

$$-\partial_{kl} A(a_0)\partial_{kl}\bar{\psi} \geq -\mathcal{M}^+_\lambda(D^2\bar{\psi}) \geq \bar{u}. \tag{2.75}$$

Combining with (2.34) and (2.74), we see that $\xi := \bar{\psi} - \bar{f} \in W^{2,n}_{\text{loc}}(B)$ satisfies

$$-\partial_{kl} A(a_0)\partial_{kl}\bar{\xi} \geq \bar{u} - \frac{1}{\theta_0}\bar{v} \geq \frac{1}{2}\bar{u} \geq 0.$$

Further from (2.42), (2.59) and (2.60)

$$\xi \geq 0 \quad \text{in } B, \quad \xi(0) = 0.$$

Then Alexandroff's Maximum Principle, see [GT, Theorem 9.6], implies $\xi \equiv 0$ that is

$$\bar{f} = \bar{\psi} \quad \text{in } B. \tag{2.76}$$

Moreover $\bar{u} - \frac{1}{\theta_0}\bar{v}, \ \frac{1}{2}\bar{u} = 0$, hence

$$\bar{u} = \bar{v} = 0, \tag{2.77}$$

and

$$-\partial_{kl} A(a_0)\partial_{kl}\bar{\psi} = -\partial_{kl} A(a_0)\partial_{kl}\bar{f} = \frac{1}{\theta_0}\bar{v} = 0.$$

Combining with (2.75), we get

$$-\mathcal{M}^+_\lambda(D^2\bar{\psi}) = -\partial_{kl} A(a_0)\partial_{kl}\bar{\psi} = 0. \tag{2.78}$$

Since $\operatorname{spec}(\partial_{kl} A(a_0)) \subseteq [\lambda(a_0), 1]$ and $\lambda < \lambda(a_0)$ by (2.72), we see that $D^2\bar{\psi}$ has no negative eigenvalues, hence $D^2\bar{\psi} \geq 0$. Then (2.78) implies $D^2\bar{\psi} = 0$, and $\bar{\psi}$ is linear. From (2.59) and (2.76), we arrive at

$$\bar{f} = \bar{\psi} = 0 \quad \text{in } B. \tag{2.79}$$

PROPOSITION 2.7. *Under the assumptions* (2.5)-(2.11), (2.50)-(2.54) *and* (2.72), (2.73), *we get for* $0 < \sigma \leq \varrho_0(\lambda_0)$ *that*

$$\limsup_{j\to\infty} \delta_j^{-1}\gamma_{j,\sigma} = 0. \tag{2.80}$$

PROOF. This follows immediately from (2.66) and (2.79). □

Since we want to prove a decay on $\sup_{0<\varrho\leq 8}\gamma_{j,\varrho}$, see our assumptions in (2.11), we have to establish a uniformity in σ in Proposition 2.7. As our assumptions hold with the same δ_j also for μ_j when rescaled to smaller balls, this follows by rescaling μ_j.

We consider $0<\varrho_j\leq 1$ and define $\mu_{j,\text{res}} := \zeta_{0,\varrho_j,\#}\mu_j$ where $\zeta_{0,\varrho}(x) := \varrho^{-1}x$, and

$$\psi_{j,\text{res}}(y) := \varrho_j^{-1}\psi_j(\varrho_j y), \qquad u_{j,\text{res}}(y) := \varrho_j u_j(\varrho_j y).$$

Since $\operatorname{spt}\mu_{j,\text{res}} = \varrho_j^{-1}\operatorname{spt}\mu_j$, (2.27) and (2.8) transform immediately from μ_j to $\mu_{j,\text{res}}$.

Now

$$\alpha_{j,\text{res}}^p = \int_{B_8^n(0)\times\mathbb{R}} |\vec{\mathbf{H}}_{\mu_{j,\text{res}}}|^p \mathrm{d}\mu_{j,\text{res}} = \varrho_j^{p-n}\int_{B_{8\varrho_j}^n(0)\times\mathbb{R}} |\vec{\mathbf{H}}_{\mu_j}|^p \mathrm{d}\mu_j \leq \alpha_j \leq \delta_j$$

by (2.11), as $p>n$ and $\varrho_j\leq 1$.

Next for $0<\varrho\leq 8$,

$$\gamma_{j,\text{res},\varrho}^2 = \text{heightex}_{\mu_{j,\text{res}}}(0,\varrho,T_j) = \text{heightex}_{\mu_j}(0,\varrho\varrho_j,T_j) = \gamma_{j,\varrho\varrho_j}^2,$$

hence

$$\max(\alpha_{j,\text{res}}, \sup_{0<\varrho\leq 8}\gamma_{j,\text{res},\varrho}) \leq \delta_j \to 0, \quad \delta_j\neq 0,$$

which is (2.11) for $\mu_{j,\text{res}}$.

Since the upper height function $\varphi_{j,\text{res},+}$ of $\mu_{j,\text{res}}$ is given by

$$\varphi_{j,\text{res},+}(y) = \varrho_j^{-1}\varphi_{j,+}(\varrho_j y),$$

and $\nabla\psi_{j,\text{res}}(y) = \nabla\psi_j(\varrho_j y)$, (2.50) and (2.51) follow immediately for the rescaled quantities and

$$\| \nabla(\psi_{j,\text{res}} - L_j) \|_{L^\infty(B_1^n(0))} \leq \varepsilon_j \to 0$$

by (2.52). Moreover since $\psi_{j,\text{res}}(0) = 0 = L_j 0$, we get

$$\| \psi_{j,\text{res}} - L_j \|_{L^\infty(B_1^n(0))} \leq \| \nabla(\psi_{j,\text{res}} - L_j) \|_{L^\infty(B_1^n(0))} \leq \varepsilon_j \to 0,$$

which is (2.52) for $\psi_{j,\text{res}}$.

By definition of $\psi_{j,\text{res}}$ and $u_{j,\text{res}}$, we get $-\mathcal{M}_\lambda^+(D^2\psi_{j,\text{res}}) = u_{j,\text{res}}\mathcal{L}^n$-almost everywhere in $B_1^n(0)$, which is (2.53) for the rescaled quantities.

Further (2.54) transforms to

$$\| u_{j,\text{res}} \|_{L^q(B_1^n(0))} = \varrho_j^{\iota'} \| u_j \|_{L^q(B_1^n(0))} \leq \delta_j \to 0$$

as $\iota' = 1 - n/q > 0$ and $\varrho_j \leq 1$.

Defining $v_{j,\text{res}}$ as in (2.14) for $\mu_{j,\text{res}}$, we see $v_{j,\text{res}}(y) := \varrho_j v_j(\varrho_j y)$, and get (2.73) for the rescaled quantities.

Clearly (2.72) is unchanged.

Thus we can strengthen Corollary 2.6 and Proposition 2.7.

COROLLARY 2.8. *Under the assumptions* (2.5)-(2.11)*, and* (2.50)-(2.54) *there exists* $C(\lambda_0, n, \theta_0, p), C(\lambda_0, n, \theta_0, q) < \infty$ *such that for any* $0 < \sigma_0 \leq \varrho_0(\lambda_0)/2$

$$\limsup_{j\to\infty} \delta_j^{-1} \sup_{0<\sigma\leq\sigma_0} \gamma_{j,\sigma} \leq C(\lambda_0, n, \theta_0, p)\sigma_0^{\iota} + C(\lambda_0, n, \theta_0, q)\sigma_0^{\iota'}. \tag{2.81}$$

Under the further assumptions (2.72), (2.73)*, we get*

$$\limsup_{j\to\infty} \delta_j^{-1} \sup_{0<\sigma\leq\varrho_0(\lambda_0)} \gamma_{j,\sigma} = 0. \tag{2.82}$$

PROOF. We choose $0 < \varrho_j \leq 1$ for (2.81) such that

$$\sup_{0<\sigma\leq\sigma_0} \gamma_{j,\sigma} \leq 2\gamma_{j,\varrho_j\sigma_0} = 2\gamma_{j,\mathrm{res},\sigma_0},$$

and for (2.82) such that

$$\sup_{0<\sigma\leq\varrho_0(\lambda_0)} \gamma_{j,\sigma} \leq 2\gamma_{j,\varrho_j\varrho_0(\lambda_0)} = 2\gamma_{j,\mathrm{res},\varrho_0(\lambda_0)},$$

rescale and apply (2.68) and (2.80). □

3. – Differential properties of the height function

In this section, we prove our key lemma which opens the path to quadratic tilt-excess decay and the maximum principle.

LEMMA 3.1. *Let* μ *be an integral* n*-varifold in* $\Omega \subseteq \mathbb{R}^{n+1}$ *with* $H_\mu \in L^p_{\mathrm{loc}}(\mu)$, $p > n$, $p \geq 2$, $\Omega := U \times \mathbb{R}$, $U \subseteq \mathbb{R}^n$ *open,* $\operatorname{spt}\mu \subseteq U\times]-1, 1[$ *and* $\varphi_+ : U \to [-\infty, \infty[$ *be the upper height function of* μ.

Then for any $n < q < p$*, there exists* $u \in L^q_{\mathrm{loc}}(U)$ *such that* φ_+ *is a* $W^{2,q}$*-viscosity subsolution of*

$$-F(\nabla\varphi_+, D^2\varphi_+) \leq u \quad \text{in } U, \tag{3.1}$$

where F *is a continuous, fully non-linear elliptic operator which is uniformly elliptic for bounded gradients and is universal in the sense that* F *is independent of* μ, n, p, q.

PROOF. First, we specify the operator F. We know that the smallest eigenvalue $(\partial_{kl}A(a))_{kl}$ is given by $\lambda(a)$, as defined in (2.70), and recall the domination of the minimal surface operator by a Pucci-extremal operator in (2.71).

To satisfy the assumption (2.72), we choose any continuous function $\tilde{\lambda} : \mathbb{R}^n \to]0, 1]$ satisfying

$$0 < \tilde{\lambda}(a) < \lambda(a) \leq 1, \qquad \tilde{\lambda}(a) \leq \tilde{\lambda}(b) \quad \text{for } |a| \geq |b|. \tag{3.2}$$

For example $\tilde{\lambda}(a) := \frac{1}{2(1+|a|^2)^{3/2}}$ will do. We define the continuous elliptic operator $F : \mathbb{R}^n \times S(n) \to \mathbb{R}$

$$F(a, X) := \mathcal{M}^+_{\tilde{\lambda}(a)}(X).$$

We turn to the right hand side, put

$$v_0(y) := \Big(\sum_{x \in \pi^{-1}(y)} |\vec{\mathbf{H}}_\mu(x)|^p J_\mu \pi(x)^{-1} \theta^n(\mu, x) \Big)^{1/p} \quad \text{for } y \in U$$

and see by Co-Area formula that

$$\int_{U'} |v_0|^p \mathrm{d}\mathcal{L}^n = \int_{U' \times]-1,1[} |\vec{\mathbf{H}}_\mu|^p \mathrm{d}\mu < \infty \quad \text{for } U' \Subset U, \tag{3.3}$$

in particular $v_0 \in L^p_{\text{loc}}(U)$. Now to prove (3.1), it suffices to consider $B^n_1(0) \Subset U$ and show that there exists $u \in L^q(B^n_1(0))$ such that

$$-F(\nabla \varphi_+, D^2 \varphi_+) \leq u \quad \text{in } B^n_1(0). \tag{3.4}$$

This can be seen by taking a countable cover of U by balls $B^n_\varrho(y) \Subset U$ which is locally finite, that is each point in U has a neighbourhood that intersects only finitely many balls.

By monotonicity formula and since $\operatorname{spt} \mu \subseteq U \times]-1, 1[$, we can choose $\theta_{\max} \in \mathbb{N}$ such that

$$\theta^n(\mu) \leq \theta_{\max} \quad \text{on } B^n_1(0) \times \mathbb{R}. \tag{3.5}$$

Defining

$$v(y) := \begin{cases} \sum_{x \in \pi^{-1}(y)} |\vec{\mathbf{H}}_\mu(x)| \theta^n(\mu, x) & \text{if } \sum_{x \in \pi(y)^{-1}} \theta^n(\mu, x) \leq \theta_{\max}, \\ 0 & \text{if } \sum_{x \in \pi(y)^{-1}} \theta^n(\mu, x) > \theta_{\max}, \end{cases} \tag{3.6}$$

we see

$$v \leq {\theta_{\max}}^{1-1/p} v_0 \quad \text{on } B^n_1(0). \tag{3.7}$$

Now, we define

$$u := 2\theta_{\max} M_{\mathcal{L}^n}((1 + v_0)^p \chi_{B^n_1(0)})^{1/p} \lfloor B^n_1(0),$$

where $M_{\mathcal{L}^n}$ denotes the maximal function, see Definition B.2, and fix $n < q < p$. From Lemma B.3, we see $u \in L^q(B^n_1(0))$, and u is q-balanced that is for any $y \in B^n_1(0)$ and $\sigma > 0$

$$\liminf_{\varrho \downarrow 0} \frac{\varrho^{-n} \parallel u \parallel_{L^1(B^n_{\sigma\varrho}(y))}}{\varrho^{-n/q} \parallel u \parallel_{L^q(B^n_\varrho(y))}} > 0. \tag{3.8}$$

Further from Proposition B.5, (3.3) and (3.7), we get

$$1 + v \leq \frac{1}{2} u, \tag{3.9}$$

$$\varrho^{-n/p} \left(\int_{B^n_{8\varrho} \times \mathbb{R}} |\vec{\mathbf{H}}_\mu|^p \mathrm{d}\mu \right)^{1/p} \leq C_n \varrho^{-n/q} \| u \|_{L^q(B^n_\varrho)} \text{ for any } B^n_{8\varrho} \subseteq B^n_1(0). \tag{3.10}$$

(3.9) will establish the assumption (2.73).

To prove (3.4) respectively (3.1), we consider $U' \Subset B^n_1(0)$, $\psi \in W^{2,q}(U')$, $\tau > 0$ satisfying

$$-F(\nabla\psi, D^2\psi) \geq u + \tau \quad \mathcal{L}^n\text{-almost everywhere in } U' \tag{3.11}$$

and assume to get a contradiction that $\varphi_+ - \psi$ has an interior maximum in U', hence, there exists a ball $B^n_{\varrho_0}(y_0) \Subset U'$ such that

$$\varphi_+ - \psi \leq (\varphi_+ - \psi)(y_0) \in \mathbb{R} \quad \text{on } B^n_{\varrho_0}(y_0). \tag{3.12}$$

In particular graph$(\psi + (\varphi_+ - \psi)(y_0))$ touches spt$\,\mu$ from above in $x_0 := (y_0, \varphi_+(y_0))$. We want to replace ψ by a function which is a solution of an elliptic equation rather than only being a supersolution as in (3.11).

We set $a_0 := \nabla\psi(y_0)$ and choose

$$\tilde{\lambda}(a_0) < \lambda < \lambda(a_0). \tag{3.13}$$

As in (A.4), there exists for $0 < \varrho < \varrho_0$ by [Caf89, Theorem 1] or [CafCab, Theorem 7.1] and the boundary estimates in [Cab00] or [Wi04], see also [Wa92, Theorem 5.8] for $p > n + 1$, a function $\psi_\varrho \in W^{2,q}(B^n_\varrho(y_0))$ such that

$$\begin{aligned} -\mathcal{M}^+_\lambda(D^2\psi_\varrho) = u \quad & \mathcal{L}^n\text{-almost everywhere in } B^n_\varrho(y_0), \\ \psi_\varrho = \psi \quad & \text{on } \partial B^n_\varrho(y_0) \end{aligned} \tag{3.14}$$

and satisfying the rescaled estimate

$$\| \varrho^{-1}(\psi_\varrho - \psi)(\varrho .) \|_{W^{2,q}(B^n_1(0))} \leq C\varrho^{\iota'},$$

where $C = C(\lambda, n, q, \varrho_0)(\| u \|_{L^q(B^n_{\varrho_0}(y_0))} + \| \psi \|_{W^{2,q}(B^n_{\varrho_0}(y_0))})$ and $\iota' := 1 - n/q \in]0, 1[$. In particular

$$\| \nabla\psi_\varrho - \nabla\psi \|_{L^\infty(B^n_\varrho(y_0))} \leq C\varrho^{\iota'}.$$

As $W^{2,q} \hookrightarrow C^{1,\iota'}$, we see

$$\| \nabla\psi - a_0 \|_{L^\infty(B^n_\varrho(y_0))} \leq C(n, q, \varrho_0)\varrho^{\iota'},$$

hence

$$\| \nabla\psi_\varrho - a_0 \|_{L^\infty(B^n_\varrho(y_0))} \leq C'\varrho^{\iota'}.$$

Choosing $0 < \varrho < \varrho_0$ small enough, we may assume that

$$\tilde{\lambda}(\nabla\psi) < \lambda < \lambda(\nabla\psi_\varrho) \quad \text{on } B^n_\varrho(y_0). \tag{3.15}$$

From (3.11), we get

$$-\mathcal{M}^+_\lambda(D^2\psi) \geq -\mathcal{M}^+_{\tilde{\lambda}(\nabla\psi)}(D^2\psi) = -F(\nabla\psi, D^2\psi) \geq u + \tau \quad \text{in } B^n_{\varrho_0}(y_0).$$

From (3.14), we see $-\mathcal{M}^+_\lambda(D^2(\psi_\varrho - \psi)) \leq -\mathcal{M}^+_\lambda(D^2\psi_\varrho) + \mathcal{M}^+_\lambda(D^2\psi) \leq -\tau$ and obtain by Alexandroff's Maximum Principle, see [GT, Theorem 9.6], that $\psi_\varrho < \psi$ on $B^n_\varrho(y_0)$. Together with (3.12), we get

$$\sup_{\partial B^n_\varrho(y_0)} (\varphi_+ - \psi_\varrho) = \sup_{\partial B^n_\varrho(y_0)} (\varphi_+ - \psi) \leq (\varphi_+ - \psi)(y_0) < \sup_{B^n_\varrho(y_0)} (\varphi_+ - \psi_\varrho),$$

and there is $\tilde{y}_0 \in B^n_\varrho(y_0) \subseteq B^n_1(0)$ such that

$$\varphi_+ - \psi_\varrho \leq (\varphi_+ - \psi_\varrho)(\tilde{y}_0) \quad \text{in } B^n_\varrho(y_0).$$

Adding a constant to ψ_ϱ, we achieve $\varphi_+(\tilde{y}_0) = \psi_\varrho(\tilde{y}_0)$. Translating by $\tilde{x}_0 := (\tilde{y}_0, \varphi_+(\tilde{y}_0))$, rescaling and abbreviating ψ_ϱ by ψ, we assume without loss of generality that

$$\varphi_+ \leq \psi \quad \text{in } B^n_1(0), \quad \varphi_+(0) = \psi(0) = 0, \tag{3.16}$$

$$-M^+_\lambda(D^2\psi) = u \quad \mathcal{L}^n\text{-almost everywhere in } B^n_1(0), \tag{3.17}$$

$$\psi(0) = a_0, \quad \frac{1}{\sqrt{1 + |\nabla\psi(0)|^2}} =: \lambda_0 > 0, \tag{3.18}$$

$$\lambda < \lambda(a_0), \tag{3.19}$$

where we have used (3.15) for the last inequality.

According to Proposition 2.1, μ has a tangent plane at 0, more precisely

$$\frac{\mu(B^{n+1}_\varrho(0))}{\omega_n \varrho^n} \to \theta^n(\mu, 0) =: \theta_0 \in \mathbb{N}, \tag{3.20}$$

$$T_0\mu = \theta_0 T, \quad T \in G(n+1, n), \tag{3.21}$$

$$J_T\pi = \lambda_0. \tag{3.22}$$

From (3.5), we get

$$\theta_0 \leq \theta_{\max}. \tag{3.23}$$

In particular $\zeta_{x_0,\varrho_j,\#}\mu \to \theta_0 T$ and $\varrho^{-1}\operatorname{spt}\mu \to T$ locally in Hausdorff distance, see for example [Sim, Lemma 17.11]. Rescaling further, we may therefore assume that

$$\operatorname{spt}\mu \subseteq \{(y,t)||t| < C(\lambda_0)|y|\} \cup \{(y,t)||t| > 2C(\lambda_0)|y|\}.$$

We define

$$\tilde{\mu} := \mu\lfloor\{(y,t)||t| < C(\lambda_0)|y|\} \tag{3.24}$$

and denote its upper height function by $\tilde{\varphi}_+$. Clearly,

$$\begin{gathered}\tilde{\varphi}_+(0) = \varphi_+(0) = \psi(0) = 0,\\ \tilde{\varphi}_+ \le \varphi_+ \le \psi \quad \text{in } B_1^n(0).\end{gathered} \tag{3.25}$$

For $0 < \varrho \le 1/8$, we define

$$\begin{gathered}\mu_\varrho := \zeta_{0,\varrho,\#}\tilde{\mu}, \qquad \psi_\varrho(y) := \varrho^{-1}\psi(\varrho y), \qquad u_\varrho(y) := \varrho u(\varrho y),\\ v_\varrho(y) := \varrho v(\varrho y), \qquad \alpha_\varrho^p := \int_{B_8^n(0)\times\mathbb{R}} |\vec{\mathbf{H}}_{\mu_\varrho}|^p \mathrm{d}\mu_\varrho,\\ \gamma_\varrho^2 := \operatorname{heightex}_{\tilde{\mu}}(0,\varrho,T), \qquad \delta_\varrho := \max(\alpha_\varrho, \sup_{0<\sigma\le 8\varrho} \gamma_\sigma, \| u_\varrho \|_{L^q(B_1^n(0))}).\end{gathered}$$

For $\varrho_j \to 0$, we set $\mu_j := \mu_{\varrho_j}$ and $T_j := T$.

From (3.16), (3.20), (3.21), (3.22) and (3.24), we see that (2.5)-(2.8) are satisfied and $\gamma_\varrho \to 0$.

Next

$$\alpha_\varrho = \varrho^\iota \left(\int_{B_{8\varrho}^n(0)\times\mathbb{R}} |\vec{\mathbf{H}}_{\tilde{\mu}}|^p \mathrm{d}\tilde{\mu} \right)^{1/p} \le C\varrho^\iota \to 0$$

and

$$0 < \| u_\varrho \|_{L^q(B_1^n(0))} = \varrho^{\iota'} \| u \|_{L^q(B_\varrho^n(0))} \le C\varrho^{\iota'} \to 0.$$

Therefore $0 < \delta_\varrho \to 0$, which is (2.11).

From (3.10), we get

$$\alpha_\varrho \le C_n \| u_\varrho \|_{L^q(B_1^n(0))} . \tag{3.26}$$

(2.50) and (2.51) are immediate from (3.16), (3.18) and (3.25). Clearly,

$$\nabla\psi_\varrho(0) = \nabla\psi(0) = a_0$$

is independent of ϱ.

Further

$$\| \nabla\psi_\varrho - a_0 \|_{L^\infty(B_1^n(0))} = \| \nabla\psi - \nabla\psi(0) \|_{L^\infty(B_\varrho^n(0))}$$
$$\leq \varrho^{\iota'} \text{höl}_{B_\varrho^n(0),\iota'}(\nabla\psi) \leq C_{n,q} \| \psi \|_{W^{2,q}(B_1^n(0))} \varrho^{\iota'} \to 0$$

and putting $L_0 y := a_0 y$, we get

$$\| \psi_\varrho - L_0 \|_{L^\infty(B_1^n(0))} \leq \varepsilon(\varrho) \to 0,$$

as $\psi_\varrho(0) = 0 = L_0 0$, which yields (2.52).
(2.53) follows (3.17) and rescaling.
(2.54) follows from $\| u_\varrho \|_{L^q(B_1^n(0))} \leq \delta_\varrho$.
(2.72) follows from (3.19).
From (3.6) and (3.23), we see that v_j as defined in (2.14) satisfies $|v_j| \leq v_{\varrho_j}$, hence from (3.9)

$$\varrho_j + |v_j| \leq \varrho_j + v_{\varrho_j} \leq \frac{1}{2} u_{\varrho_j}, \tag{3.27}$$

which implies (2.73).
Therefore we can apply Corollary 2.8 (2.82) and obtain

$$\limsup_{\varrho \downarrow 0} \delta_\varrho^{-1} \sup_{0 < \sigma \leq \varrho_0(\lambda_0)\varrho} \gamma_\sigma = 0. \tag{3.28}$$

Next, for any $\varrho_j \to 0$ such that

$$\delta_{\varrho_j}^{-1} u_{\varrho_j} \to \bar{u} \quad \text{weakly in } L^q(B_1^n(0))$$

as in (2.55), we get from (2.77) that $\bar{u} = 0$ in $B^n_{\varrho_0(\lambda_0)}(0)$. As $\| \delta_\varrho^{-1} u_\varrho \|_{L^q(B_1^n(0))} \leq 1$, any sequence $\varrho_j \to 0$ has a subsequence such that $\delta_{\varrho_j}^{-1} u_{\varrho_j}$ is weakly convergent in $L^q(B_1^n(0))$, hence $\delta_\varrho^{-1} u_\varrho \to 0$ weakly in $L^q(B^n_{\varrho_0(\lambda_0)}(0))$. Since $u_\varrho \geq 0$, we conclude

$$\| \delta_\varrho^{-1} u_\varrho \|_{L^1(B^n_{\varrho_0(\lambda_0)}(0))} = \int_{B^n_{\varrho_0(\lambda_0)}(0)} \delta_\varrho^{-1} u_\varrho \to 0. \tag{3.29}$$

As u is q-balanced in $\tilde{y}_0 \in B_1^n(0)$, here $\tilde{y}_0 = 0$, we get from (3.8) that

$$\liminf_{\varrho \downarrow 0} \frac{\| u_\varrho \|_{L^1(B^n_{\varrho_0(\lambda_0)}(0))}}{\| u_\varrho \|_{L^q(B_1^n(0))}} = \liminf_{\varrho \downarrow 0} \frac{\varrho^{1-n} \| u \|_{L^1(B^n_{\varrho_0(\lambda_0)\varrho}(0))}}{\varrho^{1-n/q} \| u \|_{L^q(B_\varrho^n(0))}} > 0.$$

Together with (3.29)

$$\liminf_{\varrho \downarrow 0} \delta_\varrho^{-1} \| u_\varrho \|_{L^q(B_1^n(0))} = 0 \tag{3.30}$$

and by (3.26)

$$\liminf_{\varrho \downarrow 0} \delta_\varrho^{-1} \alpha_\varrho = 0. \tag{3.31}$$

By definition of δ_ϱ, (3.30) and (3.31) imply that

$$\delta_\varrho = \sup_{0<\sigma\leq 8\varrho} \gamma_\sigma \tag{3.32}$$

for $\varrho > 0$ small enough.

Then (3.28) implies that $\limsup_{\varrho\downarrow 0} \delta_\varrho^{-1} \delta_{(\varrho_0(\lambda_0)/8)\varrho} = 0$, hence

$$\delta_\varrho \leq C_k \varrho^k \tag{3.33}$$

with $C_k < \infty$ for any $k > 0$.

But (3.27) and the definition of δ_ϱ imply that

$$2\omega_n^{1/q} \varrho \leq \| u_\varrho \|_{L^q(B_1^n(0))} \leq \delta_\varrho$$

contradicting (3.33) for $k > 1$. Therefore the assumption in (3.12) that $\varphi_+ - \psi$ has an interior maximum in U' leads to a contradiction, and the lemma is proved. □

4. – C^2-Approximation of the height functions

Combining the Lemmas 3.1 and A.3, we see that the height functions have approximate differentials almost everywhere.

PROPOSITION 4.1. *We keep the assumptions of Lemma* 3.1 *and consider the height functions* $\varphi_\pm : U \to [-\infty, \infty]$ *of* μ. *Then*

$$\overline{\theta}(\varphi_+), \underline{\theta}(\varphi_-) < \infty \ \mathcal{L}^n\text{-almost everywhere on } [\varphi_\pm \in \mathbb{R}], \tag{4.1}$$

$$\theta(\varphi_\pm, [\varphi_\pm \in \mathbb{R}]) < \infty \ \mathcal{L}^n\text{-almost everywhere on } [\varphi_+ = \varphi_-], \tag{4.2}$$

and $\varphi_\pm$ *are twice approximately differentiable* $\mathcal{L}^n$*-almost everywhere on* $[\varphi_\pm \in \mathbb{R}]$. *More precisely, the approximate differentials satisfy*

$$\limsup_{z\to y}(\liminf_{z\to y}) \frac{\varphi_\pm(z)-\varphi_\pm(y)-\nabla\varphi_\pm(y)(z-y) - \frac{1}{2}(z-y)^T D^2\varphi_\pm(y)(z-y)}{|z-y|^2} \leq (\geq) 0 \tag{4.3}$$

for $\mathcal{L}^n$*-almost all* $y \in [\varphi_\pm \in \mathbb{R}]$ *and*

$$\lim_{z\to y, z\in[\varphi_\pm\in\mathbb{R}]} \frac{\varphi_\pm(z)-\varphi_\pm(y)-\nabla\varphi_\pm(y)(z-y)-\frac{1}{2}(z-y)^T D^2\varphi_\pm(y)(z-y)}{|z-y|^2} = 0 \tag{4.4}$$

for $\mathcal{L}^n$*-almost all* $y \in [\varphi_+ = \varphi_-]$.

PROOF. Truncating as in Lemma A.3 by putting $\varphi_M := \max(\varphi_+, -M)$, we may assume that φ_+ is bounded from below and above.

As in [Sch01, Lemma 4.1], we see that for $y \in [\varphi_\pm \in \mathbb{R}] \subseteq U$ where $T_{(y,\varphi_+(y))}\mu$ exists and $\nu(T_{(y,\varphi_+(y))}\mu)e_{n+1} \neq 0$, which is true for almost all $y \in [\varphi_\pm \in \mathbb{R}]$, there exists $C = C_y < \infty$ such that $\varphi_+(z) \leq \varphi_+(y) + C_y|y - z|$ for all $z \in U$ and φ_+ satisfies (A.18). By Lemma 3.1, it satisfies (A.17), and (4.1) follows from Lemma A.3, first for φ_+, and then by symmetry for φ_-. (4.1) immediately implies (4.2) by observing that $\varphi_- \leq \varphi_+$ on $[\varphi_\pm \in \mathbb{R}]$.

Following the standard procedure in [CafCK96, Propositions 3.4, 3.5] and [Wa92, Theorem 4.20], we define

$$\varphi_{+,\Lambda}(y) := \varphi_+(y) - \Lambda|y|^2$$

and see that $\varphi_{+,\Lambda}$ is touched from above by its concave envelope on $[\overline{\theta}(\varphi_+) \leq \Lambda]$. By Alexandroff's Theorem, the concave envelope is twice differentiable almost everywhere, hence φ_+ is twice approximately differentiable at points where the concave envelope touches $\varphi_{+,\Lambda}$ and which have full density in the touching set. As $\varphi_{+,\Lambda}$ is touched from above, we get (4.3). The conclusion for φ_- again follows by symmetry. (4.3) immediately implies (4.4) by observing that the approximate differentials of φ_+ and φ_- coincide almost everywhere on $[\varphi_+ = \varphi_-]$ and again since $\varphi_- \leq \varphi_+$ on $[\varphi_\pm \in \mathbb{R}]$. □

For $y \in [\varphi_+ = \varphi_-]$ where the approximate differentials of φ_+ and φ_- coincide, we put

$$P_y(z) := \varphi_\pm(y) + \nabla\varphi_\pm(y)(z-y) + \frac{1}{2}(z-y)^T D^2\varphi_\pm(y)(z-y).$$

Then (4.4) states $\sup_{B^n_\varrho(y)\cap[\varphi_\pm\in\mathbb{R}]} |\varphi_\pm - P_y| = o_y(\varrho^2)$ for almost all $y \in [\varphi_+ = \varphi_-]$. Hence $\varphi_\pm$ is twice differentiable on a set whose complement in $[\varphi_+ = \varphi_-]$ is a zero set. Combining with Whitney's Extension Theorem, see Lemma C.1, we get a C^2-Approximation of the height functions.

THEOREM 4.1. *Let μ be an integral n-varifold in $\Omega \subseteq \mathbb{R}^{n+1}$ with $H_\mu \in L^p_{\mathrm{loc}}(\mu)$, $p > n, p \geq 2$, $\Omega := U \times \mathbb{R}, U \subseteq \mathbb{R}^n$ open,* $\operatorname{spt}\mu \subseteq U \times]-1,1[$ *and $\varphi_\pm : U \to [-\infty, \infty]$ be the height functions of μ.*

Then for any $U' \Subset U$ and $\varepsilon > 0$, there exists $Q \subseteq U' \cap [\varphi_+ = \varphi_-]$ such that

$$\mathcal{L}^n(U' \cap [\varphi_+ = \varphi_-] - Q) < \varepsilon$$

and there exists $\psi \in C^2(U)$ satisfying

$$D^\alpha\varphi_\pm = D^\alpha\psi \quad \text{on } Q \text{ for } |\alpha| \leq 2. \tag{4.5}$$

5. – Quadratic tilt-excess decay

Combining Proposition 4.1 with a covering argument, we obtain that the tilt-excess decays quadratically.

THEOREM 5.1 (Quadratic tilt-excess decay0. *Let μ be an integral n-varifold in $\Omega \subseteq \mathbb{R}^{n+1}$ with $H_\mu \in L^p_{\text{loc}}(\mu)$, $p > n$, $p \geq 2$. Then for μ-almost all $x \in \operatorname{spt}\mu$, the tilt-excess and the height-excess decay quadratically that is*

$$\operatorname{tiltex}_\mu(x, \varrho, T_x\mu), \operatorname{heightex}_\mu(x, \varrho, T_x\mu) = O_x(\varrho^2). \tag{5.1}$$

PROOF. We consider $x \in \operatorname{spt}\mu$ satisfying $T_x\mu$ exists, $\theta^n(\mu)$ is approximately continuous at x with respect to μ.

We know that this is satisfied μ-almost everywhere. For simplicity, we assume $x = 0, T_0\mu = \theta_0 P$.

For fixed $\varrho_0 > 0$ small enough such that $B^{n+1}_{2\varrho_0}(0) \Subset \Omega$ and

$$\operatorname{spt}\mu \cap B^{n+1}_{2\varrho_0}(0) \subseteq B^n_{2\varrho_0}(0) \times] - \varrho_0/2, \varrho_0/2[,$$

we consider

$$\tilde{\mu} := \mu\lfloor(B^n_{\varrho_0}(0) \times] - \varrho_0, \varrho_0[)$$

and its height functions $\tilde{\varphi}_\pm : B^n_{\varrho_0}(0) \to [-\infty, \infty]$.

As $\theta^n(\mu)$ is approximately continuous at 0 and $T_0\mu$ exists, putting

$$\Sigma_0 := \{x = (y, \tilde{\varphi}_\pm(y)) | y \in B^n_{\varrho_0}(0) \cap [\tilde{\varphi}_+ = \tilde{\varphi}_-], T_x\tilde{\mu} \text{ exists }, \nu(T_x\tilde{\mu})e_{n+1} \neq 0\},$$

we get by Lipschitz-Approximation, see [Bra78, Theorem 5.4] and also [Sch01, Lemma 3.4], that

$$\varrho^{-n}\mu(B^{n+1}_\varrho(0) - \Sigma_0) \leq \omega(\varrho). \tag{5.2}$$

Clearly, Proposition 4.1 (4.2) implies quadratic decay for the height-excess for $x = (y, \tilde{\varphi}_\pm(y))$ and $\mathcal{L}^n$-almost all $y \in [\tilde{\varphi}_+ = \tilde{\varphi}_-] \cap B^n_{\varrho_0}(0)$.

Since the tilt-excess is controlled by the height-excess and the mean curvature through the following estimate, see [Bra78, Theorem 5.5] or [Sim, Lemma 22.2],

$$\operatorname{tiltex}_\mu(x, \varrho/2, T) \leq C \operatorname{heightex}_\mu(x, \varrho, T) + C\varrho^{2-n} \int_{B^{n+1}_\varrho(x)} |\vec{\mathbf{H}}_\mu|^2 d\mu,$$

we obtain a quadratic tilt-excess decay

$$\operatorname{tiltex}_\mu(x, \varrho, T_x\mu) = O_x(\varrho^2)$$

when x is a Lebesgue point of $\vec{\mathbf{H}}_\mu \in L^2_{\text{loc}}(\mu)$ and $\theta^n(\mu, x) < \infty$, hence for $\mathcal{L}^n$-almost all $y \in [\tilde{\varphi}_+ = \tilde{\varphi}_-] \cap B^n_{\varrho_0}(0)$.

Putting $Q := \{x \in \Sigma | x \text{ satisfies (5.1)}\}$, this yields $\mu(B^{n+1}_{\varrho_0}(0) \cap \Sigma_0 - Q) = 0$, and by (5.2) and since $\theta^n(\mu, 0) \geq 1$ that $\theta(\mu, \Omega - Q, 0) = 0$. On the other hand this density is equal to 1 almost everywhere with respect to μ, see for example [Sim, Theorem 4.7] or consider Lebesgue points of $\chi_{\Omega - Q} \in L^1_{\text{loc}}(\mu)$. Therefore $\mu(\Omega - Q) = 0$, and the theorem is proved. □

6. – Strong maximum principle

In Proposition 4.1, we have seen that the height functions are approximately differentiable almost everywhere on $[\varphi_\pm \in \mathbb{R}]$. We relate these approximate differentials with the weak mean curvature of the varifold, first almost everywhere on the set $[\varphi_+ = \varphi_-]$ where the height functions coincide.

The following proposition essentially appeared already in the proof of [Sch01, Lemma 6.3]. We would be able to derive it just using the approximate differentials of (4.4) and (C.8) without appealing to Whitney's Extension Theorem. On the other hand, the C^2-Approximation of Theorem 4.1 simplifies the argument.

PROPOSITION 6.1. *Let μ be an integral n-varifold in $\Omega \subseteq \mathbb{R}^{n+1}$ with $H_\mu \in L^p_{loc}(\mu)$, $p > n$, $p \geq 2$, $\Omega := U \times \mathbb{R}$, $U \subseteq \mathbb{R}^n$ open,* $\operatorname{spt}\mu \subseteq U \times]-1,1[$ *and $\varphi_\pm : U \to [-\infty, \infty]$ be the height functions of μ.*

Then $\varphi_\pm$ is twice approximately differentiable $\mathcal{L}^n$-almost everywhere on $[\varphi_\pm \in \mathbb{R}]$ and the approximate differentials satisfy

$$\vec{\mathbf{H}}_\mu(y, \varphi_\pm(y)) = \nabla\left(\frac{\nabla\varphi_\pm}{\sqrt{1+|\nabla\varphi_\pm|^2}}\right)(y)\frac{(-\nabla\varphi_\pm(y), 1)}{\sqrt{1+|\nabla\varphi_\pm(y)|^2}} \tag{6.1}$$

for $\mathcal{L}^n$-almost all $y \in [\varphi_+ = \varphi_-]$.

PROOF. $\varphi_\pm$ are twice approximately differentiable almost everywhere on $[\varphi_\pm \in \mathbb{R}]$ according to Proposition 4.1. From Theorem 4.1, there exists for any $U' \Subset U$ and $\varepsilon > 0$ a set $Q \subseteq U' \cap [\varphi_+ = \varphi_-]$ such that

$$\mathcal{L}^n(U' \cap [\varphi_+ = \varphi_-] - Q) < \varepsilon \tag{6.2}$$

and there exists $\psi \in C^2(U)$ satisfying

$$D^\alpha \varphi_\pm = D^\alpha \psi \quad \text{on} \quad Q \text{ for } |\alpha| \leq 2. \tag{6.3}$$

By (6.2), it suffices to prove (6.1) almost everywhere on Q.

We consider $y \in Q$ such that $\theta^n(\mathcal{L}^n, Q, y) = 1, x := (y, \varphi_\pm(y))$ is a Lebesgue point of $\vec{\mathbf{H}}_\mu$, the tilt-excess decays quadratically at x as in (5.1), $\theta^n(\mu)$ is approximately continuous at x with respect to μ, and $\vec{\mathbf{H}}_\mu(x) \in (T_x\mu)^\perp = \operatorname{span}\nu(x)$, where $\nu(x) = \frac{(-\nabla\varphi_\pm(y),1)}{\sqrt{1+|\varphi_\pm(y)|^2}}$ is a normal at $T_x\mu$. From Theorem 5.1 and [Bra78, Theorem 5.8], we know that almost all $y \in Q$ satisfy these assumptions.

We put $\theta_0 := \theta^n(\mu, x)$ and define the integral n-varifold

$$\mu_\psi := \theta_0 \mathcal{H}^n \lfloor (\operatorname{graph} \psi | U').$$

(6.1) will be proved when we establish

$$\vec{\mathbf{H}}_\mu(x)\nu(x) = \vec{\mathbf{H}}_{\mu_\psi}(x)\nu(x). \tag{6.4}$$

We put

$$\Sigma_0 := \{(z, \varphi_\pm(z)) | z \in Q \subseteq [\varphi_+ = \varphi_- = \psi], \theta^n(\mu, (z, \varphi_\pm(z))) = \theta_0\}$$

and see

$$\mu \lfloor \Sigma_0 = \mu_\psi \lfloor \Sigma_0 \tag{6.5}$$

Since $\theta^n(\mu)$ is approximately continuous at x with respect to μ and $\nu(T_x\mu)e_{n+1} \neq 0$, we get by tilted Lipschitz-Approximation, see [Bra78, Theorem 5.4], [Sch01, Lemma 3.4] for the untilted version and Theorem D.1,

$$\varrho^{-n}\mu(B_\varrho^{n+1}(x) - \Sigma_0) \leq \omega(\varrho). \tag{6.6}$$

From (6.5), we see

$$\begin{aligned}\lim_{\varrho \downarrow 0} \varrho^{-n}\mu_\psi(B_\varrho^{n+1}(x) \cap \Sigma_0) &= \lim_{\varrho \downarrow 0} \varrho^{-n}\mu(B_\varrho^{n+1}(x) \cap \Sigma_0) \\ &= \lim_{\varrho \downarrow 0} \varrho^{-n}\mu(B_\varrho^{n+1}(x)) = \theta_0\omega_n \\ &= \lim_{\varrho \downarrow 0} \varrho^{-n}\mu_\psi(B_\varrho^{n+1}(x)),\end{aligned}$$

hence

$$\varrho^{-n}\mu_\psi(B_\varrho^{n+1}(x) - \Sigma_0) \leq \omega(\varrho). \tag{6.7}$$

We choose $\chi \in C_0^\infty(B_1^{n+1}(0))$ rotationally symmetric with

$$0 \leq \chi \leq 1 \quad \text{and} \quad \chi \equiv 1 \text{ on } B_{1/2}^{n+1}(0)$$

and put $\chi_\varrho(\xi) := \chi(\varrho^{-1}(\xi - x))$.

Since x is a Lebesgue point of $\vec{\mathbf{H}}_\mu$ and $\psi \in C^2(U)$, we calculate for $\tilde{\mu} = \mu, \mu_\psi$ that

$$\begin{aligned}\lim_{\varrho \downarrow 0}(\omega_n\varrho^n)^{-1}\delta\tilde{\mu}(\chi_\varrho) &= -\lim_{\varrho \downarrow 0}(\omega_n\varrho^n)^{-1}\int_{B_\varrho^{n+1}(x)} \chi_\varrho \vec{\mathbf{H}}_{\tilde{\mu}} \mathrm{d}\tilde{\mu} \\ &= -\omega_n^{-1}\theta_0\vec{\mathbf{H}}_{\tilde{\mu}}(x)\int_{T_x\tilde{\mu} \cap B_1^{n+1}(0)} \chi \mathrm{d}\mathcal{L}^n,\end{aligned}$$

and (6.4) will follow when we prove

$$I_\varrho := \varrho^{-n}(\delta\mu(\chi_\varrho) - \delta\mu_\psi(\chi_\varrho))\nu(x) \to 0 \quad \text{when } \varrho \downarrow 0. \tag{6.8}$$

We recall for $\tilde{\mu} = \mu, \mu_\psi$ that

$$\delta\tilde{\mu}(\chi_\varrho)\nu(x) = \int_{B_\varrho^{n+1}(x)} D\chi_\varrho(\xi)T_\xi\tilde{\mu}\nu(x)\mathrm{d}\tilde{\mu}(\xi)$$

and abbreviate

$$R_{\varrho,\tilde{\mu}} := \varrho^{-n} \int_{B_\varrho^{n+1}(x)-\Sigma_0} D\chi_\varrho(\xi)(T_\xi\tilde{\mu} - T_x\tilde{\mu})\nu(x)\mathrm{d}\tilde{\mu}(\xi).$$

Using (6.3), (6.5) and $T_x\tilde{\mu}\nu(x) = 0$, as $\nu(x)$ is normal to $T_x\tilde{\mu}$, we obtain that

$$I_\varrho = R_{\varrho,\mu} - R_{\varrho,\mu_\psi}.$$

We estimate

$$\begin{aligned}|R_{\varrho,\tilde{\mu}}| &\leq C\varrho^{-n-1} \int_{B_\varrho^{n+1}(x)-\Sigma_0} \| T_\xi\tilde{\mu} - T_x\tilde{\mu} \| \,\mathrm{d}\tilde{\mu}(\xi) \\ &\leq C\varrho^{-1}\left(\varrho^{-n}\tilde{\mu}(B_\varrho^{n+1}(x) - \Sigma_0)\right)^{\frac{1}{2}} \left(\varrho^{-n}\int_{B_\varrho^{n+1}(x)} \| T_\xi\tilde{\mu}-T_x\tilde{\mu} \|^2 \,\mathrm{d}\tilde{\mu}(\xi)\right)^{\frac{1}{2}} \\ &\leq C\varrho^{-1}\omega(\varrho)^{\frac{1}{2}}\, \mathrm{tiltex}_{\tilde{\mu}}(x,\varrho,T_x\tilde{\mu})^{\frac{1}{2}},\end{aligned}$$

where we have used (6.6) and (6.7).

Now for $\tilde{\mu} = \mu$, we have quadratic decay of the tilt-excess at x by assumption, whereas such decay is immediate for $\tilde{\mu} = \mu_\psi$, since $D^2\psi \in C^0(U)$. Therefore $|R_{\varrho,\tilde{\mu}}| \leq C\omega(\varrho)^{\frac{1}{2}}$ which proves (6.8), hence (6.4) and (6.1). □

By a covering argument, we can extend (6.1) almost everywhere on the set $[\varphi_\pm \in \mathbb{R}]$ where the height functions are finite. Combining with Lemma A.6, we obtain that the height functions are viscosity sub- and supersolutions of the minimal surface equation with right hand side given by the weak mean curvature of the varifold. By interpreting the definition of viscosity solutions as in the introduction, this can be considered as a weak maximum principle.

THEOREM 6.1. *Let μ be an integral n-varifold in $\Omega \subseteq \mathbb{R}^{n+1}$ with $H_\mu \in L^p_{\mathrm{loc}}(\mu)$, $p > n$, $p \geq 2$, $\Omega := U \times \mathbb{R}, U \subseteq \mathbb{R}^n$ open,* $\operatorname{spt}\mu \subseteq U\times]-1,1[$ *and $\varphi_+ : U \to [-\infty,\infty[$ be the upper height function of μ.*

Then φ_+ is twice approximately differentiable $\mathcal{L}^n$-almost everywhere on $[\varphi_\pm \in \mathbb{R}]$ and the approximate differentials satisfy

$$\vec{\mathbf{H}}_\mu(y,\varphi_+(y)) = \nabla\left(\frac{\nabla\varphi_+}{\sqrt{1+|\nabla\varphi_+|^2}}\right)(y)\frac{(-\nabla\varphi_+(y),1)}{\sqrt{1+|\nabla\varphi_+(y)|^2}} \tag{6.9}$$

for $\mathcal{L}^n$-almost all $y \in [\varphi_+ \in \mathbb{R}]$. Moreover φ_+ is a $W^{2,p}$-viscosity subsolution of

$$-\nabla\left(\frac{\nabla\varphi_+}{\sqrt{1+|\nabla\varphi_+|^2}}\right) \leq \vec{\mathbf{H}}_\mu(.,\varphi_+)\frac{(-\nabla\varphi_+,1)}{\sqrt{1+|\nabla\varphi_+|^2}} \quad \text{in } U, \tag{6.10}$$

where the right hand side is extended arbitrarily on $U - [\varphi_+ \in \mathbb{R}]$ to a function still in $L^p_{\mathrm{loc}}(U)$.

PROOF. φ_+ is twice approximately differentiable almost everywhere on $[\varphi_+ \in \mathbb{R}]$ according to Proposition 4.1.

We consider

$$N := \{y \in [\varphi_+ \in \mathbb{R}] | \text{ (6.9) is not satisfied for} y\}$$

and

$$Q := \{y \in [\varphi_+ \in \mathbb{R}] | x := (y, \varphi_+(y)), T_x\mu \text{ exists}, \nu(T_x\mu)e_{n+1} \neq 0, \theta^n(\mu) \text{ is approximately continuous at } x \text{ with respect to } \mu\}.$$

Clearly by Co-Area formula, $\mathcal{L}^n([\varphi_+ \in \mathbb{R}] - Q) = 0$, and (6.9) will be proved when we show that

$$\mathcal{L}^n(Q \cap N) = 0. \tag{6.11}$$

We assume $0 \in Q, \varphi_+(0) = 0, 0 \in \operatorname{spt}\mu$. Since $\nu(T_0\mu)e_{n+1} \neq 0$, there exists $1 \leq \Gamma < \infty$ such that for $\varrho_0 > 0$ small enough

$$\operatorname{spt}\mu \cap B^{n+1}_{\varrho_0}(0) \subseteq \{(y, t) | |t| < \Gamma|y|\}. \tag{6.12}$$

Since φ_+ is upper semicontinuous, choosing $0 < \varrho_1 \ll \varrho_0$, we have

$$\varphi_+ \leq \varrho_0/2 \quad \text{on } B^n_{\varrho_1}(0). \tag{6.13}$$

Choosing further $\varrho_1\sqrt{1 + 4\Gamma^2} < \varrho_0$, we define

$$\tilde{\mu} := \mu\lfloor(B^n_{\varrho_1}(0)\times] - 2\Gamma\varrho_1, 2\Gamma\varrho_1[)$$

and consider its height functions $\tilde{\varphi}_\pm : B^n_{\varrho_0}(0) \to [-\infty, \infty]$. Since $B^n_{\varrho_1}(0)\times] - 2\Gamma\varrho_1, 2\Gamma\varrho_1[\subseteq B^{n+1}_{\varrho_0}(0)$, we see from (6.12) that

$$\operatorname{spt}\tilde{\mu} \subseteq B^n_{\varrho_1}(0)\times] - \Gamma\varrho_1, \Gamma\varrho_1[, \tag{6.14}$$

and the height functions $\tilde{\varphi}_\pm$ are upper- and lower semicontinuous, respectively. Clearly, $\tilde{\varphi}_+ \leq \varphi_+$ on $B^n_{\varrho_1}(0)$. On the other hand,

$$[\tilde{\varphi}_+ \in \mathbb{R}] \subseteq [\tilde{\varphi}_+ = \varphi_+]. \tag{6.15}$$

Indeed, for $y \in B^n_{\varrho_1}(0)$ with $\tilde{\varphi}_+(y) \in \mathbb{R}$, we see from (6.13) that

$$-\Gamma\varrho_1 \leq \tilde{\varphi}_+(y) \leq \varphi_+(y) \leq \varrho_0/2,$$

hence $(y, \varphi_+(y)) \in B^{n+1}_{\varrho_0}(0)$ and $|\varphi_+(y)| \leq \Gamma\varrho_1$ by (6.12). In particular, $(y, \varphi_+(y)) \in \operatorname{spt}\tilde{\mu}$, and $\varphi_+(y) \leq \tilde{\varphi}_+(y)$.

Now (6.9) is satisfied $\mathcal{L}^n$-almost everywhere on $[\tilde\varphi_+ = \tilde\varphi_-] \subseteq [\tilde\varphi_+ \in \mathbb{R}]$ by Proposition 6.1, (6.14) and (6.15), that is

$$B^n_{\varrho_1}(0) \cap N \subseteq B^n_{\varrho_1}(0) - [\tilde\varphi_+ = \tilde\varphi_-]. \tag{6.16}$$

Since $\theta^n(\tilde\mu)$ is approximately continuous at 0 with respect to μ, as $0 \in Q$, and $\nu(T_0\mu)e_{n+1} \neq 0$, we get by tilted Lipschitz-Approximation, see [Bra78, Theorem 5.4], [Sch01, Lemma 3.4] for the untilted version and Theorem D.1,

$$\lim_{\varrho \downarrow 0} \varrho^{-n} \mathcal{L}^n(B^n_\varrho(0) - [\tilde\varphi_+ = \tilde\varphi_-]) = 0,$$

hence by (6.16)

$$\theta(\mathcal{L}^n, N, 0) = \lim_{\varrho\downarrow 0} (\omega_n \varrho^n)^{-1} \mathcal{L}^n(B^n_\varrho(0) \cap N) = 0.$$

Since $0 \in Q$ was arbitrary after translation, we get (6.11), and (6.9) is proved.

Finally, since $\vec{\mathbf{H}}_\mu \in L^p_{\mathrm{loc}}(\mu)$, we see that $(y \mapsto \vec{\mathbf{H}}_\mu(y, \varphi_+(y))\chi_{[\varphi_+\in\mathbb{R}]}(y)) \in L^p_{\mathrm{loc}}(U)$, and (6.10) follows from Lemma 3.1, (6.9) and Lemma A.6. □

Performing a perturbation argument on the minimal surface equation, we obtain the strong maximum principle from (6.10).

THEOREM 6.2 (Strong maximum principle). *Let μ be an integral n-varifold in $\Omega \subseteq \mathbb{R}^{n+1}$ with $H_\mu \in L^p_{\mathrm{loc}}(\mu)$, $p > n$, $p \geq 2$, $\Omega := U \times \mathbb{R}$, $U \subseteq \mathbb{R}^n$ open,* $\operatorname{spt}\mu \subseteq U \times]-1, 1[$ *and $\varphi_+ : U \to [-\infty, \infty[$ be the upper height function of μ.*

Then $\operatorname{spt}\mu$ *cannot be touched from above by the graph of a function $\psi \in W^{2,p}(U')$, $U' \Subset U$, open and connected, which satisfies*

$$-\nabla\left(\frac{\nabla\psi}{\sqrt{1+|\nabla\psi|^2}}\right)(y) \geq \vec{\mathbf{H}}_\mu(y, \varphi_+(y)) \frac{(-\nabla\varphi_+(y), 1)}{\sqrt{1+|\nabla\varphi_+(y)|^2}} \tag{6.17}$$

for $\mathcal{L}^n$-almost all $y \in U' \cap [\varphi_+ \in \mathbb{R}]$, unless

$$\operatorname{graph}\psi \subseteq \operatorname{spt}\mu. \tag{6.18}$$

PROOF. Let $0 \in \operatorname{spt}\mu$ be a point of touching. We have

$$\varphi_+ \leq \psi \quad \text{in } U', \quad \varphi_+(0) = \psi(0) = 0.$$

We put

$$v := -\nabla\left(\frac{\nabla\psi}{\sqrt{1+|\nabla\psi|^2}}\right) \in L^p(U')$$

and see from (6.17) and Theorem 6.1 that φ_+ is a $W^{2,p}$-viscosity subsolution of

$$-\nabla\left(\frac{\nabla\varphi_+}{\sqrt{1+|\nabla\varphi_+|^2}}\right) \leq v \quad \text{in } U'. \tag{6.19}$$

We fix an upper bound $1 \leq \Gamma_0 < \infty$

$$\| v \|_{L^p(U')}, \| \psi \|_{W^{2,p}(U')} \leq \Gamma_0.$$

We claim that there exits $\varrho_1 > 0$ such that

$$\varphi_+ = \psi \quad \text{in } B^n_{\varrho_1}(0) \subseteq U'. \tag{6.20}$$

By connectedness of U', this will prove (6.18).

If (6.20) is not satisfied, by upper semicontinuity of $\varphi_+ - \psi$, we can choose $\eta \in C^2_0(B^n_{\varrho_1}(0))$, $\eta \geq 0$, $\eta \not\equiv 0$ such that

$$\psi - \eta \geq \varphi_+ \quad \text{in } B^n_{\varrho_1}(0) \text{ and } \| \psi - \eta \|_{W^{2,p}(U')} \leq \Gamma_0 + 1.$$

We select $y_0 \in B^n_{\varrho_1}(0) - \{0\}$ such that $\eta(y_0) > 0$ and put $0 < \varrho := |y_0| < \varrho_1$.

For $\varrho_1 < \varrho_1(n, p, \Gamma_0)$, $B^n_{\varrho_1}(0) \subseteq U'$ small enough and $0 \leq \tau \leq 1$, we get by a perturbation argument that there are unique solutions $\psi_\tau \in W^{2,p}(B^n_\varrho(0))$ of

$$-\nabla \left(\frac{\nabla \psi_\tau}{\sqrt{1 + |\nabla \psi_\tau|^2}} \right) = v + \tau \quad \text{in } B^n_\varrho(0), \qquad \psi_\tau = \psi - \eta \quad \text{on } \partial B^n_\varrho(0), \tag{6.21}$$

which moreover satisfy

$$\| \psi_\tau \|_{W^{2,p}(B^n_\varrho(0))} \leq C_{n,p}(\Gamma_0, \varrho_1), \tag{6.22}$$

see [Sch01, Lemma 4.6] for the details of this perturbation argument.

Since $\psi_0 = \psi - \eta \leq \psi, \psi_0 \not\equiv \psi$ on $\partial B^n_\varrho(0)$, we see by the definition of v and the strong maximum principle, see [GT, Theorem 8.19], that $\psi_0 < \psi$ in $B^n_\varrho(0)$. By (6.22), $\psi_\tau \to \psi_0$ weakly in $W^{2,p}(B^n_\varrho(0))$, hence uniformly, and there exists $\tau > 0$ such that $\psi_\tau(0) < \psi(0) = \varphi_+(0)$.

On the other hand, $\psi_\tau = \psi - \eta \geq \varphi_+$ on $\partial B^n_\varrho(0)$, and $\varphi_+ - \psi_\tau$ has an interior maximum in $B^n_\varrho(0)$. Together with (6.21) and $\tau > 0$, this contradicts (6.19) and establishes (6.20), hence proves the theorem. □

Appendix

In this appendix, we collect for the reader's convenience some results which are consequences or adaptions of standard results.

A. Non-uniform ellipticity

We will use the following definition of viscosity solutions, see [Caf89] or [CafCK96].

DEFINITION A.1. We consider $U \subseteq \mathbb{R}^n$ open, $1 \leq p \leq \infty$, $p > n/2$ and $F : U \times \mathbb{R}^n \times S(n) \to \mathbb{R}$ which is degenerate elliptic that is $F(.,.,X) \leq F(.,.,Y)$ if $X \leq Y$.

For $u \in L^p_{\text{loc}}(U)$, we call an upper semicontinuous function $\varphi : U \to [-\infty, \infty[$ a $W^{2,p}$-viscosity subsolution of $-F(., \nabla\varphi, D^2\varphi) \leq u$ in U, if for all $\psi \in W^{2,p}(U'), U' \Subset U$ open, $\tau > 0$, such that $-F(., \nabla\psi, D^2\psi) \geq u + \tau$ pointwise almost everywhere in U', the function $\varphi - \psi$ has no interior maximum in U', that is there is no $y \in U'$ with

$$\varphi - \psi \leq (\varphi - \psi)(y) \in \mathbb{R} \quad \text{in } U'.$$

The definition of supersolutions is analogously. Solutions are functions which are both sub- and supersolutions.

In particular, we will consider elliptic operators that are uniformly elliptic only for bounded gradients.

DEFINITION A.2. Let $\mathcal{F} = \mathcal{F}_U$, for $U \subseteq \mathbb{R}^n$ open, be the class of elliptic operators $F : U \times \mathbb{R}^n \times S(n) \to \mathbb{R}$ such that $F(y, a, 0) = 0$ and F is uniformly elliptic for bounded gradients that is for $\Gamma < \infty$ there exist $0 < \lambda(\Gamma) \leq \Lambda(\Gamma) < \infty$, such that for $y \in U, |a| \leq \Gamma, X, Y \in S(n)$

$$F(y, a, X + Y) - F(y, a, X) \leq \Lambda(\Gamma) \parallel Y^+ \parallel - \lambda(\Gamma) \parallel Y^- \parallel .$$

We recall the following two results on fully non-linear elliptic equations. The first one due to Caffarelli in [Caf89] and Trudinger in [T89], see also [CafCab, Lemma 7.8] and [CafCK96], states that subsolutions of uniformly elliptic equations with right hand side in L^n are touched from above by paraboloids or equivalently have second order superdifferentials almost everywhere. Secondly, from ABP-estimate, see [Caf89, Lemma 1], [CafCab, Theorem 3.2] and [CafCK96, Proposition 3.3], see also Alexandroff's Maximum Principle for strong solutions [GT, Theorem 9.1], supersolutions of uniformly elliptic equations with right hand side in L^n which have a strict minimum coincide with their convex envelope on a set of positive measure, hence have subgradients on a set of positive measure. Actually, both these results give even quantitative estimates on the opening of the paraboloids in an integral norm and the size of the measure in terms of the ellipticity constants and the right hand side, respectively.

The main result of this section is that these properties remain true in their non-quantitative versions for sub- and supersolutions of equations which are uniformly elliptic only for bounded gradients. We slightly weaken the assumptions on the right hand side by assuming them to be in L^p, $p > n$.

For the proof, we consider sup-convolutions of order 1, defined in (A.10) below, and observe that equations of class $\mathcal{F}$ are uniformly elliptic for sup-convolutions, as sup-convolutions are lipschitz. As unbounded right hand sides do not behave well for sup-convolutions, we first have to subtract a solution of a certain elliptic equation. We will consider subsolutions.

Let $U \Subset \mathbb{R}^n$, $F \in \mathcal{F}_U$, $u \in L^p(U)$, $n < p < \infty$ and $\varphi : U \to [-\infty, \infty[$ be upper semicontinuous and be a $W^{2,p}$-subsolution of

$$-F(\nabla\varphi, D^2\varphi) \leq u \quad \text{in } U. \tag{A.1}$$

Further, we assume that φ is bounded

$$\sup_U |\varphi| < \infty. \tag{A.2}$$

Next, we may assume that $\Lambda \leq 1$ in Definition A.2 for F that is

$$|F(a, X+Y) - F(a, X)| \leq \| Y \| \tag{A.3}$$

for $a \in \mathbb{R}^n$, $X, Y \in S(n)$.

Indeed, we may choose $(\Gamma \mapsto \Lambda(\Gamma))$ to be monotone and continuous and put

$$\tilde{F}(a, X) := \max(1, \Lambda(|a|))^{-1} F(a, X)$$

and see

$$-\tilde{F}(\nabla\varphi, D^2\varphi) \leq u_+ \in L^p(U).$$

Next we choose $R_0 > 0$ large, such that $U \subseteq B^n_{R_0/2}(0)$, $0 < \varepsilon < 1, \lambda := \lambda(3/\varepsilon)/n$ and $\tilde{u} \in L^\infty(U)$ such that $\| u_+ - \tilde{u} \|_{L^p(U)} \leq \delta_1$ where we choose $0 < \delta_1 \ll \varepsilon$ below. Approximating $(u_+ - \tilde{u})\chi_U$ by smooth functions, using Perron's method, see [CIL, Theorem 4.1], and combining this with the ABP-estimate, Evans-Krylov Theorem, as $\mathcal{M}^-_\lambda$ is concave, and the $W^{2,p}$-interior estimates due to Caffarelli, see [Caf89, Theorem 1] and [CafCab, Theorems 3.2, 6.6, 7.1 and 7.4], we get a function $w \in C^0(\overline{B^n_{R_0}(0)}) \cap W^{2,p}_{\text{loc}}(B^n_{R_0}(0))$ satisfying

$$\begin{gathered} -\mathcal{M}^-_\lambda(D^2 w) = (\tilde{u} - u_+)\chi_U \quad \text{pointwise almost everywhere in } B^n_{R_0}(0), \\ w = 0 \quad \text{on } \partial B^n_{R_0}(0). \end{gathered} \tag{A.4}$$

Moreover, we get the estimates

$$\| w \|_{L^\infty(B^n_{R_0}(0))}, \| w \|_{W^{2,p} \cap C^{1,\iota}(B^n_{R_0/2}(0))} \leq C_{n,p}(R_0, \varepsilon)\delta_1. \tag{A.5}$$

Choosing $\delta_1 = \delta_1(R_0, \varepsilon, n, p)$ small such that $C_{n,p}(R_0, \varepsilon)\delta_1 \leq \varepsilon$, we get

$$\| w \|_{L^\infty(B^n_{R_0})}, \| \nabla w \|_{L^\infty(B^n_{R_0/2})} \leq \varepsilon. \tag{A.6}$$

Next, we put

$$\gamma := \varphi + w. \tag{A.7}$$

Putting

$$F_\varepsilon(a, X) := \sup_{|b|\le\varepsilon} F(a+b, X),$$

we see that $F_\varepsilon \in \mathcal{F}$, and we calculate formally

$$\begin{aligned}
-F_\varepsilon(\nabla\gamma, D^2\gamma) &\le -F(\nabla\gamma - \nabla w(y), D^2\varphi + D^2 w) \\
&\le -F(\nabla\varphi, D^2\varphi) + \| D^2 w^- \| - \lambda(3/\varepsilon) \| D^2 w^+ \| \\
&\qquad + 2\chi_{[|\nabla\gamma - \nabla w(y)| \ge 3/\varepsilon]} \| D^2 w \| \\
&\le -F(\nabla\varphi, D^2\varphi) - \mathcal{M}_\lambda^-(D^2 w) + 2\chi_{[|\nabla\gamma| \ge 2/\varepsilon]} \| D^2 w \| \\
&\le \tilde{u} + 2\chi_{[|\nabla\gamma| \ge 2/\varepsilon]} \| D^2 w \| \quad \text{in } U.
\end{aligned} \tag{A.8}$$

Replacing γ and φ by test functions η and $\xi := \eta - w \in W^{2,p}(U''), U'' \Subset U$ in (A.8) justifies the formal computation, and, since $\gamma - \eta = \varphi - \xi$, we get that γ is a $W^{2,p}$-viscosity subsolution of

$$-F_\varepsilon(\nabla\gamma, D^2\gamma) \le \tilde{u} + 2\chi_{[|\nabla\gamma| \ge 2/\varepsilon]} \| D^2 w \| \quad \text{in } U. \tag{A.9}$$

Now, we consider the sup-convolutions of order 1 for φ and γ given by

$$\begin{aligned}
&\varphi^\varepsilon(z) := \sup_{y\in U} \left(\varphi(y) - \frac{1}{\varepsilon}|y - z| \right) \\
\text{and} \qquad &\gamma^\varepsilon(z) := \sup_{y\in U} \left(\gamma(y) - \frac{1}{\varepsilon}|y - z| \right)
\end{aligned} \quad \text{for } z \in U \tag{A.10}$$

Clearly,

$$\begin{gathered}
\varphi \le \varphi^\varepsilon \text{ and } \gamma \le \gamma^\varepsilon \quad \text{on } U, \\
\operatorname{Lip}\varphi^\varepsilon, \quad \operatorname{Lip}\gamma^\varepsilon \le 1/\varepsilon, \\
\| \gamma^\varepsilon - \varphi^\varepsilon \|_{L^\infty(U)} \le \sup_U |\gamma - \varphi| = \| w \|_{L^\infty(U)} \le \varepsilon.
\end{gathered} \tag{A.11}$$

For $z \in U' \Subset U$, we consider $y \in U$ such that $\gamma(y) - \frac{1}{\varepsilon}|y - z| \ge \gamma(z)$. We get

$$|y - z| \le \varepsilon osc_U \gamma \le 2\varepsilon \sup_U |\gamma| \le 2\varepsilon(\sup_U |\varphi| + \varepsilon).$$

Since $U' \Subset U$ and $\sup_U |\varphi| < \infty$ by (A.2), we conclude for small ε that

$$\forall z \in U' : \exists y \in U : \gamma^\varepsilon(z) = \gamma(y) - \frac{1}{\varepsilon}|y - z|. \tag{A.12}$$

By standard procedure for sup-convolutions, see [CafCab, Section 5.1], we get from (A.9) that γ^ε is a $W^{2,p}$-viscosity subsolution of

$$-F_\varepsilon(\nabla\gamma^\varepsilon, D^2\gamma^\varepsilon) \leq \| \tilde{u} \|_{L^\infty(U)} < \infty \quad \text{in } U'.$$

We observe that the term $2\chi_{[|\nabla\gamma^\varepsilon| \geq \frac{2}{\varepsilon}]} \| D^2 w \|$ drops out since $|\nabla\gamma^\varepsilon| \leq \frac{1}{\varepsilon}$ by (A.11). Further, this equation is uniformly elliptic since γ^ε is lipschitz-continuous. As $\lambda = \lambda(3/\varepsilon)/n$ and $\operatorname{Lip}\gamma^\varepsilon \leq 1/\varepsilon$, we can rewrite it using the Pucci-Operator

$$-\mathcal{M}_\lambda^+(D^2\gamma^\varepsilon) \leq \| \tilde{u} \|_{L^\infty(U)} < \infty \quad \text{in } U'. \tag{A.13}$$

Putting $\tilde{\varphi}^\varepsilon := \gamma^\varepsilon - w$, we get

$$-\mathcal{M}_\lambda^+(D^2\tilde{\varphi}^\varepsilon) \leq u_\varepsilon \quad \text{in } U'. \tag{A.14}$$

where $u_\varepsilon := \| \tilde{u} \|_{L^\infty(U)} + n \| D^2 w \| \in L^p(U')$.

From (A.11), we see

$$\| \tilde{\varphi}^\varepsilon - \varphi^\varepsilon \|_{L^\infty(U)} \leq \| \gamma^\varepsilon - \varphi^\varepsilon \|_{L^\infty(U)} + \| w \|_{L^\infty(U)} \leq 2 \| w \|_{L^\infty(U)} \leq 2\varepsilon. \tag{A.15}$$

Now by standard results on convolutions, we know

$$\varphi(y) = \operatorname*{lim^*}_{\varepsilon\downarrow 0} \varphi_\varepsilon(y) := \sup\{\limsup_{k\to\infty} \varphi_{\varepsilon_k}(y_k) | \varepsilon_k \to 0, y_k \to y\} \quad \text{for } y \in U,$$

hence

$$\varphi = \operatorname*{lim^*}_{\varepsilon\downarrow 0} \tilde{\varphi}_\varepsilon \quad \text{on } U. \tag{A.16}$$

In the sequel, we will use this general construction to adapt for non-uniformly elliptic equations. First, we turn to Caffarelli's and Trudinger's result on touching subsolutions from above by paraboloids or equivalently subsolutions having second order superdifferentials. Its extension appeared already in [Sch01, Lemma 5.3].

LEMMA A.3. *Let $U \subseteq \mathbb{R}^n$, $F \in \mathcal{F}_U$, $u \in L^p(U)$, $n < p < \infty$ and $\varphi : U \to [-\infty, \infty[$ be upper semicontinuous, bounded above and a $W^{2,p}$-subsolution of*

$$-F(\nabla\varphi, D^2\varphi) \leq u \quad \textit{in } U. \tag{A.17}$$

Moreover, we assume that for $\mathcal{L}^n$-almost all $y \in U$ with $\varphi(y) \in \mathbb{R}$, there exists $C = C_y < \infty$ such that

$$\varphi(z) \leq \varphi(y) + C_y|y - z| \quad \textit{for all } z \in U. \tag{A.18}$$

Then

$$\overline{\theta}(\varphi, U) < \infty \quad \mathcal{L}^n\text{-almost everywhere on } [\varphi \in \mathbb{R}]. \tag{A.19}$$

Proof. Since φ is bounded above, the conclusion is local, and we may assume that $U \Subset \mathbb{R}^n$. First, we reduce the lemma to the case where φ is bounded from below and above. For large M, we put $\varphi_M := \max(\varphi, -M)$.

Since $F(a, 0) = 0$, we immediately get

$$-F(\nabla\varphi_M, D^2\varphi_M) \le u_+ \in L^p(U) \quad \text{in } U.$$

Clearly, φ_M satisfies (A.18) $\mathcal{L}^n$-almost everywhere on $[\varphi \ge -M]$. If $y \in [\varphi < -M]$, then for $\varrho > 0$ small enough $\varphi < -M$ on $B^n_\varrho(y) \subseteq U$, hence $\varphi_M \equiv -M$ on $B^n_\varrho(y)$, and φ_M satisfies (A.18) for $z \in B^n_\varrho(y)$ for any $C_y \ge 0$. Choosing $C_y > \varrho^{-1}(\max(\sup_U \varphi, -M) + M)$, we see that φ_M satisfies (A.18) on $U - B^n_\varrho(y)$ as well.

As we assume the lemma to be true for bounded functions which satisfy (A.17) and (A.18), we get from (A.19) that $\overline{\theta}(\varphi_M, U) < \infty$ $\mathcal{L}^n$-almost everywhere on U, in particular $\overline{\theta}(\varphi, U) < \infty$ $\mathcal{L}^n$-almost everywhere on $[\varphi \ge -M]$, as $\varphi \le \varphi_M$, which yields (A.19) for φ as $M \to \infty$.

Therefore, we may assume (A.2) and follow the construction above for fixed $U' \Subset U$ until we arrive at (A.13).

Then we can apply Caffarelli's and Trudinger's theorem, see [Caf89], [CafCab, Lemma 7.8] or [T89, Theorem 1], to conclude that $\overline{\theta}(\gamma^\varepsilon, U) < \infty$ almost everywhere in U', first locally then globally as γ_ε is bounded from above. Clearly this implies $\overline{\theta}(\gamma, U) < \infty$ almost everywhere on $[\gamma^\varepsilon = \gamma] \cap U'$, as $\gamma \le \gamma^\varepsilon$. On the other hand $\theta(w, U) < \infty$ almost everywhere on U, since $w \in W^{2,p}(U)$, $U \subseteq B^n_{R_0/2}(0)$, and (A.7) yields $\overline{\theta}(\varphi, U) < \infty$ almost everywhere on $[\gamma^\varepsilon = \gamma] \cap U'$.

Finally, we observe from (A.6) that $[\varphi^{2\varepsilon} = \varphi] \cap U' \subseteq [\gamma^\varepsilon = \gamma]$, and (A.19) follows observing $\mathcal{L}^n([\varphi^\varepsilon = \varphi] \cap U') \nearrow \mathcal{L}^n([\varphi \in \mathbb{R}] \cap U')$ by (A.18). □

Next, we turn to the existence of subgradients on a set of positive measure given by ABP-estimate.

Lemma A.4. *Let $U \subseteq \mathbb{R}^n$, $F \in \mathcal{F}_U$, $u \in L^p(U)$, $n < p < \infty$ and $\varphi : U \to [-\infty, \infty[$ be upper semicontinuous and a $W^{2,p}$-subsolution of*

$$-F(\nabla\varphi, D^2\varphi) \le u \quad \text{in } U. \tag{A.20}$$

If for some $\psi \in W^{2,p}(U)$ and $U' \Subset U$

$$\sup_{\partial U'}(\varphi - \psi) < \sup_{U'}(\varphi - \psi), \tag{A.21}$$

then for any $\delta > 0$

$$\mathcal{L}^n(\{y \in U' \mid \partial^{U'}(\psi - \varphi)(y) \cap B^n_\delta(0) \neq \emptyset\}) > 0 \tag{A.22}$$

where

$$\partial^{U'}(\psi - \varphi)(y) := \{a \in \mathbb{R}^n \mid (\psi - \varphi)(z) \ge (\psi - \varphi)(y) + a(z - y) \text{ for all } z \in U'\}$$

denotes the set of subgradients of $\psi - \varphi$ in U'.

PROOF. We may assume that $U \Subset \mathbb{R}^n$, φ is bounded from above on U and

$$\psi \in C^{1,\iota}(U) \tag{A.23}$$

where $\iota := (1 - n/p) \in]0, 1[$.

First, we reduce the lemma to the case where φ is bounded from below and above. For large M, we put $\varphi_M := \max(\varphi, -M)$.

Since $F(a, 0) = 0$, we immediately get

$$-F(\nabla\varphi_M, D^2\varphi_M) \leq u_+ \in L^p(U) \quad \text{in } U.$$

From (A.21), we can choose $y_0 \in U'$ such that $\sup_{\partial U'}(\varphi - \psi) < (\varphi - \psi)(y_0)$, in particular $\varphi(y_0) > -\infty$. Observing that $\sup_{U'} |\psi| < \infty$ by (A.23), we choose $M > 2\sup_{U'} |\psi| - \varphi(y_0)$, and get

$$\begin{aligned}\sup_{\partial U'}(\varphi_M - \psi) &\leq \max(\sup_{\partial U'}(\varphi - \psi), -M + \sup_{U'} |\psi|) \\ &< \max((\varphi - \psi)(y_0), \varphi(y_0) - \sup_{U'} |\psi|) \\ &= (\varphi - \psi)(y_0) = (\varphi_M - \psi)(y_0) \\ &\leq \sup_{U'}(\varphi_M - \psi).\end{aligned}$$

Therefore φ_M satisfies (A.20) and (A.21). As we assume the lemma to be true for bounded functions which satisfy (A.20) and (A.21), we get from (A.22) that

$$\mathcal{L}^n(\{y \in U' | \partial^{U'}(\psi - \varphi_M)(y) \cap B^n_\delta(0) \neq \emptyset\}) > 0.$$

Now for $y \in U'$ and $|a| < \delta$ such that $a \in \partial^{U'}(\psi - \varphi_M)(y)$, we have

$$(\varphi_M - \psi)(z) \leq (\varphi_M - \psi)(y) - a(z - y) \quad \text{for all } z \in U'.$$

Taking $z = y_0$, we see

$$\varphi(y_0) \leq \varphi_M(y) + 2(\sup_{U'} |\psi| + \delta \operatorname{diam}(U')),$$

and choosing

$$M > 2(\sup_{U'} |\psi| + \delta \operatorname{diam}(U')) - \varphi(y_0),$$

we obtain $-M < \varphi_M(y)$, hence $\varphi_M(y) = \varphi(y)$ and $a \in \partial^{U'}(\psi - \varphi)(y)$, since $\varphi_M \geq \varphi$.

Therefore, we may assume (A.2) and follow the construction above until we arrive at (A.16). Together with (A.21), this yields

$$\limsup_{\varepsilon \downarrow 0} \sup_{\partial U'}(\tilde{\varphi}^\varepsilon - \psi) \leq \sup_{\partial U'}(\varphi - \psi) < \sup_{U'}(\varphi - \psi) \leq \sup_{U'}(\tilde{\varphi}^\varepsilon - \psi),$$

since $\tilde{\varphi}^\varepsilon \geq \varphi$, as $\gamma^\varepsilon \geq \gamma$ by (A.11). For ε small enough, we get

$$\sup_{\partial U'}(\tilde{\varphi}^\varepsilon - \psi) < \sup_{U'}(\tilde{\varphi}^\varepsilon - \psi). \tag{A.24}$$

Putting

$$\xi := \min(\psi - \tilde{\varphi}^\varepsilon + \sup_{\partial U'}(\tilde{\varphi}^\varepsilon - \psi), 0) \quad \text{in } \overline{U'}$$

and $\xi \equiv 0$ on $\mathbb{R}^n - \overline{U'}$, we see from (A.14) and [CafCab, Proposition 2.8] that

$$-\mathcal{M}_\lambda^-(D^2\xi) \geq -\chi_{U'}(u_{\varepsilon,+} + n \parallel D^2\psi \parallel) \in L^p(\mathbb{R}^n).$$

We conclude by ABP-estimate, see [Caf89, Lemma 1], [CafCab, Theorem 3.2] and [CafCK96, Proposition 3.3], and (A.24) that

$$0 < \sup_{B_R^n(0)} \xi_- \leq C_n(\lambda, R)\left(\int_{[co(\xi|B_{2R}^n(0))=\xi]\cap U'} |u_{\varepsilon,+} + n \parallel D^2\psi \parallel|^n\right)^{1/n}$$

where $R > 0$ large , $U' \subseteq B_R^n(0)$ and $co(\xi|B_{2R}^n(0))$ denotes the convex envelope of ξ on $B_{2R}^n(0)$. In particular

$$\mathcal{L}^n([co(\xi|B_{2R}^n(0)) = \xi] \cap U') > 0. \tag{A.25}$$

For $y \in [co(\xi|B_{2R}^n(0)) = \xi] \cap U'$, there exists $a \in \mathbb{R}^n$ such that

$$\xi(z) \geq \xi(y) + a(z - y) \quad \text{for } z \in B_{2R}^n(0). \tag{A.26}$$

First, we conclude $\xi(y) < 0$.

Indeed, $\xi \leq 0$, and if $\xi(y) = 0$ then for all $z \in \partial B_{2R}^n(0)$

$$0 = \xi(z) \geq a(z - y),$$

hence $a = 0$ as $y \in B_{2R}^n(0)$. Then (A.26) yields $\xi \geq 0$, but $\sup \xi_- > 0$ by (A.24). Therefore $\xi(y) < 0$ and

$$\xi(y) = (\psi - \tilde{\varphi}^\varepsilon)(y) + \sup_{\partial U'}(\tilde{\varphi}^\varepsilon - \psi). \tag{A.27}$$

From (A.26), we conclude for $z \in \partial B_{2R}^n(0)$ that

$$0 = \xi(z) \geq \xi(y) + a(z - y),$$

hence, as $y \in U' \subseteq B_R^n(0)$,

$$|a| \leq R^{-1} \sup |\xi| \leq 2R^{-1} \parallel \tilde{\varphi}^\varepsilon - \psi \parallel_{L^\infty(U')} \leq 2R^{-1}(\sup_U |\varphi| + 2\varepsilon + \parallel \psi \parallel_{L^\infty(U')}),$$

where we have used (A.10) and (A.15). For R large enough, we get by (A.2) and (A.23) that

$$|a| < \delta. \tag{A.28}$$

By (A.27) and again by (A.26), we get

$$(\psi - \tilde{\varphi}^{\varepsilon})(z) \geq (\psi - \tilde{\varphi}^{\varepsilon})(y) + a(z - y) \quad \text{for } z \in U'$$

and, since $\tilde{\varphi}^{\varepsilon} = \gamma^{\varepsilon} - w$ that

$$(\psi + w - \gamma^{\varepsilon})(z) \geq (\psi + w - \gamma^{\varepsilon})(y) + a(z - y) \quad \text{for } z \in U'. \tag{A.29}$$

From (A.12), there exists $\tilde{y} \in U$ such that $\gamma^{\varepsilon}(y) = \gamma(\tilde{y}) - \frac{1}{\varepsilon}|\tilde{y} - y|$. From (A.29), we see

$$\begin{aligned}
\eta(z) &:= \gamma(\tilde{y}) - \frac{1}{\varepsilon}|\tilde{y} - z| - (\psi + w)(z) + az \\
&\leq \gamma^{\varepsilon}(z) - (\psi + w)(z) + az \\
&\leq \gamma^{\varepsilon}(y) - (\psi + w)(y) + ay = \gamma(\tilde{y}) - \frac{1}{\varepsilon}|\tilde{y} - y| - (\psi + w)(y) + ay \\
&= \eta(y) \quad \text{for } z \in U'.
\end{aligned}$$

Since $\psi, w \in W^{2,p}(U) \subseteq C^{1,\iota}(U')$, we see that $\eta \in C^1(U' - \{\tilde{y}\})$. Now if $y \neq \tilde{y}$, we conclude $\nabla\eta(y) = 0$, hence

$$-\frac{1}{\varepsilon}\frac{y - \tilde{y}}{|y - \tilde{y}|} = \nabla\psi(y) + \nabla w(y) - a.$$

In particular

$$\frac{1}{\varepsilon} = |\nabla\psi(y) + \nabla w(y) - a| \leq \| \nabla\psi \|_{L^{\infty}(U')} + \varepsilon + \delta$$

by (A.6) and (A.28). As $\psi \in C^{1,\iota}(U)$ by (A.23), this is impossible for ε small, hence $\tilde{y} = y$ and $\gamma^{\varepsilon}(y) = \gamma(y)$. Since $\varphi + w = \gamma \leq \gamma^{\varepsilon}$ on U' by (A.7) and (A.11), this yields together with (A.29) that

$$\begin{aligned}
(\psi - \varphi)(z) \geq (\psi + w - \gamma^{\varepsilon})(z) &\geq (\psi + w - \gamma^{\varepsilon})(y) + a(z - y) \\
&= (\psi - \varphi)(y) + a(z - y) \quad \text{for } z \in U',
\end{aligned}$$

hence

$$a \in \partial^{U'}(\psi - \varphi)(y) \cap B^n_{\delta}(0) \neq \emptyset.$$

Then (A.22) follows from (A.25). □

We use this lemma to conclude that the right hand side in a viscosity equation can be computed pointwise for subsolutions which are twice approximately differentiable almost everywhere.

LEMMA A.5. *Let* $U \subseteq \mathbb{R}^n$, $F \in \mathcal{F}_U$, $u \in L^p(U)$, $n < p < \infty$ *and* $\varphi : U \to [-\infty, \infty[$ *be upper semicontinuous and a* $W^{2,p}$*-viscosity subsolution of*

$$-F(\nabla\varphi, D^2\varphi) \leq u \quad in\ U, \tag{A.30}$$

and φ *is twice approximately differentiable* $\mathcal{L}^n$*-almost everywhere on* $[\varphi \in \mathbb{R}]$.

For $G \in \mathcal{F}_U$, *we put*

$$v(y) := -G(y, \nabla\varphi(y), D^2\varphi(y)) \quad for\ y \in [\varphi \in \mathbb{R}]$$

and extend v *on* $U - [\varphi \in \mathbb{R}]$ *arbitrarily.*

If $v \in L^p(U)$ *then* φ *is a* $W^{2,p}$*-viscosity subsolution of*

$$-G_\delta(., \nabla\varphi, D^2\varphi) \leq v \quad in\ U \tag{A.31}$$

for any $\delta > 0$ *where*

$$G_\delta(y, a, X) := \sup_{|b| \leq \delta} G(y, a + b, X).$$

PROOF. Let $U' \Subset U$ open, $\tau > 0$, $\psi \in W^{2,p}(U')$, satisfy

$$-G_\delta(., \nabla\psi, D^2\psi) \geq v + \tau \quad \text{pointwise almost everywhere in } U'. \tag{A.32}$$

We have to show that $\varphi - \psi$ has no interior maximum in U'. Assume on the contrary there is one, then there exists a ball $B^n_{\varrho_0}(y_0) \Subset U'$ such that

$$\varphi - \psi \leq (\varphi - \psi)(y_0) \in \mathbb{R} \quad \text{on } \overline{B^n_{\varrho_0}(y_0)}. \tag{A.33}$$

Putting

$$\tilde{\psi}(y) := \psi(y) + \varepsilon|y - y_0|^2,$$

we get

$$\| \nabla\tilde{\psi} - \nabla\psi \|_{L^\infty(B^n_{\varrho_0}(y_0))} \leq 2\varepsilon\varrho_0 < \delta/2$$

for ε small enough, and

$$\begin{aligned} -G_{\delta/2}(., \nabla\tilde{\psi}, D^2\tilde{\psi}) &\geq -G_\delta(., \nabla\psi, D^2\psi + 2\varepsilon I) \\ &\geq v + \tau - 2\varepsilon\Lambda_G(\| \nabla\psi \|_{L^\infty(B^n_{\varrho_0}(y_0))} + \delta) \\ &\geq v + \tau/2 \end{aligned}$$

$\mathcal{L}^n$-almost everywhere in $B^n_{\varrho_0}(y_0)$ again for ε small enough, where $\Lambda_G(.)$ is the upper ellipticity constant of G in Definition A.2. Choosing δ, τ smaller, we keep (A.32) in $B^n_{\varrho_0}(y_0)$ and strengthen (A.33) to

$$\varphi - \psi < (\varphi - \psi)(y_0) \in \mathbb{R} \quad \text{on } \overline{B^n_{\varrho_0}(y_0)} - \{y_0\}. \tag{A.34}$$

From (A.30) and (A.34), we see that φ, ψ satisfy (A.20) and (A.21) of Lemma A.4. Putting

$$Q := \{y \in B^n_{\varrho_0}(y_0) | \partial^{B^n_{\varrho_0}(y_0)}(\psi - \varphi)(y) \cap B^n_\delta(0) \neq \emptyset\},$$

Lemma A.4 yields

$$\mathcal{L}^n(Q) > 0. \tag{A.35}$$

For $y \in Q$, there is $|a| < \delta$ such that

$$(\varphi - \psi)(z) \leq (\varphi - \psi)(y) - a(z - y) \quad \text{for } z \in B^n_{\varrho_0}(y_0).$$

In particular

$$-\infty < (\varphi - \psi)(y_0) + a(y_0 - y) \leq (\varphi - \psi)(y),$$

hence $y \in [\varphi \in \mathbb{R}]$.

When φ has approximate differentials in y, we get

$$\nabla\varphi(y) = \nabla\psi(y) - a \qquad \text{and} \qquad D^2\varphi(y) \leq D^2\psi(y),$$

hence

$$v(y) = -G(y, \nabla\varphi(y), D^2\varphi(y)) \geq -G_\delta(y, \nabla\psi(y), D^2\psi(y)).$$

Since φ has approximate differentials $\mathcal{L}^n$-almost everywhere on $Q \subseteq [\varphi \in \mathbb{R}]$, inequality (A.32) cannot be satisfied $\mathcal{L}^n$-almost everywhere on Q, which is a contradiction, as Q has positive measure by (A.35). □

For the minimal surface equation, we get rid of δ in the previous lemma.

LEMMA A.6. *We keep the assumptions of Lemma* A.5 *and consider*

$$G(a, X) := \partial_{kl} A(a) X_{kl}$$

where $A(a) := \sqrt{1 + |a|^2}$. *Then* φ *is a* $W^{2,p}$*-viscosity subsolution of*

$$-\nabla\left(\frac{\nabla\varphi}{\sqrt{1 + |\nabla\varphi|^2}}\right) = -G(\nabla\varphi, D^2\varphi) \leq v \quad \textit{in } U. \tag{A.36}$$

PROOF. Let $U' \Subset U$ open, $\tau > 0$, $\psi \in W^{2,p}(U')$, satisfy

$$-G(\nabla\psi, D^2\psi) \geq v + \tau \quad \text{pointwise almost everywhere in } U' \tag{A.37}$$

and assume to get a contradiction that there exists a ball $B^n_{\varrho_0}(y_0) \Subset U'$ such that

$$\varphi - \psi \leq (\varphi - \psi)(y_0) \in \mathbb{R} \quad \text{on } B^n_{\varrho_0}(y_0). \tag{A.38}$$

Since G is linear in X, we see that G_δ defined in Lemma A.5 is convex in X. Therefore there exists for $0 < \varrho < \varrho_0$ small enough and $0 < \delta < 1$ by [Caf89, Theorem 1] or [CafCab, Theorem 7.1] and the boundary estimates in [Cab00] or [Wi04], see also [Wa92, Theorem 5.8] for $p > n + 1$, a function $\psi \in W^{2,p}(B^n_\varrho(y_0))$ such that

(A.39)

$$-G_\delta(\nabla\psi_\delta, D^2\psi_\delta) = v + \tau/2 \quad \text{pointwise almost everywhere in } B^n_\varrho(y_0),$$
$$\psi_\delta = \psi \quad \text{on } \partial B^n_\varrho(y_0),$$

and satisfying the estimate

$$\sup_{0<\delta<1} \| \psi_\delta \|_{W^{2,p}(B^n_\varrho(y_0))} < \infty.$$

For a subsequence $\delta_j \downarrow 0$, we get

$$\psi_{\delta_j} \to \tilde{\psi} \quad \text{weakly in } W^{2,p}(B^n_\varrho(y_0))$$

and strongly in $C^{1,\iota''}(B^n_\varrho(y_0))$ for $0 < \iota'' < \iota := (1 - n/p) \in]0, 1[$. Clearly $\tilde{\psi} = \psi$ on $\partial B^n_\varrho(y_0)$.

Since $\| D^3 A(a) \| \leq C_n$, we see

$$\begin{aligned} -\partial_{kl} A(\nabla\tilde{\psi})\partial_{kl}\psi_\delta &\leq -G_\delta(\nabla\psi_\delta, D^2\psi_\delta) + C_n(\| \nabla\tilde{\psi} - \nabla\psi_\delta \|_{L^\infty(B^n_\varrho(y_0))} + \delta) \| D^2\psi_\delta \| \\ &\leq v + \tau/2 + \omega(\delta) \| D^2\psi_\delta \| \quad \mathcal{L}^n\text{-almost everywhere on } B^n_\varrho(y_0), \end{aligned}$$

where

$$\omega(\delta_j) := C_n(\| \nabla\tilde{\psi} - \nabla\psi_{\delta_j} \|_{L^\infty(B^n_\varrho(y_0))} + \delta_j) \to 0.$$

Now

$$\int_{B^n_\varrho(y_0)} \omega(\delta_j) \| D^2\psi_{\delta_j} \| \, \mathrm{d}\mathcal{L}^n = \omega(\delta_j) \| D^2\psi_{\delta_j} \|_{L^1(B^n_\varrho(y_0))} \to 0,$$

hence for a further subsequence

$$\omega(\delta_j) \| D^2\psi_{\delta_j} \| \to 0 \quad \mathcal{L}^n\text{-almost everywhere on } B^n_\varrho(y_0).$$

Since

$$-\partial_{kl} A(\nabla \tilde{\psi}) \partial_{kl} \psi_{\delta_j} \to -\partial_{kl} A(\nabla \tilde{\psi}) \partial_{kl} \tilde{\psi} \quad \text{weakly in } L^p(B^n_\varrho(y_0)),$$

we get

$$-\nabla \left(\frac{\nabla \tilde{\psi}}{\sqrt{1+|\nabla \tilde{\psi}|^2}} \right) = -\partial_{kl} A(\nabla \tilde{\psi}) \partial_{kl} \tilde{\psi} \le v + \tau/2 \;\; \mathcal{L}^n\text{-almost everywhere on } B^n_\varrho(y_0).$$

Together with (A.37), this yields that $w := \psi - \tilde{\psi}$ is a local weak supersolution of

$$-\partial_l (a_{lk} \partial_k w) \ge \tau/2 > 0 \quad \text{in } B^n_\varrho(y_0),$$

where $a_{lk} := \int_0^1 \partial_{lk} A(\nabla \tilde{\psi} + t \nabla w) \mathrm{d}t$. This equation is uniformly elliptic, since $\psi, \tilde{\psi} \in W^{2,p}(B^n_\varrho(y_0)) \hookrightarrow C^{1,\iota}(B^n_\varrho(y_0))$.

Now $w = 0$ on $\partial B^n_\varrho(y_0)$, and we conclude by strong maximum principle, see [GT, Theorem 8.19], that $\psi > \tilde{\psi}$ in $B^n_\varrho(y_0)$, in particular $\psi(y_0) > \tilde{\psi}(y_0)$.

Since $\psi_{\delta_j} \to \tilde{\psi}$ uniformly on $B^n_\varrho(y_0)$ and $\psi_{\delta_j} = \psi$ on $\partial B^n_\varrho(y_0)$, we get from (A.38) that

$$\sup_{\partial B^n_\varrho(y_0)} (\varphi - \psi_{\delta_j}) = \sup_{\partial B^n_\varrho(y_0)} (\varphi - \psi) \le (\varphi - \psi)(y_0) < (\varphi - \tilde{\psi})(y_0) = \lim_{j \to \infty} (\varphi - \psi_{\delta_j})(y_0),$$

hence $\varphi - \psi_{\delta_j}$ has an interior maximum in $B^n_\varrho(y_0)$ for large j. By (A.39), this contradicts Lemma A.5 (A.31), and (A.36) is proved. □

B. Balanced functions

DEFINITION B.1. A function $u \in L^p(B^n_1(0)), 1 < p < \infty$ is called p-balanced, if for all $y \in B^n_1(0)$ and $0 < \sigma < 1$

$$\liminf_{\varrho \downarrow 0} \frac{\varrho^{-n} \parallel u \parallel_{L^1(B^n_{\sigma\varrho}(y))}}{\varrho^{-n/p} \parallel u \parallel_{L^p(B^n_\varrho(y))}} > 0. \tag{B.1}$$

The main result of this section states that maximal functions are balanced.

DEFINITION B.2. Let μ be a Radon measure on $\mathbb{R}^n$ and $v \in L^1_{\mathrm{loc}}(\mu)$. Then the maximal function of v with respect to μ is defined by

$$Mv(y) := M_\mu v(y) := \sup_{\varrho > 0} \mu(B^n_\varrho(y))^{-1} \int_{B^n_\varrho(y)} |v| \mathrm{d}\mu$$

for $y \in \operatorname{spt} \mu$.

LEMMA B.3. *Let* $v \in L^p(B^n_1(0)), v = 0$ *on* $\mathbb{R}^n - B^n_1(0), 1 < p < \infty$ *and*

$$u := M_{\mathcal{L}^n}(|v|^p)^{1/p} \llcorner B^n_1(0).$$

Then $u \in L^q(B^n_1(0))$ *and* u *is* q*-balanced for* $1 < q < p$.

Before proving Lemma B.3, we recall a well-known estimate for maximal functions which is obtained by Besicovitch's Covering Theorem. If $v \in L^1(\mu)$ then

$$\mu(Mv > t) \le C_n t^{-1} \parallel v \parallel_{L^1(\mu)} \tag{B.2}$$

for $t > 0$. We use this to prove the following statement.

Let $v \in L^1(\mu)$, $\mu \neq 0$ be finite and $0 < r < 1$. Then $Mv \in L^r(\mu)$ and

$$\parallel Mv \parallel_{L^r(\mu)} \le C_{n,r} \mu(\mathbb{R}^n)^{\frac{1}{r}-1} \parallel v \parallel_{L^1(\mu)} . \tag{B.3}$$

Indeed for any $0 < \Gamma < \infty$, we get with (B.2) that

$$\begin{aligned}\int (Mv)^r \mathrm{d}\mu &= \int_0^\infty \mu([(Mv)^r > t])\mathrm{d}t \\ &\le \int_\Gamma^\infty C_n t^{-1/r} \parallel v \parallel_{L^1(\mu)} \mathrm{d}t + \Gamma \mu(\mathbb{R}^n) \\ &\le C_{n,r} \Gamma^{1-1/r} \parallel v \parallel_{L^1(\mu)} + \Gamma \mu(\mathbb{R}^n).\end{aligned}$$

Choosing $\Gamma = (\mu(\mathbb{R}^n)^{-1} \parallel v \parallel_{L^1(\mu)})^r$, we obtain (B.3).

Lemma B.3 will follow from the following slightly more general estimate.

PROPOSITION B.4. *Let* $v \in L^p(\mathbb{R}^n)$, $1 < q < p < \infty$,

$$u := M_{\mathcal{L}^n}(|v|^p)^{1/p}$$

and $y \in \mathbb{R}^n, \varrho > 0$. *Then*

$$\varrho^{-n/p} \parallel v \parallel_{L^p(B^n_\varrho(y))} \le C_n \inf_{B^n_\varrho(y)} u \tag{B.4}$$

and

$$\varrho^{-n/q} \parallel u \parallel_{L^q(B^n_\varrho(y))} \le C_{n,p,q} \inf_{B^n_\varrho(y)} u. \tag{B.5}$$

PROOF. For $z \in B^n_\varrho(y)$, we see

$$B^n_\varrho(y) \subseteq B^n_{2\varrho}(z)$$

and

$$\fint_{B^n_\varrho(y)} |v|^p \le 2^n \fint_{B^n_{2\varrho}(z)} |v|^p \le C_n u(z)^p,$$

which yields (B.4).

To prove (B.5), we observe that for $z \in B^n_\varrho(y)$ with $u(z) < \infty$ there exists $0 < R < \infty$ such that

$$\fint_{B^n_R(z)} |v|^p \geq \frac{1}{2} u(z). \tag{B.6}$$

We define

$$Q_\geq := \{z \in B^n_\varrho(y) | \exists R \geq \varrho \text{ satisfying (B.6)}\}$$
$$Q_\leq := \{z \in B^n_\varrho(y) | \exists R \leq \varrho \text{ satisfying (B.6)}\}.$$

From (B.2), we know $u < \infty \mathcal{L}^n$-almost everywhere on $\mathbb{R}^n$. Therefore

$$\mathcal{L}^n(B^n_\varrho(y) - (Q_\geq \cup Q_\leq)) = 0. \tag{B.7}$$

We consider $z \in Q_\geq$ and $R \geq \varrho$ satisfying (B.6). Then for $z' \in B^n_\varrho(y)$, we know $B^n_R(z) \subseteq B^n_{3R}(z')$ and by (B.6) that

$$u(z)^p \leq 2 \fint_{B^n_R(z)} |v|^p \leq 2 3^n \fint_{B^n_{3R}(z')} |v|^p \leq C_n u(z')^p,$$

hence

$$\sup_{Q_\geq} u \leq C_n \inf_{B^n_\varrho(y)} u.$$

Integrating yields

$$\varrho^{-n/q} \parallel u \parallel_{L^q(Q_\geq)} \leq \omega_n^{1/q} \sup_{Q_\geq} u \leq C_n \inf_{B^n_\varrho(y)} u. \tag{B.8}$$

Now we consider $z \in Q_\leq$ and $R \leq \varrho$ satisfying (B.6). Then $B^n_R(z) \subseteq B^n_{2\varrho}(y)$ and, putting $\mu := \mathcal{L}^n \llcorner B^n_{2\varrho}(y)$,

$$u(z)^p \leq 2 \fint_{B^n_R(z)} |v|^p = 2\mu(B^n_R(z))^{-1} \int_{B^n_R(z)} |v|^p \mathrm{d}\mu \leq 2M_\mu(|v|^p)(z).$$

We apply (B.3) to $r := \frac{q}{p} < 1$ and $|v|^p$. This yields

$$\begin{aligned} \parallel u \parallel^p_{L^q(Q_\leq)} &= \parallel u^p \parallel_{L^r(Q_\leq)} \\ &\leq 2 \parallel M_\mu(|v|^p) \parallel_{L^r(B^n_\varrho(y))} \\ &\leq C_{n,p,q} \varrho^{n(\frac{1}{r}-1)} \parallel |v|^p \parallel_{L^1(\mu)} \\ &\leq C_{n,p,q} \varrho^{n(\frac{p}{q}-1)} \parallel v \parallel^p_{L^p(B^n_{2\varrho}(y))}, \end{aligned}$$

and therefore

$$\varrho^{-n/q} \parallel u \parallel_{L^q(Q_\leq)} \leq C_{n,p,q} \varrho^{-n/p} \parallel v \parallel_{L^p(B^n_{2\varrho}(y))}.$$

Combining with (B.4), we obtain

$$\varrho^{-n/q} \parallel u \parallel_{L^q(Q_\leq)} \leq C_{n,p,q} \inf_{B^n_\varrho(y)} u. \tag{B.9}$$

Now (B.7), (B.8) and (B.9) imply (B.5). □

Lemma B.3 is now an easy consequence of (B.5).

PROOF OF LEMMA B.3. Clearly $v \in L^p(\mathbb{R}^n)$, as $v = 0$ on $\mathbb{R}^n - B^n_1(0)$. First, we use Proposition B.4 (B.5) with $y = 0, \varrho = 1$ and get

$$\| u \|_{L^q(B^n_1(0))} \leq C_{n,p,q} \inf_{B^n_1(0)} u < \infty,$$

since $u < \infty \mathcal{L}^n$-almost everywhere by (B.2), as $|v|^p \in L^1(\mathbb{R}^n)$. Hence $u \in L^q(B^n_1(0))$.

Next, we consider $y \in B^n_1(0), 0 < \sigma < 1$ and $0 < \varrho < 1 - |y|$. Then we obtain from Proposition B.4 (B.5) that

$$\varrho^{-n/q} \| u \|_{L^q(B^n_\varrho(y))} \leq C_{n,p,q} \inf_{B^n_\varrho(y)} u \leq C_{n,p,q} \sigma^{-n} \varrho^{-n} \| u \|_{L^1(B^n_{\sigma\varrho}(y))},$$

hence

$$\frac{\varrho^{-n} \| u \|_{L^1(B^n_{\sigma\varrho}(y))}}{\varrho^{-n/q} \| u \|_{L^q(B^n_\varrho(y))}} \geq c_0(n, p, q, \sigma) > 0. \qquad \square$$

Finally, we relate u and v in Lemma B.3.

PROPOSITION B.5. *Let* $v \in L^p(B^n_1(0)), v = 0$ *on* $\mathbb{R}^n - B^n_1(0), 1 < p < \infty$ *and*

$$u := M_{\mathcal{L}^n}(|v|^p)^{1/p} \lfloor B^n_1(0).$$

Then

$$|v| \leq u \quad \mathcal{L}^n\text{-almost everywhere on } B^n_1(0) \tag{B.10}$$

and for any $B^n_\varrho \subseteq B^n_1(0), 0 < \sigma \leq 1, 1 \leq q < p,$

$$\varrho^{-n/p} \| v \|_{L^p(B^n_\varrho)} \leq C_n (\sigma\varrho)^{-n/q} \| u \|_{L^q(B^n_{\sigma\varrho})}. \tag{B.11}$$

PROOF. For any Lebesgue point y of $|v|^p \in L^1(B^n_1(0))$, we have

$$|v(y)|^p = \lim_{\varrho \downarrow 0} \fint_{B^n_\varrho(y)} |v|^p \leq u(y)^p$$

which yields (B.10).

For $B^n_\varrho \subseteq B^n_1(0)$, Proposition B.4 (B.4) implies

$$\varrho^{-n/p} \| v \|_{L^p(B^n_\varrho)} \leq C_n (\inf_{B^n_{\sigma\varrho}} u^q)^{1/q} \leq C_n (\sigma\varrho)^{-n/q} \| u \|_{L^q(B^n_{\sigma\varrho})}$$

which is (B.11).

C. C^2-Extension lemma

We call a function $\varphi : U \to [-\infty, \infty], U \subseteq \mathbb{R}^n$ open, twice differentiable on a set $Q \subseteq U$, if for each $y \in Q$ there exists a polynomial P_y of degree at most two such that $P_y(y) = \varphi(y)$ and

$$\lim_{z \to y, z \in Q} \frac{\varphi(z) - P_y(z)}{|z - y|^2} = 0.$$

The following C^2-Extension lemma is an easy consequence of Whitney's Extension Theorem. Unfortunately, we could not find it in literature and include therefore its proof for the reader's convenience.

LEMMA C.1. *Let $\varphi : U \to [-\infty, \infty], U \subseteq \mathbb{R}^n$ open, be $\mathcal{L}^n$-measurable and twice differentiable on a $\mathcal{L}^n$-measurable set $Q \subseteq U$.*

Then for any $U' \Subset U$ and $\varepsilon > 0$, there exists $Q_0 \subseteq U' \cap Q$ such that

$$\mathcal{L}^n(U' \cap Q - Q_0) < \varepsilon \tag{C.1}$$

and there exists $\psi \in C^2(U)$ satisfying

$$D^\alpha \varphi = D^\alpha \psi \quad \text{on } Q_0 \text{ for } |\alpha| \leq 2. \tag{C.2}$$

PROOF. Clearly, φ is twice approximately differentiable at points of full density in Q. Moreover the approximate differentials

$$\nabla \varphi : Q \to \mathbb{R}^n, \quad D^2 \varphi : Q \to S(n)$$

are $\mathcal{L}^n$-measurable. For $y \in Q$, we put

$$P_y(z) := \varphi(y) + \nabla \varphi(y)(z - y) + \frac{1}{2}(z - y)^T D^2 \varphi(y)(z - y)$$

and see

$$\sup_{B^n_\varrho(y) \cap Q} |\varphi - P_y| = o_y(\varrho^2) \tag{C.3}$$

for all $y \in Q$. By Lusin's Theorem, we can choose $Q' \subseteq U' \cap Q$ compact such that

$$\mathcal{L}^n(U' \cap Q - Q') < \varepsilon \tag{C.4}$$

and

$$(\varphi, \nabla \varphi, D^2 \varphi) | Q' \text{ is uniformly continuous.} \tag{C.5}$$

Since $\varphi|Q'$ is therefore bounded from below and above, we get from (C.3) that

$$\theta(\varphi, Q')(y) < \infty \quad \text{for all } y \in Q'. \tag{C.6}$$

For $M < \infty$, we consider the set Q_M of all $y \in Q'$ such that

$$\begin{gathered} \theta^n(\mathcal{L}^n, Q', y) = 1, \\ |D^\alpha \varphi(y)| \leq M \quad \text{for } |\alpha| \leq 2, \\ \theta(\varphi, Q')(y) \leq M. \end{gathered} \tag{C.7}$$

Clearly, the set $Q' \cap [|D^\alpha \varphi| \leq M] \cap [\theta(\varphi, Q') \leq M]$ is closed by continuity of $D^\alpha \varphi$ on the compact set Q'. Therefore Q_M is $\mathcal{L}^n$-measurable, and we see from (C.6) that

$$\mathcal{L}^n(Q' - Q_M) \to \mathcal{L}^n(Q' - \cup_{M=1}^\infty Q_M) = 0.$$

Choosing M large enough, (C.4) holds for Q_M in place of Q'.

We claim that $\nabla\varphi$ is differentiable on Q_M and $D(\nabla\varphi) = D^2\varphi$, more precisely

$$\sup_{B^n_\varrho(y) \cap Q_M} |\nabla\varphi - \nabla P_y| = o_y(\varrho). \tag{C.8}$$

First for $y \in Q_M$, we know

$$\varrho^{-n} \mathcal{L}^n(B^n_\varrho(y) - Q') \leq \omega_y(\varrho). \tag{C.9}$$

We put

$$l_y(z) := \varphi(y) + \nabla\varphi(y)(z - y)$$

and get, since $\theta(\varphi, Q')(y) \leq M$ by (C.7), that

$$\| \varphi - l_y \|_{L^\infty(B^n_\varrho(y) \cap Q')} \leq M\varrho^2. \tag{C.10}$$

Now, we fix $0 < \delta < 1$. We consider $z \in Q_M, z \neq y$, put $\varrho := |z - y|$ and

$$a = a_z := \nabla\varphi(z) - \nabla P_y(z) = \nabla\varphi(z) - \nabla\varphi(y) - (z - y)^T D^2\varphi(y).$$

For $w, w' \in B^n_{\delta\varrho}(z) \cap Q'$, we calculate

$$\begin{aligned}
&|\varphi(w') - P_y(w')| + |\varphi(w) - P_y(w)| + |\varphi(w') - l_z(w')| + |\varphi(w) - l_z(w)| \\
&\geq \Big| (\nabla\varphi(z) - \nabla\varphi(y))(w' - w) \\
&\qquad - \frac{1}{2}\big((w' - y)^T D^2\varphi(y)(w' - y) - (w - y)^T D^2\varphi(y)(w - y)\big) \Big| \\
&\geq |(\nabla\varphi(z) - \nabla\varphi(y) - (w - y)^T D^2\varphi(y))(w' - w)| \\
&\qquad - \frac{1}{2}|(w' - w)^T D^2\varphi(y)(w' - w)| \\
&\geq |a(w' - w)| - |(w - z)^T D^2\varphi(y)(w' - w)| - \frac{1}{2}|(w' - w)^T D^2\varphi(y)(w' - w)| \\
&\geq |a(w' - w)| - CM\delta^2\varrho^2.
\end{aligned}$$

Using (C.3) and (C.10), we get

$$|a(w'-w)| \leq (CM\delta^2 + \omega_y(\varrho))\varrho^2$$

Integrating yields

$$\begin{aligned} c_n|a|\delta\varrho &\leq \fint_{B^n_{\delta\varrho}(z)} \fint_{B^n_{\delta\varrho}(z)} |a(w'-w)|\mathrm{d}w\mathrm{d}w' \\ &\leq \fint_{B^n_{\delta\varrho}(z)} \fint_{B^n_{\delta\varrho}(z)} |a(w'-w)|\chi_{Q'}(w)\chi_{Q'}(w')\mathrm{d}w\mathrm{d}w' \\ &\quad + |a|2\delta\varrho\big((\omega_n\delta^n\varrho^n)^{-1}\mathcal{L}^n(B^n_{2\varrho}(y) - Q')\big)^2 \\ &\leq (CM\delta^2 + \omega_y(\varrho))\varrho^2 + C_n\delta^{-2n+1}|a|\omega_y(\varrho)\varrho \end{aligned}$$

where we have used (C.9). This yields

$$(c_n - C_n\delta^{-2n}\omega_y(\varrho))|a| \leq (CM\delta + \omega_y(\varrho)\delta^{-1})\varrho$$

and

$$\limsup_{z\to y, z\in Q_M} \frac{|\nabla\varphi(z) - \nabla\varphi(y) - (z-y)^T D^2\varphi(y)|}{|z-y|} \leq C_n M\delta.$$

which is (C.8), as δ was arbitrary.

Now, we define $\Phi : (Q_M \times Q_M) - \{(y,y) | y \in Q_M\} \to [0,\infty[$ by putting

$$\Phi(z,y) := \frac{|\varphi(z) - P_y(z)|}{|z-y|^2} + \frac{|\nabla\varphi(z) - \nabla P_y(z)|}{|z-y|} + |D^2\varphi(z) - D^2\varphi(y)|$$

and $\Delta_\varrho : Q_M \to [0,\infty]$ by

$$\Delta_\varrho(y) := \sup_{z\in Q_M, 0<|z-y|<\varrho} \Phi(z,y) = \sup_{z\in Q_M} \Phi(z,y)\chi_{]0,\varrho[}(|z-y|)$$

for $0 < \varrho < 1$. Since Φ is continuous outside the diagonal of Q_M, we see that $(y \mapsto \Phi(z,y)\chi_{]0,\varrho[}(|z-y|))$ is lower semicontinuous for all $z \in Q_M$. Therefore Δ_ϱ is lower semicontinuous, hence $\mathcal{L}^n$-measurable.

(C.3), (C.5) and (C.8) imply that

$$\lim_{\varrho\downarrow 0} \Delta_\varrho(y) = 0 \quad \text{for all } y \in Q_M.$$

By Egoroff's Theorem, there exists a compact subset $Q_0 \subseteq U' \cap Q_M$ such that (C.1) holds and $\Delta_\varrho \to 0$ uniformly on Q_0. This yields that $\varphi \in t^2(Q_0)$ in the sense of [Zie, Definition 3.5.1]. By Whitney's Extension Theorem, see [Wh34] or [Zie, Theorem 3.5.3], there exists $\psi \in C^2(U)$ satisfying (C.2). □

D. Tilted Lipschitz-Approximation

The Lipschitz-Approximation Theorem due to Brakke, see [Bra78, Theorem 5.4], allows to represent an integral varifold as a union of lipschitz graphs on a set which is large in measure. Here, we turn to a tilted version of the Lipschitz-Approximation.

For complete generality, we do this in any codimension $m \in \mathbb{N}$. To this end, we briefly setup the following notion for multivalued functions. For a metric space (M, d) and $\theta \in \mathbb{N}$ we put

$$M_\theta := (M)^\theta / \sim,$$

where $(x_1, \ldots, x_\theta) \sim (y_1, \ldots, y_\theta)$ if and only if there is a permutation $\sigma \in S_\theta$ satisfying

$$x_i = y_{\sigma(i)} \quad \text{für } i = 1, \ldots, \theta.$$

The quotient metric on M_θ is given by

$$d((x_1, \ldots, x_\theta), (y_1, \ldots, y_\theta)) := \min_{\sigma \in S_\theta} \max_{i=1}^{\theta} d(x_i, y_{\sigma(i)}).$$

We will consider two planes $P, T \in G(n+m, n)$ and their orthogonal projections π_P and π_T satisfying

$$J_T \pi_P \geq \lambda > 0$$

or equivalently apart from λ

$$T \cap P^\perp = \{0\} = P \cap T^\perp.$$

This implies that the projection $\pi_P | T : T \to P$ is invertible and λ measures the norm of the inverse

$$\| (\pi_P | T)^{-1} \| \leq C(\lambda).$$

T can be represented as graph over P by putting

$$L := (\pi_P | T)^{-1} - id : P \to P^\perp.$$

We see

$$T = \{(y, Ly) | y \in P\}.$$

Hence for $x = (y, z) \in P \times P^\perp$, we have

$$|\pi_T^\perp(x)| = \operatorname{dist}(x, T) \leq |z - Ly|. \tag{D.1}$$

On the other hand, since $\pi_T(x) \in T$, we get

$$\pi_T(x) = (\tilde{y}, L\tilde{y}) \text{ for some } \tilde{y} \in P.$$

Therefore

$$|y - \tilde{y}| = |\pi_P(x - \pi_T(x))| \leq |\pi_T^\perp(x)|$$

and

$$\begin{aligned} |z - Ly| &= |x - \pi_T(x) + (\tilde{y}, L\tilde{y}) - (y, Ly)| \\ &\leq |\pi_T^\perp(x)| + (1+ \| L \|)|y - \tilde{y}| \leq (2+ \| L \|)|\pi_T^\perp(x)|, \end{aligned}$$

hence

$$|z - Ly| \leq C(\lambda)|\pi_T^\perp(x)|. \tag{D.2}$$

After this motivation, we state a version of a tilted Lipschitz-Approximation without proof when the plane over which the graphs are considered and the plane to which the varifold is close do not coincide.

THEOREM D.1. *For* $1 \leq \Gamma, p, q < \infty, \theta, n, m \in \mathbb{N}, 0 < \delta_0, \delta, \lambda < 1, 0 \leq \iota < 1$*, there exists* $C(\lambda, n, m, \theta, \Gamma, p, q, \delta_0, \delta, \iota) < \infty$ *and* $0 < \varrho_0(\lambda) < 1/2$ *such that:*

Let μ *be an integral n-varifold in* $B_7^{n+m}(0) \subseteq \mathbb{R}^{n+m}$, $T, P \in G(n+m, n)$, $\pi_{T,P}^{(\perp)} : \mathbb{R}^{n+m} \to T^{(\perp)}, P^{(\perp)}$ *the respective orthogonal projections, satisfying*

$$J_T \pi_P \geq \lambda, \tag{D.3}$$

$$\begin{aligned} \mu(B_7^{n+m}(0)) &\leq \Gamma, \\ \mu(B_3^{n+m}(0)) &\leq (\theta + 1 - \delta_0) 3^n \omega_n, \\ (\theta - 1 + \delta_0)\omega_n &\leq \mu(B_1^{n+m}(0)), \end{aligned} \tag{D.4}$$

$$\alpha^p := \begin{cases} \int_{B_7^{n+m}(0)} |H_\mu|^p \mathrm{d}\mu & \text{if } p > 1, \\ \| \delta\mu \|_{B_7^{n+m}(0)} & \text{if } p = 1, \end{cases} \tag{D.5}$$

$$\beta^2 := \int_{B_7^{n+m}(0)} \| T_x\mu - T \|^2 \, \mathrm{d}\mu(x), \tag{D.6}$$

and

$$\gamma^q := \int_{B_7^{n+m}(0)} |\pi_T^\perp(x)|^q \mathrm{d}\mu(x). \tag{D.7}$$

Then there exists a θ*-valued lipschitz maps*

$$\begin{aligned} f = (f_1, \ldots, f_\theta) &: B_{\varrho_0(\lambda)}^n(0) \subseteq P \to P_\theta^\perp, \quad i = 1, \ldots, \theta, \\ F = (F_1, \ldots, F_\theta) &: B_{\varrho_0(\lambda)}^n(0) \subseteq P \to P \times P_\theta^\perp, \quad F_i(y) = (y, f_i(y)), \end{aligned}$$

satisfying

$$\operatorname{Lip} f \leq C(\lambda), \| f \|_{L^\infty(B_{\varrho_0(\lambda)}^n(0))} \leq 1/4, \tag{D.8}$$

and putting $\tilde{f}_i := f_i - L$, *where* $L := (\pi_P|T)^{-1} - id : P \to P^{\perp}$ *is linear,*

$$\text{(D.9)} \qquad \operatorname{Lip} \tilde{f}_i \le C(\lambda)\delta, \| \tilde{f}_i \|_{L^\infty(B^n_{\varrho_0(\lambda)}(0))} \le C(\lambda, n, q)\gamma^{\frac{q}{n+q}},$$

and there exists $Y \subseteq B^n_{\varrho_0(\lambda)}(0)$ *such that*

$$\text{(D.10)} \quad \theta^n(\mu, (y, z)) = \#\{i \,|\, f_i(y) = z\} \text{ for all } y \in Y \subseteq P, z \in B^m_{1/2}(0) \subseteq P^{\perp}$$

and

$$\text{(D.11)} \qquad X := \operatorname{spt} \mu \cap (Y \times B^m_{1/2}(0)) = \cup_{i=1}^{\theta} F_i(Y),$$

and satisfying the estimates

$$\text{(D.12)} \quad \begin{aligned} &\mu((B^n_{\varrho_0(\lambda)}(0) \times B^m(0)) - X) + \mathcal{L}^n(B^n_{\varrho_0(\lambda)}(0) - Y) \\ &\le \begin{cases} C(\alpha^{\frac{pn}{n-p\iota}} + \beta^2 + \gamma^q) & \text{for } \ p \le n, \\ C(\beta^2 + \gamma^q) & \text{for } \ p > n \text{ if } \alpha \le c_0(n, m, \theta, \Gamma, p, \delta_0, \delta). \ \square \end{cases} \end{aligned}$$

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