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Torus action on S^n and sign-changing solutions for conformally invariant equations

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Abstract. We construct sequences of sign-changing solutions for some conformally invariant semilinear elliptic equation which is defined S^n, when $n \geq 4$. The solutions we obtain have large energy and concentrate along some special submanifolds of S^n. For example, for $n \geq 4$ we obtain sequences of solutions whose energy concentrates along one great circle or finitely many great circles which are linked to each other (and they correspond to *Hopf links* embedded in $S^3 \times \{0\} \subset S^n$). In dimension $n \geq 5$ we obtain sequences of solutions whose energy concentrates along a two-dimensional torus (which corresponds to a *Clifford torus* embedded in $S^3 \times \{0\} \subset S^n$).

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1. Introduction and statement of the result

1.1. Introduction

We are interested in the existence of sign-changing solutions for the Yamabe-type equation

$$\Delta_{\mathring{g}} u - \frac{n\,(n-2)}{4}\,(1 - |u|^{\frac{4}{n-2}})\,u = 0, \tag{1.1}$$

in S^n, where $\mathring{g}$ denotes the standard metric on S^n and $n \geq 3$. Obviously $u_1 \equiv 1$ is a solution of (1.1). The classification of solutions of (1.1) which do not change sign goes back to the result of M. Obata [7] that states that all positive solutions of (1.1) arise as the functions u which appear in the identity

$$K^* \mathring{g} = u^{\frac{4}{n-2}}\, \mathring{g},$$

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where K is a conformal transformation of the sphere S^n. As far as sign-changing solutions are concerned, we recall the result of W. Ding [5] (see also [3] and [4] for related results) on the existence of solutions which are invariant under the action of the Lie group $O(k) \times O(n+1-k)$, for $k = 2, \ldots, n-1$. There is also a vast literature about the existence of sign-changing solutions using variational methods; we address to [2] and [1] for references. It is known that (1.1) has infinitely many sign-changing solutions, however the structure of these solutions is not well-understood.

Equation (1.1) has a variational structure and the associated energy reads

$$E(u) := \int_{S^n} e(u)\, \mathrm{dvol}_{\mathring{g}},$$

where the energy density is defined by

$$e(u) := \tfrac{1}{2}\, |\nabla u|^2_{\mathring{g}} + \tfrac{n(n-2)}{8}\, |u|^2 - \tfrac{(n-2)^2}{8}\, |u|^{\frac{2n}{n-2}}.$$

We introduce the constant

$$E_1 := \tfrac{n-2}{4}\, \mathrm{Vol}\,(S^n),$$

which corresponds to the energy of the solution $u_1 \equiv 1$.

In this paper we provide a wealth of sign-changing solutions of (1.1). To state precisely our result, we need to introduce some notation and definitions. We will then give the statement of a rather general result and we will provide many examples which explain how this general result can be applied. In contrast with previous existence results, we have a rather precise description of the solutions we obtain: they can be described as the superposition of the constant solution $u_1 \equiv 1$ with a large number of copies of negative solutions of (1.1) which are highly concentrated at points that in turn are evenly arranged along some special submanifolds of S^n.

As a byproduct of the main result of this paper, we have the:

Theorem 1.1. *Assume that* $1 \le d \le n-3$ *satisfy*

$$n + 1 \ge 2d.$$

Then there exists a d-dimensional flat torus $\mathbb{T}^d$ embedded in S^n, a sequence (u_k) of sign-changing solutions of (1.1) *and constants $c_1 > c_2 > 0$ and $c_3 > 0$ such that the following holds:*

(a) *The function u_k is positive away from a tubular neighborhood of radius c_1/k around $\mathbb{T}^d$ and negative in a tubular neighborhood of radius c_2/k around $\mathbb{T}^d$.*
(b) *As k tends to infinity, u_k converges uniformly on compact subsets of $S^n - \mathbb{T}^d$ to the constant function $u_1 \equiv 1$.*
(c) *As k tends to infinity, the* renormalized energy density

$$\frac{1}{k^d}\, e(u_k)\, \mathrm{dvol}_{\mathring{g}} \rightharpoonup c_3\, \mathcal{H}^d \llcorner \mathbb{T}^d,$$

in the sense of measures.

Here, if Λ is a smooth d-dimensional embedded submanifold of $\mathbb{R}^n$, $\mathcal{H}^d \llcorner \Lambda$ denotes the d-dimensional Hausdorff measure restricted to Λ, namely

$$\mathcal{H}^d \llcorner \Lambda(\Omega) := \mathcal{H}^d(\Lambda \cap \Omega).$$

1.2. Notation

To state the general result, we need to fix the notation and digress slightly. We assume that we are given integers $m, n \geq 1$ satisfying

$$n + 1 \geq 2m.$$

The integer n corresponds to the dimension of the sphere S^n over which (1.1) is defined. It will be convenient to identify the Euclidean space $\mathbb{R}^{n+1}$ with $\mathbb{C}^m \times \mathbb{R}^{n+1-2m}$, in which case we agree that the coordinates of a point $x \in \mathbb{R}^{n+1}$ are given by $(z_1, \ldots, z_m, \hat{x})$ where $z_1, \ldots, z_m \in \mathbb{C}$ and $\hat{x} \in \mathbb{R}^{n+1-2m}$. In terms of these coordinates, we define the point

$$p := \frac{1}{\sqrt{m}} \left(1, \ldots, 1, \hat{0}\right) \in S^n \subset \mathbb{C}^m \times \mathbb{R}^{n+1-2m}, \tag{1.2}$$

where $\hat{0} := (0, \ldots, 0) \in \mathbb{R}^{n+1-2m}$. Let $\mathfrak{G} \subset O(n+1)$ be the finite subgroup of isometries of S^n which is generated by the symmetries

$$\mathfrak{s}(z_1, \ldots, z_m, \hat{x}) := (z_1, \ldots, z_m, -\hat{x}), \qquad \bar{\mathfrak{s}}(z_1, \ldots, z_m, \hat{x}) := (\bar{z}_1, \ldots, \bar{z}_m, \hat{x}),$$

and

$$\mathfrak{c}(z_1, z_2, \ldots, z_{m-1}, z_m, \hat{x}) := (z_2, z_3, \ldots, z_m, z_1, \hat{x}),$$

which corresponds to a cyclic permutation of the first m-th complex coordinates. To begin with, observe that the point p is fixed under the action of any element of the group $\mathfrak{G}$, but the choice of the finite group $\mathfrak{G}$ is really motivated by the following elementary but key result:

Proposition 1.2. *Any linear form defined on $\mathbb{R}^{n+1}$ which is invariant under the action of $\mathfrak{G}$ is collinear to*

$$\mathbb{R}^{n+1} \ni x \longmapsto p \cdot x \in \mathbb{R},$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^{n+1}$.

Proof. Given the identification of $\mathbb{R}^{n+1}$ with $\mathbb{C}^m \times \mathbb{R}^{n+1-2m}$, any (real valued) linear form φ on $\mathbb{R}^{n+1}$ can be written as

$$\varphi(z_1, \ldots, z_m, \hat{x}) = \Re\,(a_1\, z_1 + \ldots + a_m\, z_m) + \hat{a} \cdot \hat{x},$$

for some $a_1, \ldots, a_m \in \mathbb{C}$ and $\hat{a} \in \mathbb{R}^{n+1-2m}$.

The fact that φ is invariant under the action of $\mathfrak{s}$ implies immediately that $\hat{a} = 0$. Next, φ is also assumed to be invariant under the action of $\mathfrak{c}$, and hence we find that all the a_j have to be equal (say to $a \in \mathbb{C}$). Therefore, we can write

$$\varphi(z_1, \ldots, z_m, \hat{x}) = \Re(a\,(z_1 + \ldots + z_m)).$$

Finally, since φ is assumed to be invariant under the action of $\bar{\mathfrak{s}}$ we conclude that $a \in \mathbb{R}$ and hence

$$\varphi(z_1, \ldots, z_m, \hat{x}) = a\,\Re(z_1 + \ldots + z_m) = a\,\sqrt{m}\; p \cdot x.$$

This completes the proof of the result. $\square$

We denote by

$$\mathbb{T}^m := \tfrac{1}{\sqrt{m}}\,(S^1 \times \ldots \times S^1) \times \{\hat{0}\} \subset S^n \subset \mathbb{C}^m \times \mathbb{R}^{n+1-2m},$$

the m-dimensional torus embedded in S^n, which is also the orbit of p through the action of the elements of the m-dimensional Lie group

$$\mathfrak{T} := (O(2) \times \ldots \times O(2)) \times \{I_{n+1-2m}\} \subset O(n+1),$$

where, for $q \geq 1$, I_q denotes the identity of $\mathbb{R}^q$. Observe that $\mathbb{T}^m$, equipped with the metric induced by $\mathring{g}$, is a flat m-dimensional torus.

It is probably worth saying a word about the terminology we will use. When $m = 1$, $\mathbb{T}^1$ is usually referred to as a *great circle* of S^n and, when $m = 2$ and $n = 3$, $\mathbb{T}^2$ is usually referred to as a *Clifford torus*. When, $n \geq 3$, $\mathbb{T}^2$ is a Clifford torus embedded in $S^3 \times \{\hat{0}\} \subset S^n$ and, with slight abuse of terminology, we shall again refer to it as a *Clifford torus*.

1.3. The assumptions

We now describe the assumptions needed in the statement of the general result and we also provide some basic examples.

(H1) We fix $1 \leq d \leq m$, and assume that we are given a d-dimensional flat torus $\Lambda \subset \mathbb{T}^m$ which is the orbit of p under the action of a d- dimensional Lie group

$$\mathfrak{L} \subset \mathfrak{T}.$$

(H2) We assume that we are given a finite subgroup

$$\mathfrak{H}_0 \subset \mathfrak{T},$$

such that, the cyclic permutation $\mathfrak{c}$ leave the group $\mathfrak{H}_0$ invariant in the sense that

$$\mathfrak{c}\,\mathfrak{H}_0 = \mathfrak{H}_0\,\mathfrak{c}.$$

Moreover, we require that, for all $\mathfrak{h} \in \mathfrak{H}_0$, either $\Lambda \cap \mathfrak{h}\,\Lambda = \emptyset$ or $\mathfrak{h} = \mathrm{Id}$.

(H3) For all $k \geq 1$, we assume that we are given a finite subgroup

$$\mathfrak{H}_k \subset \mathfrak{L},$$

which commutes with $\mathfrak{H}_0$ and such that, the cyclic permutation $\mathfrak{c}$ leaves the group $\mathfrak{H}_k$ invariant in the sense that

$$\mathfrak{c}\, \mathfrak{H}_k = \mathfrak{H}_k\, \mathfrak{c}.$$

Moreover, we require that $\Lambda_0 := \Lambda / \mathfrak{H}_k$, equipped with the metric induced by $k\, \mathring{g}$ is a d-dimensional flat torus which does not depend on k.

We will denote by Γ_0 the lattice associated to Λ_0. We will denote by $\mathfrak{H}$ the group generated by $\mathfrak{H}_0$ and $\mathfrak{H}_k$ and we will denote by O_k the orbit of p under the action of the elements of $\mathfrak{H}$, namely

$$O_k := \{\mathfrak{h}(p) \,:\, \mathfrak{h} \in \mathfrak{H}\}.$$

Observe for all $\mathfrak{h} \in \mathfrak{T}$ we have

$$\bar{\mathfrak{s}} \circ \mathfrak{h} \circ \bar{\mathfrak{s}} = \mathfrak{h}^{-1},$$

and hence O_p is invariant under the action of the elements of $\mathfrak{H}$. Also observe that, for all $\mathfrak{h} \in \mathfrak{H}_k$, we have $\mathfrak{h}\, \Lambda = \Lambda$.

We will always assume that O_k contains at least two points. Let us now briefly comment on these assumptions. Property (H3) implies that $O_k \cap \Lambda$ is uniformly distributed at the vertices of a regular lattice in Λ and, as k tends to infinity, these points becomes denser in Λ. Property (H2) implies that the elements of $\mathfrak{H}_0$ transport Λ, and hence the points of $O_k \cap \Lambda$, to $\operatorname{Card} \mathfrak{H}_0$ disjoint isometric copies of Λ in the m-dimensional torus $\mathbb{T}^m$.

It also follows from properties (H2) and (H3) that the cardinal of O_k can be computed in terms of k, the ratio between the volume of Λ and Λ_0 and the cardinal of $\mathfrak{H}_0$. More precisely, we have

$$\operatorname{Card} O_k = \frac{\mathcal{H}^d(\hat{\Lambda})}{\mathcal{H}^d(\Lambda_0)}\, k^d,$$

where

$$\hat{\Lambda} := \bigcup\nolimits_{\mathfrak{h} \in \mathfrak{H}_0} \mathfrak{h}\, \Lambda \subset \mathbb{T}^m,$$

corresponds to the submanifold of S^n over which the renormalized energy density of our solutions will concentrate.

1.4. Examples

We now illustrate this set of assumptions by giving some key examples. This will be the opportunity to become more familiar with our notation. It will be convenient to agree that, for all $n + 1 \geq 2M$ and for all $\alpha := (\alpha_1, \ldots, \alpha_m) \in S^1 \times \ldots \times S^1$, $\mathfrak{t}_\alpha \in (O(2) \times \ldots \times O(2)) \times \{I_{n+1-2m}\}$, denotes the isometry of S^n defined by

$$\mathfrak{t}_\alpha(z_1, \ldots, z_m, \hat{x}) := \left(\alpha_1\, z_1, \ldots, \alpha_m\, z_m, \hat{x}\right).$$

1.4.1. *The case where* $\mathfrak{H}_0 = \{I_{n+1}\}$

The simplest examples correspond to the case where $\mathfrak{H}_0$ reduces to the identity. Observe that, in this case, $m = d$.

Example 1.3. The simplest example is probably the one where

$$\hat{\Lambda} = \Lambda = \mathbb{T}^1 = S^1 \times \{\hat{0}\} \subset S^n,$$

is a great circle of S^n, for $n \geq 1$. According to our notation, we have identified $\mathbb{R}^{n+1}$ with $\mathbb{C} \times \mathbb{R}^{n-1}$ and we have defined $\hat{0} := (0, \ldots, 0) \in \mathbb{R}^{n-1}$. In terms of Lie groups, this situation corresponds to $m = d = 1$ and to the choice

$$\mathfrak{L} = \mathfrak{T} := O(2) \times \{I_{n-1}\} \subset O(n+1).$$

In this case,

$$p = (1, \hat{0}) \in S^n \subset \mathbb{C} \times \mathbb{R}^{n-1},$$

and, for all $k \geq 1$, we can choose $\mathfrak{H}_k \subset O(2) \times O(n-1)$ to be the group generated by $\mathfrak{t}_\alpha$, where

$$\alpha = e^{\frac{2i\pi}{k}}.$$

Therefore, the points of

$$O_k = \left\{ (e^{\frac{2ij\pi}{k}}, \hat{0}) \in S^n \; : \; j \in \mathbb{Z} \right\},$$

are regularly distributed along a great circle of S^n. It is an easy exercise to check that (H1), (H2) and (H3) are satisfied. This example corresponds to the case where $d = 1$ in Theorem 1.1.

Example 1.4. The previous example easily generalizes to the case where

$$\hat{\Lambda} = \Lambda = \mathbb{T}^d = \frac{1}{\sqrt{d}} \left(S^1 \times \ldots \times S^1 \right) \times \{\hat{0}\} \subset S^n,$$

is a d-dimensional flat torus in S^n, for $n + 1 \geq 2d$. According to our notation, we have identified $\mathbb{R}^{n+1}$ with $\mathbb{C}^d \times \mathbb{R}^{n+1-2d}$ and we have defined $\hat{0} := (0, \ldots, 0) \in \mathbb{R}^{n+1-2d}$. In terms of the Lie groups, this situation corresponds to $m = d$ and to the choice

$$\mathfrak{L} = \mathfrak{T} := (O(2) \times \ldots \times O(2)) \times \{I_{n+1-2d}\} \subset O(n+1).$$

In this case,

$$p = \frac{1}{\sqrt{d}}(1, \ldots, 1, \hat{0}) \in S^n \subset \mathbb{C}^d \times \mathbb{R}^{n+1-2d},$$

and, for all $k \geq 1$, we can choose $\mathfrak{H}_k \subset (O(2) \times \ldots \times O(2)) \times O(n+1-2d)$ to be the group generated by $\mathfrak{t}_\alpha$ and $\mathfrak{c}$, where

$$\alpha := (e^{\frac{2i\pi}{k}}, 1, \ldots, 1).$$

Then the orbit of the point p under the action of $\mathfrak{H}_k$

$$O_k = \left\{ (e^{\frac{2ij_1\pi}{k}}, \dots, e^{\frac{2ij_d\pi}{k}}, \hat{0}) \in S^n \ : \ j_1, \dots, j_d \in \mathbb{Z} \right\},$$

are points regularly distributed on $\mathbb{T}^d$. Again, it is an easy exercise to check that (H1), (H2) and (H3) are satisfied. This example corresponds to the case where $d \geq 2$ in Theorem 1.1.

In the last example, the orbit of p under the action of $\mathfrak{H}_k$ forms a regular square lattice on the torus $\mathbb{T}^d$ but other lattices can be obtained. In other words, once the submanifod Λ is chosen, there might be many different groups $\mathfrak{H}_k$ leading to non-congruent configurations of points in S^n.

Example 1.5. Keeping the notation used in Example 1.4, one can also consider $\mathfrak{H}_k \subset (O(2) \times \ldots \times O(2)) \times O(n+1-2d)$ to be the group generated by $\mathfrak{t}_\alpha$ and $\mathfrak{c}$, where

$$\alpha := (e^{\frac{2i\pi}{q_1 k}}, e^{\frac{2i\pi}{q_2 k}}, \dots, e^{\frac{2i\pi}{q_d k}}),$$

for some $q_1, \dots, q_d \in \mathbb{Z} - \{0\}$. We require that the integers q_j are chosen so that the matrix whose rows are $(\frac{1}{q_1}, \dots, \frac{1}{q_d})$ and its cyclic permutations is invertible (this will guaranty that the associated lattice in $\mathbb{T}^d$ is d-dimensional). Different choices of q_j will, in general, lead to non-congruent solutions of (1.1).

Many more examples leading to non-congruent solutions can be found by studying the sub-lattices of $\mathbb{T}^d$ which contain p and are invariant under the action of $\mathfrak{c}$ and $\bar{\mathfrak{s}}$. Let us just mention a few examples in low dimensions.

Example 1.6. Keeping the notation used in Example 1.4, when $d = 2$ and $n+1 \geq 2d = 4$ one can consider $\mathfrak{H}_k \subset (O(2) \times O(2)) \times O(n-3)$ to be the group generated by $\mathfrak{t}_\alpha$ and $\mathfrak{t}_{\tilde{\alpha}}$, where

$$\alpha := (e^{\frac{2i\pi}{k}}, e^{\frac{2i\pi}{k}}) \quad \text{and} \quad \tilde{\alpha} := (e^{\frac{2i\pi}{qk}}, e^{-\frac{2i\pi}{qk}}),$$

for some $q \geq 1$. While, when $d = 4$ and $n+1 \geq 2d = 8$ one can consider $\mathfrak{H}_k \subset (O(2) \times \ldots \times O(2)) \times O(n-7)$ to be the group generated by $\mathfrak{t}_\alpha$, $\mathfrak{t}_{\tilde{\alpha}}$, $\mathfrak{t}_{\breve{\alpha}}$ and $\mathfrak{c}$, where

$$\alpha := (e^{\frac{2i\pi}{k}}, e^{\frac{2i\pi}{k}}, e^{\frac{2i\pi}{k}}, e^{\frac{2i\pi}{k}}), \qquad \tilde{\alpha} := (e^{\frac{2i\pi}{qk}}, e^{-\frac{2i\pi}{qk}}, e^{\frac{2i\pi}{qk}}, e^{-\frac{2i\pi}{qk}}),$$

and

$$\breve{\alpha} := (e^{\frac{2i\pi}{q'k}}, e^{\frac{2i\pi}{q'k}}, e^{-\frac{2i\pi}{q'k}}, e^{-\frac{2i\pi}{q'k}}),$$

for some $q, q' \geq 1$. Different choices of q and q' will lead to non-congruent solutions of (1.1).

This last example generalizes in any dimension $d = 2^q$ and $n+1 \geq d$.

1.4.2. *The case where* $\mathfrak{H}_0 \neq \{I_{n+1}\}$

The examples corresponding to the case where $\mathfrak{H}_0$ is not reduced to the identity can be constructed using similar ideas. However their geometry are slightly more complicated to grasp.

To understand some nontrivial examples, let us recall the definition of the Hopf map
$$H : S^3 \longrightarrow S^2,$$
which can be defined as follows
$$H(z_1, z_2) := \left(2\, z_1\, \bar{z}_2, |z_1|^2 - |z_2|^2\right) \in \mathbb{C} \times \mathbb{R},$$
if we identify $\mathbb{R}^4$ with $\mathbb{C}^2$ and $\mathbb{R}^3$ with $\mathbb{C} \times \mathbb{R}$. It is easy to check that the preimage of any point of S^2 by H is a great circle of S^3. Conversely, given any $(z_1, z_2) \in S^3 \subset \mathbb{C}^2$, the image of the great circle
$$\mu \longmapsto (e^{i\mu} z_1, e^{i\mu}\, z_2) \in \mathbb{C}^2,$$
by H is a point in S^2. Also, the preimage of two distinct points of S^2 by H is the disjoint union of two great circles which are linked. For example, the preimage of $(0, 1) \in S^2 \subset \mathbb{C} \times \mathbb{R}$ is the great circle $S^1 \times \{0\}$ of S^3 while the preimage of $(0, -1) \in S^2 \subset \mathbb{C} \times \mathbb{R}$ is the great circle $\{0\} \times S^1$ of S^3 and these two circles are easily seen to be linked (this is what is usually called a *Hopf link*).

Example 1.7. To begin with, let us consider the case where $n = 3$ and $m = 2$ and $d = 1$. We define
$$\Lambda := \left\{\frac{1}{\sqrt{2}}(e^{i\theta}, e^{i\theta}) \in S^3 \ : \ \theta \in \mathbb{R}\right\} \subset \mathbb{T}^2,$$
which is a the great circle of S^3 associated to the one dimensional Lie group
$$\mathfrak{L} := \{\mathfrak{t}_\alpha \in O(2) \times O(2) \ : \ \alpha := (e^{i\theta}, e^{i\theta}), \quad \theta \in \mathbb{R}\}.$$
Observe that the image of Λ by H is equal to $(1, 0) \in S^3 \subset \mathbb{C} \times \mathbb{R}$. We set
$$\mathfrak{T} := (O(2) \times O(2)) \subset O(4).$$
We choose $\mathfrak{H}_0 \subset O(2) \times O(2)$ to be the group generated by $\mathfrak{t}_{\tilde{\alpha}} \in O(2) \times O(2)$ where
$$\tilde{\alpha} := (e^{\frac{i\pi}{q}}, e^{-\frac{i\pi}{q}}),$$
for some $q \geq 2$. Observe that, for $j = 0, \dots, q-1$, $\mathfrak{t}^j_{\tilde{\alpha}}(\Lambda)$ is again a great circle of S^3 and its image by H is given by $(e^{\frac{2ij\pi}{q}}, 0) \in S^3$. In particular, Λ, $\mathfrak{t}_{\tilde{\alpha}}(\Lambda)$, ...$\mathfrak{t}^{q-1}_{\tilde{\alpha}}(\Lambda)$ are all disjoint and in fact are linked. In this case
$$\hat{\Lambda} := \bigcup_{j=0}^{q-1} \mathfrak{t}^j_{\tilde{\alpha}}(\Lambda),$$
is the disjoint union of q great circles of S^3 which are linked.

Finally, given $k \geq 1$, we define $\mathfrak{H}_k \subset O(2) \times O(2)$ to be the group generated by $\mathfrak{t}_{\check{\alpha}} \in O(2) \times O(2)$ where

$$\check{\alpha} := (e^{\frac{2i\pi}{k}}, e^{\frac{2i\pi}{k}}).$$

It is easy to check that (H1), (H2) and (H3) are fulfilled.

This example trivially generalizes in higher dimensions. When $n \geq 3$, we just identify S^3 with $S^3 \times \{\hat{0}\}$ where $\hat{0} = (0, \ldots, 0) \in \mathbb{R}^{n-3}$. Therefore, we can consider that Λ is embedded in $S^3 \times \{\hat{0}\} \subset S^n$ and we can extend trivially any group $\mathfrak{B} \subset O(2) \times O(2)$ by $\mathfrak{B} \times \{I_{n-3}\} \subset (O(2) \times O(2)) \times O(n-3)$. This leads to examples for which $\hat{\Lambda}$ has q different connected components which are all great circles of S^n, two of which are linked. In any case, this provides examples for which $d = 1$ and $m = 2$.

We complete this list of examples with a last one for which $d = 2$ and $m = 4$, to show the flexibility of our construction.

Example 1.8. To begin with, we assume that $n = 7$, $m = 4$ and $d = 2$ and we identify $\mathbb{R}^8$ with $\mathbb{C}^4$, so that $m = 4$. We define

$$\Lambda := \left\{ \frac{1}{\sqrt{4}}(e^{i\theta}, e^{i\mu}, e^{i\theta}, e^{i\mu}) \in S^7 \ : \ \theta, \mu \in \mathbb{R} \right\},$$

which is a flat 2-torus in S^7 associated to the 2-dimensional Lie group

$$\mathfrak{L} := \{\mathfrak{t}_\alpha \in O(2) \times O(2) \ : \ \alpha := (e^{i\theta}, e^{i\mu}, e^{i\theta}, e^{i\mu}) \quad \theta, \mu \in \mathbb{R}\}.$$

We set

$$\mathfrak{T} := (O(2) \times \ldots \times O(2)) \subset O(8).$$

Given $q \geq 2$, we choose $\mathfrak{H}_0 \subset (O(2) \times \ldots \times O(2)) \subset O(8)$ to be the group generated by $\mathfrak{t}_\alpha$ and $\mathfrak{t}_{\tilde{\alpha}}$, where

$$\alpha := (e^{\frac{i\pi}{q}}, 1, e^{-\frac{i\pi}{q}}, 1), \quad \text{and} \quad \tilde{\alpha} := (1, e^{\frac{i\pi}{q}}, 1, e^{-\frac{i\pi}{q}}).$$

It is easy to check that the images of Λ by to different elements of $\mathfrak{H}_0$ are disjoint. Therefore,

$$\hat{\Lambda} := \bigcup_{\mathfrak{h} \in \mathfrak{H}_0} \mathfrak{h}(\Lambda),$$

is the disjoint union of q^2 congruent copies of a 2-dimensional flat torus in S^7.

Then for all $k \geq 1$, we denote by $\mathfrak{H}_k \subset (O(2) \times \ldots \times O(2)) \subset O(8)$ the group generated by $\mathfrak{t}_{\hat{\alpha}}$ and $\mathfrak{t}_{\check{\alpha}}$, where

$$\hat{\alpha} := (e^{\frac{i\pi}{k}}, 1, e^{\frac{i\pi}{k}}, 1) \quad \text{and} \quad \check{\alpha} := (1, e^{\frac{i\pi}{k}}, 1, e^{\frac{i\pi}{k}}).$$

Again (H1), (H2) and (H3) are fulfilled.

Again, this example also trivially generalizes in higher dimensions as Example 1.7; namely, when $n \geq 7$, we just identify S^7 with $S^7 \times \{\hat{0}\}$ where $\hat{0} = (0, \ldots, 0) \in \mathbb{R}^{n-7}$. This leads to examples for which $\hat{\Lambda}$ has q^2 different connected components which are congruent copies of a flat 2-dimensional torus of S^n. This is an example for which $d = 2$ and $m = 4$.

1.5. The main result

Having given many examples, we can now state our main result. Keeping the notation introduced in Section 1.3, we have the:

Theorem 1.9. *Assume that* $1 \leq d \leq n-3$ *and further assume that* (H1), (H2) *and* (H3) *are fulfilled. Then there exists* $k_0 \geq 1$ *and* $c_1, c_2 > 0$ *such that, for all* $k \geq k_0$ *there exists a solution* u_k *of* (1.1) *satisfying the following properties:*

(a) *The function* u_k *is invariant under the action of the elements of* $\mathfrak{G}$, $\mathfrak{H}_0$ *and* $\mathfrak{H}_k$ *but is not invariant under the action of* $\mathfrak{L}$.
(b) *The function* u_k *is positive away from a tubular neighborhood of radius* c_1/k *around* $\hat{\Lambda}$ *and is negative inside a tubular neighborhood of radius* c_2/k *around* $\hat{\Lambda}$.
(c) *As* k *tends to infinity,* u_k *converges, uniformly on compact subsets of of* $S^n - \hat{\Lambda}$, *to the constant function* $u_1 \equiv 1$.
(d) *As* k *tends to* $+\infty$, *the renormalized energy density*

$$\frac{1}{k^d}\, e(u_k)\, \mathrm{dvol}_{\mathring{g}} \rightharpoonup \frac{E_1}{\mathcal{H}^d(\Lambda_0)}\, \mathcal{H}^d \llcorner \hat{\Lambda}, \tag{1.3}$$

in the sense of measures.

Properties (a) to (d) will be consequences of the construction of the solutions and we shall not comment further on them.

The measure

$$\frac{1}{k^d}\, e(u_k)\, \mathrm{dvol}_{\mathring{g}},$$

is called the *renormalized energy density* of the function u_k. Observe that (1.3) implies that

$$E(u_k) = k^d \left(\frac{\mathcal{H}^d(\hat{\Lambda})}{\mathcal{H}^d(\Lambda_0)}\, E_1 + o(1) \right).$$

Also observe that the restriction $n-3 \geq d$ only plays a role in dimension $n = 5$ since, in higher dimensions, it is a consequence of the inequalities $2m \leq n+1$ and $d \leq m$.

Our result is closely related to a recent result of J. Wei and S. Yan [10] where sequences of solutions to the prescribed scalar curvature problem which concentrate along a circle are found. Also, we should mention the work of H.Y. Wang on the construction of sequence of Yang-Mills connections whose energy concentrates along a geodesic in $S^2 \times S^2$ or $S^1 \times S^3$. However, to our knowledge our result is the first of the kind where sequences of solutions which concentrate along higher dimensional submanifolds or disjoint union of submanifolds are exhibited.

1.6. Applications

All the examples given in Section 1.4 yield the existence of sequences of non-congruent sign changing solutions of (1.1). For example, in dimension $n \geq 4$, Theorem 1.9 applies to Example 1.3 and this yields, for k large enough, the existence of a solution u_k of (1.1), which is invariant under the action of $D_k \times O(n-1)$ where D_k is the dihedral group in $\mathbb{R}^2$ but which is not invariant under the action of $O(2) \times O(n-1)$. Moreover, as k tends to infinity, u_k converges uniformly on compact subsets of $S^n - \Lambda$ to $u_1 \equiv 1$ and the renormalized energy density of u_k concentrates uniformly along a great circle of S^n. This completes the proof of Theorem 1.1 when $d = 1$.

In dimension $n \geq 2d + 1$ with $d \geq 2$, Theorem 1.9 applies to Example 1.4, Example 1.5 and Example 1.6. In particular, for all k large enough, we obtain the existence of a solution u_k of (1.1) whose zero set is homeomorphic to $\mathbb{T}^d \times S^{n-1-2d}$. Moreover, as k tends to infinity, the sequence u_k converges uniformly to $u_1 \equiv 1$ on compact subsets of $S^n - \mathbb{T}^d$ and the renormalized energy density of u_k concentrates uniformly along $\mathbb{T}^d$. In particular, this completes the proof of Theorem 1.1 when $d \geq 2$.

In dimension $n \geq 4$, given $q \geq 2$ we can apply Theorem 1.9 to Example 1.7 and get solutions u_k of (1.1), whose renormalized energy density concentrates uniformly along the q disjoint great circles of S^n, as k tends to infinity.

Finally, in dimension $n \geq 7$, given $q \geq 2$ we can apply Theorem 1.9 to Example 1.8 and get the existence of solutionq u_k of (1.1) whose renormalized energy density concentrates uniformly along q^2 disjoint flat 2-tori of S^n, as k tends to infinity.

2. Plan of the paper

In Section 3 we recall some well-known properties of the conformal Laplacian and the conformal invariance of our problem. In particular, we use these results to describe a one parameter family u_ϵ of positive solutions of (1.1) which concentrate at one point (the North pole of S^n when ϵ tends to 0 and the South pole when ϵ tends to infinity) and for which $u_1 \equiv 1$. The next section is devoted to the definition of the approximate solutions and the derivation of precise asymptotics. We then study the linear problem associated to the linearization of (1.1) about the approximate solution. Finally, in the last section, we prove that, provided k is large enough, these approximate solutions can be perturbed into genuine solutions of (1.1) using some variant of the Liapunov- Schmidt reduction argument.

3. Conformal invariance

The conformal Laplacian on a Riemannian manifold (M^m, g) is defined by

$$L_g := \Delta_g - \tfrac{n-2}{4(n-1)}\, R_g,$$

where R_g denotes the scalar curvature of the metric g. In this section we recall some well-known properties of L_g. The following result can be found for example in [6, 8]:

Proposition 3.1. *Assume that* $f : (M^n, g) \longrightarrow (\bar{M}^n, \bar{g})$ *is a (local) conformal diffeomorphism, namely*

$$f^* \bar{g} = \phi^{\frac{4}{n-2}} \, g,$$

for some function $\phi > 0$ *defined on* M. *Then the following formula holds*

$$f^*(L_{\bar{g}} \, v) = \phi^{-\frac{n+2}{n-2}} \, L_g \, (\phi \, f^* v),$$

for any function v *defined on* $\bar{M}$.

Remark 3.2. We agree that $f^* g$ denotes the pullback of the metric g defined on TM by f and, given a function v defined on M, we agree that $f^* v$ denotes $v \circ f$.

In the particular case where the manifold is $(S^n, \mathring{g})$, the unit sphere with the standard metric, the conformal Laplacian is given by

$$L_{\mathring{g}} = \Delta_{\mathring{g}} - \tfrac{n(n-2)}{4},$$

since the scalar curvature is given by $R_{\mathring{g}} = n(n-1)$ in this case.

We will now make an intensive use of the conformal invariance of the conformal Laplacian. First of all, let $S_\circ := (0, \dots, 0, -1) \in \mathbb{R}^{n+1}$ denote the South pole of S^n. The inverse of the stereographic projection $\pi : \mathbb{R}^n \longrightarrow S^n - \{S_\circ\}$ given explicitly by

$$\pi(y) := \left(\frac{2y}{1+|y|^2}, \frac{1-|y|^2}{1+|y|^2} \right),$$

is a conformal map and we have

$$\pi^* \mathring{g} = \phi^{\frac{4}{n-2}} \, dy^2,$$

where dy^2 is the Euclidean metric in $\mathbb{R}^n$ and where

$$\phi(y) := \left(\frac{2}{1+|y|^2} \right)^{\frac{n-2}{2}}.$$

In Euclidean space, the conformal Laplacian reduces to the standard Laplacian and applying Proposition 3.1 we get

$$\pi^*(L_{\mathring{g}} \, v) = \phi^{-\frac{n+2}{n-2}} \, \Delta \, (\phi \, \pi^* v),$$

for any function v defined on the sphere S^n. Therefore, we conclude that u is a solution of (1.1) if and only if

$$w := \phi \, \pi^* u,$$

is an entire solution of

$$\Delta w + \tfrac{n\,(n-2)}{4}\,|w|^{\frac{4}{n-2}}\,w = 0, \tag{3.1}$$

in $\mathbb{R}^n$. In particular, the solution $u_1 \equiv 1$ of (1.1) is associated to the solution of (3.1), given by

$$w_1(y) := \left(\frac{2}{1+|y|^2}\right)^{\frac{n-2}{2}} = \phi(y).$$

There is yet another application of this conformal invariance which we will exploit. Recall that the group of conformal diffeomorphisms of the sphere is generated by the rotations of $\mathbb{R}^{n+1}$ and, for all $\epsilon > 0$, the Moebius transformations

$$K_\epsilon : (S^n, \mathring{g}) \longrightarrow (S^n, \mathring{g}),$$

which, given the above notation, can be defined by the identity

$$\pi^* K_\epsilon(y) = \pi(y/\epsilon).$$

We define the function $u_\epsilon > 0$ by

$$K_\epsilon^* \mathring{g} = u_\epsilon^{\frac{4}{n-2}}\,\mathring{g}.$$

Using Proposition 3.1, we conclude that

$$K_\epsilon^*\,(L_{\mathring{g}}\,v) = u_\epsilon^{-\frac{n+2}{n-2}}\,L_{\mathring{g}}(u_\epsilon\,K_\epsilon^* v), \tag{3.2}$$

for any function v defined on the sphere. Using this equality with $v \equiv 1$, we get

$$u_\epsilon^{-\frac{n+2}{n-2}}\,L_{\mathring{g}}\,u_\epsilon = -\tfrac{n(n-2)}{4}, \tag{3.3}$$

and hence u_ϵ is a solution of (1.1). As already mentioned in the introduction, the classification of solutions of (1.1) that do not change sign goes back to the result of M. Obata [7] which states that, up to the action of rotations, all positive solutions of (1.1) are given by the functions u_ϵ defined above.

Using the conformal map π we can translate this information to $\mathbb{R}^n$ and we conclude that the function

$$w_\epsilon(y) := \epsilon^{\frac{2-n}{2}}\left(\frac{2\epsilon^2}{\epsilon^2+|y|^2}\right)^{\frac{n-2}{2}},$$

is a solution of (3.1) and that all solutions of (3.1) are translations of w_ϵ. Moreover, we get an explicit formula for the function u_ϵ, given by

$$\pi^* u_\epsilon(y) := \frac{w_\epsilon(y)}{\phi(y)} = \epsilon^{\frac{n-2}{2}}\left(\frac{1+|y|^2}{\epsilon^2+|y|^2}\right)^{\frac{n-2}{2}}. \tag{3.4}$$

Observe that the sequence u_ε concentrates at the North pole S° of S^n as ϵ tends to 0, while it concentrates at the South pole $S_\circ$ of S^n as ϵ tends to infinity. Finally, $u_1 \equiv 1$ when $\epsilon = 1$. Differentiating (3.3) with respect to ϵ, we find that the function $\partial_\epsilon u_\epsilon$ satisfies

$$\mathcal{L}_\epsilon\,(\partial_\epsilon u_\epsilon) = 0,$$

where

$$\mathcal{L}_\epsilon := L_{\mathring{g}} + \tfrac{n(n+2)}{4}\, u_\epsilon^{\frac{4}{n-2}}.$$

Similarly, w_ϵ is a solution of (3.1) for all ϵ, and differentiation with respect to ϵ at $\epsilon = 1$ implies that

$$\left(\Delta + \tfrac{n(n+2)}{2}\, w_1^{\frac{4}{n-2}}\right) Z = 0,$$

where

$$Z(y) := \left(\frac{2}{1+|y|^2}\right)^{\frac{n-2}{2}} \left(\frac{1-|y|^2}{1+|y|^2}\right). \tag{3.5}$$

We end this section by the following observation: the conformal invariance of our problem implies that all our existence results for sign-changing solutions of (1.1) translate into existence results for sign-changing, entire solutions of (3.1). The reason why we have chosen to concentrate on (1.1) is simply because the description of the groups associated to our construction becomes very involved in $\mathbb{R}^n$.

4. Building the approximate solution

We use the notation introduced in Section 1.2 and Section 1.3. Let us denote by $\mathfrak{r}$ a rotation which sends the point p defined in (1.2) to S°, the North pole of S^n. The approximate solution $U_{k,\epsilon}$ we consider depends on a continuous parameter $\epsilon > 0$ as well as the discrete parameter k which appears in $\mathfrak{H}_k$. It can be described as

$$U_{k,\epsilon} := 1 - \sum_{\mathfrak{h}\in\mathfrak{H}} \mathfrak{h}^*\,(\mathfrak{r}^* u_\epsilon).$$

As will become apparent soon, we will assume that the parameter $\epsilon > 0$ is chosen so that

$$1/C \le k^2\,\epsilon \le C\,, \tag{4.1}$$

for some constant $C > 1$ which will be fixed large enough later on. Observe that, by construction, $U_{k,\epsilon}$ is invariant under the action of the elements of both $\mathfrak{G}$ and $\mathfrak{H}$. The fact that $U_{\epsilon,k}$ is invariant under the action of the elements of $\mathfrak{H}$ is standard and follows at once from the fact that $\mathfrak{H}$ is a group. Now, since u_ϵ is invariant under the action of isometries preserving the axis going through S° and $S_\circ$, we find that $r^* u_\epsilon$

is invariant under the action of isometries preserving the axis going through p and $-p$. In particular, $\mathfrak{g}^*(r^* u_\epsilon) = r^* u_\epsilon$, for all $\mathfrak{g} \in \mathfrak{G}$. We have

$$\mathfrak{g}^* U_{k,\epsilon} = 1 - \sum_{\mathfrak{h} \in \mathfrak{H}} (\mathfrak{h} \circ \mathfrak{g})^* (\mathfrak{r}^* u_\epsilon).$$

But, for $\mathfrak{g} \in \mathfrak{G}$, we have assumed that $\mathfrak{g}\,\mathfrak{H} = \mathfrak{H}\,\mathfrak{g}$. And hence

$$\begin{aligned} \mathfrak{g}^* U_{k,\epsilon} &= 1 - \sum_{\mathfrak{h} \in \mathfrak{H}} (\mathfrak{h} \circ \mathfrak{g})^* (\mathfrak{r}^* u_\epsilon) \\ &= 1 - \sum_{\mathfrak{h}' \in \mathfrak{H}} (\mathfrak{g} \circ \mathfrak{h}')^* (\mathfrak{r}^* u_\epsilon) \\ &= 1 - \sum_{\mathfrak{h}' \in \mathfrak{H}} \mathfrak{h}'^* (\mathfrak{g}^* (\mathfrak{r}^* u_\epsilon)) \\ &= 1 - \sum_{\mathfrak{h}' \in \mathfrak{H}} \mathfrak{h}'^* (\mathfrak{r}^* u_\epsilon) = U_{k,\epsilon}. \end{aligned}$$

We agree that $\tilde{\pi} : \mathbb{R}^n \longrightarrow S^n - \{-p\}$ denotes the inverse of the stereographic projection which satisfies $\tilde{\pi}(0) = p$. The expression of $\tilde{\pi}$ can be derived from π using the rotation $\mathfrak{r}$ introduced above. Indeed, we can define $\tilde{\pi}$ by

$$\mathfrak{r} \circ \tilde{\pi} = \pi.$$

We now obtain some important asymptotic expansion for $U_{k,\epsilon}$ near p (and hence, using the action of the elements of $\mathfrak{H}$, near any point of O_k). This result strongly uses the assumption $d \leq n - 3$.

Lemma 4.1. *Assume that $n \geq 4$ and $1 \leq d \leq n-3$. Then there exists a constant $\bar{\gamma} > 0$ (depending on n, d, Λ and Λ_0) and, for all k large enough, there exists a constant $\gamma_{k,\epsilon} > 0$ (depending on k, n, d, Λ and Λ_0) such that*

$$\tilde{\pi}^* (U_{k,\epsilon} - \mathfrak{r}^* u_\epsilon)(y) = 1 - \epsilon^{\frac{n-2}{2}} \gamma_{k,\epsilon} + \mathcal{O}(k^2 |y|^2), \tag{4.2}$$

in the ball of center 0 and radius c/k in $\mathbb{R}^n$, where $c > 0$ is a constant independent of k which is fixed small enough, and

$$\lim_{k \to \infty} \frac{\gamma_{k,\epsilon}}{k^{n-2}} = \bar{\gamma}.$$

Finally, $\gamma_{k,\epsilon}$ depends continuously (and in fact smoothly) on ϵ.

Proof. We first analyze a model problem. We consider Γ_0 to be a d-dimensional regular lattice in $\mathbb{R}^d \times \{0\} \subset \mathbb{R}^n$, which contains the origin. In particular $\Gamma_0 = -\Gamma_0$. For all $k \geq 1$, we consider the function

$$W_k(x) = \sum_{\bar{x} \in \Gamma_0} \left| x - \frac{\bar{x}}{k} \right|^{2-n}.$$

Then near 0, the function W_k can be expanded as

$$W_k(x)=|x|^{2-n}+k^{n-2}\sum_{\bar{x}\in\Gamma_0-\{0\}}|\bar{x}|^{2-n}+(n-2)\,k^{n-1}\left(\sum_{\bar{x}\in\Gamma_0-\{0\}}\frac{\bar{x}}{|\bar{x}|^n}\right)\cdot x+\mathcal{O}(k^n\,|x|^2),$$

since all series converge precisely when $n-d\geq 3$. Thanks to the symmetries (namely $\Gamma_0=-\Gamma_0$), we have $W_k(-x)=W_k(x)$ and hence, in the above expansion, the term which is linear in x vanishes. Therefore, we conclude that

$$W_k(x)=|x|^{2-n}+k^{n-2}\,\bar{\gamma}+\mathcal{O}(k^n\,|x|^2). \tag{4.3}$$

And this estimate holds in a ball of radius c/k centered at 0, provided $c>0$ is fixed small enough, depending on the lattice Γ_0. Similar estimates can be derived for the partial derivatives of W_k.

Now, property (H3) can be used to prove that similar computations can also be performed for the function $1-U_{k,\epsilon}$. Indeed, it is easy to check that the following expansion

$$\tilde{\pi}^*(\mathfrak{r}^*u_\epsilon)(y)=\epsilon^{\frac{n-2}{2}}\,|y|^{2-n}\left(1+\mathcal{O}(|y|^2)+\mathcal{O}(\epsilon^2\,|y|^{-2})\right),$$

is valid provided $|y|\geq\epsilon$. Using this and the analysis of the model problem, we claim that

$$\tilde{\pi}^*\left(\sum_{\mathfrak{h}\in\mathfrak{H}-\{\mathrm{Id}\}}\mathfrak{h}^*(\mathfrak{r}^*\,u_\epsilon)\right)(y)=\epsilon^{\frac{n-2}{2}}\,\gamma_{k,\epsilon}+\mathcal{O}(k^2\,|y|^2), \tag{4.4}$$

in the ball of radius c/k centered at 0 in $\mathbb{R}^n$, provided $c>0$ is fixed small enough. This is nothing but Taylor's expansion of the function

$$V_{k,\epsilon}:=\tilde{\pi}^*\left(\sum_{\mathfrak{h}\in\mathfrak{H}-\{\mathrm{Id}\}}\mathfrak{h}^*(\mathfrak{r}^*\,u_\epsilon)\right),$$

at order 2 at the origin. The constant $\gamma_{k,\epsilon}\in\mathbb{R}$ is simply given by

$$\epsilon^{\frac{n-2}{2}}\,\gamma_{k,\epsilon}=\sum_{\mathfrak{h}\in\mathfrak{H}-\{\mathrm{Id}\}}\mathfrak{h}^*(\mathfrak{r}^*\,u_\epsilon)(p). \tag{4.5}$$

Now, the function $V_{k,\epsilon}$ being invariant under the action of the elements of $\mathfrak{G}$, its gradient vanishes at the origin. This follows at once from Proposition 1.2 together with the fact that the tangent space at p can be identified with the space of vectors $x\in\mathbb{R}^{n+1}$ satisfying $p\cdot x=0$. This explains why there is no linear term in y in the expansion (4.4). Finally, it is easy to check that the norm of the second order differential of $V_{k,\epsilon}$ can be estimated by a constant times $\epsilon^{\frac{n-2}{2}}\,k^n\sim k^2$ in a ball of radius c/k centered at p, with $c>0$ fixed small enough (recall that $k^{-2}\sim\epsilon$).

Finally, we claim that

$$\sum_{\mathfrak{h}\in\mathfrak{H}-\{\mathrm{Id}\}} \mathfrak{h}^*(\mathfrak{r}^* u_\epsilon)(p) = \epsilon^{\frac{n-2}{2}} k^{n-2} \bar{\gamma} + \mathcal{O}(k^{-2}),$$

when $d < n-4$ (the $\mathcal{O}(k^{-2})$ has to be replaced by $\mathcal{O}(k^{-2} \log k)$ when $d = n-4$ and by $\mathcal{O}(k^{-1})$ when $d = n-3$). The idea is to decompose the sum of the left hand side into two parts. The first part is the sum over elements $\mathfrak{h} \in \mathfrak{H} - \{\mathrm{Id}\}$ such that the distance from p to $\mathfrak{h}(p)$ is larger than some constant $c > 0$ which is fixed small enough. Let us denote by $\mathfrak{H}^>$ this set. For any element $\mathfrak{h} \in \mathfrak{H}^>$, we can estimate

$$\mathfrak{h}^*(\mathfrak{r}^* u_\epsilon)(p) = \mathcal{O}(\epsilon^{\frac{n-2}{2}}).$$

Since there are at most a constant times k^d elements in $\mathfrak{H}^>$, summation over all $\mathfrak{h} \in \mathfrak{H}^>$ yields

$$\sum_{\mathfrak{h}\in\mathfrak{H}^>} \mathfrak{h}^*(\mathfrak{r}^* u_\epsilon)(p) = \mathcal{O}(\epsilon^{\frac{n-2}{2}} k^d) = \mathcal{O}(k^{d-2-n}).$$

Since $k^2 \sim \epsilon^{-1}$, we conclude that

$$\sum_{\mathfrak{h}\in\mathfrak{H}^>} \mathfrak{h}^*(\mathfrak{r}^* u_\epsilon)(p) = \mathcal{O}(k^{d-2-n}).$$

Let us denote by $\mathfrak{H}^<$ the set of elements $\mathfrak{h} \in \mathfrak{H} - \{\mathrm{Id}\}$ such that the distance from p to $\mathfrak{h}(p)$ is less than some constant $c > 0$ which is fixed small enough. If $s_\mathfrak{h}$ denotes the distance from p to $\mathfrak{h}(p)$, we can estimate

$$\mathfrak{h}^*(\mathfrak{r}^* u_\epsilon)(p) = \epsilon^{\frac{n-2}{2}} s_\mathfrak{h}^{2-n} (1 + \mathcal{O}(s_\mathfrak{h}^2) + \mathcal{O}(\epsilon^2 s_\mathfrak{h}^{-2})).$$

Recall that the points of O_k are arranged at the vertices of a d-dimensional regular lattice $k^{-1} \Gamma_0$. Summation over all $\mathfrak{h} \in \mathfrak{H}^<$ yields

$$\sum_{\mathfrak{h}\in\mathfrak{H}^<} \epsilon^{\frac{n-2}{2}} s_\mathfrak{h}^{2-n} = \epsilon^{\frac{n-2}{2}} k^{n-2} \bar{\gamma} + \mathcal{O}(\epsilon^{\frac{n-2}{2}} k^d),$$

where the constant $\bar{\gamma}$ corresponds to the constant which appears in (4.3). Moreover,

$$\sum_{\mathfrak{h}\in\mathfrak{H}^<} \epsilon^{\frac{n+2}{2}} s_\mathfrak{h}^{-n} = \mathcal{O}(\epsilon^{\frac{n+2}{2}} k^n)).$$

Finally,

$$\sum_{\mathfrak{h}\in\mathfrak{H}^<} \epsilon^{\frac{n-2}{2}} s_\mathfrak{h}^{4-n} = \mathcal{O}(\epsilon^{\frac{n-2}{2}} k^{n-4})),$$

when $4 - n + d < 0$ (the right had side has to be replaced by $\mathcal{O}(\epsilon^{\frac{n-2}{2}} k^{n-4} \log k))$ when $4 - n + d = 0$ and by $\mathcal{O}(\epsilon^{\frac{n-2}{2}} k^{n-3}))$ when $4 - n + d = 1$). In any case, the result follows at once by collecting these estimates and using the fact that $\epsilon \sim k^{-2}$. This completes the proof of the result. $\square$

Using similar arguments, we can also prove the weaker result which also strongly uses the assumption $n - d \geq 3$:

Lemma 4.2. *Assume that $n \geq 4$ and $1 \leq d \leq n-3$. Then for all $c > 0$, there exists a constant $C > 0$ such that, for all $x \in S^n$ satisfying $\operatorname{dist}(x, \Lambda) \geq c/k$, we have*

$$\left| \sum_{\mathfrak{h} \in \mathfrak{H} - \{\mathrm{Id}\}} \mathfrak{h}^* \, \mathfrak{r}^* \, u_\epsilon \, (x) \right| \leq C \ (k \operatorname{dist}(x, \Lambda))^{2-n+d} .$$

Proof. We follow the arguments developed in the proof of Lemma 4.1. We consider Γ_0 to be a d-dimensional regular lattice in $\mathbb{R}^d \times \{0\} \subset \mathbb{R}^n$ which contains the origin. For all $k \geq 1$, we consider the function

$$W_k(x) = \sum_{\bar{x} \in \Gamma_0} \left| x - \frac{\bar{x}}{k} \right|^{2-n} .$$

We denote by $s := \operatorname{dist}(x, \mathbb{R}^d \times \{0\})$. Then we can estimate

$$\left| \sum_{\bar{x} \in \Gamma_0 \,:\, |\bar{x}| \leq k\, s} \left| x - \frac{\bar{x}}{k} \right|^{2-n} \right| \leq c \left| \sum_{\bar{x} \in \Gamma_0 \,:\, |\bar{x}| \leq k\, s} s^{2-n} \right| \leq c\, s^{2-n+d}\, k^d ,$$

and

$$\left| \sum_{\bar{x} \in \Gamma_0 \,:\, |\bar{x}| \geq k\, s} \left| x - \frac{\bar{x}}{k} \right|^{2-n} \right| \leq c \left| \sum_{\bar{x} \in \Gamma_0 \,:\, |\bar{x}| \geq k\, s} \left| \frac{\bar{x}}{k} \right|^{2-n} \right| \leq c\, s^{2-n+d}\, k^d .$$

This implies the pointwise estimate

$$|W_k(x)| \leq c\, k^d \operatorname{dist}(x, \mathbb{R}^d \times \{0\})^{2-n+d} .$$

Now that the estimate is proven in this model situation, the estimate in the statement of the result follows from the arguments which were developed in the proof of the previous lemma. $\square$

5. Linear analysis

We fix a constant $c_0 > 0$ small enough so that the geodesic ball of radius $4\, r_k$, centered at the points of O_k are mutually disjoint, when

$$r_k := \frac{c_0}{k} .$$

We also assume that c_0 is chosen small enough so that $U_{k,\epsilon} \leq -1/2$ in the geodesic balls of radius $2\, r_k$, centered at the points of O_k. We define a cutoff function χ_k

such that $\chi_k \equiv 1$ in each geodesic ball of radius $r_k/2$ centered at p and $\chi_k \equiv 0$ away from the the geodesic ball of radius $2r_k$ centered at p. Finally, we assume that χ_k is invariant under the action of the elements of $\mathfrak{G}$.

From now on, we assume that we are working with functions which are invariant under the action of the elements of the groups $\mathfrak{H}$ and $\mathfrak{G}$. For all $\delta \in \mathbb{R}$, we define the weighted norm

$$\|w\|_{L^\infty_\delta(S^n)} := \sup_{S^n - O_k} \left((\max(\epsilon, \mathrm{dist}(\cdot, O_k))^{-\delta} \, |w| \right),$$

and we define the operator

$$\mathcal{L}_{k,\epsilon} := \Delta_{\mathring{g}} - \tfrac{n(n-2)}{4} + \tfrac{n(n+2)}{4} \, U_{k,\epsilon}^{\frac{4}{n-2}},$$

which is the linearized operator about the approximate solution $U_{k,\epsilon}$. Our construction relies on the following result which, once again, uses in a crucial way the fact that $n - d \geq 3$:

Proposition 5.1. *Assume that $n \geq 4$ and $\delta \in (2 - n + d, 0)$ are fixed. For all $f \in L^\infty(S^n)$ which is invariant under the action of both $\mathfrak{G}$ and $\mathfrak{H}$, there exists a unique $w \in L^\infty(S^n)$ and $\lambda \in \mathbb{R}$ solutions of*

$$\begin{cases} \mathcal{L}_{k,\epsilon} \, w + \lambda \displaystyle\sum_{\mathfrak{h} \in \mathfrak{H}} \mathfrak{h}^* \, \mathfrak{r}^* \, (\chi_k \, u_\epsilon^{\frac{4}{n-2}} \, \partial_\epsilon u_\epsilon) = & f, \\ \lambda \displaystyle\int_{S^n} \chi_k^2 \, u_\epsilon^{\frac{4}{n-2}} \, (\partial_\epsilon u_\epsilon)^2 \, \mathrm{dvol}_{\mathring{g}} = & \displaystyle\int_{S^n} \mathfrak{r}^* \, (\chi_k \, \partial_\epsilon u_\epsilon) \, f \, \mathrm{dvol}_{\mathring{g}} \\ & - \displaystyle\int_{S^n} w \, \mathcal{L}_{k,\epsilon} \mathfrak{r}^* \, (\chi_k \, \partial_\epsilon u_\epsilon) \, \mathrm{dvol}_{\mathring{g}}, \\ \displaystyle\int_{S^n} \mathfrak{r}^* \, (\chi_k \, u_\epsilon^{\frac{4}{n-2}} \, \partial_\epsilon u_\epsilon) \, w \, \mathrm{dvol}_{\mathring{g}} = & 0. \end{cases} \tag{5.1}$$

Moreover, we have the estimate

$$\|w\|_{L^\infty_\delta(S^n)} \leq c \, \|f\|_{L^\infty_{\delta-2}(S^n)}, \tag{5.2}$$

for some constant $c > 0$ independent of f.

Proof. We first prove some *a priori* estimates for the solutions of $\mathcal{L}_{k,\epsilon} \, w = f$.

Lemma 5.2. *Assume that $\delta \in (2 - n + d, 0)$, then there exists $k_0 > 0$ such that, for all $k \geq k_0$, the following a priori estimate holds*

$$\|w\|_{L^\infty_\delta(S^n)} \leq c \, \|\mathcal{L}_{k,\epsilon} \, w\|_{L^\infty_{\delta-2}(S^n)},$$

provided

$$\int_{S^n} \mathfrak{r}^* \, (\chi_k \, u_\epsilon^{\frac{4}{n-2}} \, \partial_\epsilon u_\epsilon) \, w \, \mathrm{dvol}_{\mathring{g}} = 0. \tag{5.3}$$

Proof. The proof is by contradiction. We assume that there exists a sequence of k_j tending to infinity and ϵ_j tending to 0 for which the result is not true (recall that $k_j^2 \sim \epsilon_j^{-1}$). In particular, there exists w_j such that

$$\|w_j\|_{L^\infty_\delta(S^n)} = 1, \tag{5.4}$$

and

$$\lim_{j\to\infty} \|\mathcal{L}_{k_j,\epsilon_j}\, w_j\|_{L^\infty_{\delta-2}(S^n)}.$$

We denote by $p_j \in S^n$ a point where (5.4) is achieved. We now distinguish different cases according to the behavior of

$$\ell_j := \max\left(\mathrm{dist}(p_j, O_{k_j}), \epsilon_j\right).$$

In each case, we rescale coordinates (using the exponential map) by $1/\ell_j$ and use elliptic estimates together Ascoli-Arzela's theorem to extract convergent subsequences. If, for some subsequence, ℓ_j remains bounded away from 0, we get in the limit a non trivial solution of

$$(\Delta_{\mathring{g}} + n)\, w = 0,$$

in $S^n - \Lambda$. Moreover, w is bounded by a constant times $(\mathrm{dist}(p, \Lambda))^\delta$ and, w_j being invariant under the action of $\mathfrak{H}$, we conclude that w is invariant under the action of $\mathfrak{L}$ and hence it does not depend on the coordinates on Λ, in other words w is invariant under the torus action associated to Λ. Since $\delta > 2 - n + d$ and since w does not depend on the coordinates in Λ, it is easy to check that the singularities of w are removable and hence w has to be a smooth solution in the kernel of $\Delta_{\mathring{g}} + n$. But w is invariant under the action of $\mathfrak{G}$ and does not depend on the coordinates on Λ, it is easy to conclude that w achieves its maximum at more than one point and hence it cannot be an element of the kernel of $\Delta_{\mathring{g}} + n$, therefore $w \equiv 0$. Which is a contradiction.

The second case we have to consider is the case where $\lim_{j\to\infty} \ell_j = 0$ and where $\lim_{j\to\infty} k_j\, \ell_j = +\infty$. In this case, we obtain a nontrivial solution of

$$\Delta\, w = 0,$$

in $\mathbb{R}^n - \{0\} \times \mathbb{R}^d$ which is bounded by a constant times $(\mathrm{dist}(\cdot, \{0\} \times \mathbb{R}^d))^\delta$ and which does not depend on the coordinates in $\{0\} \times \mathbb{R}^d$. Again, using the fact that $\delta > 2 - n + d$ we conclude that the singularities of w are removable and then it is easy to check that $w \equiv 0$ since $\delta < 0$, which is a contradiction.

Next, we consider the case where $\lim_{j\to\infty} k_j\, \ell_j$ exists and is not equal to 0. In this case, we obtain a nontrivial solution of

$$\Delta\, w = 0,$$

in $\mathbb{R}^n - \{0\} \times \Gamma_0$, where Γ_0 is a regular d-dimensional lattice in $\mathbb{R}^d$. Moreover, we know that this solution is bounded by a constant times $(\mathrm{dist}(\cdot, \Gamma_0))^\delta$ and is periodic (with period corresponding to the lattice). It is easy to check that the singularities are removable since $\delta > 2 - n$. We therefore conclude that $w \equiv 0$ since w is harmonic and $\delta < 0$. Again a contradiction.

The fourth case we have to consider is the case where $\lim_{j\to\infty} k_j\, \ell_j = 0$ and $\lim_{j\to\infty} k_j^2\, \ell_j = +\infty$. In this case, we obtain a nontrivial solution of

$$\Delta\, w = 0,$$

in $\mathbb{R}^n - \{0\}$. Moreover, this solution is bounded by a constant times $(\mathrm{dist}(\cdot, 0))^\delta$. It is easy to check that the singularities are removable since $\delta > 2 - n$. We therefore conclude that $w \equiv 0$ since w is harmonic and $\delta < 0$, which is again a contradiction.

Finally, the last case we have to consider is the case where $\lim_{j\to\infty} k_j^2\, \ell_j$ exists. In this last case, there exists $\bar\epsilon > 0$ such that w is a solution of

$$\left(\Delta + \tfrac{n(n+2)}{4}\, w_{\bar\epsilon}^{\frac{4}{n-2}}\right) w = 0,$$

in $\mathbb{R}^n$ which is bounded by a constant times $(1 + |y|)^\delta$, for $\delta < 0$. Now, the L^∞-bounded kernel of the operator $\Delta + \frac{n(n+2)}{4}\, w_{\bar\epsilon}^{\frac{4}{n-2}}$ is explicitly known and it is spanned by the functions

$$\psi_j := \left(\frac{2}{\bar\epsilon^2 + |y|^2}\right)^{\frac{n-2}{2}} \frac{2\,\bar\epsilon\, y_j}{\bar\epsilon^2 + |y|^2},$$

for all $j = 1, \ldots, n$, and

$$\psi_{n+1} := \left(\frac{2}{\bar\epsilon^2 + |y|^2}\right)^{\frac{n-2}{2}} \frac{\bar\epsilon^2 - |y|^2}{\bar\epsilon^2 + |y|^2}.$$

Passing to the limit in (5.3), we find that w is L^2-orthogonal to $w_{\bar\epsilon}^{\frac{4}{n-2}}\, \psi_{n+1}$ and hence it cannot contain any component over ψ_{n+1}. Finally, w_j being invariant under the action of the elements of $\mathfrak{G}$, the function w inherits some invariance which we now describe. We identify the tangent space of S^n at p with

$$\{(z_1, \ldots, z_m, \tilde x) \in \mathbb{C}^m \times \mathbb{R}^{n+1-2m} \,:\, \Re(z_1 + \ldots + z_m) = 0\},$$

and find that the function w is invariant under the action of $\mathfrak{s}$, $\bar{\mathfrak{s}}$ and $\mathfrak{c}$. Using this, we conclude (as in the proof of Proposition 1.2) that $w \equiv 0$, which a again a contradiction. Having reached a contradiction in each case, this completes the proof of the result. $\square$

Let us briefly comment on this result. The key idea is that, as k tends to infinity, the operator $\mathcal{L}_{k,\epsilon}$ has an eigenvalue tending to 0 and this eigenvalue is associated to an eigenfunction which is close to the function

$$\phi_{k,\epsilon} := \sum_{\mathfrak{h}\in\mathfrak{H}} \mathfrak{h}^*(\mathfrak{r}^*\,(\chi_k\, \partial_\epsilon u_\epsilon)).$$

In particular this implies that, working orthogonally to this eigenfunction, the inverse of the operator $\mathcal{L}_{k,\epsilon}$ is well behaved. Therefore, it should be natural to work orthogonally to this function and this is in essence what we will do. Now, from a technical point of view it turns out that the function $\mathfrak{r}^*(\partial_\epsilon u_\epsilon)$ does not decay fast enough to 0 when going away from p and it turns out that it is convenient to work orthogonally to the function

$$\tilde{\phi}_{k,\epsilon} := \sum_{\mathfrak{h}\in\mathfrak{H}} \mathfrak{h}^* \Big(\mathfrak{r}^* \, (\chi_k \, u_\epsilon^{\frac{4}{n-2}} \, \partial_\epsilon u_\epsilon) \Big),$$

which has a better decay properties and which is not orthogonal to $\phi_{k,\epsilon}$.

Using the above result, we can prove, for all $f \in L^\infty_{\delta-2}(S^n)$, the existence of $w \in L^\infty_{\delta-2}(S^n)$ and $\lambda \in \mathbb{R}$ solutions of (5.1) provided k is chosen large enough.

Lemma 5.3. *Assume that $\delta \in (2-n+d, 0)$, then there exists $k_0 > 0$ and, for all $k \geq k_0$, there exists a unique solution of* (5.1).

Proof. The idea is to consider the space of functions which are invariant under the action of the elements of $\mathfrak{G}$ and $\mathfrak{H}$ and which are also L^2-orthogonal to $\mathfrak{r}^* \, (\chi_k \, u_\epsilon^{\frac{4}{n-2}} \, \partial_\epsilon u_\epsilon)$. On this space, we define the operator $\tilde{\mathcal{L}}_{k,\epsilon}$ which is obtained by projecting $\mathcal{L}_{k,\epsilon} \, w$ on the L^2-orthogonal complement of $\mathfrak{r}^* \, (\chi_k \, u_\epsilon^{\frac{4}{n-2}} \, \partial_\epsilon u_\epsilon)$. This is a self adjoint elliptic operator and surjectivity boils down to injectivity. And, for k large enough, injectivity will be guarantied by the result of Lemma 5.2.

Assume that we are given w, $f \in L^\infty(S^n)$ and $\lambda \in \mathbb{R}$ solution of

$$\mathcal{L}_{k,\epsilon} \, w + \lambda \sum_{\mathfrak{h}\in\mathfrak{H}} \mathfrak{h}^* \, \mathfrak{r}^* \, (\chi_k \, u_\epsilon^{\frac{4}{n-2}} \, \partial_\epsilon u_\epsilon) = f,$$

with

$$\int_{S^n} \mathfrak{r}^* \, (\chi_k \, u_\epsilon^{\frac{4}{n-2}} \, \partial_\epsilon u_\epsilon) \, v \, \mathrm{dvol}_{\mathring{g}} = 0.$$

Taking the scalar product with $\mathfrak{r}^* \, (\chi_k \, \partial_\epsilon u_\epsilon)$ we find that $\lambda \in \mathbb{R}$ is defined by

$$\lambda \int_{S^n} \chi_k^2 u_\epsilon^{\frac{4}{n-2}} (\partial_\epsilon u_\epsilon)^2 \, \mathrm{dvol}_{\mathring{g}} = \int_{S^n} \mathfrak{r}^*(\chi_k \partial_\epsilon u_\epsilon) f \, \mathrm{dvol}_{\mathring{g}} - \int_{S^n} w \mathcal{L}_{k,\epsilon} \mathfrak{r}^*(\chi_k \, \partial_\epsilon u_\epsilon) \, \mathrm{dvol}_{\mathring{g}}.$$

It is easy to check that

$$\int_{S^n} \chi_k^2 \, u_\epsilon^{\frac{4}{n-2}} \, (\partial_\epsilon u_\epsilon)^2 \, \mathrm{dvol}_{\mathring{g}} \geq c \, k^4,$$

for k large enough. Moreover

$$\left| \int_{S^n} \mathfrak{r}^*(\chi_k \, \partial_\epsilon u_\epsilon) \, f \, \mathrm{dvol}_{\mathring{g}} \right| \leq c \, k^{4-n-2\delta} \, \|f\|_{L^\infty_{\delta-2}(S^n)}.$$

Using the fact that $\mathcal{L}_\epsilon\, \partial_\epsilon u_\epsilon = 0$, we also get

$$\left| \int_{S^n} w\, \mathcal{L}_{k,\epsilon} \mathfrak{r}^* \, (\chi_k \, \partial_\epsilon u_\epsilon) \, \mathrm{dvol}_{\mathring{g}} \right| \leq c\, k^{4-n-\delta} \, \|w\|_{L^\infty_\delta(S^n)}.$$

Observe that this estimate is essentially a consequence of the estimate of $\mathcal{L}_{k,\epsilon} \mathfrak{r}^* \, (\chi_k \, \partial_\epsilon u_\epsilon)$ in the region where the cutoff function χ_k is acting.

Finally, we have

$$\left\| u_\epsilon^{\frac{4}{n-2}} \, \partial_\epsilon u_\epsilon \right\|_{L^\infty_{\delta-2}(S^n)} \leq c\, k^{n+2\,\delta}.$$

Collecting these estimates, we conclude that

$$|\lambda| \, \left\| u_\epsilon^{\frac{4}{n-2}} \, \partial_\epsilon u_\epsilon \right\|_{L^\infty_{\delta-2}(S^n)} \leq c\, (\|f\|_{L^\infty_{\delta-2}(S^n)} + k^\delta \, \|w\|_{L^\infty_\delta(S^n)}).$$

Collecting these and using the result of Lemma 5.2, we conclude that

$$\|w\|_{L^\infty_\delta(S^n)} \leq c\, (\|f\|_{L^\infty_{\delta-2}(S^n)} + k^\delta \, \|w\|_{L^\infty_\delta(S^n)}).$$

and hence, we conclude that

$$\|w\|_{L^\infty_\delta(S^n)} \leq 2\, c\, \|f\|_{L^\infty_{\delta-2}(S^n)}, \tag{5.5}$$

for all k large enough. This *a priori* estimate implies immediately the injectivity of $\tilde{\mathcal{L}}_{k,\epsilon}$ and hence its surjectivity. In particular, given $f \in L^\infty(S^n)$, there exists a unique $w \in L^\infty(S^n)$ and $\lambda \in \mathbb{R}$ such that

$$\mathcal{L}_{k,\epsilon}\, w + \lambda \sum_{\mathfrak{h} \in \mathfrak{H}} \mathfrak{h}^* \, \mathfrak{r}^* \, (\chi_k \, u_\epsilon^{\frac{4}{n-2}} \, \partial_\epsilon u_\epsilon) = f,$$

with

$$\int_{S^n} \mathfrak{r}^* \, (\chi_k \, u_\epsilon^{\frac{4}{n-2}} \, \partial_\epsilon u_\epsilon) \, w \, \mathrm{dvol}_{\mathring{g}} = 0,$$

and this, together with (5.5) completes the proof of the result. □

The result of the proposition follows at once from the two lemma we have just proved. The first lemma gives the estimate while the second lemma yields the existence of a solution. □

6. Fixed point theorems and resolution of the nonlinear equation

From now on, we assume that the parameter $\epsilon > 0$ is chosen to fulfill

$$\left|1 - \epsilon^{\frac{n-2}{2}} \gamma_{k,\epsilon}\right| \leq 1.$$

Notice that, according to Lemma 4.1, this implies that (4.1) is satisfied provided $C > 1$ (the constant which appears in Lemma 4.1) is fixed large enough. We also assume that the hypothesis in the statement of Theorem 1.9 are satisfied. In particular, $1 \leq d \leq n-3$.

Building on the previous analysis, we now apply a fixed point argument for contraction mappings to solve the equation

$$\Delta_{\mathring{g}}(U_{k,\epsilon} + w) - \tfrac{n\,(n-2)}{4}\,(1 - |U_{k,\epsilon} + w|^{\frac{4}{n-2}})\,(U_{k,\epsilon} + w) = 0,$$

where $U_{k,\epsilon}$ is the approximate solution which has been defined in Section 4. Let us emphasize that, by construction, $U_{k,\epsilon}$ depends on a discrete parameter k and a continuous parameter ϵ. We first rewrite the equation we are trying to solve as

$$\mathcal{L}_{k,\epsilon}\, w + \mathcal{E}_{k,\epsilon} + \mathcal{Q}_{k,\epsilon}(w) = 0, \tag{6.1}$$

where

$$\mathcal{E}_{k,\epsilon} := \Delta_{\mathring{g}} U_{k,\epsilon} - \tfrac{n\,(n-2)}{4}\,(1 - |U_{k,\epsilon}|^{\frac{4}{n-2}})\, U_{k,\epsilon},$$

denotes the error we make by considering that $U_{k,\epsilon}$ is a solution of (1.1),

$$\mathcal{L}_{k,\epsilon} := \Delta_{\mathring{g}} - \tfrac{n(n-2)}{4} + \tfrac{n(n+2)}{4}\, U_{k,\epsilon}^{\frac{4}{n-2}},$$

is the linearized operator about the approximate solution $U_{k,\epsilon}$ and

$$\mathcal{Q}_{k,\epsilon}(w) := \tfrac{n(n-2)}{4}\Big(|U_{k,\epsilon} + w|^{\frac{4}{n-2}}(U_{k,\epsilon} + w) - |U_{k,\epsilon}|^{\frac{4}{n-2}} U_{k,\epsilon} - \tfrac{n+2}{n-2}|U_{k,\epsilon}|^{\frac{4}{n-2}}\, w\Big),$$

collects the nonlinear terms.

Instead of (6.1) we will first solve the equation

$$\left\{\begin{aligned} \mathcal{L}_{k,\epsilon}\, w + \lambda \sum_{\mathfrak{h}\in\mathfrak{H}} \mathfrak{h}^*\,\mathfrak{r}^*\, u_\epsilon^{\frac{4}{n-2}}\, \partial_\epsilon u_\epsilon + \mathcal{E}_{k,\epsilon} + \mathcal{Q}_{k,\epsilon}(w) &= 0,\\ \int_{S^n} w\, \mathcal{L}_{k,\epsilon}\mathfrak{r}^*\,(\chi_k\, \partial_\epsilon u_\epsilon)\, \mathrm{dvol}_{\mathring{g}} + \lambda \int_{S^n} \chi_k^2\, u_\epsilon^{\frac{4}{n-2}}\,(\partial_\epsilon u_\epsilon)^2\, \mathrm{dvol}_{\mathring{g}} & \\ + \int_{S^n} \mathfrak{r}^*\,(\chi_k\, \partial_\epsilon u_\epsilon)\,(\mathcal{E}_{k,\epsilon} + \mathcal{Q}_{k,\epsilon}(w))\, \mathrm{dvol}_{\mathring{g}} &= 0. \end{aligned}\right. \tag{6.2}$$

The solvability of this nonlinear problem relies on the result of Proposition 5.1 together with a fixed point theorem for contraction mapping.

Proposition 6.1. *Assume that the assumptions of Theorem* 1.9 *hold and that* $\delta \in (-\frac{1}{2}, 0)$ *is fixed. Further assume that* $\epsilon > 0$ *satisfies* (4.1). *Then there exist* $c_0 > 0$ *and* $k_0 \geq 0$ *and, for all* $k \geq k_0$, *there exists* $w_{k,\epsilon} \in L^\infty(S^n)$ *and* $\lambda_{k,\epsilon} \in \mathbb{R}$ *solutions of* (6.2) *such that*

$$\|w_{k,\epsilon}\|_{L^\infty_\delta(S^n)} \leq c_0\, k^{2\delta},$$

In addition, $w_{k,\epsilon}$ *and* $\lambda_{k,\epsilon}$ *depend continuously on the parameter* ϵ *satisfying* (4.1).

Before we proceed with the proof of this result, let us remark that, for $k \geq k_0$ and $\epsilon > 0$ satisfying (4.1), we have been able to find a function (not identically equal to 0)

$$u_{k,\epsilon} := U_{k,\epsilon} + w_{k,\epsilon},$$

and $\lambda_{k,\epsilon} \in \mathbb{R}$ such that

$$\Delta_{\mathring{g}} u_{k,\epsilon} - \tfrac{n\,(n-2)}{4}\,(1 - |u_{k,\epsilon}|^{\frac{4}{n-2}})\, u_{k,\epsilon} + \lambda_{k,\epsilon} \sum_{\mathfrak{h} \in \mathfrak{H}} \mathfrak{h}^*\, \mathfrak{r}^*\, (u_\epsilon^{\frac{4}{n-2}}\, \partial_\epsilon u_\epsilon) = 0.$$

Therefore, in order to complete the proof of Theorem 1.9, we will just have to solve the equation

$$\lambda_{k,\epsilon} = 0.$$

The solvability of this equation will be performed right after the proof of Proposition 6.1.

The proof of Proposition 6.1 is decomposed into two lemmas. First, we derive all necessary estimates concerned with the error term $\mathcal{E}_{k,\epsilon}$:

Lemma 6.2. *Assume that* $\delta \in (-\frac{1}{2}, 0)$ *and that the assumptions of Theorem* 1.9 *hold. Then there exist constant* $C_0, k_0 > 0$ *such that, for all* $k \geq k_0$, *the following estimates hold*

$$\left\|\mathcal{E}_{k,\epsilon}\right\|_{L^\infty_{\delta-2}(S^n)} \leq C_0\, k^{2\delta}, \tag{6.3}$$

and

$$\left| \int_{S^n} \mathfrak{r}^*(\chi_k\, \partial_\epsilon u_\epsilon)\, \mathcal{E}_{k,\epsilon}\, \mathrm{dvol}_{\mathring{g}} + c_n\, \epsilon^{\frac{n-4}{2}}\, (1 - \epsilon^{\frac{n-2}{2}}\, \gamma_{k,\epsilon}) \right| \leq C_0\, k^{2-n}\, \log k, \tag{6.4}$$

where $c_n > 0$ *only depends on* n.

Proof. Using the fact that $u_1 \equiv 1$ and $\mathfrak{h}^* \mathfrak{r}^* u_\epsilon$, for all $\mathfrak{h} \in \mathfrak{H}$, are solutions of (1.1), we compute

$$\mathcal{E}_{k,\epsilon} = \tfrac{n(n-2)}{4} \left(\left| 1 - \sum_{\mathfrak{h} \in \mathfrak{H}} \mathfrak{h}^* \mathfrak{r}^* u_\epsilon \right|^{\frac{4}{n-2}} \left(1 - \sum_{\mathfrak{h} \in \mathfrak{H}} \mathfrak{h}^* \mathfrak{r}^* u_\epsilon \right) - 1 + \sum_{\mathfrak{h} \in \mathfrak{H}} \mathfrak{h}^* \mathfrak{r}^* u_\epsilon^{\frac{n+2}{n-2}} \right).$$

Using Lemma 4.2, we estimate

$$\left| \mathcal{E}_{k,\epsilon} \right| \leq c \sum_{\mathfrak{h} \in \mathfrak{H}} \mathfrak{h}^* \mathfrak{r}^* u_\epsilon \leq c \, \left(k \operatorname{dist}(\cdot, \Lambda) \right)^{2-n+d},$$

when $\operatorname{dist}(x, \Lambda) \geq r_k$ and, using Lemma 4.1, we get

$$\left| \mathcal{E}_{k,\epsilon} \right| \leq c \, \mathfrak{r}^* u_\epsilon^{\frac{4}{n-2}},$$

when $\operatorname{dist}(x, p) \leq r_k$. These estimates imply that

$$\| \mathcal{E}_{k,\epsilon} \|_{L^\infty_\delta(S^n)} \leq c \, \left(k^{2-n+d} + k^{\delta-2} + \epsilon^{-\delta} \right).$$

The proof of the first estimate follows at once from the fact that $\epsilon^{-1} \sim k^2$ together with the fact that $\delta - 2 \leq 2\delta$ and $2 - n + d \leq -1 \leq 2\delta$ since $\delta \in (-\frac{1}{2}, 0)$.

The second estimate follows from Lemma 4.2. Indeed, in the range where $\operatorname{dist}(x, \Lambda) \leq r_k$, the result of this lemma can be restated as

$$1 - U_{k,\epsilon} = \mathfrak{r}^* \, u_\epsilon + (1 - \epsilon^{\frac{n-2}{2}} \, \gamma_{k,\epsilon}) + \mathcal{O}(\mathfrak{r}^* \, u_\epsilon^{-\frac{2}{n-2}})$$

Taylor's expansion then implies the refined estimate

$$\left| \mathcal{E}_{k,\epsilon} - \tfrac{n(n+2)}{4} \, \mathfrak{r}^* u_\epsilon^{\frac{4}{n-2}} \, (1 - \epsilon^{\frac{n-2}{2}} \, \gamma_{k,\epsilon}) \right| \leq C \, (1 + \mathfrak{r}^* u_\epsilon^{\frac{2}{n-2}} + \mathfrak{r}^* u_\epsilon^{\frac{6-n}{n-2}}),$$

in the range where $\operatorname{dist}(x, \Lambda) \leq r_k$. We leave the details to the reader. □

We now derive the necessary estimates concerned with the nonlinear terms $Q_{k,\epsilon}$. In this result, we agree that the functions, w, w' are chosen so that

$$\| w \|_{L^\infty_\delta(S^n)} + \| w' \|_{L^\infty_\delta(S^n)} \leq C_1 \, k^{2\delta},$$

for some constant C_1 which will be fixed later on. We have the:

Lemma 6.3. *Assume that $\delta \in (-\frac{1}{2}, 0)$ and that the assumptions of Theorem* 1.9 *hold. Further assume that $C_1 > 0$ is fixed. Then there exists a constant $C_0 > 0$ and there exists $k_0 > 0$, both depending on C_1, such that, for all $k \geq k_0$, the following estimates hold*

$$\left\| Q_{k,\epsilon}(w') - Q_{k,\epsilon}(w) \right\|_{L^\infty_{\delta-2}(S^n)} \leq C_0 \, k^{2\delta} \, \| \bar{w}' - \bar{w} \|_{L^\infty_\delta(S^n)}, \tag{6.5}$$

$$\left| \int_{S^n} w\, \mathcal{L}_{k,\epsilon} \mathfrak{r}^* \,(\chi_k\, \partial_\epsilon u_\epsilon)\, \mathrm{dvol}_{\mathring{g}} \right| \leq C_0\, k^{4-n+\delta}, \tag{6.6}$$

and

$$\left| \int_{S^n} \mathfrak{r}^* (\chi_k\, \partial_\epsilon u_\epsilon)\, Q_{k,\epsilon}(w)\, \mathrm{dvol}_{\mathring{g}} \right| \leq C_0\, k^{2-n}. \tag{6.7}$$

Proof. These estimates are not hard to derive but observe that some care is due to derive the first estimate since the nonlinearity $t \longmapsto |t|^{\frac{4}{n-2}}\, t$ is not $\mathcal{C}^2$ at 0 when $n \geq 7$.

Let us explain where the first estimate comes from. In the range where $U_{k,\epsilon} \geq 1/2$ (*i.e.* when $\mathrm{dist}(\cdot, \Lambda) \geq c\, r_k$, for some fixed $c > 0$ large enough), we get the pointwise estimate

$$|Q_{k,\epsilon}(w') - Q_{k,\epsilon}(w)| \leq C\, k^{2\delta} (\mathrm{dist}(\cdot, \Lambda))^{2\delta}\, \|\bar{w}' - \bar{w}\|_{L^\infty_\delta(S^n)}$$

while, when $U_{k,\epsilon} \leq -1/2$, say for example close to the point p (*i.e.* when $\mathrm{dist}(\cdot, p) \leq \tilde{c}\, r_k$, for some fixed $\tilde{c} > 0$ small enough), we get the pointwise estimate

$$|Q_{k,\epsilon}(w') - Q_{k,\epsilon}(w)| \leq C\, k^{n-6+2\delta} (\mathrm{dist}(\cdot, p))^{n-6+2\delta}\, \|\bar{w}' - \bar{w}\|_{L^\infty_\delta(S^n)}$$

Finally, in the range where $U_{k,\epsilon} \in [-1/2, 1/2]$, say for example, close to the point p, we have the pointwise estimate

$$|Q_{k,\epsilon}(w') - Q_{k,\epsilon}(w)| \leq C\, (\mathrm{dist}(\cdot, p))^{\delta}\, \|\bar{w}' - \bar{w}\|_{L^\infty_\delta(S^n)}$$

for some constant $C > 0$ which depends on C_1. Collecting these, we conclude that

$$\left\| Q_{k,\epsilon}(w') - Q_{k,\epsilon}(w) \right\|_{L^\infty_{\delta-2}(S^n)} \leq C\, (k^{2\delta} + k^{2-n} + k^{\delta-2} + k^{-2})\, \|\bar{w}' - \bar{w}\|_{L^\infty_\delta(S^n)}.$$

The estimate (6.5) then follows at once for the choice $\delta \in (-\frac{1}{2}, 0)$.

The second estimate (6.6) has already been proven in the proof of Lemma 5.3. The last estimate follows from the pointwise estimate

$$|\mathfrak{r}^* (\chi_k\, \partial_\epsilon u_\epsilon)\, Q_{k,\epsilon}(w)| \leq C\, k^{4\delta-2}\, (\mathrm{dist}(\cdot, p))^{2\delta-4}$$

when $\mathrm{dist}(\cdot, p) \leq r_k$. Therefore, we get

$$\left| \int_{S^n} \mathfrak{r}^* (\chi_k\, \partial_\epsilon u_\epsilon)\, Q_{k,\epsilon}(w)\, \mathrm{dvol}_{\mathring{g}} \right| \leq C\, k^{2-n+2\delta},$$

when $n \geq 5$, when $n = 4$, the $k^{2-n+2\delta}$ has to be replaced by k^{-2}. This completes the proof of the lemma. $\square$

We can now complete the proof of Proposition 6.1.

Proof of Proposition 6.1. We choose $\delta \in (-1/2, 0)$. First we apply a fixed point theorem for contraction mappings to solve (6.2). To this aim, we use the result of Proposition 5.1 to obtain a right inverse for the operator $\mathcal{L}_{k,\epsilon}$ and rephrase (6.2) as a fixed point problem. The above estimates (6.3) and (6.5) are precisely the one which are needed to ensure, for all $\epsilon > 0$ small enough, the existence of a fixed point w which belongs to the ball of radius $C_1 k^{2\delta}$ in $L^\infty_\delta(S^n)$, provided C_1 is fixed large enough. Reducing the range in which $\epsilon > 0$ is chosen, it is not hard to check that the solution we obtain depends continuously on ϵ by taking the difference between the equations satisfied by the solutions for two different values of ϵ and using the contraction property of the nonlinear operator. Since this is rather standard, we leave the details to the reader. This completes the proof of Proposition 6.1. $\square$

It remains to solve the equation $\lambda_{k,\epsilon} = 0$. Looking at the second equation in (6.2) we find that the equation $\lambda_{k,\epsilon} = 0$ reduces to

$$\int_{S^n} \mathfrak{r}^*(\chi_k\, \partial_\epsilon u_\epsilon)\, (\mathcal{L}_{k,\epsilon} w_{k,\epsilon} + \mathcal{E}_{k,\epsilon} + Q_{k,\epsilon}(w_{k,\epsilon}))\, \mathrm{dvol}_{\mathring{g}} = 0.$$

Using the estimates (6.4), (6.6) and (6.7), we find that this equation is equivalent to

$$(1 - \epsilon^{\frac{n-2}{2}}\, \gamma_{k,\epsilon}) = V_k(\epsilon) \tag{6.8}$$

where the function V_k depends continuously on ϵ. Then the previous analysis shows that there exists $k_0 > 0$ such that, for $k \geq k_0$, we have the estimate $|V_k(\epsilon)| \leq C\, k^\delta$, uniformly in ϵ satisfying (4.1). The existence of a solution of (6.8) then follows from the intermediate value theorem and this completes the proof of the Theorem 1.9.

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