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A structural theorem for codimension-one foliations on $\mathbb{P}^n$, $n \geq 3$, with an application to degree-three foliations

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Abstract. Let $\mathcal{F}$ be a codimension-one foliation on $\mathbb{P}^n$: for each point $p \in \mathbb{P}^n$ we define $\mathcal{J}(\mathcal{F}, p)$ as the order of the first non-zero jet $j^k_p(\omega)$ of a holomorphic 1-form ω defining $\mathcal{F}$ at p. The singular set of $\mathcal{F}$ is $\mathrm{sing}(\mathcal{F}) = \{p \in \mathbb{P}^n \mid \mathcal{J}(\mathcal{F}, p) \geq 1\}$. We prove (main Theorem 1.2) that a foliation $\mathcal{F}$ satisfying $\mathcal{J}(\mathcal{F}, p) \leq 1$ for all $p \in \mathbb{P}^n$ has a non-constant rational first integral. Using this fact we are able to prove that any foliation of degree-three on $\mathbb{P}^n$, with $n \geq 3$, is either the pull-back of a foliation on $\mathbb{P}^2$, or has a transverse affine structure with poles. This extends previous results for foliations of degree at most two.

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Notation

1. $\mathcal{O}_n$: the ring of germs at $0 \in \mathbb{C}^n$ of holomorphic functions.
   $\mathcal{O}^*_n = \{f \in \mathcal{O}_n \mid f(0) \neq 0\}$. $\mathfrak{m}_n = \{f \in \mathcal{O}_n \mid f(0) = 0\}$.
2. $f \mid g$: $f, g \in \mathfrak{m}_n \setminus \{0\}$ and f divides g.
3. $f \nmid g$: $f, g \in \mathfrak{m}_n \setminus \{0\}$ and f does not divide g.
4. $[f, g]_0$: the intersection number of $f, g \in \mathfrak{m}_2 \setminus \{0\}$, when f and g have no common factor.
5. $< f, g >$: the ideal generated by $f, g \in \mathcal{O}_p$.
6. $\mathrm{Diff}(\mathbb{C}^n, p)$: the group of germs at $p \in \mathbb{C}^n$ of biholomorphisms f with $f(p) = p$.
7. $i_X(\omega)$: the interior product of the vector field X and the form ω.
8. L_X: the Lie derivativative in the direction of the vector field X.
9. j^k_p: the k^{th}-jet at the point p.

1. Introduction

In a previous paper [10] we have proved that the space of holomorphic codimension-one foliations and degree-two on $\mathbb{P}^n$, with $n \geq 3$, has six irreducible components. A

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consequence of this classification is that we have two possibilities for a degree-two foliation $\mathcal{F}$ on $\mathbb{P}^n$, with $n \geq 3$: either $\mathcal{F}$ is defined by a meromorphic closed 1-form on $\mathbb{P}^n$, or $\mathcal{F} = g^*(\mathcal{G})$, where $g\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ is a linear map and is $\mathcal{G}$ a degree-two foliation of $\mathbb{P}^2$. A foliation defined by a meromorphic closed 1-form admits a special projective tranverse structure with poles, namely a translation structure. On the other hand, a foliation of the form $g^*(\mathcal{G})$ admits such a structure if, and only if, $\mathcal{G}$ admits one(*cf.* [4]). This is not always the case: a foliation of $\mathbb{P}^2$ which admits a projective or affine transverse structure always has algebraic leaves, whereas for any $d \geq 2$, there are degree-d foliations on $\mathbb{P}^2$ without algebraic invariant curves. The following conjecture is attributed to different authors (Brunella, Lins Neto,...):

Main Conjecture. *Any codimension-one holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^n$, with $n \geq 3$, either is a pull-back of a foliation $\mathcal{G}$ on $\mathbb{P}^2$ by a rational map $\Phi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^2$, or admits a transverse projective structure with poles on some invariant hypersurface.*

In the first case, the leaves of $\mathcal{F}$ are sub-foliated by the levels of Φ and the dynamic properties of $\mathcal{F}$ are essentialy given by $\mathcal{G}$. In the second, we can associate a triple of meromorphic 1-forms $(\omega_0, \omega_1, \omega_2)$ such that ω_0 defines $\mathcal{F}$ outside its set of poles $|\omega_0|_\infty$ and the triple satisfies the $s\ell(2, \mathbb{C})$ structural relations:

$$\begin{aligned} d\omega_0 &= \omega_0 \wedge \omega_1 \\ d\omega_1 &= \omega_0 \wedge \omega_2 \\ d\omega_2 &= \omega_1 \wedge \omega_2 \end{aligned}$$

inducing the projective structure.

For instance, when $\omega_1 = \omega_2 = 0$, that is ω_0 is closed, the integration of ω_0 on simply connected open sets $U \subset \mathbb{P}^n \setminus |\omega_0|_\infty$ gives $\omega_0 = df_U$, and defines the transverse translation structure: when $U \cap V \neq \emptyset$ we have $f_U = f_V + c_{UV}$, where $c_{UV} \in \mathbb{C}$. On the other hand, if $\omega_2 = 0$ and $\omega_1 \neq 0$ then the transverse structrure is affine.

The main conjecture seems to be reasonable (at least for foliations of small degree) for the following reasons: first of all, if $\mathbb{K}$ is a field of positive characteristic every foliation on a projective manifold over $\mathbb{K}$, in particular on $\mathbb{P}^n_{\mathbb{K}}$, is defined by a closed 1-form (*cf.* [12]). On the other hand, if $\mathcal{F}$ is a foliation on $\mathbb{P}^n$ and p is a prime number then it is possible to define $\mathcal{F}_p$, the reduction modulo p of $\mathcal{F}$. There is a conjecture of Grothendieck-Katz-type which says that if for almost all p the foliation $\mathcal{F}_p$ has a non-constant rational first integral then $\mathcal{F}$ itself has a non-constant rational first integral. Recently F. Touzet has communicated to one of the authors the following result:

Theorem. (F. Touzet) *The Grothendieck-Katz conjecture implies that any foliation of degree at most $n-1$ on $\mathbb{P}^n$ either admits a projective transverse structure, or is a pull-back of some foliation on $\mathbb{P}^k$, with $k < n$, by some rational map.*

Concerning the main conjecture, note that the first interesting case which is not covered by the above conditional result is that of foliations of degree-three on

$\mathbb{P}^3$. In fact, one of the goals of this paper is to prove that the conjecture is true for foliations of degree three.

Theorem 1.1. *Let $\mathcal{F}$ be a holomorphic codimension-one foliation of degree-three on $\mathbb{P}^n$, with $n \geq 3$. Then:*

- *either $\mathcal{F}$ has a rational first integral,*
- *or $\mathcal{F}$ has an affine transverse structure with poles on an invariant hypersurface,*
- *or $\mathcal{F} = g^*(\mathcal{G})$, where $g\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ is a rational map and $\mathcal{G}$ is a foliation on $\mathbb{P}^2$.*

One of the tools of the proof will be a result of [12] concerning foliations which admit a finite Godbillon-Vey sequence. This result essentially says that such a foliation is either a pull-back of a foliation on a surface or has a transversely projective structure. Let us explain briefly how we can apply the result.

By definition, a degree-d foliation $\mathcal{F}$ on $\mathbb{P}^n$ has d tangencies with a generic straight line of $\mathbb{P}^n$. This implies that $\mathcal{F}$ can be represented in an affine coordinate system $\mathbb{C}^n \simeq E \subset \mathbb{P}^n$ by a polynomial integrable 1-form $\omega_E = \sum_{j=0}^{d+1} \omega_j$, where the coefficients of the 1-form ω_j are homogeneous polynomials of degree j, $0 \leq j \leq d+1$, and $i_R(\omega_{d+1}) = 0$, with $R = \sum_{j=1}^{n} z_j\, \partial_{z_j}$, the radial vector field. The form ω_E can be considered as a meromorphic 1-form on $\mathbb{P}^n$ with poles of order $d+2$ at the hyperplane of infinity of E. Given $p \in E$, we set

$$\mathcal{J}(\mathcal{F}, p) = \min\{k \geq 0 \mid j_p^k(\omega_E) \neq 0\} \,.$$

It can be proved that $\mathcal{J}(\mathcal{F}, p)$ depends only on p and $\mathcal{F}$ and not on E and ω_E. The singular set of $\mathcal{F}$ is defined as

$$\operatorname{sing}(\mathcal{F}) = \{p \in \mathbb{P}^n \mid \mathcal{J}(\mathcal{F}, p) \geq 1\}\,.$$

This set is algebraic and always has irreducible components of codimension two (*cf.* [16]).

Given a degree-three foliation $\mathcal{F}$ of $\mathbb{P}^n$, we will consider two cases:

(1) There exists $p \in \operatorname{sing}(\mathcal{F})$ such that $\mathcal{J}(\mathcal{F}, p) \geq 2$.
(2) For all $p \in \operatorname{sing}(\mathcal{F})$ we have $\mathcal{J}(\mathcal{F}, p) = 1$.

Case (1) will be studied in Section 2. We will see that $\mathcal{F}$ admits a finite Godbillon-Vey sequence in this case and we can apply the result of [12]. In case (2) we will see in Section 3 that $\mathcal{F}$ has a meromorphic first integral.

In Section 3 we will introduce the *Baum-Bott index* of an irreducible component, say Γ, of codimension-two of $\operatorname{sing}(\mathcal{F})$, which we will denote $\mathrm{BB}(\mathcal{F}, \Gamma)$. As a consequence of the Baum-Bott theorem we will see that $\operatorname{sing}(\mathcal{F})$ always has a codimension-two irreducible component Γ with $\mathrm{BB}(\mathcal{F}, \Gamma) \neq 0$.

Theorem 1.2. *Let $\mathcal{F}$ be a codimension-one holomorphic foliation on $\mathbb{P}^n$, with $n \geq 3$. Assume that* $\mathrm{sing}(\mathcal{F})$ *has an irreducible component of codimension-two* Γ *such that*

(a) $\mathrm{BB}(\mathcal{F}, \Gamma) \neq 0$.
(b) *The algebraic set* $\{p \in \Gamma \,|\, \mathcal{J}(\mathcal{F}, p) > 1\}$ *has codimension at least* 4 *in* $\mathbb{P}^n$.

Then $\mathcal{F}$ has a rational first integral.

As a consequence, we will get the following:

Corollary 1.3. *Let $\mathcal{F}$ be a codimension-one holomorphic foliation on $\mathbb{P}^n$, with $n \geq 3$. If $\mathcal{J}(\mathcal{F}, p) \leq 1$ for all $p \in \mathbb{P}^n$ then $\mathcal{F}$ has a rational first integral.*

Remark 1.4. Recall that $p \in \mathrm{sing}(\mathcal{F})$ is of Kupka type if $\mathcal{F}$ is defined in a neighborhood of p by a holomorphic 1-form ω such that $d\omega(p) \neq 0$. We define $K(\mathcal{F}) = \{p \in \mathrm{sing}(\mathcal{F}) \mid p \text{ is of Kupka type}\}$. If $p \in K(\mathcal{F})$ then $\mathcal{J}(\mathcal{F}, p) = 1$. We would like to observe that if $\mathrm{sing}(\mathcal{F})$ has a smooth irreducible component, say Γ, with $\Gamma \subset K(\mathcal{F})$, then a theorem due to Calvo Andrade and M. Brunella says that $\mathcal{F}$ has a rational first integral (*cf.* [5, 6, 11] and [3]). In this sense, Theorem 1.2 is a generalization of Calvo and Brunella's theorem.

Remark 1.5. We would like to observe that the conclusion of Corollary 1.3 is not true when we consider codimension-one foliations on more general complex manifolds. For instance, let $M = \mathbb{P}^2 \times \mathbb{P}^k$, with $k \geq 1$, and $\mathcal{F} = \Pi_1^*(\mathcal{G})$, where $\Pi_1 : \mathbb{P}^2 \times \mathbb{P}^k \to \mathbb{P}^2$ is the projection on the first factor and $\mathcal{G}$ is a foliation on $\mathbb{P}^2$ of degree at least 2 without non-constant rational first integral and with $\mathcal{J}(\mathcal{G}, p) \leq 1$ for all $p \in \mathbb{P}^2$. Then $\mathcal{F}$ satisfies the hypothesis of Corollary 1.3 but not its conclusion. A natural question which arises is the following:

Problem 1.6. For which compact complex manifolds of dimension at least 3 the conclusion of Corollary 1.3 is true?

Remark 1.7. We say that a foliation admits a *purely* projective transverse structure (briefly p.p.t.s.) if it has a projective transverse structure with poles, but no affine transverse structure with poles. There are examples of foliations on $\mathbb{P}^3$, for instance the so called Hilbert modular foliations, which admit a p.p.t.s. and are not the pull-back of foliations on $\mathbb{P}^2$ (*cf.* [12]). In fact, these examples have degree at least five.

On the other hand, as a consequence of the proof of Theorem 1.1, any foliation of degree-three on $\mathbb{P}^n$, with $n \geq 3$, that admits a p.p.t.s. is the pull-back of a Riccati foliation on $\mathbb{P}^1 \times \mathbb{P}^1$ (see the third case in the proof of Lemma 2.5). For instance, there are p.p.t.s. Riccati equations on $\mathbb{C}^2$ of the form

$$x(x-1)\,dy - (a_0(x) + a_1(x)\,y + a_2(x)\,y^2)\,dx = 0\ , \tag{1.1}$$

where a_0, a_1 and a_2 are degree-one polynomials. If $\mathcal{G}$ is a p.p.t.s. foliation defined by (1.1) on $\mathbb{P}^2$ then it has degree-three. In particular, if $\Pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^2$ is linear then $\Pi^*(\mathcal{G})$ is a p.p.t.s. degree-three foliation on $\mathbb{P}^n$.

It seems reasonable to hope that Theorem 1.1 will give a clue to a classification of the irreducible components of the space of degree-three foliations on $\mathbb{P}^n$ which are not the pull-back by rational maps of foliations on $\mathbb{P}^2$. However, the analysis of the components of rational pull-back type seems to be more delicate, since we have no control on the degrees of the objects that appear in our proofs.

Problem 1.8. Classify the irreducible components of the space of foliations of degree-three on $\mathbb{P}^n$, with $n \geq 3$.

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2. Proof of Theorem 1.1

The aim of this section is to prove Theorem 1.1 by admitting Theorem 1.2. In Section 2.1 we will analyse the case where there exists $p \in \mathbb{P}^n$ such that $\mathcal{J}(\mathcal{F}, p) \geq 2$ and in Section 2.2 we will finish the proof.

2.1. The case $\mathcal{J}(\mathcal{F}, p) \geq 2$ for some p

Let $\mathcal{F}$ be a codimension-one holomorphic foliation on a complex manifold X. A Godbillon-Vey sequence (briefly G-V-S) associated to $\mathcal{F}$ is a sequence of meromorphic 1-forms on X, say $(\omega_j)_{j \geq 0}$, such that

1. $\mathcal{F}$ is defined by ω_0 outside its set of poles, $|\omega_0|_\infty$. In particular, ω_0 is integrable, that is $\omega_0 \wedge d\omega_0 = 0$.
2. The 1-form defined by the formal power series

$$\Omega = dz + \omega_0 + z\,\omega_1 + \frac{z^2}{2}\,\omega_2 + \sum_{j \geq 3} \frac{z^j}{j!}\,\omega_j \tag{2.1}$$

   is integrable.

When there exists N such that $\omega_N \neq 0$ but $\omega_j = 0$ for all $j > N$ then we say that $\mathcal{F}$ admits a finite G-V-S of lenght N. In this case, the form in (2.1) is meromorphic and can be extended meromorphically to $X \times \mathbb{P}^1$. Since it is integrable, it defines a codimension-one foliation $\mathcal{H}$ on $X \times \mathbb{P}^1$ such that $\mathcal{H}|_{X \times \{0\}} = \mathcal{F}$.

Remark 2.1. Let $\mathcal{F}$ and $\mathcal{G}$ be foliations on complex manifolds X and Y, respectively. Assume that $\mathcal{G}$ admits a finite G-V-S of lenght N and that $\mathcal{F} = \Phi^*(\mathcal{G})$, where $\Phi \colon X \longrightarrow Y$ is a rational map. Then $\mathcal{F}$ also admits a G-V-S of lenght N (*cf.* [12]).

Remark 2.2. When $\mathcal{F}$ admits a G-V-S of lenght $N \leq 2$ then $\mathcal{F}$ has a transverse projective structure with poles in a hypersurface. When $N = 1$ then the structure is in fact affine (*cf.* [14] and [23]).

On the other hand, if it admits a finite G-V-S of lenght $N \geq 3$ then we have the following

Theorem 2.3 ([12]). *Let $\mathcal{F}$ be a foliation on a compact holomorphic manifold X admiting a finite G-V-S of lenght $N \geq 3$. Then*

- *either $\mathcal{F}$ is transversely affine,*
- *or there exist a compact Riemann surface S, meromorphic 1-forms $\alpha_0, ..., \alpha_N$ on S and a rational map $\phi\colon X \dashrightarrow S \times \mathbb{P}^1$ such that $\mathcal{F}$ is defined by the meromorphic 1-form $\omega = \phi^*(\eta)$, where*

$$\eta = dz + \alpha_0 + z\,\alpha_1 + ... + z^N\,\alpha_N \ . \tag{2.2}$$

When $X = \mathbb{P}^n$, with $n \geq 3$, necessarilly $S = \mathbb{P}^1$ and (2.2) can be written as

$$\eta = dz - P(t, z)\,dt \ ,$$

where $P \in \mathbb{C}(t)[z]$ and $\mathcal{F} = \phi^*(\mathcal{G})$, where $\mathcal{G}$ is defined on $\mathbb{P}^1 \times \mathbb{P}^1$ by the differential equation $\frac{dz}{dt} = P(t, z)$. Since $\mathbb{P}^1 \times \mathbb{P}^1$ is birrational to $\mathbb{P}^2$ we get the following consequence:

Corollary 2.4. *If $\mathcal{F}$ is a codimension-one holomorphic foliation on $\mathbb{P}^n$, with $n \geq 3$, which admits a finite G-V-S then, either it has a tranverse projective structure, or it is a pull-back of foliation on $\mathbb{P}^2$ by a rational map.*

Now, let us consider a degree-three codimension-one foliation $\mathcal{F}$ on $\mathbb{P}^n$ and assume that there exists $p \in \mathbb{P}^n$ such that $\mathcal{J}(\mathcal{F}, p) \geq 2$. In this case, if we take an affine coordinate system $\mathbb{C}^n \subset \mathbb{P}^n$ such that $p = 0 \in \mathbb{C}^n$ then $\mathcal{F}|_{\mathbb{C}^n}$ is defined by a polynomial 1-form $\omega = \alpha_2 + \alpha_3 + \alpha_4$, where the coefficients of α_j are homogeneous polynomials of degree j, $2 \leq j \leq 4$, and $i_R(\alpha_4) = 0$, R the radial vector field.

Lemma 2.5. *In the above situation we have three possibilities:*

(a) *$\mathcal{F}$ has an affine transverse structure with poles in a hypersurface.*
(b) *$\mathcal{F}$ is the pull-back by a rational map of a foliation on $\mathbb{P}^2$.*
(c) *$\mathcal{F}$ is the pull-back by a linear map $\pi : \mathbb{P}^n \longrightarrow \mathbb{P}^{n-1}$ of a foliation of degree-three on $\mathbb{P}^{n-1}$.*

In particular, if $n = 3$ then $\mathcal{F}$ satisfies (a) *or* (b).

Proof. With the previous notations, set $\alpha_j = \sum_{i=1}^n P_{ji}(z) dz_i$ and $F_j(z) = i_R(\alpha_{j-1}) = \sum_{i=1}^n z_i . P_{ji}(z)$, $j = 3, 4$. We will divide the proof in three cases:

1$^{\text{st}}$. $i_R(\omega) \equiv 0$, which is equivalent to $F_3 \equiv F_4 \equiv 0$. In this case, we will prove that $\omega = \alpha_4$ and we will get (c).

2nd. $F_3 \not\equiv 0$. In this case, we will prove that there exists a birrational map $\Phi\colon \mathbb{P}^n \longrightarrow \mathbb{P}^n$, an affine coordinate system $(x, z) \in \mathbb{C}^{n-1} \times \mathbb{C} \simeq \mathbb{C}^n \subset \mathbb{P}^n$ and meromorphic 1-forms $\beta_1, \beta_2\, \beta_3$, with $i_{\partial_z}(\beta_j) = 0$ and $L_{\partial_z}(\beta_j) = 0$, $1 \leq j \leq 3$, such that $\Phi^*(\omega) = g.\,\eta$, where

$$\eta = dz + z\,\beta_1 + z^2\,\beta_2 + z^3\,\beta_3\ . \tag{2.3}$$

We will show that we can apply Theorem 2.3 to prove that $\mathcal{F}$ satisfies (a) or (b).

3rd. $F_3 \equiv 0$ and $F_4 \not\equiv 0$. In this case, we will prove that, either $\mathcal{F}$ is the pull-back of a Ricatti equation on $\mathbb{P}^1 \times \mathbb{P}^1$, or it admits an affine transverse structure, or ω has an integrating factor, that is there exists a meromorphic function $h \not\equiv 0$ such that $d(h\,\omega) = 0$.

Analysis of the 1st case. First of all, note that $\alpha_4 \not\equiv 0$, for otherwise $\mathcal{F}$ would have degree ≤ 2. The integrability condition gives

$$\omega \wedge d\omega = 0 \quad \Longrightarrow \quad \omega \wedge i_R(d\omega) = 0\ .$$

On the other hand,

$$i_R(d\omega) = L_R(\omega) - d(i_R(\omega)) = L_R(\omega) = 3\,\alpha_2 + 4\,\alpha_3 + 5\,\alpha_4$$

$$\Longrightarrow\ 0 \equiv (\alpha_2 + \alpha_3 + \alpha_4) \wedge (3\alpha_2 + 4\alpha_3 + 5\alpha_4) = \alpha_2 \wedge \alpha_3 + 2\,\alpha_2 \wedge \alpha_4 + \alpha_3 \wedge \alpha_4\ .$$

Since the coefficients of α_j are homogeneous of degree j, $2 \leq j \leq 4$, we get

$$\alpha_2 \wedge \alpha_3 = \alpha_2 \wedge \alpha_4 = \alpha_3 \wedge \alpha_4 = 0\ . \tag{2.4}$$

Since $\alpha_4 \not\equiv 0$, (2.4) implies that there are meromorphic functions f_j, $j = 2, 3$, such that $\alpha_j = f_j.\alpha_4$. If $f_j \not\equiv 0$ for some $j \in \{2, 3\}$ then the foliation $\mathcal{F}$ would have degree less than three. Therefore, $\alpha_2 = \alpha_3 = 0$.

In particular, we get $\omega = \alpha_4$. Since α_4 is integrable, it defines a foliation of degree-three, say $\mathcal{F}_{n-1}$, on $\mathbb{P}^{n-1}$. If we consider $\mathbb{P}^{n-1}$ as the set of lines through $0 \in \mathbb{C}^n \subset \mathbb{P}^n$ and $\pi : \mathbb{P}^n \setminus \{0\} \to \mathbb{P}^{n-1}$ is the natural projection then $\mathcal{F} = \pi^*(\mathcal{F}_{n-1})$. This finishes the analysis of the 1^{st} case.

In the analysis of the two other cases, we consider first a blowing-up $\pi : \tilde{\mathbb{P}}^n \to \mathbb{P}^n$ at $0 \in \mathbb{C}^n \subset \mathbb{P}^n$. Let us compute the foliation $\pi^*(\mathcal{F})$. In the chart

$$(\tau_1, ..., \tau_{n-1}, x) = (\tau, x) \in \mathbb{C}^{n-1} \times \mathbb{C} \overset{\pi}{\mapsto} (x.\tau, x) = (z_1, ..., z_n) \in \mathbb{C}^n \subset \mathbb{P}^n$$

we get

$$\pi^*(\omega) = x^2 \left[x\,\theta_2 + x^2\,\theta_3 + x^3\,\theta_4 + (F_3(\tau, 1) + x\,F_4(\tau, 1))dx\right]\ , \tag{2.5}$$

where

$$\theta_j = \sum_{i=1}^{n-1} P_{ji}(\tau, 1)\,d\tau_i\ ,\ 2 \leq j \leq 4\ ,$$

depends only on τ.

Analysis of the 2nd case. Since $F_3 \not\equiv 0$ we have two possibilities:

2a. $F_4 \equiv 0$. In this case, if we set $\beta_j = \frac{1}{F_3(\tau,1)}\,\theta_{j+2}$ then we can write $\pi^*(\omega) = x^2.\,F_3(\tau,1).\,\eta$, where $i_{\partial_x}(\beta_j) = 0$, $L_{\partial_x}(\beta_j) = 0$, $1 \le j \le 3$, and

$$\eta = dx + x\,\beta_1 + x^2\,\beta_2 + x^3\,\beta_3\ .$$

Therefore, we get (2.3).

2b. $F_4 \not\equiv 0$. In this case, $\pi^*(\mathcal{F})$ is defined in this chart by

$$\nu := x^{-2}.\pi^*(\omega) = x\,\theta_2 + x^2\,\theta_3 + x^3\,\theta_4 + (F_3(\tau,1) + x\,F_4(\tau,1))dx$$

and we need one more birrational transformation. Consider the birrational map

$$\psi(\tau,z) = \left(\tau,\ \frac{\frac{F_3(\tau,1)}{F_4(\tau,1)}z}{1-z}\right) = (\tau,x)\ \text{with inverse}\ \psi^{-1}(\tau,x) = \left(\tau,\ \frac{x}{x+\frac{F_3(\tau,1)}{F_4(\tau,1)}}\right).$$

A straightforward computation gives $\psi^*(\nu) = \frac{F_3^2(\tau,1)}{F_4(\tau,1)}\,\eta$, where

$$\eta = dz + z\,\beta_1 + z^2\,\beta_2 + z^3\,\beta_3$$

with

$$\beta_1 = \frac{1}{F_3(\tau,1)}\,\theta_2 + \frac{dF_3(\tau,1)}{F_3(\tau,1)} - \frac{dF_4(\tau,1)}{F_4(\tau,1)}$$

$$\beta_2 = \frac{1}{F_4(\tau,1)}\,\theta_3 - 2\frac{1}{F_3(\tau,1)}\,\theta_2 + \frac{dF_4(\tau,1)}{F_4(\tau,1)} - \frac{dF_3(\tau,1)}{F_3(\tau,1)}$$

$$\beta_3 = \frac{F_3(\tau,1)}{F_4^2(\tau,1)}\,\theta_4 - \frac{1}{F_4(\tau,1)}\,\theta_3 + \frac{1}{F_3(\tau,1)}\,\theta_2\,.$$

Therefore, in both sub-cases we obtain (2.3).

At this point we should mention that, in order to use Theorem 2.3, we have to assure that the G-V-S is of lenght at least 3. If η is like in (2.3) then the integrability condition of the form η implies that it admits the G-V-S $(\eta = \eta_0, \eta_1, \eta_2, \eta_3)$, where $\eta_j = L^{(j)}_{\partial_z}(\eta)$ (*cf.* [12]). The lenght is three if $\eta_3 = 6\,\beta_3 \not\equiv 0$. On the other hand, if $\beta_3 \equiv 0$ then $\mathcal{F}$ has an affine transverse structure.

In fact, if we set $z = 1/w$ in (2.3) with $\beta_3 = 0$ then we get $\eta = -w^{-2}\Omega$, where

$$\Omega = dw - \beta_2 - w\,\beta_1\ .$$

Therefore, Ω admits the G-V-S of lenght one $(\Omega, -\beta_1)$. Note that $d\Omega = \beta_1 \wedge \Omega$ and $d\beta_1 = 0$. Hence, $\mathcal{F}$ has an affine transverse structure with poles in some hypersurface.

Analysis of the 3rd case. In this case, after the blowing-up π, we get $\pi^*(\omega) = x^3 . F_4(\tau, 1) . \eta$, where

$$\eta = dx + \beta_0 + x\,\beta_1 + x^2\,\beta_2\ ,\ \ \beta_j = \frac{1}{F_4(\tau,1)}\,\theta_{j+2}\ ,\ \ 0 \leq j \leq 2\ . \tag{2.6}$$

In particular, η admits a G-V-S of lenght ≤ 2, $(\eta = \eta_0, \eta_1, \eta_2)$, where $\eta_j = L^{(j)}_{\partial_x}(\eta)$, $j = 1, 2$. A foliation defined by a meromorphic form like in (2.6) has always a projective transverse structure, but in general has no affine transverse structure. Therefore, we have to work more to conclude the proof in this case.

We begin by recalling that $\pi^*(\alpha_2) = x^2(x\,\theta_2 + F_3(\tau, 1)\,dx) = x^3\,\theta_2$. When $\alpha_2 \equiv 0$ we get $\beta_0 \equiv 0$ and (2.6) becomes similar to (2.3) with $\beta_3 \equiv 0$, which we have already considered; in that case we have an affine transverse structure. Let us assume $\alpha_2 \not\equiv 0$. The integrability condition $\omega \wedge d\omega = 0$ implies that $\alpha_2 \wedge d\alpha_2 = 0$. In particular, either $\mathrm{cod}(\mathrm{sing}(\alpha_2)) \geq 2$ and α_2 defines a foliation of degree-one on $\mathbb{P}^{n-1}$, or $\alpha_2 = h . \alpha_1$, where α_1 defines a foliation of degree zero on $\mathbb{P}^{n-1}$. In both cases, it is known that α_2 has an integrating factor. In other words, there exists a homogeneous polynomial $f \not\equiv 0$ of degree-three such that $d(f^{-1}\,\alpha_2) = 0$ (*cf.* [10]). This implies

$$\pi^*\left(\frac{\alpha_2}{f}\right) = \frac{\theta_2}{f(\tau,1)} \implies d\left(\frac{\theta_2}{f(\tau,1)}\right) = 0 \implies d\left(\frac{F_4(\tau,1)}{f(\tau,1)}\beta_0\right) = 0\,. \tag{2.7}$$

Set $F(\tau) := f(\tau,1)/F_4(\tau,1)$ and consider the birrational map $\Phi(\tau,z) = (\tau, F(\tau), z) = (\tau, x)$. If η is like in (2.6) then a straightforward computation gives $\Phi^*(\eta) = F . \tilde{\eta}$, where

$$\tilde{\eta} = dz + \tilde{\beta}_0 + z\,\tilde{\beta}_1 + z^2\,\tilde{\beta}_2\ ,\ \ \tilde{\beta}_0 := F^{-1} . \beta_0\ ,\ \ \tilde{\beta}_1 = \beta_1 + \frac{dF}{F}\ ,\ \ \tilde{\beta}_2 = F . \beta_2\ .$$

In this situation it is convenient to consider the birrational map $\Psi(\tau, w) = (\tau, 1/w) = (\tau, z)$. We have $\Psi^*(\tilde{\eta}) = -w^{-2} . \hat{\eta}$, where

$$\hat{\eta} = dw - \tilde{\beta}_2 - w\,\tilde{\beta}_1 - w^2\,\tilde{\beta}_0\ .$$

Since $i_{\partial_w}(\tilde{\beta}_j) = 0$ and $L_{\partial_w}(\tilde{\beta}_j) = 0$, $0 \leq j \leq 2$, the integrability of $\hat{\eta}$ implies

$$\begin{cases} d\tilde{\beta}_0 = \tilde{\beta}_0 \wedge \tilde{\beta}_1 \\ d\tilde{\beta}_1 = 2\,\tilde{\beta}_0 \wedge \tilde{\beta}_2 \\ d\tilde{\beta}_2 = \tilde{\beta}_1 \wedge \tilde{\beta}_2\,. \end{cases} \tag{2.8}$$

From (2.7) we get $d\tilde{\beta}_0 = 0$, and so the first relation in (2.8) gives $\tilde{\beta}_0 \wedge \tilde{\beta}_1 = 0$.

Let us denote by $\mathcal{M}_k$ the set of meromorphic functions on $\mathbb{P}^k$. Since $\tilde{\beta}_0 \wedge \tilde{\beta}_1 = 0$ there exists $g \in \mathcal{M}_{n-1}$ such that $\tilde{\beta}_1 = g . \tilde{\beta}_0$. The second relation in (2.8) gives

$$d\tilde{\beta}_1 = dg \wedge \tilde{\beta}_0 = 2\,\tilde{\beta}_0 \wedge \tilde{\beta}_2 \implies (dg + 2\,\tilde{\beta}_2) \wedge \tilde{\beta}_0 = 0$$

$$\implies \exists\ h \in \mathcal{M}_{n-1} \text{ such that } \tilde{\beta}_2 = h . \tilde{\beta}_0 - \frac{1}{2}\,dg\ .$$

The third relation in (2.8) implies

$$\begin{aligned} dh \wedge \tilde{\beta}_0 = d\tilde{\beta}_2 = \tilde{\beta}_1 \wedge \tilde{\beta}_2 = -g\, \tilde{\beta}_0 \wedge \left(h\, \tilde{\beta}_0 - \frac{1}{2}\, dg \right) \\ = \frac{1}{2}\, g\, dg \wedge \tilde{\beta}_0 \implies d\left(h - \frac{1}{4}\, g^2 \right) \wedge \tilde{\beta}_0 = 0 \ . \end{aligned} \tag{2.9}$$

Let us denote by $\mathcal{G}$ the foliation defined by $\tilde{\beta}_0$ on $\mathbb{P}^{n-1}$. We have two possibilities:

3a. *$\mathcal{G}$ has no non-constant meromorphic first integral.* We assert that ω has an integrating factor.

In fact, relation (2.9) implies that

$$d\left(h - \frac{1}{4}\, g^2 \right) = 0 \implies h = \frac{1}{4}\, g^2 + c \ , \ c \in \mathbb{C} \ ,$$

for otherwise $h - \frac{1}{4}\, g^2$ would be a non-constant first integral of $\mathcal{G}$. From the above relations we get

$$\hat{\eta} = dw + \frac{1}{2}\, dg - (g^2/4 + c + g.\, w + w^2)\, \tilde{\beta}_0 = d(w + g/2) - ([w + g/2]^2 + c)\, \tilde{\beta}_0 \ .$$

In particular, if we set $\mu := ((w + g/2)^2 + c)^{-1}.\, \hat{\eta}$ then

$$\mu = \frac{d(w + g/2)}{(w + g/2)^2 + c} - \tilde{\beta}_0 \implies d\mu = 0 \ .$$

This implies that ω has an integrating factor, as asserted.

3b. *$\mathcal{G}$ has a non-constant meromorphic first integral.* We assert that $\mathcal{F}$ is the pull-back of a Riccati equation on $\mathbb{P}^1 \times \mathbb{P}^1$ by a birrational map.

In fact, by Stein's fatorization theorem $\mathcal{G}$ has a meromorphic first integral, say f, with connected fibers: if $\phi \in \mathcal{M}_{n-1}$ and $d\phi \wedge df = 0$ then there exists $\psi \in \mathcal{M}_1$ such that $\phi = \psi(f)$, where $\psi(f) := \psi \circ f$. Since f is a first integral of $\mathcal{G}$ we have $\tilde{\beta}_0 = \phi_1.\, df$, for some $\phi_1 \in \mathcal{M}_{n-1}$. This implies $d\phi_1 \wedge df = 0$, and so $\phi_1 = \psi_1(f)$, where $\psi_1 \in \mathcal{M}_1 \setminus \{0\}$. On the other hand, relation (2.9) implies that there exists $\psi_2 \in \mathcal{M}_1$ such that $h = \frac{1}{4}\, g^2 + \psi_2(f)$. In particular,

$$\begin{aligned} \hat{\eta} &= d(w + g/2) - (g^2/4 + g\, w + w^2 + \psi_2(f))\, \psi_1(f)\, df \\ &= d(w + g/2) - ([w + g/2]^2 + \psi_2(f))\, \psi_1(f)\, df \ . \end{aligned}$$

Consider the rational map $\Phi_1 \colon \mathbb{P}^{n-1} \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ given by $\Phi_1(\tau, w) = (f(\tau), w - g(\tau)) := (x, y)$. Then $\hat{\eta} = \Phi^*(\theta)$, where

$$\theta = dy - (y^2 + \psi_2(x))\, \psi_1(x)\, dx \ .$$

Since $\theta = 0$ defines a Riccati equation on $\mathbb{P}^1 \times \mathbb{P}^1$ this finishes the proof of 3.b and of Lemma 2.5. $\square$

2.2. End of the proof of Theorem 1.1

The proof is by induction on $n \geq 3$. If $n = 3$ then Theorem 1.1 follows from Lemma 2.5. Let us assume that Theorem 1.1 is true for $n-1 \geq 3$ and prove that it is true for n.

Let $\mathcal{F}$ be a codimension-one foliation of degree-three on $\mathbb{P}^n$, $n \geq 4$. It follows from Lemma 2.5 and Theorem 1.2 that, either $\mathcal{F}$ satisfies one of the conclusions of Theorem 1.1, or $\mathcal{F}$ is the pull-back by a linear map $\pi : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$ of a foliation of degree-three on $\mathbb{P}^{n-1}$. In this last case, since Theorem 1.1 is true for $n-1$, then one of the three possilities bellow is true:

(i) $\mathcal{F}_{n-1}$ has a rational first integral, say $F\colon \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^1$. In this case, $F \circ \pi$ is a rational first integral of $\mathcal{F}$ and we are done.
(ii) $\mathcal{F}_{n-1} = \Phi^*(\mathcal{G})$, where $\mathcal{G}$ is a foliation on $\mathbb{P}^2$ and $\Phi\colon \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^2$ a rational map. In this case, we get $\mathcal{F} = (\Phi \circ \pi)^*(\mathcal{G})$ and we are done.
(iii) $\mathcal{F}_{n-1}$ has an affine transverse structure. In this case, $\mathcal{F}_{n-1}$ admits a G-V-S of lenght one. Therefore, $\mathcal{F}$ also admits a G-V-S of lenght one by Remark 2.1.

This finishes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

In Section 3.1 we state some general facts about the Baum-Bott index that will be used in the proof. After that we give the proof of Theorem 1.2 in dimension three and at the end we will see the proof in dimension $n \geq 4$.

3.1. The Baum-Bott index

We begin by recalling briefly the proof of [16] that sing($\mathcal{F}$) has components of codimension-two. This proof is based in the Baum-Bott theorem for foliations on compact holomorphic surfaces.

Theorem 3.1. *Let $\mathcal{G}$ be a foliation on a compact surface with isolated singularites. Then*

$$\sum_{p \in \mathrm{sing}(\mathcal{G})} \mathrm{BB}(\mathcal{G}, p) = N_{\mathcal{G}}^2 \,, \tag{3.1}$$

where $N_{\mathcal{G}}$ is the normal bundle of the foliation $\mathcal{G}$ and $\mathrm{BB}(\mathcal{G}, p)$ the Baum-Bott index of $\mathcal{G}$ at the point p.

A proof of Theorem 3.1 and the definition of $N_{\mathcal{G}}$ can be found in [2]. The Baum-Bott index $\mathrm{BB}(\mathcal{G}, p)$ is defined as follows: let $(U, (x, y))$ be a holomorphic chart around p such that $x(p) = y(p) = 0$ and $\mathrm{sing}(\mathcal{G}) \cap U = \{0\}$ and $\omega = P(x, y)\, dy - Q(x, y)\, dx$ be a holomorphic 1-form representing $\mathcal{G}|_U$. Let η be a

C^∞ $(1,0)$-form on $U \setminus \{0\}$ such that $d\omega = \eta \wedge \omega$. For instance, one can take

$$\eta = \frac{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)}{|P|^2 + |Q|^2}\left(\overline{P}\,dx + \overline{Q}\,dy\right) \ .$$

Then the 3-form $\eta \wedge d\eta$ is closed and

$$\mathrm{BB}(\mathcal{G}, p) = \frac{1}{(2\pi i)^2}\int_{\partial B} \eta \wedge d\eta \tag{3.2}$$

where B is a closed ball containing $p = 0$ in its interior with $\mathrm{sing}(\mathcal{G}) \cap B = \{p\}$ (*cf.* [2]). In particular, the integral in (3.2) does not depend on the form η chosen. We will also use the notation $\mathrm{BB}(\omega, p) := \mathrm{BB}(\mathcal{G}, p)$.

Remark 3.2. We would like to point out some consequences of (3.2).

1. $\mathrm{BB}(\mathcal{G}, p)$ is invariant by local analytic equivalences.
2. If the foliation $\mathcal{G}$ has a holomorphic first integral in a neighborhood of the singular point p with an isolated singularity at p then $\mathrm{BB}(\mathcal{G}, p) = 0$.
3. If the foliation $\mathcal{G}$ is represented in a neighborhood of p by a vector field X such that $DX(p)$ is non-singular then

$$\mathrm{BB}(\mathcal{G}, p) = \frac{\mathrm{tr}(DX(p))^2}{\det(DX(p))} \ ,$$

   where tr denotes trace and det determinant.
4. If $(\mathcal{G}_t)_{t\in(\mathbb{C}^k,0)}$ is a germ of holomorphic deformation of $\mathcal{G}$ such that $\mathcal{G}_0 = \mathcal{G}$ and $\mathrm{sing}(\mathcal{G}_t) \cap \mathrm{int}(B) = \{p_1(t), ..., p_k(t)\}$ then

$$\lim_{t\to 0}\left(\sum_{j=1}^{k} \mathrm{BB}(\mathcal{G}_t, p_j(t))\right) = \mathrm{BB}(\mathcal{G}, p) \, .$$

Let us prove that the singular set of the codimension-one foliation $\mathcal{F}$ with $dg(\mathcal{F}) = d$ on $\mathbb{P}^n$ has at least one irreducible component of codimension-two. If not, then there exists a linear embedding $i : \mathbb{P}^2 \to \mathbb{P}^n$ such that:

(i) $E \cap \mathrm{sing}(\mathcal{F}) = \emptyset$.
(ii) The tangencies of $\mathcal{F}$ with $E := i(\mathbb{P}^2)$ are generic (of Morse type, see [LN]).

Let $\mathcal{G} = i^*(\mathcal{F})$. Note that (ii) implies that $dg(\mathcal{G}) = d$. Moreover, (i) and (ii) imply that if $p \in \mathrm{sing}(\mathcal{G})$ then $\mathcal{G}$ has a local holomorphic first integral of Morse type in a neighborhood of p. In particular, we get from (2) in Remark 3.2 that

$$\sum_{p\in\mathrm{sing}(\mathcal{G})} \mathrm{BB}(\mathcal{G}, p) = 0$$

On the other hand, $N_{\mathcal{G}} = (d+2)H$, where H is the class of a hyperplane, so that the Baum-Bott theorem gives (*cf.* [2]):

$$\sum_{p \in \mathrm{sing}(\mathcal{G})} \mathrm{BB}(\mathcal{G}, p) = (d+2)^2 > 0 \ ,$$

a contradiction.

We will denote $\mathrm{sing}_2(\mathcal{F})$ the union of the codimension-two irreducible components of $\mathrm{sing}(\mathcal{F})$. Let Γ be an irreducible component of $\mathrm{sing}_2(\mathcal{F})$. Given a smooth point $p \in \Gamma$ and a germ of embedding $i\colon (\mathbb{C}^2, 0) \to (\mathbb{P}^n, p)$, transverse to Γ, define $\mathrm{BB}(\mathcal{F}, \gamma, i, p) := \mathrm{BB}(i^*(\mathcal{F}), 0)$. The following result can be proved:

Theorem 3.3. *There exists a proper analytic subset $\Gamma_1 \subset \Gamma$ such that:*

(a) *If $p \in \Gamma \setminus \Gamma_1$ then $\mathrm{BB}(\mathcal{F}, \Gamma, i, p)$ does not depend on the embedding $i\colon (\mathbb{D}^2, 0) \to (\mathbb{P}^n, p)$, transverse to Γ. We then denote $\mathrm{BB}(\mathcal{F}, \Gamma, p) := \mathrm{BB}(\mathcal{F}, \Gamma, i, p)$.*
(b) *The map $p \in \Gamma \setminus \Gamma_1 \mapsto \mathrm{BB}(\mathcal{F}, \Gamma, p) \in \mathbb{C}$ is constant.*

We then denote $\mathrm{BB}(\mathcal{F}, \Gamma) := \mathrm{BB}(\mathcal{F}, \Gamma, p)$, where $p \in \Gamma \setminus \Gamma_1$.

The proof in the general case can be done by using the results of J. F. Mattei about the equiresolution of integrable families of foliations of $(\mathbb{C}^2, 0)$ (*cf.* [19]) and also the fact that the Baum-Bott indexes of two germs of foliations on $(\mathbb{C}^2, 0)$ are the same if their Seidenberg resolutions of singularities are C^∞ isomorphic with corresponding singularities with the same Baum-Bott index (*cf.* [2]). We give the proof of Theorem 3.3 in the case we are interested.

Lemma 3.4. *Let ω be a holomorphic integrable 1-form in a neighborhood of $0 \in U \subset \mathbb{C}^n$, with $n \geq 3$, such that $\omega(0) = 0$ and $j^1_p(\omega) \neq 0$ for all $p \in U$. Assume that $\mathrm{sing}(\omega)$ is connected and smooth of codimension-two. Then for any $p, q \in \mathrm{sing}(\omega)$ and any two transverse sections to $\mathrm{sing}(\omega)$ through p and q, say Σ_p and Σ_q, then $\mathrm{BB}(\omega, \mathrm{sing}(\omega), \Sigma_p, p) = \mathrm{BB}(\omega, \mathrm{sing}(\omega), \Sigma_q, q)$.*

Proof. Denote by $\mathcal{F}$ the foliation defined by ω on U. We will prove that for any $p \in \mathrm{sing}(\omega)$ there is a neighborhood V of p in $\mathrm{sing}(\omega)$ such that for any two transverse sections Σ_p and Σ_q through p and $q \in V$, respectively, then $\mathrm{BB}(\omega, \mathrm{sing}(\omega), \Sigma_p, p) = \mathrm{BB}(\omega, \mathrm{sing}(\omega), \Sigma_q, q)$.

Fix $p \in \mathrm{sing}(\omega)$. Assume first that p is a Kupka singularity, that is $d\omega(p) \neq 0$. In this case, the distribution defined by $E_q = \{v \mid i_v(d\omega(q)) = 0\}$ has codimension-two and is integrable in some neighborhood W of p, defining a codimension-two foliation $\mathcal{E}$ on W. Moreover, $\mathrm{sing}(\omega) \cap W$ is a leaf of $\mathcal{E}$. If Σ is a germ of embedded two plane transverse to $\mathrm{sing}(\omega)$ at p, we can define a germ of submersion $g\colon (\mathbb{C}^n, p) \to (\Sigma, p)$ by following the leaves of $\mathcal{E}$. It can be proved that $\omega = g^*(\omega|_\Sigma)$, so that $\mathcal{F}|_W$ is product of a singular foliation on Σ by the regular foliation of codimension-two $\mathcal{E}$ (*cf.* [15]). This implies that if Σ' is another transverse section through a point $p' \in \mathrm{sing}(\omega)$ near p then $\mathrm{BB}(\omega, \mathrm{sing}(\omega), \Sigma, p) = \mathrm{BB}(\omega, \mathrm{sing}(\omega), \Sigma', p')$.

When $d\omega(p) = 0$ the 1-jet $\omega_1 = j^1_p(\omega)$ is exact ($d\omega_1 = 0$). Since $\omega_1 \neq 0$ and $\mathrm{codim}(\mathrm{sing}(\omega)) = 2$ we must have $1 \leq \mathrm{codim}(\mathrm{sing}(\omega_1)) \leq 2$. Hence, after a change of variables we can suppose that $p = 0$ and, either $\omega_1 = x\,dy + y\,dx$, or $\omega_1 = x\,dx$. In the case $\omega_1 = x\,dy + y\,dx$ with $\mathrm{cod}(\mathrm{sing}(\omega)) = 2$, the situation is similar to the Kupka case. It is proved in [13] that $\mathcal{F}$ is equivalent in a neighborhood of p to the product of a dimension one foliation in a transversal section by regular foliation of codimension-two. Hence, if Σ and Σ' are transverse sections to $\mathrm{sing}(\omega)$ we have again $\mathrm{BB}(\mathcal{F}, \mathrm{sing}(\omega), \Sigma, p) = \mathrm{BB}(\mathcal{F}, \mathrm{sing}(\omega), \Sigma', p')$.

In the case $\omega_1 = x\,dx$ we use a result due to F. Loray. Since this case will appear before in the proofs, we give a formal definition.

Definition 3.5. We say that a singularity p of the foliation $\mathcal{F}$ is *nilpotent* if there exists an integrable holomorphic 1-form ω defining $\mathcal{F}$ in a neighborhood of p such that $j^1_p(\omega) = x\,dx$, in some coordinate chart $(x, y) \in \mathbb{C} \times \mathbb{C}^{n-1}$ around p such that $x(p) = 0$.

The next result is a consequence of corollary 3 in [17, page 710].

Theorem 3.6 ([17]). *Let θ be a germ at $(0,0) \in \mathbb{C}\times\mathbb{C}^m$ of holomorphic integrable 1-form, where*

$$\theta = g(w,z)\,dw + \sum_{j=1}^{m} f_j(w,z)\,dz_j \ , \ \ (w,z) = (w, z_1, ..., z_m) \in \mathbb{C} \times \mathbb{C}^m \ .$$

Denote by $\mathcal{F}$ the germ of foliation defined by θ. Assume that $j^1_0(\theta) = w\,dw$. Then there exist local analytic coordinates $(x,\zeta) \in \mathbb{C} \times \mathbb{C}^m$, a germ $f \in \mathcal{O}_m$, with $f(0) = 0$, and germs $g, h \in \mathcal{O}_1$ such that $\mathcal{F}$ is defined in the chart (x,ζ) by the 1-form

$$\omega = x\,dx + [g(f(\zeta)) + x\,h(f(\zeta))]\,df(\zeta) \ . \tag{3.3}$$

In particular, $\mathcal{F} = \varphi^*(\mathcal{G})$ where $\varphi\colon (\mathbb{C} \times \mathbb{C}^m, 0) \to (\mathbb{C}^2, 0)$ is given by $\varphi(x,\zeta) = (x, f(\zeta))$ and $\mathcal{G}$ is the germ at $(\mathbb{C}^2, 0)$ of foliation defined by

$$\eta := x\,dx + [g(t) + x\,h(t)]\,dt \ .$$

Let us finish the proof of Lemma 3.4. Note that if ω is like in (3.3) then $\mathrm{sing}(\omega) \subset (x = 0)$. Since we are assuming that $\mathrm{sing}(\omega)$ is smooth and has codimension-two, after a holomorphic change of variables involving only ζ, we can assume that $\mathrm{sing}(\omega)_p = (x = \zeta_1 = 0)$, where $\mathrm{sing}(\omega)_p$ is the germ of $\mathrm{sing}(\omega)$ at $p = 0$. Therefore,

$$\mathrm{sing}(\omega)_p = (x = \zeta_1 = 0) = (x = g(f(\zeta)) = 0) \cup (x = \partial f/\partial\zeta_1 = ... = \partial f/\partial\zeta_{n-1} = 0) \, .$$

Hence, either $g(0) = 0$ and $\zeta_1 \mid f$, or $g(0) \neq 0$ and $\zeta_1 \mid \partial f/\partial\zeta_j$ for all $j = 1, ..., n-1$. In any case, we get $\zeta_1 \mid f$ and so $f(\zeta) = \zeta_1^k.G(\zeta)$, where $G \in \mathcal{O}_n$, $k \in \mathbb{N}$ and $\zeta_1 \nmid G$. We have two possibilities:

1st. $G(0) \neq 0$. In this case, after the holomorphic change of variables

$$\Phi(x,\zeta) = (x, \zeta_1 . G^{1/k}(\zeta), \zeta_2, ..., \zeta_n) = (x, y, \zeta_2, ..., \zeta_n) \ ,$$

where $G^{1/k}$ is a branch of the k^{th} rooth of G, we get $f \circ \Phi^{-1} = y^k$ and

$$\Phi_*(\omega) = x\,dx + [g(y^k) + x\,h(y^k)]\,k\,y^{k-1}\,dy := vx\,dx + [g_1(y) + x\,h_1(y)]\,dy \,. \quad (3.4)$$

Hence, in this case $\mathcal{F}$ is locally the product of a singular foliation on $(\mathbb{C}^2, 0)$ by a regular foliation of codimension-two and the argument is similar to the preceding cases.

2nd. $G(0) = 0$. Since $\text{sing}(\omega)_p = (x = \zeta_1 = 0)$ and

$$\omega = x\,dx + (g(\zeta_1^k . G) + x\,h(\zeta_1^k . G))\,\zeta_1^{k-1}\,(\zeta_1 . dG + k\,G . d\zeta_1)$$

we get

2.1. $g(0) \neq 0$, for otherwise $\text{sing}(\omega)_p \supset (x = \zeta_1 . G(\zeta) = 0) \supsetneq (x = \zeta_1 = 0)$.
2.2. $k \geq 2$, for otherwise $\zeta_1 \mid G$.

Recall that $\omega = \varphi^*(\eta)$, where $\eta = x\,dx + (g(t) + x\,h(t))\,dt$ and $\varphi(x,\zeta) = (x, f(\zeta))$. Since $g(0) \neq 0$ we have $\eta(0,0) \neq 0$ and the foliation defined by η has a non-constant holomorphic first integral, say $H(x,t)$, in a neighborhood of $0 \in \mathbb{C}^2$, with $H(0,0) = 0$, $\frac{\partial H}{\partial t}(0,0) \neq 0$, $\frac{\partial H}{\partial x}(0,0) = 0$ and $\frac{\partial^2 H}{\partial x^2}(0,0) \neq 0$. This implies that $H_1(x,\zeta) := H(x, \zeta_1^k . G(\zeta))$ is a non-constant holomorphic first integral of ω in a neighborhood of $0 \in \mathbb{C}^n$. By using that $\frac{\partial H}{\partial t}(0,0) \neq 0$, $\frac{\partial H}{\partial x}(0,0) = 0$ and $\frac{\partial^2 H}{\partial x^2}(0,0) \neq 0$, it can be checked that for any $q \in \text{sing}(\omega)_p$ and any transverse section Σ_q through q then $H_1|_{\Sigma_q}$ has an isolated singularity at q. It follows from (2) in Remark 3.2 that $\text{BB}(\omega, \text{sing}(\omega), \Sigma_q, q) = 0$. This finishes the proof of Lemma 3.4. $\square$

Remark 3.7. Let $\mathcal{F}$ be a codimension-one foliation on $\mathbb{P}^n$, with $n \geq 3$. It follows from the argument of [16] that $\text{sing}_2(\mathcal{F})$ has at least one irreducible component of codimension-two, say Γ, such that $\text{BB}(\mathcal{F}, \Gamma) \neq 0$.

Assume that $\mathcal{J}(\mathcal{F}, p) = 1$ for all $p \in \Gamma$. Denote by $\tilde{\Gamma}$ the smooth part of $\text{sing}(\omega)$ contained in Γ. We would like to remark that for any $p \in \tilde{\Gamma}$ then the germ $\mathcal{F}_p$, of $\mathcal{F}$ at p, is equivalent to the product of a singular foliation on $(\mathbb{C}^2, 0)$ by a regular foliation of codimension-two. In fact, we have seen in the proof of Lemma 3.4 that the unique case in which perhaps this fact is not true is the 2^{nd}, where $\text{BB}(\mathcal{F}, \Gamma) = 0$.

Note that the irreducibility of Γ implies that $\tilde{\Gamma}$ is connected. In particular, there exists a germ of holomorphic 1-form η at $(\mathbb{C}^2, 0)$ such that for any $p \in \tilde{\Gamma}$ then there is a germ of submersion $\varphi\colon (\mathbb{P}^n, p) \to (\mathbb{C}^2, 0)$ such that $\mathcal{F}_p$ is defined by $\varphi^*(\eta)$.

Definition 3.8. The normal type of $\mathcal{F}$ along $\tilde{\Gamma}$ is, by definition, the equivalent class of the foliation defined by η on $(\mathbb{C}^2, 0)$.

Since in the proof of Theorem 1.2 we will deal with nilpotent 1-forms, before closing this section we would like to state a result in which we compute the Baum-Bott index for this type of form.

Lemma 3.9. *Let $U \subset \mathbb{C}$ be an open set and $g_1, h_1 \in \mathcal{O}(U)$, $g_1 \not\equiv 0$. Consider the foliation $\mathcal{G}$ of $\mathbb{C} \times U$ defined by $\eta = 0$, where*

$$\eta = x\,dx + (g_1(t) + x\,h_1(t))\,dt\ .$$

Then for any $t_o \in U$ such that $g_1(t_o) = 0$ we have

$$\mathrm{BB}(\mathcal{G}, (0, t_o)) = \mathrm{Res}\left(\frac{(h_1(t))^2}{g_1(t)}\,dt, t = t_o\right)\ . \tag{3.5}$$

Proof. The vector field $X = -(x\,h_1(t) + g_1(t))\,\partial_x + x\,\partial_t$ also defines $\mathcal{G}$. Let $\ell \geq 1$ be the multiplicity of g_1 at t_o, so that $g_1(t) = (t - t_o)^\ell.\phi(t)$ and $\phi(t_o) \neq 0$.

Assume first that $\ell = 1$. In this case $g_1'(t_o) = \phi(t_o) \neq 0$ and $(0, t_o)$ is a non-degenerate singularity of X. Therefore, by computing the jacobian matrix of $DX(0, t_o)$ we get from (3) in Remark 3.2 that

$$\mathrm{BB}(\mathcal{G}, (0, t_o)) = \frac{(\mathrm{tr}(DX(0, t_o))^2}{\det(DX(0, t_o)} = \frac{h_1(t_o)^2}{g_1'(t_o)} = \mathrm{Res}\left(\frac{(h_1(t))^2}{g_1(t)}\,dt, t_o\right)\ .$$

Suppose now that $\ell > 1$. Consider the family $(\mathcal{G}_s)_{s\in\mathbb{C}}$ of foliations defined by $\eta_s = x\,dx + (g_1(t) - s^\ell + x\,h_1(t))\,dt$ and set $\theta_s = \frac{(h_1(t))^2}{g_1(t) - s^\ell}\,dt$. For small $|s| \neq 0$, the equation $g_1(t) = s^\ell$ has exactly ℓ roots near t_o, say $t_1(s), ..., t_\ell(s)$, such that $\lim\limits_{t\to 0} t_j(s) = t_o$ and $g_1'(t_j(s)) \neq 0$ for $s \neq 0$. Therefore, the first case implies that $\mathrm{BB}(\mathcal{G}_s, (0, t_j(s))) = \mathrm{Res}(\theta_s, t_j(s))$, $1 \leq j \leq \ell$, $|s| \neq 0$ and small. On the other hand, by (4) in Remark 3.2 we have

$$\begin{aligned}\mathrm{BB}(\mathcal{G}, (0, t_o)) &= \lim_{s\to 0}\left(\sum_{j=1}^{\ell} \mathrm{BB}(\mathcal{G}_s, (0, t_j(s)))\right) \\ &= \lim_{s\to 0}\left(\sum_{j=1}^{\ell} \mathrm{Res}(\theta_s, t_j(s))\right) = \mathrm{Res}\left(\frac{(h_1(t))^2}{g_1(t)}\,dt, t = t_o\right)\ . \quad \square\end{aligned}$$

3.2. Proof of Theorem 1.2 in dimension three

Let $\mathcal{F}$ be a codimension-one holomorphic foliation on $\mathbb{P}^3$ and assume that $\mathrm{sing}_2(\mathcal{F})$ has an irreducible component Γ with $\mathrm{BB}(\mathcal{F}, \Gamma) \neq 0$ (see Remark 3.7) and $\mathcal{J}(\mathcal{F}, p) = 1$ for all $p \in \Gamma$ (which is equivalent to hypothesis (b) of Theorem 1.2 in the case $n = 3$). Since we are working in dimension three, the irreducible components of $\mathrm{sing}(\mathcal{F})$ are either curves, or points. As a consequence, the connected component Δ of $\mathrm{sing}(\mathcal{F})$ which contains Γ is of pure dimension one, and so is a finite union of irreducible algebraic curves. We denote $\mathrm{sing}(\Delta)$ the singular set of Δ and $\tilde{\Gamma} = \Gamma \setminus \mathrm{sing}(\Delta)$. Note that any point of $\tilde{\Gamma}$ is a smooth point of Γ. Let η a germ at $0 \in \mathbb{C}^2$ of 1-form representing the normal type of $\mathcal{F}$ along $\tilde{\Gamma}$.

Remark 3.10. Any point $p \in \Gamma \setminus \tilde{\Gamma}$ is a nilpotent singularity of $\mathcal{F}$. Moreover, the normal type η of $\mathcal{F}$ along $\tilde{\Gamma}$ is, either Kupka, or nilpotent. In other words, either $d\eta(0) \neq 0$, or η is nilpotent.

Proof of the remark. As we have seen in the proof of Lemma 3.4, for any $q \in \Gamma$ we have two possibilities:

(i) The germ of $\mathcal{F}$ at q is equivalent to a product of a singular foliation on $(\mathbb{C}^2, 0)$ by a regular foliation of dimension one.
(ii) q is a nilpotent singularity of $\mathcal{F}$.

In case (i) the germ of $\mathrm{sing}(\mathcal{F})$ at q is smooth of codimension-two and so $q \in \tilde{\Gamma}$. This proves the first assertion.

On the other hand, if the normal type is not Kupka then $d\eta(0) = 0$ and $\eta_1 = j_0^1(\eta) \neq 0$ is exact. If η_1 is not nilpotent, then $\eta_1 = x\,dy + y\,dx$ in some chart. But, this implies that $\mathrm{BB}(\mathcal{F}, \Gamma) = 0$, which contradicts $\mathrm{BB}(\mathcal{F}, \Gamma) \neq 0$. $\square$

Definition 3.11. A separatrix of $\mathcal{F}$ along Γ is a germ of hypersurface Σ along Γ which is $\mathcal{F}$-invariant. In other words, given $p \in \Gamma$ there exists a germ $u_p \in \mathfrak{m}_p \setminus \{0\}$ such that:

(a) The ideal of the germ Σ_p of Σ at p is generated by u_p.
(b) The germ Γ_p of Γ at p is contained in Σ_p.
(c) If $\mathcal{F}$ is represented by a holomorphic 1-form ω in a neighborhood of p then $du_p \wedge \omega = u_p . \Theta$, where Θ is a germ of holomorphic 2-form. This condition is equivalent to the $\mathcal{F}$-invariance of Σ.
(d) If u is a representative of u_p in a small neighborhood U of p then, for any $q \in \Gamma \cap U$ there exists $g \in \mathcal{O}_q^*$ such that $u_q = g.(u)_q$, where $(u)_q$ denotes the germ of u at q.

We say that Σ is smooth if $du_p(p) \neq 0$ for all $p \in \Gamma$.

Consider now the normal type $\eta = P(x, y)\,dy - Q(x, y)\,dx$ of $\mathcal{F}$ along $\tilde{\Gamma}$. Assume that η has a smooth separatrix $\sigma = (u(x, y) = 0)$, $u \in \mathfrak{m}_2 \setminus \{0\}$, $du(0) \neq 0$. Let $\Pi \colon (M, D) \to (\mathbb{C}^2, 0)$ be the minimal Seidenberg's resolution of singularities of η, in the sense of [7] or [2]. Denote by $\mathcal{G}$ be the foliation on (M, D) defined by the strict transform of $\Pi^*(\eta)$. We would like to recall that:

(A) $D = \bigcup_{j=1}^{k} D_j$, where each divisor D_j is biholomorphic to $\mathbb{P}^1$. Moreover, if $i \neq j$ and $D_i \cap D_j \neq \emptyset$ then $D_i \cap D_j = \{p\}$ and D_i cuts D_j tranversely at p. The divisor D_j is *dicritical* if it is not $\mathcal{G}$-invariant. Otherwise, it is *non-dicritical*.

(B) All singularities of $\mathcal{G}$ in D are simple, in the sense that if $p \in \operatorname{sing}(\mathcal{G}) \cap D$ and $\mathcal{G}$ is represented by a holomorphic vector field X in a neighborhood of p then the eigenvalues λ_1, λ_2 of $DX(p)$ satisfy one the conditions bellow:

(B.1) If one of the eigenvalues is zero then the other is non-zero. In this case, p is a saddle-node of $\mathcal{G}$.

(B.2) $\lambda_1, \lambda_2 \neq 0$ and $\lambda_2/\lambda_1 \notin \mathbb{Q}_+$.

Definition 3.12. Let σ be a smooth separatrix of η and $\Pi\colon (M, D) \to (\mathbb{C}^2, 0)$, $D = \bigcup_i D_i$, and $\mathcal{G}$ be as above. Let $\hat{\sigma}$ be the strict transform of σ by Π, where $\hat{\sigma} \cap D = \{p\}$. We say that σ is a *distinguished* separatrix of η if for any other smooth separatrix, say σ_1, of η, with strict transform $\hat{\sigma}_1$ and $\hat{\sigma}_1 \cap D = \{q\}$ $(p \neq q)$ then there is no local biholomorphism $\Phi\colon (M, p) \to (M, q)$ such that $\Phi^*(\mathcal{G}_q) = \mathcal{G}_p$, where $\mathcal{G}_x$ denotes the germ of $\mathcal{G}$ at $x \in D$.

Remark 3.13. We would like to remark the following facts:

(I) When 0 is already a simple singularity of η then η has at least one and at most two analytic separatrices through 0, all smooth (*cf.* [9]). In the case (B.2) it has exactly two, each one tangent to an eigendirection of $DX(0)$. In the case (B.1) it has always one, which is tangent to the non-zero eigenvalue of $DX(0)$. Sometimes it has also another tangent to the eigendirection of the eigenvalue 0. We would like to observe that all separatrices of η are distingueshed, except when $\lambda_1 = -\lambda_2 \neq 0$. However, in this last case we have $\mathrm{BB}(\eta, 0) = 0$.

(II) If $\Phi \in \operatorname{Diff}(\mathbb{C}^2, 0)$ preserves the foliation defined by η and σ is a distinguished separatrix of η then $\Phi(\sigma) = \sigma$. In other words, if $\Phi^*(\eta) = h.\eta$, where $h \in \mathcal{O}_2^*$ then $\Phi(\sigma) = \sigma$. This follows from the fact that there is a germ of biholomorphism $\hat{\Phi}\colon (M, D) \to (M, D)$ such that $\Pi \circ \hat{\Phi} = \Phi \circ \Pi$.

(III) When the strict transform $\hat{\sigma}$ of σ cuts transversely some dicritical divisor in a regular point q of $\mathcal{G}$ then it is not distinguished. This follows from the fact that there exists a chart $(W, (u, v))$ around q such that $W \cap D = (v = 0)$, $W \cap \hat{\sigma} = (u = 0)$ and $\mathcal{G}|_W$ is defined by $du = 0$.

We will say that a separatrix Σ of $\mathcal{F}$ along Γ *extends* a separatrix σ of η, if for some transverse section Λ through a point $p \in \tilde{\Gamma}$, where $\mathcal{F}|_\Lambda$ is defined by η, then σ coincides with $\Sigma \cap \Lambda$. We will say also that σ can be extended to Σ.

Lemma 3.14. *If the normal type $\eta = P(x, y)\,dy - Q(x, y)\,dx$ has a dintinguished smooth separatrix σ then it can be extended to a smooth separatrix Σ of $\mathcal{F}$ along Γ.*

Proof. Let us prove first that σ extends to a germ of separatrix $\tilde{\Sigma}$ of $\mathcal{F}$ along $\tilde{\Gamma}$. It follows from the definition of the normal type that there exists a covering $(W_\alpha)_{\alpha \in A}$ of $\tilde{\Gamma}$ by polydiscs with the following properties:

(i) $W_\alpha \cap \tilde{\Gamma}$ is connected and non-empty for all $\alpha \in A$. If $W_{\alpha\beta} := W_\alpha \cap W_\beta \neq \emptyset$ then $W_{\alpha\beta} \cap \tilde{\Gamma}$ is connected and non-empty.

(ii) For all $\alpha \in A$ there is a chart $(x_\alpha, y_\alpha, z_\alpha)\colon W_\alpha \to \mathbb{C}^3$ such that $\mathcal{F}|_W$ is represented by $\eta_\alpha = P(x_\alpha, y_\alpha)\, dy_\alpha - Q(x_\alpha, y_\alpha)\, dx_\alpha$ and $\tilde{\Gamma} \cap W_\alpha = (x_\alpha = y_\alpha = 0)$.

Let $u(x, y) = 0$ be an equation of σ and define $u_\alpha(x_\alpha, y_\alpha, z_\alpha) := u(x_\alpha, y_\alpha) \in \mathcal{O}(W_\alpha)$. Set $\Sigma_\alpha = (u_\alpha = 0)$. Since σ is smooth we have $du(0) \neq 0$, which implies $du_\alpha(0, 0, z_\alpha) \neq 0$, so that Σ_α is smooth along $\tilde{\Gamma} \cap W_\alpha$.

Fix $W_{\alpha\beta} \neq \emptyset$. Since $\mathcal{F}|_{W_{\alpha\beta}}$ is represented by $\eta_\alpha|_{W_{\alpha\beta}}$ and by $\eta_\beta|_{W_{\alpha\beta}}$ there exists $\varphi \in \mathcal{O}^*(W_{\alpha\beta})$ such that $\eta_\alpha = \varphi.\, \eta_\beta$. Let Λ be a transverse section through a point $q \in \tilde{\Gamma} \cap W_{\alpha\beta}$. Then $\Sigma_\alpha \cap \Lambda$ and $\Sigma_\beta \cap \Lambda$ are separatrices of $\eta_\alpha|_\Lambda$ and $\eta_\beta|_\Lambda$, respectively. Since they correspond to σ, which is distinguished, they must coincide, by (II) in Remark 3.13. This implies $\Sigma_\alpha \cap W_{\alpha\beta} = \Sigma_\beta \cap W_{\alpha\beta}$. In particular, there exists a germ of hypersurface $\tilde{\Gamma}$, which extends σ, and such that $\tilde{\Gamma} \cap W_\alpha = \Gamma_\alpha$ for all $\alpha \in A$.

This finishes the proof when $\tilde{\Gamma} = \Gamma$. Assume that $\Gamma \setminus \tilde{\Gamma} \neq \emptyset$ and let us prove that $\tilde{\Sigma}$ extends to a smooth separatrix Σ of $\mathcal{F}$ along Γ.

Fix a point $p \in \Gamma \cap \operatorname{sing}(\Delta) = \Gamma \setminus \tilde{\Gamma}$. Since p is a nilpotent singularity of $\mathcal{F}$, by Loray's normal form, we can find a chart $(x, s, t)\colon U \to \mathbb{C}^3$ such that $\mathcal{F}|_U$ is represented by

$$\omega = x\, dx + (g(f(s, t)) + x\, h(f(s, t)))\, df(s, t)$$

and $\Gamma \cap U = (x = f(s, t) = 0)$. As we have seen before, given $q \in \tilde{\Gamma} \cap U$ there is a local chart $(W, (x, y, z))$ with $W \subset U$, $x(q) = y(q) = z(q) = 0$, $f|_W = y^k$, $k \geq 1$, and

$$\omega|_W = x\, dx + (g(y^k) + x\, h(y^k))\, k\, y^{k-1}\, dy := x\, dx + (\tilde{g}(y) + x\, \tilde{h}(y))\, dy\ .$$

Let Λ be the transverse section $(z = 0)$ and set

$$\theta = \omega|_\Lambda = x\, dx + (\tilde{g}(y) + x\, \tilde{h}(y))\, dy\ .$$

Note that $\tilde{g}(0) = 0$, because $(0, 0)$ is a singularity of θ.

Let $\alpha \in A$ be such that $q \in W_\alpha$ and $x(q) = y(q) = z(q) = 0$. If we cut $\Sigma_\alpha = \tilde{\Sigma} \cap W_\alpha$ by the transverse section $\Lambda = (z = 0)$, then we find a smooth separatrix $\tilde{\sigma} := \Sigma_\alpha \cap \Lambda$ of the differential equation $\eta_\alpha|_\Lambda = 0$, which is also a separatrix of $\theta = 0$ and corresponds to the separatrix σ of η.

The idea is to prove that $\tilde{\sigma}$ admits an equation in the chart (x, y) of the form $x = \psi(y^k)$, $\psi \in \mathfrak{m}_1$. Since $f|_W = y^k$, this will imply that the form ω has a smooth separatrix with equation $x = \psi(f(s, t))$ which extends $\tilde{\Sigma}$ to a neighborhood of p. This will finish the proof of Lemma 3.14.

We assert that $\tilde{\sigma}$ is not tangent to the x-axis. This will imply that $\tilde{\sigma}$ admits an equation of the form $x = \phi(y)$, $\phi \in \mathfrak{m}_1$, because it is smooth.

In fact, assume by contradiction that $\tilde{\sigma}$ is tangent to the x-axis. In this case, it admits an equation of the form $y = \phi(x)$, where $\phi(0) = \phi'(0) = 0$. Since $\tilde{\sigma}$ is a solution of $\theta = 0$, we get

$$x + (\tilde{g}(\phi(x)) + x\,\tilde{h}(\phi(x)))\,\phi'(x) \equiv 0\,.$$

Since $\tilde{g}(0) = \phi(0) = \phi'(0) = 0$, the above relation implies

$$x = j_0^1\,[x + (\tilde{g}(\phi(x)) + x\,\tilde{h}(\phi(x)))\,\phi'(x)] = 0\ ,$$

a contradiction. Therefore, $\tilde{\sigma}$ admits an equation of the form $x = \phi(y)$ with $\phi \in \mathfrak{m}_1$. When $k = 1$ this already proves that $\tilde{\Sigma}$ can be extended to a smooth surface in a neighborhood of p. When $k > 1$ we consider the automorphism $\Phi\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ given by $\Phi(x, y) = (x, \zeta.\, y)$, where ζ is a primitive k^{th}-root of unity. Since $\theta = x\,dx + (g(y^k) + x\,h(y^k))\,d(y^k)$ we get $\Phi^*(\theta) = \theta$. This implies $\Phi(\tilde{\sigma}) = \tilde{\sigma}$, because $\Phi(\tilde{\sigma})$ is a separatrix of θ and $\tilde{\sigma}$ is distinguished. On the other hand,

$$\begin{aligned} \Phi(\tilde{\sigma}) = \Phi(x - \phi(y) = 0) &= (x - \phi(\zeta.\, y) = 0) \\ \Longrightarrow\quad & (x - \phi(y) = 0) = (x - \phi(\zeta.\, y) = 0) \\ \Longrightarrow\quad & \phi(\zeta.\, y) = \phi(y)\ ,\ \ \forall\, y \in (\mathbb{C}, 0) \\ \Longrightarrow\quad & \phi(y) = \psi(y^k)\ ,\ \ \psi \in \mathfrak{m}_1\ . \end{aligned}$$

This finishes the proof of Lemma 3.14. □

The next result will be used several times in the rest of the proof.

Proposition 3.15. *Let γ be an irreducible curve of $\mathbb{P}^3$. Assume that there exists a germ Σ of smooth surface along γ. If N_Σ denotes the normal bundle of Σ and $c_1(N_\Sigma)$ its first Chern class then*

$$\int_\gamma c_1(N_\Sigma) > 0\ .$$

In particular, the above integral is a positive integer.

Proof of Proposition 3.15. According to the definition, by taking a representative of Σ in a sufficiently small neighborhood W of γ and a covering $\mathcal{U} = (U_\alpha)_{\alpha\in A}$ of W by polydiscs, we can say that there exist

I. A collection $(u_\alpha)_{\alpha\in A}$, where $u_\alpha \in \mathcal{O}(U_\alpha)$, $\Sigma_\alpha := \Sigma \cap U_\alpha = (u_\alpha = 0)$ and $du_\alpha(q) \neq 0$ for all $q \in \Sigma_\alpha$,
II. A multiplicative cocycle $(A_{\alpha\beta})_{U_{\alpha\beta}\neq\emptyset}$, $U_{\alpha\beta} := U_\alpha \cap U_\beta$, such that $u_\alpha = A_{\alpha\beta}.u_\beta$ for any $U_{\alpha\beta} \neq \emptyset$.

The cocycle $(A_{\alpha\beta})_{U_{\alpha\beta}\neq\emptyset}$ represents the normal bundle N_Σ of Σ in $H^1(\mathcal{U}, \mathcal{O}^*)$. Let $c_1(N_\Sigma)$ be the first Chern class of N_Σ, considered as an element of $H^2_{DR}(W)$.

Denote by $\mathcal{X}_3$ the set of holomorphic vector fields on $\mathbb{P}^3$. It is well known that $\dim(\mathcal{X}_3) = 15$. Given $X \in \mathcal{X}_3$ let $\mathrm{Tang}(X, \Sigma) \subset \Sigma$ be the divisor of tangencies of X with Σ. This divisor can be expressed as follows in the covering $\mathcal{U}$: if $q \in U_\alpha \cap \Sigma$ then $q \in |\mathrm{Tang}(X, \Sigma)| \cap U_\alpha$ if, and only if, $X(u_\alpha)(q) = u_\alpha(q) = 0$. Set $\Sigma_\alpha := U_\alpha \cap \Sigma$ and $g_\alpha := X(u_\alpha)|_{\Sigma_\alpha}$.

Let $B = \{\alpha \in A \mid \Sigma_\alpha \neq \emptyset\}$. If $\alpha, \beta \in B$ and $\Sigma_{\alpha\beta} \neq \emptyset$ then

$$X(u_\alpha) = X(A_{\alpha\beta}.u_\beta) = A_{\alpha\beta}.X(u_\beta) + u_\beta.X(A_{\alpha\beta}) \implies g_\alpha = a_{\alpha\beta}.g_\beta \ ,$$

where $a_{\alpha\beta} = A_{\alpha\beta}|_\Sigma$. Hence, $(a_{\alpha\beta})_{\Sigma_{\alpha\beta}\neq\emptyset}$ is a multiplicative cocycle and $(g_\alpha)_{\alpha\in B}$ defines the divisor $\mathrm{Tang}(X, \Sigma)$ of Σ.

If X is not completely tangent to Σ then $(g_\alpha)_{\alpha\in B}$ is effective ($g_\alpha \not\equiv 0$ for all α), which implies that $\mathrm{Tang}(X, \Sigma).[\gamma] \geq 0$.

A straightforward computation in affine coordinates shows that, given $p \neq q \in \gamma$ there exists $X \in \mathcal{X}_3$ such that $X(p) \in T_p\Sigma$ and $X(q) \notin T_q\Sigma$, where $T_x\Sigma$ denotes the tangent space of Σ at $x \in \Sigma$. Let us fix such vector field. Since $X(q) \notin T_q\Sigma$, $\mathrm{Tang}(X, \Sigma)$ is effective. Since $X(p) \in T_p\Sigma$ we have $p \in |\mathrm{Tang}(X, \Sigma)| \cap \gamma$, which implies $\mathrm{Tang}(X, \Sigma).[\gamma] > 0$. On the other hand, it is known that

$$\mathrm{Tang}(X, \Sigma).[\gamma] = \int_\gamma c_1(\mathrm{Tang}(X, \Sigma)) \implies \int_\gamma c_1(\mathrm{Tang}(X, \Sigma)) > 0 \ .$$

Since the cocycle associated to $\mathrm{Tang}(X, \Sigma)$ in the covering $(\Sigma_\alpha)_{\alpha\in B}$ is $(a_{\alpha\beta} = A_{\alpha\beta}|_\Sigma)_{\alpha\beta}$, we get

$$c_1(\mathrm{Tang}(X, \Sigma)) = c_1(N_\Sigma)|_\Sigma \implies \int_\gamma c_1(N_\Sigma) > 0 \ . \qquad \square$$

Lemma 3.16. *The normal type η is not nilpotent.*

Proof. The proof will be by contradicition. Assuming that all points of Γ are nilpotent, we will prove that $\mathcal{F}$ has a smooth separatrix Σ along Γ and that

$$\int_\Gamma c_1(N_\Sigma) = 0 \ ,$$

which contradicts Proposition 3.15. $\square$

In the proof of the existence of the smooth separatrix we will need the resolution of singularities of a nilpotent 1-form η on $(\mathbb{C}^2, 0)$ with $\mathrm{BB}(\eta, 0) \neq 0$. The following consequence of Lemma 3.9 will be usefull.

Corollary 3.17. *Let*

$$\eta = x\,dx + (g(y) + x\,h(y))\,dy \ , \tag{3.6}$$

where $\mathrm{BB}(\eta, 0) \neq 0$. *Then* $h \not\equiv 0$ *and*

$$\nu(g, 0) \geq 2\,\nu(h, 0) + 1 \ , \tag{3.7}$$

where $\nu(.\,, 0)$ *denotes the multiplicity at* $0 \in \mathbb{C}$.

Proof. It follows from Lemma 3.9 that

$$0 \neq \mathrm{BB}(\eta, 0) = \mathrm{Res}\left(\frac{h(y)^2}{g(y)}\,dy,\, y = 0\right) \ .$$

This implies $h \not\equiv 0$ and $\nu(g, 0) > 2\,\nu(h, 0)$, because otherwise $\frac{h(y)^2}{g(y)}\,dy$ would be holomorphic at $0 \in \mathbb{C}$ and the residue would vanish. $\square$

Existence of the smooth separatrix along Γ. We can assume that the normal type is given by η as in (3.6). Since $\mathrm{BB}(\eta, 0) = \mathrm{BB}(\mathcal{F}, \Gamma) \neq 0$ we get from Corollary 3.17 that $h \not\equiv 0$ and $m \geq 2n + 1 \geq 3$, where $\nu(g, 0) := m$ and $\nu(h, 0) := n$. Note that $n \geq 1$, because otherwise η would not be nilpotent. According to Lemma 3.14 it is sufficient to prove that η has a dintinguished smooth separatrix.

Let us give a brief description of the Seidenberg resolution of η (*cf.* [21]). Write $g(y) = y^m . \zeta_1(y)$ and $h(y) = y^n . \zeta_2(y)$, where $\zeta_j(0) \neq 0$, $j = 1, 2$, so that

$$\eta = x\,dx + (y^m . \zeta_1(y) + x\,y^n . \zeta_2(y))\,dy \ .$$

After the $(n + 1)^{\text{th}}$ step of this resolution we find a chain of divisors $D_1, \ldots, D_{n+1}$ and a blowing-up map $\Pi\colon (M, D) \to (\mathbb{C}^2, 0)$, $D = \bigcup_j D_j = \Pi^{-1}(0)$, where:

(I) $D_j.D_i = 0$ if $j < i - 1$ and $D_j.D_{j+1} = 1$, $1 \leq j < i \leq n + 1$.
(II) $D_j^2 = -2$ if $1 \leq j \leq n$ and $D_{n+1}^2 = -1$.

Let us denote by $\mathcal{G}$ the strict transform of the foliation defined by $\Pi^*(\eta)$. It can be proved that (*cf.* [21]):

(III) All the divisors $D_1, \ldots, D_{n+1}$ are $\mathcal{G}$-invariant.
(IV) If $j < n + 1$ then $\mathrm{sing}(\mathcal{G}) \cap D_j = D_j \cap D_{j+1} := \{p_j\}$. Moreover, if X_j is a vector field representing $\mathcal{G}$ around p_j then $DX_j(p_j)$ has eigenvalues $\lambda_1^j, \lambda_2^j \neq 0$ with $\lambda_1^j/\lambda_2^j \in \mathbb{Q}_-$. In particular, p_j is a simple singularity of X_j and $\mathcal{G}$ has only two separatrices through p_j, which are contained in the divisors D_j and D_{j+1}.

(V) The divisor D_{n+1} appears after the $(n+1)^{\text{th}}$ blowing-up. Moreover, there is a chart $(u, y) \in \mathbb{C}^2$ around $D_{n+1} \setminus \{p_n\}$, where $\Pi(u, y) = (y^{n+1}.u, y) = (x, y)$. In this chart, we get $D_{n+1} \setminus \{p_n\} = (y = 0)$ and $\Pi^*(\eta) = y^{2n+1}.\alpha$, with

$$\alpha = u\, y\, du + \Big((n+1)\, u^2 + \zeta_2(y)\, u + y^{m-(2n+1)}\, \zeta_1(y)\Big)\, dy \ .$$

The idea is to prove that $\mathcal{G}$ has a distinguished smooth separatrix $\hat{\sigma}$ transverse to D_{n+1} with an equation of the form $u = \zeta(y)$, where $\zeta \in \mathcal{O}_1^*$ and $(\zeta(0), 0)$ is a singularity of $\mathcal{G}$. In this case, if $\sigma = \Pi(\hat{\sigma})$ then σ admits an equation of the form $x = y^{n+1}.u = y^{n+1}.\zeta(y)$. In particular, σ will be a smooth distinguished separatrix of η.

The foliation $\mathcal{G}$ is defined around $D_{n+1} \setminus \{p_n\}$, in the chart (u, y), by the vector field

$$Z = \Big((n+1)\, u^2 + \zeta_2(y)\, u + y^{m-(2n+1)}\, \zeta_1(y)\Big)\, \partial_u - u\, y\, \partial_y \ .$$

If we set $a = \zeta_2(0) \neq 0$, $b = 0$ if $m > 2n+1$ and $b = \zeta_1(0)$ if $m = 2n+1$, then, in this chart, the singularities of Z along $(y = 0) \subset D_{n+1}$ are $q_1 = (u_1, 0)$ and $q_2 = (u_2, 0)$, where u_1, u_2 are the roots of $(n+1)\, u^2 + a\, u + b = 0$. The eigenvalues of $DZ(q_i)$ are $\lambda_t^i = 2(n+1)\, u_i + a$ and $\lambda_n^i = -u_i$, where λ_t^i corresponds to the eigendirection of the separatrix $(y = 0)$, $i = 1, 2$. Since Z is not nilpotent at q_i, $i = 1, 2$, we can apply the classification of non-nilpotent singularities. According to the values of a and b, we have three possibilities:

1st. $b \neq 0$ and $a^2/b = 4(n+1)$. In this case, $q_1 = q_2 = (-a/2(n+1), 0)$. The singularity is a saddle-node, $\lambda_t^1 = 0$ and $\lambda_n^1 = a/2(n+1) \neq 0$. It follows that $\mathcal{G}$ has an unique separatrix $\hat{\sigma}$ through q_1, which is smooth and transverse to the divisor. The separatrix $\sigma = \Pi(\hat{\sigma})$ is the unique one of η and so it is distinguished.

2nd. $b \neq 0$ and $a^2/b \neq 4(n+1)$. In this case, $q_1 \neq q_2$ and $\lambda_t^i, \lambda_n^i \neq 0$, $i = 1, 2$. Since $\lambda_t^i = 2(n+1)u_i + a$ and $\lambda_n^i = -u_i$, it follows that $\lambda_n^1/\lambda_t^1 \neq \lambda_n^2/\lambda_t^2$. On the other hand, a straightforward computation, using the values of λ_t^i and λ_n^i, $i = 1, 2$ (or the Camacho-Sad theorem [9]), shows that:

$$\frac{\lambda_n^1}{\lambda_t^1} + \frac{\lambda_n^2}{\lambda_t^2} = -\frac{1}{n+1} \ .$$

Since $\lambda_n^1/\lambda_t^1 \neq \lambda_n^2/\lambda_t^2$, either $\lambda_n^1/\lambda_t^1 \notin \mathbb{Q}_+$, or $\lambda_n^2/\lambda_t^2 \notin \mathbb{Q}_+$. If, for instance, $\lambda_n^1/\lambda_t^1 \notin \mathbb{Q}_+$ then $\mathcal{G}$ has an unique smooth separatrix $\hat{\sigma}$ through q_1, transverse to the divisor $(y = 0)$, with

$$CS(\mathcal{G}, \hat{\sigma}) = \lambda_t^1/\lambda_n^1 \notin \mathbb{Q}_+ \ ,$$

where $CS(\mathcal{G}, \hat{\sigma})$ denotes the Camacho-Sad index of the separatrix $\hat{\sigma}$ with respect to $\mathcal{G}$ (*cf.* [9]). If η has another smooth separatrix, say σ', then its strict transform $\hat{\sigma}'$ must satisfy $\hat{\sigma}' \cap (y = 0) = \{q_2\}$ and

$$CS(\mathcal{G}, \sigma') = \lambda_t^2/\lambda_n^2 \neq CS(\mathcal{G}, \hat{\sigma}) \ .$$

Since the Camacho-Sad index is an analytic invariant of the pair ($\mathcal{G}$, separatrix), it follows that $\sigma = \Pi(\hat{\sigma})$ is a smooth dinguished separatrix of η.

3rd. $b = 0$. In this case, we can take $u_2 = 0$ and $u_1 = -a/(n+1) \neq 0$, which give $\lambda_n^2 = 0$, $\lambda_t^2 = a \neq 0$ and $\lambda_t^1/\lambda_n^1 = -(n+1) \notin \mathbb{Q}_+$. In particular, q_2 is a saddle-node and $\mathcal{G}$ has an unique separatrix $\hat{\sigma}$ through q_1. In this case, $\sigma = \Pi(\hat{\sigma})$ is a distinguished separatrix of η because q_2 is saddle-node and q_1 is not.

This proves the existence of the smooth separatrix Σ of $\mathcal{F}$ along Γ.

Proof of $\int_{\gamma_j} c_1(N_\Sigma) = 0$. We have seen that given $p \in \Gamma$ then:

(i) There exists a germ of local chart $\psi = (x, y, z) \colon (\mathbb{P}^3, p) \to (\mathbb{C}^3, 0)$ such that the germ Γ_p, of Γ at p, satisfies $\Gamma_p \subset (x = 0)$.
(ii) There exist $\zeta_1, \zeta_2 \in \mathcal{O}_1^*$ and $f \in \mathfrak{m}_2$, depending only on (y, z), such that $\mathcal{F}_p$ is represented by

$$\omega = x\,dx + (f^m.\,\zeta_1(f) + x\,f^n.\,\zeta_2(f))\,df\ .$$

In particular Γ_p is defined by $(x = f = 0)$.
(iii) The germ Σ_p, of Σ at p, is defined by $x - f^{n+1}(y, z)\,\zeta(f(y, z)) = 0$, where $\zeta \in \mathcal{O}_1^*$. Set $\phi(t) = t^{n+1}\,\zeta(t)$.
(iv) When $p \in \tilde{\Gamma}$ then we can choose the chart in such a way that $\Gamma_p = (x = y = 0)$ and $f(y, z) = y^k$, $k \geq 1$.

When we consider the change of variables $\Psi(u, y, z) = (u + \phi(f(y, z)), y, z)$ then a straightforward computation shows that:

$$\Psi^*(\omega) = (u + \phi(f(y, z)))\,du + u\,\left[f^n(y, z).\,\zeta_2(f(y, z)) + \phi'(f(y, z))\right]\,df(y, z)\ .$$

In particular, in the new chart we have $\Sigma_p = (u = 0)$. This implies that, if we choose a small neighborhood U of Γ, where the germ Σ has a representative, then we can find a finite covering $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ of U by polydiscs with the following properties

(v) $U_\alpha \cap \Gamma \neq \emptyset$ for all α, and $U_{\alpha\beta} \cap \Gamma \neq \emptyset$ for all $U_{\alpha\beta} \neq \emptyset$.
(vi) If $U_\alpha \cap (\Gamma \setminus \tilde{\Gamma}) \neq \emptyset$ then $U_\alpha \cap (\Gamma \setminus \tilde{\Gamma})$ contains just one point. Moreover, if $U_\alpha \cap (\Gamma \setminus \tilde{\Gamma}) = \{p\}$ then $p \notin U_\beta$ for all $\beta \in A$ with $\beta \neq \alpha$.
(vii) For all $\alpha \in A$ there exists a holomorphic chart $\Psi_\alpha = (u_\alpha, y_\alpha, z_\alpha) \colon U_\alpha \to \mathbb{C}^3$, such that $\Psi_\alpha(U_\alpha) = \mathbb{D}^3$ and $\Sigma \cap U_\alpha = (u_\alpha = 0)$, so that $\Psi_a(U_\alpha \cap \Sigma) = \{0\} \times \mathbb{D}^2$.
(viii) For all $\alpha \in A$ there exist $f_\alpha \in \mathcal{O}(\mathbb{D}^2)$ and $\phi_\alpha \in \mathcal{O}(f_\alpha(\mathbb{D}^2))$, with $\phi_\alpha(t) = t^{n+1}\,\zeta_\alpha(t)$, $\zeta_\alpha \in \mathcal{O}^*(f_\alpha(\mathbb{D}^2))$, such that $\mathcal{F}|_{U_\alpha}$ is represented in the chart (U_α, Ψ_α) by

$$\omega_\alpha = (u_\alpha + \phi_\alpha(f_\alpha))\,du_\alpha + u_\alpha\,\left(f_\alpha^n.\,\zeta_2(f_\alpha) + \phi_\alpha'(f_\alpha)\right)\,df_\alpha\ .$$

(ix) If $U_\alpha \cap (\Gamma \setminus \tilde{\Gamma}) = \emptyset$ then $f_\alpha(y_\alpha, z_\alpha) = y_\alpha^k$.
(x) If $U_\alpha \cap (\Gamma \setminus \tilde{\Gamma}) = \{p\}$ then $\Psi_a(p) = 0$. Moreover, if $q \in \Gamma \setminus \{p\}$ then there exists a chart $(W, (u_\alpha, v_\alpha, w_\alpha))$ around q such that $f_\alpha|_W = v_\alpha^k$.

It follows from (vii) that there exists a multiplicative cocycle $G = (g_{\alpha\beta})_{U_{\alpha\beta} \neq \emptyset}$ such that $u_\alpha = g_{\alpha\beta}.u_\beta$ on $U_{\alpha\beta} \neq \emptyset$. The cocycle G represents N_Σ in $H^1(\mathcal{U}, \mathcal{O}^*)$. The idea is to prove that $g_{\alpha\beta}|_{\Gamma \cap U_{\alpha\beta}}$ is locally constant for all $U_{\alpha\beta} \neq \emptyset$. This will imply that $\int_\Gamma c_1(N_\Sigma) = 0$.

Since ω_α represents $\mathcal{F}|_{U_\alpha}$, there exists a multiplicative cocycle $(\varphi_{\alpha\beta})_{U_{\alpha\beta} \neq \emptyset}$ such that $\omega_\alpha = \varphi_{\alpha\beta} . \omega_\beta$ on $U_{\alpha\beta} \neq \emptyset$. Fix $U_{\alpha\beta} \neq \emptyset$ and $q \in \Gamma \cap U_{\alpha\beta}$. Let us prove that $g_{\alpha\beta}|_{\Gamma \cap U_{\alpha\beta}}$ is constant in a neighborhood of q.

Note that $q \in \tilde{\Gamma}$, because $(\Gamma \setminus \tilde{\Gamma}) \cap U_{\alpha\beta} = \emptyset$ by (vi). On the other hand, (ix) and (x) imply that we can find germs of charts $(u_\alpha, v_\alpha, w_\alpha) \colon (\mathbb{P}^3, q) \to (\mathbb{C}^3, 0)$ and $(u_\beta, v_\beta, w_\beta) \colon (\mathbb{P}^3, q) \to (\mathbb{C}^3, 0)$ such that

(xi) $\Gamma_q = (u_\alpha = v_\alpha = 0) = (u_\beta = v_\beta = 0)$ and $\Sigma_q = (u_\alpha = 0) = (u_\beta = 0)$.
(xii) If $i \in \{\alpha, \beta\}$ then $\mathcal{F}_q$ is represented by the germ at $0 \in \mathbb{C}^3$ of

$$\omega_i = (u_i + \phi_i(v_i^k))\, du_i + u_i\, (v_i^n . \zeta_2(v_i^k) + \phi_i'(v_i^k))\, d(v_i^k) \ .$$

In particular, we get from (xi) that $u_\alpha = g_{\alpha\beta}.u_\beta$ and $v_\alpha = h_{\alpha\beta}.u_\beta + k_{\alpha\beta}.v_\beta$, where $h_{\alpha\beta}, k_{\alpha\beta} \in \mathcal{O}_q$ and $g_{\alpha\beta}.k_{\alpha\beta} \in \mathcal{O}_q^*$. If we substitute these relations in ω_α then we get the expression of ω_α in the other coordinate system

$$\omega_\alpha = (g_{\alpha\beta}.u_\beta + \phi_\alpha((h_{\alpha\beta}.u_\beta + k_{\alpha\beta}.v_\beta)^k))\,(g_{\alpha\beta}\, du_\beta + u_\beta\, dg_{\alpha\beta}) \tag{3.8}$$

$$+ g_{\alpha\beta}\, u_\beta((h_{\alpha\beta}.u_\beta + k_{\alpha\beta}.v_\beta)^s . \zeta_2(v_\alpha^k) + \phi_i'((h_{\alpha\beta}.u_\beta + k_{\alpha\beta}.v_\beta)^k))\, d(h_{\alpha\beta}.u_\beta + k_{\alpha\beta}.v_\beta)^k$$

$$:= A(u_\beta, v_\beta, w_\beta)\, du_\beta + B(u_\beta, v_\beta, w_\beta)\, dv_\beta + C(u_\beta, v_\beta, w_\beta)\, dw_\beta \ .$$

Since $\omega_\alpha = \varphi_{\alpha\beta}.\omega_\beta$ and ω_β has no term with dw_β, we get $C \equiv 0$. Write

$$C(u_\beta, v_\beta, w_\beta) = \sum_{i,j \geq 0} C_{ij}(w_\beta)\, u_\beta^i\, v_\beta^j \ .$$

It follows from (3.8) that $C_{00}(w_\beta) = C_{10}(w_\beta) = C_{01}(w_\beta) = 0$ and

$$C_{20}(w_\beta) = g_{\alpha\beta}(0, 0, w_\beta). \frac{\partial g_{\alpha\beta}(0, 0, w_\beta)}{\partial w_\beta} = 0 \implies \frac{\partial g_{\alpha\beta}(0, 0, w_\beta)}{\partial w_\beta} = 0$$

$$\implies \ g_{\alpha\beta}|_{U_{\alpha\beta} \cap \Gamma} \text{ is locally constant.}$$

Recall that $c_1(N_\sigma)|_\Gamma$ can be obtained from the additive cocycle of $\left(\frac{dg_{\alpha\beta}}{g_{\alpha\beta}} |_{U_{\alpha\beta}} \right)_{U_{\alpha\beta} \neq \emptyset}$ by taking a fine resolution. Since $\frac{dg_{\alpha\beta}}{g_{\alpha\beta}}|_{\Gamma \cap U_{\alpha\beta}} = 0$ we get $\int_\Gamma c_1(N_\Sigma) = 0$. This finishes the proof of Lemma 3.16. $\square$

Remark 3.18. Lemma 3.16 implies that $\tilde{\Gamma} \subset K(\mathcal{F})$, the set of singularities of Kupka type of $\mathcal{F}$.

Corollary 3.19. *If $\Gamma \setminus \tilde{\Gamma} = \emptyset$ then Theorem 1.2 is true in dimension three.*

Proof. If $\Gamma = \tilde{\Gamma}$ then it is smooth and $\Gamma \subset K(\mathcal{F})$. Therefore, $\mathcal{F}$ has a meromorphic first integral by [6] and [3]. □

In view of Corollary 3.19, from now on we will assume that $\Gamma \setminus \tilde{\Gamma} \neq \emptyset$. Another consequence of Lemma 3.16 is the following:

Corollary 3.20. *Fix $p \in \Gamma \setminus \tilde{\Gamma}$ and consider a germ of holomorphic chart $(x, y, z)\colon (\mathbb{P}^3, p) \to (\mathbb{C}^3, 0)$ such that $\mathcal{F}_p$ is represented in this chart by the form*

$$\omega = x\,dx + (f^m(y,z).\,\zeta_1(f(y,z)) + x\,f^n(y,z).\,\zeta_2(f(y,z)))\,df(y,z) \ ,$$

where $f \in \mathfrak{m}_2$ and $\zeta_1, \zeta_2 \in \mathcal{O}_1^*$. Then:

(a) *$n = 0$ and $m \geq 1$.*
(b) *$0 \in \mathbb{C}^2$ is a singularity of f and f is reduced in $\mathcal{O}_2$.*

Proof. Note that

$$\begin{aligned} d\omega &= f^n(y,z).\,\zeta_2(f(y,z))\,dx \wedge df(y,z) \\ &= f^n(y,z).\,\zeta_2(f(y,z)) \left[\frac{\partial f(y,z)}{\partial y}\,dx \wedge dy + \frac{\partial f(y,z)}{\partial z}\,dx \wedge dz \right] . \end{aligned}$$

Since $\tilde{\Gamma} \subset K(\mathcal{F})$ we must have $(\omega = d\omega = 0) = \{0\}$. Therefore, $n = 0$, $m \geq 2n + 1 = 1$ and $\left(\frac{\partial f(y,z)}{\partial y} = \frac{\partial f(y,z)}{\partial z} = 0 \right) = \{0\}$, which implies that f is reduced in $\mathcal{O}_2$. Finally, 0 must be a singular point of f because $p \in \Gamma \setminus \tilde{\Gamma}$ is a nilpotent singularity of $\mathcal{F}$. □

Lemma 3.21. *The normal type of $\mathcal{F}$ along $\tilde{\Gamma}$ is linearizable and can be defined by the germ of 1-form on $(\mathbb{C}^2, 0)$:*

$$\eta = m\,x\,dy - n\,y\,dx \ ,$$

where $m, n \in \mathbb{N}$, $gcd(m,n) = 1$ and $n > m \geq 1$.

Proof. Let $\eta = P(x,y)\,dy - Q(x,y)\,dx$ be a germ at $0 \in \mathbb{C}^2$ of holomorphic 1-form defining the normal type. Set $\eta_1 = j_0^1(\eta)$. Since $d\eta(0) \neq 0$ we get $d\eta_1 \neq 0$ and, after a linear change of variables, we have three possibilities:

(a) $\eta_1 = x\,dy$ (saddle-node).
(b) $\eta_1 = x\,dy - \lambda\,y\,dx$, where $\lambda \notin \{0, -1\}$.
(c) $\eta_1 = x\,dy - (x + y)\,dx$.

We will show first that $\lambda \in \mathbb{Q}_+$ in case (b). With the same type of argument we will show that case (a) is not possible in our situation. As a consequence, we will get that always $\lambda \neq 0$ and $\lambda \in \mathbb{Q}_+$. Concerning the linearization, we will use Poincaré-Dulac's normal form: when $\lambda, 1/\lambda \in \mathbb{Q}_+ \setminus \mathbb{N}$ then η is linearizable, whereas if $\lambda = n \in \mathbb{N}$, for instance, then η is equivalent to $\beta_n = x\,dy - (n\,y + a\,x^n)\,dx$. When $a = 0$ the form β_n is linear, whereas if $a \neq 0$ then it is not linearizable and we can assume that $a = 1$. However, in our situation, we will prove that $a = 0$. The same argument will imply that case (c) is not possible.

Let us examine the existence of distinguished separatrices. In case (a) the following normal form is known (*cf.* [18])

$$\eta = [x(1 + \mu\, y^n) + h.o.t.]\,dy - y^{n+1}\,dx \ ,$$

where $n \geq 1$. The separatrix $\sigma := (y = 0)$ is the unique one tangent to the direction of $y = 0$ and it is distinguished. Therefore, by Lemma 3.14 it can be extended to a smooth separatrix Σ_1 of $\mathcal{F}$ along Γ. In this case, we will see that $\int_\Gamma c_1(N_{\Sigma_1}) = 0$, a contradicion with Proposition 3.15.

On the other hand, in case (b), if X is the dual vector field of the normal type η then the eigenvalues of $DX(0)$ are 1 and λ. When $\lambda, 1/\lambda \notin \mathbb{N}$ then the vector field X has only two smooth separatrices through 0: one, say σ_1, tangent to the eigenspace correspondent to 1, and the other, say σ_2, tangent to the eigenspace correspondent to λ. Both separatrices are distinguished because

$$CS(X, \sigma_1) = \lambda \neq 1/\lambda = CS(X, \sigma_2) \ .$$

Therefore, σ_j extends to a smooth separatrix Σ_j of $\mathcal{F}$ along Γ, $j = 1, 2$. We will see that

$$\int_\Gamma c_1(N_{\Sigma_2}) = \lambda \int_\Gamma c_1(N_{\Sigma_1}) \ . \tag{3.9}$$

This will imply $\lambda \in \mathbb{Q}_+$, because $\int_\Gamma c_1(N_{\Sigma_i}) \in \mathbb{N}$, $i = 1, 2$, by Proposition 3.15.

In both cases, we will consider a covering $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ of Γ satisfying (v) and (vi) of the proof of Lemma 3.16. In particular, if $U_{\alpha\beta} \neq \emptyset$ then $U_{\alpha\beta} \cap (\Gamma \setminus \tilde{\Gamma}) = \emptyset$. Let us analyse first case (b) with $\lambda, 1/\lambda \notin \mathbb{N}$.

We can assume that $\sigma_1 = (y = 0)$ and $\sigma_2 = (x = 0)$. Dividing η by some $\phi \in \mathcal{O}_2^*$, the normal type becomes

$$\theta := \phi^{-1}.\,\eta = x\,dy - \lambda\,y(1 + R(x, y))\,dx \ , \ \ \nu(R, 0) \geq 1 \ .$$

Therefore, we can suppose that:

(b.1). If $U_\alpha \cap \Gamma \setminus \tilde{\Gamma} = \emptyset$ then there is a chart $(x_\alpha, y_\alpha, z_\alpha) \colon U_\alpha \to \mathbb{C}^3$ such that

$$\omega_a = x_\alpha\,dy_\alpha - \lambda\,y_\alpha\,(1 + R(x_\alpha, y_\alpha))\,dx_\alpha \ . \tag{3.10}$$

In particular, $\Sigma_1 \cap U_\alpha = (x_\alpha = 0)$ and $\Sigma_2 \cap U_\alpha = (y_\alpha = 0)$. We take x_α and y_α as defining equations of $\Sigma_1 \cap U_\alpha$ and $\Sigma_2 \cap U_\alpha$, respectively.

Now, given $p \in \Gamma \setminus \tilde{\Gamma}$, by Corollary 3.20 there is a chart $(U, (u, v, w))$, around p, where $\mathcal{F}|_U$ is defined by

$$\omega = u\,du + (f^r . \zeta_1(f) + u . \zeta_2(f))\,df \ , \ f = f(v, w) \ , \tag{3.11}$$

where f is reduced, $r \geq 1$ and $\zeta_i(0) \neq 0$, $i = 1, 2$. If we fix some point $q \in U \cap \tilde{\Gamma}$ then there is a chart $(W, (u, y, z))$ such that $f|_W = y$, and so

$$\omega|_W = u\,du + (y^r\,\zeta_1(y) + u\,\zeta_2(y))\,dy \ .$$

Since $q \in \tilde{\Gamma}$, the form $\theta_1 := \omega|_W$ is analytically equivalent to θ. This implies $r = 1$ and

$$\frac{\zeta_2(0)^2}{\zeta_1(0)} = \mathrm{Res}\left(\frac{\zeta_2^2(y)\,dy}{y\,\zeta_1(y)}, 0\right) = \mathrm{BB}(\theta_1, 0) = \mathrm{BB}(\theta, 0) = \frac{(\lambda+1)^2}{\lambda} \ .$$

In particular, we can assume that:

(b.2). If $\emptyset \neq U_\alpha \cap (\Gamma \setminus \tilde{\Gamma}) = \{p\}$ then there is a chart $(u_\alpha, v_\alpha, w_\alpha) \colon U_\alpha \to \mathbb{C}^3$ such that $p = (0, 0, 0)$ and $\mathcal{F}|_{U_\alpha}$ is defined by

$$\omega_\alpha = u_\alpha\,du_\alpha + (f_\alpha . \zeta_1(f_\alpha) + u_\alpha . \zeta_2(f_\alpha))\,df_\alpha \ , \ f_\alpha = f_\alpha(v_\alpha, w_\alpha) \ . \tag{3.12}$$

As we have seen before, in this chart we can set $\Sigma_i \cap U_\alpha = (u_\alpha - \phi_i(f_\alpha(v_\alpha, w_\alpha)) = 0)$, where $\phi_i(t) = t\,\psi_i(t)$, $i = 1, 2$, and $\psi_1(0)$ and $\psi_2(0)$ are the roots of $z^2 + \zeta_2(0)\,z + \zeta_1(0) = 0$. We take $x_\alpha := u_\alpha - \phi_1(f_a(v_\alpha, w_\alpha))$ and $y_\alpha := u_\alpha - \phi_2(f_\alpha(v_\alpha, w_\alpha))$ as the defining equations of $\Sigma_1 \cap U_\alpha$ and $\Sigma_2 \cap U_\alpha$, respectively.

Let $(g_{\alpha\beta})_{U_{\alpha\beta}\neq\emptyset}$, $(k_{\alpha\beta})_{U_{\alpha\beta}\neq\emptyset}$ and $(\varphi_{\alpha\beta})_{U_{\alpha\beta}\neq\emptyset}$ be the multiplicative cocycles such that $x_\alpha = g_{\alpha\beta} . x_\beta$, $y_\alpha = k_{\alpha\beta} . y_\beta$ and $\omega_\alpha = \varphi_{\alpha\beta} . \omega_\beta$ on $U_{\alpha\beta} \neq \emptyset$. We assert that

$$\left.\frac{dk_{\alpha\beta}}{k_{\alpha\beta}} - \lambda\frac{dg_{\alpha\beta}}{g_{\alpha\beta}}\right|_{U_{\alpha\beta}\cap\Gamma} \equiv 0 \ , \ \forall\ U_{\alpha\beta} \cap \Gamma \neq \emptyset. \tag{3.13}$$

Note that (3.13) implies (3.9).

Proof of (3.13) Fix $\alpha, \beta \in A$ such that $U_{\alpha\beta} \cap \Gamma \neq \emptyset$. Since the covering satisfies (vi), we can assume that $U_\alpha \cap (\Gamma \setminus \tilde{\Gamma}) = \emptyset$, so that ω_α is like in (3.10). When we substitute $x_\alpha = g_{\alpha\beta} . x_\beta$ and $y_\alpha = k_{\alpha\beta} . y_\beta$ in ω_α, we get the expression of $\omega_\alpha|_{U_{\alpha\beta}}$ in the other chart:

$$\omega_\alpha = g_{\alpha\beta} . x_\beta\, d(k_{\alpha\beta} . y_\beta) - \lambda . k_{\alpha\beta} . y_\beta\,(1 + R(g_{\alpha\beta} . x_\beta, k_{\alpha\beta} . y_\beta))\,d(g_{\alpha\beta} . x_\beta) \ . \tag{3.14}$$

We have two possibilities:

1st. $U_\beta \cap (\Gamma \setminus \tilde{\Gamma}) = \emptyset$. In this case, ω_β is also like in (3.10) and we get:

$$\begin{aligned} &g_{\alpha\beta} . x_\beta\, d(k_{\alpha\beta} . y_\beta) - \lambda . k_{\alpha\beta} . y_\beta\,(1 + R(g_{\alpha\beta} . x_\beta, k_{\alpha\beta} . y_\beta))\,d(g_{\alpha\beta} . x_\beta) \\ &\qquad := A\,dx_\beta + B\,dy_\beta + C\,dz_\beta = \varphi_{\alpha\beta}\,\big(x_\beta\,dy_\beta - \lambda\,y_\beta\,(1 + R(x_\beta, y_\beta))\,dx_\beta\big) \ . \end{aligned}$$

Since ω_β does not contain terms with dz_β, we get $C \equiv 0$. If we set $C(x_\beta, y_\beta, z_\beta) = \sum_{ij\geq 0} C_{ij}(z_\beta).x_\beta^i.y_\beta^j$ then

$$C_{11}(z_\beta) = g_{\alpha\beta}(0,0,z_\beta).\frac{\partial k_{\alpha\beta}(0,0,z_\beta)}{\partial z_\beta} - \lambda\, k_{\alpha\beta}(0,0,z_\beta).\frac{\partial g_{\alpha\beta}(0,0,z_\beta)}{\partial z_\beta} = 0 \ .$$

Since $U_{\alpha\beta} \cap \Gamma = (x_\beta = y_\beta = 0)$, the above relation implies (3.13).

2nd. $U_\beta \cap (\Gamma \setminus \tilde{\Gamma}) = \{p\}$. In this case, ω_β is like in (3.12) and the substitution of $x_\alpha = g_{\alpha\beta}.x_\beta$ and $y_\alpha = k_{\alpha\beta}.y_\beta$ in (3.14) becomes more complicated. However, if we fix $q \in U_{\alpha\beta} \cap \tilde{\Gamma}$ then we can find a chart $(W, (u,s,t))$ around q such that $u_\beta = u$ and $f_\beta(v_\beta, w_\beta)|_W = s$. In this new chart, ω_β does not contain terms in dt. Since $x_\alpha = g_{\alpha\beta}.(u - \phi_1(s))$ and $y_\beta = k_{\alpha\beta}.(u - \phi_2(s))$ in this chart, a direct computation shows that the term in dt of ω_α, after the substitution, is

$$\begin{aligned} &C(u,s,t) \\ &= (u-\phi_1(s))(u-\phi_2(s)) \left\{ g_{\alpha\beta}\frac{\partial k_{\alpha\beta}}{\partial t} - \lambda[1 + R(x_\alpha, y_\alpha)]k_{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial t} \right\} \equiv 0 \\ &\Longrightarrow\ g_{\alpha\beta}(0,0,t)\frac{\partial k_{\alpha\beta}(0,0,t)}{\partial t} - \lambda\, k_{\alpha\beta}(0,0,t)\frac{\partial g_{\alpha\beta}(0,0,t)}{\partial t} \equiv 0 \ \Longrightarrow\ (3.13)\ . \end{aligned}$$

This proves that in case (b) we must have $\lambda \in \mathbb{Q}_+$. Moreover, Poincaré's linearization theorem implies that if $\lambda, 1/\lambda \notin \mathbb{N}$ then the normal type is equivalent to $m\,x\,dy - n\,y\,dx$, $m, n \in \mathbb{N}$.

Let us analyse case (a), in which, a priori, $\mathcal{F}$ has just the separatrix Σ_1. We can take the covering $\mathcal{U}$ in such a way that:

(a.1). If $U_\alpha \cap (\Gamma \setminus \tilde{\Gamma}) = \emptyset$ then there is a chart $(x_\alpha, y_\alpha, z_\alpha)\colon U_\alpha \to \mathbb{C}^3$ such that $\mathcal{F}|_{U_\alpha}$ is represented by:

$$\omega_\alpha = [y_\alpha(1 + \mu\, x_\alpha^n) + R(x_\alpha, y_\alpha)]\, dx_\alpha - x_\alpha^{n+1}\, dy_\alpha \ , \tag{3.15}$$

where $\nu(R,0) \geq n+2$. In particular, $\Sigma_1 \cap U_\alpha = (x_\alpha = 0)$ and we take x_α as the defining equation of $\Sigma \cap U_\alpha$.

(a.2). If $U_\alpha \cap (\Gamma \setminus \tilde{\Gamma}) \neq \emptyset$ and $U_\alpha \cap (\Gamma \setminus \tilde{\Gamma}) = \{p\}$ then there is a chart $(u_\alpha, v_\alpha, w_\alpha)\colon U_\alpha \to \mathbb{C}^3$ such that $\mathcal{F}|_{U_\alpha}$ is represented by a form like in (3.11):

$$\omega_\alpha = u_\alpha\, du_\alpha + \left(f_\alpha^r.\,\zeta_1(f_\alpha) + u_\alpha.\,\zeta_2(f_\alpha)\right) df_\alpha \ , \ \ f_\alpha = f_\alpha(v_\alpha, w_\alpha)\ ,$$

where $r \geq 1$, f_α is reduced and $\zeta_j(0) \neq 0$, $j = 1, 2$.

Let $(\varphi_{\alpha\beta})_{U_{\alpha\beta}\neq\emptyset}$ be the multiplicative cocycle such that $\omega_\alpha = \varphi_{\alpha\beta}.\omega_\beta$ on $U_{\alpha\beta} \neq \emptyset$.

We would like to observe that $r = n+1 > 1$ in the situation (a.2). In fact, if we take $\beta \in A$ such that $\beta \neq \alpha$ with $\tilde{\Gamma} \cap U_{\alpha\beta} \neq \emptyset$ and $q = (0, v_o, w_o) \in \tilde{\Gamma} \cap U_{\alpha\beta}$, then $f_\alpha(v_o, w_o) = 0$ and there is a chart $(W, (u, y, z))$ around q such that $u = u_\alpha$ and $f_\alpha|_W = y$. In particular, in this chart

$$\omega_\alpha = u\,du + (y^r\,\zeta_1(y) + u.\,\zeta_2(y))\,dy\ .$$

Therefore, the multiplicity (Milnor number) of the singularity 0 of ω_α (in a transverse section) is $\mu(\omega_\alpha, 0) = [u, y^r\,\zeta_{1\alpha}(y) + u\,\zeta_{2\alpha}(y)]_0 = r$. Since $\omega_\alpha = \varphi_{\alpha\beta}.\omega_\beta$, where ω_β is like in (3.15), we get $r = \mu(\omega_\beta, 0) = n + 1$.

In particular, a straightforward computation shows that the equation of Σ_1 in the chart (u, y, z) is of the form $u + \phi(y) = 0$, where $\phi(0) = 0$ and $\phi'(0) = \zeta_2(0) \neq 0$. Therefore, the equation of $\Sigma_1 \cap U_\alpha$ is $u_\alpha + \phi(f_\alpha(v_\alpha, w_\alpha)) = 0$. We take $x_\alpha := u_\alpha + \phi(f_\alpha(v_\alpha, w_\alpha))$ as the defining equation of $\Sigma_\alpha \cap U_\alpha$ in the situation (a.2).

Note that $< u_\alpha, x_\alpha > = < u_\alpha, f_\alpha >$, because $\phi(0) = 0$ and $\phi'(0) \neq 0$. In particular, $\Gamma \cap U_\alpha$ is defined by the ideal $< u_\alpha, x_\alpha >$.

Let $G = (g_{\alpha\beta})_{U_{\alpha\beta}}$ be the multiplicative cocycle such that $x_\alpha = g_{\alpha\beta}.\,x_\beta$ on $U_{\alpha\beta} \neq \emptyset$. We will see that $g_{\alpha\beta}|_{\Gamma\cap U_{\alpha\beta}}$ is locally constant and this will imply $\int_\Gamma c_1(N_{\Sigma_1}) = 0$.

Fix $\alpha, \beta \in A$ such that $\Gamma \cap U_{\alpha\beta} \neq \emptyset$. By the construction of the covering $\mathcal{U}$, we can assume that $(\Gamma \setminus \tilde{\Gamma}) \cap U_\alpha = \emptyset$, so that ω_α is like in (3.15) and $\Gamma \cap U_\alpha = (x_\alpha = y_\alpha = 0)$. We have two possibilities:

1st. $U_\beta \cap (\Gamma \setminus \tilde{\Gamma}) = \emptyset$. In this case, ω_β is also like in (3.15) and $\Gamma \cap U_{\alpha\beta} = (x_\alpha = y_\alpha = 0) = (x_\beta = y_\beta = 0)$. This implies that $y_\alpha = h_{\alpha\beta}.x_\beta + k_{\alpha\beta}.y_\beta$ on $U_{\alpha\beta}$, where $g_{\alpha\beta}.k_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$.

2nd. $U_\beta \cap (\Gamma \setminus \tilde{\Gamma}) \neq \emptyset$. In this case, ω_β is like in (3.11) and $\Gamma \cap U_{\alpha\beta} = (x_\alpha = y_\alpha = 0) = (x_\beta = f_\beta = 0)$. This implies that $y_\alpha = h_{\alpha\beta}.x_\beta + k_{\alpha\beta}.f_\beta$ on $U_{\alpha\beta}$, where $g_{\alpha\beta}.k_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$.

In the 2nd case, if $q \in U_{\alpha\beta} \cap \Gamma$ then $dx_\beta \wedge df_\beta(q) \neq 0$. Hence, we can find a chart $(W, (x_\beta, y_\beta, z_\beta))$ around some point $q \in U_{\alpha\beta} \cap \Gamma$ such that $f_\beta|_W = y_\beta$. In both cases, we have $y_\alpha = h_{\alpha\beta}.x_\beta + k_{\alpha\beta}.y_\beta$ and ω_β do not contain terms with dz_β. On the other hand, if we write ω_α in the coordinates $(x_\beta, y_\beta, z_\beta)$, using the relations $x_\alpha = g_{\alpha\beta}.x_\beta$ and $y_\alpha = h_{\alpha\beta}.x_\beta + k_{\alpha\beta}.y_\beta$, we get $\omega_\alpha = A(x_\beta, y_\beta, z_\beta)\,dx_\beta + B(x_\beta, y_\beta, z_\beta)\,dy_\beta + C(x_\beta, y_\beta, z_\beta)\,dz_\beta$, where we can write

$$C(x_\beta, y_\beta, z_\beta) = \sum_{i,j\geq 0} C_{ij}(z_\beta)\,x_\beta^i\,y_\beta^j \equiv 0 \implies C_{ij}(z_\beta) \equiv 0\ ,\ \forall i, j \geq 0\ .$$

By substituting explicitly $x_\alpha = g_{\alpha\beta}.x_\beta$ and $y_\alpha = h_{\alpha\beta}.x_\beta + k_{\alpha\beta}.y_\beta$ in (3.15) we get

$$C_{11}(z_\beta) = k_{\alpha\beta}(0, 0, z_\beta).\frac{\partial g_{\alpha\beta}(0, 0, z_\beta)}{\partial z_\beta} = 0\ ,$$

which implies $\frac{\partial g_{\alpha\beta}(0,0,z_\beta)}{\partial z_\beta} = 0$. Hence, $g_{\alpha\beta}|_{\Gamma\cap U_{\alpha\beta}}$ is locally constant and this finishes the analysis of case (a).

Next, we prove that in our situation the normal type cannot be equivalent to

$$\beta_n = x\,dy - (n\,y + x^n)\,dx \ , \ n \geq 1 \, .$$

This will imply that case (c) is not possible: if $\eta_1 = x\,dy - (x + y)\,dx$ then by Poincaré's linearization theorem η is equivalent to η_1 and so to β_1, a contradiction.

Let us suppose by contradiction that the normal type of $\mathcal{F}$ at Γ is equivalent to β_n.

Remark 3.22. We would like to observe the following facts about the foliation defined by β_n:

(a) It has no meromorphic non-constant first integral in a neighborhood of $0 \in \mathbb{C}^2$. This implies that $\mathcal{F}$ has no meromorphic non-constant meromorphic first integral in a neighborhood of any point $p \in \Gamma$.
(b) The separatrix $\sigma = (x = 0)$ is the unique analytic separatrix of η through $0 \in \mathbb{C}^2$. In particular, it is distinguished and can be extended to a smooth separatrix Σ of $\mathcal{F}$ along Γ.
(c) β_n has an integrating factor: if we set $\theta := x^{-(1+n)}.\,\beta_n$ then:

$$\theta = d\left(\frac{y}{x^n}\right) - \frac{dx}{x} \quad \Longrightarrow \quad d\theta = 0 \, .$$

Let us sketch the proof. First of all, we will prove that the closed meromorphic 1-form θ, on some transverse section Λ, can be extended from the transverse section to a closed meromorphic 1-form Θ on some connected neighborhood U of Γ, in such a way that:

(i) The divisor $(\Theta)_\infty$, of poles of Θ, is $(\Theta)_\infty = (\Sigma)^{n+1}$.
(ii) Θ defines $\mathcal{F}$ on $U \setminus \Sigma$.
(iii) $\mathrm{Res}(\Theta, \Sigma) = -1$.

Using an extension theorem of meromorphic functions on U (*cf.* [1] and [22]), the form Θ can be extended to a global closed meromorphic 1-form. The contradiction will be a consequence of (i) and (iii), as we will see.

Extension of θ to a neighborhood of $\tilde{\Gamma}$. Fix a covering $(U_\alpha)_{\alpha\in A}$ of $\tilde{\Gamma}$ such that, for all $\alpha \in A$, there is a chart $(x_\alpha, y_\alpha, z_\alpha)\colon U_\alpha \to \mathbb{C}^3$ such that $\mathcal{F}|_{U_\alpha}$ is represented by $\eta_\alpha = x_\alpha\,dy_\alpha - (n\,y_\alpha + x_\alpha^n)\,dx_\alpha$. We can assume also that, if $U_{\alpha\beta} \neq \emptyset$ then $U_{\alpha\beta} \cap \tilde{\Gamma}$ is connected and non-empty. Note that $\Sigma \cap U_\alpha = (x_\alpha = 0)$ for all $\alpha \in A$.

Set $\Theta_\alpha = d\left(\frac{y_\alpha}{x_\alpha^n}\right) - \frac{dx_\alpha}{x_\alpha}$, $\alpha \in A$. We assert that, if $U_{\alpha\beta} \neq \emptyset$ then $\Theta_\alpha = \Theta_\beta$ on $U_{\alpha\beta}$.

In fact, fix $U_{\alpha\beta} \neq \emptyset$ and $g, \varphi \in \mathcal{O}^*(U_{\alpha\beta})$ such that $x_\alpha = g.x_\beta$ and $\eta_\alpha = \varphi.\eta_\beta$ on $U_{\alpha\beta}$. From $\Theta_\alpha = x_\alpha^{-(n+1)}.\eta_\alpha$ and $\Theta_\beta = x_\beta^{-(n+1)}.\eta_\beta$ we get $\Theta_\alpha = \phi.\Theta_\beta$, where $\phi = \varphi/g^{n+1} \in \mathcal{O}^*(U_{\alpha\beta})$. Since Θ_α and Θ_β are closed, we get

$$0 = d\Theta_\alpha = d\phi \wedge \Theta_\beta \implies d\phi \wedge \eta_\beta = 0 \implies$$
$$\phi \text{ is a holomorphic first integral of } \mathcal{F}|_{U_{\alpha\beta}}.$$

This implies that ϕ is a constant, because $U_{\alpha\beta} \cap \tilde{\Gamma} \neq \emptyset$ and $\mathcal{F}$ has no non-constant holomorphic first integral in a neighborhood of any point $q \in \tilde{\Gamma}$ (see Remark 3.22). Now, observe that

$$\mathrm{Res}(\Theta_\alpha, \Sigma) = \mathrm{Res}\left(-\frac{dx_\alpha}{x_\alpha}, \Sigma\right) = \mathrm{Res}\left(-\frac{dx_\alpha}{x_\alpha}, (x_a = 0)\right) = -1 \ .$$

Similarly, $\mathrm{Res}(\Theta_\beta, \Sigma) = -1$. Since ϕ is a constant, we get

$$-1 = \mathrm{Res}(\Theta_\alpha, \Sigma) = \phi.\mathrm{Res}(\Theta_\beta, \Sigma) = -\phi \implies \phi \equiv 1 \ ,$$

which proves the assertion.

It follows that there exists a meromorphic 1-form $\tilde{\Theta}$ on the neighborhood $\tilde{U} = \bigcup_\alpha U_\alpha$ of $\tilde{\Gamma}$, such that $\tilde{\Theta}|_{U_\alpha} = \Theta_\alpha$ for all $\alpha \in A$. Let us extend $\tilde{\Theta}$ to a neighborhood $U \supset \tilde{U}$ of Γ.

Extension of $\tilde{\Theta}$ to a neighborhood of Γ. Fix $p \in \Gamma \setminus \tilde{\Gamma}$ and a chart $(u, s, t)\colon V \to \mathbb{C}^3$, around p, such that $\mathcal{F}|_V$ is defined by

$$\omega = u\,du + (f.\zeta_1(f) + u\,\zeta_2(f))\,df \ , \ f = f(s,t) \ .$$

Choose a point $q \in \tilde{\Gamma} \cap V$ and a chart $(W, (u, v, w))$, around q, such that $v = f(s,t)$, $\tilde{\Gamma} \cap V = (u = v = 0)$ and $\omega = u\,du + (v\,\zeta_1(v) + u\,\zeta_2(v))\,dv$. Since $\tilde{\Theta}|_W$ defines $\mathcal{F}$ on $W \setminus \Sigma$, there exists a meromorphic function $h = h(u, v, w)$ on W such that $\tilde{\Theta} = h.\omega$. We assert that $\frac{\partial h}{\partial w} = 0$, so that $h = h(u, v)$.

In fact, since $\tilde{\Theta}$ is closed, we get

$$0 = d\tilde{\Theta} = dh \wedge \omega + h\,d\omega \implies \zeta_2(v)\,du \wedge dv = d\omega = -\frac{dh}{h} \wedge \omega$$
$$\implies -\zeta_2(v)\,du \wedge dv = \frac{dh}{h} \wedge [u\,du + (v\,\zeta_1(v) + u\,\zeta_2(v))\,dv] \implies \frac{\partial h}{\partial w} = 0,$$

which proves the assertion. It follows that the meromorphic 1-form $h(u, f(s,t)).\omega$ is closed and extends $\tilde{\Theta}$ to some neighborhood of p. Hence, $\tilde{\Theta}$ can be extended to a closed meromorphic 1-form Θ, on some connected neighborhood U of Γ, which satisfies (i), (ii) and (iii).

Now, we use the following result:

Theorem 3.23 ([1,22]). *Let Y be a connected analytic subset of $\mathbb{P}^n$, $n \geq 2$, with $\dim(Y) \geq 1$. Then any meromorphic function in a connected neighborhood of Y extends to a meromorphic function on all of $\mathbb{P}^n$.*

As a consequence of Theorem 3.23, the form Θ can be extended to a global meromorphic 1-form on $\mathbb{P}^3$. In fact, consider an affine coordinate system $(x, y, z) \in \mathbb{C}^3 \subset \mathbb{P}^3$, such that $\Gamma \not\subset L_\infty$, where L_∞ denotes the plane at infinity of $\mathbb{C}^3$. We can write

$$\Theta|_{\mathbb{C}^3 \cap U} = A\,.dx + B\,dy + C\,dz\ ,$$

where A, B and C are meromorphic functions on $\mathbb{C}^3 \cap U$. Since dx, dy and dz are global meromorphic forms on $\mathbb{P}^3$, the functions A, B and C can be extended to meromorphic functions on U, and, as a consequence, to meromorphic functions on $\mathbb{P}^3$, by Theorem 3.23, which proves the assertion. Denote the extension again by Θ. Let $|\Theta|_\infty$ be the set of poles of Θ.

Note that $|\Theta|_\infty$ is an algebraic hypersurface of $\mathbb{P}^3$. Since $|\Theta|_\infty \cap U = \Sigma$, the separatrix Σ extends to an irreducible algebraic hypersurface of $\mathbb{P}^3$, which we still denote Σ. On the other hand, if S is an irreducible component of $|\Theta|_\infty$ then

$$S \cap \Gamma \neq \emptyset \implies S \cap U \neq \emptyset \implies S \cap U = \Sigma \cap U \implies S = \Sigma \implies |\Theta|_\infty = \Sigma.$$

Now, we arrive to a contradiction: we have seen that $\mathrm{Res}(\Theta, \Sigma) = -1$. If we take a line $\mathbb{P}^1 \simeq \ell \subset \mathbb{P}^3$ cutting Σ transversely in $dg(\Sigma)$ points, then $\sum_{p \in \ell} \mathrm{Res}(\Theta|_\ell, p) = -dg(\Sigma) \neq 0$, which is not possible by the residue theorem. Therefore, the normal type is equivalent to $m\,x\,dy - n\,y\,dx$, where $m, n \in \mathbb{N}$, where we can assume that $n \geq m \geq 1$ and $gcd(m, n) = 1$.

It remains to prove that in our situation we don't have $m = n = 1$. This is a consequence of the fact that we are assuming $\Gamma \setminus \tilde{\Gamma} \neq \emptyset$.

In fact, suppose by contradiction that the normal type was $\eta = x\,dy - y\,dx$. Fix a point $q \in \tilde{\Gamma}$ and a transverse section Λ to $\tilde{\Gamma}$ through q. Let X be a vector field on Λ defining $\mathcal{F}|_\Lambda$. Since the normal type is η, we have $\eta(X) = 0$, which implies that the linear part $DX(q)$ of X at q, must be of the form $\delta\, I$, where I is the identity and $\delta \neq 0$. On the other hand, if $p \in \Gamma \setminus \tilde{\Gamma}$ then there is a chart $(U, \psi = (x, y, z))$ around p such that $\psi(p) = 0$ and $\mathcal{F}|_U$ is represented by

$$\omega = x\,dx + (f^r\,\zeta_1(f) + x\,\zeta_2(f))\,df\ ,\ \ f = f(y, z)\ ,$$

where f is reduced, $r \geq 1$ and $\zeta_1(0), \zeta_2(0) \neq 0$. If $q \in U \cap \tilde{\Gamma}$ then we can find a chart $(W, (u, v, w))$ around q such that $x = u$ and $f(y, z) = v$. In this chart $\mathcal{F}|_W$ is represented, in a normal section to $\tilde{\Gamma}$ through q by $u\,du + (v^r\,\zeta_1(v) + u\,\zeta_2(v))\,dv$. The dual vector field of this form is $X = (v^r\,\zeta_1(v) + u\,\zeta_2(v))\,\partial_u - u\,\partial_v$. As the reader can check, $DX(0) \neq \delta\, I$. This finishes the proof of Lemma 3.21. $\square$

End of the proof of Theorem 1.2 in dimension three. We have proved that the normal type of $\mathcal{F}$ along $\tilde{\Gamma}$ is given by $\eta = m\,x\,dy - n\,y\,dx$, where $m, n \in \mathbb{N}$, $gcd(m, n) = 1$ and $n > m \geq 1$. Let us give an idea of the proof.

Consider a meromorphic integrable 1-form Ω on $\mathbb{P}^3$ representing $\mathcal{F}$ outside its set of poles. By using the normal type, we will see that there exists a closed meromorphic 1-form $\tilde{\Lambda}$, on some connected neighborhood U of Γ, such that $d\Omega = \tilde{\Lambda} \wedge \Omega$ on U. The extension theorem of [1] and [22] will imply that $\tilde{\Lambda}$ can be extended to a closed meromorphic 1-form Λ on $\mathbb{P}^3$ with $d\Omega = \Lambda \wedge \Omega$. Next, working with the pole divisors and residues of Λ, we will see that $\Lambda = \frac{dF}{F}$, where F is meromorphic on $\mathbb{P}^3$. In particular, we will get $d\left(\frac{\Omega}{F}\right) = 0$, that is, F is an integrating factor of Ω. Finally, by studing $\frac{\Omega}{F}$ around Γ, we will show that $\mathcal{F}$ has a rational first integral of the form f_2^m/f_1^n, where $m.dg(f_2) = n.dg(f_1)$.

Remark 3.24. Since $n > m$, the separatrix $\sigma = (x = 0)$ is distinguished. In particular, it extends to a smooth separatrix Σ_1 of $\mathcal{F}$ along Γ. When $n > m > 1$ the other separatrix, $\sigma_2 = (y = 0)$, is also distinguished and can be extended to another separatrix, say Σ_2, of $\mathcal{F}$ along Γ.

Another fact that we would like to observe is that $f(x, y) := y^m/x^n$ is a mermorphic first integral of η. On the other hand, η has no non-constant holomorphic first integral in a neighborhood $0 \in \mathbb{C}^2$.

Fix an affine chart $(x, y, z) \in \mathbb{C}^3 \subset \mathbb{P}^3$ and a polynomial integrable 1-form Ω on $\mathbb{C}^3$ which represents $\mathcal{F}|_{\mathbb{C}^3}$. Without lost of generality, we can assume that Γ is transverse to the line at infinity $L_\infty = \mathbb{P}^3 \setminus \mathbb{C}^3$.

Construction of $\tilde{\Lambda}$ in a neighborhood of $\tilde{\Gamma}$. Let $(U_\alpha)_{\alpha\in A}$ be a covering of $\tilde{\Gamma}$ with the following properties:

(a) $U_\alpha \cap \tilde{\Gamma}$ is connected and non-empty for all $\alpha \in A$.
(b) If $U_{\alpha\beta} \neq \emptyset$ then $U_{\alpha\beta} \cap \tilde{\Gamma}$ is connected and non-empty.
(c) For all $\alpha \in A$ there is a chart $(x_\alpha, y_\alpha, z_\alpha)\colon U_\alpha \to \mathbb{C}^3$ such that $\tilde{\Gamma} \cap U_\alpha = (x_\alpha = y_\alpha = 0)$ and $\mathcal{F}|_{U_\alpha}$ is represented by $\eta_\alpha = m\, x_\alpha\, dy_\alpha - n\, y_\alpha\, dx_\alpha$.

In particular, $\Sigma_1 \cap U_\alpha = (x_\alpha = 0)$, $f_\alpha := y_\alpha^m/x_\alpha^n$ is a meromorphic first integral of $\mathcal{F}|_{U_\alpha}$ and

$$df_\alpha = \frac{y_\alpha^{m-1}}{x_\alpha^{n+1}} . \eta_\alpha \ , \ \forall \alpha \in A \ . \tag{3.16}$$

Fix $U_{\alpha\beta} \neq \emptyset$ and let $\varphi_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$ and $g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$ be such that $\eta_\alpha = \varphi_{\alpha\beta} . \eta_\beta$ and $x_\alpha = g_{\alpha\beta} . x_\beta$ on $U_{\alpha\beta}$. From (3.16) we get

$$df_\alpha = a_{\alpha\beta} . df_\beta \ , \ a_{\alpha\beta} = \frac{(y_\alpha/y_\beta)^{m-1}}{g_{\alpha\beta}^{n+1}} \varphi_{\alpha\beta} \ .$$

Note that $a_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$. In fact, if $m = 1$ this is clear. On the other hand, if $m > 1$ then by Remark 3.24, there is a separatrix Σ_2 along Γ such that

$$\Sigma_2 \cap U_{\alpha\beta} = (y_\alpha = 0) \cap U_\beta = (y_\beta = 0) \cap U_\alpha \ .$$

As a consequence, there exists $h_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$ such that $y_\alpha = h_{\alpha\beta} . y_\beta$. Hence, $a_{\alpha\beta} = h_{\alpha\beta}^{m-1} . \varphi_{\alpha\beta}/g_{\alpha\beta}^{n+1} \in \mathcal{O}^*(U_{\alpha\beta})$.

From $df_\alpha = a_{\alpha\beta}.df_\beta$ we get

$$da_{\alpha\beta} \wedge df_\beta = 0 \implies da_{\alpha\beta} \wedge \eta_\beta = 0$$

and $a_{\alpha\beta}$ is a holomorphic first integral of $\mathcal{F}$ in a neighborhood of $U_{\alpha\beta} \cap \Gamma$. This implies that $a_{\alpha\beta} \in \mathbb{C}^*$, because the normal type has no non-constant holomorphic first integral.

Given $\alpha \in A$, $\Omega|_{U_\alpha}$ and df_α represent $\mathcal{F}$ in the complement of their poles. Hence, there is a meromorphic function g_α on U_α such that $\Omega = g_\alpha . df_\alpha$. Since $df_\alpha = a_{\alpha\beta}.df_\beta$ on $U_{\alpha\beta} \neq \emptyset$, we get

$$\Omega = g_\alpha.df_\alpha = g_\alpha.a_{\alpha\beta}.df_\beta = g_\beta.df_\beta \implies g_\beta = a_{\alpha\beta}.g_\alpha \text{ , on } U_{\alpha\beta} \ .$$

Since $a_{\alpha\beta} \in \mathbb{C}^*$, we get

$$\frac{dg_\alpha}{g_\alpha} = \frac{dg_\beta}{g_\beta} \text{ , on } U_{\alpha\beta}$$

and this implies that there exists a meromorphic 1-form $\tilde{\Lambda}$ on $\tilde{U} := \bigcup_\alpha U_\alpha$ such that $\tilde{\Lambda}|_{U_\alpha} = \frac{dg_\alpha}{g_\alpha}$ for all $\alpha \in A$. Finally, $d\Omega = \tilde{\Lambda} \wedge \Omega$ because

$$d\Omega|_{U_\alpha} = dg_\alpha \wedge df_\alpha = \frac{dg_\alpha}{g_\alpha} \wedge \Omega|_{U_\alpha} = \tilde{\Lambda} \wedge \Omega|_{U_\alpha} \ .$$

Extension of $\tilde{\Lambda}$ to a neighborhood of $\Gamma \setminus \tilde{\Gamma}$. Fix $p \in \Gamma \setminus \tilde{\Gamma}$ and a local chart $(V, (u, s, t))$ around p such that $\mathcal{F}|_V$ is represented by

$$\omega = u\,du + (h.\,\zeta_1(h) + u.\,\zeta_2(h))\,dh \ , \ \ h = h(s,t)$$

and $\Gamma \cap V = (u = h(s,t) = 0)$. Choose $q \in V \cap \tilde{\Gamma}$ and a chart $(W, (u, v, w))$ around q with $W \subset V$ and $h(s,t) = v$, so that

$$\omega|_W = u\,du + (v\,\zeta_1(v) + u\,\zeta_2(v))\,dv \ .$$

Let $\alpha \in A$ be such that $q \in U_\alpha$. We can assume that $W \subset U_\alpha$. Since $\eta_\alpha|_W$ and $\omega|_W$ represent $\mathcal{F}|_W$ there is $\varphi = \varphi(u, v, w) \in \mathcal{O}^*(W)$ sucht that $\eta_\alpha = \varphi.\omega$ on W. This implies $df_\alpha|_W = h.\omega|_W$, where $h(u, v, w) = \varphi.y_\alpha^{m-1}/x_\alpha^{n+1}$ is meromorphic on W. In particular, $d(h.\,\omega|_W) = 0$, which implies $d\omega|_W = -\frac{dh}{h} \wedge \omega|_W$. Since $d\omega|_W$ do not contain terms with $du \wedge dw$ and $dv \wedge dw$, from the last relation we get

$$\frac{\partial h}{\partial w} \equiv 0 \implies h = h(u, v) \ .$$

Therefore, the closed 1-form $\theta := h(u, f(s,t)).\omega$ is meromorphic in some neighborhood U_p of p and extends df_α to this neighborhood. As before, we have $\Omega = g.\theta$, where g is meromorphic on U_p and is an extension of g_α to U_p. This implies that $\frac{dg}{g}$ extends $\tilde{\Lambda}$ to U_p. In particular, $\tilde{\Lambda}$ can be extended meromorphically to some connected neighborhood U of Γ. Finally, Theorem 3.23 implies that $\tilde{\Lambda}$ can be extended to a closed meromorphic 1-form Λ on $\mathbb{P}^3$ with $d\Omega = \Lambda \wedge \Omega$.

Poles and residues of Λ. Let $|\Lambda|_\infty$ be the set of poles of Λ. Fix $p \in \tilde{\Gamma}$ and $\alpha \in A$ such that $p \in U_\alpha$. Note that $L_\infty = \mathbb{P}^3 \setminus \mathbb{C}^3$ is a pole of Ω of order $d+2$, where $d = dg(\mathcal{F})$ (*cf.* [2]). Let $(u_\alpha = 0)$ be a reduced equation of $L_\infty \cap U_\alpha$. Since $\Omega|_{U_\alpha}$ and η_α represent $\mathcal{F}|_{U_\alpha}$ there is $\phi_\alpha \in \mathcal{O}^*(U_\alpha)$ such that

$$\Omega|_{U_\alpha} = \frac{\phi_\alpha}{u_\alpha^{d+2}} . \eta_\alpha = \frac{\phi_\alpha . x_\alpha^{n+1}}{u_\alpha^{d+2} . y_\alpha^{m-1}} . df_\alpha \implies g_\alpha = \frac{\phi_\alpha . x_\alpha^{n+1}}{u_\alpha^{d+2} . y_\alpha^{m-1}} .$$

From the above expression, we get

$$\Lambda|_{U_\alpha} = \frac{dg_\alpha}{g_\alpha} = (n+1)\frac{dx_\alpha}{x_\alpha} - (m-1)\frac{dy_\alpha}{y_\alpha} - (d+2)\frac{du_\alpha}{u_\alpha} + \frac{d\phi_\alpha}{\phi_\alpha} . \tag{3.17}$$

We have two possibilities:

1st. $1 < m < n$. In this case, $|\Lambda|_\infty \cap U_\alpha = (x_\alpha = 0) \cup (y_\alpha = 0) \cup (u_\alpha = 0)$. Since $\Sigma_1 \cap U_\alpha = (x_\alpha = 0)$ and $\Sigma_2 \cap U_\alpha = (y_\alpha = 0)$, they extend to global algebraic irreducible surfaces, which we call again Σ_1 and Σ_2, respectively. Moreover, we get $|\Lambda|_\infty \supset \Sigma_1 \cup \Sigma_2 \cup L_\infty$. We assert that $|\Lambda|_\infty = \Sigma_1 \cup \Sigma_2 \cup L_\infty$.

Let S be an irreducible component of $|\Lambda|_\infty$, $S \neq L_\infty$, and let us prove that $S \subset \Sigma_1 \cup \Sigma_2$. We assert that S is $\mathcal{F}$-invariant.

In fact, fix a smooth point $p \in S \setminus (L_\infty \cup \mathrm{sing}(\mathcal{F}))$. Consider a local chart $\psi = (x_1, x_2, x_3)\colon W \to \mathbb{C}^3$ around p such that $\psi(p) = 0$, $W \cap (L_\infty \cup \mathrm{sing}(\mathcal{F})) = \emptyset$ and $S \cap W = |\Lambda|_\infty \cap W = (x_3 = 0)$. We can write

$$\Lambda|_W = \frac{\theta}{x_3^k} , \ \theta = A_1\, dx_1 + A_2\, dx_2 + A_3\, dx_3 ,$$

where $A_i \in \mathcal{O}(W)$, $i = 1, 2, 3$, $x_3 \nmid A_i$ for some $i = 1, 2, 3$, and $k \geq 1$. From $d\Lambda = 0$, we get

$$x_3^{-k}\, d\theta - k\, x_3^{-(k+1)}\, dx_3 \wedge \theta = 0 \implies d\theta = k\,\frac{dx_3}{x_3} \wedge \theta ,$$

which implies that $x_3 \mid A_1, A_2$ and $x_3 \nmid A_3$.Therefore, we can write $\theta = x_3\,\alpha + A_3\, dx_3$, where α is holomorphic on W. Since $d\Omega = \Lambda \wedge \Omega$, we get

$$x_3^k\, d\Omega|_W = \theta \wedge \Omega|_W \implies A_3\, dx_3 \wedge \Omega|_W = x_3(x_3^{k-1} d\Omega|_W - \alpha \wedge \Omega|_W) .$$

From the last relation above, we obtain that $\frac{dx_3}{x_3} \wedge \Omega|_W := \beta$ is holomorphic. Hence, S is $\mathcal{F}$-invariant, because $dx_3 \wedge \Omega|_W = x_3\,\beta$, where β is holomorphic.

Since S is $\mathcal{F}$-invariant and $\Gamma \cap S \neq \emptyset$, S must contain some separatrix of $\mathcal{F}$ along Γ. In particular, $S \cap U_\alpha \neq \emptyset$, which implies that $S \cap U_\alpha \subset (x_\alpha = 0) \cup (y_\alpha = 0)$. Therefore, either $S = \Sigma_1$, or $S = \Sigma_2$.

Let f_1, f_2, f_3 be irreducible homogeneous polynomials on $\mathbb{C}^4$, f_3 of degree-one, such that $f_i = 0$ is an equation of Σ_i, $i = 1, 2$, and $f_3 = 0$ is an equation of L_∞ (in homogeneous coordinates). By (3.17) the residues of Λ are $n + 1$ (on Σ_1), $-(m - 1)$ (on Σ_2) and $-(d + 2)$ (on L_∞). Therefore, Λ can be written in homogeneous coordinates as dF/F, where

$$F = \frac{f_1^{n+1}}{f_2^{m-1} . f_3^{d+2}} \ .$$

2nd. $n > m = 1$. In this case, $|\Lambda|_\infty \cap U_\alpha = (x_\alpha = 0) \cup (u_\alpha = 0)$. With the same argument of the 1st case, we get $|\Lambda|_\infty = \Sigma_1 \cup L_\infty$. Let f_1, f_3 be irreducible homogeneous polynomials on $\mathbb{C}^4$, f_3 of degree-one, such that $f_1 = 0$ is an equation of Σ_1 and $f_3 = 0$ is an equation of L_∞ (in homogeneous coordinates). By (3.17) the residues of Λ are $n + 1$ (on Σ_1) and $-(d + 2)$ (on L_∞). Therefore, Λ can be written in homogeneous coordinates as dF/F, where

$$F = \frac{f_1^{n+1}}{f_3^{d+2}} \ .$$

The first integral. Let $\Pi \colon \mathbb{C}^4 \setminus \{0\} \to \mathbb{P}^3$ be the canonical projection and (x_0, x_1, x_2, x_3) be homogeneous coordinates such that $L_\infty = (f_3 = x_0 = 0)$ and the previous affine chart $\mathbb{C}^3 \subset \mathbb{P}^3$ is $(x_0 = 1)$. In this chart,

$$\Pi(x_0, x_1, x_2, x_3) = \left(\frac{x_1}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_0} \right) \ .$$

Since $dg(\mathcal{F}) = d$ we can write $\Pi^*(\Omega) = \frac{1}{x_0^{d+2}}\, \omega$, where the coefficients of ω are homogenous of degree $d + 1$ and $i_R(\omega) = 0$, $R = \sum_i x_i \partial_{x_i}$. If $m = 1$ we set $f_2^{m-1} := 1$. With this convention, we can write $F = \frac{f_1^{n+1}}{f_2^{m-1} . x_0^{d+2}}$. On the other hand, the relation $d\Omega = \frac{dF}{F} \wedge \Omega$ is equivalent to $d(F^{-1}\, \Omega) = 0$, and so the form

$$\mu := \frac{\Omega}{F} = \frac{f_2^{m-1}\, \omega}{f_1^{n+1}}$$

is closed. Since it is closed and its pole divisor is f_1^{n+1}, it can written as

$$\mu = \lambda \, \frac{df_1}{f_1} + d \left(\frac{h}{f_1^n} \right)$$

where $\lambda \in \mathbb{C}$, h is a homogeneous polynomial and $dg(h) = n \, dg(f_1)$.

Since $0 = i_R(\mu) = \lambda\, dg(f_1)$, we get $\lambda = 0$. It follows that h/f_1^n is a rational first integral of $\mathcal{F}$. If $m > 1$ then $\Sigma_2 = (f_2 = 0)$ is $\mathcal{F}$-invariant. Hence, there exists $b \in \mathbb{C}$ such that $(f_2 = 0) \subset (h + b\, f_1^n = 0)$. In particular, there exist $k \in \mathbb{N}$ and a homogeneous polynomial g such that $g.f_2^k = h + b\, f_1^n$, where $f_1, f_2 \nmid g$ and $dg(g) + k\, dg(f_2) = n\, dg(f_1)$. This implies

$$\frac{f_2^{m-1}}{f_1^{n+1}}\,\omega = d\left(\frac{h}{f_1^n}\right) = d\left(\frac{g.f_2^k}{f_1^n}\right)$$
$$\Longrightarrow\ f_2^{m-1}\,\omega = f_2^{k-1}\,(f_1\, f_2\, dg + k\, f_1\, g\, df_2 - n\, g\, f_2\, df_1) \ \Longrightarrow\ m = k$$

and g is a constant, because otherwise in a point $q \in (g = f_1 = f_2 = 0) \cap \Gamma$ we would have $j_q^1(\omega) > 1$. This implies that $\mathcal{F}$ has a first integral of the form f_2^m/f_1^n. When $m = 1$, we have that h is irreducible and we take $f_2 = h$. This finishes the proof of Theorem 1.2 in dimension three.

3.3. Proof of Theorem 1.2 in dimension $n \geq 4$

The idea is to use the case of dimension three and the following known result (*cf.* [8]):

Theorem 3.25. *Let $\mathcal{G}$ be a codimension-one holomorphic foliation on $\mathbb{P}^n$, $n \geq 3$. Assume that there is a k-plane $E \simeq \mathbb{P}^k$, $2 \leq k < n$ such that E is in general position with $\mathcal{G}$ and $\mathcal{G}|_E$ is represented by a closed meromorphic 1-form ω on E outside its poles. Then ω can be extended to a closed meromorphic 1-form Ω on $\mathbb{P}^n$ representing $\mathcal{G}$ outside its poles. In particular, if $\mathcal{G}|_E$ has a rational first integral then it can be extended to rational first integral of $\mathcal{G}$.*

Recall that E is in general position with $\mathcal{G}$ if:

(a) E is not $\mathcal{G}$-invariant.
(b) The divisor of tangencies between $\mathcal{G}$ and E has codimension at least 2 in E.

Moreover, the set of k-planes in general position with $\mathcal{G}$ is a Zariski open and dense subset of the respective grassmanian (*cf.* [8]).

Let $\mathcal{F}$ be a codimension-one foliation on $\mathbb{P}^n$, $n \geq 4$, such that $\mathrm{sing}_2(\mathcal{F})$ has an irreducible component Γ with $\mathrm{BB}(\mathcal{F}, \Gamma) \neq 0$ and the set $X := \{p \in \Gamma \mid \mathcal{J}(\mathcal{F}, p) > 1\}$ has codimension at least 4 in $\mathbb{P}^n$. Set $\mathcal{N}_\Gamma = \{p \in \Gamma \mid p$ is a nilpotent singularity of $\mathcal{F}\}$ and $K_\Gamma = \{p \in \Gamma \mid p$ is a singularity of Kupka type of $\mathcal{F}\}$. Since $\mathrm{cod}_{\mathbb{P}^n}(X) \geq 4$ and $\mathrm{cod}_{\mathbb{P}^n}(\Gamma) = 2$, we have $\Gamma = \mathcal{N}_\Gamma \cup K_\Gamma \cup X$ and

- Either $\Gamma = \mathcal{N}_\Gamma \cup X$, or K_Γ is a Zariski open and dense subset of Γ.

When $\mathcal{N}_\Gamma \cup X = \emptyset$ then $\Gamma \subset K(\mathcal{F})$ and so Theorem 1.2 is true by [6, 11] and [3]. Therefore, from now on we will assume that $\mathcal{N}_\Gamma \cup X \neq \emptyset$. In view of Theorem 3.25, the next result will reduce the problem to the case $n = 3$.

Lemma 3.26. *In the above situation, there is a $(n-1)$-plane $\mathbb{P}^{n-1} \simeq E \subset \mathbb{P}^n$ in general position with $\mathcal{F}$ and such that:*

(a) $\Gamma \cap E \subset \mathrm{sing}_2(\mathcal{F}|_E)$.
(b) *The set* $X_E := \{p \in \Gamma \cap E \mid \mathcal{J}(\mathcal{F}|_E, p) > 1\}$ *has codimension at least* 4 *in* E.
(c) *If* Γ' *is an irreducible component of* $\Gamma \cap E$ *then* $\mathrm{BB}(\mathcal{F}|_E, \Gamma') \neq 0$.

Proof. Fix an affine chart $(z_1, ..., z_n) \in \mathbb{C}^n \subset \mathbb{P}^n$ and a polynomial 1-form Ω representing $\mathcal{F}$ in this chart. Given $p \in \mathbb{C}^n \cap \mathcal{N}_\Gamma$ there is $\ell_p \in \mathbb{C}[z_1, ..., z_n]$, of degree-one, such that $\ell_p(p) = 0$ and

$$j^1_p(\Omega) = \ell_p \, d\ell_p \ .$$

Note that the hyperplane $H_p = \overline{(\ell_p = 0)} \in \check{\mathbb{P}}^n$ does not depend on the affine chart containing p. As a consequence, the correspondence $p \mapsto H_p$ defines an analytic map $H : \mathcal{N}_\Gamma \to \check{\mathbb{P}}^n$. Since $\dim(\mathcal{N}_\Gamma) \leq n-2$, we get $\dim(H(\mathcal{N}_\Gamma)) \leq n-2$. In particular, the set

$$A := \check{\mathbb{P}}^n \setminus \overline{H(\mathcal{N}_\Gamma)}$$

is a Zariski open and dense subset of $\check{\mathbb{P}}^n$. Let $B = \{E \in A \mid E$ is in general position with $\mathcal{F}\}$.

Note that B is a Zariski open and dense subset of $\check{\mathbb{P}}^n$. Moreover, if $E \in B$ then all points of $\mathcal{N}_\Gamma \cap E$ are nilpotent singularities of $\mathcal{F}|_E$. In fact, fix $p \in \mathcal{N}_\Gamma \cap E$, an affine coordinate system $z = (z_1, ..., z_n) \in \mathbb{C}^n \subset \mathbb{P}^n$ and a polynomial 1-form Ω representing $\mathcal{F}$ in this chart, such that $z(p) = 0$ and $E \cap \mathbb{C}^n = (z_n = 0)$. Let ℓ_p be a degree-one polynomial with $\ell_p(p) = 0$, $H_p \cap \mathbb{C}^n = (\ell_p = 0)$ and $j^1_p(\Omega) = \ell_p \, d\ell_p$. Since $\ell_p(0) = 0$ and $E \neq H_p$, we can set $\ell_p(z) = \sum_{j=1}^n a_j z_j$, where $a_j \neq 0$ for some $j \in \{1, ..., n-1\}$. The polynomial $\tilde{\ell}_p := \ell_p|_{E \cap \mathbb{C}^n}$ is non-constant. In particular,

$$j^1_0(\Omega|_E) = \tilde{\ell}_p \, d\tilde{\ell}_p \neq 0 \ .$$

Therefore, p is a nilpotent singularity of $\mathcal{F}|_E$.

Now, consider an algebraic stratification $\mathrm{sing}(\mathcal{F}) := S_0 \supset S_1 \supset ... \supset S_r = \emptyset$, where $\dim(S_0) = n-2$, $\dim(S_{j+1}) < \dim(S_j)$ and $S_j \setminus S_{j+1}$ is a smooth manifold, for all $0 \leq j < r$. By transversality theory, there exists $E \in B$ transverse to all manifolds $S_j \setminus S_{j+1}$, $0 \leq j < r$. We assert that E satisfies properties (a), (b) and (c).

In fact, since $\Gamma \subset \mathrm{sing}_2(\mathcal{F})$ we must have $\Gamma \setminus S_1 \neq \emptyset$, and so $\mathrm{cod}(\Gamma \cap E) = 2$, which implies (a), because $\Gamma \cap E \subset \mathrm{sing}(\mathcal{F}|_E)$. On the other hand, since K_Γ is smooth of codimension-two, we get $K_\Gamma \subset S_0 \setminus S_1$. In particular, E is transverse to K_Γ and this implies that $K_\Gamma \cap E \subset K(\mathcal{F}|_E)$. Therefore, $\mathcal{J}(\mathcal{F}|_E, p) \leq 1$ for all $p \in (\Gamma \setminus X) \cap E$. This implies also that $X_E = X \cap E$. Since $X \subset S_1$, by transversality we get $\mathrm{cod}_E(X_E) \geq 4$.

Finally, if Γ' is an irreducible component of $\Gamma \cap E$ then $\mathrm{BB}(\mathcal{F}|_E, \Gamma')$ can be computed in any dimension two transverse section, say Λ, through any point in the smooth part of $\Gamma \cap E$. If we take such a point in the smooth part of Γ then we see that Λ is also transverse to Γ at this point, which implies

$$\mathrm{BB}(\mathcal{F}|_E, \Gamma') = \mathrm{BB}(\mathcal{F}, \Gamma) \neq 0 \ . \qquad \square$$

By using Lemma 3.26 inductively $n-3$ times we get:

Corollary 3.27. *In the situation of Lemma* 3.26 *there is a 3-plane* $\mathbb{P}^3 \simeq E \subset \mathbb{P}^n$*, in general position with* $\mathcal{F}$*, with* $\mathcal{J}(\mathcal{F}|_E, p) \leq 1$*, for all* $p \in \Gamma \cap E$*, and* $\mathrm{BB}(\mathcal{F}|_E, \Gamma') \neq 0$*, for all irreducible components of* Γ' *of* $\Gamma \cap E$.

In particular, $\mathcal{F}|_E$ has a rational first integral of the form f_1^m/f_2^n, where $gcd(m,n) = 1$, $1 \leq m < n$, $m\,dg(f_1) = n\,dg(f_2)$ and f_1, f_2 are irreducible. By Theorem 3.25 this first integral can be extended to a rational first integral of $\mathcal{F}$. This finshes the proof of Theorem 1.2. $\square$

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