Quantitative uniqueness estimates for the shallow shell system and their application to an inverse problem

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Abstract. In this paper we derive some quantitative uniqueness estimates for the shallow shell equations. Our proof relies on appropriate Carleman estimates. For applications, we consider the size estimate inverse problem.

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1. Introduction

In this work we study a quantitative uniqueness for the shallow shell system and its application to the inverse problem of estimating the size of an embedded inclusion by boundary measurements. To begin, we let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \). Without loss of generality, we assume \( 0 \in \Omega \). Let \( \tilde{\theta} : \overline{\Omega} \to \mathbb{R} \) satisfy an appropriate regularity assumption which will be specified later. For a shallow shell, its middle surface is described by \( \{(x_1, x_2, \varepsilon \rho_0 \tilde{\theta}(x_1, x_2)) : (x_1, x_2) \in \overline{\Omega}\} \) for \( \varepsilon > 0 \), where \( \rho_0 > 0 \) is the characteristic length of \( \Omega \) (see Section 3.1). From now on, we set \( \theta = \rho_0 \tilde{\theta} \). Let \( u = (u_1, u_2, u_3) = (u', u_3) : \Omega \to \mathbb{R}^3 \) represent the displacement vector of the middle surface. Then \( u \) satisfies the following equations:

\[
\begin{cases}
-\partial_j n^\theta_{ij}(u) = 0 & \text{in } \Omega, \\
\partial_j^2 m_{ij}(u_3) - \partial_j(n^\theta_{ij}(u)\partial_i \theta) = 0 & \text{in } \Omega,
\end{cases}
\tag{1.1}
\]

where

\[
\begin{align*}
    m_{ij}(u_3) &= \rho_0^2 \left\{ \frac{4\lambda \mu}{3(\lambda + 2\mu)} (\Delta u_3) \delta_{ij} + \frac{4\mu}{3} \partial^2_{ij} u_3 \right\}, \\
    n^\theta_{ij}(u) &= \frac{4\lambda \mu}{\lambda + 2\mu} e^\theta_{kk}(u) \delta_{ij} + 4\mu e^\theta_{ij}(u), \\
    e^\theta_{ij}(u) &= \frac{1}{2} (\partial_i u_j + \partial_j u_i + (\partial_i \theta) \partial_j u_3 + (\partial_j \theta) \partial_i u_3).
\end{align*}
\]

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and $\lambda$, $\mu$ are Lamé coefficients. Hereafter, the Roman indices (except $n$) belong to $\{1, 2\}$ and the Einstein summation convention is used for repeated indices.

Assume that $D$ is a measurable subdomain of $\Omega$ with $\overline{D} \subset \Omega$. We consider Lamé parameters

$$\tilde{\lambda} = \lambda + \chi_D \lambda_0 \quad \text{and} \quad \tilde{\mu} = \mu + \chi_D \mu_0,$$

where $\chi_D$ is the characteristic function of $D$. The domain $D$ represents the inclusion inside of $\Omega$. With such parameters $\tilde{\lambda}$, $\tilde{\mu}$, we denote the displacement field $\tilde{u} = (\tilde{u}', \tilde{u}_3)'$ satisfying (1.1) and the Neumann boundary conditions on $\partial \Omega$:

$$\begin{cases}
\tilde{m}_{ij}^0 v_j = \rho_0^{-1} \tilde{T}_i, \\
\tilde{m}_{ij} v_j = \tilde{M}_v, \\
(\partial_t \tilde{m}_{ij} - \tilde{m}_{ij}^0 \partial_t \theta) v_j + \partial_s (\tilde{m}_{ij} v_j \tau_j) = - \partial_s \tilde{M}_\tau,
\end{cases} \quad (1.3)$$

where $\tilde{m}_{ij} = \tilde{m}_{ij}(\tilde{u}_3)$ and $\tilde{m}_{ij}^0 = \tilde{m}_{ij}^0(\tilde{u})$ are defined in (1.2) with $\lambda$, $\mu$, $u$ replaced by $\tilde{\lambda}$, $\tilde{\mu}$, $\tilde{u}$. Hereafter, $v = (v_1, v_2)$, $\tau = (\tau_1, \tau_2)$ are, respectively, the normal and the tangent vectors along $\partial \Omega$, and $s$ is the arclength parameter of $\partial \Omega$. Precisely, the tangent vector $\tau$ is obtained by rotating $v$ counterclockwise of angle $\pi/2$. The boundary field $M = \tilde{M}_\tau v + \tilde{M}_v \tau$, i.e., $M_\tau = \tilde{M} \cdot \tau$ and $M_v = \tilde{M} \cdot \nu$. We remark that in the plate theory, $\tilde{M}_\tau$ and $\tilde{M}_v$ are the twisting and bending moments applied on $\partial \Omega$. The field $\tilde{T}$ satisfies the compatibility condition which will be specified in the following section. An interesting inverse problem is to determine geometric information on $D$ from a pair $\{\tilde{T}, \tilde{M}; \tilde{u}'|_{\partial \Omega}, (\tilde{u}_3)|_{\partial \Omega}, \partial_\nu \tilde{u}_3|_{\partial \Omega}\}$, i.e., from the Cauchy data of the solution $\tilde{u}$. Despite its practical value, the fundamental global uniqueness, even for the scalar equation, is yet to be proved. For the development of the uniqueness issue for this kind of inverse problems, we refer to [13] and references therein for details.

In this paper we are interested in estimating the size of the area of $D$ in terms of the Cauchy data of $\tilde{u}$. This type of problem has been studied for the scalar equation and for systems of equations such as the isotropic elasticity and plate. We refer to the survey article [3] for the early developments and [20, 21] for the latest results on the plate equations. Specifically, the size of $D$ is estimated by the following two quantities:

$$\tilde{W} = \int_{\partial \Omega} \rho_0^{-1} \tilde{T} \cdot \tilde{u}' + \tilde{M}_v \partial_\nu \tilde{u}_3 + \partial_s \tilde{M}_\tau \tilde{u}_3$$

and

$$W = \int_{\partial \Omega} \rho_0^{-1} \tilde{T} \cdot u' + \tilde{M}_v \partial_\nu u_3 + \partial_s \tilde{M}_\tau u_3,$$

where $u = (u', u_3)'$ is the displacement vector satisfying (1.1) and (1.3) with $D = \emptyset$, i.e., $\tilde{\lambda} = \lambda$ and $\tilde{\mu} = \mu$. Here we assume that $\lambda$, $\mu$ are given a priori, thus, both $\tilde{W}$ and $W$ are known. To be more precise, in this paper, we will show that under some
\textit{a priori} assumptions, there exist positive constants $C_1$, $C_2$ such that

$$C_1 \left| \frac{\tilde{W} - W}{W} \right| \leq \text{area}(D) \leq C_2 \left| \frac{\tilde{W} - W}{W} \right|,$$

where $C_1$, $C_2$ depend on the \textit{a priori} data.

The derivation of the volume bounds on $D$ relies on the following integral inequalities

$$\frac{1}{K} \int_D \sum_{ij} |e_{ij}^\theta(u)|^2 + \rho_0^2 |\partial_{ij} u_3|^2 \leq |W - \tilde{W}| \leq K \int_D \sum_{ij} |e_{ij}^\theta(u)|^2 + \rho_0^2 |\partial_{ij} u_3|^2,$$

where the constant $K$ depends on the \textit{a priori} data. The lower bound for $\text{area}(D)$ is a consequence of the second inequality of (1.5) and the elliptic regularity estimate for $u$. To derive the upper bound for $\text{area}(D)$, we shall use the first inequality of (1.5). As indicated in all previous related results, we need to estimate $\int_D \sum_{ij} |e_{ij}^\theta(u)|^2 + \rho_0^2 |\partial_{ij} u_3|^2$ from below. This can be achieved by the quantitative uniqueness estimates of solutions $u$ solving (1.1), which is one of the themes of the paper.

For the second order elliptic operator, using the Carleman or the frequency functions methods, quantitative estimates for the strong unique continuation under different assumptions on coefficients were derived in [8–11, 14, 16, 18]. For the isotropic elasticity, similar estimates can be found in [1, 4, 19]. Further, for the elastic plate, quantitative uniqueness estimates were derived in [20, 21]. Note that global versions of quantitative uniqueness estimates, in the form of doubling inequality, were given in [4] and [21], where their arguments rely on a local version for the power of Laplacian derived in [17].

In this paper, we will derive three-ball inequalities and doubling inequalities for the shallow shell system (1.1) with $\lambda, \mu \in C^{1,1}(\Omega)$. Since the first and the second equations in (1.1) have different orders, it seems that the Carleman method is the most efficient way to derive those quantitative uniqueness estimates for (1.1). We will give detailed derivations of quantitative uniqueness estimates based on the Carleman estimates in Section 4. The investigation of the inverse problem is given in Section 5. Since the Neumann boundary value problem for (1.1) is not standard, we will first study this forward problem in Section 3.

\section{Notation}

\textbf{Definition 2.1.} Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $n \geq 2$. Given $k \in \mathbb{Z}^+$, we say that $\partial \Omega$ is of class $C^{k,1}$ with constants $\rho_0$, $A_0$, if, for any point $z \in \partial \Omega$, there
exists a rigid coordinate transformation under which $z = 0$ and
\[
\Omega \cap B_{\rho_0}(0) = \{ x = (x_1, \ldots, x_{n-1}, x_n) = (x', x_n) \in B_{\rho_0}(0) : x_n > \varphi(x') \},
\]
where $\varphi(x')$ is a $C^{k,1}$ function on $B'_{\rho_0}(0) = B_{\rho_0}(0) \cap \{ x_n = 0 \}$ satisfying $\varphi(0) = 0$ and $\nabla \varphi(0) = 0$ if $k \geq 1$ and
\[
\| \varphi \|_{C^{k,1}(B'_{\rho_0}(0))} \leq A_0 \rho_0.
\]
Throughout the paper, we will normalize all norms such that they are dimensionally homogeneous and coincide with the standard definitions when the dimensional parameter is one. With this in mind, we define
\[
\| \varphi \|_{C^{k,1}(B'_{\rho_0}(0))} = \sum_{j=0}^{k} \rho_0^j \| \nabla^j \varphi \|_{L^\infty(B'_{\rho_0}(0))} + \rho_0^{k+1} \| \nabla^{k+1} \varphi \|_{L^\infty(B'_{\rho_0}(0))}.
\]
Similarly, when $\Omega$ with $\partial \Omega$ defined above and $w : \Omega \to \mathbb{R}$, we define
\[
\| w \|_{C^{k,1}(\Omega)} = \sum_{j=0}^{k} \rho_0^j \| \nabla^j w \|_{L^\infty(\Omega)} + \rho_0^{k+1} \| \nabla^{k+1} w \|_{L^\infty(\Omega)},
\]
\[
\| w \|_{L^2(\Omega)}^2 = \rho_0^{-n} \int_{\Omega} w^2,
\]
\[
\| w \|_{H^k(\Omega)}^2 = \rho_0^{-n} \sum_{j=0}^{k} \rho_0^{2j} \int_{\Omega} |\nabla^j w|^2, \quad k \geq 1.
\]
In particular, if $\Omega = B_{\rho}(0)$, then $\Omega$ satisfies Definition 2.1 with $\rho_0 = \rho$.

Let $\mathcal{A}$ be an open connected component of $\partial \Omega$. For any given point $z_0 \in \mathcal{A}$, we define the positive orientation of $\mathcal{A}$ associated with an arclength parametrization $\zeta(s) = (x_1(s), x_2(s)), s \in [0, \text{length}(\mathcal{A})]$ such that $\zeta(0) = z_0$ and $\zeta'(s) = \tau(\zeta(s))$. Finally, we define for any $h > 0$
\[
\Omega_h = \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) > h \}.
\]

3. The forward problem

3.1. The Neumann boundary value problem for the shallow shell equation

At this moment, we assume $\partial \Omega \in C^{1,1}$ with constants $A_0$, $\rho_0$. Also, let $\Omega$ satisfy
\[
|\Omega| \leq A_1 \rho_0^2
\]
throughout the article, and
\[
\| \nabla \theta \|_{L^\infty(\Omega)} = \rho_0 \| \nabla \tilde{\theta} \|_{L^\infty(\Omega)} \leq A_2
\]
for some positive constants $A_1$ and $A_2$. We will investigate the Neumann boundary value problem, the forward problem, for the shallow shell system. To begin, let us assume that Lamé coefficients $\lambda, \mu \in L^\infty(\Omega)$ satisfying
\begin{equation}
0 < \delta_0 \leq \mu(x), \quad \delta_0 \leq \lambda(x), \quad \forall \; x \in \Omega. \tag{3.3}
\end{equation}

We aim to find $u = (u_1, u_2, u_3) = (u', u_3)$ satisfying
\begin{equation}
\begin{cases}
-\partial_j n_{ij}^\theta(u) = 0 & \text{in } \Omega, \\
\partial_j^2 m_{ij} + \partial_j (n_{ij}^\theta(u) \partial_i \theta) = 0 & \text{in } \Omega,
\end{cases} \tag{3.4}
\end{equation}
with boundary conditions
\begin{equation}
\begin{cases}
n_{ij}^\theta(u) v_j = \rho_0^{-1} \hat{T}_i, \\
m_{ij}(u_3) v_i v_j = \hat{M}_v, \\
(\partial_i m_{ij}(u_3) - n_{ij}^\theta(u) \partial_i \theta) v_j + \partial_s (m_{ij}(u_3) v_i \tau_j) = -\partial_s \hat{M}_\tau.
\end{cases} \tag{3.5}
\end{equation}

Now assume that $u = (u', u_3)$ satisfies (3.4)-(3.5). Let $v = (v', v_3) \in (H^1(\Omega))^2 \times H^2(\Omega)$, then multiplying the first and second equations of (3.4) by $v'$ and $v_3$, respectively, and using the standard integration by parts, we can obtain that
\begin{equation}
\int_\Omega \sum_{ij} (n_{ij}^\theta(u) e_{ij}^\theta(v) + m_{ij}(u_3) \partial_j^2 v_3) = \int_{\partial\Omega} \rho_0^{-1} \hat{T} \cdot v' + \partial_s \hat{M}_\tau v_3 + \hat{M}_v \partial_v v_3. \tag{3.6}
\end{equation}

The boundary field $\hat{M} = \hat{M}_\tau v + \hat{M}_v \tau$ in the cartesian coordinates is written as
\begin{equation}
\hat{M} = \hat{M}_1 e_2 + \hat{M}_2 e_1.
\end{equation}

In view of the relation
\begin{equation}
\partial_s \hat{M}_\tau v_3 = \partial_s (\hat{M}_\tau v_3) - \hat{M}_\tau \partial_s v_3,
\end{equation}

one can see that the right-hand side of (3.6) becomes
\begin{equation}
\int_{\partial\Omega} \rho_0^{-1} \hat{T} \cdot v' - \hat{M}_\tau \partial_s v_3 + \hat{M}_v \partial_v v_3.
\end{equation}

Recall that $\partial_j v_3 = \partial_s v_3 \tau_j + \partial_v v_3 v_j$ for $j = 1, 2$. Using the relation $\tau = (-v_2, v_1)$ if $v = (v_1, v_2)$, we get that
\begin{align*}
\hat{M}_1 \partial_1 v_3 - \hat{M}_2 \partial_2 v_3 &= \hat{M}_1 (\partial_s v_3 \tau_1 + \partial_v v_3 v_1) - \hat{M}_2 (\partial_s v_3 \tau_2 + \partial_v v_3 v_2) \\
&= (\hat{M}_1 \tau_1 - \hat{M}_2 \tau_2) \partial_s v_3 + (\hat{M}_1 v_1 - \hat{M}_2 v_2) \partial_v v_3 \\
&= -\hat{M}_\tau \partial_s v_3 + \hat{M}_v \partial_v v_3
\end{align*}
In view of the above computations, we deduce that
\[
\int_\Omega \sum_{ij} (n^\theta_{ij}(u)e^\theta_{ij}(v) + m_{ij}(u_3)\partial^2_{ij}v_3) = \int_{\partial\Omega} \rho_0^{-1} \hat{T} \cdot v' + \hat{M}_1 \partial_1 v_3 - \hat{M}_2 \partial_2 v_3 \quad (3.7)
\]
(see the similar derivation for the plate equation in [20]). Let \( v' = a + W \cdot x + b\theta \) and \( v_3 = c - b \cdot x \), where \( a = (a_1, a_2) \), \( b = (b_1, b_2) \) are two-dimensional vectors, \( W \) is a \( 2 \times 2 \) skew-symmetric matrix, and \( c \) is a scalar. Then \( e^\theta_{ij}(v) = \partial^2_{ij}v_3 = 0 \) for all \( i, j \). Thus, to solve (3.4) and (3.5), the pair \( (\hat{T}, \hat{M}) \) must satisfy the compatibility condition
\[
\int_{\partial\Omega} \rho_0^{-1} \hat{T} \cdot (a + W \cdot x + b\theta) - b_1 \hat{M}_1 + b_2 \hat{M}_2 = 0. \quad (3.8)
\]
Note that taking \( b = 0 \), we have the usual compatibility condition for the traction of the elasticity equation, i.e.,
\[
\int_{\partial\Omega} \hat{T} \cdot (a + W \cdot x) = 0.
\]
On the other hand, to guarantee uniqueness for the forward problem, we impose the following normalization conditions
\[
\int_{\Omega} u = 0, \quad \int_{\Omega} \nabla u_3 = 0, \quad \int_{\Omega} (\partial_1 u_2 - \partial_2 u_1) + (\partial_1 \theta \partial_2 u_3 - \partial_2 \theta \partial_1 u_3) = 0. \quad (3.9)
\]
To solve the forward problem, the following Poincaré-Korn inequality is very important.

**Proposition 3.1.** There exists an absolute constant \( C > 0 \), depending on \( A_0, A_1, A_2 \), such that for all \( u = (u', u_3) \in (H^1(\Omega))^2 \times H^2(\Omega) \) satisfying (3.9) we have
\[
\|u'\|^2_{H^1(\Omega)} + \|u_3\|^2_{H^2(\Omega)} \leq C \int_{\Omega} \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij}u_3|^2. \quad (3.10)
\]

**Proof.** The inequality (3.10) is a combination of Poincaré’s and Korn’s inequalities. By abuse of notation, the variable \( x \) in our proof stands for \( (x_1, x_2, x_3) = (x', x_3) \). Now let \( \Gamma = \Omega \times (-\rho_0, \rho_0) \subset \mathbb{R}^3 \) and introduce new variables \( \tilde{x}' = x' \) and \( \tilde{x}_3 = x_3 + \theta(x') \). Denote by \( \Gamma \) the domain of \( \Omega \) under the coordinate transformation \( x \mapsto \tilde{x} \), i.e., \( \tilde{\Gamma} = \Omega \times (-\rho_0 + \theta, \rho_0 + \theta) \). Both domains \( \Gamma \) and \( \tilde{\Gamma} \) are clearly Lipschitz. On \( \tilde{\Gamma} \), we have the standard Korn’s inequality: there exists a constant \( K_0 > 0 \) such that for any 3 vector \( v \in H^1(\tilde{\Gamma}) \) satisfying
\[
\int_{\tilde{\Gamma}} v d\tilde{x} = 0, \quad \int_{\tilde{\Gamma}} (\nabla_{\tilde{x}} v - (\nabla_{\tilde{x}} v)^T) d\tilde{x} = 0, \quad (3.11)
\]
we have
\[
\rho_0^{-2} \|v\|^2_{L^2(\tilde{\Gamma})} + \|\nabla_{\tilde{x}} v\|^2_{L^2(\tilde{\Gamma})} \leq K_0 \|\nabla_{\tilde{x}} v\|^2_{L^2(\tilde{\Gamma})}, \quad (3.12)
\]
where \( \hat{\nabla}_x v = (\nabla_x v + (\nabla_x v)^t)/2 \) and \( K_0 \) depends on \( A_0, A_1, A_2 \). Let \( w(x) = w(x_1, x_2, x_3) \in H^1(\Gamma) \), then \( v(\tilde{x}) := w(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 - \theta(\tilde{x}')) \in H^1(\tilde{\Gamma}) \). By observing that the Jacobian of the coordinate transformation \( x \mapsto \tilde{x} \) is 1, we can write (3.11), (3.12) in terms of \( x \) and get that for all \( w \in H^1(\Gamma) \) satisfying

\[
\begin{align*}
\int_{\Gamma} wd x &= 0, \\
\int_{\Gamma} (\partial_1 w_2 - \partial_1 \theta \partial_3 w_2 - \partial_2 w_1 + \partial_2 \theta \partial_3 w_1) d x &= 0, \\
\int_{\Gamma} (\partial_1 w_3 - \partial_1 \theta \partial_3 w_3 - \partial_3 w_1) d x &= 0, \\
\int_{\Gamma} (\partial_2 w_2 - \partial_2 \theta \partial_3 w_2 - \partial_3 w_2) d x &= 0,
\end{align*}
\]

we have

\[
\rho_0^{-2}\|w\|_{L^2(\Gamma)}^2 + \|\nabla^\theta_x w\|_{L^2(\Gamma)}^2 \leq K_0 \|\hat{\nabla}_x^\theta w\|_{L^2(\Gamma)}^2,
\]

where

\[
\nabla^\theta_x w = \begin{pmatrix}
(\partial_1 - \partial_1 \theta \partial_3)w_1 & (\partial_1 - \partial_1 \theta \partial_3)w_2 & (\partial_1 - \partial_1 \theta \partial_3)w_3 \\
(\partial_2 - \partial_2 \theta \partial_3)w_1 & (\partial_2 - \partial_2 \theta \partial_3)w_2 & (\partial_2 - \partial_2 \theta \partial_3)w_3 \\
\partial_3 w_1 & \partial_3 w_2 & \partial_3 w_3
\end{pmatrix}
\]

and the symmetric part \( \hat{\nabla}_x^\theta w \) of \( \nabla^\theta_x w \) is defined similarly. In fact, by the form of \( \nabla^\theta_x w \), (3.14) can be improved to

\[
\rho_0^{-2}\|w\|_{L^2(\Gamma)}^2 + \|\nabla_x w\|_{L^2(\Gamma)}^2 \leq K_1 \|\hat{\nabla}_x^\theta w\|_{L^2(\Gamma)}^2
\]

for some constant \( K_1 \), also depending on \( A_0, A_1, A_2 \). Now let \( u = (u', u_3)^t \in (H^1(\Omega))^2 \times H^2(\Omega) \), we apply (3.13) and (3.15) to

\[
w(x) = (u_1(x') - x_3 \partial_1 u_3(x'), u_2(x') - x_3 \partial_2 u_3(x'), u_3(x')),
\]

where \((x', x_3) \in \Omega \times (-\rho_0, \rho_0)\). It is easy to check that the constraints (3.13) are reduced to the normalization conditions (3.9). On the other hand, easy computations show that (3.15) becomes

\[
\rho_0 \int_{\Omega} \left( \rho_0^{-2}|u|^2 + |\nabla u|^2 + \rho_0^2 \sum_{ij} \partial^2_{ij} u_3^2 \right) \leq C \int_{\Omega} \sum_{ij} \left( \rho_0 |e_{ij}^\theta(u)|^2 + \rho_0^3 |\partial^2_{ij} u_3|^2 \right)
\]

with \( C \) only depending on \( A_0, A_1, A_2 \).

\( \square \)

### 3.2. Existence and uniqueness

We will use the variational method to solve the forward problem. This seems to be standard. But we could not find any literature discussing the Neumann boundary value problem for the shallow shell. For the sake of completeness, we give a proof
of this forward problem. The arguments used here are adapted from [20]. To begin, let us introduce
\[ H(u, v) = \int_\Omega \sum_{ij} n_{ij}^\theta (u) \partial_j v_i + m_{ij}(u_3) \partial^2_{ij} v_3 + n_{ij}^\theta (u) \partial_i \theta \partial_j v_3 \]
\[ = \int_\Omega \sum_{ij} n_{ij}^\theta (u) e_{ij}^\theta (v) + m_{ij}(u_3) \partial^2_{ij} v_3 \]
and
\[ L(v) = \int_{\partial \Omega} \rho_0^{-1} \tilde{T} \cdot v' + \partial_3 \tilde{M}_\tau v_3 + \tilde{M}_v \partial_v v_3. \]
We now give a weak formulation of the Neumann boundary value problem (3.4)-(3.5).

**Definition 3.2.** A vector valued function \( u = (u', u_3)' \in (H^1(\Omega))^2 \times H^2(\Omega) \) is a weak solution to (3.4)-(3.5) if and only if
\[ H(u, v) = L(v) \quad \text{for all } v = (v', v_3)' \in H^1(\Omega) \times H^2(\Omega). \]

From the above computations, we know that
\[ L(v) = \int_{\partial \Omega} \rho_0^{-1} \tilde{T} \cdot v' + \tilde{M}_1 \partial_1 v_3 - \tilde{M}_2 \partial_2 v_3 := \tilde{L}(v). \]
In other words, (3.16) is equivalent to
\[ H(u, v) = \tilde{L}(v) \quad \text{for all } v = (v', v_3)' \in (H^1(\Omega))^2 \times H^2(\Omega). \]

**Theorem 3.3.** Assume that \( \theta \) satisfies (3.2) and \( \lambda, \mu \in L^\infty(\Omega) \) satisfy (3.3). Given any boundary field \((\tilde{T}, \tilde{M}) \in H^{-1/2}(\partial \Omega) \) and the compatibility condition (3.8) holds. Then (3.4)-(3.5) admits a unique weak solution \( u = (u', u_3)' \) satisfying the conditions (3.9) and
\[ \|u'\|_{H^1(\Omega)} + \|u_3\|_{H^2(\Omega)} \leq C \|H(\tilde{T}, \tilde{M})\|_{(H^{-1/2}(\partial \Omega))^3}, \]
where \( C \) depends on \( A_0, A_1, A_2, \delta_0. \)

**Proof.** Let \( V \) be the subspace of \((H^1(\Omega))^2 \times H^2(\Omega) \) characterized by
\[ V = \{w = (w', w_3) \in (H^1(\Omega))^2 \times H^2(\Omega) : w \text{ satisfies (3.9)}\}. \]
In view of (3.10), we have that
\[ \int_\Omega \sum_{ij} |e_{ij}^\theta (u)|^2 + \rho_0^2 |\partial_{ij}^2 u_3|^2 \leq \|u'\|^2_{H^1(\Omega)} + \|u_3\|^2_{H^2(\Omega)} \]
\[ \leq C \int_\Omega \sum_{ij} |e_{ij}^\theta (u)|^2 + \rho_0^2 |\partial_{ij}^2 u_3|^2 \]
\[ \leq C \int_\Omega \sum_{ij} |e_{ij}^\theta (u)|^2 + \rho_0^2 |\partial_{ij}^2 u_3|^2 \]
for all $u \in V$. We now define a functional $J : V \to \mathbb{R}$ by

$$J(u) = \frac{1}{2} H(u, u) - \tilde{L}(u).$$

We first want to prove that $J$ has a unique minimizer on $V$. To this end, it suffices to show that $J$ is coercive and strictly convex on $V$. It is easy to see that

$$H(u, u) = \int_{\Omega} \sum_{i,j} n_{ij}^0 e_{ij}^0(u) + m_{ij}(u_3) \partial_{ij}^2 u_3$$

$$= \int_{\Omega} \frac{4\lambda \mu}{\lambda + 2\mu} \left| \sum_k e_{kk}^0(u) \right|^2 + 4\mu \sum_{ij} |e_{ij}^0(u)|^2$$

$$+ \frac{4\lambda \mu \rho_0^2}{3(\lambda + 2\mu)} |\Delta u_3|^2 + \frac{4\mu \rho_0^2}{3} \sum_{ij} |\partial_{ij}^2 u_3|^2$$

$$\geq \frac{4\delta_0}{3} \int_{\Omega} \sum_{ij} |e_{ij}^0(u)|^2 + \rho_0^2 |\partial_{ij}^2 u_3|^2.$$

Thus, (3.19) implies

$$H(u, u) \geq C (\|u'\|_{H^1(\Omega)}^2 + \|u_3\|_{H^2(\Omega)}^2)$$

(3.20)

with $C$ depending only on $A_0, A_1, A_2, \delta_0$. On the other hand, the trace inequality leads to

$$\tilde{L}(u) \leq C \|\hat{T}, \hat{M}\|_{(H^{-1/2}(\partial \Omega))^3} (\|u'\|_{H^1(\Omega)} + \|u_3\|_{H^2(\Omega)}).$$

Consequently, we obtain that

$$J(u) \geq C \left( \|u'\|_{H^1(\Omega)}^2 + \|u_3\|_{H^2(\Omega)}^2$$

$$- \|\hat{T}, \hat{M}\|_{(H^{-1/2}(\partial \Omega))^3} (\|u'\|_{H^1(\Omega)} + \|u_3\|_{H^2(\Omega)}) \right),$$

which shows that $J$ is coercive and bounded from below on $V$.

Now for $t \in [0, 1]$ and $u, v \in V$, we have that

$$H(tu + (1 - t)v, tu + (1 - t)v) - tH(u, u) - (1 - t)H(v, v)$$

$$= -t(1 - t)H(u - v, u - v) \leq 0$$

and for $t \in (0, 1)$

$$H(tu + (1 - t)v, tu + (1 - t)v) = tH(u, u) + (1 - t)H(v, v)$$

if and only if

$$H(u - v, u - v) = 0.$$
Since \( u, v \in V \), we see that \( H(u - v, u - v) = 0 \) if and only if \( u = v \) in \((H^1(\Omega))^2 \times H^2(\Omega)\). In other words, we have shown that \( H(u, u) \) is strictly convex on \( V \). Taking into account that \( \tilde{L}(u) \) is linear, we have that \( J(u) \) is strictly convex on \( V \). Therefore, \( J(u) \) has a unique minimizer, denoted by \( w \), on \( V \). In other words, \( J'(w)[v] = 0 \) for all \( v \in V \), i.e.,

\[
H(w, v) = \tilde{L}(v)
\]

for all \( v \in V \). Now we need to show that (3.21) is valid for all \( v \in (H^1(\Omega))^2 \times H^2(\Omega) \), that is, \( w \) indeed a weak solution. Given any \( z = (z', z_3) \in H^1(\Omega) \times H^2(\Omega) \), one can easily check that \( \tilde{z} \) satisfies (3.9), where

\[
\begin{align*}
\tilde{z}' &= z' - \frac{1}{|\Omega|} \int_{\Omega} z' - \left[ \frac{1}{|\Omega|} \int_{\Omega} \frac{(\nabla z' - (\nabla z')')}{2} \right] (x - x_\Omega) + (\theta - \theta_\Omega) \frac{1}{|\Omega|} \int_{\Omega} \nabla z_3, \\
\tilde{z}_3 &= z_3 - \frac{1}{|\Omega|} \int_{\Omega} z_3 - \left( \frac{1}{|\Omega|} \int_{\Omega} \nabla z_3 \right) \cdot (x - x_\Omega),
\end{align*}
\]

and

\[
\theta_\Omega = \frac{1}{|\Omega|} \int_{\Omega} \theta, \quad x_\Omega = \frac{1}{|\Omega|} \int_{\Omega} x.
\]

Since \((\tilde{T}, \tilde{M})\) satisfies the compatibility condition (3.8), we conclude that

\[
H(w, z) = H(w, \tilde{z}) = \tilde{L}(z) = \tilde{L}(\tilde{z}) \quad \forall \ z \in (H^1(\Omega))^2 \times H^2(\Omega).
\]

The estimate (3.18) is an easy consequence of (3.20) and the trace inequality.

### 3.3. Global regularity

To study the inverse problem, we also need a global regularity theorem for the shallow shell equations. To simplify our presentation, we impose a technical assumption on \( \tilde{\theta} \) (or \( \theta \)) in this section. Assume that \( \tilde{\theta} \) satisfies

\[
\tilde{\theta} = \nabla \tilde{\theta} = 0 \quad \text{on} \quad \partial \Omega.
\]

We shall prove the following theorem.

**Theorem 3.4.** Assume that \( \Omega \) is a bounded domain in \( \mathbb{R}^2 \) satisfying (3.1) whose boundary \( \partial \Omega \) is of class \( C^{4,1} \) with constants \( A_0 \) and \( \rho_0 \). Let \( \lambda, \mu \in C^{1,1}(\bar{\Omega}) \) satisfy (3.3) and \( \tilde{\theta} \in C^{2,1}(\bar{\Omega}) \) satisfy (3.22) and

\[
\|\lambda\|_{C^{1,1}(\bar{\Omega})} + \|\mu\|_{C^{1,1}(\bar{\Omega})} + \|\tilde{\theta}\|_{C^{2,1}(\bar{\Omega})} \leq A_2.
\]

Let \( u \in (H^1(\Omega))^2 \times H^2(\Omega) \) be the weak solution of (3.4), (3.5) with Neumann boundary condition \((\tilde{T}, \tilde{M}) \in (H^{1/2}(\partial \Omega))^2 \times H^{3/2}(\partial \Omega)\) satisfying (3.8). Assume that \( u \) satisfies the normalization conditions (3.9). Then there exists a constant \( C > 0 \), depending on \( A_0, A_1, A_2, \delta_0 \) such that

\[
\|u'\|_{H^2(\Omega)} + \|u\|_{H^4(\Omega)} \leq C \|((\tilde{T}, \tilde{M}))(H^{1/2}(\partial \Omega))^2 \times H^{3/2}(\partial \Omega)\).
\]
Proof. To prove this theorem, it suffices to consider (3.4) with homogeneous Neumann boundary conditions. In view of (3.22), the boundary conditions (3.5) are simplified to

\[
\begin{cases}
  n_{ij}(u')v_j = \rho_0^{-1}\hat{T}_i, \\
  m_{ij}(u_3)v_i v_j = \hat{M}_v, \\
  \partial_i m_{ij}(u_3)v_j + \partial_s(m_{ij}(u_3)v_i\tau_j) = -\partial_s\hat{M}_r,
\end{cases}
\]  

(3.25)

where

\[ n_{ij}(u') = \frac{4\lambda\mu}{\lambda + 2\mu} e_{kk}(u')v_i + 4\mu e_{ij}(u')v_j = \rho_0^{-1}\hat{T}_i \quad \text{on} \quad \partial\Omega \]

with \( e_{ij}(u') = \frac{1}{4}(\partial_i u_j + \partial_j u_i) \). It is clear that boundary conditions (3.25) are decoupled. Using the result in [20, Proposition 8.1], one can find \( \tilde{w}_3 \) satisfying

\[
\begin{cases}
  m_{ij}(\tilde{w}_3)v_i v_j = \hat{M}_v, \\
  \partial_i m_{ij}(\tilde{w}_3)v_j + \partial_s(m_{ij}(\tilde{w}_3)v_i\tau_j) = -\partial_s\hat{M}_r
\end{cases}
\]

on \( \partial\Omega \) and the estimate

\[ \|\tilde{w}_3\|_{H^4(\Omega)} \leq C\|\hat{M}\|_{H^{3/2}(\partial\Omega)}. \]  

(3.26)

Similarly, we can choose \( \tilde{w}' \) such that

\[ n_{ij}(\tilde{w}')v_j = \rho_0^{-1}\hat{T}_i \quad \text{on} \quad \partial\Omega \]

and

\[ \|\tilde{w}'\|_{H^2(\Omega)} \leq C\|\hat{T}\|_{(H^{1/2}(\partial\Omega))^2}. \]  

(3.27)

The constant \( C \) in (3.26) and (3.27) depend on \( A_0, A_1, A_2, \delta_0 \). By setting

\[ w' = \tilde{w}' - \frac{1}{|\Omega|} \int_\Omega \tilde{w}' - \left[ \frac{1}{|\Omega|} \int_\Omega \frac{(\nabla \tilde{w}' - (\nabla \tilde{w}')^t)}{2} \right] (x - x_\Omega) + (\theta - \theta_\Omega) \frac{1}{|\Omega|} \int_\Omega \nabla \tilde{w}_3 \]

and

\[ w_3 = \tilde{w}_3 - \frac{1}{|\Omega|} \int_\Omega \tilde{w}_3 - \left( \frac{1}{|\Omega|} \int_\Omega \nabla \tilde{w}_3 \right) \cdot (x - x_\Omega), \]

we can see that \((w', w_3)\) satisfies the boundary condition (3.25), the normalization conditions (3.9), and the estimate

\[ \|w'\|_{H^2(\Omega)} + \|w_3\|_{H^4(\Omega)} \leq C\|\hat{T}, \hat{M}\|_{(H^{1/2}(\partial\Omega))^2 \times H^{3/2}(\partial\Omega)}, \]

(3.28)

where \( C \) depends on \( A_0, A_1, A_2, \delta_0 \).
So now by letting $u = w + v$, we obtain that $v$ satisfies

\[
\begin{aligned}
-\partial_j n_{ij}^0(v) &= f_i \quad \text{in } \Omega, \\
\partial_j^2 m_{ij}(v_3) - \partial_j(n_{ij}^0(v)\partial_i \theta) &= f_3 \quad \text{in } \Omega,
\end{aligned}
\]

with homogeneous Neumann boundary conditions on $\partial \Omega$

\[
\begin{aligned}
n_{ij}(v')v_j &= 0, \\
m_{ij}(v_3)v_jv_j &= 0, \\
\partial_j m_{ij}(v_3)v_j + \partial_i(m_{ij}(v_3)v_j \tau_j) &= 0,
\end{aligned}
\]

where $f = (f_1, f_2, f_3) = (f', f_3)$ is given by

\[
\begin{aligned}
f_i &= \partial_j n_{ij}^0(w), \ i = 1, 2, \\
f_3 &= -\partial_j^2 m_{ij}(w_3) + \partial_j(n_{ij}^0(w)\partial_i \theta).
\end{aligned}
\]

Using the integration by parts, it is not hard to check that $f$ satisfies the following compatibility conditions

\[
\int \Omega f = 0, \quad \int \Omega (f_1 x_2 - f_2 x_1) = 0, \quad \int \Omega (f_1 \theta + f_3 x_1) = 0, \quad \int \Omega (f_2 \theta + f_3 x_2) = 0.
\]

Now to obtain a global estimate for $v$, we decouple (3.29) as follows

\[
\begin{aligned}
-\partial_j n_{ij}(v') &= f_i + \frac{1}{2} \partial_j(\partial_i \theta \partial_j v_3 + \partial_j \theta v_3) := \tilde{f}_i \quad \text{in } \Omega, \\
\partial_j^2 m_{ij}(v_3) &= f_3 + \partial_j(n_{ij}^0(v)\partial_i \theta) := \tilde{f}_3 \quad \text{in } \Omega.
\end{aligned}
\]

By (3.31) and straightforward computations, we can deduce that $\tilde{f} = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) = (f', f_3)$ satisfy

\[
\int \Omega \tilde{f} = 0, \quad \int \Omega (\tilde{f}_1 x_2 - \tilde{f}_2 x_1) = 0, \quad \int \Omega \tilde{f}_3 x_1 = \int \Omega \tilde{f}_3 x_2 = 0,
\]

which are the compatibility conditions for the existence of the boundary value problem (3.32) and (3.30). Recall the global estimate for the isotropic elasticity with homogeneous Neumann boundary condition, we have

\[
\|v'\|_{H^2(\Omega)} \leq C(\rho_0^2 \|\tilde{f}\|_{L^2(\Omega)} + \|v'\|_{H^4(\Omega)}) \leq C(\rho_0^2 \|f'\|_{L^2(\Omega)} + \|v_3\|_{H^2(\Omega)} + \|v'\|_{H^4(\Omega)}).
\]

where $C$ depend on $A_0$, $A_1$, $A_2$, $\delta_0$. For $v_3$, we use [20, Proposition 8.2] to obtain that

\[
\|v_3\|_{H^4(\Omega)} \leq C(\rho_0^2 \|\tilde{f}_3\|_{L^2(\Omega)} + \|v_3\|_{H^2(\Omega)}) \leq C(\rho_0^2 \|\tilde{f}_3\|_{L^2(\Omega)} + \|v_3\|_{H^2(\Omega)} + \|v'\|_{H^4(\Omega)}).
\]
The dependence of $C$ is the same as above. Putting (3.34) and (3.35) together yields
\[
\|v\|_{H^2(\Omega)} + \|v_3\|_{H^4(\Omega)} \leq C(\rho_0^2 \|f\|_{L^2(\Omega)} + \|v_3\|_{H^2(\Omega)} + \|v'\|_{H^1(\Omega)}).
\] (3.36)

Now using the weak formulation of the boundary value problem (3.32), (3.30), the Poincaré-Korn inequality (3.10), (3.28), we get from (3.36) that
\[
\|v\|_{H^2(\Omega)} + \|v_3\|_{H^4(\Omega)} \leq C(\rho_0^2 \|f\|_{L^2(\Omega)} + \|v_3\|_{H^2(\Omega)} + \|v'\|_{H^1(\Omega)})
\leq C\rho_0^2 \|f\|_{L^2(\Omega)}
\leq C(\hat{T}, \hat{M}) \|_{(H^{1/2(\partial\Omega)})^2 \times H^{3/2(\partial\Omega)}}.
\] (3.37)

Finally, combining (3.28) and (3.37) gives (3.24). \qed

4. Quantitative uniqueness estimates

4.1. Main theorems

In this section, we would like to derive the three-ball inequalities for (1.1), which is a form of quantitative uniqueness estimate. The regularity of $\partial\Omega$ is irrelevant for the estimates derived here. But to make the paper consistent, we assume that $\Omega$ is at least a Lipschitz domain with constant $A_0$ and $\rho_0$. Let $\lambda(x), \mu(x)$ satisfy (3.3) and $\lambda, \mu, \hat{\theta}$ satisfy estimate (3.23). We now first state the main results of this section. Assume that $B_{\rho_0\bar{R}_0} \subset \Omega$ with $\bar{R}_0 \leq 1$. Let us denote $U_r = (ru', u_3) = (ru_1, ru_2, u_3)$. Then the following local estimates hold.

**Theorem 4.1.** There exists a positive number $R_1$, depending on $\delta_0, K_1, K_2$, such that if $0 < r_1 < r_2 < r_3 \leq \rho_0\bar{R}_0$ and $r_1/r_3 < r_2/r_3 < R_1$, then
\[
\int_{|x| < r_2} |U_{r_2}|^2 dx \leq C_1 \left( \int_{|x| < r_1} |U_{r_1}|^2 dx \right)^{\tau} \left( \int_{|x| < r_3} |U_{r_3}|^2 dx \right)^{1-\tau}
\] (4.1)

for $(u', u_3) \in (H^1(B_{\rho_0\bar{R}_0}))^2 \times H^3(B_{\rho_0\bar{R}_0})$ satisfying (1.1) in $B_{\rho_0\bar{R}_0}$, where $C_1 > 0$ and $0 < \tau < 1$ depend on $r_1/r_3, r_2/r_3, \delta_0, A_2$.

**Remark 4.2.** The estimate (4.1) is the three-ball inequality. Constants $C_1$ and $\tau$ appeared above can be explicitly written as $\tau = B/(E + B)$ and

\[
C_1 = \max\{C_0[(\log(r_1/r_3))^2/(\log(r_2/r_3))^2](r_2/r_1)^2, \exp(B\beta_0))(r_3/r_1)^{2\tau},
\]

where $C_0 > 1$ and $\beta_0$ are constants depending on $\delta_0, A_2$ and
\[
E = E(r_1/r_3, r_2/r_3) = (\log(r_1/r_3) - 1)^2 - (\log(r_2/r_3))^2,
B = B(r_2/r_3) = -1 - 2\log(r_2/r_3).
\]
Remark 4.3. If \( r_3 \leq 1 \), then (4.1) is reduced to

\[
\int_{|x|<r_2} |U|^2 \, dx \leq \frac{C_1}{r_2^\tau} \left( \int_{|x|<r_1} |U|^2 \, dx \right)^\tau \left( \int_{|x|<r_3} |U|^2 \, dx \right)^{1-\tau}.
\] (4.2)

By abuse of notation, we denote \( U = (u', u_3) \).

Using the three-ball inequality, we can prove

**Theorem 4.4.** If \((u', u_3) \in (H^1(B_{\rho_0 \tilde{R}_0}))^2 \times H^3(B_{\rho_0 \tilde{R}_0})\) is a nontrivial solution to (1.1), then we can find a constant \( R_2 \) depending on \( \delta_0, A_2 \) and a constant \( m_1 \) depending on \( \delta_0, A_2 \) and \( \|U_{R_2^2}\|_{L^2(|x|<\rho_0 R_2^2)} / \|U_{R_2^4}\|_{L^2(|x|<\rho_0 R_2^4)} \) such that

\[
\int_{|x|<R} |U|^2 \, dx \geq K R^{m_1},
\] (4.3)

where \( R \) is sufficiently small and the constant \( K \) depends on \( R_2 \) and \( U \).

In view of the standard unique continuation property for (1.1) in a connected domain containing the origin, if \( u \) vanishes in a neighborhood of the origin then it vanishes identically in \( \Omega \). Theorem 4.4 provides an upper bound on the vanishing order of a nontrivial solution to (1.1). The following doubling inequality is another quantitative estimate of the strong unique continuation for (1.1).

**Theorem 4.5.** Let \((u', u_3) \in (H^1(B_{\rho_0 \tilde{R}_0}))^2 \times H^3(B_{\rho_0 \tilde{R}_0})\) be a nonzero solution to (1.1). Then there exist positive constants \( R_3 \), depending on \( \delta_0, A_2 \), \( \|U_{R_3^2}\|_{L^2(|x|<\rho_0 R_3^2)} / \|U_{R_3^4}\|_{L^2(|x|<\rho_0 R_3^4)} \), and \( C_2 \), depending on \( \delta_0, A_2, m_1 \), such that if \( 0 < r \leq \rho_0 \tilde{R}_3 \), then

\[
\int_{|x| \leq 2r} |U|^2 \, dx \leq C_2 \int_{|x| \leq r} |U|^2 \, dx,
\] (4.4)

where \( R_2 \) and \( m_1 \) are the constants obtained in Theorem 4.4.

The rest of this section is devoted to the proofs of Theorem 4.1, 4.4, and 4.5.

### 4.2. Preliminaries

From now on, it suffices to take \( \rho_0 = 1 \). The first step is to transform the system (1.1) into a new system with uncoupled principal parts. To simplify the notation in the following proofs, we denote \( u = u' = (u_1, u_2) \) (suppress the prime), \( w = u_3 \), and \( v = \text{div} u' = \text{div} u \). Putting (1.1) and the equation obtained by taking the divergence of the first system of (1.1) together, we come to the following new system

\[
\begin{align*}
\Delta u &= P_1(Du, Dv) + P_2(D^2w, Dw), \\
\Delta v &= P_3(Du, Dv) + P_4(D^3w, D^2w, Dw), \\
\Delta^2 w &= P_5(D^3w, D^2w, Dw) + P_6(Du),
\end{align*}
\] (4.5)
where $P_1 - P_6$ are zeroth order operators with at least $L^\infty$ coefficients which are bounded by a constant depending on $\delta_0, A_2$.

To prove Theorem 4.1, the following interior estimate is useful. From now on, the notation $X \lesssim Y$ or $X \gtrsim Y$ means that $X \leq CY$ or $X \geq CY$ with some constant $C$ which could only depend on $\delta_0, A_2$.

**Lemma 4.6.** Let $(u, w) \in (H^1_{\text{loc}}(B_{\tilde{R}_0}))^2 \times H^3_{\text{loc}}(B_{\tilde{R}_0})$ be a solution of (1.1). Then for any $0 < a_3 < a_1 < a_2 < a_4$ there exists a constant $r_0$ with $a_4 r_0 < \tilde{R}_0(<1)$ such that if $r \leq r_0$

\[
\sum_{|\alpha| \leq 2} \int_{a_1 r < |x| < a_2 r} |x|^{2|\alpha|} |D^\alpha u|^2 \, dx + \sum_{|\alpha| \leq 4} \int_{a_1 r < |x| < a_2 r} |x|^{2|\alpha|} |D^\alpha w|^2 \, dx \\
\leq C_3 \int_{a_3 r < |x| < a_4 r} (|u|^2 + |w|^2) \, dx,
\]

(4.6)

where $C_3$ is independent of $r$ and $(u, w)$.

**Proof.** The proof here is motivated by the ideas used in [12, Corollary 17.1.4]. Let $X = B_{a_4r} \setminus B_{a_3r}$ and $d(x)$ be the distant from $x \in X$ to $\mathbb{R}^2 \setminus X$. We obtain from (1.1) that $u \in (H^2_{\text{loc}}(B_{\tilde{R}_0} \setminus [0]))^2$ and $w \in H^4_{\text{loc}}(B_{\tilde{R}_0} \setminus [0])$. Denote

\[
\mathcal{L}(x, D) u := \frac{4\lambda \mu}{\lambda + 2\mu} \nabla (\text{div} u) + 4\mu \text{div} (\text{Sym} (\nabla u)).
\]

Since $\mathcal{L}(x, D)$ and $\Delta^2$ are uniformly elliptic, it is obvious that

\[
\left\{ \begin{array}{ll}
\|f\|_{H^2(\mathbb{R}^n)} \lesssim \|\mathcal{L}(y, D)f\|_{L^2(\mathbb{R}^n)} + \|f\|_{L^2(\mathbb{R}^n)} \\
\|g\|_{H^4(\mathbb{R}^n)} \lesssim \|\Delta^2 g\|_{L^2(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)}
\end{array} \right.
\]

(4.7)

for all $f \in H^2(\mathbb{R}^n), g \in H^4(\mathbb{R}^n)$ and any fixed $y$ in $\Omega$. Note that the absolute constant appearing in the first estimate of (4.7) can be chosen to be uniformly in $y \in \Omega$. By changing variables $x \rightarrow B^{-1}x$ in (4.7), we will have

\[
\left\{ \begin{array}{ll}
\sum_{|\alpha| \leq 1} B^{2-|\alpha|} \|f\|_{H^2(\mathbb{R}^n)} \lesssim \|\mathcal{L}(y, D)f\|_{L^2(\mathbb{R}^n)} + B^2 \|f\|_{L^2(\mathbb{R}^n)} \\
\sum_{|\alpha| \leq 3} B^{4-|\alpha|} \|D^\alpha g\|_{L^2(\mathbb{R}^n)} \lesssim \|\Delta^2 g\|_{L^2(\mathbb{R}^n)} + B^4 \|g\|_{L^2(\mathbb{R}^n)}
\end{array} \right.
\]

(4.8)

for all $f \in H^2(\mathbb{R}^n)$ and $g \in H^4(\mathbb{R}^n)$. To apply (4.8) on $(u, w)$, we need to cut-off $(u, w)$. So let $\xi(x) \in C^\infty_0(\mathbb{R}^n)$ satisfy $0 \leq \xi(x) \leq 1$ and

\[
\xi(x) = \begin{cases} 
1, & |x| < 1/4, \\
0, & |x| \geq 1/2.
\end{cases}
\]
Let us denote $\xi_y(x) = \xi((x - y)/d(y))$. For $y \subset X$, we apply (4.8) to $\xi_y(x)u(x)$ and use the first equation of (1.1) to get that

$$
\sum_{|\alpha| \leq 2} B^{4-2|\alpha|} \int_{|x-y| \leq d(y)/4} |D^\alpha u|^2 dx
$$

$$
\lesssim \sum_{|\alpha| \leq 1} \int_{|x-y| \leq d(y)/2} d(y)^{-4+2|\alpha|} |D^\alpha u|^2 dx + \int_{|x-y| \leq d(y)/2} |\mathcal{L}(x, D)u|^2 dx
$$

$$
+ \sum_{|\alpha| \geq 2} \int_{|x-y| \leq d(y)/2} |D^\alpha w|^2 dx
$$

$$
\lesssim \sum_{|\alpha| \leq 1} \int_{|x-y| \leq d(y)/2} d(y)^{-4+2|\alpha|} |D^\alpha u|^2 dx + r \sum_{|\alpha| \geq 2} \int_{|x-y| \leq d(y)/2} |D^\alpha u|^2 dx
$$

$$
+ B^4 \int_{|x-y| \leq d(y)/2} |u|^2 dx + \sum_{|\alpha| \leq 2} \int_{|x-y| \leq d(y)/2} |D^\alpha w|^2 dx.
$$

(4.9)

Now taking $B = Md(y)^{-1}$ for some positive constant $M$ and multiplying $d(y)^4$ on both sides of (4.9), we have

$$
\sum_{|\alpha| \leq 2} M^{4-2|\alpha|} \int_{|x-y| \leq d(y)/4} d(y)^{2|\alpha|} |D^\alpha u|^2 dx
$$

$$
\lesssim \sum_{|\alpha| \leq 1} \int_{|x-y| \leq d(y)/2} d(y)^{2|\alpha|} |D^\alpha u|^2 dx + \sum_{|\alpha| \geq 2} \int_{|x-y| \leq d(y)/2} d(y)^4 |D^\alpha u|^2 dx
$$

$$
+ M^4 \int_{|x-y| \leq d(y)/2} |u|^2 dx + \sum_{|\alpha| \leq 2} \int_{|x-y| \leq d(y)/2} d(y)^4 |D^\alpha w|^2 dx.
$$

(4.10)

Integrating $d(y)^{-2}dy$ over $X$ on both sides of (4.10) and using Fubini’s Theorem, we get that

$$
\sum_{|\alpha| \leq 2} M^{4-2|\alpha|} \int_{X} \int_{|x-y| \leq d(y)/4} d(y)^{2|\alpha|-2} |D^\alpha u|^2 dy dx
$$

$$
\lesssim \sum_{|\alpha| \leq 1} \int_{X} \int_{|x-y| \leq d(y)/2} d(y)^{2|\alpha|-2} |D^\alpha u|^2 dy dx
$$

$$
+ M^4 \int_{X} \int_{|x-y| \leq d(y)/2} d(y)^{-2} |u|^2 dy dx + r \sum_{|\alpha| \geq 2} \int_{X} \int_{|x-y| \leq d(y)/2} d(y)^2 |D^\alpha u|^2 dy dx
$$

$$
+ \sum_{|\alpha| \leq 2} \int_{X} \int_{|x-y| \leq d(y)/2} d(y)^2 |D^\alpha w|^2 dy dx.
$$

(4.11)
Note that $|d(x) - d(y)| \leq |x - y|$. If $|x - y| \leq d(x)/3$, then
\[ 2d(x)/3 \leq d(y) \leq 4d(x)/3. \tag{4.12} \]
On the other hand, if $|x - y| \leq d(y)/2$, then
\[ d(x)/2 \leq d(y) \leq 3d(x)/2. \tag{4.13} \]

By (4.12) and (4.13), we have
\[
\begin{align*}
\int_{|x-y| \leq d(y)/4} d(y)^{-2} dy &\geq 9/16 \int_{|x-y| \leq d(x)/6} d(x)^{-2} dy \\
\int_{|x-y| \leq d(y)/2} d(y)^{-2} dy &\leq 4 \int_{|x-y| \leq 3d(x)/4} d(x)^{-2} dy \\
&\leq 9/4 \int_{|y| \leq 1} dy
\end{align*}
\tag{4.14}
\]

Combining (4.11)–(4.14), we obtain
\[
\sum_{|\alpha| \leq 2} M^{4-2|\alpha|} \int_X d(x)^{2|\alpha|} |D^\alpha u|^2 dx
\lesssim \sum_{|\alpha| \leq 1} \int_X d(x)^{2|\alpha|} |D^\alpha u|^2 dx + r \sum_{|\alpha| = 2} \int_X d(x)^4 |D^\alpha u|^2 dx
\]
\[
+ M^4 \int_X |u|^2 dx + \sum_{|\alpha| \leq 2} \int_X d(x)^4 |D^\alpha w|^2 dx.
\tag{4.15}
\]

We can take $M$ large enough and $r$ small enough to absorb the first two terms on the right-hand side of (4.15). Thus we conclude that
\[
\sum_{|\alpha| \leq 2} M^{4-2|\alpha|} \int_X d(x)^{2|\alpha|} |D^\alpha u|^2 dx
\lesssim M^4 \int_X |u|^2 dx + \sum_{|\alpha| \leq 2} \int_X d(x)^4 |D^\alpha w|^2 dx.
\tag{4.16}
\]

Similarly, we can apply (4.8) to $\xi_j(x) w(x)$ and use the second equation of (1.1) to get that
\[
\sum_{|\alpha| \leq 4} M^{8-2|\alpha|} \int_X d(x)^{2|\alpha|} |D^\alpha w|^2 dx
\lesssim M^8 \int_X |w|^2 dx + \sum_{|\alpha| \leq 1} \int_X d(x)^8 |D^\alpha u|^2 dx.
\tag{4.17}
\]

Combining (4.16), (4.17) and letting $M$ be sufficiently large, we can eliminate the last terms of (4.16) and (4.17). After that we fix $M$ and obtain
\[
\sum_{|\alpha| \leq 2} \int_X d(x)^{2|\alpha|} |D^\alpha u|^2 dx + \sum_{|\alpha| \leq 4} \int_X d(x)^{2|\alpha|} |D^\alpha w|^2 dx
\lesssim \int_X |u|^2 dx + \int_X |w|^2 dx.
\tag{4.18}
We recall that \( X = B_{a_4 \ell} \setminus B_{a_3 \ell} \) and note that \( d(x) \geq \hat{C}r \) if \( x \in B_{a_3 \ell} \setminus B_{a_4 \ell} \), where \( \hat{C} \) is independent of \( r \). Hence, (4.6) is an easy consequence of (4.18).

The next result follows from Lemma 4.6:

**Corollary 4.7.** Let \( (u, w) \in (H^1_{\text{loc}}(B_{\tilde{R}_0}))^2 \times H^3_{\text{loc}}(B_{\tilde{R}_0}) \) be a solution of (1.1) and \( v = \text{div} \, u \). Then for any \( 0 < a_3 < a_1 < a_2 < a_4 \), there exists a constant \( r_0 \) such that if \( r \leq r_0 \), we have

\[
\sum_{|\alpha| \leq 1} \int_{a_1 \ell < |x| < a_2 \ell} |x|^{2|\alpha|+2} |D^\alpha v|^2 \, dx 
\leq C_3 \int_{a_3 \ell < |x| < a_4 \ell} (|u|^2 + |w|^2) \, dx, \tag{4.19}
\]

where the constant \( C_3 \) is independent of \( r \) and \( (u, w) \).

### 4.3. Proof of Theorem 4.1

To begin, we recall a Carleman estimate with weight \( \varphi_\beta = \varphi_\beta(x) = \exp\left(\frac{\beta}{2} (\log |x|)^2\right) \) given in [15].

**Lemma 4.8.** [15, Corollary 3.2] Given \( \sigma_1 \in \mathbb{Z} \) and \( \sigma_2 \in \mathbb{Z} \) there exist a sufficiently large number \( \beta_0 > 0 \) and a sufficiently small number \( r_0 > 0 \) depending on \( n, l, \sigma_1 \) and \( \sigma_2 \) such that for all \( u \in U_{r_0} \) with \( 0 < r_0 < e^{-1} \), \( \beta \geq \beta_0 \), we have that

\[
\sum_{|\alpha| \leq 2^l} \beta^{2l-2|\alpha|} \int |\varphi_\beta^2 |x|^{2\sigma_1 + 2|\alpha| - n} (\log |x|)^{2\sigma_2 + 2l - 2|\alpha|} |D^\alpha u|^2 \, dx 
\leq \tilde{C}_0 \int |\varphi_\beta^2 |x|^{2\sigma_1 + 4l - n} (\log |x|)^{2\sigma_2} |\Delta^l u|^2 \, dx, \tag{4.20}
\]

where \( U_{r_0} = \{ u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) : \text{supp}(u) \subset B_{r_0} \} \) and \( \tilde{C}_0 \) is a positive constant depending on \( n \) and \( l \). Here \( e = \exp(1) \).

**Remark 4.9.** The estimate (4.20) in Lemma 4.8 remains valid if we assume \( u \in H^2_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \) with compact support. This can be easily seen by cutting off \( u \) for small \( |x| \) and regularizing.

We first consider the case where \( 0 < r_1 < r_2 < R < 1/e \) and \( B_R \subset \Omega \). The constant \( R \) will be chosen later. To use the estimate (4.20), we need to cut-off \( u \). So let \( \xi(x) \in C_0^\infty(\mathbb{R}^n) \) satisfy \( 0 \leq \xi(x) \leq 1 \) and

\[
\xi(x) = \begin{cases} 
0, & |x| \leq r_1/e, \\
1, & r_1/2 < |x| < er_2, \\
0, & |x| \geq 3r_2.
\end{cases}
\]
It is easy to check that for all multi-index $\alpha$
\[
\begin{align*}
|D^\alpha \xi| &= O(r_1^{-|\alpha|}) \text{ for all } r_1/e \leq |x| \leq r_1/2 \\
|D^\alpha \xi| &= O(r_2^{-|\alpha|}) \text{ for all } er_2 \leq |x| \leq 3r_2.
\end{align*}
\]  
(4.21)

Noting that the commutator $[\Delta', \xi]$ is a $2l - 1$ order differential operator and using the estimate (4.20) on $\xi u$ with parameters $\sigma_1 = 0, \sigma_2 = 0, l = 1, n = 2$, we can derive from the first equations of (4.5) and (4.21) that
\[
\sum_{|\alpha| \leq 1} B^{3-2|\alpha|} \int_{r_1/2 < |x| < er_2} \varphi^2_\mu |x|^{2|\alpha|} (\log |x|)^{2-2|\alpha|} |D^\alpha u|^2 dx 
\leq \sum_{|\alpha| \leq 2} B^{3-2|\alpha|} \int_{r_1/2 < |x| < er_2} \varphi^2_\mu |x|^{2|\alpha|} (\log |x|)^{2-2|\alpha|} |D^\alpha (\xi u)|^2 dx 
\leq \int \varphi^2_\mu |x|^2 |\Delta (\xi u)|^2 dx
\]
(4.22)
\[
\leq \int \varphi^2_\mu |x|^2 \left( |\Delta u|^2 + \sum_{|\alpha| \leq 1} |[\Delta, \xi] u|^2 \right) dx 
\]
\[
\leq \int_{r_1/2 < |x| < er_2} \varphi^2_\mu |x|^2 \left[ \sum_{|\alpha| \leq 1} (|D^\alpha u|^2 + |D^\alpha v|^2) + \sum_{|\alpha| \leq 2} |D^\alpha w|^2 \right] dx 
\]
\[
\leq \int_{r_1/2 < |x| < r_1/2} \varphi^2_\mu |x|^2 \left[ \sum_{|\alpha| \leq 1} (|x|^{2|\alpha|} |D^\alpha u|^2 + |D^\alpha v|^2) + \sum_{|\alpha| \leq 2} |D^\alpha w|^2 \right] dx 
\]
\[
+ \int_{er_2 < |x| < 3r_2} \varphi^2_\mu |x|^2 \left[ \sum_{|\alpha| \leq 1} (|x|^{2|\alpha|} |D^\alpha u|^2 + |D^\alpha v|^2) + \sum_{|\alpha| \leq 2} |D^\alpha w|^2 \right] dx.
\]  
(4.23)
Finally, applying (4.20) to $\xi w$ with parameters $\sigma_1 = 0, \sigma_2 = 1, l = 2, n = 2$, we obtain from the third equation of (4.5) and (4.21) that

$$
\sum_{|\alpha| \leq 3} \beta^{6-2|\alpha|} \int_{r_1/2 < |x| < e r_2} \varphi^2_{\beta} |x|^{2|\alpha|-2} (\log |x|)^{6-2|\alpha|} |D^\alpha w|^2 \, dx
$$

$$
< \int_{r_1/2 < |x| < e r_2} \varphi^2_{\beta} |x|^6 (\log |x|)^2 \left[ \sum_{|\alpha| \leq 1} |D^\alpha u|^2 + \sum_{|\alpha| \leq 3} |D^\alpha w|^2 \right] \, dx
$$

$$
+ \int_{r_1/e < |x| < r_1/2} \varphi^2_{\beta} |x|^6 (\log |x|)^2 \left[ \sum_{|\alpha| \leq 1} |D^\alpha u|^2 + \sum_{|\alpha| \leq 3} |x|^{2|\alpha|-8} |D^\alpha w|^2 \right] \, dx
$$

$$
+ \int_{e r_2 < |x| < 3 e r_2} \varphi^2_{\beta} |x|^6 (\log |x|)^2 \left[ \sum_{|\alpha| \leq 1} |D^\alpha u|^2 + \sum_{|\alpha| \leq 3} |x|^{2|\alpha|-8} |D^\alpha w|^2 \right] \, dx.
$$

(4.24)

Putting (4.22), (4.23), (4.24) together, we can take $\beta \geq \tilde{\beta}_0 \gg 1$ and $R \leq \tilde{R}_0 \ll 1$ such that the terms $\int_{r_1/2 < |x| < e r_2} (\cdots) \, dx$ on the right-hand side are absorbed by the corresponding terms on the left-hand side. In other words, for $\beta \geq \beta_0$ and $R \leq \tilde{R}_0$, we have that

$$
\sum_{|\alpha| \leq 1} \beta^{3-2|\alpha|} \int_{r_1/2 < |x| < e r_2} \varphi^2_{\beta} |x|^{2|\alpha|-2} (\log |x|)^{2-2|\alpha|} |D^\alpha u|^2 \, dx
$$

$$
+ \sum_{|\alpha| \leq 1} \beta^{3-2|\alpha|} \int_{r_1/2 < |x| < e r_2} \varphi^2_{\beta} |x|^{2|\alpha|-2} (\log |x|)^{2-2|\alpha|} |D^\alpha v|^2 \, dx
$$

$$
+ \sum_{|\alpha| \leq 3} \beta^{6-2|\alpha|} \int_{r_1/2 < |x| < e r_2} \varphi^2_{\beta} |x|^{2|\alpha|-2} (\log |x|)^{6-2|\alpha|} |D^\alpha w|^2 \, dx
$$

$$
< \int_{r_1/e < |x| < r_1/2} \varphi^2_{\beta} |x|^{-2} \left( \sum_{|\alpha| \leq 1} |x|^{2|\alpha|} |D^\alpha u|^2 + |x|^{2|\alpha|+2} |D^\alpha v|^2 \right) \, dx
$$

(4.25)

$$
+ \int_{r_1/e < |x| < r_1/2} \varphi^2_{\beta} (\log |x|)^2 |x|^{-2} \sum_{|\alpha| \leq 3} |x|^{2|\alpha|} |D^\alpha w|^2 \, dx
$$

$$
+ \int_{e r_2 < |x| < 3 e r_2} \varphi^2_{\beta} |x|^{-2} \left( \sum_{|\alpha| \leq 1} |x|^{2|\alpha|} |D^\alpha u|^2 + |x|^{2|\alpha|+2} |D^\alpha v|^2 \right) \, dx
$$

$$
+ \int_{e r_2 < |x| < 3 e r_2} \varphi^2_{\beta} (\log |x|)^2 |x|^{-2} \sum_{|\alpha| \leq 3} |x|^{2|\alpha|} |D^\alpha w|^2 \, dx.
$$
Now using (4.6) and (4.19) in (4.25) leads to

\[
(\log r_2)^2 r_2^{-2} \varphi_\beta^2(r_2) \int_{r_1/2 < |x| < r_2} |U|^2 \, dx \\
\lesssim (\log r_1)^2 r_1^{-2} \varphi_\beta^2(r_1/e) \int_{r_1/4 < |x| < r_1} |U|^2 \, dx \\
+ (\log r_2)^2 r_2^{-2} \varphi_\beta^2(\epsilon r_2) \int_{2r_2 < |x| < 4r_2} |U|^2 \, dx.
\]

Here \( U = (u_1, u_2, w) = (u_1, u_2, u_3) \). Note that we have used the restriction \( r_1 < r_2 < 1/e \) in the above computations. Dividing by \( (\log r_2)^2 r_2^{-2} \varphi_\beta^2(r_2) \) on both sides of (4.26) implies

\[
\int_{r_1/2 < |x| < r_2} |U|^2 \, dx \\
\lesssim [(\log r_1)^2 / (\log r_2)^2] (r_2 / r_1)^2 \varphi_\beta^2(r_1/e) \varphi_\beta^2(r_2) \int_{r_1/4 < |x| < r_1} |U|^2 \, dx \\
+ [\varphi_\beta^2(\epsilon r_2) / \varphi_\beta^2(r_2)] \int_{2r_2 < |x| < 4r_2} |U|^2 \, dx
\]

(4.27)

\[
\lesssim [(\log r_1)^2 / (\log r_2)^2] (r_2 / r_1)^2 \varphi_\beta^2(r_1/e) \varphi_\beta^2(r_2) \int_{|x| < r_1} |U|^2 \, dx \\
+ [(\log r_1)^2 / (\log r_2)^2] (r_2 / r_1)^2 \varphi_\beta^2(\epsilon r_2) \varphi_\beta^2(r_2) \int_{|x| < 1} |U|^2 \, dx.
\]

Adding \( \int_{|x| < r_1/2} |U|^2 \, dx \) to both sides of (4.27), we get for \( \beta \geq \beta_0 \) that

\[
\int_{|x| < r_2} |U|^2 \, dx \\
\lesssim [(\log r_1)^2 / (\log r_2)^2] (r_2 / r_1)^2 \varphi_\beta^2(r_1/e) \varphi_\beta^2(r_2) \int_{|x| < r_1} |U|^2 \, dx
\]

(4.28)

\[
+ [(\log r_1)^2 / (\log r_2)^2] (r_2 / r_1)^2 \varphi_\beta^2(\epsilon r_2) \varphi_\beta^2(r_2) \int_{|x| < 1} |U|^2 \, dx.
\]

By denoting

\[
E = \beta^{-1} \log[\varphi_\beta^2(r_1/e) / \varphi_\beta^2(r_2)] = (\log r_1 - 1)^2 - (\log r_2)^2 > 0,
\]

\[
B = -\beta^{-1} \log[\varphi_\beta^2(\epsilon r_2) / \varphi_\beta^2(r_2)] = -1 - 2 \log r_2 > 0,
\]
(4.28) becomes
\[
\int_{|x|<r_2} |U|^2 dx \\
\lesssim [(\log r_1)^2/(\log r_2)^2](r_2/r_1)^2 \times \\
\left( \exp(E\beta) \int_{|x|<r_1} |U_{r_1}|^2 dx + \exp(-B\beta) \int_{|x|<1} |U|^2 dx \right). 
\]

To further simplify the terms on the right-hand side of (4.29), we consider two cases. If
\[
\int_{|x|<r_1} |U|^2 dx \neq 0
\]
and
\[
\exp(E\beta_0) \int_{|x|<r_1} |U|^2 dx < \exp(-B\beta_0) \int_{|x|<1} |U|^2 dx,
\]
then we can pick a \( \beta > \beta_0 \) such that
\[
\exp(E\beta) \int_{|x|<r_1} |U|^2 dx = \exp(-B\beta) \int_{|x|<1} |U|^2 dx.
\]
Using such \( \beta \), we obtain from (4.29) that
\[
\int_{|x|<r_2} |U|^2 dx \\
\lesssim [(\log r_1)^2/(\log r_2)^2](r_2/r_1)^2 \exp(E\beta) \int_{|x|<r_1} |U|^2 dx 
\]
(4.30)
\[
\lesssim [(\log r_1)^2/(\log r_2)^2](r_2/r_1)^2 \left( \int_{|x|<r_1} |U|^2 dx \right)^{\frac{R}{E+B}} \left( \int_{|x|<1} |U|^2 dx \right)^{\frac{E}{E+B}}.
\]
If
\[
\int_{|x|<r_1} |U|^2 dx = 0,
\]
then it follows from (4.29) that
\[
\int_{|x|<r_2} |U|^2 dx = 0
\]
since we can take \( \beta \) arbitrarily large. The three-sphere inequality obviously holds.
On the other hand, if
\[
\exp(-B\beta_0) \int_{|x|<1} |U|^2 dx \leq \exp(E\beta_0) \int_{|x|<r_1} |U|^2 dx,
\]
then we have
\[
\int_{|x|<r_2} |U|^2 dx 
\leq \left( \int_{|x|<1} |U|^2 dx \right)^{\frac{B}{E+B}} \left( \int_{|x|<1} |U|^2 dx \right)^{\frac{E}{E+B}} \tag{4.31}
\]
\[
\leq \exp(B\beta_0) \left( \int_{|x|<r_1} |U|^2 dx \right)^{\frac{B}{E+B}} \left( \int_{|x|<1} |U|^2 dx \right)^{\frac{E}{E+B}}.
\]
Putting together (4.30), (4.31), we arrive at
\[
\int_{|x|<r_2} |U|^2 dx \leq \tilde{C}_3 \left( \int_{|x|<r_1} |U|^2 dx \right)^{\frac{B}{E+B}} \left( \int_{|x|<1} |U|^2 dx \right)^{\frac{E}{E+B}}, \tag{4.32}
\]
where \( \tilde{C}_3 = \max\{ \tilde{C}_2[(\log r_1)^2/(\log r_2)^2](r_2/r_1)^2, \exp(B\beta_0) \} \) for some positive constant \( \tilde{C}_2 \), depending on \( \delta_0, A_2 \).

Now for the general case, we take \( R_1 = \tilde{R}_0 \) and consider \( 0 < r_1 < r_2 < r_3 \leq \rho_0 \tilde{R}_0 < 1 \) with \( r_1/r_3 < r_2/r_3 \leq R_1 \). By defining \( \tilde{u}(y) := r_3 u(r_3y), \tilde{w}(y) := w(r_3y), \hat{\lambda}(y) := \lambda(r_3y), \hat{\mu}(y) := \mu(r_3y), \hat{\theta}(y) = \theta(r_3y) \), we can see that the system (1.1) is invariant under this scaling. On the other hand, \( \hat{\lambda}(y), \hat{\mu}(y) \) and \( \hat{\theta}(y) \) satisfy (3.23), respectively, with the same constants. Therefore, from (4.32), we get that
\[
\int_{|y|<r_2/r_3} |\tilde{U}|^2 dy \leq \tilde{C}_1 \left( \int_{|y|<r_1/r_3} |\tilde{U}|^2 dy \right)^{\tau} \left( \int_{|y|<1} |\tilde{U}|^2 dy \right)^{1-\tau}, \tag{4.33}
\]
where \( |\tilde{U}|^2 = |\tilde{u}|^2 + |\tilde{w}|^2 \), \( \tau = B/(E+B) \) with
\[
E = E(r_1/r_3, r_2/r_3) = (\log(r_1/r_3) - 1)^2 - (\log(r_2/r_3))^2, \\
B = B(r_2/r_3) = -1 - 2\log(r_2/r_3),
\]
and \( \tilde{C}_1 = \max\{ \tilde{C}_2[(\log r_1/r_3)^2/(\log r_2/r_3)^2](r_2/r_1)^2, \exp(B\beta_0) \} \). Rewriting (4.33) with the original variables yields
\[
\int_{|x|<r_2} |U_{r_2}|^2 dx \leq C_1 \left( \int_{|x|<r_1} |U_{r_1}|^2 dx \right)^{\tau} \left( \int_{|x|<1} |U_{r_3}|^2 dx \right)^{1-\tau}
\]
with \( C_1 = \tilde{C}_1(r_3/r_1)^{2\tau} \).

\[ \square \]

4.4. Proof of Theorems 4.4 and 4.5

In this section we prove Theorem 4.4 and 4.5. We begin with another Carleman estimate derived in [15, Lemma 2.1]: for any \( f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) \) and for any \( m \in \mathbb{N} \), we have

\[
\int_{|x|<r_2} |U|^2 dx \leq \sup_{|x|<1} |U|^2 dx \left( \int_{|x|<1} |U|^2 dx \right)^{\frac{B}{E+B}} \left( \int_{|x|<1} |U|^2 dx \right)^{\frac{E}{E+B}} \tag{4.34}
\]

where \( B = |E|, E \leq (\log r_1)^2 \). This is a Carleman estimate for the shallow shell system (1.1), and it will be used in the proof of Theorems 4.4 and 4.5. The proof of Theorem 4.4 and 4.5 is similar to that of Theorem 4.3. The key point is to use the Carleman estimate (4.34) and to apply a suitable extension criterion. The details are left to the reader.
we have
\[
\sum_{|\alpha| \leq 2l} \int m^{2l-2|\alpha|} |x|^{-2m+2|\alpha|} |\partial^\alpha f|^2 \, dx \leq C \int |x|^{-2m+4l-n} |\Delta^l f|^2 \, dx, \tag{4.34}
\]
where \(C\) depends only on the dimension \(n\) and the power \(l\).

**Remark 4.10.** Using the cut-off function and regularization, estimate (4.34) remains valid for any fixed \(m\) if \(f \in H^{2l}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})\) with compact support.

In view of Remark 4.10, we define \(\chi(x) \in C^\infty_0(\mathbb{R}^2 \setminus \{0\})\) such that
\[
\chi(x) = \begin{cases} 
0 & \text{if } |x| \leq \delta/3, \\
1 & \text{if } \delta/2 \leq |x| \leq (R_0 + 1)R_0R/4 = r_4R, \\
0 & \text{if } 2r_4R \leq |x|,
\end{cases}
\]
where \(\delta \leq R_0^2/4, R_0 > 0\) is a small number which will be chosen later and \(R\) is given by \(R = (\gamma m)^{-1}\), where \(\gamma > 0\) is a large constant which will be chosen later.

In view of the definition of \(\chi\), it is easy to see that for all multi-index \(\alpha\)
\[
\begin{align*}
|D^\alpha \chi| &= O(\delta^{-|\alpha|}) \quad \text{for all } \delta/3 < |x| < \delta/2, \\
|D^\alpha \chi| &= O((r_4R)^{-|\alpha|}) \quad \text{for all } r_4R < |x| < 2r_4R. 
\end{align*}
\tag{4.35}
\]

Using the estimate (4.34) to \(\chi u\) with parameters \(l = 1, n = 2\) and the equations (4.5), (4.35), the same arguments as (4.22) arrive that
\[
\sum_{|\alpha| \leq 2} m^{2-2|\alpha|} \int_{|x| \leq r_4R} |x|^{-2m+2|\alpha|} |\partial^\alpha u|^2 \, dx \\
\leq \int_{|x| \leq r_4R} |x|^{-2m+2} \left[ \sum_{|\alpha| \leq 1} (|\partial^\alpha u|^2 + |\partial^\alpha v|^2) + \sum_{|\alpha| \leq 2} |\partial^\alpha w|^2 \right] \, dx \\
+ \int_{\delta/3 < |x| < \delta/2} |x|^{-2m+2} \left[ \sum_{|\alpha| \leq 1} (|x|^{2|\alpha|-4} |\partial^\alpha u|^2 + |\partial^\alpha v|^2) + \sum_{|\alpha| \leq 2} |\partial^\alpha w|^2 \right] \, dx \\
+ \int_{r_4R < |x| < 2r_4R} |x|^{-2m+2} \left[ \sum_{|\alpha| \leq 1} (|x|^{2|\alpha|-4} |\partial^\alpha u|^2 + |\partial^\alpha v|^2) + \sum_{|\alpha| \leq 2} |\partial^\alpha w|^2 \right] \, dx. \tag{4.36}
\]
Similarity, applying (4.34) to $\chi v$ with parameters $m = m - 1, l = 1, n = 2$, we can derive from (4.5) and (4.35) that

$$
\sum_{|\alpha| \leq 2} (m - 1)^{2 - 2|\alpha|} \int_{\delta/2 \leq |x| \leq r_4 R} |x|^{-2m + 2|\alpha|} |D^\alpha v|^2 \, dx
$$

\[ \geq \int_{\delta/2 \leq |x| \leq r_4 R} |x|^{-2m + 4} \left[ \sum_{|\alpha| \leq 1} (|D^\alpha u|^2 + |D^\alpha v|^2) + \sum_{|\alpha| \leq 3} |D^\alpha w|^2 \right] \, dx \tag{4.37} \]

$$
+ \int_{\delta/3 < |x| < \delta/2} |x|^{-2m + 4} \left[ \sum_{|\alpha| \leq 1} (|x|^{2|\alpha| - 4} |D^\alpha v|^2 + |D^\alpha u|^2) + \sum_{|\alpha| \leq 3} |D^\alpha w|^2 \right] \, dx
$$

$$
+ \int_{r_4 R < |x| < 2r_4 R} |x|^{-2m + 4} \left[ \sum_{|\alpha| \leq 1} (|x|^{2|\alpha| - 4} |D^\alpha v|^2 + |D^\alpha u|^2) + \sum_{|\alpha| \leq 3} |D^\alpha w|^2 \right] \, dx.
$$

Next applying (4.34) to $\chi w$ with parameters $l = 2, n = 2$, we get from (4.5) and (4.35) that

$$
\sum_{|\alpha| \leq 3} m^{4 - 2|\alpha|} \int_{\delta/2 \leq |x| \leq r_4 R} |x|^{-2m + 2|\alpha| - 2} |D^\alpha w|^2 \, dx
$$

\[ \geq \int_{\delta/2 \leq |x| \leq r_4 R} |x|^{-2m + 6} \left[ \sum_{|\alpha| \leq 1} |D^\alpha u|^2 + \sum_{|\alpha| \leq 3} |D^\alpha w|^2 \right] \, dx \tag{4.38} \]

$$
+ \int_{\delta/3 < |x| < \delta/2} |x|^{-2m + 6} \left[ \sum_{|\alpha| \leq 1} |D^\alpha u|^2 + \sum_{|\alpha| \leq 3} |x|^{2|\alpha| - 8} |D^\alpha w|^2 \right] \, dx
$$

$$
+ \int_{r_4 R < |x| < 2r_4 R} |x|^{-2m + 6} \left[ \sum_{|\alpha| \leq 1} |D^\alpha u|^2 + \sum_{|\alpha| \leq 3} |x|^{2|\alpha| - 8} |D^\alpha w|^2 \right] \, dx.
$$
Adding (4.36), $K_1 \times (4.37)$ and $K_2 m^2 \times (4.38)$ together, we obtain that

$$
\sum_{|\alpha| \leq 2} m^{2-2|\alpha|} \int_{\delta/2 \leq |x| \leq r_4 R} |x|^{-2m+2|\alpha|-2} |D^{\alpha} u|^2 dx
$$

$$
+ K_1 \sum_{|\alpha| \leq 2} (m-1)^{2-2|\alpha|} \int_{\delta/2 \leq |x| \leq r_4 R} |x|^{-2m+2|\alpha|} |D^{\alpha} v|^2 dx
$$

$$
+ K_2 \sum_{|\alpha| \leq 3} m^{4-2|\alpha|} \int_{\delta/2 \leq |x| \leq r_4 R} |x|^{-2m+2|\alpha|-2} |D^{\alpha} w|^2 dx
$$

$$
+ \int_{|\alpha| \leq 1} \bigg[ \sum_{|\alpha| \leq 1} (|D^{\alpha} u|^2 + |D^{\alpha} v|^2) + \sum_{|\alpha| \leq 2} |D^{\alpha} w|^2 \bigg] dx \quad (4.39)
$$

$$
+ K_1 \int_{\delta/2 \leq |x| \leq r_4 R} |x|^{-2m+4} \left( \sum_{|\alpha| \leq 1} (|D^{\alpha} u|^2 + |D^{\alpha} v|^2) + \sum_{|\alpha| \leq 3} |D^{\alpha} w|^2 \right) dx
$$

$$
+ K_2 \int_{\delta/2 \leq |x| \leq r_4 R} |x|^{-2m+6} \left( \sum_{|\alpha| \leq 1} |D^{\alpha} u|^2 + \sum_{|\alpha| \leq 3} |D^{\alpha} w|^2 \right) dx
$$

$$
+ \int_{\delta/3 < |x| < \delta/2} (\cdots) + \int_{r_4 R < |x| < 2r_4 R} (\cdots).
$$

Now we choose $K_1$ sufficiently large such that the terms

$$
\int_{\delta/2 \leq |x| \leq r_4 R} |x|^{-2m+2} \sum_{|\alpha| = 1} |D^{\alpha} v|^2 dx
$$

on the right-hand side of (4.39) are absorbed by its left-hand side. After that the constant $K_1$ is fixed. We continue to choose $K_2$ large enough such that

$$
K_1 \int_{\delta/2 \leq |x| \leq r_4 R} |x|^{-2m+4} \sum_{|\alpha| = 3} |D^{\alpha} w|^2 dx
$$

on the right-hand side of (4.39) are absorbed by its left-hand side. Then we fix $K_2$. To eliminate other terms inside the integral $\int_{\delta/2 \leq |x| \leq r_4 R}$ on the right-hand side of (4.39), we recall that $R = (\gamma m)^{-1}$. So by choosing $\gamma \geq \gamma_0$ and $m \geq m_0'$ with large
\( \gamma_0 \) and \( m'_0 \), we have that

\[
\sum_{|\alpha| \leq 2} m^{-2|\alpha|} \int_{\delta/2 \leq |x| \leq r_4 R} |x|^{-2m+2|\alpha|-2} |D^\alpha u|^2 \, dx
\]

\[
+ \sum_{|\alpha| \leq 2} (m - 1)^{-2|\alpha|} \int_{\delta/2 \leq |x| \leq r_4 R} |x|^{-2m+2|\alpha|} |D^\alpha v|^2 \, dx
\]

\[
+ \sum_{|\alpha| \leq 2} m^{-2|\alpha|} \int_{\delta/2 \leq |x| \leq r_4 R} |x|^{-2m+2|\alpha|-2} |D^\alpha w|^2 \, dx
\]

\[
\lesssim \int_{\delta/3 < |x| < \delta/2} |x|^{-2m-2} \left( \sum_{|\alpha| \leq 1} |x|^{2|\alpha|} |D^\alpha u|^2 + |x|^{2|\alpha|+2} |D^\alpha v|^2 \right) \, dx \quad (4.40)
\]

\[
+ \int_{\delta/3 < |x| < \delta/2} |x|^{-2m-2} m^2 \sum_{|\alpha| \leq 3} |x|^{2|\alpha|} |D^\alpha w|^2 \, dx
\]

\[
+ \int_{r_4 R < |x| < 2r_4 R} |x|^{-2m-2} \left( \sum_{|\alpha| \leq 1} |x|^{2|\alpha|} |D^\alpha u|^2 + |x|^{2|\alpha|+2} |D^\alpha v|^2 \right) \, dx
\]

\[
+ \int_{r_4 R < |x| < 2r_4 R} |x|^{-2m-2} m^2 \sum_{|\alpha| \leq 3} |x|^{2|\alpha|} |D^\alpha w|^2 \, dx.
\]

Note that \( R_0^2 \leq r_4 \) provided \( R_0 \leq 1/3 \). Also, if \( R_0 \leq 1/3 \), it is obvious that \( 2r_4 \leq R_0 \). Dividing \( m^2 \) on both sides of (4.40) and using (4.19) and (4.6) in (4.40), it obtains that

\[
(2\delta)^{-2m-2} \int_{\delta/2 < |x| \leq 2\delta} |U|^2 \, dx + (R_0^2 R)^{-2m-2} \int_{2\delta < |x| \leq R_0^2 R} |U|^2 \, dx
\]

\[
\lesssim \int_{\delta/2 \leq |x| \leq r_4 R} |x|^{-2m-2} |U|^2 \, dx \quad (4.41)
\]

\[
\leq C'(\delta/3)^{-2m-2} \int_{|x| \leq \delta} |U|^2 \, dx + C''(r_4 R)^{-2m-2} \int_{|x| \leq R_0 R} |U|^2 \, dx,
\]

where \( C' \) and \( C'' \) absolute constants. From now on, we need to trace the constants to make the estimates more clearly. Adding \( (2\delta)^{-2m-2} \int_{|x| \leq \delta/2} |U|^2 \, dx \) to both sides
of (4.41), we have that
\[
\frac{1}{2} (2\delta)^{-2m-2} \int_{|x| \leq 2\delta} |U|^2 \, dx + \int_{|x| \leq R_0^2 R} |U|^2 \, dx
\]
\[
= \frac{1}{2} (2\delta)^{-2m-2} \int_{|x| \leq 2\delta} |U|^2 \, dx + (R_0^2 R)^{-2m-2} \int_{|x| \leq R_0^2 R} |U|^2 \, dx
\]
\[
+ (R_0^2 R)^{-2m-2} \int_{2\delta < |x| \leq R_0^2 R} |U|^2 \, dx
\]
\[
\leq \frac{1}{2} (2\delta)^{-2m-2} \int_{|x| \leq 2\delta} |U|^2 \, dx + \frac{1}{2} (2\delta)^{-2m-2} \int_{|x| \leq 2\delta} |U|^2 \, dx
\]
\[
+ (R_0^2 R)^{-2m-2} \int_{2\delta < |x| \leq R_0^2 R} |U|^2 \, dx
\]
\[
\leq (C' + 1)(\delta/3)^{-2m-2} \int_{|x| \leq \delta} |U|^2 \, dx + C''(r_4 R)^{-2m-2} \int_{|x| \leq R_0 R} |U|^2 \, dx
\]
\[
= (C' + 1)(\delta/3)^{-2m-2} \int_{|x| \leq \delta} |U|^2 \, dx
\]
\[
+ (R_0^2 R)^{-2m-2} C'' \left( \frac{R_0^2}{r_4} \right)^{2m+2} \int_{|x| \leq R_0 R} |U|^2 \, dx.
\]
We now observe that
\[
C'' \left( \frac{R_0^2}{r_4} \right)^{2m+2} = C'' \left( \frac{4R_0}{R_0 + 1} \right)^{2m+2}
\]
\[
\leq C'' (4R_0)^{2m+2} \leq \exp(-2m)
\]
for all $R_0 < e^{-1}/4$ and $m \geq m_0$, where $m_0$ depends on $C''$ and $R_0$. Thus, we obtain that
\[
\frac{1}{2} (2\delta)^{-2m-2} \int_{|x| \leq 2\delta} |U|^2 \, dx + (R_0^2 R)^{-2m-2} \int_{|x| \leq R_0^2 R} |U|^2 \, dx
\]
\[
\leq (C' + 1)(\delta/3)^{-2m-2} \int_{|x| \leq \delta} |U|^2 \, dx
\]
\[
+ (R_0^2 R)^{-2m-2} \exp(-2m) \int_{|x| \leq R_0 R} |U|^2 \, dx.
\]
It should be noted that (4.43) is valid for all $m = j + \frac{1}{2}$ with $j \in \mathbb{N}$ and $j \geq j_0$, where $j_0$ depends on $R_0$. Setting $R_j = (\gamma(j + \frac{1}{2}))^{-1}$ and using the relation $m =$
\((\gamma R)^{-1}\), we get from (4.43) that
\[
\frac{1}{2}(2\delta)^{-2m-2} \int_{|x| \leq 2\delta} |U|^2 dx + (R_0^2 R_j)^{-2m-2} \int_{|x| \leq R_0^2 R_j} |U|^2 dx
\]
\[
\leq (C' + 1)(\delta/3)^{-2m-2} \int_{|x| \leq \delta} |U|^2 dx
\]
\[
+(R_0^2 R_j)^{-2m-2} \exp(-2c R_j^{-1}) \int_{|x| \leq R_0 R_j} |U|^2 dx
\]
for all \(j \geq j_0\) and \(c = \gamma^{-1}\). We now observe that
\[
R_{j+1} < R_j < 2R_{j+1} \quad \text{for all } j \in \mathbb{N}.
\]
Thus, if \(R_{j+1} < r \leq R_j\), we can conclude that
\[
\left\{ \begin{array}{l}
\int_{|x| \leq R_0^2 r} |U|^2 dx \leq \int_{|x| \leq R_0^2 R_j} |U|^2 dx, \\
\exp(-2c R_j^{-1}) \int_{|x| \leq R_0 R_j} |U|^2 dx \leq \exp(-cr^{-1}) \int_{|x| \leq r} |U|^2 dx,
\end{array} \right.
\]
where we have used the inequality \(R_0 R_j < 2R_0 R_{j+1} \leq R_{j+1}/(2c) < R_j\) to derive the second inequality above. Namely, we have from (4.44) and (4.45) that
\[
\frac{1}{2}(2\delta)^{-2m-2} \int_{|x| \leq 2\delta} |U|^2 dx + (R_0^2 R_j)^{-2m-2} \int_{|x| \leq R_0^2 r} |U|^2 dx
\]
\[
\leq (C' + 1)(\delta/3)^{-2m-2} \int_{|x| \leq \delta} |U|^2 dx
\]
\[
+(R_0^2 R_j)^{-2m-2} \exp(-cr^{-1}) \int_{|x| \leq r} |U|^2 dx.
\]
If there exists \(s \in \mathbb{N}\) such that
\[
R_{j+1} < R_0^{2s} \leq R_j \quad \text{for some } j \geq j_0,
\]
then replacing \(r\) by \(R_0^{2s}\) in (4.46) leads to
\[
\frac{1}{2}(2\delta)^{-2m-2} \int_{|x| \leq 2\delta} |U|^2 dx + (R_0^2 R_j)^{-2m-2} \int_{|x| \leq R_0^{2s+2}} |U|^2 dx
\]
\[
\leq (C' + 1)(\delta/3)^{-2m-2} \int_{|x| \leq \delta} |U|^2 dx
\]
\[
+(R_0^2 R_j)^{-2m-2} \exp(-c R_0^{-2s}) \int_{|x| \leq R_0^{2s}} |U|^2 dx.
\]
Here $s$ and $R_0$ are yet to be determined. The trick now is to find suitable $s$ and $R_0$ satisfying (4.47) and the inequality

$$\exp(-cR_0^{-2x}) \int_{|x| \leq R_0^{2x}} |U|^2 \, dx \leq \frac{1}{2} \int_{|x| \leq R_0^{2x+2}} |U|^2 \, dx$$

(4.49)

holds with such choices of $s$ and $R_0$.

It is time to use the three-ball inequality (4.1). To this end, we choose $r_1 = R_0^{2k+2}$, $r_2 = R_0^{2k}$ and $r_3 = R_0^{2k-2}$ for $k \geq 1$ and require $R_0^2 \leq \min\{(1/4e)^2, R_1\}$. Thus (4.1) implies

$$\int_{|x| < R_0^{2k}} |U_{R_0^{2k}}|^2 \, dx / \int_{|x| < R_0^{2k+2}} |U_{R_0^{2k+2}}|^2 \, dx \leq C^{1/\tau} \left( \frac{\int_{|x| < R_0^{2k-2}} |U_{R_0^{2k-2}}|^2 \, dx / \int_{|x| < R_0^{2k}} |U_{R_0^{2k}}|^2 \, dx}{C^{1/\tau}} \right)^a,$$

(4.50)

where

$$C = \max\{4C_0R_0^{-4}, \exp(b_0(-1 - 4 \log R_0))\}R_0^{-8\tau}$$

and

$$a = \frac{1 - \tau}{\tau} = \frac{A}{B} = \frac{(\log(r_1/r_3) - 1)^2 - (\log(r_2/r_3))^2}{-1 - 2 \log(r_2/r_3)}$$

$$= \frac{(4 \log R_0 - 1)^2 - (2 \log R_0)^2}{-1 - 4 \log R_0}.$$

It is not hard to see that

$$\begin{cases} 
1 < C \leq C_0 R_0^{-\beta_1}, \\
2 < a \leq -4 \log R_0,
\end{cases}$$

(4.51)

where $\beta_1 = 32 \max\{1, \beta_0\}$ (note $\tau < 1$). Combining (4.51) and using (4.50) recursively, we have that

$$\int_{|x| \leq R_0^{2s}} |U_{R_0^{2s}}|^2 \, dx / \int_{|x| \leq R_0^{2s+2}} |U_{R_0^{2s+2}}|^2 \, dx$$

$$\leq C^{1/\tau} \left( \frac{\int_{|x| < R_0^{2s-2}} |U_{R_0^{2s-2}}|^2 \, dx / \int_{|x| < R_0^{2s}} |U_{R_0^{2s}}|^2 \, dx}{C^{1/\tau}} \right)^a$$

(4.52)

$$\leq C^{\frac{a^{s-1} - 1}{\tau(a - 1)}} \left( \frac{\int_{|x| < R_0^2} |U_{R_0^2}|^2 \, dx / \int_{|x| < R_0^4} |U_{R_0^4}|^2 \, dx}{C^{1/\tau}} \right)^{a^{s-1}}$$

for all $s \geq 1$. Now from the definition of $a$, we have $\tau = 1/(a + 1)$ and thus

$$\frac{a^{s-1} - 1}{\tau(a - 1)} = \frac{a + 1}{a - 1}(a^{s-1} - 1) \leq 3a^{s-1}.$$
Then it follows from (4.52) that
\[
\int_{|x| \leq R_0^{2s}} |U_{R_0^{2s}}|^2 dx / \int_{|x| \leq R_0^{2s+2}} |U_{R_0^{2s+2}}|^2 dx 
\leq C^3(-4 \log R_0)^{s-1} \left( \int_{|x| < R_0^4} |U_{R_0^4}|^2 dx / \int_{|x| < R_0^4} |U_{R_0^4}|^2 dx \right)^{a^s-1} 
\]
\[
\leq (C_0^3(R_0)^{-3\beta_1})(-4 \log R_0)^{s-1} \left( \int_{|x| < R_0^4} |U_{R_0^4}|^2 dx / \int_{|x| < R_0^4} |U_{R_0^4}|^2 dx \right)^{a^s-1}. 
\]

Note that
\[
\int_{|x| \leq R_0^{2s}} |U|^2 dx \leq R_0^{-4s} \int_{|x| \leq R_0^{2s}} |U_{R_0^{2s}}|^2 dx, 
\]
\[
\int_{|x| \leq R_0^{2s+2}} |U_{R_0^{2s+2}}|^2 dx \leq \int_{|x| \leq R_0^{2s+2}} |U|^2 dx. 
\]

Thus, by (4.53) and (4.54), we can get that
\[
\exp(-cR_0^{-2s}) \int_{|x| \leq R_0^{2s}} |U|^2 dx 
\leq \exp(-cR_0^{-2s}) R_0^{-4s} (C_0^3(R_0)^{-3\beta_1})(-4 \log R_0)^{s-1} \left( \int_{|x| < R_0^4} |U_{R_0^4}|^2 dx / \int_{|x| < R_0^4} |U_{R_0^4}|^2 dx \right)^{a^s-1} \int_{|x| \leq R_0^{2s+2}} |U|^2 dx. 
\]

Let \( \mu = -\log R_0 \), then if \( R_0 \) is sufficiently small, i.e., \( \mu \) is sufficiently large, we can see that
\[
\frac{c}{4} \exp(2\mu t) > 4t\mu + (4\mu)^t(-1) (\log C_0^3 + 3\beta_1\mu) 
\]
for all \( t \in \mathbb{N} \). In other words, we have that for \( R_0 \) small
\[
R_0^{-4t}(C_0^3(R_0)^{-3\beta_1})(-4 \log R_0)^{t-1} < \exp(c R_0^{-2t}/4) < (1/2) \exp(c R_0^{-2t}/2) 
\]
for all \( t \in \mathbb{N} \). We now fix a \( R_0 \leq \min\{1/4e, \sqrt{R_1}\} \) so that (4.56) holds. The constants \( m_0(R_0) \) and \( f_0(R_0) \) are then fixed as well. It is a key step in our proof that we can find a universal constant \( R_0 \). After fixing \( R_0 \), we then define a number \( t_0 \), depending on \( R_0 \) and \( U \), by
\[
t_0 = \inf \left\{ t \in \mathbb{R} : t \geq \left( \log 2 - \log(ac) 
\right. 
+ \log \log \left( \int_{|x| < R_0^4} |U_{R_0^4}|^2 dx / \int_{|x| < R_0^4} |U_{R_0^4}|^2 dx \right) \left( -2 \log R_0 - \log a \right)^{-1} \right\}. 
\]
By (4.51), one can easily check that $-2 \log R_0 - \log a > 0$ for all $R_0 \leq 1/16$. With the choice of $t_0$, we can see that

$$\left( \int_{|x| < R_0^2} |U_{R_0^2}|^2 dx / \int_{|x| < R_0^4} |U_{R_0^4}|^2 dx \right)^{a^{-1}} \leq \exp(cR_0^{-2t}/2)$$

(4.57)

for all $t \geq t_0$.

Let $s_1$ be the smallest positive integer such that $s_1 \geq t_0$. If

$$R_0^{2s_1} \leq R_{j_0} = (\gamma(j_0 + 1/2))^{-1},$$

(4.58)

then we can find a $j_1 \in \mathbb{N}$ with $j_1 \geq j_0$ such that (4.47) holds, i.e.,

$$R_{j_1 + 1} < R_0^{2s_1} \leq R_{j_1}.$$  

On the other hand, if

$$R_0^{2s_1} > R_{j_0},$$

(4.59)

then we pick the smallest positive integer $s_2 > s_1$ such that $R_0^{2s_2} \leq R_{j_0}$ and thus we can also find a $j_1 \in \mathbb{N}$ with $j_1 \geq j_0$ for which (4.47) holds. We now define

$$s = \begin{cases} 
    s_1 & \text{if (4.58) holds,} \\
    s_2 & \text{if (4.59) holds.}
\end{cases}$$

It is important to note that with such an $s$, (4.47) is satisfied for some $j_1$ and (4.56), (4.57) hold. Now we set $m_1 = 2 + 2(j_1 + 1/2)$ and $m = (m_1 - 2)/2$. Combining (4.55), (4.56) and (4.57) yields that

$$\exp(-cR_0^{-2s}) \int_{|x| \leq R_0^{2s}} |U|^2 dx$$

$$\leq \exp(-cR_0^{-2s}) R_0^{-4s} (C_0^2(R_0)^{-3\beta_1})(-3 \log R_0)^{s-1}$$

$$\left( \int_{|x| < R_0^2} |U_{R_0^2}|^2 dx / \int_{|x| < R_0^4} |U_{R_0^4}|^2 dx \right)^{a^{(s-1)}} \int_{|x| \leq R_0^{2s+2}} |U|^2 dx.$$  

$$\leq \frac{1}{2} \int_{|x| \leq R_0^{2s+2}} |U|^2 dx$$

which is (4.49). Using (4.49) in (4.48), we have that

$$\frac{1}{2}(2\delta)^{-2m-2} \int_{|x| \leq 2\delta} |U|^2 dx + \frac{1}{2}(R_0^2 R_{j_1})^{-2m-2} \int_{|x| \leq R_0^{2s+2}} |U|^2 dx$$

$$\leq (C' + 1)(\delta/3)^{-2m-2} \int_{|x| \leq \delta} |U|^2 dx.$$  

(4.60)
It follows from (4.60) that
\[
\frac{1}{2(C' + 1)} (3R_0^2 R_{j_1})^{-m_1} \int_{|x| \leq R_0^{2^{s+2}}} |U|^2 dx \leq \delta^{-m_1} \int_{|x| \leq \delta} |U|^2 dx \quad (4.61)
\]
and
\[
\frac{1}{2(2\delta)^{-2m-2}} \int_{|x| \leq 2\delta} |U|^2 dx \leq (C' + 1)(\delta/3)^{-2m-2} \int_{|x| \leq \delta} |U|^2 dx
\]
which implies
\[
\int_{|x| \leq 2\delta} |U|^2 dx \leq \frac{1}{2(C' + 1)} 6^{m_1} \int_{|x| \leq \delta} |U|^2 dx. \tag{4.62}
\]

The estimates (4.61) and (4.62) are valid for all \( \delta \leq R_0^{2^{s+2}}/4 \). Now we choose \( R_2 = R_0 \) in Theorem 4.4 and \( R_3 = R_0^{2^{s+2}}/4 \) in Theorem 4.5. The proof is complete. \( \square \)

4.5. Lipschitz propagation of smallness

To study our inverse problem, we need to obtain three-ball inequalities in terms of \( \sum_{ij} |e_{ij}^\rho(u)|^2 + \rho_0^2 |\partial_{ij} u_3|^2 \) instead of \( |u'|^2 + |u_3|^2 \). To this end, the following Caccioppoli-type inequality is useful.

**Lemma 4.11.** Assume that \( \lambda(x), \mu(x) \in L^\infty(B_\rho) \) satisfying (3.3) and there exists \( K_3 > 0 \) such that
\[
\|\lambda\|_{L^\infty(B_\rho)} + \|\mu\|_{L^\infty(B_\rho)} + \|\nabla \theta\|_{L^\infty(B_\rho)} \leq K_3.
\]

Let \( (u', u_3) \in (H^1(B_\rho))^2 \times H^2(B_\rho) \) be a solution of (1.1) in \( B_\rho \). Then there exists a constant \( C > 0 \), depending on \( \delta_0, K_3 \) such that
\[
\int_{B_{\rho/2}} \sum_{ij} |e_{ij}^\rho(u)|^2 + \rho_0^2 |\partial_{ij} u_3|^2 \leq \frac{C}{\rho^2} \int_{B_\rho} |u'|^2 + C \left( \frac{1}{\rho^4} + \frac{1}{\rho^2} \right) \int_{B_\rho} |u_3|^2. \tag{4.63}
\]

**Proof.** The proof of this lemma is adopted from [21]. Let \( \eta \in C^4_0(B_\rho) \) with \( 0 \leq \eta \leq 1 \) and \( \eta \equiv 1 \) on \( B_{\rho/2} \) satisfying
\[
\sum_{|\alpha| \leq 3} \rho^{|\alpha|} |\partial^\alpha \eta| \leq C_1 \quad \text{in} \quad B_\rho \quad (4.64)
\]
for some positive constant \( C_1 \). Multiplying the first equation of (1.1) by \( \eta^4 u' \) and the second equation of (1.1) by \( \eta^4 u_3 \) and performing integration by parts, we can get that
\[
\int_{B_\rho} n_{ij}^\theta(u) \partial_j (\eta^4 u_i) + \int_{B_\rho} m_{ij}(u_3) \partial_j^2 (\eta^4 u_3) + \int_{B_\rho} n_{ij}^\rho(u) \partial_i \theta \partial_j (\eta^4 u_3) = 0. \tag{4.65}
\]
It is easy to see that (4.65) is equivalent to
\[
\int_{B_\rho} n_{ij}^\theta(u)[\eta^4 \partial_j u_i + 4(\partial_j \eta)\eta^3 u_i]
+ m_{ij}(u_3)[\eta^4 \partial_i^2 u_3 + 8(\partial_j \eta)\eta^3 \partial_i u_3 + (\partial_j \eta)^4 u_3]
+ n_{ij}^\theta(u) \partial_i \theta[\eta^4 \partial_j u_3 + 4(\partial_j \eta)\eta^3 u_3] = 0.
\]

(4.66)

It follows from (4.66) that
\[
\int_{B_\rho} \eta^4 \sum_{ij} \left( n_{ij}^\theta(u) e_i^j(u) + m_{ij}(u_3) \partial_i^2 u_3 \right)
= -4 \int_{B_\rho} n_{ij}^\theta(u)(\partial_j \eta)\eta^3 u_i - 8 \int_{B_\rho} m_{ij}(u_3)(\partial_j \eta)\eta^3 \partial_i u_3
\]
\[
- \int_{B_\rho} m_{ij}(u_3)(\partial_j^2 \eta^4) u_3
- 4 \int_{B_\rho} n_{ij}^\theta(u)(\partial_i \theta)(\partial_j \eta)\eta^3 u_3.
\]

(4.67)

Observe that
\[
\left| \int_{B_\rho} n_{ij}^\theta(u)(\partial_j \eta)\eta^3 u_i \right| \leq \frac{\varepsilon}{2} \int_{B_\rho} \eta^4 \sum_{ij} |n_{ij}^\theta|^2 + \frac{C_2}{\varepsilon \rho^2} \int_{B_\rho} |u'|^2
\]

(4.68)

for some $C_2 > 0$, depending on $C_1$. Likewise, we can obtain that
\[
\left| \int_{B_\rho} m_{ij}(u_3)(\partial_j^2 \eta^4) u_3 \right| \leq \frac{\varepsilon}{2} \int_{B_\rho} \eta^4 \sum_{ij} |m_{ij}(u_3)|^2 + \frac{C_3}{\varepsilon \rho^4} \int_{B_\rho} |u_3|^2
\]

(4.69)

for some $C_3 > 0$, and
\[
\left| \int_{B_\rho} n_{ij}^\theta(u)(\partial_i \theta)(\partial_j \eta)\eta^3 u_3 \right| \leq \frac{\varepsilon}{2} \int_{B_\rho} \eta^4 \sum_{ij} |n_{ij}^\theta(u)|^2 + \frac{C_4}{\varepsilon \rho^4} \int_{B_\rho} |u_3|^2
\]

(4.70)

for some $C_4 > 0$, also depending on $K_3$. Finally, we have
\[
\left| \int_{B_\rho} m_{ij}(u_3)(\partial_j \eta)\eta^3 \partial_i u_3 \right| \leq \frac{\varepsilon}{2} \int_{B_\rho} \eta^4 \sum_{ij} |m_{ij}(u_3)|^2
+ \frac{1}{2\varepsilon} \int_{B_\rho} \eta^2 \sum_{ij} (\partial_j \eta \partial_i u_3)^2.
\]

(4.71)
Using the same computations on page 10-11 of [21], we get that
\[
\int_{B_{R}} \eta^{2} \sum_{ij} (\partial_{j} \eta \partial_{i} u_{3})^{2} \leq \frac{C_{5}}{\rho^{4}} \left( 1 + \frac{1}{\varepsilon^{2}} \right) \int_{B_{R}} |u_{3}|^{2} + \frac{\varepsilon^{2}}{2} \int_{B_{R}} \eta^{4} \sum_{ij} |\partial_{ij} u_{3}|^{2}
\]
(4.72)
for some $C_{5} > 0$. Putting (4.67)-(4.72) together and taking $\varepsilon$ sufficiently small, we immediately arrive at the desired estimate (4.63).

\[\square\]

**Remark 4.12.** If $\rho \leq 1$, then (4.63) can be written as
\[
\int_{B_{R}/2} \sum_{ij} |e_{ij}^{0}(u)|^{2} + \rho_{0}^{2} |\partial_{ij} u_{3}|^{2} \leq \frac{C}{\rho^{4}} \int_{B_{R}} |u'|^{2} + |u_{3}|^{2}.
\]
(4.73)

Our aim here is to derive another version of the three sphere inequality. We will make use of arguments introduced in [5]. For any scalar or vector valued function $f$, we denote $(f)_{r} = \frac{1}{|B_{R}|} \int_{B_{r}} f$. Then we define an operator $T : (H^{1}(B_{R}))^{2} \times H^{2}(B_{R}) \rightarrow (H^{1}(B_{R}))^{2} \times H^{2}(B_{R})$ by $Tu = T(u', u_{3}) = (v'(x; r), v_{3}(x; r))$, where
\[
v'(x; r) = (u')_{r} + \frac{1}{2} (\nabla u' - (\nabla u')_{r}) \cdot x
\]
\[
+ \frac{1}{2} [((\nabla \theta)_{r} \otimes (\nabla u_{3})_{r} - ((\nabla \theta)_{r} \otimes (\nabla u_{3})_{r})'] x - (\theta - (\theta)_{r})(\nabla u_{3})_{r}
\]
(4.74)
and
\[
v_{3}(x; r) = (u_{3})_{r} + (\nabla u_{3})_{r} \cdot x.
\]
(4.75)
Here the tensor product of two vectors $\xi$ and $\eta$ is defined as
\[
\xi \otimes \eta = \begin{pmatrix} \xi_{1} \eta_{1} & \xi_{1} \eta_{2} \\ \xi_{2} \eta_{1} & \xi_{2} \eta_{2} \end{pmatrix}.
\]

Note that $(x)_{r} = 0$.

We now denote the space
\[
\mathcal{R} = \{ w = (w', w_{3}) \mid w' = a + Wx - \theta c, \ w_{3} = b + c \cdot x \},
\]
where $a, c$ are two-dimensional vectors, $b$ is a scalar, and $W$ is a $2 \times 2$ skew-symmetric matrix. It is readily seen that $Tu = u$ for all $u \in \mathcal{R}$. Denote $Lu = \left( (e_{ij}^{0}(u))_{1 \leq i \leq 2, 1 \leq j \leq 2}, (\partial_{ij}^{2} u_{3})_{1 \leq i \leq 2, 1 \leq j \leq 2} \right)$. It is also easy to check that $\mathcal{R}$ is the null space of $L$. We need to compute the norm of $T$. Recall that
\[
\|v'\|_{H^{1}(B_{R})}^{2} + \|v_{3}\|_{H^{2}(B_{R})}^{2}
= \frac{1}{R^{2}} \int_{B_{R}} |u'|^{2} + \int_{B_{R}} |\nabla u'|^{2} + \frac{1}{R^{2}} \int_{B_{R}} |u_{3}|^{2} + \int_{B_{R}} |\nabla u_{3}|^{2} + R^{2} \int_{B_{R}} |\nabla^{2} u_{3}|^{2}.
\]
In view of (4.74), (4.75) and the assumption on \( \theta \), we obtain that

\[
\| T \| \leq C \left( 1 + \frac{R^2}{r^2} + \frac{1}{r^2} \right)^{1/2}
\]

with an absolute constant \( C > 0 \). Assume that \( B_R \subset \Omega \) and so (3.2) holds on \( B_R \). Now if we take \( r = R \), then \((u' - v'(\cdot; R), u_3 - v_3(\cdot; R))\) satisfies the normalization conditions (3.9) with \( \Omega \) be replaced by \( B_R \) and therefore (3.10) becomes

\[
\| u' - v'(\cdot; R) \|^2_{H^1(B_R)} + \| u_3 - v_3(\cdot; R) \|^2_{H^2(B_R)} \leq C \int_{B_R} \sum_{ij} |e^\theta_{ij}(u)|^2 + R^2 |\partial^2_{ij}u_3|^2,
\]

where \( C \) depends on \( A_2 \). Using Lemma 2.1 in [5], we conclude that

\[
\| u' - v'(\cdot; r) \|^2_{H^1(B_r)} + \| u_3 - v_3(\cdot; r) \|^2_{H^2(B_r)} \leq C (1 + \| T \|)^2 \int_{B_r} \sum_{ij} |e^\theta_{ij}(u)|^2 + R^2 |\partial^2_{ij}u_3|^2
\]

\[
\leq C (1 + \frac{R^2}{r^2} + \frac{1}{r^2}) \int_{B_r} \sum_{ij} |e^\theta_{ij}(u)|^2 + R^2 |\partial^2_{ij}u_3|^2.
\]

In particular, we have that

\[
\int_{B_r} |u' - v'(\cdot; r)|^2 + |u_3 - v_3(\cdot; r)|^2 \leq CR^2 \left( 1 + \frac{R^2}{r^2} + \frac{1}{r^2} \right) \left( 1 + \frac{R^2}{\rho_0^2} \right) \int_{B_r} \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij}u_3|^2.
\] (4.76)

We now prove the following three-ball inequalities.

**Theorem 4.13.** Assume that \( \tilde{R}_0 \) and \( R_1 \) are given in Theorem 4.1. If \( 0 < r_1 < r_2 < 2r_2 < r_3 \leq \min(\rho_0 \tilde{R}_0, 1) \) and \( r_1/r_3 < 2r_2/r_3 < R_1 \), then

\[
\int_{B_{r_2}} \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij}u_3|^2 \leq \frac{C}{r_2^6} \left( \frac{r_3}{r_1} \right)^{2-2\tau} \left( \int_{B_{r_1}} \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij}u_3|^2 \right)^\tau \times \left( \int_{B_{r_3}} \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij}u_3|^2 \right)^{1-\tau}
\] (4.77)

for \((u', u_3) \in (H^1(B_{\rho_0 \tilde{R}_0}))^2 \times H^3(B_{\rho_0 \tilde{R}_0})\) satisfying (1.1), where \( C > 0 \) and \( 0 < \tau < 1 \) depend on \( r_1/r_3, r_2/r_3, \delta_0, A_2 \).
Proof. Let \( \tilde{u}' = u' - v'(x; r_1) \), \( \tilde{u}_3 = u_3 - v_3(x; r_1) \). Note that \( (\tilde{u}', \tilde{u}_3) \) satisfies the normalization condition (3.9) on \( B_{r_1} \). Recall that \( \tilde{R}_0 \leq 1 \). Now combining (4.73), (4.76), (4.2), and (3.10) implies that

\[
\int_{B_{r_2}} \sum_{ij} |e_{ij}^0(u)|^2 + \rho_0^2 |\partial_{ij}^2 u_3|^2 \\
= \int_{B_{r_2}} \sum_{ij} |e_{ij}^0(\tilde{u})|^2 + \rho_0^2 |\partial_{ij}^2 \tilde{u}_3|^2 \\
\leq \frac{C}{r_2^4} \int_{B_{r_2}} |\tilde{u}'|^2 + |\tilde{u}_3|^2 \\
\leq \frac{C'}{r_2^6} \left( \int_{B_{r_1}} |\tilde{u}'|^2 + |\tilde{u}_3|^2 \right)^\tau \left( \int_{B_{r_3}} |\tilde{u}'|^2 + |\tilde{u}_3|^2 \right)^{1-\tau} \\
\leq \frac{C''}{r_2^6} \left( \int_{B_{r_1}} |e_{ij}^0(u)|^2 + \rho_0^2 |\partial_{ij}^2 u_3|^2 \right)^\tau \left( \int_{B_{r_3}} |e_{ij}^0(u)|^2 + \rho_0^2 |\partial_{ij}^2 u_3|^2 \right)^{1-\tau} \\
\leq C'' \frac{1}{r_2^6} \left( \frac{r_3}{r_1} \right)^{2-2\tau} \\
\times \left( \int_{B_{r_1}} \sum_{ij} |e_{ij}^0(u)|^2 + \rho_0^2 |\partial_{ij}^2 u_3|^2 \right)^\tau \left( \int_{B_{r_3}} \sum_{ij} |e_{ij}^0(u)|^2 + \rho_0^2 |\partial_{ij}^2 u_3|^2 \right)^{1-\tau}.
\]

A key ingredient in solving our inverse problem is a continuation estimate from the interior for the solution \( u \) of (3.4), (3.5). To do this, we need some assumptions the coupled field \( (\hat{T}, \hat{M}) \). We assume that \( (\hat{T}, \hat{M}) \) satisfies

\[
\text{supp} (\hat{T}, \hat{M}) \subset \Gamma_0,
\]

where \( \Gamma_0 \) is an open subarc of \( \partial \Omega \) with

\[
|\Gamma_0| \leq (1 - \gamma_0) |\partial \Omega|
\]

for some \( \gamma_0 > 0 \). We first prove the following lemma.

**Lemma 4.14.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) with \( C^{2,1} \) boundary \( \partial \Omega \) characterized by constants \( A_0 \) and \( \rho_0 \). Assume that \( \lambda, \mu \in L^\infty(\Omega) \) satisfy (3.3), \( \nabla \theta \in L^\infty(\Omega) \), and

\[
\|\lambda\|_{L^\infty(\Omega)} + \|\mu\|_{L^\infty(\Omega)} + \|\nabla \theta\|_{L^\infty(\Omega)} \leq A_2
\]
for some $A_2 > 0$. Let $(u', u_3) \in (H^1(\Omega))^2 \times H^2(\Omega)$ be the unique weak solution of (3.4), (3.5) satisfying (3.9), with $(\hat{T}, \hat{M}) \in H^{-1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ satisfying (4.78), (4.79), and (3.8). Then we have

$$\| (\hat{T}, \hat{M}) \|_{(H^{-1/2}(\partial\Omega))^3} \leq C \left( \int_{\Omega} \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2 \right)^{1/2}, \quad (4.80)$$

where $C$ depends on $\delta_0$, $A_0$, $A_1$, $A_2$, $\gamma_0$.

Proof. We follow the arguments used in the proof of [20, Lemma 7.1]. For any $(f, g) \in (H^{1/2}(\partial\Omega))^3$, one can find $(v', v_3) \in (H^1(\Omega))^2 \times H^2(\Omega)$ such that $v'|_{\partial\Omega} = f$, $v_3|_{\partial\Omega} = 0$, $\partial_v v_3|_{\partial\Omega} = g$ and

$$\| (v', v_3) \|_{(H^1(\Omega))^2 \times H^2(\Omega)} \leq C \| (f, \rho_0 g) \|_{(H^{1/2}(\partial\Omega))^3}, \quad (4.81)$$

where $C$ depends on $A_0$ and $A_1$. In view of the weak formulation of the solution, we can compute

$$\begin{align*}
\int_{\partial\Omega} \frac{1}{\rho_0} \hat{T} \cdot f + \hat{M}_v g &= \frac{1}{\rho_0} \int_{\partial\Omega} \hat{T} \cdot v' + \hat{M}_v \rho_0 g \\
&= \frac{1}{\rho_0} \int_{\partial\Omega} \hat{T} \cdot v' + \rho_0 \hat{M}_v \partial_v v_3 + \rho_0 \partial_s \hat{M}_\tau v_3 \\
&= \frac{1}{\rho_0^2} \int_{\Omega} \sum_{ij} (\rho_0^2 \theta_{ij}(u)e^\theta_{ij}(v) + \rho_0^4 m_{ij}(u_3) \partial^2_{ij} v_3) \\
&\leq C \left( \frac{1}{\rho_0^2} \int_{\Omega} \sum_{ij} \rho_0^2 |e^\theta_{ij}(u)|^2 + \rho_0^4 |\partial^2_{ij} u_3|^2 \right)^{1/2} \\
&\times \left( \frac{1}{\rho_0^2} \int_{\Omega} \sum_{ij} \rho_0^2 |e^\theta_{ij}(v)|^2 + \rho_0^4 |\partial^2_{ij} v_3|^2 \right)^{1/2} \\
&\leq C \left( \frac{1}{\rho_0^2} \int_{\Omega} \sum_{ij} \rho_0^2 |e^\theta_{ij}(u)|^2 + \rho_0^4 |\partial^2_{ij} u_3|^2 \right)^{1/2} \| (v', v_3) \|_{(H^1(\Omega))^2 \times H^2(\Omega)} \\
&\leq C \left( \frac{1}{\rho_0^2} \int_{\Omega} \sum_{ij} \rho_0^2 |e^\theta_{ij}(u)|^2 + \rho_0^4 |\partial^2_{ij} u_3|^2 \right)^{1/2} \| (f, \rho_0 g) \|_{(H^{1/2}(\partial\Omega))^3},
\end{align*}$$

where $C$ depends on $\delta_0$, $A_0$, $A_1$, $A_2$. Consequently, we obtain

$$\| (\hat{T}, \hat{M}_v) \|_{(H^{-1/2}(\partial\Omega))^3} \leq C \left( \int_{\Omega} \sum_{ij} |e^\theta_{ij}(u)|^2 + \rho_0^2 |\partial^2_{ij} u_3|^2 \right)^{1/2}. \quad (4.82)$$
On the other hand, for any given $g \in H^{1/2}(\partial \Omega)$, let $h \in H^{3/2}(\Gamma_0)$ satisfy $\partial_\nu h = g$ on $\Gamma_0$ and $\|h\|_{H^{3/2}(\Gamma_0)} \leq C\|\rho_0 g\|_{H^{1/2}(\Gamma_0)} \leq C\|\rho_0 g\|_{H^{1/2}(\partial \Omega)}$, where $C$ here depends on $A_0$ and $A_1$. Now let $\tilde{h} \in H^{3/2}(\partial \Omega)$ be such that $\tilde{h} = h$ on $\Gamma_0$ and $\|\tilde{h}\|_{H^{3/2}(\partial \Omega)} \leq C\|h\|_{H^{3/2}(\Gamma_0)}$, where $C$ depends on $A_0$, $A_1$, $\gamma_0$. Moreover, let $v_3 \in H^2(\Omega)$ satisfy $v_3 = \tilde{h}$, $\partial_\nu v_3 = 0$ on $\partial \Omega$ and $\|v_3\|_{H^2(\Omega)} \leq C\|	ilde{h}\|_{H^{3/2}(\partial \Omega)}$, where $C$ depends on $A_0$ and $A_1$. Let $f \in (H^{1/2}(\partial \Omega))^2$ be the same function given above. Now we can derive that

$$
\int_{\partial \Omega} \frac{1}{\rho_0} \tilde{T} \cdot (-f) + \tilde{M}_t g = \frac{1}{\rho_0} \int_{\Gamma_0} \tilde{T} \cdot (-f) + \rho_0 \tilde{M}_t \partial_\nu h
$$

$$= -\frac{1}{\rho_0} \int_{\Gamma_0} \tilde{T} \cdot f + \rho_0 \partial_\nu \tilde{M}_t \tilde{h}
$$

$$= -\frac{1}{\rho_0} \int_{\partial \Omega} \tilde{T} \cdot f + \rho_0 \partial_\nu \tilde{M}_t v_3 + \rho_0 \tilde{M}_t \partial_\nu v_3
$$

$$\leq C \left( \frac{1}{\rho_0^2} \int_{\Omega} \sum_{ij} \rho_0^2 |e_{ij}^\theta(u)|^2 + \rho_0^4 |\partial_\nu^2 u_3|^2 \right)^{1/2} \| (f, \rho_0 g) \|_{(H^{1/2}(\partial \Omega))^3}$$

with $C$ depending on $\delta_0$, $A_0$, $A_1$, $A_2$, $\gamma_0$, which implies

$$\| (\tilde{T}, \tilde{M}) \|_{(H^{-1/2}(\partial \Omega))^3} \leq C \left( \int_{\Omega} \sum_{ij} |e_{ij}^\theta(u)|^2 + \rho_0^2 |\partial_\nu^2 u_3|^2 \right)^{1/2} . \quad (4.83)$$

Finally, combining (4.82) and (4.83) leads to (4.80). \qed

We are now ready to prove the following theorem.

**Theorem 4.15 (Lipschitz propagation of smallness).** Assume that $\Omega$ is a bounded domain having boundary $\partial \Omega \in C^{4,1}$ with constants $A_0$, $\rho_0$. Let $\lambda$, $\mu \in C^{1,1}(\bar{\Omega})$ satisfy (3.3) and $\bar{\theta} \in C^{2,1}(\bar{\Omega})$ satisfy (3.22) and (3.23) hold. Let $u \in (H^1(\Omega))^2 \times H^2(\Omega)$ be the weak solution of (3.4), (3.5) satisfying (3.9) with Neumann boundary condition $(\tilde{T}, \tilde{M}) \in (H^{1/2}(\partial \Omega))^2 \times H^{3/2}(\partial \Omega)$ satisfying (3.8), (4.78), (4.79). Then for every $\rho > 0$ and every $x \in \Omega_{R_1 / \rho}$, we have

$$\int_{B_{\rho \rho_0}(x)} \sum_{ij} |e_{ij}^\theta(u)|^2 + \rho_0^2 |\partial_\nu^2 u_3|^2 \geq C_\rho \int_{\Omega} \sum_{ij} |e_{ij}^\theta(u)|^2 + \rho_0^2 |\partial_\nu^2 u_3|^2 , \quad (4.84)$$

where $C_\rho$ depends on $A_0$, $A_1$, $A_2$, $\delta_0$, $\gamma_0$, $\rho$, and $\| (\tilde{T}, \tilde{M}) \|_{L^2(\partial \Omega)^2 \times H^{1/2}(\partial \Omega)} / \| (\tilde{T}, \tilde{M}) \|_{(H^{-1/2}(\partial \Omega))^3}$. Here $\bar{\theta} = R_1$ and $R_1$ is the constant given in Theorem 4.1.

**Proof.** There is no restriction to take $\rho_0 = 1$. The general case can be proved by a simple scaling argument. Note that $\Omega_{\rho \bar{\rho}}$ is connected for all $0 < \rho \leq \zeta$, where $\zeta$
depends on $\delta_0, A_0, A_2$. It suffices to prove the result for small $\rho$. We now choose
a $\rho$ such that $7\rho/\vartheta \leq \tilde{R}_0$. Let $y \in \Omega_{\vartheta}\rho$ and $\gamma(t)$ be an arc in $\Omega_{\vartheta}\rho$ joining $y$ and
$x$. We now define $\{x_i\}$, $i = 1, \ldots, L$, as follows: $x_1 = x, x_{i+1} = \gamma(t)$ with $t_i = \max\{|t| \mid \gamma(t) - x_i| > 2\rho\}$ if $|x_i - y| > 2\rho$, otherwise, let $i = L$ and stop the
process. By construction, we can see that the spheres $B_\rho(x_i)$ are pairwise disjoint
and $|x_i - x| = 2\rho$ for $i = 1, \ldots, L - 1, |x_L - y| \leq 2\rho$.
Since $x_i \in \Omega_{\vartheta}\rho$, we use the three-sphere inequality (4.77) with $x = x_i, r_1 = \rho, r_2 = 3\rho, r_3 = \frac{7\rho}{\vartheta} \leq \tilde{R}_0 < 1$ for $i = 1, \ldots, L - 1$ to obtain
\[
\frac{\int_{B_\rho(x_{i+1})} \sum_{ij} |e_{ij}^0(u)|^2 + |\partial^2_{ij} u_3|^2}{\int_{\Omega} \sum_{ij} |e_{ij}^0(u)|^2 + |\partial^2_{ij} u_3|^2} \leq C \left( \frac{1}{\rho} \right)^6 \left( \frac{\int_{B_\rho(x_i)} \sum_{ij} |e_{ij}^0(u)|^2 + |\partial^2_{ij} u_3|^2}{\int_{\Omega} \sum_{ij} |e_{ij}^0(u)|^2 + |\partial^2_{ij} u_3|^2} \right)^{\vartheta}
\]
where $C > 0$ depends on $\delta_0, A_2$. Induction on $i$ implies
\[
\frac{\int_{B_\rho(y)} \sum_{ij} |e_{ij}^0(u)|^2 + |\partial^2_{ij} u_3|^2}{\int_{\Omega} \sum_{ij} |e_{ij}^0(u)|^2 + |\partial^2_{ij} u_3|^2} \leq C^{1+\frac{1}{\vartheta}} \left( \frac{1}{\rho} \right)^{\vartheta} \left( \frac{\int_{B_\rho(x)} \sum_{ij} |e_{ij}^0(u)|^2 + |\partial^2_{ij} u_3|^2}{\int_{\Omega} \sum_{ij} |e_{ij}^0(u)|^2 + |\partial^2_{ij} u_3|^2} \right)^{\vartheta'^L}.
\]
Note that $L \leq |\Omega|/(\pi \rho^2) \leq A_1/\pi$.

Let us now cover $\Omega_{\vartheta}\rho$ with internally nonoverlapping closed cubes of side
$\ell = \sqrt{2\rho}/\vartheta$. It is clear that any such cube is contained in a sphere of radius $\rho$ with
center in $\Omega_{\vartheta}\rho$ and the number of such cube is controlled by $N = 2|\Omega|\vartheta^2/(4\rho^2)$. It follows from (4.85) that
\[
\frac{\int_{\Omega_{\vartheta}\rho}} \sum_{ij} |e_{ij}^0(u)|^2 + |\partial^2_{ij} u_3|^2}{\int_{\Omega} \sum_{ij} |e_{ij}^0(u)|^2 + |\partial^2_{ij} u_3|^2} \leq C \left( \frac{1}{\rho} \right)^{\vartheta - \frac{2}{\vartheta}} \left( \frac{\int_{B_\rho(x)} \sum_{ij} |e_{ij}^0(u)|^2 + |\partial^2_{ij} u_3|^2}{\int_{\Omega} \sum_{ij} |e_{ij}^0(u)|^2 + |\partial^2_{ij} u_3|^2} \right)^{\vartheta'^L}.
\]
Here $C$ depends on $\delta_0, A_2, |\Omega|$.

Now we want to estimate the left-hand side of (4.86) from below by a positive constant. Obviously, we have
\[
\frac{\int_{\Omega_{\vartheta}\rho}/} \sum_{ij} |e_{ij}^0(u)|^2 + |\partial^2_{ij} u_3|^2}{\int_{\Omega} \sum_{ij} |e_{ij}^0(u)|^2 + |\partial^2_{ij} u_3|^2} = 1 - \frac{\int_{\Omega \setminus \Omega_{\vartheta}\rho}} \sum_{ij} |e_{ij}^0(u)|^2 + |\partial^2_{ij} u_3|^2}{\int_{\Omega} \sum_{ij} |e_{ij}^0(u)|^2 + |\partial^2_{ij} u_3|^2}.
\]
It suffices to show that there exists $\overline{\rho} > 0$ such that
\[
\frac{\int_{\Omega \setminus \Omega_{\vartheta}\rho}} \sum_{ij} |e_{ij}^0(u)|^2 + |\partial^2_{ij} u_3|^2}{\int_{\Omega} \sum_{ij} |e_{ij}^0(u)|^2 + |\partial^2_{ij} u_3|^2} \leq \frac{1}{2},
\]
for every $\rho, 0 < \rho \leq \overline{\rho}$. The proposition, then, follows from (4.86) and (4.88).
By Hölder’s inequality and Sobolev’s inequality

\[ \| w \|_{L^4(\Omega)}^2 \leq C \| w \|_{H^{1/2}(\Omega)}^2 \]

with C depending on \( A_0, A_1 \), we have

\[ \int_{\Omega \setminus \Omega_{8\rho/\theta}} \sum_{ij} |e_{ij}^\theta (u)|^2 + |\partial_{ij}^2 u_3|^2 \]

\[ \leq |\Omega \setminus \Omega_{8\rho/\theta}| \sum_{ij} \left( \int_{\Omega \setminus \Omega_{8\rho/\theta}} \left( |e_{ij}^\theta (u)|^2 + |\partial_{ij}^2 u_3|^2 \right) \right)^{1/2} \]

\[ \leq \sqrt{2} |\Omega \setminus \Omega_{8\rho/\theta}| \sum_{ij} \left( \left( \int_{\Omega \setminus \Omega_{8\rho/\theta}} |e_{ij}^\theta (u)|^4 \right)^{1/2} + \left( \int_{\Omega \setminus \Omega_{8\rho/\theta}} |\partial_{ij}^2 u_3|^4 \right)^{1/2} \right) \]

\[ \leq C |\Omega \setminus \Omega_{8\rho/\theta}| (u', u_3)_{(H^{3/2}(\Omega))^2 \times H^{5/2}(\Omega)}^2 \]

Interpolating the global estimates (3.18) and (3.24) yields

\[ \| (u', u_3) \|_{(H^{3/2}(\Omega))^2 \times H^{5/2}(\Omega)} \]

\[ \leq \| (u', u_3) \|_{(H^{3/2}(\Omega))^2 \times H^3(\Omega)} \]

\[ \leq C \| (\widehat{T}, \widehat{\mathcal{M}}) \|_{(L^2(\partial \Omega))^2 \times H^{1/2}(\partial \Omega)} \]

(4.90)

where C depends on \( A_0, A_1, A_2, \delta_0 \).

Following the argument of [6] (see [6, A.3] for details), there exists a positive constant C, depending on \( A_0, A_1, A_2, \delta_0, \delta_0 \), such that

\[ |\Omega \setminus \Omega_{8\rho/\theta}| \leq C \rho. \]

(4.91)

It follows from (4.89), (4.90), and (4.91) that

\[ \int_{\Omega \setminus \Omega_{8\rho/\theta}} \sum_{ij} |e_{ij}^\theta (u)|^2 + |\partial_{ij}^2 u_3|^2 \leq C \rho^{1/2} \| (\widehat{T}, \widehat{\mathcal{M}}) \|_{(L^2(\partial \Omega))^2 \times H^{1/2}(\partial \Omega)}^2. \]

(4.92)

From (4.80), we can obtain that

\[ \frac{\int_{\Omega \setminus \Omega_{8\rho/\theta}} \sum_{ij} |e_{ij}^\theta (u)|^2 + |\partial_{ij}^2 u_3|^2}{\int_{\Omega} \sum_{ij} |e_{ij}^\theta (u)|^2 + |\partial_{ij}^2 u_3|^2} \leq C \rho^{1/2} \frac{\| (\widehat{T}, \widehat{\mathcal{M}}) \|_{(L^2(\partial \Omega))^2 \times H^{1/2}(\partial \Omega)}^2}{\| (\widehat{T}, \widehat{\mathcal{M}}) \|_{(H^{-1/2}(\partial \Omega))^3}}. \]

where C depends on \( A_0, A_1, A_2, \delta_0, \gamma_0 \). Finally, we can choose \( \bar{\rho} \), depending on \( A_0, A_1, A_2, \delta_0, \gamma_0 \), and \( \| (\widehat{T}, \widehat{\mathcal{M}}) \|_{(L^2(\partial \Omega))^2 \times H^{1/2}(\partial \Omega)}^2 / \| (\widehat{T}, \widehat{\mathcal{M}}) \|_{(H^{-1/2}(\partial \Omega))^3} \), such that

\[ \frac{\int_{\Omega \setminus \Omega_{8\rho/\theta}} \sum_{ij} |e_{ij}^\theta (u)|^2 + |\partial_{ij}^2 u_3|^2}{\int_{\Omega} \sum_{ij} |e_{ij}^\theta (u)|^2 + |\partial_{ij}^2 u_3|^2} \leq \frac{1}{2} \]

for all \( 0 < \rho < \bar{\rho} \). The proof now is complete.
5. The inverse problem

In this section, we would like to study the problem of estimating the size of inclusion embedded in a shallow shell by one boundary measurement. Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain with boundary $\partial \Omega$, which is of class $C^{4,1}$ with constants $A_0, \rho_0$. Assume that (3.1) holds. Now let $D$ be a possibly disconnected measurable subdomain of $\Omega$ satisfying

$$\text{dist}(D, \partial \Omega) \geq d_0 \rho_0$$

(5.1)

for some given constant $d_0$. Let $\lambda, \mu \in C^{1,1}(\tilde{\Omega})$ satisfy (3.3) and $\tilde{\theta} \in C^{2,1}(\tilde{\Omega})$ satisfy (3.22). Besides, assume that the estimate (3.23) holds. For measurable functions $\lambda_0, \mu_0$, we define

$$\tilde{\lambda} = \lambda + \chi_D \lambda_0 \quad \text{and} \quad \tilde{\mu} = \mu + \chi_D \mu_0,$$

where $\chi_D$ is the characteristic function of $D$. To guarantee the well-posedness of the forward problem, we assume

$$0 < \tilde{\delta}_0 \leq \tilde{\lambda} \quad \text{and} \quad \tilde{\delta}_0 \leq \tilde{\mu} \quad \forall \ x \in \Omega.$$

To describe the jump condition, we introduce some shorthand notations. We set

$$a = \frac{4\lambda \mu}{\lambda + 2 \mu}, \ b = 4 \mu, \ c = \frac{4\lambda \mu}{3(\lambda + 2 \mu)}, \ d = \frac{4\mu}{3}$$

(5.2)

and the corresponding $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ replacing $\lambda, \mu$ with $\tilde{\lambda}, \tilde{\mu}$ respectively. We assume the following condition on the jump at the interface $\partial D$. There exists a constant $k_0 > 0$ such that

$$(\tilde{f} - f) \leq k_0 f \quad \forall \ x \in \partial D,$$

(5.3)

where $f = a, b, c, d$ and $\tilde{f} = \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$. On the prescribed boundary field $(\tilde{T}, \tilde{M})$, we assume

$$(\tilde{T}, \tilde{M}) \in (H^{1/2}(\partial \Omega))^2 \times H^{3/2}(\partial \Omega) \quad \text{and} \quad \text{supp} (\tilde{T}, \tilde{M}) \subset \Gamma_0,$$

(5.4)

where $\Gamma_0$ is an open subarc of $\partial \Omega$ with

$$|\Gamma_0| \leq (1 - \gamma_0)|\partial \Omega|$$

(5.5)

for some $\gamma_0 > 0$ and satisfies the compatibility condition (3.8). We consider two boundary value problems. Let $u = (u', u_3)$ satisfy

$$
\begin{align*}
\partial_j n_{ij}^\theta (u) &= 0 \quad \text{in} \quad \Omega, \\
\partial_j^2 m_{ij}(u_3) - \partial_j (n_{ij}^\theta(u) \partial_i \theta) &= 0 \quad \text{in} \quad \Omega,
\end{align*}
$$

(5.6)
with boundary conditions

\[
\begin{align*}
&n_{ij}^{\theta}(u)v_j = \rho_0^{-1} \hat{T}_i, \\
&m_{ij}(u_3)v_i v_j = \hat{M}_v, \\
&(\partial_t m_{ij}(u_3) - n_{ij}^{\theta}(u)\partial_t \theta)v_j + \partial_s (m_{ij}(u_3)v_i \tau_j) = -\partial_s \hat{M}_\tau.
\end{align*}
\]  

(5.7)

Next we let \( \tilde{u} = (\tilde{u}', \tilde{u}_3) \) satisfy

\[
\begin{align*}
&\partial_j \tilde{n}_{ij}^{\theta}(\tilde{u}) = 0 \quad \text{in} \quad \Omega, \\
&\partial^2_{ij} \tilde{m}_{ij}(\tilde{u}_3)v_i v_j = 0 \quad \text{in} \quad \Omega,
\end{align*}
\]

(5.8)

with boundary conditions

\[
\begin{align*}
&\tilde{n}_{ij}^{\theta}(\tilde{u})v_j = \rho_0^{-1} \hat{T}_i, \\
&\tilde{m}_{ij}(\tilde{u}_3)v_i v_j = \hat{M}_v, \\
&(\partial_t \tilde{m}_{ij}(\tilde{u}_3) - \tilde{n}_{ij}^{\theta}(\tilde{u})\partial_t \theta)v_j + \partial_s (\tilde{m}_{ij}(\tilde{u}_3)v_i \tau_j) = -\partial_s \hat{M}_\tau.
\end{align*}
\]  

(5.9)

To ensure the uniqueness of the solution, we impose the normalization conditions (3.9). Let

\[
\tilde{W} = \int_{\Omega} \rho_0^{-1} \hat{T} \cdot \tilde{u}' + \hat{M}_v \partial_v \tilde{u}_3 + \partial_s \hat{M}_\tau \tilde{u}_3,
\]

\[
W = \int_{\Omega} \rho_0^{-1} \hat{T} \cdot u' + \hat{M}_v \partial_v u_3 + \partial_s \hat{M}_\tau u_3 = \int_{\Omega} \sum_{ij} n_{ij}^{\theta}(u)e_{ij}^{\theta}(u) + m_{ij}(u_3)\partial^2_{ij}u_3,
\]

(5.10)

represent the work exerted by the boundary field when the inclusion is present or absent, respectively. For \( r > 0 \) we shall use the notation

\[
D_r = \{x \in D : \text{dist}(x, \partial D) > r\}.
\]

We can now state our main result.

**Theorem 5.1.** Suppose that all the hypotheses stated in this section hold. Moreover, we assume \( \rho_0 < 1 \). Let \( D \) be an inclusion satisfying the following fatness condition

\[
|D_{h_1\rho_0}| \geq \frac{1}{2} |D|
\]  

(5.11)

for a given positive constant \( h_1 \). Then we have the estimate

\[
C_1 \rho_0^2 \left| \frac{W - \tilde{W}}{W} \right| \leq |D| \leq C_2 \rho_0^2 \left| \frac{W - \tilde{W}}{W} \right|,
\]

(5.12)

where \( C_1 \) depends on \( A_0, A_1, A_2, d_0, k_0 \) and \( \delta_0 \) and \( C_2 \) depends on \( A_0, A_1, A_2, \delta_0, \gamma_0, d_0, h_1 \) and the ratio

\[
\|(\tilde{M}, \hat{T})\|_{(L^2(\partial\Omega))^2 \times H^{1/2}(\partial\Omega)^3} / \|(\tilde{M}, \hat{T})\|_{(H^{-1/2}(\partial\Omega))^3}.
\]
The key ingredients in the proof of Theorem 5.1 are the energy estimate for the Neumann problems (5.6)-(5.7), (5.8)-(5.9), and the Lipschitz propagation of smallness.

Lemma 5.2. Let \((\mathcal{M}, \mathcal{T}) \in L^2(\partial \Omega)\) satisfy (5.4), (5.5) and (3.8). Let \(\lambda, \mu, \tilde{\lambda}, \tilde{\mu} \in L^\infty(\Omega)\) satisfy (3.3) and (5.3). Let \(u, \tilde{u} \in (H^1(\Omega))^2 \times H^2(\Omega)\) solutions to (5.6)-(5.7) and (5.8)-(5.9) respectively. Then there exist positive constants \(\tilde{C}_1, \tilde{C}_2\) depending on \(A_0, A_1, A_2, \delta_0, k_0\) such that

\[
\tilde{C}_1 \int_D \sum_{ij} |e_{ij}^\theta(u)|^2 + \rho_0^2 |\partial_{ij}^2 u_3|^2 \leq W - \tilde{W} \leq \tilde{C}_2 \int_D \sum_{ij} |e_{ij}^\theta(u)|^2 + \rho_0^2 |\partial_{ij}^2 u_3|^2. \tag{5.13}
\]

With the help of Lemma 5.2 and Theorem 4.15, we first prove Theorem 5.1.

Proof of Theorem 5.1. By the interior regularity theorem and the Sobolev embedding, we have that

\[
\left\| \sum_{ij} |e_{ij}^\theta(u)|^2 + \rho_0^2 |\partial_{ij}^2 u_3|^2 \right\|_{L^\infty(D)} \leq \frac{C}{\rho_0^2} \left( \|u\|^2_{H^1(\Omega)} + \|u_3\|^2_{H^2(\Omega)} \right), \tag{5.14}
\]

with \(C\) depending on \(A_2, d_0, \delta_0\). From (5.14), Proposition 3.1, (5.10), we obtain that

\[
\left\| \sum_{ij} |e_{ij}^\theta(u)|^2 + \rho_0^2 |\partial_{ij}^2 u_3|^2 \right\|_{L^\infty(D)} \leq \frac{C}{\rho_0^2} W, \tag{5.15}
\]

where \(C\) depends on \(A_0, A_1, A_2, d_0, \delta_0\). The lower bound on \(|D|\) in (5.12) follows from the right-hand side of (5.13) and from (5.15).

Let us prove the upper bound for \(|D|\). Let \(\varepsilon = \min\{2d_0 \delta / 7, h_1 / \sqrt{2}\}\) and let us cover \(D_{h_1, \rho_0}\) with internally non overlapping closed squares \(Q_l\) of side \(\varepsilon \rho_0\) for \(l = 1, \ldots, L\). By choice of \(\varepsilon\) the squares \(Q_l\) are contained in \(D\). Let \(\tilde{l}\) be such that

\[
\int_{Q_{\tilde{l}}} \sum_{ij} |e_{ij}^\theta(u)|^2 + \rho_0^2 |\partial_{ij}^2 u_3|^2 = \min_l \int_{Q_l} \sum_{ij} |e_{ij}^\theta(u)|^2 + \rho_0^2 |\partial_{ij}^2 u_3|^2.
\]

Noticing that \(|D_{h_1, \rho_0}| \leq L \varepsilon^2 \rho_0^2\), we have

\[
\int_D \sum_{ij} |e_{ij}^\theta(u)|^2 + \rho_0^2 |\partial_{ij}^2 u_3|^2 \geq \int_{\bigcup_{l=1}^{\tilde{l}} Q_l} \sum_{ij} |e_{ij}^\theta(u)|^2 + \rho_0^2 |\partial_{ij}^2 u_3|^2 \geq L \int_{Q_{\tilde{l}}} \sum_{ij} |e_{ij}^\theta(u)|^2 + \rho_0^2 |\partial_{ij}^2 u_3|^2 \tag{5.16}
\]

\[
\geq \frac{|D_{h_1, \rho_0}|}{\rho_0^2 \varepsilon^2} \int_{Q_{\tilde{l}}} \sum_{ij} |e_{ij}^\theta(u)|^2 + \rho_0^2 |\partial_{ij}^2 u_3|^2.
\]
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Let \( \bar{x} \) be the center of \( Q_\tau \). From (5.16), estimate (4.84) with \( x = \bar{x} \) and \( \rho = \varepsilon/2 \), and hypothesis (5.11), we conclude that

\[
\int_D \sum_{ij} |e_{ij}^\theta(u)|^2 + \rho_0^2 |\theta_{ij}^\theta u_3|^2 \geq \frac{K|D|}{\rho_0^2} W,
\]

where \( K \) is a positive constant depending on \( A_0, A_1, A_2, d_0, \delta_0, \gamma_0, h_1 \) and \( \|(\hat{\mathbf{M}}, \hat{T})\|_{L^2(\partial\Omega)^2 \times H^{1/2}(\partial\Omega)} / \|(\hat{\mathbf{M}}, \hat{T})\|_{H^{-1/2}(\partial\Omega)^3} \). The upper bound for \( D \) follows from the left-hand side of (5.13) and from (5.17).

To end this section, we give a proof of Lemma 5.2.

**Proof of Lemma 5.2.** Let \( w = (w', w_3) \). From the first equation of (5.6), we have that

\[
\int_\Omega n_{ij}^\theta(u) \partial_j w_i' = \int_{\partial\Omega} \rho_0^{-1} \hat{T}_i w_i'.
\]

On the other hand, by the second equation of (5.6), the integration by parts leads to

\[
\int_\Omega m_{ij}(u_3) \partial_{ij}^2 w_3 + n_{ij}^\theta(u) \partial_i \theta \partial_j w_3
\]

\[
= \int_{\partial\Omega} -\partial_i m_{ij}(u_3) v_j w_3 + m_{ij}(u_3) v_i \partial_j w_3 + n_{ij}^\theta(u) \partial_i \theta v_j w_3. \tag{5.19}
\]

Replacing \( \partial_j w_3 \) by \( v_j \partial_v w_3 + \tau_j \partial_\tau w_3 \) in (5.19) gives

\[
\int_\Omega m_{ij}(u_3) \partial_{ij}^2 w_3 + n_{ij}^\theta(u) \partial_i \theta \partial_j w_3
\]

\[
= \int_{\partial\Omega} -\partial_i m_{ij}(u_3) v_j w_3 + m_{ij}(u_3) v_i (v_j \partial_v w_3 + \tau_j \partial_\tau w_3) + n_{ij}^\theta(u) \partial_i \theta v_j w_3
\]

\[
= \int_{\partial\Omega} -\partial_i m_{ij}(u_3) v_j w_3 - \partial_\tau (m_{ij}(u_3) v_i \tau_j) w_3 + n_{ij}^\theta(u) \partial_i \theta v_j w_3
\]

\[
+ m_{ij}(u_3) v_i v_j \partial_v w_3
\]

\[
= \int_{\partial\Omega} \partial_\tau \hat{\mathbf{M}}_\tau w_3 + \hat{\mathbf{M}}_v \partial_v w_3. \tag{5.20}
\]

Thus, by combining (5.18) and (5.19), we get

\[
\int_\Omega n_{ij}^\theta(u) \partial_j w_i' + m_{ij}(u_3) \partial_{ij}^2 w_3 + n_{ij}^\theta(u) \partial_i \theta \partial_j w_3
\]

\[
= \int_{\partial\Omega} \rho_0^{-1} \hat{T}_i w_i' + \partial_\tau \hat{\mathbf{M}}_\tau w_3 + \hat{\mathbf{M}}_v \partial_v w_3. \tag{5.21}
\]
Likewise, we can deduce

\[
\int_{\Omega} \tilde{n}^0_{ij}(\tilde{u}) \partial_j w'_i + \tilde{m}_{ij}(\tilde{u}_3) \partial^2_{ij} w'_3 + \tilde{n}^0_{ij}(\tilde{u}) \partial_i \theta \partial_j w'_3
\]

\[
= \int_{\partial\Omega} \rho_0^{-1} \tilde{T}_i w'_i + \partial_s \tilde{M}_\tau w'_3 + \tilde{M}_\nu \partial_i w'_3.
\]

(5.22)

and therefore,

\[
\int_{\Omega} n^0_{ij}(u) \partial_j w'_i + m_{ij}(u_3) \partial^2_{ij} w'_3 + n^0_{ij}(u) \partial_i \theta \partial_j w'_3
\]

\[
= \int_{\Omega} \tilde{n}^0_{ij}(\tilde{u}) \partial_j w'_i + \tilde{m}_{ij}(\tilde{u}_3) \partial^2_{ij} w'_3 + \tilde{n}^0_{ij}(\tilde{u}) \partial_i \theta \partial_j w'_3.
\]

In turn, we obtain

\[
\int_{\Omega} \tilde{n}^0_{ij}(\tilde{u} - u) \partial_j \tilde{w}'_i + \tilde{m}_{ij}(\tilde{u}_3 - u_3) \partial^2_{ij} \tilde{w}'_3 + \tilde{n}^0_{ij}(\tilde{u} - u) \partial_i \theta \partial_j \tilde{w}'_3
\]

\[
= \int_{\Omega} (n^0_{ij} - \tilde{n}^0_{ij})(u) \partial_j w'_i + (m_{ij} - \tilde{m}_{ij})(u) \partial^2_{ij} w'_3
\]

\[
+ (n^0_{ij} - \tilde{n}^0_{ij})(u) \partial_i \theta \partial_j w'_3.
\]

(5.23)

Substituting \( w = \tilde{u} \) into (5.23), we get that

\[
\int_{\Omega} \tilde{n}^0_{ij}(\tilde{u} - u) \partial_j \tilde{w}_i + \tilde{m}_{ij}(\tilde{u}_3 - u_3) \partial^2_{ij} \tilde{w}_3 + \tilde{n}^0_{ij}(\tilde{u} - u) \partial_i \theta \partial_j \tilde{w}_3
\]

\[
= \int_{\Omega} (n^0_{ij} - \tilde{n}^0_{ij})(u) \partial_j \tilde{w}_i + (m_{ij} - \tilde{m}_{ij})(u) \partial^2_{ij} \tilde{w}_3
\]

\[
+ (n^0_{ij} - \tilde{n}^0_{ij})(u) \partial_i \theta \partial_j \tilde{w}_3.
\]

(5.24)

By straightforward computations, we can see that

\[
\int_{\Omega} \tilde{n}^0_{ij}(\tilde{u} - u) \partial_j \tilde{w}_i + \tilde{m}_{ij}(\tilde{u}_3 - u_3) \partial^2_{ij} \tilde{w}_3 + \tilde{n}^0_{ij}(\tilde{u} - u) \partial_i \theta \partial_j \tilde{w}_3
\]

\[
= \int_{\partial\Omega} \rho_0^{-1} \tilde{T}_i (\tilde{u}'_i - u'_i) + \partial_s \tilde{M}_\tau (\tilde{u}_3 - u_3) + \tilde{M}_\nu \partial_i (\tilde{u}_3 - u_3)
\]

and it follows from (5.24) that

\[
\int_{\Omega} (n^0_{ij} - \tilde{n}^0_{ij})(u) \partial_j \tilde{w}_i + (m_{ij} - \tilde{m}_{ij})(u) \partial^2_{ij} \tilde{w}_3 + (n^0_{ij} - \tilde{n}^0_{ij})(u) \partial_i \theta \partial_j \tilde{w}_3.
\]

\[
= \int_{\partial\Omega} \rho_0^{-1} \tilde{T}_i (\tilde{u}'_i - u'_i) + \partial_s \tilde{M}_\tau (\tilde{u}_3 - u_3) + \tilde{M}_\nu \partial_i (\tilde{u}_3 - u_3).
\]

(5.25)
Now replacing $w = \tilde{u} - u$ in (5.23) and using (5.25), we obtain that

$$\int_{\Omega} \tilde{n}_{ij}^\theta (\tilde{u} - u) \partial_j (\tilde{u}'_i - u'_i) + \tilde{m}_{ij} (\tilde{u}_3 - u_3) \partial^2_{ij} (\tilde{u}_3 - u_3)$$

$$+ \tilde{n}_{ij}^\theta (\tilde{u} - u) \partial_i \theta \partial_j (\tilde{u}_3 - u_3)$$

$$= \int_{\Omega} (n_{ij}^\theta - \tilde{n}_{ij}^\theta) (u) \partial_j (\tilde{u}'_i - u'_i) + (m_{ij} - \tilde{m}_{ij})(u) \partial^2_{ij} (\tilde{u}_3 - u_3)$$

$$+ (n_{ij}^\theta - \tilde{n}_{ij}^\theta) (u) \partial_i \theta \partial_j (\tilde{u}_3 - u_3)$$

$$= \int_{\partial \Omega} \rho_0^{-1} \tilde{T}_i (\tilde{u}'_i - u'_i) + \partial_s \tilde{M}_\tau (u_3 - \tilde{u}_3) + \tilde{M}_\nu \partial_v (u_3 - \tilde{u}_3) + \int_D (n_{ij}^\theta - \tilde{n}_{ij}^\theta)(\tilde{u}) \partial_j \tilde{u}'_i$$

$$+ (m_{ij} - \tilde{m}_{ij})(\tilde{u}) \partial^2_{ij} \tilde{u}_3 + (n_{ij}^\theta - \tilde{n}_{ij}^\theta)(\tilde{u}) \partial_i \theta \partial_j \tilde{u}_3.$$  

Exchanging the role of $\tilde{u}$ and $u$, we can deduce that

$$\int_{\Omega} n_{ij}^\theta (u - \tilde{u}) \partial_j (u'_i - \tilde{u}'_i) + m_{ij} (u_3 - \tilde{u}_3) \partial^2_{ij} (u_3 - \tilde{u}_3)$$

$$+ n_{ij}^\theta (u - \tilde{u}) \partial_i \theta \partial_j (u_3 - \tilde{u}_3)$$

$$= \int_{\partial \Omega} \rho_0^{-1} \tilde{T}_i (u'_i - \tilde{u}'_i) + \partial_s \tilde{M}_\tau (u_3 - \tilde{u}_3) + \tilde{M}_\nu \partial_v (u_3 - \tilde{u}_3)$$

$$+ \int_D (n_{ij}^\theta - \tilde{n}_{ij}^\theta)(u) \partial_j \tilde{u}'_i$$

$$+ (m_{ij} - \tilde{m}_{ij})(\tilde{u}) \partial^2_{ij} \tilde{u}_3 + (n_{ij}^\theta - \tilde{n}_{ij}^\theta)(\tilde{u}) \partial_i \theta \partial_j \tilde{u}_3.$$  

Finally, plugging $w = \tilde{u}$ into (5.21) and $w = u$ into (5.22), respectively, we have that

$$\int_D (n_{ij}^\theta - n_{ij}^\theta)(\tilde{u}) \partial_j u_i + (m_{ij} - m_{ij})(\tilde{u}_3) \partial^2_{ij} u_3 + (n_{ij}^\theta - n_{ij}^\theta)(\tilde{u}) \partial_i \theta \partial j u_3$$

$$= \int_{\partial \Omega} \rho_0^{-1} \tilde{T}_i (u'_i - \tilde{u}'_i) + \partial_s \tilde{M}_\tau (u_3 - \tilde{u}_3) + \tilde{M}_\nu \partial_v (u_3 - \tilde{u}_3).$$  

The following identity is useful in our further arguments. Let $a, b, c$ and $d$ be any functions. It is easy to compute that

$$\left( a e_{kk}^\theta (w) \delta_{ij} + b e_{ij}^\theta (w) \right) \partial_j w_i + \left( c (\Delta w_3) \delta_{ij} + d \partial^2_{ij} w_3 \right) \partial^2_{ij} w_3$$

$$+ \left( a e_{kk}^\theta (w) \delta_{ij} + b e_{ij}^\theta (w) \right) \partial_i \theta \partial_j w_3$$

$$= a |\nabla \cdot w'| + \nabla \theta \cdot \nabla w_3|^2 + b \sum_{ij} |e_{ij}^\theta (w)|^2 + c |\Delta w_3|^2 + d \sum_{ij} |\partial^2_{ij} w_3|^2$$

(5.29)
Now let $a, b, c, d$ be given in (5.2) and $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ be defined similarly. Putting $w = \tilde{v} - u$, using (5.29), (5.26), we get that

\[
W - \tilde{W} \leq \int_D (\tilde{n}_{ij}^\theta - n_{ij}^\theta)(u)\partial_j u_i + (\tilde{m}_{ij} - m_{ij})(u)\partial_j u_i + (\tilde{n}_{ij}^\theta - n_{ij}^\theta)(u)\partial_i \theta \partial_j u_3 \\
= \int_D (\tilde{a} - a)|\nabla \cdot u' + \nabla \theta \cdot \nabla u_3|^2 + (\tilde{b} - b) \sum_{ij} |e_{ij}^\theta(u)|^2 \\
+ (\tilde{c} - c)|\Delta u_3|^2 + (\tilde{d} - d) \sum_{ij} |\partial^2_{ij} u_3|^2.
\]

(5.30)

On the other hand, for any $\varepsilon > 0$, one can easily compute that

\[
\int_D (\tilde{a} - a)|\nabla \cdot u' + \nabla \theta \cdot \nabla u_3|^2 + (\tilde{b} - b) \sum_{ij} |e_{ij}^\theta(u)|^2 + (\tilde{c} - c)|\Delta u_3|^2 \\
+ (\tilde{d} - d) \sum_{ij} |\partial^2_{ij} u_3|^2 \\
\leq (1 + \varepsilon^{-1}) \int_D (\tilde{a} - a)|\nabla \cdot (\tilde{u}' - u') + \nabla \theta \cdot \nabla (\tilde{u}_3 - u_3)|^2 \\
+ (\tilde{b} - b) \sum_{ij} |e_{ij}^\theta(\tilde{u} - u)|^2 + (\tilde{c} - c)|\Delta (\tilde{u}_3 - u_3)|^2 \\
+(\tilde{d} - d) \sum_{ij} |\partial^2_{ij} (\tilde{u}_3 - u_3)|^2 + (1 + \varepsilon) \int_D (\tilde{a} - a)|\nabla \cdot \tilde{u}' + \nabla \theta \cdot \nabla \tilde{u}_3|^2 \\
+ (\tilde{b} - b) \sum_{ij} |e_{ij}^\theta(\tilde{u})|^2 + (\tilde{c} - c)|\Delta \tilde{u}_3|^2 + (\tilde{d} - d) \sum_{ij} |\partial^2_{ij} \tilde{u}_3|^2.
\]

(5.31)

By (5.3) we can choose $\varepsilon_* > 0$ such that

\[
\frac{1 + \varepsilon_*^{-1}}{1 + \varepsilon_*} = \frac{1}{k_0}.
\]

Therefore, from (5.31) and (5.29), we deduce that

\[
\int_D (\tilde{a} - a)|\nabla \cdot u' + \nabla \theta \cdot \nabla u_3|^2 + (\tilde{b} - b) \sum_{ij} |e_{ij}^\theta(u)|^2 + (\tilde{c} - c)|\Delta u_3|^2 \\
+ (\tilde{d} - d) \sum_{ij} |\partial^2_{ij} u_3|^2 \\
\leq (1 + \varepsilon_*) \left( \int_D n_{ij}^\theta(u - \tilde{u})\partial_j (u_i' - \tilde{u}_i') + m_{ij}(u_3 - \tilde{u}_3)\partial^2_{ij}(u_3 - \tilde{u}_3) \\
+ n_{ij}(u - \tilde{u})\partial_i \theta \partial_j (u_3 - \tilde{u}_3) \\
+ \int_D (\tilde{n}_{ij}^\theta - n_{ij}^\theta)(u)\partial_j \tilde{u}_i + (\tilde{m}_{ij} - m_{ij})(\tilde{u})\partial^2_{ij} \tilde{u}_3 + (\tilde{n}_{ij}^\theta - n_{ij}^\theta)(\tilde{u})\partial_i \theta \partial_j \tilde{u}_3 \right).
\]

(5.32)
Now combining (5.27) and (5.32) immediately yields

\[
\frac{1}{1 + \varepsilon_*} \int_D (\tilde{a} - a) |\nabla \cdot u' + \nabla \theta \cdot \nabla u_3|^2 + (\tilde{b} - b) \sum_{ij} |e_{ij}^\theta(u)|^2 \\
+ (\tilde{c} - c) |\Delta u_3|^2 + (\tilde{d} - d) \sum_{ij} |\partial_{ij}^2 u_3|^2 \leq W - \tilde{W}
\]

(5.33)

and we obtain (5.13). \qed

Remark 5.3. It is tempting to estimate the size of \( D \) without the \textit{a priori} fatness condition (5.11) as in [2, 7], and [21]. The important tool in these papers is a global doubling inequality. It seems possible to derive the size estimate without the fatness condition for the shallow shell system since we have derived local doubling inequalities (4.4). Like Theorem 4.13, to investigate this inverse problem, we actually need global doubling inequalities in terms of \( \sum_{ij} |e_{ij}^\theta(u)|^2 + \rho_0^2 |\partial_{ij}^2 u_3|^2 \) instead of \( |u'|^2 + |u_3|^2 \). However, attempts to derive such global doubling inequalities were unsuccessful. The difficulty is due to the fact that \( u' \) and \( u_3 \) in (3.4) have different scalings.

References


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