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Abstract framework for John-Nirenberg inequalities and applications to Hardy spaces

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Abstract. In this paper, we develop an abstract framework for John-Nirenberg inequalities associated to BMO-type spaces. This work can be seen as the sequel of [6], where the authors introduced a very general framework for atomic and molecular Hardy spaces. Moreover, we show that our assumptions allow us to recover some already known John-Nirenberg inequalities. We give applications to the atomic Hardy spaces too.

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1. Introduction

The first BMO space (space of functions satisfying a *Bounded Mean Oscillation*) was originally introduced by F. John and L. Nirenberg in [17]. This space naturally arises as the class of functions whose deviation from their means over cubes is bounded. From a point of view of Harmonic Analysis, this space is strictly including the L^∞ space, and is a good extension of the Lebesgue spaces scale $(L^p)_{1<p<\infty}$ for $p \to \infty$. For example, it plays an important role for boundedness of Calderón-Zygmund operators, real interpolation, Carleson measure, study of paraproducts, ... Moreover the BMO space can be characterized as the dual space of the Coifman Weiss space H^1. This observation was announced by C. Fefferman in [12] and then proved in [13].

Here we are interested in one of the most important properties of the BMO space: the so-called John-Nirenberg inequality (see [17]). This property describes the exponential integrability of the oscillations for a BMO-function. More precisely, for Q a ball of the Euclidean space $\mathbb{R}^n$ then a function $f \in \mathrm{BMO}$ satisfies

$$\left| \left\{ x \in Q, \ \left| f(x) - \fint_Q f \right| > \lambda \right\} \right| \leq c_1 |Q| e^{-c_2\lambda/\|f\|_{\mathrm{BMO}}},$$

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for some constants c_1, c_2 only dependent on the dimension n. Consequently the oscillation $f - \fint_Q f$, which was initially supposed to belong to $L^1(Q)$ (by the definition of BMO), is indeed exponentially integrable on Q. The BMO-norm gives rise to a self-improvement of the integrability of the oscillation.

The first consequence of such inequalities was the equivalence between the spaces BMO_q for $q \in (1, \infty)$ (where BMO_q is based on a control of the oscillations in L^q norm). A second important consequence concerns the Hardy space H^1. Since the duality $(H^1)^* = \mathrm{BMO}$ holds, it turns out that the Hardy space defined by p-atoms, does not depend on the exponent $p \in (1, \infty)$ (see the work of R. Coifman and G. Weiss about Hardy spaces [7]).

Our aim in this paper is to extend these properties in an abstract framework.

BMO space has many other properties, and the use of this space in real Harmonic Analysis has given rise to many works (related to Calderón-Zygmund operators, Carleson measures, $T(1)$-theorems, ...). However, there are situations where the John-Nirenberg space BMO is not the right substitute to L^∞, and there have been recently numerous works whose goal is to define an adapted BMO space according to the context (see [10, 14] ...). For example the classical space BMO is not well adapted to operators such as the Riesz transform on Riemannian manifolds. That is why in [14], S. Hofmann and S. Mayboroda develop the theory of Hardy and BMO spaces associated to second order divergence form elliptic operators, which also includes the corresponding John-Nirenberg inequality. In recent works [10] and [9], X. T. Duong and L. Yan studied some new BMO type spaces and proved an associated version of the John-Nirenberg inequality on these spaces (with duality results).

In [6], the authors developed an abstract framework for atomic Hardy spaces (and proved some results about interpolation with Lebesgue spaces). Without more precise assumptions, it seems to be impossible to get a full characterization of their dual space as a BMO-type space, although the last one is well-defined. This framework can cover the classical space BMO and those defined in [10] and [14]. Now let us detail how the setting of Hardy and BMO spaces can be extended to more abstract situations. The main idea is as follows. In the classical case, Hardy and BMO spaces are related to the "oscillation" of a function f over a ball Q, given by

$$f - \fint_Q f \, d\mu.$$

In the setting of a semigroup e^{-tL} (associated to a second order divergence form operator L, or a positive Laplacian $L = -\Delta$ on a Riemannian manifold), it is not clear how to describe the action of the semigroup on this oscillation. So in many situations (recently studied in numerous papers), the oscillation was replaced by another quantity:

$$f - e^{-r_Q^m L} f \quad \text{or} \quad (1 - e^{-r_Q^m L})^M f,$$

where m is the order of the operator L, r_Q is the radius of the ball Q and M a large enough integer. In this case, the semigroup $e^{-r_Q^m L}$ can be thought as a "smooth version" of the mean-value operator related to the generator L.

Aiming to generalize these two situations, the idea in [6] was to define Hardy and BMO spaces related to a general collection of "oscillation operators" $(B_Q)_Q$ indexed by the balls (for details, see Section 2 in the following). According to the two previous examples, it seems that the operators $A_Q := I - B_Q$ play an important role and should be seen as "approaching" the mean-value operators. For applications, we recover the classical case when A_Q is the mean-value operator on Q and the cases relatively to a semigroup with $A_Q = e^{-r_Q^m L}$ or $A_Q = 1 - (1 - e^{-r_Q^m L})^M$. For more detailed examples, see Section 3 and [6].

In order to unify all the existing results concerning Hardy and BMO spaces, this abstract setting also seems to be natural. So there is a large schedule: for each property concerning classical Hardy and BMO spaces, find what are the good assumptions to require about these abstract "oscillation operators" B_Q to extend the property in more general situations. For example, we refer the reader to [3, 6] for the interpolation results between Hardy and Lebesgue spaces. In [4], a new version of the famous $T(1)$-theorem is described in this point of view. In [5,15,16], general self-improving properties for Poincaré type inequalities are proved with this same approach.

This work fits into this program and aims to build a unified theory. More precisely, we describe general assumptions implying John-Nirenberg inequalities relatively to these new BMO spaces. In detail, our paper is organized as follows:

In Section 2, we define our framework of Hardy and BMO spaces, then state our main results concerning John-Nirenberg inequalities (see Theorems 2.7 and 2.9). We postpone their proofs to Subsection 2.3. In Section 3, we check that our assumptions are reasonable, therefore our results generalize some already known particular cases such as the John-Nirenberg inequalities in [13, 17], and [10]. In Section 4, we present an application of our John-Nirenberg inequalities to the corresponding Hardy spaces.

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2. An abstract framework for John-Nirenberg inequalities

2.1. Hardy and BMO spaces

Let (X, d, μ) be a space of homogeneous type. So d is a quasi-distance on the space X, and μ is a Borel measure which satisfies the doubling property: there exist constants $A, \delta > 0$, such that for all $x \in X, r > 0$ and $t \geq 1$

$$\mu(B(x, tr)) \leq At^{\delta} \mu(B(x, r)), \tag{2.1}$$

where $B(x, r)$ is the open ball with center $x \in X$ and radius $r > 0$. We call δ the homogeneous dimension of X. For Q a ball, and $i \geq 0$, we write $C_i(Q)$ the scaled corona around the ball Q:

$$C_i(Q) := \left\{x,\ 2^i \leq 1 + \frac{d(x, c(Q))}{r_Q} < 2^{i+1}\right\},$$

where r_Q is the radius of the ball Q and $c(Q)$ its center. Note that $C_0(Q)$ corresponds to the ball Q and $C_i(Q) \subset 2^{i+1}Q$ for $i \geq 1$, where λQ is as usual the ball with center $c(Q)$ and radius λr_Q. For $p \in [1, \infty]$, we denote the Lebesgue space by $L^p = L^p(X)$. We write $\mathcal{M}$ for the Hardy-Littlewood maximal operator and for $p \in [1, \infty)$, we denote its L^p-version by

$$\mathcal{M}_p(f)(x) := \mathcal{M}(|f|^p)(x)^{1/p}.$$

Let us denote by $\mathcal{Q}$ the collection of all balls:

$$\mathcal{Q} := \{B(x, r),\ x \in X, r > 0\}.$$

Let $\mathbb{B} := (B_Q)_{Q \in \mathcal{Q}}$ be a collection of linear operators, indexed by the collection $\mathcal{Q}$. We write $A_Q := Id - B_Q$ and B_Q^* for its adjoint operator. We assume that these operators B_Q are uniformly bounded in some Lebesgue space: there exist two exponents $p_1 < p_0$ belonging to $(1, \infty]$ and a constant $0 < A' < \infty$ so that: for all $p \in [p_1, p_0]$

$$\forall f \in L^p,\ \forall Q \text{ ball}, \qquad \|B_Q(f)\|_{L^p} \leq A'\|f\|_{L^p}. \tag{2.2}$$

In the rest of the paper, we allow the constants to depend on A, A' and δ.

For convenience, we first recall the definition of *atoms*, *molecules* and the corresponding Hardy spaces introduced in [6].

Definition 2.1. ([6]) Let $\epsilon > 0$ and $p \in [p_1, p_0]$ be fixed parameters. A function $m \in L^1_{\text{loc}}$ is called an (ϵ, p)-molecule associated to a ball Q if there exists a real function f_Q such that

$$m = B_Q(f_Q),$$

with

$$\forall i \geq 0, \qquad \|f_Q\|_{L^p(C_i(Q))} \leq \left(\mu(2^i Q)\right)^{-1+1/p} 2^{-\epsilon i}.$$

We call $m = B_Q(f_Q)$ a p-atom if in addition we have $supp(f_Q) \subset Q$. So a p-atom is exactly an (∞, p)-molecule.

Definition 2.2. ([6]) A measurable function h belongs to the molecular Hardy space $H^1_{p,\epsilon,mol}$ if there exists a decomposition:

$$h = \sum_{i \in \mathbb{N}} \lambda_i m_i \qquad \mu - a.e, \tag{2.3}$$

where for all i, m_i is an (ϵ, p)-molecule and $(\lambda_i)_i$ are real numbers satisfying

$$\sum_{i \in \mathbb{N}} |\lambda_i| < \infty.$$

Here, we assume that the infinite sum in (2.3) is absolutely convergent for almost every point $x \in X$.

We define the norm:

$$\|h\|_{H^1_{p,\epsilon,mol}} := \inf_{h=\sum_{i\in\mathbb{N}} \lambda_i m_i} \sum_i |\lambda_i|.$$

Similarly we define the atomic space $H^1_{p,\mathrm{ato}}$ replacing (ϵ, p)-molecules by p-atoms.

For more details about these notations and these abstract Hardy spaces, see [6] (and [2] for extension about Hardy-Sobolev spaces). Also in [6] and [3], interpolation results are described. Moreover in [6] the authors have described weak results concerning duality results $H^1-\mathrm{BMO}$. In the following, we define the BMO spaces.

Definition 2.3. For $q \in [p_0', p_1']$, a function $f \in L^q$ belongs to the space Bmo_q if

$$\|f\|_{\mathrm{BMO}_q} := \sup_{Q \text{ ball}} \left(\frac{1}{\mu(Q)} \int_Q \left|B_Q^*(f)\right|^q d\mu \right)^{1/q} < \infty,$$

where we denote B_Q^* for the adjoint operator of B_Q. We write BMO_q for the completion of Bmo_q with the corresponding norm.

We will see that it could be interesting to define other "molecular" BMO spaces as follows:

Definition 2.4. For $\epsilon > 0$ and $q \in [p_0', p_1']$, a function $f \in L^q$ belongs to the molecular space $\mathrm{Bmo}_{\epsilon,q}$ if

$$\|f\|_{\mathrm{BMO}_{\epsilon,q}} := \sup_{Q \text{ ball}} \sup_{j\geq 0} 2^{j\epsilon} \left(\frac{1}{\mu(2^j Q)} \int_{C_j(Q)} \left|B_Q^*(f)\right|^q d\mu \right)^{1/q} < \infty.$$

Remark 2.5. Obviously we have $\mathrm{BMO}_{\epsilon,q} \hookrightarrow \mathrm{BMO}_q$ for every $\epsilon > 0$. The question of a reverse property is open in such an abstract framework.

After these definitions, we can state our main results in the following subsection.

2.2. John-Nirenberg inequalities

In order to get John-Nirenberg inequalities, first we have to make some assumptions on the operators $(B_Q)_Q$. Now let us describe the required properties.

Assumptions. We set $q_0 = p_0'$ and $q_1 = p_1'$ such that $1 \leq q_0 < q_1 < \infty$.

We assume $L^{q_0} - L^{q_1}$ off-diagonal decay for operators A_Q^*: there exist coefficients γ_j such that for every ball Q and every function $h \in L^{q_0}$,

$$\left(\frac{1}{\mu(Q)} \int_Q \left|A_Q^*(h)\right|^{q_1} d\mu \right)^{1/q_1} \lesssim \sum_j \gamma_j \left(\frac{1}{\mu(2^j Q)} \int_{C_j(Q)} |h|^{q_0} d\mu \right)^{1/q_0}, \quad (2.4)$$

with the property

$$\sum_j \gamma_j < \infty. \tag{2.5}$$

Remark 2.6. Let us note that the off-diagonal decay (2.4) is slightly stronger than the following maximal inequality: for all $x \in X$

$$\sup_{Q \ni x} \left(\frac{1}{\mu(Q)} \int_Q \left| A_Q^*(h) \right|^{q_1} d\mu \right)^{1/q_1} \lesssim \mathcal{M}_{q_0}(|f|)(x).$$

This comparison between maximal operators is exactly the assumption, required in our previous works [3, 6], in order to get interpolation results between our Hardy spaces and L^{p_0}. It is interesting that the similar assumption seems to be important for both interpolation results and John-Nirenberg inequalities.

Moreover we make the following assumption: there exists a constant c such that

$$\sup_{\substack{R \text{ ball} \\ R \subset 2Q}} \left(\frac{1}{\mu(R)} \int_R \left| B_R^*(A_Q^*(f)) \right|^{q_1} d\mu \right)^{1/q_1} \leq c \|f\|_{\mathrm{BMO}_{q_0}}. \tag{2.6}$$

In some cases, we will require the following stronger assumption, which describes that the operators A_Q^* continuously act on the Bmo spaces: there is a constant c such that

$$\|A_Q^*(f)\|_{\mathrm{BMO}_{q_1}} \leq c \|f\|_{\mathrm{BMO}_{q_0}}. \tag{2.7}$$

Then we have the following results.

Theorem 2.7. *Let us first assume that the operators $(B_Q)_Q$ depend only on the radius r_Q of the ball and that* (2.4), (2.5) *and* (2.6) *hold. Then the space* Bmo_{q_0} *satisfies a John-Nirenberg inequality: there exist constants $\rho_1, \rho_2 > 0$ such that for all function $f \in \mathrm{Bmo}_{q_0}$ and every ball $Q \subset X$,*

$$\mu\left(\left\{ x \in Q,\ \left| B_Q^*(f)(x) \right| > \lambda \|f\|_{\mathrm{BMO}_{q_0}} \right\} \right) \leq \rho_1 \mu(Q) \left[e^{-\rho_2 \lambda} + \frac{\mathbf{1}_{q_1 < \infty}}{\lambda^{q_1}} \right].$$

Corollary 2.8. *If we are working on the Euclidean space $\mathbb{R}^n$, we can just require*

$$\sup_{\substack{R \text{ ball} \\ R \subset Q}} \left(\frac{1}{\mu(R)} \int_R \left| B_R^*(A_Q^*(f)) \right|^{q_1} d\mu \right)^{1/q_1} \leq c \|f\|_{\mathrm{BMO}_{q_0}}. \tag{2.8}$$

instead of (2.6)*, see Remark* 2.14.

Let us now consider general operators B_Q, which could depend on the ball Q.

Theorem 2.9. *Let $\epsilon > 0$ and suppose that the coefficients $(\gamma_j)_j$ given by* (2.4) *satisfy*

$$\sup_j \gamma_j < \infty$$

instead of the stronger inequality (2.5).

Under (2.7)*, the space* $\mathrm{Bmo}_{\epsilon,q_0}$ *satisfies a John-Nirenberg inequality: there exist constants $\rho_1, \rho_2 > 0$ such that for all function $f \in \mathrm{Bmo}_{\epsilon,q_0}$ every ball $Q \subset X$ and integer $l \geq 0$*

$$\mu\left(\left\{x \in C_l(Q),\ \left|B_Q^*(f)(x)\right| > \lambda \|f\|_{\mathrm{BMO}_{\epsilon,q_0}}\right\}\right) \leq \rho_1 2^{-\epsilon l} \mu(2^l Q)\left[e^{-\rho_2 \lambda} + \frac{\mathbf{1}_{q_1<\infty}}{\lambda^{q_1}}\right]. \tag{2.9}$$

Remark 2.10. Indeed, we prove a more accurate result in the particular case of non-increasing coefficients γ_j. In this case, we can work with the smaller norm

$$\|f\|_{\widetilde{\mathrm{BMO}}_{\epsilon,q}} := \sup_{Q \text{ ball}} \sup_{\substack{j\geq 0\\ \gamma_j \neq 0}} 2^{j\epsilon}\left(\frac{1}{\mu(2^j Q)}\int_{C_j(Q)} \left|B_Q^*(f)\right|^q d\mu\right)^{1/q}$$

and prove (2.9). We leave the details to the reader as it just suffices to follow the contribution of coefficients γ_j in the proof.

As usual, a John-Nirenberg inequality allows us to prove some equivalence in BMO spaces (with estimating the oscillation in L^q for different exponents q).

Corollary 2.11. *Under the assumptions of Theorem* 2.7 *(respectively Theorem* 2.9*), the norms* BMO_q *(respectively* $\mathrm{BMO}_{\epsilon,q}$ *for some $\epsilon > 0$) for $q \in (q_0, q_1)$ are equivalent and consequently the spaces* BMO_q *are equal.*

Proof. We only treat the case of the BMO_q spaces, as the proof for the spaces $\mathrm{BMO}_{\epsilon,q}$ is similar, on using Theorem 2.9 instead of Theorem 2.7. We take two exponents $r > q$ belonging to the range (q_0, q_1) and a function $\phi \in \mathrm{Bmo}_r \cap \mathrm{Bmo}_q$. First using Hölder inequality, we have for every ball Q

$$\left(\frac{1}{\mu(Q)}\int_Q \left|B_Q^*(\phi)\right|^q d\mu\right)^{1/q} \leq \left(\frac{1}{\mu(Q)}\int_Q \left|B_Q^*(\phi)\right|^r d\mu\right)^{1/r} \leq \|\phi\|_{\mathrm{BMO}_r},$$

therefore we deduce the inclusion $\mathrm{Bmo}_r \subset \mathrm{Bmo}_q$.

Then it remains to check that $\mathrm{Bmo}_q \subset \mathrm{Bmo}_r$. Using John-Nirenberg inequality (obtained in Theorem 2.7), we get a weak inequality for every ball Q

$$\frac{1}{\mu(Q)^{1/q_1}}\left\|B_Q^*(\phi)\right\|_{L^{q_1,\infty}} \lesssim \|\phi\|_{\mathrm{BMO}_{q_0}} \lesssim \|\phi\|_{\mathrm{BMO}_q}.$$

By invoking Kolmogorov's inequality, we get

$$\left(\frac{1}{\mu(Q)}\int_Q \left|B_Q^*(\phi)\right|^r d\mu\right)^{1/r} \leq \left(\frac{q_1}{q_1-r}\right)^{1/r} \frac{1}{\mu(Q)^{1/q_1}} \left\|B_Q^*(\phi)\right\|_{L^{q_1,\infty}},$$

which finally yields

$$\left(\frac{1}{\mu(Q)}\int_Q \left|B_Q^*(\phi)\right|^r d\mu\right)^{1/r} \lesssim \|\phi\|_{\mathrm{BMO}_q}. \qquad \square$$

2.3. Proof of Theorems 2.7 and 2.9

The following proof has been written with abstract operators B_Q (depending on the ball), as we will refer to it in Theorem 2.9 which requires this abstract framework.

Proof of Theorem 2.7. We follow the ideas of [10].

Without loss of generality, we can assume that $\|f\|_{\mathrm{BMO}_{q_0}} = 1$ and we also have to prove that for any fixed ball Q

$$\mu\left(\left\{x \in Q,\ \left|B_Q^*(f)(x)\right| > \lambda\right\}\right) \leq \rho_1 \mu(Q)\left[e^{-\rho_2\lambda} + \frac{\mathbf{1}_{q_1<\infty}}{\lambda^{q_1}}\right]. \tag{2.10}$$

Obviously (2.10) holds for $\lambda \leq 1$ with $\rho_2 = 1$ and $\rho_1 = e$. So we only consider $\lambda > 1$ and set

$$f_0 := \mathbf{1}_Q B_Q^*(f).$$

Then we get

$$\|f_0\|_{L^1} \leq \int_Q |B_Q^*(f)| d\mu \leq \|f\|_{\mathrm{BMO}_{q_0}} \mu(Q) \leq \mu(Q).$$

Using a constant $\beta > 1$ (later chosen) and the Hardy-Littlewood maximal operator $\mathcal{M}$, we set

$$F := \{x,\ \mathcal{M}(f_0)(x) \leq \beta\} \quad \text{and} \quad \Omega := F^c = \{x,\ \mathcal{M}(f_0)(x) > \beta\}.$$

Let us consider a Whitney decomposition of Ω: a collection of balls $(Q_{1,i})_i$ such that

a−) $\Omega = \bigcup_i Q_{1,i}$

b−) each point is contained in at most a finite number N of balls $Q_{1,i}$

$$\sum_i \mathbf{1}_{Q_{1,i}} \leq N$$

c−) there exists $\kappa > 1$ such that for all i, $\kappa Q_{1,i} \cap F \neq \emptyset$.

From $a-$), it comes for all $x \in Q \setminus (\cup_i Q_{1,i})$

$$|B_Q^*(f)(x)| = |f_0(x)| \le \mathcal{M}(f_0)(x) \le \beta. \tag{2.11}$$

The weak-type (1, 1) of the Hardy-Littlewood maximal operator yields

$$\sum_i \mu(Q_{1,i}) \le N\mu(\Omega) \lesssim \frac{1}{\beta}\|f_0\|_{L^1} \le \frac{c_1}{\beta}\mu(Q) \tag{2.12}$$

for some numerical constant $c_1 > 0$.

We choose β such that for every ball $Q_{1,i}$ with $Q_{1,i} \cap Q \neq \emptyset$, then $Q_{1,i} \subset 2Q$. Let us check that this is possible. If $r_{Q_{1,i}} \ge r_Q$ then $Q \subset 2Q_{1,i}$ thus (2.12) implies

$$\mu(Q) \le \mu(2Q_{1,i}) \lesssim \mu(Q_{1,i}) \le \frac{c_3}{\beta}\mu(Q),$$

for some numerical constant c_3. So we can choose $\beta > c_3$, then the previous inequality does not hold, because every ball has a non vanishing measure (due to the doubling property of the space X). Consequently, we deduce that the condition $r_{Q_{1,i}} \le r_Q$ yields $Q_{1,i} \subset 2Q$.

Then we will use the following Lemma.

Lemma 2.12. *There exists a numerical constant $c_2 \ge 1$, such that for all i:*

$$\left(\frac{1}{\mu(Q_{1,i})}\int_{Q_{1,i}} \left|B_Q^*(f) - B_{Q_{1,i}}^*(f)\right|^{q_1} d\mu\right)^{1/q_1} \le c_2\beta. \tag{2.13}$$

For the readability, we postpone the proof to the end of this theorem. Now let us come back to the proof of our main Theorem. For each index i, we repeat the procedure as follows: denote

$$f_{1,i} := \mathbf{1}_{Q_{1,i}} B_{Q_{1,i}}^*(f).$$

Then we consider a collection of balls $(Q_{2,i,m})_m$ such that

- for all $x \in Q_{1,i} \setminus \left(\cup_m Q_{2,i,m}\right)$, we have

$$\left|B_{Q_{1,i}}^*(f)(x)\right| \le \beta$$

- the weak-type (1, 1) of the maximal function yields

$$\sum_m \mu(Q_{2,i,m}) \le \frac{c_1}{\beta}\mu(Q_{1,i})$$

- and for all balls $Q_{2,i,m}$ intersecting $Q_{1,i}$, it comes that

$$\left(\frac{1}{\mu(Q_{2,i,m})}\int_{Q_{2,i,m}}\left|B^*_{Q_{1,i}}(f)-B^*_{Q_{2,i,m}}(f)\right|^{q_1}d\mu\right)^{1/q_1}\leq c_2\beta.$$

Then we put together all families $(Q_{2,i,m})_m$ for all indices i and we get a new family

$$(Q_{2,m})_m := \bigcup_i (Q_{2,i,m})_m.$$

We also have

$$\sum_m \mu(Q_{2,m}) \leq \frac{c_1}{\beta}\sum_i \mu(Q_{1,i}) \leq \left(\frac{c_1}{\beta}\right)^2 \mu(Q).$$

Moreover for all $x \in Q \setminus \left(\cup_i Q_{1,i}\right)$, we already know from (2.11) that

$$\left|B^*_Q(f)(x)\right| \leq \beta.$$

For all x belonging to $Q \cap Q_{1,i}$ but not in the associated collection $(Q_{2,i,m})_m$, we have

$$\left|B^*_{Q_{1,i}}(f)(x)\right| \leq \beta,$$

consequently

$$\left|B^*_Q(f)(x)\right| \leq \beta + \left|B^*_Q(f)(x) - B^*_{Q_{1,i}}(f)(x)\right|.$$

According to (2.13), we get:

$$\left(\frac{1}{\mu(Q_{1,i})}\int_{Q_{1,i}\setminus(\cup_m Q_{2,i,m})}\left|B^*_Q(f)\right|^{q_1}d\mu\right)^{1/q_1}\leq (c_2+1)\beta \leq 2c_2\beta.$$

We iterate this procedure and then having obtain the collection $(Q_{k,i})_i$ for some integer $k \geq 1$, we build for all i a collection $(Q_{k+1,i,m})_m$ and also a collection $(Q_{k+1,m})_m = \cup_i (Q_{k+1,i,m})_m$ satisfying:

- for all $x \in Q_{k,i} \setminus \left(\cup_m Q_{k+1,i,m}\right)$

$$\left|B^*_{Q_{k,i}}(f)(x)\right| \leq \beta \tag{2.14}$$

- we have

$$\sum_m \mu(Q_{k+1,i,m}) \leq \frac{c_1}{\beta}\mu(Q_{k,i})$$

- for all balls $Q_{k+1,i,m}$ intersecting $Q_{k,i}$, we have

$$\left(\frac{1}{\mu(Q_{k+1,i,m})}\int_{Q_{k+1,i,m}}\left|B^*_{Q_{k,i}}(f)-B^*_{Q_{k+1,i,m}}(f)\right|^{q_1}d\mu\right)^{1/q_1}\leq c_2\beta. \quad (2.15)$$

So for all integer $k\geq 1$,

$$\sum_m \mu(Q_{k,m})\leq \frac{c_1}{\beta}\sum_i \mu(Q_{k-1,i})\leq\cdot\leq\left(\frac{c_1}{\beta}\right)^k\mu(Q). \quad (2.16)$$

First case: If $q_1<\infty$.
We have seen at the beginning of the proof that (2.10) holds for $\lambda\lesssim 1$, so we are only interested in large enough λ. We choose an integer K_0 and a constant $\gamma<1$, such that for all large enough λ, there exists an integer $K\geq K_0$ satisfying

$$\gamma^K\lambda\simeq 2\beta \quad \text{and} \quad \left(\frac{c_1}{\beta}\right)^K\leq\lambda^{-q_1}.$$

This is possible by choosing for example $\gamma^{q_1}=lc_1/\beta$ with $l\geq(2\beta)^{q_1/K_0}$ ($\beta>c_1$, β being chosen large enough, we can find an integer K_0 such that $l_0:=(2\beta)^{q_1/K_0}>1$ satisfies $l_0c_1/\beta<1$).

So the integer K allows us to parameterize the scale of λ by a logarithmic scale $K\simeq\log(\lambda)$.

In particularly, we have

$$\frac{1}{\gamma^{q_1}}\frac{c_1}{\beta}<1. \quad (2.17)$$

Then we can obtain

$$\begin{aligned}\mu\left(\{x\in Q,|B^*_Q(f)(x)|>\lambda\}\right)&\leq\sum_i\mu\left(\{x\in Q_{1,i},\ |B^*_Q(f)(x)|>\lambda\}\right)\\&\leq\sum_i\mu\left(\left\{x\in Q_{1,i},|B^*_Q(f)(x)-B^*_{Q_{1,i}}(f)(x)|>(1-\gamma)\lambda\right\}\right)\\&\quad+\sum_i\mu\left(\left\{x\in Q_{1,i},\ |B^*_{Q_{1,i}}(f)(x)|>\gamma\lambda\right\}\right).\end{aligned}$$

The first term is bounded by

$$\begin{aligned}&\sum_i\mu\left(\left\{x\in Q_{1,i},\ |B^*_Q(f)(x)-B^*_{Q_{1,i}}(f)(x)|>(1-\gamma)\lambda\right\}\right)\\&\qquad\leq(1-\gamma)^{-q_1}\lambda^{-q_1}\sum_i\int_{Q_{1,i}}|B^*_Q(f)-B^*_{Q_{1,i}}(f)|^{q_1}d\mu\\&\qquad\leq\left(\frac{c_2\beta}{(1-\gamma)\lambda}\right)^{q_1}\sum_i\mu(Q_{1,i})\\&\qquad\leq\left(\frac{c_2\beta}{(1-\gamma)\lambda}\right)^{q_1}\frac{c_1}{\beta}\mu(Q).\end{aligned}$$

Then we repeat the procedure with $B^*_{Q_{1,i}}(f)$ instead of $B^*_Q(f)$:

$$\begin{aligned}
&\sum_i \mu\left(\left\{x\in Q_{1,i},\ |B^*_{Q_{1,i}}(f)(x)|>\gamma\lambda\right\}\right)\\
&\quad\le \sum_{i,j} \mu\left(\left\{x\in Q_{2,i,j},\ |B^*_{Q_{1,i}}(f)(x)|>\gamma\lambda\right\}\right)\\
&\quad\le \sum_{i,j} \mu\left(\left\{x\in Q_{2,i,j},\ |B^*_{Q_{1,i}}(f)(x)-B^*_{Q_{2,i,j}}(f)(x)|>(1-\gamma)\gamma\lambda\right\}\right)\\
&\qquad+\sum_{i,j} \mu\left(\left\{x\in Q_{2,i,j},\ |B^*_{Q_{2,i,j}}(f)(x)|>\gamma^2\lambda\right\}\right).
\end{aligned}$$

The first term (in the last inequality) is controlled by

$$\begin{aligned}
&\sum_{i,j} \mu\left(\left\{x\in Q_{2,i,j},\ |B^*_{Q_{1,i}}(f)(x)-B^*_{Q_{2,i,j}}(f)(x)|>(1-\gamma)\gamma\lambda\right\}\right)\\
&\qquad\le [(1-\gamma)\gamma\lambda]^{-q_1}\sum_{i,j}\int_{Q_{2,i,j}} |B^*_{Q_{1,i}}(f)-B^*_{Q_{2,i,j}}(f)|^{q_1}d\mu\\
&\qquad\le \left(\frac{c_2\beta}{(1-\gamma)\gamma\lambda}\right)^{q_1}\sum_{i,j}\mu(Q_{2,i,j})\\
&\qquad\le \left(\frac{c_2\beta}{(1-\gamma)\gamma\lambda}\right)^{q_1}\left(\frac{c_1}{\beta}\right)^2\mu(Q)
\end{aligned}$$

and the second one is equal to

$$\sum_i \mu\left(\left\{x\in Q_{2,i},\ |B^*_{Q_{2,i}}(f)(x)|>\gamma^2\lambda\right\}\right).$$

Thus, by iterating this reasoning, we deduce that

$$\begin{aligned}
\mu\left(\left\{x\in Q,\ |B^*_Q(f)(x)|>\lambda\right\}\right)\le \frac{\mu(Q)}{\lambda^{q_1}}\sum_{k=0}^{K-1}\left(\frac{c_2\beta}{(1-\gamma)\gamma^k}\right)^{q_1}\left(\frac{c_1}{\beta}\right)^{k+1}\\
+\sum_i \mu\left(\left\{x\in Q_{K,i},\ |B^*_{Q_{K,i}}(f)(x)|>\gamma^K\lambda\right\}\right).
\end{aligned}$$

Consequently, (2.16) yields

$$\begin{aligned}
\mu\left(\left\{x\in Q,\ |B^*_Q(f)(x)|>\lambda\right\}\right)\le \frac{\mu(Q)}{\lambda^{q_1}}\sum_{k=0}^{K-1}\left(\frac{c_2\beta}{(1-\gamma)\gamma^k}\right)^{q_1}\left(\frac{c_1}{\beta}\right)^{k+1}\\
+\left(\frac{c_1}{\beta}\right)^{K}\mu(Q).
\end{aligned}$$

With the choice of the constant γ and the integer K, we deduce from (2.17) that

$$\mu\left(\left\{x \in Q,\ |B_Q^*(f)(x)| > \lambda\right\}\right) \lesssim \lambda^{-q_1}\mu(Q)$$

which corresponds to the desired inequality (2.10) when $q_1 < \infty$.

Second case: If $q_1 = \infty$.
In this case, we repeat the proof of [10]. For λ large enough, we denote $K \geq K_0$ an integer such that

$$Kc_2\beta < \lambda \leq (K+1)c_2\beta.$$

Then, since (2.13) and (2.15), it follows that on $Q_{1,i} \setminus (\cup_j Q_{2,i,j})$, $|B_Q^*(f)| \leq \beta$, on $Q_{1,i_1} \cap Q_{2,i_2} \setminus (\cup_j Q_{2,i_2,j})$

$$|B_Q^*(f)| \leq |B_Q^*(f) - B_{Q_{1,i}}^*(f)| + |B_{Q_{1,i_1}}^*(f)| \leq (1+c_2)\beta \leq 2c_2\beta$$

and by iterating on $Q_{1,i_1} \cap \cdots \cap Q_{K-1,i_{K-1}} \setminus (\cup_j Q_{K,i_{K-1},j})$,

$$\begin{aligned}|B_Q^*(f)| &\leq |B_Q^*(f) - B_{Q_{1,i}}^*(f)| + \sum_{l=1}^{K-2} |B_{Q_{l+1,i_l}}^*(f) - B_{Q_{l+1,i_{l+1}}}^*(f)| + |B_{Q_{K-1,i_{K-1}}}^*(f)| \\ &\leq (1+(K-1)c_2)\beta \leq Kc_2\beta < \lambda.\end{aligned}$$

Hence

$$\left\{x \in Q,\ |B_Q^*(f)(x)| > \lambda\right\} \subset \bigcup_i Q_{K,i},$$

which yields (thanks to (2.16))

$$\mu\left(\left\{x \in Q,\ |B_Q^*(f)(x)| > \lambda\right\}\right) \leq \left(\frac{c_1}{\beta}\right)^K.$$

As c_1/β is a constant smaller than 1 and $K \simeq \lambda$, this allows us to obtain the desired inequality. □

It remains us to prove Lemma 2.12. We recall the statement with the notations of the previous proof.

Lemma 2.13. *There exists a numerical constant $c_2 \geq 1$, such that for all i with $Q_{1,i} \cap Q \neq \emptyset$:*

$$\left(\frac{1}{\mu(Q_{1,i})}\int_{Q_{1,i}} \left|B_Q^*(f) - B_{Q_{1,i}}^*(f)\right|^{q_1} d\mu\right)^{1/q_1} \leq c_2\beta. \tag{2.18}$$

Proof. The desired result corresponds to a "local version" of inequality (3.3) in [10] (extended in our abstract framework), which essentially rests on Proposition 2.6 of [10]. We know that $Q_{1,i} \subset 2Q$ and we have

$$\begin{aligned} B_Q^*(f) - B_{Q_{1,i}}^*(f) &= A_Q^*(f) - A_{Q_{1,i}}^*(f) \\ &= \left[A_Q^*(f) - A_{Q_{1,i}}^* A_Q^*(f)\right] \\ &\quad + \left[A_{Q_{1,i}}^* A_Q^*(f) - A_{Q_{1,i}}^*(f)\right] \\ &= B_{Q_{1,i}}^* A_Q^*(f) - A_{Q_{1,i}}^* B_Q^*(f). \end{aligned} \tag{2.19}$$

Let us study the first term. As $Q_{1,i} \subset 2Q$, Assumption (2.6) implies:

$$\left(\frac{1}{\mu(Q_{1,i})}\int_{Q_{1,i}} \left|B_{Q_{1,i}}^* A_Q^*(f)(f)\right|^{q_1} d\mu\right)^{1/q_1} \lesssim \|f\|_{\mathrm{BMO}_{q_0}} \lesssim 1 \leq \beta,$$

as β is chosen large enough.

So it remains to study the second term $A_{Q_{1,i}}^* B_Q^*(f)$. To estimate it, we have to use $(L^{q_0} - L^{q_1})$-off diagonal decays of $A_{Q_{1,i}}$ as follows. Let P be the first integer such that $2Q \subset 2^{P+1}Q_{1,i}$ and $2Q \cap (2^P Q_{1,i})^c \neq \emptyset$. Then with Assumption (2.4), we have

$$\left(\frac{1}{\mu(Q_{1,i})}\int_{Q_{1,i}} \left|A_{Q_{1,i}}^* B_Q^*(f)\right|^{q_1} d\mu\right)^{1/q_1} \lesssim I + II$$

where

$$I := \sum_{j=0}^{P+1} \gamma_j \left(\frac{1}{\mu(2^j Q_{1,i})}\int_{C_j(Q_{1,i})} \left|B_Q^*(f)\right|^{q_0} d\mu\right)^{1/q_0}$$

and

$$II := \sum_{j=P+2}^{\infty} \gamma_j \left(\frac{1}{\mu(2^j Q_{1,i})}\int_{C_j(Q_{1,i})} \left|B_Q^*(f)\right|^{q_0} d\mu\right)^{1/q_0}.$$

It follows from the property of the ball $Q_{1,i}$ (property $c-$) in the proof of Theorem 2.7), that there exists another constant κ', for $j \leq P+1$,

$$\begin{aligned} \left(\frac{1}{\mu(2^j Q_{1,i})}\int_{C_j(Q_{1,i})} \left|B_Q^*(f)\right|^{q_0} d\mu\right)^{1/q_0} &\leq \kappa' \left(\frac{1}{\mu(2^j\kappa Q_{1,i})}\int_{2^j\kappa Q_{1,i}} |f_0|^{q_0}\, d\mu\right)^{1/q_0} \\ &\leq \kappa'\beta. \end{aligned}$$

So it yields

$$I \leq \sum_{j=0}^{P+1} \gamma_j \kappa' \beta \lesssim \beta.$$

To estimate the second term II. For any $j \geq P+1$, we know that $2^j Q_{1,i}$ contains the ball Q and so $2^j r_{Q_{1,i}} \geq r_Q$. So we choose $(\tilde{Q}_k^j)_k$ a bounded covering of $C_j(Q_{1,i})$ with balls of radius r_Q and as previously, we get (using that the operators B_Q only depend on the radius of the ball Q)

$$\begin{aligned} II &\leq \sum_{j=P+2}^{\infty} \gamma_j \left(\frac{1}{\mu(2^j Q_{1,i})} \sum_k \int_{\tilde{Q}_k^j} \left|B_Q^*(f)\right|^{q_0} d\mu \right)^{1/q_0} \\ &\lesssim \sum_{j=P+2}^{\infty} \gamma_j \left(\frac{1}{\mu(2^j Q_{1,i})} \sum_k \int_{\tilde{Q}_k^j} \left|B_{\tilde{Q}_k^j}^*(f)\right|^{q_0} d\mu \right)^{1/q_0} \qquad (2.20) \\ &\lesssim \beta \sum_{j=P+2}^{\infty} \gamma_j \left(\frac{1}{\mu(2^j Q_{1,i})} \sum_k \mu(\tilde{Q}_k^j) \right)^{1/q_0} \\ &\lesssim \beta \sum_{j=P+2}^{\infty} \gamma_j \lesssim \beta. \end{aligned}$$

So finally, the estimates of I and II imply (2.18), which concludes the proof. □

Remark 2.14. Let us show how we can obtain Corollary 2.8. As explained in the proof of Theorem 2.7, we use a Whitney decomposition of the set

$$\Omega := \{x,\ \mathcal{M}\left(\mathbf{1}_Q B_Q^*(f)\right)(x) > \beta\},$$

for a ball Q and some fixed parameter $\beta > 1$. Using the dyadic structure of $\mathbb{R}^d$, let us deal with a dyadic cube Q. We can choose a Whitney decomposition of Ω with dyadic (relatively to Q) cubes $Q_{1,i}$ – see Theorem 5.2 of [15] for a detailed construction –. Then the proof is based on such balls $Q_{1,i}$ such that

$$Q_{1,i} \cap Q \neq \emptyset, \quad \text{and} \quad Q_{1,i} \subset cQ,$$

for some constant $c > 1$. We have chosen $c = 2$ for simplicity, but we can consider $c = 3/2$ for example. Then the dyadic structure of the Euclidean space, implies that $Q_{1,i}$ is included in Q. We can now reproduce the same arguments and Assumption (2.8) is sufficient to conclude.

Let us now consider general operators B_Q, which could depend on the ball Q.

To get a result concerning abstract operators B_Q (they now depend on the ball and not only on the radius), we have to require some extra properties. In the previous proof, the only one point where we used the property of dependence (on the radii) of the operators B_Q is the inequality (2.20). So let us just take the notations of the previous proof and recall the problem: for $j \geq P+1, C_j(Q_{1,i}) \subset C_{j-P}(Q)$, and we have to estimate

$$\int_{C_j(Q_{1,i})} \left|B_Q^*(f)\right|^{q_0} d\mu.$$

From the BMO-norm, we only have information about $B_Q^*(f)$ on the ball Q, so we do not know how we can control this term. In order to get around this lack of information, we use the BMO_ϵ associated to *molecules* (see Theorem 2.9).

Proof of Theorem 2.9. We have to prove that

$$\mu\left(\left\{x \in C_l(Q), \left|B_Q(f)(x)\right| > \lambda \|f\|_{\mathrm{BMO}_{\epsilon,q_0}}\right\}\right) \leq \rho_1 2^{-\epsilon l} \mu(2^l Q) \left[e^{-\rho_2 \lambda} + \frac{\mathbf{1}_{q_1<\infty}}{\lambda^{q_1}}\right] \tag{2.21}$$

for all integer $l \geq 0$ and Q a ball of X.

So let us fix the ball Q and the function f.

For $l = 0$, we follow the proof of Theorem 2.7. In this case, the only one point where we used the property of dependence (on the radii) of the operators B_Q is the inequality (2.20). So let us keep the same notations of the previous proof. We have an integer $j \geq P + 1$ (so $C_j(Q_{1,i}) \subset C_{j-P}(Q)$) and we have to estimate

$$\int_{C_j(Q_{1,i})} \left|B_Q^*(f)\right|^{q_0} d\mu.$$

We also have

$$\int_{C_j(Q_{1,i})} \left|B_Q^*(f)\right|^{q_0} d\mu \leq \int_{C_{j-P}(Q)} \left|B_Q^*(f)\right|^{q_0} d\mu \lesssim 2^{-\epsilon(j-P)}.$$

Consequently, we get

$$\begin{aligned} II &\leq \sum_{j=P+1}^{\infty} \gamma_j 2^{-\epsilon(j-P)} \\ &\lesssim \sum_{j=P+2}^{\infty} 2^{-\epsilon(j-P)} \lesssim 1 \lesssim \beta. \end{aligned}$$

This estimate permits to conclude the proof of Lemma 2.13 and by this way the proof of (2.21) for $l = 0$.

For $l \geq 1$, we produce the same reasoning by starting with the function $f_0 := \mathbf{1}_{C_l(Q)} B_Q^*(f)$ which satisfies

$$\|f_0\|_{L^1} \leq \int_{C_l(Q)} |B_Q^*(f)| d\mu \leq \|f\|_{\mathrm{BMO}_{\epsilon,q_0}} 2^{-\epsilon l} \mu(2^l Q).$$

We reproduce the same proof (using Assumption (2.7)), which we leave it to the reader. □

3. Verification of our assumptions in some usual cases

In this section we verify that our results generalize the already known particular cases, and more precisely that our assumptions are satisfied.

3.1. The John-Nirenberg space

Consider the Euclidean space $X = \mathbb{R}^n$ and the usual BMO space. In [17], the first John-Nirenberg inequalities was proved using a Calderón-Zygmund decomposition. We refer the reader to [19, Chapter IV 1.3] for another proof based on the duality $H^1 - \text{BMO}$.

The first BMO space is defined by the operators:

$$B_Q^*(f) := f - \left(\frac{1}{\mu(Q)} \int_Q f\right) \mathbf{1}_Q.$$

So A_Q^* is the mean value operator and obviously off diagonal decays (2.4) hold with $q_1 = \infty$ and $q_0 = 1$. In this case, Assumption (2.7) does not hold. However for such operators, the coefficients $\gamma_j = 0$ as soon as $j \geq 1$. So to apply Theorem 2.9, we only have to check Assumption $\widetilde{(2.8)}$, thanks to Corollary 2.8 and Remark 2.10 (in this particular case, the spaces $\widetilde{\text{BMO}}_{\epsilon,q}$ are equal to BMO_q because $\gamma_j = 0$ for $j \geq 1$), which is

$$\sup_{\substack{R \text{ ball} \\ R \subset Q}} \left(\frac{1}{\mu(R)} \int_R \left|B_R^*(A_Q^*(f))\right|^{q_1} d\mu\right)^{1/q_1} \leq c \|f\|_{\text{BMO}_{q_0}}.$$

This last property is true since for $R \subset Q$, we have:

$$\mathbf{1}_R B_R^*(A_Q^*(f)) = \left(\frac{1}{\mu(Q)} \int_Q f\right) [\mathbf{1}_R - \mathbf{1}_R] = 0.$$

Conclusion: In the framework of the classical BMO space, our Assumptions (2.4) and (2.8) are satisfied. We recover the John-Nirenberg inequality (see [17]).

3.2. The Morrey-Campanato spaces

Consider the set $X = [0, 1]$ with its Euclidean structure. For works related to Morrey-Campanato spaces and associated John-Nirenberg inequalities, we refer the reader to [18] and [8].

Let us first define these spaces.

Definition 3.1. For $\beta \geq 0$, $s \in \mathbb{N}$ and $q \in (1, \infty)$, we say that a locally integrable function $f \in L^1(X)$ belongs to the Morrey-Campanato spaces $L(\beta, q, s)$ if

$$\|f\|_{L(\beta,q,s)} := \sup_{Q \in \mathcal{Q}} |Q|^{-\beta} \left[\fint_Q \left|f(x) - P_Q(f)(x)\right|^q dx \right]^{1/q} < \infty,$$

where for Q a ball (an interval) of X, $P_Q(f)$ is the only one polynomial function of degree at most s such that for all $i \in \{0, .., s\}$

$$\int_Q x^i \left(f(x) - P_Q(f)(x)\right) dx = 0.$$

Remark 3.2. $L(\beta, q, 0)$ exactly corresponds to the previous BMO space (of John and Nirenberg) as in this case $P_Q(f) = \fint_Q f$.

In this framework, we set $A_Q := P_Q^*$ in order that $L(0, q, s)$ can be identified to our BMO space. An easy computation shows that $P_Q(f)$ is a polynomial function whose coefficients are given by the quantities

$$\int_Q f(x)x^i dx,$$

for $i \in \{0, .., s\}$. So $A_Q^* = P_Q$ can be written as follows

$$A_Q^*(f)(x) = \sum_{j=0}^{s} c_j x^j \mathbf{1}_Q(x),$$

with coefficients c_j satisfying

$$|c_j| \lesssim \int_Q |f(x)| dx,$$

since we are working on $X = [0, 1]$. It comes out that off- diagonal decays (2.4) hold with $q_1 = \infty$ and $q_0 = 1$. As previously since we are working on the Euclidean space and coefficients $\gamma_j = 0$ as soon as $j \geq 1$, it is sufficient to check that

$$\sup_{\substack{R \text{ ball} \\ R \subset Q}} \left(\frac{1}{\mu(R)} \int_R \left| B_R^*(A_Q^*(f)) \right|^{q_1} d\mu \right)^{1/q_1} \leq c \|f\|_{\mathrm{BMO}_{q_0}}.$$

This property is satisfied since for $R \subset Q$, we have

$$B_R^*(A_Q^*(f)) = P_Q(f) - P_R P_Q(f) = 0.$$

The last equality is due to the fact that $P_Q(f)$ is a polynomial function of degree at most s, so by uniqueness (in the definition of P_R): $P_R[P_Q(f)] = P_Q(f)$.

Conclusion: In the framework of the classical Morrey-Campanato spaces $L(0,q,s)$, our Assumptions (2.4) and (2.8) are satisfied. We also recover the John-Nirenberg inequality for all $q \in (1, \infty)$ (see [18]). For $\beta > 0$, we refer the reader to a forthcoming work of the first author and J. M. Martell [5] (and [15, 16]), dealing with more general self-improvement properties of inequalities.

3.3. General case of semigroup

Let us recall the framework of [10].

Consider a space of homogeneous type (X, d, μ) with a family of operators $(\mathcal{A}_r)_{r>0}$ satisfying the following properties:

- For every $r > 0$, the linear operator $\mathcal{A}_r$ is given by a kernel a_r satisfying

$$|a_r(x, y)| \lesssim \frac{1}{\mu(B(x, r^{1/m}))} \left(1 + \frac{d(x, y)}{r^{1/m}}\right)^{-n-2N-\epsilon},$$

  where m is a parameter, n is the homogeneous dimension of the space X, and N another "dimensional parameter" due to the homogeneous type ($N \geq 0$ could be equal to 0).
- $(\mathcal{A}_r)_{r>0}$ is a semigroup: for all $t, s > 0$ then $\mathcal{A}_s \mathcal{A}_t = \mathcal{A}_{s+t}$.

Related to such a collection, we build the following operator: for Q a ball

$$B_Q^*(f) = f - \mathcal{A}_{r_Q^m}(f).$$

Let us check that our assumptions hold with $q_1 = \infty$ and $q_0 = 1$.

By considering a ball Q, it comes that

$$\begin{aligned}
\left\| A_Q^*(f) \right\|_{L^\infty(Q)} &\lesssim \frac{1}{\mu(Q)} \sup_{x \in Q} \sum_{j \geq 0} \int_{C_j(Q)} \left(1 + \frac{d(x, y)}{r_Q}\right)^{-n-2N-\epsilon} |f(y)| d\mu(y) \\
&\lesssim \frac{1}{\mu(Q)} \sum_{j \geq 0} 2^{-(n+2N+\epsilon)j} \int_{C_j(Q)} |f(y)| d\mu(y) \\
&\lesssim \sum_{j \geq 0} \gamma_j \left(\frac{1}{\mu(2^j Q)} \int_{C_j(Q)} |f| d\mu \right)
\end{aligned}$$

with

$$\gamma_j \lesssim 2^{-(n+2N+\epsilon)j} \frac{\mu(2^j Q)}{\mu(Q)} \lesssim 2^{-(2N+\epsilon)j} \lesssim 2^{-\epsilon j}.$$

So Assumption (2.4) is satisfied.

Then it remains to check Assumption (2.6). Indeed it corresponds to a local version of Proposition 2.6 of [10] for $K = 2$. Let us consider a ball Q and another one $R \subset 2Q$, we have to estimate

$$\left\| B_R^*(A_Q^*(f)) \right\|_{L^\infty(R)}.$$

By writing r_R and r_Q for the radii of R and of Q, the semigroup property yields

$$B_R^* A_Q^*(f) = \mathcal{A}_{r_Q^m}(f) - \mathcal{A}_{r_R^m + r_Q^m}(f).$$

Then as $r_R \leq 2r_Q$, Proposition 2.6 of [10] proves that this quantity is bounded by $\|f\|_{\mathrm{BMO}_1}$. Consequently we deduce that Assumption (2.6) is satisfied too.

Conclusion: In the framework of [10], our Assumptions (2.4) and (2.7) are satisfied. We can apply Theorem 2.7 and obtain a John-Nirenberg inequality for the BMO_q spaces with $q \in (1, \infty)$. We also recover the results of Section 3 in [10].

3.4. The BMO space for a second order divergence form operator

The aim of this subsection is to compare the John-Nirenberg inequalities of [14] (associated to a divergence form operator) to ours.

Let us first recall the framework of [14]. Consider the Euclidean space $X = \mathbb{R}^n$ and A be an $n \times n$ matrix-valued function satisfying the ellipticity condition: there exist two constants $\Lambda \geq \lambda > 0$ such that

$$\forall \xi, \zeta \in \mathbb{C}^n, \qquad \lambda |\xi|^2 \leq Re\left(A\xi \cdot \overline{\xi}\right) \quad \text{and} \quad |A\xi \cdot \overline{\zeta}| \leq \Lambda |\xi||\zeta|.$$

We define the second order divergence form operator

$$L(f) := -\mathrm{div}(A\nabla f).$$

The semigroup associated to such operators satisfies some "Gaffney estimates":

Proposition 3.3 (Lemma 2.5 [14] and [1]). *There exist exponents* $1 \leq p_L < 2 < \widetilde{p_L} < \infty$, *such that for every* p *and* q *with* $p_L < p \leq q < \widetilde{p_L}$, *the semigroup* $(e^{-tL})_{t>0}$ *satisfies* $L^p - L^q$ *off-diagonal estimates,* i.e. *for arbitrary closed sets* $E, F \subset \mathbb{R}^n$*:*

$$\|e^{-tL} f\|_{L^q(F)} \lesssim t^{\frac{n}{2}\left(\frac{1}{q}-\frac{1}{p}\right)} e^{-\frac{d(E,F)^2}{t}} \|f\|_{L^p(E)},$$

for every $t > 0$ *and every function* $f \in L^p(E)$.

In [14], S. Hofmann and S. Mayboroda define a Hardy space $H^1_{L,p}$ associated to this operator and give several characterizations, where $p \in (p_L, \tilde{p_L})$. For $f \in L^1$, we have the equivalence of the following norms:

$$\|f\|_{H^1_{L,p}} := \|f\|_{L^1} + \left\| \left(\iint_{\substack{t>0,\ y\in\mathbb{R}^n \\ |x-y|\leq t}} \left| t^2 L e^{-t^2 L} f(y) \right|^2 \frac{dt dy}{t^{n+1}} \right)^{1/2} \right\|_{L^1}$$

$$\simeq \|f\|_{L^1} + \left\| \sup_{\substack{t>0,\ y\in\mathbb{R}^n \\ |x-y|\leq t}} \left(\frac{1}{t^n} \int_{B(y,t)} \left| e^{-t^2 L} f(z) \right|^2 dz \right)^{1/2} \right\|_{L^1}.$$

In addition, they prove a molecular decomposition with the following definition: let $\epsilon > 0$ and $M > n/4$ be fixed, a function $m \in L^2$ is a (p, ϵ, M)-molecule, if there exists a ball $Q \subset \mathbb{R}^n$ such that:

$$\forall i \geq 0, \forall k \in \{0, ..., M\}, \qquad \left\| \left(r_Q^{-2} L^{-1} \right)^k m \right\|_{L^p(C_i(Q))} \leq 2^{-i\epsilon} |2^{i+1} Q|^{-1+\frac{1}{p}}. \quad (3.1)$$

Moreover they prove that these spaces $H^1_{L,p}$ do not depend on $p \in (p_L, \widetilde{p_L})$, and they identify the dual spaces as a BMO space. For a ball Q, they consider operators $(B_Q)_Q$ given by the radius of the balls according to

$$B_Q(f) := \left(I - e^{-r_Q^2 L} \right)^M (f),$$

with a large enough integer $M > n/4$.

Let us check that our assumptions are satisfied in this case.
The operator A_Q is given by the semigroup as follows

$$A_Q^*(f) := f - \left(I - e^{-r_Q^2 L^*}\right)^M (f).$$

By expanding the power M, it comes that A_Q^* is a finite sum of semigroups:

$$A_Q^*(f) = \sum_{k=0}^{M} \binom{M}{k} (-1)^k e^{-kr_Q^2 L^*}(f).$$

So Gaffney estimates (see Proposition 3.3) give us some coefficients γ_j (depending on M) such that Assumption (2.4) holds, for $q_1 = p_L' - \epsilon$, $q_0 = \widetilde{p_L}' + \epsilon$ with $\epsilon > 0$ as small as we want. It remains to check Assumption (2.6), which is the goal of the following proposition (in fact we prove the stronger assumption (2.7)).

Proposition 3.4. *In this framework, for $t > 0$ the semigroup e^{-tL^*} acts continuously in* BMO *spaces. Let us write for a parameter (as small as we want) $\tau > 0$, $q_1 = p_L' - \tau$ and $q_0 = \widetilde{p_L}' + \tau$.*

- *There is a constant c such that for all $t > 0$ and all exponent $q \in [q_0, q_1]$,*

$$\left\| e^{-tL^*} f \right\|_{\mathrm{BMO}_q} \leq c \|f\|_{\mathrm{BMO}_q}$$

- *Assumption* (2.6) *holds:*

$$\sup_{\substack{R \text{ ball} \\ r_R^2 \leq 4t}} \left(\frac{1}{|R|} \int_R \left| B_R^*(e^{-tL^*}) \right|^{q_1} \right)^{1/q_1} \leq c \|f\|_{\mathrm{BMO}_{q_0}}.$$

Proof. Indeed we can prove a more precise result using duality. Thanks to [14, Theorem 1.3], the desired result is equivalent to the following: there is a constant c such that for all $t > 0$ and $p \in [q_1', q_0']$

$$\left\| e^{-tL} f \right\|_{H^1_{L,p}} \leq c \|f\|_{H^1_{L,p}}$$

and

$$\left\| e^{-tL} f \right\|_{H^1_{L,q_0'}} \leq c \|f\|_{H^1_{L,q_1',t}},$$

where $H^1_{L,q_1',t}$ is the Hardy space built by (q_1', ϵ) molecules associated to balls of radius lower than $2\sqrt{t}$. This will be achieved by invoking the following lemma and the fact that the space $H^1_{L,q}$ does not depend on the integer M. □

Lemma 3.5. *If $p \in [q_1', q_0']$ and f is a $(p, \epsilon, 2M)$-molecule adapted to a ball Q, then for every $t \geq 0$, $e^{-tL}(f)$ is a (p, ϵ, M) molecule adapted to the same cube. If f is a $(q_1', \epsilon, 2M)$-molecule adapted to a ball Q (with $r_Q \lesssim \sqrt{t}$), then $e^{-tL}(f)$ is a (q_0', ϵ, M) molecule adapted to the same cube.*

Proof. Let us first check the first claim. We need to show that $e^{-tL}(f)$ satisfies (3.1) for p, up to some multiplicative constant that is uniform for t. We fix the indices i and k and consider two cases.

Case 1: $t \leq 2^i r_Q^2$.
If $i \leq 3$, then we have

$$\left\|(r_Q^2 L)^{-k} e^{-tL^2}(f)\right\|_{L^p(C_i(Q))} = \left\|e^{-tL^2}(r_Q^2 L)^{-k}(f)\right\|_{L^p(C_i(Q))} \lesssim \left\|(r_Q^2 L)^{-k}(f)\right\|_{L^p} \lesssim |Q|^{-1+\frac{1}{p}},$$

as desired, where we have used the L^p-boundedness of the semigroup and the normalization of the molecule f in (3.1).

Suppose $i > 3$. We split

$$(r_Q^2 L)^{-k}(f) = g_1 + g_2$$

with

$$g_1 := (r_Q^2 L)^{-k}(f)\mathbf{1}_{2^{i-2}Q} \qquad g_2 := (r_Q^2 L)^{-k}(f)\mathbf{1}_{(2^{i-2}Q)^c}.$$

We then have

$$\begin{aligned}\left\|(r_Q^2 L)^{-k} e^{-tL^2}(f)\right\|_{L^p(C_i(Q))} &= \left\|e^{-tL^2}(r_Q^2 L)^{-k}(f)\right\|_{L^p(C_i(Q))} \\ &\leq \left\|e^{-tL^2} g_1\right\|_{L^p(C_i(Q))} + \left\|e^{-tL^2} g_2\right\|_{L^p(C_i(Q))} \\ &\lesssim e^{-\frac{4^i r_Q^2}{t}}|Q|^{-1+1/p} + \sum_{j=i-1}^{\infty} 2^{-j(n/2+\epsilon)}|Q|^{-1+1/p}\end{aligned}$$

where in the last step we have used the Gaffney estimate (Proposition 3.3), L^p-boundedness of the semigroup and (3.1). The desired bound follows in the present case.

Case 2: $t \geq 2^i r_Q^2$.

In this case, we have

$$\begin{aligned}\left\|(r_Q^2 L)^{-k} e^{-tL^2}(f)\right\|_{L^p(C_i(Q))} &= \left\|(r_Q^2 L)^{M} e^{-tL^2}(r_Q^2 L)^{-k-M}(f)\right\|_{L^p(C_i(Q))} \\ &= \left(\frac{r_Q^2}{t}\right)^M \left\|(tL)^{M} e^{-tL^2}(r_Q^2 L)^{-k-M}(f)\right\|_{L^p(C_i(Q))} \\ &\lesssim 2^{-iM}|Q|^{-1+1/p}\end{aligned}$$

where in the last line we have used L^p-boundedness of $(tL)^M e^{-tL^2}$, along the fact that (3.1) holds with $k+M \leq 2M$ instead of k, for f is a $(p,\epsilon,2M)$-molecule. Since we can choose $M > n/p$, the desired bound follows.

Second claim.
It remains us to check the second claim, stated in the lemma. The proof is the same one as the first claim with the following modification. Now we have to use the off-diagonal decay $L^{q_1'} - L^{q_0'}$ of the semigroup and the global boundedness (instead of the $L^p - L^p$ ones used before). Using Proposition 3.3, this operation makes appear an extra factor:

$$\left(\frac{t^{n/2}}{|Q|}\right)^{\frac{1}{q_1'}-\frac{1}{q_0'}}.$$

The exponent $\frac{1}{q_1'}-\frac{1}{q_0'}$ is negative, so $\frac{t^{n/2}}{|Q|}$ should be bounded below in order that this new coefficient be bounded. That is why we require $r_Q \lesssim \sqrt{t}$. □

Remark 3.6. To prove that the semigroup of operators e^{-tL} continuously acts on the Hardy space $H^1_{L,p}$, we refer the reader to the work [11] of J. Dziubański and M. Preisner. It is obvious that the function $x \to e^{-tx}$ satisfies the main assumption in [11] and so the associated multiplier e^{-tL} is bounded on the Hardy space (or at least on the molecules).

Conclusion: In the framework of [14], our Assumptions (2.4) and (2.7) are satisfied. We can also apply Theorem 2.7 and obtain John-Nirenberg inequalities for the BMO_q spaces with $q \in (\widetilde{p_L}', p_L')$. The precise inequality seems to be new, however we emphasize that the authors in [14] have already obtained an implicit John-Nirenberg inequalities in order to identify their BMO spaces, with various exponents $q \in (\widetilde{p_L}', p_L')$ (see [14, Section 10]).

4. Application to Hardy spaces

We devote this section to an application of John-Nirenberg inequalities in the theory of Hardy spaces. We refer the reader to Subsection 2.1 for definitions of *atoms* and *Hardy spaces*. We only deal with the atomic Hardy spaces for simplicity but a molecular version of the following results can be obtained too.

First let us give a "Hardy spaces"-point of view of our main Assumption (2.7).

Remark 4.1. Assumption (2.7) is equivalent to a $H^1_{p_1,\mathrm{ato}} - H^1_{p_0,\mathrm{ato}}$ boundedness of operators A_Q.

Now we assume that $\mathbb{B} = (B_Q)_Q$ satisfies some $L^p - L^p$ decay estimates: for $p \in [p_1, p_0]$ and M'' a large enough exponent, there exists a constant C such that

$$\forall k \geq 0,\ \forall f \in L^p,\ \mathrm{supp}(f) \subset Q \qquad \left\| B_Q(f) \right\|_{L^p(C_k(Q))} \leq C 2^{-Mk} \|f\|_{L^p(Q)}. \tag{4.1}$$

Using (4.1) we get the following properties about Hardy spaces and BMO spaces.

Proposition 4.2. *The Hardy space $H^1_{p,\mathrm{ato}}$ is included in L^1 .*

Proof. Since its atomic decomposition, we only have to control the L^1-norm of each atom $m = B_Q(f_Q) \in H^1_{p,\mathrm{ato}}$, by a uniform bound.

By (4.1), the estimates for f_Q, the doubling property of μ and the fact that M is large enough ($M > \delta/p'$ works), we have

$$\|B_Q(f_Q)\|_{L^1} \leq \sum_{k\geq 0} \|B_Q(f_Q)\|_{L^1(C_k(Q))} \lesssim \sum_{k\geq 0} \mu(Q)^{1/p'} 2^{-Mk} \mu(Q)^{1/p-1} 2^{k\delta/p'} \lesssim 1.$$

So we obtain that each p-atom is bounded in L^1, which permits to complete the proof. □

Corollary 4.3. *For $p \in [p_1, p_0]$, the space $H^1_{p,\mathrm{ato}}$ is a Banach space.*

Proof. The proof is already written in [6]. We reproduce it here for an easy reference.

We only verify the completeness: $H^1_{p,\mathrm{ato}}$ is a Banach space if for all sequences $(h_i)_{i\in\mathbb{N}}$ of $H^1_{p,\mathrm{ato}}$ satisfying

$$\sum_{i\geq 0} \|h_i\|_{H^1_{p,\mathrm{ato}}} < \infty,$$

the series $\sum h_i$ converges in the Hardy space $H^1_{p,\mathrm{ato}}$.

For such sequence in $H^1_{p,\mathrm{ato}}$, we say that $\sum_i h_i \in L^1$, because each atom decomposition is absolutely convergent in L^1-sense (since the previous proposition). If we denote $f = \sum_i h_i \in L^1$, then using the condition that $\sum_{i\geq 0} \|h_i\|_{H^1_{p,\mathrm{ato}}} < \infty$, we have

$$\|f - \sum_{i=0}^{n} h_i\|_{H^1_{p,\mathrm{ato}}} \leq \sum_{i=n+1}^{\infty} \|h_i\|_{H^1_{p,\mathrm{ato}}} \xrightarrow[n\to\infty]{} 0.$$ □

We now come to our main result of this section: the Hardy space $H^1_{p,\mathrm{ato}}$ does not depend on $p \in (p_1, p_0]$.

Theorem 4.4. *Under the above assumptions of Theorem* 2.7 *and* (4.1)*, the Hardy space $H^1_{p,\mathrm{ato}}$ does not depend on the exponent $p \in (p_1, p_0]$.*

Proof. The proof is based on duality and the property that the BMO spaces are not depending on the exponent, see Corollary 2.11.

We recall duality results and we refer to [6, Section 8] for more details. Fix an exponent $p \in (p_1, p_0]$. We cannot have a precise characterization of the dual space of our atomic Hardy space. However, we have the following results. Since the operator B_Q are acting on a function (with a bounded support) to the Hardy space, we know that we can define and extend B_Q^* from $(H^1_{p,\text{ato}})^*$ to $L^{p'}_{\text{loc}}$. Then, we claim that for $\phi \in (H^1_{p,\text{ato}})^*$

$$\|\phi\|_{\text{BMO}_{p'}} \simeq \|\phi\|_{(H^1_{p,\text{ato}})^*}. \tag{4.2}$$

First step: Proof of (4.2).

For each $\phi \in \text{BMO}_{p'} \cap (H^1_{p,\text{ato}})^*$, for each atom $m \in \left(H^1_{p,\text{ato}}\right)$, where $m = B_Q(f_Q)$, by Hölder inequality, we have

$$\begin{aligned} |\langle \phi, m\rangle| &= \left|\int_Q B_Q^*(\phi) f_Q d\mu\right| \\ &\leq \left(\int_Q |B_Q^*(\phi)|^{p'} d\mu\right)^{\frac{1}{p'}} \left(\int_Q |f|^p d\mu\right)^{\frac{1}{p}} \leq \|\phi\|_{\text{BMO}_{p'}}. \end{aligned} \tag{4.3}$$

Therefore by atomic decomposition, we deduce the first inequality $\|\phi\|_{(H^1_{p,\text{ato}})^*} \leq \|\phi\|_{\text{BMO}_{p'}}$. It remains us to check the reverse inequality.

For arbitrary $f \in L^p(Q)$ satisfying $\|f\|_{L^p(Q)} = 1$, we set $g_Q := \mu(Q)^{\frac{1}{p}-1} f$, then $m = B_Q(g_Q)$ is a p-atom. Therefore

$$\left|\int_Q B_Q^*(\phi) f d\mu\right| = \mu(Q)^{1-\frac{1}{p}} |\langle \phi, B_Q(g_Q)\rangle| \leq \mu(Q)^{\frac{1}{p'}} \|\phi\|_{(H^1_{p,\text{ato}})^*}.$$

This holds for every $L^p(Q)$-normalized function f. By duality, we deduce the reverse inequality, which concludes the proof of (4.2).

Second step: End of the proof.

Let choose two exponents p, r in the above range $[p_1, p_0]$. By symmetry, it is just sufficient to prove that

$$\|f\|_{H^1_{p,\text{ato}}} \lesssim \|f\|_{H^1_{r,\text{ato}}} \tag{4.4}$$

for every function $f \in H^1_{p,\text{ato}} \cap H^1_{r,\text{ato}}$ (since it is easy to check that $H^1_{p,\text{ato}} \cap H^1_{r,\text{ato}}$ is dense into both Hardy spaces). So let us fix such a function f. The Hardy spaces are Banach spaces (Corollary 4.3), so Hahn-Banach Theorem implies that there is $\phi \in (H^1_{p,\text{ato}})^*$ normalized such that

$$\|f\|_{H^1_{p,\text{ato}}} = \langle \phi, f\rangle.$$

We know that for every ball Q, $B_Q^*(\phi)$ belongs to $L^{p'}(Q)$ and satisfies

$$\left(\frac{1}{\mu(Q)}\int_Q |B_Q^*(\phi)|^{p'} d\mu\right)^{\frac{1}{p'}} \leq 1.$$

We can apply John-Nirenberg inequality: Theorem 2.7 [1]. We also obtain that for all ball Q, $B_Q^*(\phi)$ belongs to $L^{r'}(Q)$ and satisfies

$$\left(\frac{1}{\mu(Q)}\int_Q |B_Q^*(\phi)|^{r'} d\mu\right)^{\frac{1}{r'}} \lesssim 1.$$

Then as $f \in H^1_{p,\mathrm{ato}} \cap H^1_{r,\mathrm{ato}} \subset H^1_{r,\mathrm{ato}}$, it follows by the previous reasoning (step 1) that

$$\|f\|_{H^1_{p,\mathrm{ato}}} = \langle \phi, f\rangle \lesssim \sum_i |\lambda_i \langle \phi, m_i\rangle| \lesssim \|f\|_{H^1_{r,\mathrm{ato}}},$$

where we have used an "extremizing" decomposition $f = \sum_i \lambda_i m_i$ with r-atoms. The proof of (4.4) is also achieved. □

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