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Non-divergence form parabolic equations associated with non-commuting vector fields: boundary behavior of nonnegative solutions

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Abstract. In a cylinder $\Omega_T = \Omega \times (0, T) \subset \mathbb{R}^{n+1}_+$ we study the boundary behavior of nonnegative solutions of second order parabolic equations of the form

$$Hu = \sum_{i,j=1}^{m} a_{ij}(x,t)X_i X_j u - \partial_t u = 0, \ (x,t) \in \mathbb{R}^{n+1}_+,$$

where $X = \{X_1, \dots, X_m\}$ is a system of C^∞ vector fields in $\mathbb{R}^n$ satisfying Hörmander's rank condition (1.2), and Ω is a non-tangentially accessible domain with respect to the Carnot-Carathéodory distance d induced by X. Concerning the matrix-valued function $A = \{a_{ij}\}$, we assume that it is real, symmetric and uniformly positive definite. Furthermore, we suppose that its entries a_{ij} are Hölder continuous with respect to the parabolic distance associated with d. Our main results are: 1) a backward Harnack inequality for nonnegative solutions vanishing on the lateral boundary (Theorem 1.1); 2) the Hölder continuity up to the boundary of the quotient of two nonnegative solutions which vanish continuously on a portion of the lateral boundary (Theorem 1.2); 3) the doubling property for the parabolic measure associated with the operator H (Theorem 1.3). These results generalize to the subelliptic setting of the present paper, those in Lipschitz cylinders by Fabes, Safonov and Yuan in [20, 39]. With one proviso: in those papers the authors assume that the coefficients a_{ij} be only bounded and measurable, whereas we assume Hölder continuity with respect to the intrinsic parabolic distance.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and consider the cylinder $\Omega_T = \Omega \times (0, T) \subset \mathbb{R}^{n+1}_+$, where $T > 0$ is fixed. In this paper we establish a number of results concern-

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ing the boundary behavior of nonnegative solutions in Ω_T of second order parabolic equations of the type

$$Hu = Lu - \partial_t u = \sum_{i,j=1}^{m} a_{ij}(x,t) X_i X_j u - \partial_t u = 0. \tag{1.1}$$

Here, $X=\{X_1,\dots,X_m\}$ is a system of C^∞ vector fields in $\mathbb{R}^n$ satisfying Hörmander's rank condition, see [25]:

$$\text{rank Lie}\,[X_1,\dots,X_m] \equiv n. \tag{1.2}$$

Concerning the $m \times m$ matrix-valued function $A(x,t) = \{a_{ij}(x,t)\}$ we assume that it is symmetric, with bounded and measurable entries, and that there exists $\lambda \in [1,\infty)$ such that for every $(x,t) \in \mathbb{R}^{n+1}$, and $\xi \in \mathbb{R}^m$,

$$\lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^{m} a_{ij}(x,t)\xi_i\xi_j \leq \lambda|\xi|^2. \tag{1.3}$$

When $m = n$ and $\{X_1,\dots,X_m\} = \{\partial_{x_1},\dots,\partial_{x_n}\}$, the operator H in (1.1) coincides with that studied in [20, 39]. However, in contrast with these papers, in which the coefficients were assumed only bounded and measurable, we will also assume that the entries of the matrix $A(x,t)$ are Hölder continuous with respect to the intrinsic parabolic distance associated with the system X. More precisely, we indicate with $d(x,y)$ the Carnot-Carathéodory distance, between $x, y \in \mathbb{R}^n$, induced by $\{X_1,\dots,X_m\}$. We let

$$d_p(x,t,y,s) = (d(x,y)^2 + |t-s|)^{1/2}$$

denote the parabolic distance associated with the metric d. Then, we assume that there exist $C > 0$, and $\sigma \in (0,1)$, such that for $(x,t),\ (y,s) \in \mathbb{R}^{n+1}$,

$$|a_{ij}(x,t) - a_{ij}(y,s)| \leq C d_p(x,t,y,s)^\sigma, \quad i,j \in \{1,..,m\}. \tag{1.4}$$

The reason for imposing (1.4) will be discussed below.

Concerning the domain Ω we will assume that Ω is a NTA domain (non-tangentially accessible domain), with parameters M, r_0, in the sense of [7, 11], see Definition 2.6 below. Under this assumption we can prove that all points on the parabolic boundary

$$\partial_p\Omega_T = S_T \cup (\Omega \times \{0\}), \quad S_T = \partial\Omega \times (0,T),$$

of the cylinder Ω_T are regular for the Dirichlet problem for the operator H in (1.1). In particular, for any $f \in C(\partial_p\Omega_T)$, there exists a unique Perron-Wiener-Brelot-Bauer solution $u = u_f^{\Omega_T} \in C(\overline{\Omega}_T)$ to the Dirichlet problem

$$Hu = 0 \text{ in } \Omega_T, \quad u = f \text{ on } \partial_p\Omega_T. \tag{1.5}$$

Moreover, one can conclude that for every $(x,t) \in \Omega_T$ there exists a unique probability measure $d\omega^{(x,t)}$ on $\partial_p\Omega_T$ such that

$$u(x,t) = \int_{\partial_p\Omega_T} f(y,s)\,d\omega^{(x,t)}(y,s). \tag{1.6}$$

Henceforth, we refer to $\omega^{(x,t)}$ as the *H-parabolic measure* relative to (x,t) and Ω_T.

The metric ball centered at $x \in \mathbb{R}^n$ with radius $r > 0$ will be indicated with

$$B_d(x,r) = \{y \in \mathbb{R}^n : d(x,y) < r\}.$$

For $(x,t) \in \mathbb{R}^{n+1}$ and $r > 0$ we let

$$C_r^-(x,t) = B_d(x,r) \times (t - r^2, t), \quad C_r(x,t) = B_d(x,r) \times (t - r^2, t + r^2),$$

and we define

$$\Delta(x,t,r) = S_T \cap C_r(x,t). \tag{1.7}$$

By Definition 2.6 below, if Ω is a given NTA domain with parameters M and r_0, then for any $x_0 \in \partial\Omega$, $0 < r < r_0$, there exists a non-tangential corkscrew, *i.e.*, a point $A_r(x_0) \in \Omega$, such that

$$M^{-1}r < d(x_0, A_r(x_0)) < r, \quad \text{and} \quad d(A_r(x_0), \partial\Omega) \geq M^{-1}r.$$

In the following we let $A_r(x_0,t_0) = (A_r(x_0), t_0)$ whenever $(x_0,t_0) \in S_T$ and $0 < r < r_0$. When we say that a constant c depends on the operator H we mean that c depends on the dimension n, the number of vector fields m, the vector fields $\{X_1, \ldots, X_m\}$, the constant λ in (1.3) and the parameters C, σ in (1.4). We let $\operatorname{diam}(\Omega) = \sup\{d(x,y) \mid x, y \in \Omega\}$ denote the diameter of Ω. The following theorems represent the main results of this paper.

Theorem 1.1 (Backward Harnack inequality). *Let u be a nonnegative solution of $Hu = 0$ in Ω_T vanishing continuously on S_T. Let $0 < \delta \ll \sqrt{T}$ be a fixed constant, let $(x_0,t_0) \in S_T$, $\delta^2 \leq t_0 \leq T - \delta^2$, and assume that $r < \min\{r_0/2, \sqrt{(T - t_0 - \delta^2)/4}, \sqrt{(t_0 - \delta^2)/4}\}$. Then, there exists a constant $c = c(H, M, r_0, \operatorname{diam}(\Omega), T, \delta)$, $1 \leq c < \infty$, such that for every $(x,t) \in \Omega_T \cap C_{r/4}(x_0,t_0)$ one has*

$$u(x,t) \leq c\,u(A_r(x_0,t_0)).$$

Theorem 1.2 (Boundary Hölder continuity of quotients of solutions). *Let u,v be nonnegative solutions of $Hu = 0$ in Ω_T. Given $(x_0,t_0) \in S_T$, assume that $r < \min\{r_0/2, \sqrt{(T - t_0)/4}, \sqrt{t_0/4}\}$. If u, v vanish continuously on $\Delta(x_0,t_0,2r)$, then the quotient v/u is Hölder continuous on the closure of $\Omega_T \cap C_r^-(x_0,t_0)$.*

Theorem 1.3 (Doubling property of the H-parabolic measure). *Let $K \geq 100$ and $\nu \in (0,1)$ be fixed constants. Let $(x_0, t_0) \in S_T$, and suppose that $r < \min\{\nu r_0/2, \sqrt{(T-t_0)/4}, \sqrt{t_0/4}\}$. Then, there exists a constant $c = c(H, M, \nu, K, r_0)$, $1 \leq c < \infty$, such that for every $(x,t) \in \Omega_T$, with $d(x_0, x) \leq K|t - t_0|^{1/2}$, $t - t_0 \geq 16r^2$, one has*

$$\omega^{(x,t)}(\Delta(x_0, t_0, 2r)) \leq c\omega^{(x,t)}(\Delta(x_0, t_0, r)).$$

Concerning Theorems 1.1, 1.2 and 1.3, we note that the study of the type of problems considered in this paper has a long and rich history which, for uniformly parabolic equations in $\mathbb{R}^{n+1}$ (*i.e.*, when in (1.1) one has $m = n$ and $\{X_1, \ldots, X_m\} = \{\partial_{x_1}, \ldots, \partial_{x_n}\}$), culminated with the celebrated papers of Fabes, Safonov and Yuan [19, 20, 39]. In these works the authors proved Theorem 1.1-1.3 for uniformly parabolic equations, both in divergence and non-divergence form, whose coefficients are only bounded and measurable. We remark that, while these authors work in Lipschitz cylinders, one can easily see that their proofs can be generalized to the setting of bounded NTA domains in the sense of [26]. While the works [20, 39] completed this line of research for parabolic operators in non-divergence form, prior contributions by other researchers are contained in [17, 21, 22, 29]. For the corresponding developments for second order parabolic operators in divergence form we refer to [16, 19, 34]. For the elliptic theory, for both operators in divergence and non-divergence form, we refer to [1, 6, 15, 26]. Finally, and for completion, we also note that second order elliptic and parabolic operators in divergence form with singular lower order terms were studied in [24, 27].

In the subelliptic setting of the present paper, *i.e.*, when $m < n$ and $X = \{X_1, \ldots, X_m\}$ is assumed to satisfy (1.2), much less is known. Several delicate new issues arise in connection with the intricate (sub-Riemannian) geometry associated with the vector fields, and the interplay of such geometry with the so-called characteristic points on the boundary of the relevant domain. In addition, the derivatives along the vector fields do not commute, and the commutators are effectively derivatives of higher order. For all these aspects we refer the reader to the works [7, 10, 11, 13, 14, 30–32, 35], but this only represents a partial list of references.

In the stationary case, and for operators in divergence form, results similar to those in the present paper have been obtained in [7, 10, 11], see also [8, 9], whereas for parabolic operators in divergence form the reader is referred to the recent paper by one of us [33]. The methods in [33], however, extensively exploit the divergence structure of the operator and do not apply to the setting of the present paper.

We stress that for non-divergence form operators such as those treated in this paper, results such as Theorems 1.1-1.3 are new even for the case of stationary equations such as

$$Lu = \sum_{i,j=1}^{m} a_{ij}(x) X_i X_j u = 0.$$

In view of these considerations our paper provides a novel contribution to the understanding of the boundary behavior of solutions to parabolic equations arising from a system of non-commuting vector fields.

Concerning the proofs of Theorems 1.1-1.3 our approach is modeled on the ideas developed by Fabes, Safonov and Yuan in [20, 39]. In fact, the ideas in those papers have provided an important guiding line for our work. Yet, the arguments in [20,39] use mainly elementary principles like comparison principles, interior regularity theory, the (interior) Harnack inequality, Hölder continuity type estimates and decay estimates at the lateral boundary, for solutions which vanish on a portion of the lateral boundary, as well as estimates for the Cauchy problem and the fundamental solution associated to the operator at hand. In this connection it is important that the reader keep in mind that when the matrix $A(x,t) = \{a_{ij}(x,t)\}$ in (1.1) has entries which are just bounded and measurable, then most of these results presently represent in our setting *terra incognita*. More specifically, the counterparts of the Harnack inequality of Krylov and Safonov [29] and the Alexandrov-Bakel'man-Pucci type maximum principle due to Krylov [28] presently constitute fundamental open questions.

With this being said, our work uses heavily the recent important results of Bramanti, Brandolini, Lanconelli and Uguzzoni [5], see also [4], concerning the (interior) Harnack inequality, the Cauchy problem and the existence and Gaussian estimates for fundamental solutions for the non-divergence form operators H defined in (1.1). In fact, we assume (1.4) precisely in order to be able to use results from [5]. We want to stress, however, that we have strived throughout the whole paper to provide proofs which are "purely metrical". By this we mean that, should the above mentioned counterpart of the results in [28, 29] become available, then our proofs would carry to the more general setting of bounded and measurable coefficients in (1.1) with minor changes.

In closing we mention that the rest of the paper is organized as follows. Section 2 is of a preliminary nature. In it we collect some notation and results concerning basic underlying principles, and we also introduce the notion of NTA domains following [7]. In Section 3 we prove a number of basic estimates concerning the boundary behavior of nonnegative solutions of (1.1). In addition we prove a number of technical lemmas which allow us to present the proofs of Theorems 1.1-1.3 in a quite condensed manner. Finally, the proofs of Theorems 1.1 and 1.2 will be presented in Section 4, whereas that of Theorem 1.3 will be given in Section 5.

2. Preliminaries

In this section we introduce some notation and state a number of preliminary results for the operator H defined in (1.1). Specifically, we will discuss the Cauchy problem and Gaussian estimates for the fundamental solution, the Harnack inequality and comparison principle, and the Dirichlet problem in bounded domains. In particular, we also justify the notion of H-parabolic measure and introduce the notion of NTA domain.

2.1. Notation

In $\mathbb{R}^n$, with $n \geq 3$, we consider a system $X = \{X_1, \ldots, X_m\}$ of C^∞ vector fields satisfying Hörmander's rank condition (1.2). As in [18], a piecewise C^1 curve $\gamma : [0, \ell] \to \mathbb{R}^n$ is called subunitary if at every $t \in [0, \ell]$ at which $\gamma'(t)$ exists one has for every $\xi \in \mathbb{R}^n$

$$< \gamma'(t), \xi >^2 \leq \sum_{j=1}^{m} < X_j(\gamma(t)), \xi >^2 .$$

We note explicitly that the above inequality forces $\gamma'(t)$ to belong to the span of $\{X_1(\gamma(t)), \ldots, X_m(\gamma(t))\}$. The subunit length of γ is by definition $l_s(\gamma) = \ell$. If we fix an open set $\Omega \subset \mathbb{R}^n$, then given $x, y \in \Omega$, denote by $\mathcal{S}_\Omega(x, y)$ the collection of all subunitary $\gamma : [0, \ell] \to \Omega$ which join x to y. The accessibility theorem of Chow and Rashevsky, [12,37], states that, if Ω is connected, then for every $x, y \in \Omega$ there exists $\gamma \in \mathcal{S}_\Omega(x, y)$. As a consequence, if we define

$$d_\Omega(x, y) = \inf \{l_s(\gamma) \mid \gamma \in \mathcal{S}_\Omega(x, y)\},$$

we obtain a distance on Ω, called the Carnot-Carathéodory distance, associated with the system X. When $\Omega = \mathbb{R}^n$, we write $d(x, y)$ instead of $d_{\mathbb{R}^n}(x, y)$. It is clear that $d(x, y) \leq d_\Omega(x, y)$, $x, y \in \Omega$, for every connected open set $\Omega \subset \mathbb{R}^n$. In [36] it was proved that, given $\Omega \subset\subset \mathbb{R}^n$, there exist $C, \epsilon > 0$ such that

$$C|x - y| \leq d_\Omega(x, y) \leq C^{-1}|x - y|^\epsilon, \qquad x, y \in \Omega. \tag{2.1}$$

This gives $d(x, y) \leq C^{-1}|x - y|^\epsilon$, $x, y \in \Omega$, and therefore

$$i : (\mathbb{R}^n, |\cdot|) \to (\mathbb{R}^n, d) \qquad \text{is continuous.}$$

Furthermore, it is easy to see that also the continuity of the opposite inclusion holds [23], and therefore the metric and the Euclidean topologies are equivalent.

For $x \in \mathbb{R}^n$ and $r > 0$, we let $B_d(x, r) = \{y \in \mathbb{R}^n \mid d(x, y) < r\}$. The basic properties of these balls were established by Nagel, Stein and Wainger in their seminal paper [36]. These authors proved in particular that, given bounded open set $U \subset \mathbb{R}^n$, there exist constants $C, R_0 > 0$ such that, for any $x \in U$, and $0 < r \leq R_0$,

$$C \leq \frac{B_d(x, r)}{\Lambda(x, r)} \leq C^{-1},$$

where $\Lambda(x, r) = \sum_I |a_I(x)| r^{d_I}$ is a polynomial function with continuous coefficients. As a consequence, one has with $C_1 > 0$,

$$|B_d(x, 2r)| \leq C_1 |B_d(x, r)| \qquad \text{for every} \quad x \in U \quad \text{and} \quad 0 < r \leq R_0. \tag{2.2}$$

In what follows, given $\beta \in (0, 1)$, we let $\Gamma^\beta(\Omega_T)$ denote the space of functions $u : \Omega_T \to \mathbb{R}$ such that

$$\begin{aligned} ||u||_{\Gamma^\beta(\Omega_T)} :&= \sup_{\Omega_T} |u| \\ &+ \sup_{(x,t),(x',t')\in\Omega_T,\ (x,t)\neq(x',t')} \frac{|u(x,t) - u(x',t')|}{d_p(x,t,x',t')^\beta} < \infty. \end{aligned} \tag{2.3}$$

We say that u has a Lie derivative along X_j, at $(x, t) \in \Omega_T$, if $u \circ \gamma$ is differentiable at 0, where γ is the integral curve of X_j such that $\gamma(0) = (x, t)$. Moreover, we indicate with $\Gamma^{2+\beta}(\Omega_T)$ the space of functions $u \in \Gamma^\beta(\Omega_T)$ which admit Lie derivatives up to second order along $X_1, \ldots, X_m$, and up to order one with respect to t, in $\Gamma^\beta(\Omega_T)$. If $u \in \Gamma^{2+\beta}(\Omega_T)$ then we let $||u||_{\Gamma^{2+\beta}(\Omega_T)}$ denote the naturally defined norm of u. Furthermore, $u \in \Gamma^\beta_{\text{loc}}(\Omega_T)$ if $u \in \Gamma^\beta(D)$ for any compact subset D of Ω_T. The space $\Gamma^{2+\beta}_{\text{loc}}(\Omega_T)$ is defined analogously. Finally, if $\beta = 0$ then we simply write $\Gamma^2(\Omega_T)$ for $\Gamma^{2+0}(\Omega_T)$. Throughout the paper we will use the following notation:

$$\begin{aligned} C_r(x,t) &= B_d(x,r) \times (t - r^2, t + r^2), \\ C_r^+(x,t) &= B_d(x,r) \times (t, t + r^2), \\ C_r^-(x,t) &= B_d(x,r) \times (t - r^2, t), \\ C_{r_1,r_2}(x,t) &= B_d(x,r_1) \times (t - r_2^2, t + r_2^2), \\ C^+_{r_1,r_2}(x,t) &= B_d(x,r_1) \times (t, t + r_2^2), \\ C^-_{r_1,r_2}(x,t) &= B_d(x,r_1) \times (t - r_2^2, t), \end{aligned} \tag{2.4}$$

for $(x, t) \in \mathbb{R}^{n+1}$ and $r, r_1, r_2 > 0$. Furthermore, if $\Omega \subset \mathbb{R}^n$ is a bounded domain, and $T > 0$ and $\delta > 0$ are given, then we let

$$\Omega^\delta = \{x \in \Omega |\ d(x, \partial\Omega) > \delta\},\ \ \Omega^\delta_T = \Omega^\delta \times (0, T). \tag{2.5}$$

2.2. The Cauchy problem

Let H be defined as in (1.1), with the hypothesis (1.2), (1.3) and (1.4) in place. These assumptions allow us to use some basic results established in [5]. In particular, for what concerns the existence of a fundamental solution of the operator H, and Gaussian estimates, we will henceforth suppose, as it is done in [5], that the sub-Laplacian $\sum_{i=1}^m X_i^2$ associated with X coincides with the standard Laplacian $\Delta = \sum_{j=1}^n \partial^2_{x_j}$ in $\mathbb{R}^n$ outside of a fixed compact set in $\mathbb{R}^n$.

In [5] it is proved that, under such hypothesis, there exists a fundamental solution, Γ, for H, with a number of important properties. In particular, Γ is a continuous function away from the diagonal of $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ and $\Gamma(x, t, \xi, \tau) = 0$ for $t \leq$

τ. Moreover, $\Gamma(\cdot,\cdot,\xi,\tau)\in\Gamma^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^{n+1}\setminus\{(\xi,\tau)\})$ for every fixed $(\xi,\tau)\in\mathbb{R}^{n+1}$ and $H(\Gamma(\cdot,\cdot,\xi,\tau))=0$ in $\mathbb{R}^{n+1}\setminus\{(\xi,\tau)\}$. For every $\psi\in C^\infty_0(\mathbb{R}^{n+1})$ the function

$$w(x,t)=\int_{\mathbb{R}^{n+1}}\Gamma(x,t,\xi,\tau)\psi(\xi,\tau)d\xi d\tau$$

belongs to $\Gamma^{2+\alpha}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and we have $Hw=\psi$ in $\mathbb{R}^{n+1}$. Furthermore, let $\mu\geq 0$ and $T_2>T_1$ be such that $(T_2-T_1)\mu$ is small enough, let $0<\beta\leq\alpha$, let $g\in C^{0,\beta}(\mathbb{R}^n\times[T_1,T_2])$ and $f\in C(\mathbb{R}^n)$ be such that $|g(x,t)|,|f(x)|\leq c\exp(\mu d(x,0)^2)$ for some constant $c>0$. Then, for $x\in\mathbb{R}^n$, $t\in(T_1,T_2]$, the function

$$u(x,t)=\int_{\mathbf{R}^n}\Gamma(x,t,\xi,T_1)f(\xi)d\xi+\int_{T_1}^{t}\int_{\mathbf{R}^n}\Gamma(x,t,\xi,\tau)g(\xi,\tau)d\xi d\tau, \qquad (2.6)$$

belongs to the class $\Gamma^{2+\beta}_{\mathrm{loc}}(\mathbb{R}^n\times(T_1,T_2))\cap C(\mathbb{R}^n\times[T_1,T_2])$. Moreover, u solves the Cauchy problem

$$Hu=g \text{ in } \mathbb{R}^n\times(T_1,T_2),\ u(\cdot,T_1)=f(\cdot) \text{ in } \mathbb{R}^n. \qquad (2.7)$$

One also has the following Gaussian bounds.

Lemma 2.1. *There exist a positive constant C and, for every $T>0$, a positive constant $c=c(T)$ such that, if $0<t-\tau\leq T$, $x,\xi\in\mathbb{R}^n$, then*

$$c^{-1}\frac{e^{-Cd(x,\xi)^2/(t-\tau)}}{|B(x,\sqrt{t-\tau})|}\leq\Gamma(x,t,\xi,\tau)\leq c\frac{e^{-C^{-1}d(x,\xi)^2/(t-\tau)}}{|B(x,\sqrt{t-\tau})|}. \qquad (2.8)$$

Furthermore, one also has

$$|X_i\Gamma(\cdot,t,\xi,\tau)(x)|\leq c(t-\tau)^{-1/2}\frac{e^{-C^{-1}d(x,\xi)^2/(t-\tau)}}{|B(x,\sqrt{t-\tau})|}, \qquad (2.9)$$

and

$$|X_iX_j\Gamma(\cdot,t,\xi,\tau)(x)|+|\partial_t\Gamma(x,\cdot,\xi,\tau)(t)|\leq c(t-\tau)^{-1}c\frac{e^{-C^{-1}d(x,\xi)^2/(t-\tau)}}{|B(x,\sqrt{t-\tau})|}. \qquad (2.10)$$

2.3. The Harnack inequality and strong maximum principle

We next state the Harnack inequality and the strong maximum principle for the operator H, see [4] and also [5].

Theorem 2.2. *Let $R > 0$, $0 < h_1 < h_2 < 1$ and $\gamma \in (0,1)$. Then, there exists a positive constant $C = C(h_1, h_2, \gamma, R)$ such that the following holds for every $(\xi, \tau) \in \mathbb{R}^{n+1}$, $r \in (0, R]$. If*

$$u \in \Gamma^2(C_r^-(\xi,\tau)) \cap C(\overline{C_r^-(\xi,\tau)})$$

satisfies $Hu = 0$, $u \geq 0$, in $C_r^-(\xi,\tau)$, then

$$u(x,t) \leq Cu(\xi,\tau) \text{ whenever } (x,t) \in \overline{C^-_{\gamma r, h_2 r}(\xi,\tau)} \setminus C^-_{\gamma r, h_1 r}(\xi,\tau).$$

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a connected, bounded open set, and let $T > 0$. Let $u \in \Gamma^2(\Omega_T)$ and assume that $Lu \geq 0$, $u \leq 0$ in Ω_T. Assume that $u(x_0, t_0) = 0$ for some $(x_0, t_0) \in \Omega_T$. Then $u(x,t) \equiv 0$ whenever $(x,t) \in \Omega_T \cap \{t : t \leq t_0\}$.*

2.4. The Dirichlet problem

In the following we let D be any bounded open subset of $\mathbb{R}^{n+1}$ and we study the Dirichlet problem

$$Hu = 0 \text{ in } D, u = f \text{ on } \partial_p D, \tag{2.11}$$

with $f \in C(\partial_p D)$. Here, $\partial_p D$ denotes the parabolic boundary of D. If $u : D \to \mathbb{R}$ is a smooth function satisfying $Hu = 0$ in D, then we say that u is H-parabolic in D. We denote by $P(D)$ the linear space of functions which are H-parabolic in D.

We say that D is H-regular if for any $f \in C(\partial_p D)$ there exists a unique function $H_f^D \in P(D)$ such that $\lim_{(x,t)\to(x_0,t_0)} H_f^D(x,t) = f(x_0,t_0)$ for every $(x_0,t_0) \in \partial_p D$. Following the arguments in [30], see in particular Theorems 6.5 and 10.1, we can easily construct a basis for the Euclidean topology of $\mathbb{R}^{n+1}$ which is made of cylindrical H-regular sets. Furthermore, if D is H-regular, then in view of Theorem 2.3 (one actually only needs the weak maximum principle) for every fixed $(x,t) \in D$ the map $f \mapsto H_f^D(x,t)$ defines a positive linear functional on $C(\partial_p D)$. By the Riesz representation theorem there exists a unique Borel measure $\omega = \omega_D$, supported in $\partial_p D$, such that

$$H_f^D(x,t) = \int_{\partial_p D} f(y,s)\, d\omega^{(x,t)}(y,s), \qquad \text{for every } f \in C(\partial_p D). \tag{2.12}$$

We will refer to $\omega^{(x,t)} = \omega_D^{(x,t)}$ as the *H-parabolic measure* relative to D and (x,t).

A lower semi-continuous function $u : D \to]-\infty, \infty]$ is said to be H-superparabolic in D if $u < \infty$ in a dense subset of D and if

$$u(x,t) \geq \int_{\partial V} u(y,s)\, d\omega_V^{(x,t)}(y,s),$$

for every open H-regular set $V \subset \overline{V} \subset D$ and for every $(x,t) \in V$. We denote by $\overline{S}(D)$ the set of H-superparabolic functions in D, and by $\overline{S}^+(D)$ the set of the functions in $\overline{S}(D)$ which are nonnegative. A function $v : D \to [-\infty, \infty[$ is said to be H-subparabolic in D if $-v \in \overline{S}(D)$ and we write $\underline{S}(D) := -\overline{S}(D)$. As the collection of H-regular sets is a basis for the Euclidean topology, it follows that $\overline{S}(D) \cap \underline{S}(D) = P(D)$. Finally, we recall that H_f^D can be realized as the generalized solution in the sense of Perron-Wiener-Brelot-Bauer to the problem in (2.11). In particular,

$$\inf \overline{\mathcal{U}}_f^D = \sup \underline{\mathcal{U}}_f^D = H_f^D, \tag{2.13}$$

where we have indicated with $\overline{\mathcal{U}}_f^D$ the collection of all $u \in \overline{S}(D)$ such that $\inf_D u > -\infty$, and

$$\liminf_{(x,t)\to(x_0,t_0)} u(x,t) \geq f(x_0,t_0), \ \forall\,(x_0,t_0) \in \partial_p D,$$

and with $\underline{\mathcal{U}}_f^D$ the collection of all $u \in \underline{S}(D)$ for which $\sup_D u < \infty$, and

$$\limsup_{(x,t)\to(x_0,t_0)} u(x,t) \leq f(x_0,t_0), \ \forall\,(x_0,t_0) \in \partial_p D.$$

Lemma 2.4. *Let $D \subset \mathbb{R}^{n+1}$ be a bounded open set, let $f \in C(\partial_p D)$ and let u be the generalized Perron-Wiener-Brelot-Bauer solution to the problem in* (2.11)*, i.e., $u = H_f^D$ where H_f^D be defined as in* (2.13)*. Then $u \in \Gamma^2(D)$.*

Proof. This follows from Theorem 1.1 in [40]. □

In the following we are concerned with the issue of regular boundary points and we note, concerning the solvability of the Dirichlet problem for the operator H, that in [40] Uguzzoni developes what he refers to as a "cone criterion" for non-divergence equations modeled on Hörmander vector fields. This is a generalization of the well-known positive density condition of classical potential theory. We next describe his result in the setting of domains of the form $\Omega_T = \Omega \times (0,T)$, where $\Omega \subset \mathbb{R}^n$ is assumed to be a bounded domain. In [40] a bounded open set Ω is said to have *outer positive d-density at $x_0 \in \partial\Omega$* if there exist $r_0, \theta > 0$ such that

$$|B_d(x_0,r) \setminus \bar{\Omega}| \geq \theta |B_d(x_0,r)|, \quad \text{for all } r \in (0,r_0). \tag{2.14}$$

Furthermore, if r_0 and θ can be chosen independently of x_0 then one says that Ω satisfies the outer positive d-density condition. The following lemma is a special case of [40, Theorem 4.1].

Lemma 2.5. *Assume that Ω satisfies the outer positive d-density condition. Given $f \in C(\partial_p\Omega_T)$ and $g \in \Gamma^\beta(\Omega_T)$ for some $0 < \beta \leq \sigma$, where σ is the Hölder exponent in* (1.4)*, there exists a unique solution $u \in \Gamma^{2+\beta}(\Omega_T) \cap C(\Omega_T \cup \partial_p\Omega_T)$ to the problem*

$$Hu = g \text{ in } \Omega_T, \quad u = f \text{ on } \partial_p\Omega_T.$$

In particular, Ω_T is H-regular for the Dirichlet problem (2.11).

2.5. NTA domains

In this section we recall the notion of NTA domain with respect to the control distance $d(x, y)$ induced by the system $X = \{X_1, \dots, X_m\}$. We recall that, when $d(x, y) = |x - y|$, the notion of NTA domain was introduced in [26] in connection with the study of the boundary behavior of nonnegative harmonic functions. The first study of NTA domains in a sub-Riemannian context was conducted in [7], where a large effort was devoted to the nontrivial question of the construction of examples. In that paper the relevant Fatou theory was also developed and, in particular, the doubling condition for harmonic measure, and the comparison theorem for quotients of nonnegative solutions of sub-Laplacians. Subsequently, in the papers [10, 11] the notion of NTA domain was combined with an intrinsic outer ball condition to obtain the complete solvability of the Dirichlet problem.

Given a bounded open set $\Omega \subset \mathbb{R}^n$, we recall that a ball $B_d(x, r)$ is M-nontangential in Ω (with respect to the metric d) if

$$M^{-1}r < d(B_d(x, r), \partial\Omega) < Mr.$$

Furthermore, given $x, y \in \Omega$ a sequence of M-non-tangential balls in Ω, $B_d(x_1, r_1), \dots, B_d(x_p, r_p)$, is called a Harnack chain of length p joining x to y if $x \in B_d(x_1, r_1)$, $y \in B_d(x_p, r_p)$, and $B_d(x_i, r_i) \cap B_d(x_{i+1}, r_{i+1}) \neq \varnothing$ for $i \in \{1, \dots, p-1\}$. We note that in this definition consecutive balls have comparable radii.

Definition 2.6. We say that a connected, bounded open set $\Omega \subset \mathbb{R}^n$ is a *non-tangentially accessible domain* with respect to the system $X = \{X_1, \dots, X_m\}$ (NTA domain, hereafter) if there exist $M, r_0 > 0$ for which:

(i) (Interior corkscrew condition) For any $x_0 \in \partial\Omega$ and $r \leq r_0$ there exists $A_r(x_0) \in \Omega$ such that $M^{-1}r < d(A_r(x_0), x_0) \leq r$ and $d(A_r(x_0), \partial\Omega) > M^{-1}r$. (This implies that $B_d(A_r(x_0), (2M)^{-1}r)$ is $(3M)$-nontangential.)

(ii) (Exterior corkscrew condition) $\Omega^c = \mathbb{R}^n \setminus \Omega$ satisfies property (i).

(iii) (Harnack chain condition) There exists $C(M) > 0$ such that for any $\epsilon > 0$ and $x, y \in \Omega$ such that $d(x, \partial\Omega) > \epsilon$, $d(y, \partial\Omega) > \epsilon$, and $d(x, y) < C\epsilon$, there exists a Harnack chain joining x to y whose length depends on C but not on ϵ.

We observe that the Chow-Rashevski accessibility theorem implies that the metric space $(\mathbb{R}^n, d)$ be locally compact, see [23]. Furthermore, for any bounded set $\Omega \subset \mathbb{R}^n$ there exists $R_0 = R_0(\Omega) > 0$ such that the closure of balls $B(x_0, R)$ with $x_0 \in \Omega$ and $0 < R < R_0$ are compact. We stress that metric balls of large radii fail to be compact in general, see [23]. In view of these observations, for a given NTA domain $\Omega \subset \mathbb{R}^n$ with constant M and r_0 we will always assume, following [7], that the constant r_0 has been adjusted in such a way that the closure of balls $B(x_0, R)$, with $x_0 \in \Omega$ and $0 < R < r_0$, be compact.

We note the following lemma which will prove useful in the sequel and which follows directly from Lemma 2.5 and Definition 2.6. In its statement the number σ denotes the Hölder exponent in (1.4).

Lemma 2.7. *Let $\Omega \subset \mathbb{R}^n$ be* NTA *domain, then there exist constants C, R_1, depending on the* NTA *parameters of Ω, such that for every $y \in \partial\Omega$ and every $0 < r < R_1$ one has,*

$$C|B_d(y, r)| \leq \min\{|\Omega \cap B_d(y, r)|, |\Omega^c \cap B_d(y, r)|\} \leq C^{-1}|B_d(y, r)|.$$

In particular, every NTA *domain has outer positive d-density and therefore, in view of Lemma* 2.5*, given $f \in C(\partial_p\Omega_T)$, there exists a unique solution $u \in \Gamma^{2+\sigma}(\Omega_T) \cap C(\Omega_T \cup \partial_p\Omega_T)$ to the Dirichlet problem* (2.11)*. In particular, Ω_T is H-regular.*

Assume that $\Omega \subset \mathbb{R}^n$ is a non-tangentially accessible domain with respect to the system $X = \{X_1, \dots, X_m\}$ and with parameters M, r_0. Let $T > 0$ and define $\Omega_T = \Omega \times (0, T)$. Based on Definition 2.6, for every $(x_0, t_0) \in S_T$, $0 < r < r_0$, we introduce the following points of reference whenever

$$\begin{aligned} A_r^+(x_0, t_0) &= (A_r(x_0), t_0 + 2r^2),\\ A_r^-(x_0, t_0) &= (A_r(x_0), t_0 - 2r^2),\\ A_r(x_0, t_0) &= (A_r(x_0), t_0). \end{aligned} \tag{2.15}$$

We note here that according to [30, Lemma 6.4], $\mathbb{R}^n \setminus B_d(x_0, R)$ satisfies condition (ii) in Definition 2.6, and thus it also satisfies the uniform outer positive d-density condition, and one can solve the Dirichlet problem there. Also note that the same is true of the intersection of two sets that satisfy condition (ii) in Definition 2.6. This is used to prove the following lemma ([30, Theorem 6.5]) which states that one can approximate any bounded open set with a set where one can solve the Dirichlet problem (2.11).

Lemma 2.8. *Let $D \subset \mathbb{R}^n$ be a bounded open set. Then, for every $\delta > 0$ there exists a set D_δ such that $\{x \in D : d(x, \partial D) > \delta\} \subset D_\delta \subset D$, and D_δ satisfies the uniform outer positive d-density condition.*

To apply the Harnack inequality to the equation (1.1) in a cylinder Ω_T, we will need to connect two points of Ω_T with a suitable Harnack chain of parabolic cylinders. We thus introduce the relevant geometric definition.

Definition 2.9. Let (y_1, s_1), $(y_2, s_2) \in \Omega_T$, with $s_2 > s_1$. Suppose that $(s_2 - s_1)^{1/2} \geq \eta^{-1} d(y_1, y_2)$ for some $\eta > 1$, and that $d(y_1, \partial\Omega) > \epsilon$, $d(y_2, \partial\Omega) > \epsilon$, $(T - s_2) > \epsilon^2$, $s_1 > \epsilon^2$ and $d_p((y_1, s_1), (y_2, s_2)) < c\epsilon$ for some $\epsilon > 0$. We say that $\{C_{\hat{r}_i,\hat{\rho}_i}(\hat{y}_i, \hat{s}_i)\}_{i=1}^{\ell}$ is a *parabolic Harnack chain of length ℓ connecting* (y_1, s_1) *to* (y_2, s_2), if $\hat{r}_i$, $\hat{\rho}_i$, $\hat{y}_i$, $\hat{s}_i$ satisfy the following:

(i) $c(\eta)^{-1} \leq \frac{\hat{\rho}_i}{\hat{r}_i} \leq c(\eta)$ for $i = 1, 2, \dots, \ell$,
(ii) $\hat{s}_{i+1} - \hat{s}_i \geq c(\eta)^{-1}\hat{r}_i^2$, for $i = 1, 2, \dots, \ell - 1$,
(iii) $B_d(\hat{y}_i, \hat{r}_i)$ is M-nontangential in Ω for $i = 1, 2, \dots, \ell$,
(iv) $(y_1, s_1) \in C_{\hat{r}_1,\hat{\rho}_1}(\hat{y}_1, \hat{s}_1)$, $(y_2, s_2) \in C_{\hat{r}_\ell,\hat{\rho}_\ell}(\hat{y}_\ell, \hat{s}_\ell)$,
(v) $C_{\hat{r}_{i+1},\hat{\rho}_{i+1}}(\hat{y}_{i+1}, \hat{s}_{i+1}) \cap C_{\hat{r}_i,\hat{\rho}_i}(\hat{y}_i, \hat{s}_i) \neq \varnothing$ for $i = 1, 2, \dots, \ell - 1$.

Lemma 2.10. *Let $\Omega \subset \mathbb{R}^n$ be a* NTA-*domain. Given $T > 0$ and (y_1, s_1), $(y_2, s_2) \in \Omega_T$, suppose that $s_2 > s_1$, $(s_2 - s_1)^{1/2} \geq \eta^{-1} d(y_1, y_2)$ for some $\eta > 1$, that $d(y_1, \partial\Omega) > \epsilon$, $d(y_2, \partial\Omega) > \epsilon$, $(T - s_2) > \epsilon^2$, $s_1 > \epsilon^2$ and that $d_p((y_1, s_1), (y_2, s_2)) < c\epsilon$ for some $\epsilon > 0$. Then, there exists a parabolic Harnack chain $\{C_{\hat{r}_i,\hat{\rho}_i}(\hat{y}_i, \hat{s}_i)\}_{i=1}^{\ell}$, connecting (y_1, s_1) to (y_2, s_2) in the sense of Definition* 2.9. *Furthermore, the length ℓ of the chain can be chosen to depend only on η and c, but not on ϵ.*

Proof. Since Ω is a NTA domain and since $d(y_1, \partial\Omega) > \epsilon$, $d(y_2, \partial\Omega) > \epsilon$,

$$d(y_1, y_2) \leq d_p((y_1, s_1), (y_2, s_2)) < c\epsilon,$$

it follows that we can use Definition 2.6 to conclude the existence of a Harnack chain of length $\hat{\ell} = \hat{\ell}(c)$, $\{B_d(\hat{y}_i, \hat{r}_i)\}_{i=1}^{\hat{\ell}}$, connecting y_1 and y_2. In the following we let β be a degree of freedom to be fixed below. Using β we define $\hat{\rho}_i = \beta \hat{r}_i$, we let $\hat{s}_i = s_1 + \frac{1}{\beta} \sum_{j=1}^{i} \hat{r}_j^2$ for $i \in \{1, .., \hat{\ell}\}$, and we consider the sequence of cylinders

$$\{C_{\hat{r}_i,\hat{\rho}_i}(\hat{y}_i, \hat{s}_i)\}_{i=1}^{\hat{\ell}}.$$

If we now choose $\beta > 1$, and if we assume that β is chosen as a function of η, then (i), (ii), (iii), (v) and the first part of (iv) in Definition 2.9 are satisfied. In particular, it only remains to ensure that the second part of (iv) in Definition 2.9 is satisfied. To do this we first note that we can assume, without loss of generality, that $\hat{r}_i \leq d(y_1, y_2)$ for all $i \in \{1, \ldots, \hat{\ell}\}$. Hence, $\sum_1^{\hat{l}} \hat{r}_i^2 \leq \hat{\ell} \cdot d(y_1, y_2)^2$. Furthermore, since $d(y_1, y_2)^2 \leq \eta^2 (s_2 - s_1)$, we have

$$\hat{s}_{\hat{\ell}} - s_1 = \frac{1}{\beta} \sum_{1}^{\hat{\ell}} \hat{r}_j^2 \leq \frac{\hat{\ell}}{\beta} d(y_1, y_2)^2 \leq \frac{\hat{\ell}}{\beta} \eta^2 (s_2 - s_1). \tag{2.16}$$

We now let $\beta = \hat{\ell} \cdot \eta^2$ and we can conclude that $\hat{s}_{\hat{\ell}} \leq s_2$. If $\hat{s}_{\hat{\ell}} = s_2$ we are done. Otherwise, we only step up in time with cylinders $C_j = \{C_{\hat{r}_{\hat{\ell}},\hat{r}_{\hat{\ell}}}(y_2, \hat{s}_{\hat{\ell}} + j\hat{r}_{\hat{\ell}})\}$ until we reach (y_2, s_2). The time that is left depends on η, and, in particular, we have that $s_2 - \hat{s}_{\hat{\ell}} \leq c^2 \epsilon^2$. Furthermore, since $\hat{r}_\ell \leq c\epsilon$, the number of steps we need to reach (y_2, s_2) only depends on c. In particular, it is clear that the length of the entire parabolic Harnack chain only depends on c and η. □

Lemma 2.11. *Let u be a nonnegative solution to the equation $Hu = 0$ in Ω_T. Furthermore, let (y_1, s_1), $(y_2, s_2) \in \Omega_T$, suppose that $s_2 > s_1$, $(s_2 - s_1)^{1/2} \geq \eta^{-1} d(y_1, y_2)$ for some $\eta > 1$, that $d(y_1, \partial\Omega) > \epsilon$, $d(y_2, \partial\Omega) > \epsilon$, $(T - s_2) > \epsilon^2$, $s_1 > \epsilon^2$ and that $d_p((y_1, s_1), (y_2, s_2)) < c\epsilon$ for some $\epsilon > 0$. Then, there exists a constant $\hat{c} = \hat{c}(H, \eta, c, r_0)$, $1 \leq \hat{c} < \infty$, such that*

$$u(y_1, s_1) \leq \hat{c} u(y_2, s_2).$$

Proof. To prove the lemma we simply use the parabolic Harnack chain from Lemma 2.10 and apply Theorem 2.2 in each cylinder. Note that the dependence of constant $\hat{c}$ on r_0 enters through the size parameter R in the statement of Theorem 2.2. □

3. Basic estimates

The purpose of this section is to establish a number of basic technical estimates that will be used in the proof of Theorems 1.1-1.3. We mention that, using the notion of NTA domain and Lemma 2.11, several of the proofs previously established in the literature in the classical case $m = n$ and $\{X_1, \dots, X_m\} = \{\partial_{x_1}, \dots, \partial_{x_n}\}$ can be extended to our setting. As a consequence, wherever appropriate, we will either omit details or be brief. As previously, unless otherwise stated, c will denote a positive constant ≥ 1, not necessarily the same at each occurrence, depending only on H and M. In general, $c(a_1, \dots, a_m)$ denotes a positive constant ≥ 1, which may depend only on H, M and $a_1, \dots, a_m$, and which is not necessarily the same at each occurrence. When we write $A \approx B$ we mean that A/B is bounded from above and below by constants which, unless otherwise stated, only depend on H, M.

Lemma 3.1. *Let* $(x_0, t_0) \in S_T$ *and*

$$r < \min\left\{r_0/2, \sqrt{(T - t_0)/4}, \sqrt{t_0/4}\right\}.$$

Let u be a nonnegative solution to $Hu = 0$ in $\Omega_T \cap C_{2r}(x_0, t_0)$ which vanishes continuously on $\Delta(x_0, t_0, 2r)$. Then, there exist $c = c(H, M, r_0)$, $1 \leq c < \infty$, and $\gamma = \gamma(H, M) > 0$, such that for every $(x, t) \in \Omega_T \cap C_r(x_0, t_0)$,

$$u(x, t) d_p(x, t, S_T)^\gamma \leq c r^\gamma u(A_r^+(x_0, t_0)).$$

Proof. The proof of this lemma is based on Lemma 2.11. In particular, let $P_0 = (x, t) \in \Omega_T \cap C_r(x_0, t_0)$ and let $a = d_p(P_0, S_T)$. Note that, without loss of generality, we can assume that $a < r/c_1$ for some large c_1 since otherwise we are done immediately by a simple application of Lemma 2.11. Now, take $Q_0 \in S_T$ such that $d_p(Q_0, P_0) = a$ and define $P_i = A^+_{2^i a}(Q_0)$ for all $i \geq 1$ such that $A^+_{2^i a}(Q_0)$ is well-defined. We intend to use Lemma 2.11 to prove that $u(P_i) \leq c u(P_{i+1})$ for some constant $c = c(H, M, r_0)$. In the following we write $P_i = (P_i^x, P_i^t)$, $Q_0 = (Q_0^x, Q_0^t)$ to indicate the spatial and time coordinate of P_i and Q_0 respectively. Then, for $i = 0$ we have

$$d(P_0^x, P_1^x) \leq d(P_0^x, Q_0^x) + d(Q_0^x, P_1^x) \leq a + 2a = \frac{3}{2\sqrt{2}}(P_0^t - P_1^t)^{1/2}.$$

Since $3/2\sqrt{2} > 1$, using Lemma 2.11 we can conclude that $u(P_0) \leq c u(P_1)$. To continue, for $i \geq 1$ we first note that

$$P_{i+1}^t - P_i^t = 2(2^{i+1}a)^2 - 2(2^i a)^2 = 3 \cdot 2^{2i+1} a^2.$$

Furthermore, we also have

$$d(P_{i+1}^x, P_i^x) \leq d(P_{i+1}^x, Q_0^x) + d(Q_0^x, P_i^x) \leq 2^{i+1}a + 2^i a = \sqrt{\frac{3}{2}}(P_{i+1}^t - P_i^t)^{1/2}.$$

Let $\epsilon = 2^i a/M$. Then $d(P_i^x, \partial\Omega) > \epsilon$, $d(P_{i+1}^x, \partial\Omega) > \epsilon$ and $d_p(P_{i+1}, P_i) = (3\cdot 2^{2i}a^2 + 3\cdot 2^{2i+1}a^2)^{1/2} = \sqrt{15}M\cdot\epsilon$. Since $\sqrt{\frac{3}{2}}$ and $\sqrt{15}M$ are both independent of i and since $\sqrt{\frac{3}{2}} > 1$, we can again conclude, using Lemma 2.11, that $u(P_i) \leq Cu(P_{i+1})$ for all $i > 0$ such that P_i and P_{i+1} lie in $\Omega_T \cap C_{2r}(x_0, t_0)$. In particular, to complete the proof it is now enough to consider the largest k such that $2^k a \leq r$ and then iterate the above inequalities in a standard fashion. We omit further details. □

Lemma 3.2. *Let* $(x_0, t_0) \in S_T$ *and*

$$r < \min\left\{r_0/2, \sqrt{(T-t_0)/4}, \sqrt{t_0/4}\right\}.$$

Let u be a nonnegative solution to $Hu = 0$ in $\Omega_T \cap C_{2r}(x_0, t_0)$ vanishing continuously on $\Delta(x_0, t_0, 2r)$. Then, there exist $c = c(H, M, r_0)$, $1 \leq c < \infty$, and $\gamma = \gamma(H, M, r_0) > 0$, such that

$$u(A_r^-(x_0, t_0)) \leq c\left(\frac{r}{d_p(x, t, \partial_p\Omega_T)}\right)^\gamma u(x, t),$$

whenever $(x, t) \in \Omega_T \cap C_r(x_0, t_0)$.

Proof. To prove this lemma one can proceed similarly to the proof of Lemma 3.1. □

Lemma 3.3. *There exists a $\hat{K} \gg 1$, $\hat{K} = \hat{K}(H, M)$, such that the following is true whenever $(x_0, t_0) \in \mathbb{R}^{n+1}$ and $r < r_0/(2\hat{K})$. Assume that D is a domain in $\mathbb{R}^n$ such that $D \subset B_d(x_0, \hat{K}r)$ and assume that there exist $\hat{x}_0 \in B_d(x_0, \hat{K}r)$ and $\rho > 0$ such that $B_d(\hat{x}_0, 2\rho) \subset B_d(x_0, r)$, $B_d(\hat{x}_0, 2\rho) \cap D = \varnothing$ and $M^{-1}r < \rho < r$. Let u be a function in $D \times (t_0 - 4r^2, t_0)$ which satisfies $Hu \geq 0$ in $D \times (t_0 - 4r^2, t_0)$, $u \leq 0$ on $\partial_p(D \times (t_0 - 4r^2, t_0)) \setminus \partial_p C^-_{\hat{K}r,2r}(x_0, t_0)$ and $\sup_{D\times(t_0-4r^2,t_0)} u > 0$. Then, there exists a constant $\theta = \theta(H, M, r_0)$, $0 < \theta < 1$, such that*

$$\sup_{(D\times(t_0-4r^2,t_0))\cap C_r^-(x_0,t_0)} u \leq \theta \sup_{D\times(t_0-4r^2,t_0)} u. \tag{3.1}$$

Proof. Let $\hat{K} \gg 1$ be a constant to be fixed below. We let $\phi_1 \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \leq \phi_1 \leq 1$, $\phi_1 \equiv 1$ on $B_d(x_0, \hat{K}r + r) \setminus B_d(x_0, \hat{K}r - r)$, $\phi_1 \equiv 0$ on $B_d(x_0, \hat{K}r - 2r) \cup (\mathbb{R}^n \setminus B_d(x_0, \hat{K}r + 2r))$. Similarly, we let $\phi_2 \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \leq \phi_2 \leq 1$, $\phi_2 \equiv 1$ on $B_d(x_0, \hat{K}r) \setminus B_d(\hat{x}_0, 2\rho)$, $\phi_2 \equiv 0$ on $B_d(\hat{x}_0, \rho) \cup (\mathbb{R}^n \setminus B_d(x_0, \hat{K}r + 2r))$. Using ϕ_1 and ϕ_2 we define

$$\Phi_1(\hat{x}, \hat{t}) = \int_{\mathbb{R}^n} \Gamma(\hat{x}, \hat{t}, \xi, t_0 - 4r^2)\phi_1(\xi)d\xi,$$
$$\Phi_2(\hat{x}, \hat{t}) = \int_{\mathbb{R}^n} \Gamma(\hat{x}, \hat{t}, \xi, t_0 - 4r^2)\phi_2(\xi)d\xi,$$

whenever $(\hat{x}, \hat{t}) \in \mathbb{R}^{n+1}$, $\hat{t} \geq t_0 - 4r^2$. To preceed we first prove that there exist a constant c such that

$$1 \leq c\Phi_1(\hat{x}, \hat{t}) \text{ for } (\hat{x}, \hat{t}) \in \partial_p(C^-_{\hat{K}r,2r}(x_0, t_0) \cap \{(x, t) : t_0 - 4r^2 < t < t_0\}). \quad (3.2)$$

To establish this, let $(\hat{x}, \hat{t})$ be as in (3.2), and for simplicity assume that $t_0 - 4r^2 = 0$. Then, using Lemma 2.1 and (2.2) we see that

$$\begin{aligned}
\Phi_1(\hat{x}, \hat{t}) &\geq \int_{B_d(\hat{x}, \sqrt{\hat{t}}/2)} \Gamma(\hat{x}, \hat{t}, \xi, 0)\phi_1(\xi) d\xi \\
&\geq \int_{B_d(\hat{x}, \sqrt{\hat{t}}/2)} c^{-1} \left|B(\hat{x}, \sqrt{\hat{t}})\right|^{-1} e^{-Cd(\hat{x},\xi)^2/\hat{t}} d\xi \\
&= e^{-C\hat{t}/4\hat{t}} \int_{B_d(\hat{x}, \sqrt{\hat{t}}/2)} c^{-1} \left|B(\hat{x}, \sqrt{\hat{t}})\right|^{-1} e^{-C\left(4d(\hat{x},\xi)^2 - \hat{t}\right)/4\hat{t}} d\xi \\
&\geq e^{-C/4} c^{-1} \left|B(\hat{x}, \sqrt{\hat{t}})\right|^{-1} \int_{B_d(\hat{x}, \sqrt{\hat{t}}/2)} d\xi \geq e^{-C/4} c^{-1} \hat{C}^{-1}.
\end{aligned}$$

We conclude that (3.2) holds provided that we choose $c \leq e^{-C/4}\hat{C}^{-1}$. Now, let

$$M = \sup_{D \times (t_0 - 4r^2, t_0)} u. \quad (3.3)$$

Using (3.2) and the maximum principle on $D \times (t_0 - 4r^2, t_0)$ we thus see that the estimate

$$u(\hat{x}, \hat{t}) \leq cM\Phi_1(\hat{x}, \hat{t}) + M\Phi_2(\hat{x}, \hat{t}) \quad (3.4)$$

holds in $D \times (t_0 - 4r^2, t_0)$, and thus in particular in $\big(D \times (t_0 - 4r^2, t_0)\big) \cap C^-_r(x_0, t_0)$. Further, if $(\hat{x}, \hat{t}) \in \big(D \times (t_0 - 4r^2, t_0)\big) \cap C^-_r(x_0, t_0)$, then

$$\begin{aligned}
\Phi_1(\hat{x}, \hat{t}) &\leq \int_{B_d(x_0, \hat{K}r + r) \setminus B_d(x_0, \hat{K}r - r)} |B(\hat{x}, \sqrt{\hat{t}})|^{-1} e^{-C^{-1}d(\hat{x},\xi)^2/\hat{t}} d\xi \\
&\leq \int_{B_d(x_0, \hat{K}r + r) \setminus B_d(x_0, \hat{K}r - r)} |B(\hat{x}, r)|^{-1} e^{-c^{-1}d(\hat{x},\xi)^2/r^2} d\xi \\
&\leq ce^{-c^{-1}\hat{K}^2} |B_d(x_0, \hat{K}r + r) \setminus B_d(x_0, \hat{K}r - r)||B(\hat{x}, r)|^{-1} \\
&\leq ce^{-c^{-1}\hat{K}^2} |B_d(x_0, \hat{K}r + r)||B(\hat{x}, r)|^{-1}.
\end{aligned}$$

Iterating (2.2) and using that $r < r_0/(2\hat{K})$ we see that

$$ce^{-c^{-1}\hat{K}^2} |B_d(x_0, \hat{K}r + r)||B(\hat{x}, r)|^{-1} \leq ce^{-c^{-1}\hat{K}^2} \hat{K}^\eta$$

for some integer $\eta \gg 1$ which is independent of $\hat{K}$, x_0, $\hat{x}$ and r. In particular

$$\Phi_1(\hat{x}, \hat{t}) \leq ce^{-c^{-1}\hat{K}^2} \hat{K}^\eta.$$

To estimate $\Phi_2(\hat{x}, \hat{t})$ we note that

$$\Phi_2(\hat{x}, \hat{t}) = 1 - \hat{\Phi}_2(\hat{x}, \hat{t}), \text{ where}$$
$$\hat{\Phi}_2(\hat{x}, \hat{t}) = \int_{\mathbb{R}^n} \Gamma(\hat{x}, \hat{t}, \xi, t_0 - 4r^2)(1 - \phi_2(\xi))d\xi,$$

and by construction,

$$\hat{\Phi}_2(\hat{x}, \hat{t}) \geq \int_{B_d(\hat{x},\rho)} \Gamma(\hat{x}, \hat{t}, \xi, t_0 - 4r^2)d\xi.$$

As before we then prove that

$$\hat{\Phi}_2(\hat{x}, \hat{t}) \geq c^{-1},$$

and actually, for ε small enough,

$$\hat{\Phi}_2(\hat{x}_0, t_0 - 4r^2 + \varepsilon^2\rho^2) \geq c^{-1}.$$

Hence, by using the Harnack inequality we can conclude that $\Phi_2(\hat{x}, \hat{t}) = 1 - \hat{\Phi}_2(\hat{x}, \hat{t}) \leq \left(1 - c^{-1}\right)$, whenever $(\hat{x}, \hat{t}) \in \left(D \times (t_0 - 4r^2, t_0)\right) \cap C_r^-(x_0, t_0)$ for some $c = c(H, M, r_0) > 1$. In particular, for every $(\hat{x}, \hat{t}) \in \left(D \times (t_0 - 4r^2, t_0)\right) \cap C_r^-(x_0, t_0)$, we have

$$u(\hat{x}, \hat{t}) \leq cM\Phi_1(\hat{x}, \hat{t}) + M\Phi_2(\hat{x}, \hat{t}) \leq M\left(ce^{-c^{-1}\hat{K}^2}\hat{K}^\eta + \left(1 - \hat{c}^{-1}\right)\right),$$

for some $\hat{c} = \hat{c}(H, M, r_0)$. Given $\hat{c}$, we choose $\hat{K}$ so that $ce^{-c^{-1}\hat{K}^2}\hat{K}^\eta \leq \hat{c}^{-1}/2$, and we let $\theta = \left(1 - \hat{c}^{-1}/2\right) < 1$. Then, the following inequality holds

$$u(\hat{x}, \hat{t}) \leq \theta M, \tag{3.5}$$

with M as in (3.3). This establishes (3.1), thus completing the proof. □

We will also need a few variations on the theme of Lemma 3.3.

Corollary 3.4. *There exists a $\hat{K} \gg 1$, $\hat{K} = \hat{K}(H, M, r_0)$, such that the following is true whenever $(x_0, t_0) \in S_T$ and*

$$r < \min\{r_0/(2\hat{K}), \sqrt{(T - t_0)/4}, \sqrt{t_0/4}\}.$$

Let u be a nonnegative solution to $Hu = 0$ in $\Omega_T \cap C^-_{\hat{K}r,2r}(x_0, t_0)$ vanishing continuously on $S_T \cap C^-_{\hat{K}r,2r}(x_0, t_0)$. Then, there exists a constant $\theta = \theta(H, M)$, $0 < \theta < 1$, such that

$$\sup_{\Omega_T \cap C_r^-(x_0,t_0)} u \leq \theta \sup_{\Omega_T \cap C^-_{\hat{K}r,2r}(x_0,t_0)} u.$$

Proof. This is an obvious consequence of the NTA character of Ω and of Lemma 3.3. We omit further details. □

Lemma 3.5. *There exists a* $\hat K \gg 1$, $\hat K = \hat K(H, M, r_0)$, *such that the following is true whenever* $(x_0, t_0) \in S_T$ *and*

$$r < \min\left\{r_0/(2\hat K), \sqrt{(T-t_0)/(4\hat K)^2}, \sqrt{t_0/(4\hat K)^2}\right\}.$$

Let u *be a solution to* $Hu = 0$ *in* $\Omega_T \cap C^-_{\hat K r, 2r}(x_0, t_0)$ *which vanishes continuously on* $S_T \cap C^-_{\hat K r, 2r}(x_0, t_0)$. *Then, there exists a constant* $\theta = \theta(H, M)$, $0 < \theta < 1$, *such that*

$$\sup_{\Omega_T \cap C^-_r(x_0,t_0)} u^{\pm} \le \theta \sup_{\Omega_T \cap C^-_{\hat K r, 2r}(x_0,t_0)} u^{\pm},$$

where $u^+(x, t) = \max\{0, u(x, t)\}$, $u^-(x, t) = -\min\{0, u(x, t)\}$.

Proof. We first prove Lemma 3.5 for u^+. In fact, in this case the argument is essentially the same as that in the proof of Lemma 3.3. In particular, if we let

$$M^+ = \sup_{\Omega_T \cap C^-_{\hat K r, 2r}(x_0,t_0)} u^+,$$

then we see that (3.4) still holds but with M replaced by M^+. Furthermore, repeating the argument in (3.3)-(3.5), we see that

$$u(\hat x, \hat t) \le \theta M^+,$$

whenever $(\hat x, \hat t) \in \Omega_T \cap C^-_r(x_0, t_0)$. Obviously this completes the proof of Lemma 3.5 for u^+. Concerning the same estimate for u^- we see, by analogy, that

$$-u(\hat x, \hat t) \le \theta M^-, \; M^- = \sup_{\Omega_T \cap C^-_{\hat K r, 2r}(x_0,t_0)} (-u) = \sup_{\Omega_T \cap C^-_{\hat K r, 2r}(x_0,t_0)} u^-, \tag{3.6}$$

whenever $(\hat x, \hat t) \in \Omega_T \cap C^-_r(x_0, t_0)$ and from (3.6) we deduce Lemma 3.5 for u^-. This completes the proof of the lemma. □

Lemma 3.6. *Let* $(x_0, t_0) \in S_T$ *and let* $r < \min\{r_0/2, \sqrt{(T-t_0)/4}, \sqrt{t_0/4}\}$. *Let* u *be a nonnegative solution to* $Hu = 0$ *in* $\Omega_T \cap C_{2r}(x_0, t_0)$ *which vanishes continuously on* $\Delta(x_0, t_0, 2r)$. *Then, there exist a constant* $c = c(H, M, r_0)$, $1 \le c < \infty$, *and* $\alpha = \alpha(H, M) \in (0, 1)$, *such that*

$$u(x, t) \le c\left(\frac{d_p(x, t, x_0, t_0)}{r}\right)^{\alpha} \sup_{\Omega_T \cap C_{2r}(x_0,t_0)} u \tag{3.7}$$

whenever $(x, t) \in \Omega_T \cap C_{r/c}(x_0, t_0)$.

Proof. This lemma is a simple consequence of Corollary 3.4. □

Lemma 3.7. *Let $(x_0, t_0) \in S_T$ and let $r < \min\{r_0/2, \sqrt{(T-t_0)/4}, \sqrt{t_0/4}\}$. Let u be a nonnegative solution to $Hu = 0$ in $\Omega_T \cap C_{2r}(x_0, t_0)$ vanishing continuously on $\Delta(x_0, t_0, 2r)$. Then, there exists a constant $c = c(H, M, r_0)$, $1 \le c < \infty$, such that*

$$u(x,t) \le cu(A_r^+(x_0, t_0))$$

whenever $(x,t) \in \Omega_T \cap C_{r/c}(x_0, t_0)$.

Proof. This lemma is a consequence of Lemma 3.6, the Harnack inequality and a classical argument developed in [6] and [38]. □

Remark 3.8. Note that if u is a nonnegative solution to $Hu = 0$ in all of Ω_T then Lemma 3.7 can be improved in the following way. Let $(x_0, t_0) \in S_T$ and r be as in the statement of Lemma 3.7. Let u be a nonnegative solution to $Hu = 0$ in Ω_T vanishing continuously on $\Delta(x_0, t_0, 2r)$. Then, there exists a constant $c = c(H, M, r_0)$, $1 \le c < \infty$, such that

$$u(x,t) \le cu(A_r^+(x_0, t_0))$$

whenever $(x,t) \in \Omega_T \cap C_r(x_0, t_0)$. In fact, the restriction $(x,t) \in \Omega_T \cap C_{r/c}(x_0, t_0)$ in Lemma 3.7 is simply a result of the fact that we in Lemma 3.7 are only assuming that u is a nonnegative solution in $\Omega_T \cap C_{2r}(x_0, t_0)$.

Lemma 3.9. *Let u be a nonnegative solution to $Hu = 0$ in Ω_T which vanishes continuously on S_T. Let $0 < \delta \ll \sqrt{T}$ be given. Then, there exists a constant $c = c(H, M, \mathrm{diam}(\Omega), T, \delta, r_0)$, $1 \le c < \infty$, such that*

$$\sup_{(x,t)\in\Omega^\delta \times (\delta^2, T)} u(x,t) \le c \inf_{(x,t)\in\Omega^\delta \times (\delta^2, T)} u(x,t).$$

Proof. To prove this we can proceed, using the lemmas given above, exactly as in the proof of [34, Lemma 2.7]. □

Lemma 3.10. *Let $K \gg 1$ be given, let $(x_0, t_0) \in S_T$ and assume that $r < \min\{r_0/(8K), \sqrt{(T-t_0)/64}, \sqrt{t_0/64}\}$. Let u be a nonnegative solution to the equation $Hu = 0$ in Ω_T vanishing continuously on S_T. Let $\gamma = \gamma(H, M) \in (0,1)$ be as in Lemma 3.1 and Lemma 3.2. Assume that*

$$\sup_{\Omega_T \cap C^-_{2Kr,2r}(x_0,t_0)} u \ge (2K)^{-\gamma} \sup_{\Omega_T \cap C^-_{4Kr,8r}(x_0,t_0)} u.$$

Then, provided $K = K(H, M)$ is chosen large enough, there exists $c = c(H, M, r_0) \ge 1$, such that

$$\sup_{\Omega_T \cap C^-_{4Kr}(x_0,t_0) \cap \{(x,t): t = t_0 - 64r^2\}} u \ge c^{-1} \sup_{\Omega_T \cap C^-_{2Kr,2r}(x_0,t_0)} u.$$

Proof. The proof of this lemma is similar to that of Lemma 3.4. In particular, we let $\phi_1 \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \le \phi_1 \le 1$, $\phi_1 \equiv 1$ on $B_d(x_0, 4Kr+2r)\setminus B_d(x_0, 4Kr-2r)$, $\phi_1 \equiv 0$ on $B_d(x_0, 4Kr-4r)\cup(\mathbb{R}^n\setminus B_d(x_0, 4Kr+4r))$. Since Ω is NTA we see that there exist $\hat{x}_0$ and $\rho > 0$ such that $r/M < 4\rho < r$ and such that $B(\hat{x}_0, 2\rho) \subset (\mathbb{R}^n \setminus \Omega) \cap B(x_0, r)$. Based on this we let $\phi_2 \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \le \phi_2 \le 1$, $\phi_2 \equiv 1$ on $B_d(x_0, 4Kr)\setminus B(\hat{x}_0, 2\rho)$, $\phi_2 \equiv 0$ on $(\mathbb{R}^n\setminus B_d(x_0, 4Kr+4r))\cup B(\hat{x}_0, \rho)$. Using ϕ_1 and ϕ_2 we define

$$\begin{aligned}\Phi_1(\hat{x}, \hat{t}) &= \int_{\mathbb{R}^n} \Gamma(\hat{x}, \hat{t}, \xi, t_0 - 64r^2)\phi_1(\xi)d\xi,\\ \Phi_2(\hat{x}, \hat{t}) &= \int_{\mathbb{R}^n} \Gamma(\hat{x}, \hat{t}, \xi, t_0 - 64r^2)\phi_2(\xi)d\xi,\end{aligned}$$

whenever $(\hat{x}, \hat{t}) \in \mathbb{R}^{n+1}$, $\hat{t} \ge t_0 - 64r^2$. Let $\Gamma_1 = \Omega_T \cap C^-_{4Kr}(x_0, t_0)\cap\{(x, t) : t = t_0 - 64r^2\}$, $\Gamma_2 = \partial_p(\Omega_T \cap C^-_{4Kr,8r}(x_0, t_0)) \setminus \Gamma_1 \setminus S_T$. In the following we let

$$M = \sup_{\Omega_T\cap C^-_{4Kr,8r}(x_0,t_0)} u,\ \ \hat{M} = \sup_{\Omega_T\cap C^-_{4Kr}(x_0,t_0)\cap\{(x,t):t=t_0-64r^2\}} u.$$

Then, by arguing as in the proof of Lemma 3.4, we first see that there exists c such that

$$1 \le c\Phi_1(\hat{x}, \hat{t}) \text{ for } (\hat{x}, \hat{t}) \in \partial_p(C^-_{4Kr,8r}(x_0, t_0)) \cap \{(x, t) : t_0 - 64r^2 < t < t_0\},$$

and then, by the maximum principle we see, that

$$u(\hat{x}, \hat{t}) \le cM\Phi_1(\hat{x}, \hat{t}) + \hat{M}\Phi_2(\hat{x}, \hat{t})$$

for $(\hat{x}, \hat{t}) \in \Omega_T \cap C^-_{2Kr,2r}(x_0, t_0)$. As in the proof of Lemma 3.4 we can then deduce that

$$u(\hat{x}, \hat{t}) \le cMe^{-c^{-1}\,K^2}K^\eta + \hat{M}\Phi_2(\hat{x}, \hat{t}),$$

for $(\hat{x}, \hat{t}) \in \Omega_T \cap C^-_{2Kr,2r}(x_0, t_0)$. Next, using the assumption stated in the lemma we see that

$$(2K)^{-\gamma}M \le cMe^{-c^{-1}\,K^2}K^\eta + \hat{M} \sup_{\Omega_T\cap C^-_{2Kr,2r}(x_0,t_0)} \Phi_2(\hat{x}, \hat{t}).$$

Hence, assuming that K is so large that $(2K)^{-\gamma} > ce^{-c^{-1}\,K^2}K^\eta$, we have that

$$((2K)^{-\gamma} - e^{-c^{-1}\,K^2}K^\eta)M \le \hat{M} \sup_{\Omega_T\cap C^-_{2Kr,2r}(x_0,t_0)} \Phi_2(\hat{x}, \hat{t}) \le \hat{M}.$$

In particular, we can conclude, for $K = K(H, M)$ large enough, that

$$\frac{1}{2}(2K)^{-\gamma} \sup_{\Omega_T\cap C^-_{2Kr,2r}(x_0,t_0)} u \le \frac{1}{2}(2K)^{-\gamma}M \le \hat{M}.$$

This completes the proof. □

Lemma 3.11. *Let $\hat{K}$ be as in the statement of Lemma 3.3, let $K \gg \hat{K}$ be a constant to be suitably chosen, $(x_0, t_0) \in S_T$ and assume*

$$r < \min\Big\{r_0/(2K\hat{K}), \sqrt{(T-t_0)/(4K^2)}, \sqrt{t_0/(4K^2)}\Big\}.$$

Let u be a solution to $Hu = 0$ in $(\Omega_T \setminus \Omega_T^r) \cap C_{Kr}^-(x_0, t_0)$ which is continuous on the closure of $(\Omega_T \setminus \Omega_T^r) \cap C_{Kr}^-(x_0, t_0)$. Moreover, assume that

(i) $u(x,t) \le 1$ *whenever* $(x,t) \in (\Omega_T \setminus \Omega_T^r) \cap C_{Kr}^-(x_0, t_0)$,

(ii) $u(x,t) \le 0$ *whenever* $(x,t) \in [(\partial\Omega \cup \partial\Omega^r) \times (t_0 - (Kr)^2, t_0)] \cap C_{Kr}^-(x_0, t_0)$.

Then, there exists a constant $c = c(H, M, r_0)$, $1 \le c < \infty$, such that

$$u(x,t) \le e^{-cK}$$

whenever $(x,t) \in (\Omega_T \setminus \Omega_T^r) \cap C_{\hat{K}r}^-(x_0, t_0)$.

Proof. In the following we consider odd integers $2j+1$ where $j \in [0, (K/\hat{K} - 1)/2]$. For each such j we define a point $(\hat{X}_j, \hat{t}_j) \in (\Omega_T \setminus \Omega_T^r) \cap C_{(2j+1)\hat{K}r}^-(x_0, t_0)$ through the relation

$$\sup_{(\Omega_T \setminus \Omega_T^r) \cap C_{(2j+1)\hat{K}r}^-(x_0,t_0)} u = u(\hat{X}_j, \hat{t}_j). \tag{3.8}$$

We then note, using the maximum principle, that $(\hat{X}_j, \hat{t}_j) \in \partial_p[(\Omega_T \setminus \Omega_T^r) \cap C_{(2j+1)\hat{K}r}^-(x_0, t_0)]$. By construction we also see that there exists $(\tilde{X}_j, \hat{t}_j) \in \partial\Omega \times [t_0 - ((2j+1)\hat{K}r)^2, t_0)$ such that $d_p(\tilde{X}_j, \hat{t}_j, \hat{X}_j, \hat{t}_j) = d(\tilde{X}_j, \hat{X}_j) \le r$. In particular, $(\hat{X}_j, \hat{t}_j)$ is in the closure of $C_r^-(\tilde{X}_j, \hat{t}_j)$. We next note that

$$C_{\hat{K}r,2r}^-(\tilde{X}_j, \hat{t}_j) \cap [(\Omega_T \setminus \Omega_T^r) \cap C_{Kr}^-(x_0, t_0)] \subset (\Omega_T \setminus \Omega_T^r) \cap C_{(2j+3)\hat{K}r}^-(x_0, t_0). \tag{3.9}$$

Let D be defined through the relation $D \times (\hat{t}_j - 4r^2, \hat{t}_j) = C_{\hat{K}r,2r}^-(\tilde{X}_j, \hat{t}_j) \cap [(\Omega_T \setminus \Omega_T^r) \cap C_{Kr}^-(x_0, t_0)]$. Then, applying Lemma 3.3 we see that there exists $\theta = \theta(H, M)$, $0 < \theta < 1$, such that

$$\sup_{(D \times (\hat{t}_j - 4r^2, \hat{t}_j)) \cap C_r^-(\tilde{X}_j, \hat{t}_j)} u \le \theta \sup_{D \times (\hat{t}_j - 4r^2, \hat{t}_j)} u. \tag{3.10}$$

In particular, since $(\hat{X}_j, \hat{t}_j)$ is in the closure of the set $(D \times (\hat{t}_j - 4r^2, \hat{t}_j)) \cap C_r^-(\tilde{X}_j, \hat{t}_j)$ we can use continuity of u, (3.10) and (3.9) to conclude that

$$\begin{aligned} u(\hat{X}_j, \hat{t}_j) &\le \theta \sup_{D \times (\hat{t}_j - 4r^2, \hat{t}_j)} u \\ &\le \theta \sup_{(\Omega_T \setminus \Omega_T^r) \cap C_{(2j+3)\hat{K}r}^-(x_0,t_0)} u = \theta u(\hat{X}_{j+1}, \hat{t}_{j+1}). \end{aligned} \tag{3.11}$$

Let j_0 be the largest positive integer such that $(2j_0+3)\hat{K} \le K$. Then, by iteration we see that

$$\sup_{(\Omega_T\setminus\Omega_T^r)\cap C^-_{\hat{K}r}(x_0,t_0)} u = u(\hat{X}_1,\hat{t}_1) \le \theta^{j_0} u(\hat{X}_{j_0+1},\hat{t}_{j_0+1}) \le \theta^{j_0} \tag{3.12}$$

where we, at the last step, has used that $u(x,t)\le 1$. Hence

$$\sup_{(\Omega_T\setminus\Omega_T^r)\cap C^-_{\hat{K}r}(x_0,t_0)} u \le \theta^{j_0}. \tag{3.13}$$

Obviously (3.13) implies the statement in Lemma 3.11 and the proof is complete. □

Lemma 3.12. *Let $\hat{K}$ be as in the statement of Lemma 3.3, let $K \gg \hat{K}$ be given, $(x_0,t_0)\in S_T$ and assume that*

$$r < \min\left\{r_0/(2K), \sqrt{(T-t_0)/(4K^2)}, \sqrt{t_0/(4K^2)}\right\}.$$

Let u and v be two solutions to $Hu=0$ in $(\Omega_T\setminus\Omega_T^r)\cap C^-_{Kr}(x_0,t_0)$. Moreover, assume that

(i) $u(x,t)\ge 0,\ v(x,t)\le 1$ *whenever* $(x,t)\in(\Omega_T\setminus\Omega_T^r)\cap C^-_{Kr}(x_0,t_0)$,

(ii) $u(x,t)\ge 1$ *whenever* $(x,t)\in[\partial\Omega^r\times(t_0-(Kr)^2,t_0)]\cap C^-_{Kr}(x_0,t_0)$,

(iii) $v(x,t)\le 0$ *whenever* $(x,t)\in[(\partial\Omega\cup\partial\Omega^r)\times(t_0-(Kr)^2,t_0)]\cap C^-_{Kr}(x_0,t_0)$.

Then, for any $(x,t)\in\Omega_T\cap C^-_r(x_0,t_0)$ one has

$$v(x,t)\le u(x,t),$$

provided $K=K(H,M)$ is chosen large enough.

Proof. To start the proof of Lemma 3.12 we claim that if u as in the statement of the lemma, then

$$u(x,t)\ge 2\epsilon\left(\frac{d_p(x,t,S_T)}{r}\right)^\eta \text{ whenever } (x,t)\in\Omega_T\cap C^-_r(x_0,t_0), \tag{3.14}$$

where ϵ and η are positive constants depending only on H, M. However, we postpone the proof of this claim until the end. We thus establish the lemma assuming (3.14). To do this we first note that (3.14) implies that

$$u(x,t)\ge 2\epsilon K^{-\eta} \text{ whenever } (x,t)\in\Omega_T^{r/K}\cap C^-_r(x_0,t_0). \tag{3.15}$$

Furthermore, since v satisfies the assumptions stated in Lemma 3.11, from this result we see that

$$v(x,t) \le e^{-cK} \le \epsilon K^{-\eta} \text{ whenever } (x,t) \in \Omega_T \cap C_r^-(x_0,t_0), \tag{3.16}$$

provided $K = K(H, M)$ is large enough. In particular,

$$v(x,t) \le \epsilon K^{-\eta} \le u(x,t) \text{ whenever } (x,t) \in \Omega_T^{r/K} \cap C_r^-(x_0,t_0).$$

We now define for $(x,t) \in (\Omega_T \setminus \Omega_T^{r/K}) \cap C_r^-(x_0,t_0)$,

$$u_1(x,t) = \frac{K^\eta}{2\epsilon} u(x,t),\ \ v_1(x,t) = \frac{K^\eta}{2\epsilon}(2v(x,t) - u(x,t)).$$

Then, using (3.15), (3.16), we see that

(i$_1$) $u_1(x,t) \ge 0,\ v_1(x,t) \le 1$ whenever $(x,t) \in (\Omega_T \setminus \Omega_T^{r/K}) \cap C_r^-(x_0,t_0)$,

(ii$_1$) $u_1(x,t) \ge 1$ whenever $(x,t) \in [\partial\Omega^{r/K} \times (t_0 - (r)^2, t_0)] \cap C_r^-(x_0,t_0)$,

(iii$_1$) $v_1(x,t) \le 0$ whenever $(x,t) \in [(\partial\Omega \cup \partial\Omega^{r/K}) \times (t_0 - r^2, t_0)] \cap C_r^-(x_0,t_0)$.

Moreover, u_1, v_1 are solutions to $Hu = 0$ in $(\Omega_T \setminus \Omega_T^{r/K}) \cap C_r^-(x_0,t_0)$. In particular, the pair (u_1, v_1) satisfies the assumptions stated in Lemma 3.12 with r replaced by r/K. Furthermore, by construction we have that

$$u(x,t) - v(x,t) = \frac{\epsilon}{K^\eta}(u_1(x,t) - v_1(x,t)) \ge 0,$$

whenever $(x,t) \in \Omega_T^{r/K^2} \cap C_{r/K}^-(x_0,t_0)$. Hence, by iteration of this argument we see that we can construct functions u_j and v_j, for $j = 1, 2, \ldots,$ such that

$$u(x,t) - v(x,t) = \left(\frac{\epsilon}{K^\eta}\right)^j (u_j(x,t) - v_j(x,t)) \ge 0$$

whenever $(x,t) \in (\Omega_T \setminus \Omega_T^{r/K^{j+1}}) \cap C_{r/K^j}^-(x_0,t_0)$. As a consequence we obtain that

$$u(x,t) - v(x,t) \ge 0 \text{ whenever } (x,t) \in I(x_0,t_0),$$

where $I(x_0,t_0) = \bigcup_{j=1}^\infty \Omega_T^{r/K^j} \cap C_{r/K^{j-1}}^-(x_0,t_0)$. Finally, for arbitrary $(\hat{x}_0, \hat{t}_0) \in \Omega_T \cap C_r^-(x_0,t_0)$ one can choose $(\tilde{x}_0, \tilde{t}_0) \in S_T$ such that $d_p(\hat{x}_0, \hat{t}_0, S_T) = d_p(\hat{x}_0, \hat{t}_0, \tilde{x}_0, \tilde{t}_0) = d(\hat{x}_0, \tilde{x}_0)$. Then $(\hat{x}_0, \hat{t}_0) \in I(\tilde{x}_0, \tilde{t}_0)$ and $d(x_0, \tilde{x}_0) < r$, *i.e.*,

$$(\Omega_T \setminus \Omega_T^r) \cap C_{Kr}^-(\tilde{x}_0, \tilde{t}_0) \subset (\Omega_T \setminus \Omega_T^r) \cap C_{(K+2)r}^-(x_0,t_0),$$
$$\partial_p\big((\Omega_T \setminus \Omega_T^r) \cap C_{Kr}^-(\tilde{x}_0, \tilde{t}_0)\big) \cap \partial_p\Omega_T^r \subset \partial_p\big((\Omega_T \setminus \Omega_T^r) \cap C_{(K+2)r}^-(x_0,t_0)\big) \cap \partial_p\Omega_T^r.$$

Hence, by replacing K with $K+2$ in the original assumptions and repeating the proof up to here with $(\tilde{x}_0, \tilde{t}_0)$ instead of (x_0, t_0), we can conclude that $u(x,t) - v(x,t) \geq 0$ on $I(\tilde{x}_0, \tilde{t}_0)$ and in particular, $u(\hat{x}_0, \hat{t}_0) - v(\hat{x}_0, \hat{t}_0) \geq 0$. Since $(\hat{x}_0, \hat{t}_0) \in \Omega_T \cap C_r^-(x_0, t_0)$ is arbitrary we can hence conclude that $u - v \geq 0$ on $\Omega_T \cap C_r^-(x_0, t_0)$. In particular, to complete the proof of Lemma 3.12 we are only left with proving the claim in (3.14).

To do this we proceed as follows. Let $\hat{K}$ be as in the statement of Lemma 3.3 and let $\Lambda \gg 1$ be given. Assume that $K \gg \Lambda \hat{K}$. Given $x_0 \in \partial\Omega$, according to Lemma 2.8 we can find a set U such that

$$B_d(x_0, \Lambda \hat{K} r) \subset U \subset B_d(x_0, (\Lambda + 1)\hat{K} r),$$

and such that we can solve the Dirichlet problem (2.11) in

$$\Omega_T \cap [U \times (t_0 - 16t^2, t_0)].$$

Furthermore, we choose $\tilde{x}_0 \in \Omega$ and Λ so that $\tilde{x}_0 \in \partial B_d(x_0, \Lambda \hat{K} r)$ and $B_d(\tilde{x}_0, 2\hat{K} r) \subset \Omega$. We note that, since Ω is an NTA domain, this can always be accomplished by choosing Λ large enough. We next introduce an auxiliary function $\tilde{u}$ as follows. We let $\tilde{u}$ be such that $H\tilde{u} = 0$ in $\Omega_T \cap [U \times (t_0 - 16t^2, t_0)]$, $\tilde{u} = 1$ on $\partial_p(\Omega_T \cap [U \times (t_0 - 16t^2, t_0)]) \cap C^-_{\hat{K}r,2r}(\tilde{x}_0, t_0 - 4r^2)$ and $\tilde{u} = 0$ on the rest of $\partial_p(\Omega_T \cap [U \times (t_0 - 16t^2, t_0)])$. We then have $0 \leq \tilde{u} \leq 1$, and $\tilde{u} \leq u$ where u and $\tilde{u}$ are both defined. Also, $\tilde{u}$ is not identical to 1 in $\Omega_T \cap [U \times (t_0 - 16t^2, t_0)]$.

Let $D = U \cap B_d(\tilde{x}_0, \hat{K} r)$ and define $\hat{u} = 1 - \tilde{u}$ in $D \times (t_0 - 8r^2, t_0 - 4r^2)$. Then $\hat{u}$ satisfies $H\hat{u} = 0$ in $D \times (t_0 - 8r^2, t_0 - 4r^2)$, $\hat{u} \leq 0$ on $\partial_p(D \times (t_0 - 8r^2, t_0 - 4r^2)) \setminus \partial_p C^-_{\hat{K}r,2r}(\tilde{x}_0, t_0 - 4r^2)$ and $\sup_{D \times (t_0 - 8r^2, t_0 - 4r^2)} \hat{u} > 0$. Because of the construction of U, there exists $\hat{x}_0 \in B_d(\tilde{x}_0, \hat{K} r)$ and ρ such that $B_d(\hat{x}_0, \rho) \subset B_d(\tilde{x}_0, r)$, $B_d(\hat{x}_0, \rho) \cap D = \varnothing$ and $\hat{M}^{-1} r < \rho < r$ for some $\hat{M}$ independent of r. We can now apply Lemma 3.3 to conclude that there exists a constant θ, $0 < \theta < 1$, independent of r, such that

$$\sup_{(D \times (t_0 - 8r^2, t_0 - 4r^2)) \cap C_r^-(\tilde{x}_0, t_0 - 4r^2)} \hat{u} \leq \theta \sup_{D \times (t_0 - 8r^2, t_0 - 4r^2)} \hat{u} \leq \theta. \tag{3.17}$$

In particular, by continuity we see from (3.17) that

$$u(\tilde{x}_0, t_0 - 4r^2) \geq \tilde{u}(\tilde{x}_0, t_0 - 4r^2) \geq 1 - \theta > 0. \tag{3.18}$$

Furthermore, using (3.18), the Harnack inequality and Lemma 3.2 we see that

$$1 - \theta \leq c u(A_r^-(x_0, t_0)) \leq c^2 r^\gamma u(x,t) d_p(x,t,S_T)^{-\gamma} \tag{3.19}$$

whenever $(x,t) \in \Omega_T \cap C_r^-(x_0, t_0)$. Obviously this gives (3.14) with $\eta = \gamma$ and $2\epsilon = (1-\theta)/c^2$. This completes the proof. □

4. Proof of Theorem 1.1 and Theorem 1.2

The purpose of this section is proving Theorems 1.1 and 1.2.

4.1. Proof of Theorem 1.1

To begin the proof we let $0<\delta\ll\sqrt{T}$ be a fixed constant, we let $(x_0,t_0)\in S_T$, $\delta^2\leq t_0\leq T-\delta^2$, and we assume that $r<\min\{r_0/2,\sqrt{(T-t_0-\delta^2)/4},\sqrt{(t_0-\delta^2)/4}\}$. For $\hat{r}>0$ we define

$$f(\hat{r})=\hat{r}^{-\gamma}\sup_{\Omega_T\cap C^-_{2\hat{r}}(x_0,t_0)}u(x,t) \tag{4.1}$$

where γ is the constant appearing in Lemma 3.1. Furthermore, we let

$$\rho=\max\{\hat{r}:\ r\leq\hat{r}\leq\delta,\ f(\hat{r})\geq f(r)\}. \tag{4.2}$$

By the definition of ρ in (4.2) we see that

$$\sup_{\Omega_T\cap C^-_{2r}(x_0,t_0)}u(x,t)\leq(r/\rho)^\gamma\sup_{\Omega_T\cap C^-_{2\rho}(x_0,t_0)}u(x,t). \tag{4.3}$$

Furthermore, using Lemma 3.2 we see that

$$u(A^-_{2\rho}(x_0,t_0))\leq c(\rho/r)^\gamma u(A^-_r(x_0,t_0)). \tag{4.4}$$

In the following we prove that

$$\sup_{\Omega_T\cap C^-_{2\rho}(x_0,t_0)}u(x,t)\leq cu(A^-_{2\rho}(x_0,t_0)) \tag{4.5}$$

for this particular choice of ρ. In fact, combining (4.3), (4.4) and (4.5) we see that

$$\sup_{\Omega_T\cap C^-_{2r}(x_0,t_0)}u(x,t)\leq cu(A^-_r(x_0,t_0)). \tag{4.6}$$

To prove (4.5) we let $K\gg 1$ be given as in Lemma 3.10, and we divide the proof into two cases. First, we assume that $\delta/(2K)<\rho$. In this case ρ is large and combining Lemma 3.7 and Lemma 3.9 we see that

$$\sup_{\Omega_T\cap C^-_{2\rho}(x_0,t_0)}u(x,t)\leq cu(A^+_{2\rho}(x_0,t_0))\leq c^2u(A^-_{2\rho}(x_0,t_0)), \tag{4.7}$$

for some $c=c(H,M,\mathrm{diam}(\Omega),T,\delta,K)$, $1\leq c<\infty$. Hence, the proof is complete in this case. Second, we assume that $r\leq\rho\leq\delta/(2K)$ and we then first note, by the definition of ρ, that $f(2K\rho)\leq f(\rho)$, *i.e.*,

$$\sup_{\Omega_T\cap C^-_{2\rho}(x_0,t_0)}u\geq(2K)^{-\gamma}\sup_{\Omega_T\cap C^-_{4K\rho}(x_0,t_0)}u.$$

Obviously the above inequality implies

$$\sup_{\Omega_T\cap C^-_{2K\rho,2\rho}(x_0,t_0)} u \geq (2K)^{-\gamma} \sup_{\Omega_T\cap C^-_{4K\rho,8\rho}(x_0,t_0)} u,$$

and hence we can use Lemma 3.10 to conclude that

$$\sup_{\Omega_T\cap C^-_{4K\rho}(x_0,t_0)\cap\{(x,t):t=t_0-64\rho^2\}} u \geq c^{-1} \sup_{\Omega_T\cap C^-_{2K\rho,2\rho}(x_0,t_0)} u. \tag{4.8}$$

In particular, using if necessary Lemma 3.7, and the Harnack inequality in Theorem 2.2, we can now use (4.8) to conclude (4.5). This completes the proof of (4.5). Furthermore, Theorem 1 now follows readily from (4.5).

□

4.2. Proof of Theorem 1.2

To prove Theorem 1.2 we first establish a few lemmas.

Lemma 4.1. *Let $K \gg 1$ be the constant appearing in Lemma 3.12, let $(x_0, t_0) \in S_T$ and assume that*

$$r < \min\Big\{r_0/(2K), \sqrt{(T-t_0)/(4K^2)}, \sqrt{t_0/(4K^2)}\Big\}.$$

Let u and v be two nonnegative solutions to $Hu = 0$ in Ω_T, and assume that $v = 0$ continuously on $\Delta(x_0, t_0, 2Kr)$. Then, there exists a constant $c = c(H, M, r_0)$ such that

$$\sup_{\Omega_T\cap C^-_r(x_0,t_0)} \frac{v}{u} \leq c\frac{v(A^+_{Kr}(x_0,t_0))}{u(A^-_{Kr}(x_0,t_0))}.$$

Proof. We first note that if we choose K large enough then, since $(\Omega_T \setminus \Omega^r_T) \cap C^-_{Kr}(x_0,t_0) \subset \Omega_T \cap C^-_{Kr}(x_0,t_0)$, we can use Remark 3.8 to conclude that

$$v(x,t) \leq c_1 v(A^+_{Kr}(x_0,t_0)) \text{ whenever } (x,t) \in (\Omega_T \setminus \Omega^r_T) \cap C^-_{Kr}(x_0,t_0). \tag{4.9}$$

Furthermore, by the Harnack inequality we have that

$$u(x,t) \geq c_2^{-1} u(A^-_{Kr}(x_0,t_0)),$$

for every $(x,t) \in \partial_p\big((\Omega_T \setminus \Omega^r_T) \cap C^-_{Kr}(x_0,t_0)\big) \cap \partial_p\Omega^r_T$. For $(x,t) \in (\Omega_T \setminus \Omega^r_T) \cap C^-_{Kr}(x_0,t_0)$ let

$$\tilde v(x,t) = v(x,t)/v(A^+_{Kr}(x_0,t_0)),$$

$$\tilde u(x,t) = u(x,t)/u(A^-_{Kr}(x_0,t_0)),$$

$$\hat v(x,t) = c_1^{-1}\tilde v(x,t) - c_2\tilde u(x,t),$$

and

$$\hat{u}(x,t) = c_2\tilde{u}(x,t).$$

Then, we can apply Lemma 3.12 with u, v replaced by $\hat{u}$, $\hat{v}$ to first conclude that $\hat{v}(x,t) \le \hat{u}(x,t)$, for $(x,t) \in \Omega_T \cap C_r^-(x_0,t_0)$, and then that

$$\begin{aligned}\frac{v(x,t)}{u(x,t)} &\le \frac{v(A_{Kr}^+(x_0,t_0))}{u(A_{Kr}^-(x_0,t_0))}\frac{c_2}{c_1}\left(\frac{\hat{v}(x,t)}{\hat{u}(x,t)}+1\right)\\ &\le c_3\frac{v(A_{Kr}^+(x_0,t_0))}{u(A_{Kr}^-(x_0,t_0))}\end{aligned}$$

whenever $(x,t) \in \Omega_T \cap C_r^-(x_0,t_0)$. This completes the proof of Lemma 4.1. □

Lemma 4.2. *Let $K \gg 1$ be the constant appearing in Lemma 3.12, let $(x_0,t_0) \in S_T$ and assume that*

$$r < \min\left\{r_0/(2K), \sqrt{(T-t_0)/(4K^2)}, \sqrt{t_0/(4K^2)}\right\}.$$

Let u and v be two nonnegative solutions to $Hu = 0$ in Ω_T, assume that $u = 0$ continuously on S_T, that $v = 0$ continuously on $\Delta(x_0,t_0,4Kr)$, and that u and v are not identically zero. Then, the quotient v/u is Hölder continuous on the closure of $\Omega_T \cap C_r^-(x_0,t_0)$.

Proof. To prove this lemma we proceed similarly to [20]. Given (x,t) in the closure of Ω_T and $\rho > 0$ we define

$$\omega(x,t,\rho) = \sup_{\Omega_T\cap C_\rho^-(x,t)}\frac{v}{u} - \inf_{\Omega_T\cap C_\rho^-(x,t)}\frac{v}{u}. \tag{4.10}$$

Then, to start with, we note that Lemma 4.1 implies that

$$\omega(x_0,t_0,2r) \le 2c\frac{v(A_{Kr}^+(x_0,t_0))}{u(A_{Kr}^-(x_0,t_0))} \le C < \infty. \tag{4.11}$$

In the following we let (x,t) be an arbitrary point in $\Omega_T \cap C_r^-(x_0,t_0)$ and we consider $0 < \rho \le r$. Let $d := d(x,\partial\Omega) = d_p(x,t,S_T)$. We divide the proof into the cases $\rho \le d$ and $\rho > d$.

The case $\rho \le d$. Assume first that, in addition, $\rho \le d/2$. We note that we can assume, without loss of generality, that

(i) $0 \le \frac{v(y,s)}{u(y,s)} \le 1$, for $(y,s) \in C_\rho^-(x,t)$,
(ii) $\omega(x,t,\rho) = 1$,
(iii) $\frac{v(x,t-\rho^2/2)}{u(x,t-\rho^2/2)} \ge \frac{1}{2}$.

To see this notice that to achieve (i) and (ii) we can replace v by

$$\hat{v} \equiv \omega(x,t,\rho)^{-1}\left(v - \left(\inf_{\Omega_T \cap C_\rho^-(x,t)} v/u\right) u\right).$$

Furthermore, if (iii) does not hold, then we can replace v by $\bar{v} \equiv u - \hat{v} \geq 0$ to achieve (iii). Next, using the Harnack inequality we first see that

$$v(x,t-\rho^2/2) \leq cv(y,s),\ u(y,s) \leq cu(x,t+\rho^2/2)$$

whenever $(y,s) \in C_{\rho/2}^-(x,t)$. Moreover, as in the proof of Theorem 1, we derive that

$$u(x,t+\rho^2/2) \leq cu(x,t-\rho^2/2).$$

Thus

$$\frac{1}{2} \leq \frac{v(x,t-\rho^2/2)}{u(x,t-\rho^2/2)} \leq c\frac{v(y,s)}{u(y,s)} \leq c,$$

whenever $(y,s) \in C_{\rho/2}^-(x,t)$, and hence

$$\omega(x,t,\rho/2) \leq \tilde{\theta}_1 \omega(x,t,\rho) \tag{4.12}$$

where $\tilde{\theta}_1 = 1 - 1/(2c) \in (0,1)$. Furthermore, iterating the estimate in (4.12) we deduce that

$$\omega(x,t,\rho) \leq \left(\frac{2\rho}{d}\right)^{\sigma_1} \omega(x,t,d) \tag{4.13}$$

whenever $\rho \leq d/2$ and where $\sigma_1 = -\log_2 \tilde{\theta}_1$. Obviously this estimate also holds whenever $d/2 < \rho \leq d$.

The case $\rho > d$. Assume first, in addition, that $\rho < r/2$. Note that $C_\rho^-(x,t) \subset C_{2\rho}^-(\tilde{x}_0,t)$ for some $\tilde{x}_0 \in \partial\Omega$ such that $d = d(x,\tilde{x}_0)$. Then by arguing as in the proof in the case $\rho \leq d$, using Lemma 4.1, it follows as in [20] that

$$\begin{aligned} \omega(x,t,\rho) \leq \omega(\tilde{x}_0,t,2\rho) &\leq \left(\frac{4K\rho}{r}\right)^{\sigma_2} \omega(\tilde{x}_0,t,r) \\ &\leq \left(\frac{4K\rho}{r}\right)^{\sigma_2} \omega(x_0,t,2r), \end{aligned} \tag{4.14}$$

for some $\sigma_2 \in (0,1)$. Obviously (4.14) also holds in the case $r/2 \leq \rho \leq r$.

Combining (4.13) and (4.14) we see that

$$\omega(x,t,\rho) \leq \left(\frac{2\rho}{d}\right)^{\sigma_1} \omega(x,t,d) \leq \left(\frac{2\rho}{d}\right)^{\sigma_1} \left(\frac{4Kd}{r}\right)^{\sigma_2} \omega(x_0,t,2r)$$

also when $\rho \le d < r/2$. Finally, using (4.11) we obtain for some $\sigma_3 \in (0, 1)$

$$\omega(x, t, \rho) \le cK\left(\frac{\rho}{r}\right)^{\sigma_3}\omega(x, t, 2r) \le cK\left(\frac{\rho}{r}\right)^{\sigma_3} C,$$

whenever $\rho \le d < r$. Combining these estimates completes the proof of the lemma. □

Proof of Theorem 1.2. The interior case is straightforward since both u and v are Hölder continuous and since we only consider solutions which are nonnegative and not identically zero. Hence, we have that the quotient v/u is Hölder continuous in $\Omega_T' \subset\subset \Omega_T \cap C_r^-(x_0, t_0)$. Then to prove Theorem 1.2 we first assume that $u = 0$ continuously on S_T. In this case, using Lemma 4.2, we see that v/u is Hölder continuous on the closure of $\Omega_T \cap C_{r_1}^-(\hat{x}, \hat{t})$, for some small $r_1 > 0$, whenever $(\hat{x}, \hat{t}) \in S_T \cap C_r^-(x_0, t_0)$. Combining this fact with the interior argument we see that v/u is Hölder continuous on the closure of $\Omega_T \cap C_r^-(x_0, t_0)$. In the general case we represent u in the form $u = u_0 + u_1$, where $Lu = Lu_1 = 0$ in Ω_T, $u_0 = 0$, $u_1 = u$ on S_T and $u_0 = u$, $u_1 = 0$ on $\Omega \times \{t = 0\}$. Then by the argument above we see that v/u_0 as well as u_1/u_0 are Hölder continuous on the closure of $\Omega_T \cap C_r^-(x_0, t_0)$. Using this we can conclude, as

$$\frac{v}{u} = \frac{v}{u_0}\frac{1}{1 + \dfrac{u_1}{u_0}},$$

that also v/u is Hölder continuous on the closure of $\Omega_T \cap C_r^-(x_0, t_0)$. This completes the proof of Theorem 1.2. □

5. Proof of Theorem 1.3

The objective of this section is proving Theorem 1.3. With this in mind, we first need to introduce some additional notation. In particular, for $(x_0, t_0) \in \mathbb{R}^{n+1}$, $0 < r_1 < r_2$, and $K \gg 100$, we define,

$$\begin{aligned}\Gamma_K^+(x_0, t_0, r_1, r_2) =&\{(x, t) \mid d(x, x_0) \le K|t - t_0|^{1/2},\\ &r_1 \le |t - t_0|^{1/2} \le r_2,\ t > t_0\}.\end{aligned} \tag{5.1}$$

Furthermore, given $(x_0, t_0) \in S_T$, $0 < \rho$, $\mu \in (0, 1)$, and a function u, we let

$$f_1^u(x_0, t_0, \rho, \mu) = \inf_{\{(x,t):\ x\in\Omega^{\mu\rho'}\cap B_d(x_0,\rho),\ t=t_0+\rho^2\}} u(x, t) \tag{5.2}$$

where $\rho' = \min\{\rho, r_0\}$. Similarly, given $(x_0, t_0) \in S_T$ and a function u we define

$$f_2^u(x_0, t_0, \rho, K) = \sup_{\{(x,t)\in\Omega_T\cap\partial_p C_{K\rho,\rho}(x_0,t_0)\cap\{(x,t):\ |t-t_0|<\rho^2\}\}} u^-(x, t), \tag{5.3}$$

where $u^-(x, t) = -\min\{0, u(x, t)\}$.

To establish Theorem 1.3 we will first prove four lemmas.

Lemma 5.1. *Let $\omega^{(x,t)}$ be the H-parabolic measure at $(x,t) \in \Omega_T$. Let $(x_0,t_0) \in \partial_p\Omega_T$ and assume that $r < \min\{r_0/2\}$. Then, there exists a constant $c = c(H, M, r_0)$, $1 \le c < \infty$, such that*

$$\omega^{(x,t)}(C_{2r}(x_0,t_0) \cap \partial_p\Omega_T) \ge c^{-1},$$

whenever $(x,t) \in \Omega_T \cap C_r(x_0,t_0)$.

Proof. First, let $(x_0,t_0) \in S_T$. By Lemma 2.8, we can choose a set U which is regular for the Dirichlet problem and such that

$$B_d(x_0, 3r/2) \subset U \subset B_d(x_0, 2r).$$

Since Ω is NTA, there exists a point $A_r'(x_0) \in \mathbb{R}^n \setminus \Omega$ such that

$$\frac{r}{M} < d(A_r'(x_0), x_0) \le r, \quad \text{and } d(A_r'(x_0), \partial\Omega) > \frac{r}{M}.$$

Furthermore, using Lemma 2.8 once again we can also find a set U', which is H-regular for the Dirichlet problem, such that

$$B_d(A_r'(x_0), r/4M) \subset U' \subset B_d(A_r'(x_0), r/2M) \subset U \setminus \Omega.$$

Using this notation we let

$$C := U \times [t_0 - 4r^2, t_0 + 4r^2], \quad C' := U' \times [t_0 - 4r^2, t_0 + 4r^2],$$

and $B = U' \times \{t_0 - 4r^2\}$. We also let $v(x,t) = \omega_C^{(x,t)}(B)$ and $v'(x,t) = \omega_{C'}^{(x,t)}(B)$. By the maximum principle, we have $\omega^{(x,t)}(\Delta(x_0,t_0,2r)) \ge v(x,t)$ in $C_r(x_0,t_0) \cap \Omega_T$, and $v(x,t) \ge \tilde{v}(x,t)$ in C'. By the Harnack principle applied in C, we have

$$\begin{aligned}\inf_{C_r(x_0,t_0)\cap\Omega_T} \omega^{(x,t)}(\Delta(x_0,t_0,2r)) &\ge \inf_{C_r(x_0,t_0)\cap\Omega_T} v(x,t)\\ &\ge c^{-1} v(A_r'(x_0), t_0 - 2r^2)\\ &\ge c^{-1} v'(A_r'(x_0), t_0 - 2r^2).\end{aligned}$$

We can extend the function v' to the cylinder $C'' = U' \times [t_0 - 5r^2, t_0 + 4r^2]$ by setting

$$v'(x,t) = \omega_{C''}^{(x,t)}(\partial_p(C'' \cap \{t \le t_0 - 4r^2\})),$$

that is, letting $v' = 1$ below B. We now apply the Harnack inequality to v' in C'' and obtain

$$v'(A_r'(x_0), t_0 - 2r^2) \ge c^{-1} v'(A_r'(x_0), t_0 - 4r^2) = c^{-1},$$

and we are finished. The case when (x_0,t_0) is on the bottom of $\partial_p\Omega_T$ is similar, but simpler. □

Lemma 5.2. *Let $K \gg 1$ be given, $(x_0, t_0) \in S_T$, and assume that u be a solution to $Hu = 0$ in Ω_T such that $u \geq 0$ in $\Omega_T \cap \Gamma_K^+(x_0, t_0, \rho_0, R)$ for some ρ_0 and R such that $0 < 2\rho_0 \leq R \leq \nu r_0$, where $\nu > 0$ is a fixed constant. Then, for every $\mu \in (0, 1)$ there exists a $\gamma_1 > 0$ depending on H, μ, K, ν, and r_0, such that*

$$\inf_{\rho_0 \leq r \leq 2\rho_0} f_1^u(x_0, t_0, r, \mu) \leq \left(\frac{\rho}{\rho_0}\right)^{\gamma_1} f_1^u(x_0, t_0, \rho, \mu)$$

for all ρ such that $0 < 2\rho_0 \leq \rho \leq R$.

Proof. To prove Lemma 5.2 we let ρ satisfy $0 < 2\rho_0 \leq \rho \leq R$. We define

$$D(x_0, t_0, \rho, \mu) := \{(x, t) : \ x \in \Omega^{\mu\rho'} \cap B_d(x_0, \rho), \ t = t_0 + \rho^2\},$$

and note that we can apply the Harnack inequality to conclude that

$$\begin{aligned} f_1^u(x_0, t_0, \rho, \mu) &= \inf_{D(x_0, t_0, \rho, \mu)} u(x, t) \\ &\geq 2^{-\gamma_1} \sup_{D(x_0, t_0, \rho/2, \mu)} u(x, t) \geq 2^{-\gamma_1} f_1^u(x_0, t_0, \rho/2, \mu), \end{aligned} \tag{5.4}$$

for some $\gamma_1 = \gamma_1(H, M, r_0, \mu) > 0$. In particular, iterating k times the inequality in (5.4), where k satisfies $2\rho_0 > 2^{-k}\rho \geq \rho_0$, we see that

$$\begin{aligned} f_1^u(x_0, t_0, \rho, \mu) &\geq 2^{-k\gamma_1} f_1^u(x_0, t_0, 2^{-k}\rho, \mu) \\ &\geq \left(\frac{\rho_0}{\rho}\right)^{\gamma_1} f_1^u(x_0, t_0, 2^{-k}\rho, \mu). \end{aligned}$$

This latter inequality implies the statement in Lemma 5.2, thus completing the proof. □

Lemma 5.3. *Let $K \gg 1$ be given, $(x_0, t_0) \in S_T$, and assume that u is a solution to $Hu = 0$ in Ω_T such that $u \geq 0$ in $\Omega_T \cap \Gamma_K^+(x_0, t_0, \rho_0, R)$ for some ρ_0 and R such that $0 < \rho_0 \leq R$. Furthermore, assume that*

$$u(x, t) = 0 \text{ whenever } (x, t) \in \partial_p\Omega_T \setminus C_{\rho_0/2}(x_0, t_0).$$

Then, there exists $\gamma_2 > 0$, which depends on H, M, K and r_0, such that

$$f_2^u(x_0, t_0, \rho, K) \leq \left(\frac{2\rho_0}{\rho}\right)^{\gamma_2} f_2^u(x_0, t_0, \rho_0, K)$$

for all ρ such that $0 < \rho_0 \leq \rho \leq R$. Moreover, $\gamma_2 \to \infty$ as $K \to \infty$.

Proof. By simply using the maximum principle, we first note that since $u = 0$ continuously on $\partial_p\Omega_T \cap \{t : t \leq t_0 - \rho_0^2\}$, we also have that $u \equiv 0$ on $\Omega_T \cap \{t : t \leq t_0 - \rho_0^2\}$. Furthermore, again by the maximum principle, applied to u in $\Omega_T \setminus C_{K\rho,\rho}(x_0, t_0)$, we see that as a function of $\rho \in [\rho_0, R]$, the $f_2^u(x_0, t_0, \rho, K)$ decreases, and therefore the conclusion of Lemma 5.3 holds for $\rho \in [\rho_0, 2\rho_0]$. Hence it remains to consider $\rho \in (2\rho_0, R]$. For such ρ we see that there exists $(\hat{x}, \hat{t}) \in \Omega_T \cap \partial_p C_{K\rho,\rho}(x_0, t_0) \cap \{(x, t) : |t - t_0| < \rho^2\}$ such that $f_2^u(x_0, t_0, \rho, K) = u^-(\hat{x}, \hat{t})$. Note, in particular, that $d(x_0, \hat{x}) = K\rho$ and that $(\hat{x}, \hat{t}) \in C_{2\rho}^+(\hat{x}, t_0 - \rho^2)$. Hence,

$$f_2^u(x_0, t_0, \rho, K) = u^-(\hat{x}, \hat{t}) \leq \sup_{\Omega_T \cap C_{2\rho}(\hat{x}, t_0 - \rho^2)} u^-. \tag{5.5}$$

We claim that that there exists $\hat{K} \gg 1$, $\hat{K} \ll K$, $\hat{K} = \hat{K}(H, M)$, such that

$$\sup_{\Omega_T \cap C_{2\rho}(\hat{x}, t_0 - \rho^2)} u^- \leq \theta \sup_{\Omega_T \cap C_{\hat{K}\rho,2\rho}(\hat{x}, t_0 - \rho^2)} u^- \tag{5.6}$$

for some $\theta \in (0, 1)$. This is proved by arguing as in Lemma 3.3, except that the proof is simpler: we only need the function Φ_1, and can omit Φ_2, since u^- vanishes on the bottom of $\Omega_T \cap C_{\hat{K}\rho,2\rho}(\hat{x}, t_0 - \rho^2)$.

To proceed with the proof of Lemma 5.3 we note that (5.5) and (5.6) imply that

$$f_2^u(x_0, t_0, \rho, K) \leq \theta \sup_{\Omega_T \cap C_{\hat{K}\rho,2\rho}(\hat{x}, t_0 - \rho^2)} u^-.$$

We next note that:

1) the sets $\Omega_T \cap C_{\hat{K}\rho,2\rho}(\hat{x}, t_0 - \rho^2)$ and

$$\Omega_T \cap \partial_p C_{K\rho_0,\rho_0}(x_0, t_0) \cap \{(x, t) : |t - t_0| < \rho_0^2\}$$

are separated by the cylindrical surface

$$S = \{d(x_0, x) = (K - \hat{K})\rho\} = \{d(x_0, x) = qK\rho\},$$

where $q = (K - \hat{K})/K \in [1/2, 1)$, provided $K \geq 2\hat{K}$;

2) that

$$\Omega_T \cap \partial_p C_{Kq\rho,q\rho}(x_0, t_0) \cap \{(x, t) : |t - t_0| < (q\rho)^2\} \subset S;$$

3) and that $u \geq 0$ in

$$S \setminus (\Omega_T \cap \partial_p C_{Kq\rho,q\rho}(x_0,t_0) \cap \{(x,t) : \ |t-t_0| < (q\rho)^2\}).$$

In particular, by the maximum principle, we obtain that

$$\begin{aligned} f_2^u(x_0,t_0,\rho,K) &\leq \theta \sup_{\{\Omega_T \cap \partial_p C_{Kq\rho,q\rho}(x_0,t_0) \cap \{(x,t):\ |t-t_0|<(q\rho)^2\}\}} u^- \\ &= \theta f_2^u(x_0,t_0,q\rho,K) = q^{\gamma_2} f_2^u(x_0,t_0,q\rho,K) \end{aligned}$$

where $\gamma_2 = \log_q \theta > 0$. Next, we choose $k \geq 1$ so that $\rho_0 \leq q^k \rho \leq 2\rho_0$, and by iteration we derive

$$\begin{aligned} f_2^u(x_0,t_0,\rho,K) &\leq q^{k\gamma_2} f_2^u(x_0,t_0,q^k\rho,K) \\ &\leq \left(\frac{2\rho_0}{\rho}\right)^{\gamma_2} f_2^u(x_0,t_0,\rho_0,K). \end{aligned}$$

Finally, for $K \geq 2\hat{K}$ we have

$$\begin{aligned} \frac{1}{q} &= 1 + \frac{\hat{K}}{K-\hat{K}} \leq 1 + \frac{2\hat{K}}{K}, \ \ln q^{-1} \leq \frac{2\hat{K}}{K}, \\ \gamma_2 &= \log_q \theta \geq \frac{K \ln(\theta^{-1})}{2\hat{K}} \to \infty \text{ as } K \to \infty. \end{aligned}$$

In particular, this completes the proof of Lemma 5.3. □

In what follows we let

$$\Omega_{[t_0+\rho^2,T]} = \Omega_T \cap \{(y,s) \in \mathbb{R}^{n+1} \mid t_0 + \rho^2 < s < T\}.$$

For a given Borel set $E \subset \partial_p \Omega_{[t_0+\rho^2,T]}$, we will denote by $\omega^{(x,t)}_{\Omega_{[t_0+\rho^2,T]}}(E)$ the value in $(x,t) \in \Omega_{[t_0+\rho^2,T]}$ of the H-parabolic measure of E.

Lemma 5.4. *Let $K \gg 1$ be given, $(x_0,t_0) \in S_T$, and suppose that $0 < 2\rho \leq \nu r_0$, where $\nu > 0$ is a constant. Then, there exist constants $\mu \in (0,1)$ depending on M, and $\hat{c}$ depending on H, M, ν and K, such that*

$$\begin{aligned} &\omega^{(x,t)}_{\Omega_{[t_0+\rho^2,T]}}(\Omega_T \cap (\mathbb{R}^n \times \{t_0+\rho^2\})) \\ &\leq \hat{c}\, \omega^{(x,t)}_{\Omega_{[t_0+\rho^2,T]}}(\Omega_T^{\mu\rho'} \cap (B_d(x_0,\rho) \times \{t_0+\rho^2\})), \end{aligned}$$

whenever $(x,t) \in \Omega_T \cap C_{2K\rho}(x_0,t_0) \cap (\mathbb{R}^n \times \{t_0+4\rho^2\})$, where $\rho' = \min(p, r_0)$.

Proof. Follows just as the proof of [39, Lemma 4.5]. One also needs to prove the equivalent of [39, Theorem 2.4], which also follows just as in that article. These proofs make use of Lemma 4.1, Lemma 5.1 and the Harnack inequality. We omit the details. □

We are finally in a position to establish the main result of this section.

Proof of Theorem 1.3. To start the proof we choose $K \gg 1$ large enough to guarantee that $\gamma_1 < \gamma_2$, where γ_1 and γ_2 are the constants of Lemma 5.2 and Lemma 5.3 respectively. Moreover, for this choice of γ_1, γ_2, and given the constant in Lemma 5.4, $\hat{c} = \hat{c}(H, M, \nu, K)$, $1 \leq \hat{c} < \infty$, we let $\hat{r} = \hat{r}(H, M, \nu, K)$ be

$$\text{the smallest } \hat{r} \text{ which satisfies } 4^{-\gamma_1}(\hat{r}/4r)^{\gamma_2-\gamma_1} \geq \hat{c}. \tag{5.7}$$

Below we will, in the end, distinguish between the cases $\nu r_0 \leq \hat{r}$ and $\nu r_0 > \hat{r}$. Let μ be the constant in Lemma 5.4. To prove Theorem 1.3 we intend to prove that there exists a constant $c = c(H, M, \nu, K)$ such that

$$u(x,t) := c\omega^{(x,t)}(\Delta(x_0,t_0,r)) - \omega^{(x,t)}(\Delta(x_0,t_0,2r)) \geq 0, \tag{5.8}$$

whenever $(x,t) \in \Gamma_K^+(x_0,t_0,4r,\nu r_0)$. To start the proof of (5.8) we first note, using Lemma 5.1 and the Harnack inequality, that

$$\omega^{(x,t)}(\Delta(x_0,t_0,r)) \geq \tilde{c}^{-1}, \tag{5.9}$$

whenever $(x,t) \in \Omega_T^{\mu\rho'} \cap (B_d(x_0, 2K\rho) \times \{t_0 + 4\rho^2\})$, $0 < 2\rho \leq R \leq \nu r_0$, for some $\tilde{c} = \tilde{c}(H, M, \nu, K, R)$, $1 \leq \tilde{c} < \infty$. Let $\hat{c}$ be the constant in Lemma 5.4. Then, using (5.9) and Lemma 5.4 we see that

$$\begin{aligned} \hat{c}\tilde{c}\omega^{(x,t)}(\Delta(x_0,t_0,r)) &\geq \hat{c}\omega^{(x,t)}_{\Omega_{[t_0+\rho^2,T]}}(\Omega_T^{\mu\rho'} \cap (B_d(x_0,\rho) \times \{t_0+\rho^2\})) \\ &\geq \omega^{(x,t)}_{\Omega_{[t_0+\rho^2,T]}}(\Omega_T \cap \{t : t = t_0 + \rho^2\}) \end{aligned} \tag{5.10}$$

when $(x,t) \in \Omega_T^{\mu\rho'} \cap \{(x,t) : x \in B_d(x_0, 2K\rho), t = t_0 + 4\rho^2\}$. Note that the first inequality in (5.10) uses (5.9), the trivial inequality $1 \geq \omega(x, t, \Omega_T^{\mu\rho'} \cap \{(x,t) : x \in B_d(x_0,\rho), t = t_0 + \rho^2\}, \Omega_{[t_0+\rho^2,T]})$ and the maximum principle on $\Omega_T \cap \{t \geq t_0 + \rho^2\}$. Furthermore, let $4r \leq 2\rho \leq R \leq \nu r_0$. Then, and this is a simple consequence of the maximum principle,

$$\omega^{(x,t)}_{\Omega_{[t_0+\rho^2,T]}}(\Omega_T \cap \{t : t = t_0 + \rho^2\}) \geq \omega^{(x,t)}(\Delta(x_0,t_0,2r)), \tag{5.11}$$

whenever $(x,t) \in \Gamma_K^+(x_0,t_0,4r,\nu r_0) \cap \{t : t = t_0 + 4\rho^2\}$. In particular, combining (5.10) and (5.11) we can conclude that

$$\hat{c}\tilde{c}\omega^{(x,t)}(\Delta(x_0,t_0,r)) \geq \omega^{(x,t)}(\Delta(x_0,t_0,2r)), \tag{5.12}$$

whenever $(x, t) \in \Gamma_K^+(x_0, t_0, 4r, \nu r_0) \cap \{t : t = t_0 + 4\rho^2\}$. Therefore the function u in (5.8), defined with constant $c = \hat{c}\tilde{c}$, satisfies $u \geq 0$ in $\Gamma_K^+(x_0, t_0, 4r, \nu r_0)$. In particular, if $\nu r_0 \leq \hat{r}$, where $\hat{r} = \hat{r}(H, M, \nu, K)$ is as in (5.7), then the constant $\tilde{c}$, and hence c, can be chosen to only depend on H, M, ν, K, and we are done. Hence, it only remains to consider the case $\nu r_0 > \hat{r}$. However, by arguing as above, we see in this case that there exists $c = c(H, M, \nu, K)$ such that, if we consider the function u in (5.8) with this c, then

(i) $u(x, t) \geq 1$, for $(x, t) \in \Omega_T^{\mu\rho'} \cap \{(x, t) : x \in B_d(x_0, \rho), t = t_0 + \rho^2\}$, for $2r \leq \rho \leq 4r$;
(ii) $u(x, t) \geq 0$ for $(x, t) \in \Gamma_K^+(x_0, t_0, 4r, \hat{r})$.

In the following we prove that (i) and (ii) imply (5.8) for all $(x,t) \in \Gamma_K^+(x_0, t_0, 4r, \nu r_0)$. To do this we argue by contradiction. Hence, we assume that there exist $\rho > 4r$ such that $u \geq 0$ whenever $(x, t) \in \Gamma_K^+(x_0, t_0, 4r, \rho)$ and that $u(\hat{x}, \hat{t}) < 0$ at some point

$$(\hat{x}, \hat{t}) \in \Omega_T \cap \{(x, t) : x \in B_d(x_0, 2K\rho), t = t_0 + 4\rho^2\} \subset \Gamma_K^+(x_0, t_0, 4r, 2\rho).$$

Let $\omega_{\Omega_T \setminus C_{K\rho,\rho}(x_0,t_0)}$ denote the H-parabolic measure with respect to $\Omega_T \setminus C_{K\rho,\rho}(x_0, t_0)$. Then, we first note that

$$\begin{aligned} u(\hat{x}, \hat{t}) \geq & \int_{\Omega_T^{\mu\rho'} \cap (B_d(x_0,\rho)\times\{t_0+\rho^2\})} u\, d\omega_{\Omega_T \setminus C_{K\rho,\rho}(x_0,t_0)}^{(\hat{x},\hat{t})} \\ & + \int_{\Omega_T \cap \partial_p C_{K\rho,\rho}(x_0,t_0) \cap \{(x,t):\ |t-t_0|<\rho^2\}} u\, d\omega_{\Omega_T \setminus C_{K\rho,\rho}(x_0,t_0)}^{(\hat{x},\hat{t})}. \end{aligned} \tag{5.13}$$

Let

$$E_1 = \omega_{\Omega_T \setminus C_{K\rho,\rho}(x_0,t_0)}^{(\hat{x},\hat{t})} \big(\Omega_T^{\mu\rho'} \cap (B_d(x_0, \rho) \times \{t_0 + \rho^2\})\big),$$

and

$$E_2 = \omega_{\Omega_T \setminus C_{K\rho,\rho}(x_0,t_0)}^{(\hat{x},\hat{t})} \big(\Omega_T \cap \partial_p C_{K\rho,\rho}(x_0, t_0) \cap \{(x, t) :\ |t - t_0| < \rho^2\}\big).$$

Using (5.13), Lemma 5.2 and Lemma 5.3 we deduce that

$$u(\hat{x}, \hat{t}) \geq E_1 \left(\frac{2r}{\rho}\right)^{\gamma_1} - E_2 \left(\frac{8r}{\rho}\right)^{\gamma_2}. \tag{5.14}$$

Furthermore, by the maximum principle and Lemma 5.4 we see that

$$\begin{aligned} & \omega_{\Omega_T \setminus C_{K\rho,\rho}(x_0,t_0)}^{(\hat{x},\hat{t})} (\Omega_T \cap \partial_p C_{K\rho,\rho}(x_0, t_0) \cap \{(x, t) :\ |t - t_0| < \rho^2\}) \\ & \leq \omega_{\Omega_{[t_0+\rho^2,T]}}^{(\hat{x},\hat{t})} (\Omega_T \cap \{t :\ t = t_0 + \rho^2\}) \\ & \leq \hat{c}\omega_{\Omega_{[t_0+\rho^2,T]}}^{(\hat{x},\hat{t})} (\Omega_T^{\mu\rho'} \cap \{(x, t) : x \in B_d(x_0, \rho), t = t_0 + \rho^2\}), \end{aligned} \tag{5.15}$$

and so $E_2 \leq \hat{c}E_1$. In particular, using that $0 > u(\hat{x}, \hat{t})$ and combining (5.14) and (5.15) we can conclude that

$$4^{-\gamma_2}(\rho/2r)^{\gamma_2-\gamma_1} < \hat{c} \leq 4^{-\gamma_2}(\hat{r}/4r)^{\gamma_2-\gamma_1},$$

and hence that $2\rho < \hat{r}$. This implies that $\Gamma_K^+(x_0, t_0, 4r, 2\rho) \subset \Gamma_K^+(x_0, t_0, 4r, \hat{r})$. Since $(\hat{x}, \hat{t}) \in \Gamma_K^+(x_0, t_0, 4r, 2\rho)$ we can therefore conclude from (B) that $u(\hat{x}, \hat{t}) \geq 0$. This contradicts our choice of $(\hat{x}, \hat{t})$ and hence (5.8) must be true. This completes the proof of Theorem 1.3. □

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