Quantitative isoperimetric inequalities
and homeomorphisms with finite distortion

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Abstract. We prove quantitative isoperimetric inequalities for images of the unit ball under homeomorphisms of exponentially integrable distortion. We show that the metric distortions of such domains can be controlled by their Fraenkel asymmetries. An application of the quantitative isoperimetric inequality proved by Hall and Fusco, Maggi, and Pratelli then shows that for these domains a version of Bonnesen's inequality holds in all dimensions.

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1. Introduction

The classical Bonnesen inequality states that for a planar Jordan domain Ω the inequality

$$\ell(\partial\Omega)^2 - 4\pi|\Omega| \geq 4\pi(R - \rho)^2 \tag{1.1}$$

holds, where R and ρ are the circumradius and the inradius of Ω, respectively, see Osserman [13]. There are several related inequalities which show that if a planar Jordan domain is almost a disk in the sense of the isoperimetric inequality, then it is also geometrically close to a disk, with quantitative bounds. Such inequalities are called Bonnesen-style inequalities in [13].

By considering cusp domains, one can see that inequalities like (1.1) do not hold in dimensions higher than two. However, Hall [5] (see also [6]) showed that another natural quantitative isoperimetric inequality holds in all dimensions. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. The Fraenkel asymmetry $\lambda(\Omega)$ is

$$\lambda(\Omega) = \min_{x \in \mathbb{R}^n} \frac{|\Omega \setminus B(x, r)|}{r^n},$$

where r is defined by $|B(x, r)| = |\Omega|$. The isoperimetric deficit of Ω is

$$\delta(\Omega) = \frac{\mathcal{H}^{n-1}(\partial\Omega)}{n\alpha_n^{1/n}|\Omega|^{(n-1)/n}} - 1 \geq 0, \quad \alpha_n = |B^n|.$$

Here $\mathcal{H}^{n-1}$ is the Hausdorff $(n-1)$-measure. Hall proved that the isoperimetric deficit controls the Fraenkel asymmetry, and conjectured that the sharp inequality is

$$\lambda(\Omega) \leq C(n)\sqrt{\delta(\Omega)}. \tag{1.2}$$

A beautiful solution to this problem was given in [3], where it was shown that (1.2) indeed holds. Recently, progress has been made in understanding related inequalities, *cf.* the references in [3]. For convex domains, Fuglede [2] proved essentially sharp higher-dimensional versions of (1.1).

In [14] we applied Hall's result in our study of the branching of quasiregular mappings in space. In particular, we showed that an inequality like (1.1) holds for images of the unit ball under global K-quasiconformal maps. This was done by proving that the Fraenkel asymmetry of such a domain Ω controls its metric distortion

$$\beta(\Omega) = \min\left\{\frac{R-r}{r} : \exists x \in \mathbb{R}^n \text{ so that } B(x, r) \subset \Omega \subset B(x, R)\right\} :$$

$$\beta(\Omega)^n \leq C(n, K)\lambda(\Omega), \tag{1.3}$$

and then applying Hall's theorem. In this paper we consider the more general case where the class of quasiconformal maps is replaced by the class of homeomorphisms with exponentially integrable distortion. From the point of view of conformal analysis, this is essentially the largest class for which inequalities like (1.3), and, consequently, inequalities like (1.1) hold true. Our main objectives are to prove fairly sharp extensions of (1.3), and to demonstrate, again relying on (1.2), that besides convex domains there are also other natural classes of domains in the n-space which satisfy Bonnesen-style inequalities.

Denote by $|Df|$ and J_f the operator norm and the Jacobian determinant of the distributional differential of a $W^{1,1}$-homeomorphism f, respectively, and assume that $J_f \geq 0$ almost everywhere. Then $K(x) = K_f(x) = |Df(x)|^n/J_f(x)$ if $J_f(x) > 0$, $K(x) = 1$ if $|Df(x)| = J_f(x) = 0$, and $K(x) = \infty$ otherwise. Our main theorem reads as follows.

Theorem 1.1. *Let* $f : B(2) \to fB(2) \subset \mathbb{R}^n$ *be a* $W^{1,1}$-*homeomorphism so that* $J_f \geq 0$ *almost everywhere, and*

$$\int_{B(2)} \exp(\mu K(x)) \, \mathrm{d}x \leq K \tag{1.4}$$

for some K and $\mu > 0$. Then

$$\beta(fB^n)^{n+n^2/\mu} \leq C(n, \mu, K)\lambda(fB^n). \tag{1.5}$$

In Section 6 we prove a similar result for the preimage of the unit ball under a polynomial integrability condition on K. The following example demonstrates that, except for the constant n^2 in (1.5), there is not much room for improvement in Theorem 1.1.

Theorem 1.2. *There exists $c(n) > 0$ so that if $n \geq 2$ and $\mu > 0$, there exist $K = K(n, \mu) > 0$ and a sequence (f_j), so that each f_j satisfies the assumptions of Theorem* 1.1, $\lambda(f_j B^n) \to 0$ *as $j \to \infty$, and*

$$\beta(f_j B^n)^{n+c(n)/\mu} \geq \lambda(f_j B^n)$$

for every j.

By combining Theorem 1.1 with the sharp inequality (1.2) proved in [3], we have the following Bonnesen-style inequality. Recall that a homeomorphism f is by definition K-quasiconformal, $1 \leq K < \infty$, if the distortion $K(x)$ defined above satisfies $K(x) \leq K$ almost everywhere.

Corollary 1.3. *Let f be as in Theorem* 1.1. *Then*

$$\beta(f B^n)^{2n+2n^2/\mu} \leq C(n, \mu, K)\delta(f B^n). \tag{1.6}$$

If f is K-quasiconformal, then

$$\beta(f B^n)^{2n} \leq C(n, K)\delta(f B^n). \tag{1.7}$$

In [15] we prove a sharp form of (1.7). It is not difficult to prove (1.3) for quasiconformal maps using the local-to-global principle (quasisymmetry) which says that the distortions of images of balls are bounded in all scales, at least away from the boundary, see Remark 4.5 below. One can also prove versions of (1.3) for non-homeomorphic mappings in small scales, but this is more technical, see [14].

In the case of unbounded distortion considered in this paper, results like Theorem 1.1 cannot be proved using the local-to-global principle, because the corresponding estimates are quite inefficient unless the distortion is assumed to be uniformly bounded. Therefore, in order to prove Theorem 1.1 we have to use a different method. We briefly outline the proof of Theorem 1.1.

We first fix a ball B which realizes the Fraenkel asymmetry of $f B^n$. Our goal then is to estimate $\max_{y \in f S^{n-1}} \operatorname{dist}(y, \partial B)$ in terms of $|f B^n \setminus B|$. This is done using the conformal invariance of the modulus of concentric spheres and their preimages, and coarea-type estimates. On the image side, we get a lower bound for the modulus depending on τ and λ. We then need to prove an efficient upper bound on the domain side. Here we need the exponential integrability of the distortion, and the sharp modulus of continuity, proved in [12]. Combining the two estimates shows that $f B^n$ cannot be too cusplike, which is the main property needed to prove the theorem.

Notation

We denote an n-ball with center x and radius r by $B(x, r)$, and $B(r) = B(0, r)$, $B^n = B(0, 1)$. The corresponding notations for $(n-1)$-spheres are $S(x, r)$ and $S(r) = S(0, r)$. The Lebesgue measure of $E \subset \mathbb{R}^n$ is $|E|$, and $\alpha_n = |B^n|$. The matrix $D^\sharp f^{-1}(x)$ is the adjoint matrix of the differential $Df^{-1}(x)$. Under the assumptions of Theorem 1.1 we have

$$|D^\sharp f^{-1}|^n \leq K J_f^{n-1} \tag{1.8}$$

almost everywhere, see [8].

   We note that inequality (1.2) holds for general Borel sets, and it is stated, as isoperimetric inequalities usually are, in terms of the perimeter measure. One can apply [1, Proposition 3.62], to show that the estimates above can be stated in terms of the Hausdorff $(n-1)$-measure.

2. Conformal modulus of surface families

The main technical tool used in the proofs of Theorem 1.1 and Theorem 6.1 below is the conformal modulus of suitable $(n-1)$-dimensional surfaces. Let first $f : B(2) \to f B(2)$ be a homeomorphism. Moreover, let $G \subset f B(2)$ be open, $y \in \mathbb{R}^n$, $E \subset (0, \infty)$ a Borel set, and

$$\Lambda = \{U_t : t \in E\} = \{G \cap S(y, t) : t \in E\}.$$

We denote by Y the family of all Borel functions $\rho : \mathbb{R}^n \to [0, \infty]$ for which

$$\int_{U_t} \rho(y) \, d\mathcal{H}^{n-1}(y) \geq 1 \quad \text{for every } t \in E,$$

and by X the corresponding family with the requirement

$$\int_{f^{-1}(U_t)} \rho(x) \, d\mathcal{H}^{n-1}(x) \geq 1 \quad \text{for every } t \in E.$$

Lemma 2.1. *Let* $f : B(2) \to f B(2)$ *be as in Theorem* 1.1. *Then*

$$M\Lambda := \inf_{\rho \in Y} \int_{\mathbb{R}^n} \rho(y)^{n/(n-1)} \, dy$$

$$\leq \inf_{\rho \in X} \int_{\mathbb{R}^n} \rho(x)^{n/(n-1)} K(x)^{1/(n-1)} \, dx =: M_K f^{-1}\Lambda.$$

Proof. Let $\rho \in X$. Under our assumptions we have $f^{-1} \in W^{1,n}_{\text{loc}}(f B(2), \mathbb{R}^n)$, see [7]. In particular, the restriction of f^{-1} to U_t locally belongs to $W^{1,n}$ for almost every t. In such U_t, the change of variables inequality

$$\int_{U_t} |D^\sharp f^{-1}(y)| \rho(f^{-1}(y)) \, d\mathcal{H}^{n-1}(y) \geq \int_{f^{-1}(U_t)} \rho(x) \, d\mathcal{H}^{n-1}(x) \geq 1$$

holds, see [11]. Thus the function

$$y \mapsto |D^{\sharp} f^{-1}(y)| \rho(f^{-1}(y))$$

belongs to Y (after redefining the function in a set of measure zero). The n-dimensional change of variables formula holds under our assumptions, see [8], and by (1.8) we have

$$\int_{fB(2)} \left(|D^{\sharp} f^{-1}(y)| \rho(f^{-1}(y)) \right)^{n/(n-1)} \mathrm{d}y$$

$$= \int_{fB(2)} \frac{|D^{\sharp} f^{-1}(y)|^{n/(n-1)}}{J_{f^{-1}}(y)} \rho(f^{-1}(y))^{n/(n-1)} J_{f^{-1}}(y) \, \mathrm{d}y$$

$$\leq \int_{fB(2)} K(f^{-1}(y))^{1/(n-1)} \rho(f^{-1}(y))^{n/(n-1)} J_{f^{-1}}(y) \, \mathrm{d}y$$

$$= \int_{B(2)} K(x)^{1/(n-1)} \rho(x)^{n/(n-1)} \, \mathrm{d}x.$$

The lemma follows by taking the infimum with respect to $\rho \in X$. $\qquad\square$

For the proof of Theorem 6.1, we need an inequality as above for the inverse of f. Let $f : f^{-1}B(2) \to B(2)$ be a homeomorphism. Let $G \subset f^{-1}B(2)$ be open, $x \in \mathbb{R}^n$, and $E \subset (0, \infty)$ a Borel set. We consider the family

$$\Lambda = \{U_t : t \in E\} = \{S(x, t) \cap G : t \in E\}, \tag{2.1}$$

and define the quantities

$$M_{1/K} \Lambda = \inf_{\rho \in X} \int_{\mathbb{R}^n} \rho(x)^{n/(n-1)} K(x)^{-1} \, \mathrm{d}x,$$

where X is the family of all Borel functions for which

$$\int_{U_t} \rho(x) \, \mathrm{d}\mathcal{H}^{n-1}(x) \geq 1 \quad \text{for every } t \in E,$$

and $Mf \Lambda$ as before.

Lemma 2.2. *Let $f : f^{-1}B(2) \to B(2)$ be a $W^{1,1}_{\mathrm{loc}}$-homeomorphism so that $J_f \geq 0$ almost everywhere and $K_f \in L^p_{\mathrm{loc}}$ for some $p > n - 1$. Then*

$$M_{1/K} \Lambda \leq Mf \Lambda.$$

Proof. We may assume that $K = K_f$ is globally p-integrable. By Hölder's inequality and the change of variables inequality

$$\int_U g(f(x)) J_f(x) \, \mathrm{d}x \leq \int_{fU} g(y) \, \mathrm{d}y, \tag{2.2}$$

valid for all $W^{1,1}$-homeomorphisms,

$$
\begin{aligned}
\int_{f^{-1}B(2)} |Df(x)|^q \, dx &= \int_{f^{-1}B(2)} K(x)^{q/n} J_f(x)^{q/n} \, dx \\
&\leq \left(\int_{f^{-1}B(2)} K(x)^p \, dx \right)^{(n-q)/n} \left(\int_{f^{-1}B(2)} J_f(x) \, dx \right)^{q/n} \quad (2.3) \\
&\leq \left(\int_{f^{-1}B(2)} K(x)^p \, dx \right)^{(n-q)/n} |B(2)|^{q/n},
\end{aligned}
$$

where $q = np/(p+1) > n - 1$ when $p > n - 1$.

We conclude that $f \in W^{1,q}(f^{-1}B(2), \mathbb{R}^n)$ for some $q > n - 1$, and therefore the $(n-1)$-dimensional change of variables formula

$$
\int_{U_t} |Df(x)|^{n-1} \rho(x) \, d\mathcal{H}^{n-1}(x) \geq \int_{fU_t} \rho(y) \, d\mathcal{H}^{n-1}(y)
$$

holds on almost every U_t, see [11]. The proof can now be carried out as the proof of Lemma 2.1. Notice that the n-dimensional change of variables formula is not needed here; inequality (2.2) suffices. $\qquad\square$

3. Distortion estimates

We first prove a version of the local-to-global principle for mappings with exponentially integrable distortion. As mentioned in the introduction, this geometric estimate does not by itself lead to results like Theorem 1.1, and it will not play a major role in the proof. It will be used in a qualitative way; for instance, it will used in the proof of Theorem 1.1 to show that we can assume the Fraenkel asymmetry to be as small as we wish. Similar estimates have been proved in [9].

Lemma 3.1. *Let f be as in Theorem 1.1. If $B(x, t) \subset B(3/2)$, then*

$$
\frac{L}{\ell(x, t)} = \frac{\max_{y \in S(3/2)} |f(x) - f(y)|}{\min_{y \in S(x,t)} |f(x) - f(y)|} \leq \exp\left(C(n, \mu, K) t^{-1/(n-3/2)} \right).
$$

Proof. We show that

$$
\frac{L}{L(x, t)} \leq \exp\left(C(n, \mu, K) t^{-1/(n-3/2)} \right), \tag{3.1}
$$

where $L(x, t) = \max_{y \in S(x,t)} |f(x) - f(y)|$. We choose a point $x_0 \in S(3/2)$ such that $|f(x_0) - f(x)| = L$, and

$$
I = f^{-1}(\{f(x_0) + T(f(x_0) - f(x)) : T \geq 0\}).
$$

Then $|f(z) - f(x)| \geq L$ for $z \in I$. Now a simple geometric argument shows that there exist $p \in B(2)$ and $t/4 \leq r \leq 3/2$ such that, for every $r < s < r + t/8$, the sphere $S(p, s)$ contains a spherical cap $Q(s) \subset B(2)$ such that

$$Q(s) \cap I \neq \emptyset \quad \text{and} \quad Q(s) \cap B(x, t) \neq \emptyset. \tag{3.2}$$

Let
$$g(z) = \max\{\min\{\log|f(z) - f(x)|, \log L\}, \log L(x, t)\},$$

and $E_t = \{z : L(x, t) \leq |f(z) - f(x)| \leq L\}$. Then, by (3.2) and the Sobolev embedding theorem on spheres,

$$\frac{t}{8}\left(\log \frac{L}{L(x, t)}\right)^{n-1/2} \leq C(n) \int_r^{r+t/8} \int_{Q(s) \cap E_t} |\nabla g(z)|^{n-1/2} \, d\mathcal{H}^{n-1}(z) \, ds$$

$$\leq C(n) \int_{E_t} \frac{|Df(z)|^{n-1/2}}{|f(z) - f(x)|^{n-1/2}} \, dz.$$

By Hölder's inequality and the distortion inequality $|Df|^n \leq K J_f$, the last integral is bounded from above by

$$\left(\int_{B(2)} K(z)^{2n-1} \, dz\right)^{1/(2n)} \left(\int_{E_t} \frac{J_f(z)}{|f(z) - f(x)|^n} \, dz\right)^{(n-1/2)/n}.$$

By Jensen's inequality and (1.4), the first integral is bounded by $C(n, \mu, K)$. Also, by change of variables, the second term is bounded by

$$\left(\log \frac{L}{L(x, t)}\right)^{(n-1/2)/n}.$$

Combining the estimates gives (3.1). We also have

$$\frac{L(x, t)}{\ell(x, t)} \leq \exp\left(C(n, \mu, K)t^{-1/(n-3/2)}\right). \tag{3.3}$$

Inequality (3.3) is proved in a similar way as (3.1), and we thus omit the proof. See [9, Theorem 3.6] for a more general result. The lemma follows by combining (3.1) and (3.3). □

The validity of inequality (1.5) will ultimately depend on the sharp local modulus of continuity of mappings with exponentially integrable distortion.

Theorem 3.2 ([12]). *Let f be as in Theorem 1.1. If x and $y \in B(5/4)$, then*

$$|f(x) - f(y)| \leq \frac{C(n, \mu, K)}{\log^{\mu/n} \frac{1}{|x-y|}} |f B(3/2)|^{1/n}.$$

Similarly, we use a sharp continuity estimate for the inverse mapping to prove Theorem 6.1. In the case $n = 2$ this was proved in [10], and the case $n \geq 3$ is proved similarly; we omit the proof.

Theorem 3.3. *Let f be as in Theorem* 6.1 *below, and x and y in $B(5/4)$. Then*

$$|f^{-1}(x) - f^{-1}(y)| \leq \frac{C(n, p, K, |f^{-1}B(3/2)|)}{\log^\alpha \frac{1}{|x-y|}},$$

where $\alpha = p(n-1)/n$.

4. Proof of Theorem 1.1

We now begin to prove Theorem 1.1. We may assume that $|f B^n| = \alpha_n$. We choose x_0 so that $\lambda(f B^n) = |f B^n \setminus B(x_0, 1)|$. Without loss of generality, $x_0 = 0$. Lemma 3.1 applied to the unit ball now shows that $\beta(f B^n) \leq C(n, \mu, K)$. Thus we may assume that

$$\lambda = \lambda(f B^n) < \epsilon = \epsilon(n, \mu, K), \tag{4.1}$$

where $\epsilon > 0$ is to be determined later. Let

$$R = \min\{s : f B^n \subset B(s)\}, \tag{4.2}$$

and

$$r = \max\{s : B(s) \subset f B^n\}. \tag{4.3}$$

Then $\beta(f B^n) \leq R/r - 1$. Theorem 1.1 now follows if we can bound both $R - 1$ and $1 - r$ in terms of $|f B^n \setminus B^n| = |B^n \setminus f B^n|$. We first give an estimate for $R - 1$.

4.1. Estimate for the outer radius

We choose $a \in S^{n-1}$ so that $|f(a)| = R$. Without loss of generality, $a = e_1$ and $f(e_1) = Re_1$.

Lemma 4.1. *There exists $\kappa' = \kappa'(n, \mu, K) > 0$ such that*

$$f^{-1}B(Re_1, \kappa') \subset B(e_1, 1/4).$$

Proof. Since $|f B^n| = \alpha_n$, we have $|f(e_1) - f(x)| \geq 1$ for some $x \in B(3/2)$. Thus by Lemma 3.1,

$$|f(x) - f(e_1)| = |f(x) - Re_1| \geq \kappa'(n, \mu, K)$$

for every $x \notin B(e_1, 1/4)$. $\square$

Now let

$$\kappa = \min\{R - 1, \kappa'\}, \quad \text{and} \quad U_t = f B^n \cap S(Re_1, t).$$

We may assume that κ is so small that $s < 1/10$ in Lemma 4.2 below.

Lemma 4.2. *There exists $C = C(n, \mu, K) > 0$ so that if $s = \exp(-C\kappa^{-n/\mu})$, then $f^{-1}U_t$ separates $B^n \setminus B(e_1, 1/4)$ and $B^n \cap B(e_1, s)$ in B^n for every $\kappa/2 < t < \kappa$.*

Proof. Let $\kappa/2 < t < \kappa$. From Lemma 3.1 it follows that

$$|f B(3/2)| \leq C(n, \mu, K)|f B^n| = C(n, \mu, K)\alpha_n.$$

Combining this with Theorem 3.2 shows that $|x - e_1| \geq s$ whenever $x \in f^{-1}U_t$. Lemma 4.1 then shows that

$$s \leq |x - e_1| \leq 1/4 \tag{4.4}$$

for every $x \in f^{-1}U_t$. Since U_t separates $B(Re_1, \kappa/2)$ and any point $y \in f B^n \setminus B(Re_1, t)$ in $f B^n$, the lemma follows by (4.4). $\qquad\square$

We now estimate the conformal moduli of the family $\{U_t\}$ and its preimage. We first show that the conformal modulus of $\{U_t\}$ gets larger the more cusplike $f B^n$ is.

Lemma 4.3. *Let κ and U_t be as in Lemma 4.2, and*

$$\Lambda = \{U_t : \kappa/2 < t < \kappa\}.$$

Then

$$M\Lambda \geq \frac{\kappa^{n/(n-1)}}{2^{n/(n-1)}\lambda^{1/(n-1)}}.$$

Proof. Let $\rho \in Y$, see Lemma 2.1. Now

$$\kappa/2 \leq \int_{\kappa/2}^{\kappa} \int_{U_t} \rho(z) \, d\mathcal{H}^{n-1}(z) \, dt \leq \int_{f B^n \setminus B^n} \rho(y) \, dy$$

$$\leq |f B^n \setminus B^n|^{1/n} \left(\int_{f B^n \setminus B^n} \rho(y)^{n/(n-1)} \, dy \right)^{(n-1)/n}$$

by Hölder's inequality. The lemma follows, since $\lambda = |f B^n \setminus B^n|$. $\qquad\square$

Lemma 4.2 now shows that the preimages of the sets U_t lie on on annulus whose fatness is controlled. This together with the exponential integrability shows that the corresponding (weighted) conformal modulus on the domain is not very large, depending on κ.

Lemma 4.4. *Let Λ be as in Lemma 4.3. Then*

$$M_K f^{-1} \Lambda \leq C(n, \mu, K) \kappa^{-n^2/((n-1)\mu)}.$$

Proof. By the separation property in Lemma 4.2 and a simple calculation,

$$\rho(x) = 10^n |x - e_1|^{1-n} \chi_{B^n \setminus B(e_1, s)}(x)$$

satisfies $\rho \in X$, where s is as in Lemma 4.2. We may assume that $\log \frac{1}{s}$ is an integer. Then

$$M_K f^{-1} \Lambda \leq \int_{\mathbb{R}^n} \rho(x)^{n/(n-1)} K(x)^{1/(n-1)} \, dx$$

$$\leq C(n) \sum_{j=0}^{\log \frac{1}{s}} |B_j|^{-1} \int_{B_j} K(x)^{1/(n-1)} \, dx,$$

where $B_j = B(e_1, \exp(-j))$. Since the function $t \mapsto \exp(\mu t^{n-1})$ is convex, we can use Jensen's inequality as follows:

$$|B_j|^{-1} \int_{B_j} K(x)^{\frac{1}{n-1}} \, dx \leq \mu^{\frac{-1}{n-1}} \log^{\frac{1}{n-1}} \left(|B_j|^{-1} \int_{B_j} \exp(\mu K(x)) \, dx \right)$$

$$\leq \mu^{\frac{-1}{n-1}} \log^{\frac{1}{n-1}} (\alpha_n^{-1} \exp(nj) K) \leq C(n, \mu, K) j^{\frac{1}{n-1}}.$$

Thus

$$M_K f^{-1} \Lambda \leq C(n, \mu, K) \sum_{j=0}^{\log \frac{1}{s}} j^{\frac{1}{n-1}} \leq C(n, \mu, K) \log^{\frac{n}{n-1}} \frac{1}{s}$$

$$= C(n, \mu, K) \kappa^{\frac{-n^2}{(n-1)\mu}}. \qquad \square$$

Combining Lemmas 2.1, 4.3 and 4.4 yields

$$\kappa^{n+n^2/\mu} \leq C(n, \mu, K) \lambda.$$

When we assume that ϵ in (4.1) is small enough depending on n, μ and K, we have $\kappa = R - 1$. Then

$$(R - 1)^{n+n^2/\mu} \leq C(n, \mu, K) \lambda, \tag{4.5}$$

as desired.

Remark 4.5. In the case of quasiconformal homeomorphisms, it is not difficult to prove (4.5), with exponent n on the left, using the local-to-global principle. Indeed,

let f be as above, but K-quasiconformal. Then, for every $x \in B(3/2)$ and $0 < r < 1/4$,

$$\frac{L(x,r)}{\ell(x,r)} =: \frac{\max_{y \in S(x,r)} |f(y) - f(x)|}{\min_{y \in S(x,r)} |f(y) - f(x)|} \leq H, \tag{4.6}$$

where H depends only on n and K, $cf.$ [16]. By Lemma 4.1, $|Re_1 - f(0)| > \kappa$, with the above notation. Therefore we can choose $0 < t < 1$ such that $f B(te_1, 1 - t) \subset f B^n \setminus B^n$ and diam $f B(te_1, 1 - t) = \kappa$. Then, by (4.6), $f B(te_1, 1 - t)$ and, consequently, $f B^n \setminus B^n$, contains a ball of radius κ/H. Thus, when λ is assumed to be small enough depending on n and K, $(R - 1)^n \leq C(n, K)\lambda$. Using similar arguments for the inner radius, one can prove (1.3).

4.2. Estimate for the inner radius

We now give an estimate for $1 - r$. Notice that we could have $r = 0$ in general, $i.e.$ $0 \notin f B^n$. However, we will soon see that this does not happen when λ is small enough. We denote

$$a' = \inf\{|f(x)| : x \in B(2) \setminus B(3/2)\}.$$

Lemma 4.6. *We have*

$$1 - a' \leq C(n, \mu, K)\lambda^{1/n}.$$

Proof. We may assume that $a' < 1$. Let $\eta = \min\{1 - a', 1/2\}$. Since $\max\limits_{x \in \overline{B}(3/2)} |f(x)| > 1$, for every $1 - \eta < t < 1$ there exists

$$p(t) \in S(t) \cap f(B(2) \setminus B(3/2)).$$

We may assume that $\lambda < \epsilon(n)$, so that $S(t) \cap \overline{f B^n} \neq \emptyset$ for every $1 - \eta < t < 1$. We choose a point

$$q(t) \in S(t) \cap \overline{f B^n}$$

so that

$$s(t) = |p(t) - q(t)| = \min\{|p(t) - y| : y \in S(t) \cap \overline{f B^n}\}.$$

Since f^{-1} belongs to $W^{1,n}$ by [7], also the restriction of f^{-1} to $S(t) \cap f B(2)$ belongs to $W^{1,n}$ for almost every t. We denote

$$Q(t) = S(t) \cap B(p(t), s(t)).$$

Then, by the Sobolev embedding theorem in $Q(t)$,

$$\begin{aligned} \frac{1}{2} &\leq |f^{-1}(p(t)) - f^{-1}(q(t))| \\ &\leq C(n)s(t)^{1/n} \left(\int_{Q(t)} |\nabla|f^{-1}|(y)|^n \, d\mathcal{H}^{n-1}(y) \right)^{1/n} \tag{4.7} \\ &\leq C(n)s(t)^{1/n} \left(\int_{Q(t)} K_{f^{-1}}(y) J_{f^{-1}}(y) \, d\mathcal{H}^{n-1}(y) \right)^{1/n} \end{aligned}$$

for almost every t, $1 - \eta < t < 1$. Since $K_{f^{-1}}(y) \leq K(f^{-1}(y))^{n-1}$ almost everywhere, see [8], (4.7) yields

$$C(n)^{-1} s(t)^{-1} \leq \int_{Q(t)} K(f^{-1}(y))^{n-1} J_{f^{-1}}(y) \, d\mathcal{H}^{n-1}(y).$$

By integrating over t, using Fubini's theorem and changing variables (the change of variables formula holds under our assumptions, see [8]), we have

$$\int_{1-\eta}^{1} \frac{dt}{s(t)} \leq C(n) \int_{B(2)} K(x)^{n-1} \, dx \leq C(n, \mu, K). \tag{4.8}$$

Here the last inequality follows from Jensen's inequality and our assumption on K. By Hölder's inequality, and (4.8),

$$
\begin{aligned}
\eta &= \int_{1-\eta}^{1} \frac{s(t)^{(n-1)/n}}{s(t)^{(n-1)/n}} \, dt \\
&\leq \left(\int_{1-\eta}^{1} s(t)^{n-1} \, dt \right)^{1/n} \left(\int_{1-\eta}^{1} \frac{dt}{s(t)} \right)^{(n-1)/n} \\
&\leq C(n, \mu, K) \left(\int_{1-\eta}^{1} \mathcal{H}^{n-1}(S(t) \cap B^n \setminus f B^n) \, dt \right)^{1/n} \tag{4.9} \\
&\leq C(n, \mu, K) \lambda^{1/n}.
\end{aligned}
$$

Thus, when λ is small enough depending on n, μ and K, $\eta = 1 - a'$ and the lemma follows from (4.9). $\qquad\square$

Now we continue in a similar way as in the proof of (4.5). We claim that

$$(1 - r)^{n + n^2/\mu} \leq C(n, \mu, K) \lambda. \tag{4.10}$$

We may assume that $f(e_1) = \min\{|f(x)| : x \in S(1)\} =: r'$. Notice that if $r' < 1$ is close to one and if λ is small, then $r' = r$. The argument given below will show that r' is indeed close to one. Hence we will from now on abuse notation and denote r' by r. By Lemma 4.6, we may assume that $a' \geq (1 + r)/2$. Let $b = \min\{1, a'\}$. Then, if we denote

$$r_0 = r + (1 - r)/4,$$

$b - r_0 \geq (1 - r)/4$. Let

$$W_t = S(t) \cap f(B(2) \setminus B^n).$$

Then

$$f^{-1} W_t \subset B(3/2) \setminus B(e_1, s)$$

for every $r_0 < t < b$ by Theorem 3.2 and Lemma 4.6, where

$$s = \exp(-C(n, \mu, K)(1 - r)^{-n/\mu}).$$

Therefore, $f^{-1}W_t$ separates $B(e_1, s) \setminus B^n$ and $S(3/2)$ in $B(3/2) \setminus B^n$ for every such t. We denote $\Lambda = \{W_t : r_0 < t < b\}$. Then Lemma 2.1 gives

$$M\Lambda \le M_K f^{-1}\Lambda. \tag{4.11}$$

We estimate $M\Lambda$ from below as follows: if $\rho \in Y$, then

$$\frac{1 - r}{4} \le \int_{r_0}^{b} \int_{W_t} \rho(z) \, d\mathcal{H}^{n-1}(z) \, dt \le \int_{B^n \setminus f B^n} \rho(y) \, dy$$

$$\le \lambda^{1/n} \left(\int_{\mathbb{R}^n} \rho(y)^{n/(n-1)} \right)^{(n-1)/n},$$

and so

$$M\Lambda \ge \frac{(1 - r)^{n/(n-1)}}{4^{n/(n-1)} \lambda^{1/(n-1)}}. \tag{4.12}$$

In order to give an upper bound for $M_K f^{-1}\Lambda$, we notice that the separation property mentioned above implies that the function $\rho : B(2) \to [0, \infty)$,

$$\rho(x) = 100^n |x - e_1|^{1-n} \chi_{B(3/2) \setminus B(e_1, s)}(x)$$

belongs to the test function space X. By calculating as in Lemma 4.4, we see that

$$M_K f^{-1}\Lambda \le C(n, \mu, K)(1 - r)^{-n^2/((n-1)\mu)}. \tag{4.13}$$

Combining (4.11), (4.12) and (4.13) then yields (4.10). Since $r \ge 1/2$ for small enough λ, Theorem 1.1 follows by combining (4.5) and (4.10).

5. Proof of Theorem 1.2

The example we construct below maps the unit ball onto the union of a ball and a narrow cone. The shape of the cone will then determine the metric distortion and the Fraenkel asymmetry of the image set. The narrowness of the cone is controlled by the exponential integrability requirement of the map. In order to get a sequence of maps we take a sequence of such cones whose diameters tend to zero. For technical convenience we will work on half-spaces instead of balls and then use compositions with Möbius transformations that map half-spaces onto balls.

We fix $\mu > 0$ and a small $a > 0$, and denote $\epsilon = a^{\epsilon_1(n)/\mu} \ll 1$, where $\epsilon_1(n)$ will be determined later. The main part of the proof will be the construction of a

Lipschitz continuous homeomorphism $g : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ with the following properties. Denote $\mathbb{H}_a = \{x_1 \leq -a\}$, and by V the truncated cone

$$\{-a \leq x_1 \leq -|x| \cos \epsilon\}.$$

Then we require that

$$g(\mathbb{H}_0) = \mathbb{H}_a \cup V \quad \text{and} \quad g(0) = 0. \tag{5.1}$$

Also, we require that

$$\int_{B^n} \exp(\mu K_g(x)) \, \mathrm{d}x \leq C(\mu, n), \tag{5.2}$$

and that g is $K(n)$-quasiconformal in $\overline{R}^n \setminus \overline{B}^n$.

Suppose for the moment that such a g exists. Denote $\tau_a(x) = x + ae_1$, and let $\mathcal{M} : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be a Möbius transformation that maps B^n onto $\mathbb{H}_0$ and e_1 to 0. Then consider

$$f = \mathcal{M}^{-1} \circ \tau_a \circ g \circ \mathcal{M}.$$

Then, since $\mathcal{M}_{|B(e_1, 1/2)}$ is bi-Lipschitz continuous, f maps B^n onto the union of B^n and a bi-Lipschitz image of V when a is small enough. Thus

$$\lambda(f B^n) \leq C(n) a^n \epsilon^{n-1} = C(n) a^{n+(n-1)\epsilon_1(n)/\mu} \leq C(n) \beta(f B^n)^{n+(n-1)\epsilon_1(n)/\mu},$$

which is the desired estimate (when $a \to 0$). Also, assuming (5.2), we can use the conformality and the local bi-Lipschitz property of $\mathcal{M}$ to show that

$$\int_{B(2)} \exp(\mu K_f(x)) \, \mathrm{d}x \leq C(\mu, n).$$

We conclude that in order to prove Theorem 1.2 it suffices to construct, for any given small $a > 0$, a homeomorphism g as above.

Let $\mu_0 = C_0(n)\mu$, where $C_0(n)$ is determined later. We will consider the case $n \geq 3$; the case $n = 2$ is an easy modification. Let $x = (r, \varphi, \phi)$, where $r = |x|$ and $0 \leq \varphi \leq \pi$ is the angle $\angle(x, 0, e_1)$. Also, $\phi \in S^{n-2}$, $\phi = \hat{x}/|\hat{x}|$ when $\hat{x} \neq 0$, where $\hat{x} = (x_2, \ldots, x_n)$. Then the map g is of the form

$$g(r, \varphi, \phi) = (g_r, \eta(r, \varphi), \phi)$$

when $\hat{x} \neq 0$, $g(0) = 0$, and $g(x) = g_r x_1/|x_1|$ otherwise. Here

$$g_r = \begin{cases} r \exp(1) & \exp(-1) \leq r \leq \infty, \\ \log^{-\mu_0} \frac{1}{r}, & \exp(-(2a)^{-1/\mu_0}) \leq r \leq \exp(-1), \\ 2a \exp((2a)^{-1/\mu_0}) r & 0 \leq r \leq \exp(-(2a)^{-1/\mu_0}). \end{cases}$$

In order to define η, we first set

$$\eta_0 = \begin{cases} \pi - \arccos\left(\dfrac{a}{g_r}\right), & r \geq \exp\left(-\left(\dfrac{a}{\cos\epsilon}\right)^{-1/\mu_0}\right), \\ \pi - \epsilon & \text{otherwise.} \end{cases}$$

Then

$$\eta = \begin{cases} \dfrac{2\eta_0\varphi}{\pi}, & 0 \leq \varphi \leq \pi/2, \\ (2 - 2\eta_0/\pi)\varphi - \pi + 2\eta_0, & \pi/2 \leq \varphi \leq \pi. \end{cases}$$

Now g is a Lipschitz homeomorphism and satisfies (5.1). We next estimate K_g. The Jacobian determinant J_g is given by

$$J_g = \partial_r g_r \cdot \frac{g_r \partial_\varphi \eta}{r} \cdot \left(\frac{g_r \sin\eta}{r \sin\varphi}\right)^{n-2},$$

and

$$|Dg| \leq C(n) \max\left\{\partial_r g_r, \frac{g_r \partial_\varphi \eta}{r}, \frac{g_r \sin\eta}{r \sin\varphi}\right\}.$$

Thus g is $K(n)$-quasiconformal when $r \geq \exp(-1)$. Let A_1 be the set where $g_r = \log^{-\mu_0}\frac{1}{r}$. Then

$$\partial_r g_r = \mu_0 \log^{-1}\frac{1}{r} \cdot \frac{g_r}{r},$$

$$|Dg| \leq C(n)\frac{g_r}{r}$$

in A_1. Also, since $\eta_0 \leq 2\pi/3$ in A_1,

$$\partial_\varphi \eta \left(\frac{\sin\eta}{\sin\varphi}\right)^{n-2} \geq C(n)^{-1}.$$

Therefore,

$$K_g \leq \max\left\{C(n), \frac{C(n)}{\mu_0}\log\frac{1}{r}\right\},$$

and

$$\int_{A_1} \exp(\mu K_g(x))\, dx \leq 100^{n+\mu} + \int_{A_1} |x|^{-\alpha}\, dx,$$

where $\alpha = C(n)/C_0(n)$. Thus, when C_0 is chosen to be large enough so that $\alpha \leq 1$, the integral is bounded by $200^{n+\mu}$.

Let A_2 be the set where $g_r = 2a \exp((2a)^{-1/\mu_0})r$. Then

$$|Dg| \leq \frac{C(n)g_r}{r}, \quad \partial_r g_r = \frac{g_r}{r}, \quad \text{and} \quad \partial_\varphi \eta \left(\frac{\sin\eta}{\sin\varphi}\right)^{n-2} \geq C(n)^{-1}\epsilon^{n-1}$$

in A_2. Therefore,
$$K_g \leq C(n)\epsilon^{1-n},$$
and
$$\int_{A_2} \exp(\mu K_g(x)) \, dx \leq \exp\left(C(n)\mu\epsilon^{1-n}\right)|A_2|$$
$$\leq \exp\left(C(n)\mu\epsilon^{1-n} - C(n,\mu)a^{-1/\mu_0}\right).$$

If we now choose $\epsilon_1(n)$ to be small enough depending on $C_0(n)$, then the integral is smaller than 1 for small a. By combining the estimates we see that (5.2) holds when a is small. The proof is complete.

6. Theorem 1.1 for inverse images

In this section we show that, when a suitable polynomial integrability condition on K is assumed, an estimate similar to Theorem 1.1 holds for the inverse of a ball under f.

Theorem 6.1. *Let* $f : f^{-1}B(2) \to B(2)$ *be a* $W^{1,1}$-*homeomorphism so that* $J_f \geq 0$ *almost everywhere, and*
$$\int_{f^{-1}B(2)} K(x)^p \, dx \leq K|f^{-1}B^n| \tag{6.1}$$

for some $K > 0$ *and* $p > n-1$. *Then there exists* $\epsilon > 0$, *depending only on n, such that*
$$\beta(f^{-1}B^n)^{n+n^2/(p-n+1)} \leq C(n,p,K)\lambda(f^{-1}B^n)$$

whenever
$$\lambda(f^{-1}B^n) < \epsilon. \tag{6.2}$$

We do not know if assumption (6.2) is really needed. Recall that in Theorem 1.1 such assumption is not needed thanks to Lemma 3.1.

The proof of Theorem 6.1 is similar to the proof of Theorem 1.1. Therefore, we will leave out some details to avoid unnecessary repetition.

We may assume that $|f^{-1}B^n| = \alpha_n$ and that
$$\lambda = \lambda(f^{-1}B^n) = |f^{-1}(B^n) \setminus B^n|.$$

Let
$$R = \min\{s : f^{-1}B^n \subset B(s)\},$$
and
$$r = \max\{s : B(s) \subset f^{-1}B^n\}.$$

We first claim that

$$(R-1)^{n+n^2/(p-n+1)} \le C(n, p, K)\lambda. \tag{6.3}$$

We may assume that $Re_1 = f^{-1}(e_1)$. We choose U_t and Λ in (2.1) so that $x = 0$, $G = f^{-1}B^n$, and $E = (1, 1+(R-1)/2)$.

We first establish the existence of a continuum with fixed diameter. This auxiliary result is later needed in to guarantee efficient modulus estimates.

Lemma 6.2. *There exist $\kappa = \kappa(n, p, K) > 0$ and a continuum γ in $B^n \cap f^{-1}B^n$ so that* $\operatorname{diam} f\gamma \ge \kappa$.

Proof. If λ is small enough, then there exists $p \in B(1/2) \cap f^{-1}B^n$. Consequently, p and $S(1) \cap f^{-1}B^n$ can be connected in $\overline{B^n} \cap f^{-1}B^n$ by a continuum γ. Since $\operatorname{diam} \gamma \ge 1/2$, $\operatorname{diam} f\gamma \ge \kappa(n, p, K)$ by Theorem 3.3. $\square$

Next we use the continuity estimate 3.3 for the inverse of f in order to guarantee that the image of the family $\{U_t\}$ lies on annulus whose fatness is controlled by the data. This will later lead to an upper bound for the corresponding conformal modulus.

Lemma 6.3. *There exists $C = C(n, p, K) > 0$ so that if*

$$s = \exp(-C(R-1)^{-n/((n-1)p)}),$$

then fU_t separates $f\gamma$ and $F = B^n \cap B(e_1, s)$ in B^n for every $1 < t < 1 + (R-1)/2$.

Proof. Apply Theorem 3.3 as in the proof of Lemma 4.2. $\square$

Now we show that the weighted conformal modulus detects the cusplike behavior of $f^{-1}B^n$ in a suitable manner when the distortion is p-integrable.

Lemma 6.4. *We have*

$$C(n, p, K)M_{1/K}\Lambda \ge (R-1)^{n/(n-1)}\lambda^{(n-1-p)/(p(n-1))}.$$

Proof. Let $\rho \in X$, where X is the test function space in Section 2. Then, by polar coordinates and Hölder's inequality,

$$
\begin{aligned}
\frac{R-1}{2} &\le \int_{f^{-1}(B^n) \setminus B^n} \rho(x) K(x)^{-(n-1)/n} K(x)^{(n-1)/n} \, dx \\
&\le \left(\int_{f^{-1}B(2)} \rho(x)^{n/(n-1)} K(x)^{-1} \, dx \right)^{(n-1)/n} \\
&\quad \times \left(\int_{f^{-1}B(2)} K(x)^p \, dx \right)^{(n-1)/(np)} |f^{-1}(B^n) \setminus B^n|^{\tau},
\end{aligned}
$$

where $\tau = (p-n+1)/(np)$. The lemma follows. $\square$

Next we prove an upper bound for the modulus on the image side. Instead of giving the full calculation, which is a bit technical, we refer to some basic properties of surface and path families. In particular, we use the Loewner estimate which gives a lower bound for the conformal modulus of paths joining two disjoint continua in terms of the diameters and distances of the continua, see [4].

Lemma 6.5. *We have*

$$Mf\Lambda \leq C(n, p, K) \min\{1, (R-1)^{-n/((n-1)p)}\}.$$

Proof. We use the duality of the modulus $M\Lambda$ of separating surfaces and conformal capacity, and apply a classical estimate for conformal capacity. Namely, by Lemma 6.3 and [17], and the so-called Loewner property of the unit ball (*cf.* [4]), we have

$$Mf\Lambda \leq C(n) \log \frac{1}{s} \tag{6.4}$$

when $s < \kappa$ (s is as in Lemma 6.3 and κ as in Lemma 6.2), and

$$Mf\Lambda \leq C(n, p, K)$$

otherwise. The lemma follows. $\qquad\square$

The estimate (6.3) now follows by combining Lemmas 2.2, 6.4 and 6.5. Now we give an estimate for $1 - r$. We claim that

$$(1-r)^{n+n^2/(p-n+1)} \leq C(n, p, K)\lambda. \tag{6.5}$$

Again, we need to assume that λ is small in order to guarantee that $0 \in f^{-1}B^n$. We denote

$$a = \inf\{|f^{-1}(x)| : x \in B(2) \setminus B(3/2)\}.$$

Lemma 6.6. *We have*

$$1 - a \leq C(n, p, K)\lambda^{(p-n+1)/(np)}.$$

Proof. We may assume that $a < 1$. Let $\eta = \min\{1-a, 1/2\}$ and $s(t) = \mathcal{H}^{n-1}(S(t) \setminus f^{-1}B^n)$. As in the proof of Lemma 4.6, we apply the Sobolev embedding theorem on spheres to conclude that

$$1 \leq C(n)s(t)^{q-n+1} \int_{S(t) \cap f^{-1}B(2)} |\nabla|f|(x)|^q \, d\mathcal{H}^{n-1}(x)$$

for almost every $1 - \eta \leq t \leq 1$, where $q = np/(p+1)$. Therefore, by integration with respect to t, and (2.3),

$$\int_{1-\eta}^1 s(t)^{n-1-q} \, dt \leq C(n, p, K).$$

By Hölder's inequality,

$$\eta \leq \left(\int_{1-\eta}^{1} s(t)^{n-1-q} \, dt \right)^{1/\tau} \left(\int_{1-\eta}^{1} s(t)^{n-1} \, dt \right)^{(\tau-1)/\tau}$$
$$\leq C(n, p, K)\lambda^{(\tau-1)/\tau},$$

where $\tau = q/(n-1)$. The lemma follows. $\qquad\qquad\qquad\square$

We now prove (6.5). Notice that

$$\frac{np}{p-n+1} \leq n + \frac{n^2}{p-n+1}.$$

Thus we may assume that $a \geq (1+r)/2$ by Lemma 6.6. Also, as in the proof of (4.10), we may assume that

$$r = \min\{|f^{-1}(y)| : y \in S(1)\}.$$

Let $b = \min\{1, a\}$ and $r_0 = r + (1-r)/4$. Moreover, let

$$\Lambda = \{W_t : r_0 < t < b\},$$

where

$$W_t = S(t) \cap f^{-1}(B(2) \setminus B^n).$$

Then, as in the proofs of (4.10) and (6.3), we have

$$C(n, p, K)M_{1/K}\Lambda \geq (1-r)^{n/(n-1)}\lambda^{(n-1-p)/(p(n-1))},$$

and

$$Mf\Lambda \leq C(n, p, K)(1-r)^{-n/(p(n-1))}.$$

Combining these estimates with Lemma 2.2 gives (6.5). The proof is complete.

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