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Multiple constant sign and nodal solutions for nonlinear Neumann eigenvalue problems

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Abstract. We consider a nonlinear Neumann eigenvalue problem driven by a possibly nonhomogeneous differential operator which incorporates as a special case the p-Laplacian. We assume that the right-hand side nonlinearity is $(p-1)$-superlinear, but need not satisfy the Ambrosetti-Rabinowitz condition or to be monotone. We show that, for all values of the parameter λ in an upper half line, the problem has two positive and two negative solutions. Subsequently, for the case of the p-Laplacian, we also produce a nodal solution. Finally, for the semilinear case we show that the problem has two nodal solutions.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with a C^2-boundary $\partial\Omega$. We study the following nonlinear eigenvalue Neumann problem:

$$
(\mathrm{P}_\lambda) \qquad \begin{cases} -\operatorname{div} a(x, \nabla u(x)) + \lambda |u(x)|^{p-2} u(x) = f(x, u(x)) & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}
$$

where $n(x)$ denotes the outward unit normal at $x \in \partial\Omega$ and $\frac{\partial u}{\partial n}$ stands for the normal derivative of u on $\partial\Omega$. Here $a : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ is a map which is strictly monotone in the second variable and satisfies certain regularity conditions (see hypotheses H(a) in Section 3). The p-Laplacian, with any $p > 1$, is a particular case of the differential operator $\operatorname{div} a(x, \nabla(\cdot)(x))$, but generally this operator need not be homogeneous, which is a source of difficulties in the application of minimax

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methods. Different such nonlinear operators in divergence form are discussed in Example 3.4. Our hypotheses on the map $a(x, y)$ make the normal derivative the right one to be considered for the Neumann boundary condition of problem (P_λ) (see the nonlinear Green's identity in [11]). In fact, nonlinear regularity theory allows to interpret the boundary condition in a pointwise sense. In the formulation of problem (P_λ), the nonlinearity $f(x, u)$ is a Carathéodory function and $\lambda \in \mathbb{R}$ is a parameter. Under our hypotheses, the nonlinear term $f(x, u)$ is $(p-1)$-superlinear, but it is not required to satisfy the Ambrosetti-Rabinowitz condition or to be monotone.

Eigenvalue problems for nonlinear elliptic boundary value problems were investigated primarily in the context of Dirichlet equations driven by the p-Laplacian differential operator. In this respect, we mention the works [3,16,17,19,20], where the authors consider problems which involve the combined effects of $(p-1)$-superlinear and $(p-1)$-sublinear terms, so problems with convex and concave nonlinearities if $p = 2$, and prove the existence of a pair of positive solutions. A multiplicity result for nonlinear eigenvalue Dirichlet problems guaranteeing three nontrivial solutions, two of which having opposite constant sign, was proven by the authors [26]. Other results regarding multiple and positive solutions for Dirichlet eigenvalue problems can be found in [28]. Nodal solutions were obtained for Dirichlet problems with the p-Laplacian in the works [1, 6, 7, 9, 10, 27, 36, 37]. To the best of our knowledge, no such results exist for the Neumann problems. Only recently the authors [29] examined a class of equations with the Neumann p-Laplacian, and under conditions of near resonance proved multiplicity results, but without providing information about the sign of solutions.

In the present paper, we first obtain in Theorem 4.11 multiple solutions of constant sign for problem (P_λ) when the parameter $\lambda > 0$ is in an upper half-line. Specifically, we show that, under our hypotheses and for $\lambda > 0$ sufficiently large, problem (P_λ) possesses two positive solutions and two negative solutions. This theorem is based on two auxiliary results that are of independent interest. The first one, stated as Proposition 3.5, sets forth that the nonlinear operator associated in an appropriate Sobolev setting to the principal part $-\mathrm{div}\, a(x, \nabla u(x))$ of the equation in (P_λ) is maximal monotone, strictly monotone and satisfies the $(S)_+$-property. The second basic auxiliary result, which is formulated as Proposition 3.6, points out the relationship between $W^{1,p}$- and C^1- local minimizers for functionals related to the variational structure of Neumann problem (P_λ). Then, in the case where the differential operator is the p-Laplacian, we produce in Theorem 5.5 a nodal (sign changing) solution of (P_λ), whenever $\lambda > 0$ is large enough, in addition to the four constant sign solutions already found. The argument leading to the existence of the sign changing solution strongly relies on Proposition 5.4 ensuring the existence of extremal solutions provided $\lambda > 0$ is sufficiently large, namely, a smallest positive solution and a biggest negative solution. Finally, in the semilinear case (*i.e.*, $p = 2$), we generate an additional nodal solution. The existence of the second nodal solution is established by combining the minimax methods with Morse theoretic techniques.

The rest of the paper is organized as follows. Section 2 contains mathematical background which is needed in the sequel. Section 3 presents some auxiliary results. Section 4 is devoted to constant sign solutions. Section 5 focuses on the existence of a sign changing solution. Section 6 studies the existence of a second sign changing solution in the semilinear case.

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2. Mathematical background

In this section we briefly present the related basic definitions and facts which we will use in the sequel.

Let X be a Banach space and let X^* be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Let $\varphi \in C^1(X)$. We say that φ satisfies the *Cerami condition at level* $c \in \mathbb{R}$ (the $(\mathrm{C})_c$-condition, for short), if every sequence $\{x_k\}_{k\ge1} \subset X$ such that $\varphi(x_k) \to c$ and $(1 + \|x_k\|)\varphi'(x_k) \to 0$ in X^* as $k \to \infty$ has a strongly convergent subsequence. It is said that φ satisfies the *Cerami condition* (the (C)-condition, for short) if it verifies the $(\mathrm{C})_c$-condition at every level $c \in \mathbb{R}$. As shown in [4], the deformation theorem and consequently the minimax theory of a function $\varphi \in C^1(X)$ holds if we employ the (C)-condition.

For later use, we recall the *mountain pass theorem*, which provides a minimax characterization for certain critical values of a C^1-functional.

Theorem 2.1. *If $\varphi \in C^1(X)$, $x_0, x_1 \in X$ and $r > 0$ satisfy*

$$\max\{\varphi(x_0), \varphi(x_1)\} \le \inf\{\varphi(x) : \ \|x - x_0\| = r\} =: \xi_r\,, \quad \|x_0 - x_1\| > r,$$

and φ fulfills the $(\mathrm{C})_c$-condition, where

$$c := \inf_{\gamma_0\in\Gamma_0} \max_{-1\le t\le 1} \varphi(\gamma_0(t)),$$
$$\Gamma_0 := \{\gamma_0 \in C([-1, 1], X) : \ \gamma_0(-1) = x_0,\ \gamma_0(1) = x_1\},$$

then $c \ge \xi_r$ and c is a critical value of φ. Moreover, if $c = \xi_r$, there exists a critical point x of φ such that $\varphi(x) = c$ and $\|x - x_0\| = r$.

The following notion will help us verify the (C)-condition.

Definition 2.2. Let X be a reflexive Banach space and $V : X \to X^*$. We say that V is of type $(S)_+$ if for every sequence $\{x_k\}_{k\ge1} \subset X$ such that $x_k \overset{\mathrm{w}}{\to} x$ and

$$\limsup_{k\to\infty} \langle V(x_k), x_k - x\rangle \le 0,$$

one has that $x_k \to x$ in X.

For every $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we use the notation: $\varphi^c = \{x \in X : \varphi(x) \leq c\}$ and $K = \{x \in X : \varphi'(x) = 0\}$. The critical groups of $\varphi \in C^1(X)$ at an isolated critical point $x_0 \in X$ with $\varphi(x_0) = c$ are defined by

$$C_k(\varphi, x_0) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{x_0\}) \text{ for every } k \geq 0,$$

where U is a neighborhood of x_0 such that $K \cap \varphi^c \cap U = \{x_0\}$ and $H_k(V, W)$ denotes the kth singular homology group with coefficients in $\mathbb{Z}$ for the topological pair (V, W) (see, *e.g.*, [12, 23]). The definition of critical groups is independent of U.

Suppose that φ satisfies the (C)-condition and that $\inf \varphi(K) > -\infty$. Fix $c < \inf \varphi(K)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \text{ for all } k \geq 0$$

(see [5]). The deformation theorem implies that the definition of critical groups of φ at infinity is independent of the choice of $c < \inf \varphi(K)$.

If K is finite, the Morse-type numbers of φ are defined by

$$M_k = \sum_{x \in K} \operatorname{rank} C_k(\varphi, x) \text{ for all } k \geq 0.$$

The Betti-type numbers of φ are introduced by

$$\beta_k = \operatorname{rank} C_k(\varphi, \infty) \text{ for all } k \geq 0.$$

The Poincaré-Hopf formula holds

$$\sum_{k \geq 0} (-1)^k M_k = \sum_{k \geq 0} (-1)^k \beta_k \tag{2.1}$$

if all M_k, β_k are finite and the series converge.

In the analysis of problem (P_λ), we will use the spaces

$$C_n^1(\overline{\Omega}) = \left\{ u \in C^1(\overline{\Omega}) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}$$

and $W_n^{1,p}(\Omega)$ defined as the closure of $C_n^1(\overline{\Omega})$ in the norm $\|\cdot\|$ of $W^{1,p}(\Omega)$ (see also [29, 31]). Actually, we have $W^{1,p}(\Omega) = W_n^{1,p}(\Omega)$. As it is usual, we write $H_n^1(\Omega) = W_n^{1,2}(\Omega)$. The Banach space $C_n^1(\overline{\Omega})$ is an ordered Banach space with the positive cone

$$C_+ = \{u \in C_n^1(\overline{\Omega}) : u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \{u \in C_+ : u(x) > 0 \text{ for all } x \in \overline{\Omega}\}.$$

Finally, we recall a few basic facts about the spectrum of the negative Neumann p-Laplacian, with $1 < p < \infty$, defined by

$$\langle -\Delta_p u, v\rangle = \int_\Omega \|\nabla u\|^{p-2}(\nabla u, \nabla v)_{\mathbb{R}^N}\, dx \quad \text{for all } u, v \in W_n^{1,p}(\Omega).$$

Consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_p u(x) = \lambda |u(x)|^{p-2}u(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

A number $\lambda \in \mathbb{R}$ is an eigenvalue of $(-\Delta_p, W_n^{1,p}(\Omega))$ if problem (2.2) has a nontrivial solution. An eigenvalue satisfies $\lambda \geq 0$. Note that $\lambda_0 = 0$ is an eigenvalue with the corresponding eigenspace $\mathbb{R}$ and it is isolated. By virtue of the Ljusternik–Schnirelmann theory, we have a whole strictly increasing sequence of eigenvalues $\{\lambda_k\}_{k\geq 0}$, $\lambda_k \to +\infty$ as $k \to \infty$. If $p = 2$, these are all the eigenvalues of $(-\Delta, H_n^1(\Omega))$.

In what follows, we will use the notation $r^\pm = \max\{\pm r, 0\}$ for all $r \in \mathbb{R}$, and $|\cdot|_N$ for the Lebesgue measure on $\mathbb{R}^N$.

3. Auxiliary results

The hypotheses on the map $a(x, y)$ are the following:

H(a) $a(x, y) = h(x, \|y\|)y$, where $h(x, t) > 0$ for all $(x, t) \in \overline{\Omega} \times (0, +\infty)$ and

(i) $a \in C^{0,\alpha}_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N) \cap C^1(\overline{\Omega} \times (\mathbb{R}^N \setminus \{0\}), \mathbb{R}^N)$ with $0 < \alpha \leq 1$;

(ii) for every $x \in \overline{\Omega}$ and $y \in \mathbb{R}^N \setminus \{0\}$, we have

$$\|D_y a(x, y)\| \leq c_1 \|y\|^{p-2} \quad \text{for some } c_1 > 0,\ 1 < p < \infty;$$

(iii) for every $x \in \overline{\Omega}$ and $y \in \mathbb{R}^N \setminus \{0\}$, we have

$$(D_y a(x, y)\xi, \xi)_{\mathbb{R}^N} \geq c_0 \|y\|^{p-2}\, \|\xi\|^2$$

whenever $\xi \in \mathbb{R}^N$, for some $c_0 > 0$;

(iv) the function $G(x, y)$ determined by $\nabla_y G(x, y) = a(x, y)$ for all $x \in \overline{\Omega}$, $y \in \mathbb{R}^N$ and $G(x, 0) = 0$ for all $x \in \overline{\Omega}$ satisfies

$$p\, G(x, y) - (a(x, y), y)_{\mathbb{R}^N} \geq \eta(x) \quad \text{for a.a. } x \in \Omega,\ \ y \in \mathbb{R}^N,$$

with some $\eta \in L^1(\Omega)$.

Remark 3.1. Such hypotheses are used frequently when dealing with quasilinear elliptic problems (see [14, 24, 35]). Here assumption H(a)(iv) is weakened with respect to the usual condition. Setting $g(x,t) = h(x,t)t$ for all $x \in \overline{\Omega}$ and $t \geq 0$, hypotheses H(a) imply the following unidimensional estimate

$$c_0 t^{p-2} \leq g'_t(x,t) \leq c_1 t^{p-2} \text{ for all } x \in \overline{\Omega}, t > 0.$$

These inequalities follow from

$$a(x,(t,0,\ldots,0)) = (g(x,t),0,\ldots,0) \quad \text{for all } x \in \overline{\Omega}, t > 0,$$

by differentiation with respect to $t \in \mathbb{R}$ and making use of H(a)(ii)-(iii). Denoting $G_0(x,t) = \int_0^t g(x,s)\,ds$ for all $(x,t) \in \overline{\Omega} \times \mathbb{R}_+$, it is seen that $G_0(x,\cdot)$ is strictly convex and strictly increasing on $\mathbb{R}_+$. Now let $G(x,y) = G_0(x,\|y\|)$. Then $G(x,\cdot)$ is convex for all $x \in \overline{\Omega}$, $G(x,0) = 0$ and

$$\nabla_y G(x,y) = (G_0)'_t(x,\|y\|)\frac{y}{\|y\|} = g(x,\|y\|)\frac{y}{\|y\|} = h(x,\|y\|)y = a(x,y)$$

for all $(x,y) \in \overline{\Omega} \times (\mathbb{R}^N \setminus \{0\})$. Therefore G is uniquely defined in H(a)(iv). Since $G(x,\cdot)$ is convex and $\nabla_y G(x,y) = a(x,y)$, it turns out that

$$(a(x,y),y)_{\mathbb{R}^N} \geq G(x,y) \text{ for all } (x,y) \in \overline{\Omega} \times \mathbb{R}^N. \tag{3.1}$$

Using H(a) and (3.1), it is straightforward to justify the lemma below.

Lemma 3.2. *If hypotheses* H(a) *hold, then*

(a) *for all* $x \in \overline{\Omega}$, $y \mapsto a(x,y)$ *is maximal monotone and strictly monotone*;
(b) *for all* $(x,y) \in \overline{\Omega} \times \mathbb{R}^N$, $\|a(x,y)\| \leq \frac{c_1}{p-1}\|y\|^{p-1}$;
(c) *for all* $(x,y) \in \overline{\Omega} \times \mathbb{R}^N$, $(a(x,y),y)_{\mathbb{R}^N} \geq \frac{c_0}{p-1}\|y\|^p$.

Lemma 3.2 leads to:

Corollary 3.3. *If* H(a) *hold, then for all* $(x,y) \in \overline{\Omega} \times \mathbb{R}^N$, *we have*

$$\frac{c_0}{p(p-1)}\|y\|^p \leq G(x,y) \leq \frac{c_1}{p(p-1)}\|y\|^p.$$

Example 3.4. A typical example of a map a satisfying hypotheses H(a) is $a(x,y) = \theta(x)\|y\|^{p-2}y$ for any $\theta \in C^1(\overline{\Omega})$, $\theta > 0$. As different examples satisfying H(a), we indicate in the case $p > 2$ and with $\theta(x)$ as above, $a(x,y) = \theta(x)(\|y\|^{p-2}y + \ln(1+\|y\|^{p-2})y)$ and

$$a(x,y) = \begin{cases} \theta(x)(\|y\|^{p-2}y + \|y\|^{q-2}y), & \|y\| \leq 1 \\ \theta(x)(\|y\|^{p-2}y + c\|y\|^{\tau-2}y - (c-1)y), & \|y\| > 1, \end{cases}$$

for $c = \frac{q-2}{\tau-2}$, $1 < \tau < p \leq q$, $\tau \neq 2$. An example different of $a(x, y) = \theta(x)\|y\|^{p-2}y$ for any $\theta \in C^1(\overline{\Omega})$, $\theta > 0$, for the situation $1 < p < 2$ is

$$a(x, y) = \theta(x)\left(\|y\|^{p-2}y + c\,\frac{\|y\|^{p-2}y}{1 + \|y\|^p}\right),$$

with $0 < c < 4p(p-1)$ and $\theta(x)$ as above. In the case $p \geq 2$, it satisfies hypotheses H(a) provided $0 < c < \frac{4p}{(p-1)^2}$. For instance, taking $c = 1$ then we need to have $\frac{1+\sqrt{2}}{2} < p < 3 + 2\sqrt{2}$.

Let $V : W_n^{1,p}(\Omega) \to W_n^{1,p}(\Omega)^*$ be the map defined by

$$\langle V(u), v\rangle = \int_\Omega (a(x, \nabla u), \nabla v)_{\mathbb{R}^N}\, dx \quad \text{for all } u, v \in W_n^{1,p}(\Omega).$$

Proposition 3.5. *If hypotheses* H(a) *hold, then V is maximal monotone, strictly monotone and of type $(S)_+$ (see Definition* 2.2).

Proof. By hypotheses H(a) and Lemma 3.2, V is demicontinuous, strictly monotone, hence it is maximal monotone (see, *e.g.*, [18, page 310]). To show that V is of type $(S)_+$, let $u_k \xrightarrow{w} u$ in $W_n^{1,p}(\Omega)$ with $\limsup_{k\to\infty}\langle V(u_k), u_k - u\rangle \leq 0$. Then, by virtue of the monotonicity of V, it follows that

$$\lim_{k\to\infty} \langle V(u_k) - V(u), u_k - u\rangle = 0. \tag{3.2}$$

Let $w_k(x) = (a(x, \nabla u_k(x)) - a(x, \nabla u(x)), \nabla u_k(x) - \nabla u(x))_{\mathbb{R}^N}$. We have $w_k(x) \geq 0$ for a.a. $x \in \Omega$, all $k \geq 1$ and, from (3.2), we infer that $w_k \to 0$ in $L^1(\Omega)$ as $k \to \infty$. By passing to a subsequence, we may assume that

$$w_k(x) \to 0 \text{ a.e. on } \Omega \quad \text{and} \quad 0 \leq w_k(x) \leq \zeta(x) \quad \text{for a.a. } x \in \Omega, k \geq 1, \tag{3.3}$$

with $\zeta \in L^1(\Omega)$. Using (3.3) together with Lemma 3.2 (b), (c), we obtain

$$\begin{aligned}\zeta(x) \geq{}& \frac{c_0}{p-1}\,(\|\nabla u_k(x)\|^p + \|\nabla u(x)\|^p) - \frac{c_1}{p-1}\,\|\nabla u_k(x)\|^{p-1}\,\|\nabla u(x)\| \\ &- \frac{c_1}{p-1}\,\|\nabla u(x)\|^{p-1}\,\|\nabla u_k(x)\| \quad \text{for a.a. } x \in \Omega, \text{ all } k \geq 1.\end{aligned} \tag{3.4}$$

From (3.4), we see that we can find $S \subset \Omega$ with $|S|_N = 0$ such that $\{\nabla u_k(x)\}_{k\geq 1}$ is bounded in $\mathbb{R}^N$ for all $x \in \Omega \setminus S$. Passing to an appropriate subsequence (which, in general, depends on $x \in \Omega \setminus S$), we have

$$\nabla u_k(x) \to \xi(x) \quad \text{in } \mathbb{R}^N \quad \text{as } k \to \infty.$$

Since $w_k(x) \to 0$ for a.a. $x \in \Omega$, in the limit as $k \to \infty$, we have

$$(a(x, \xi(x)) - a(x, \nabla u(x)), \xi(x) - \nabla u(x))_{\mathbb{R}^N} = 0 \quad \text{for a.a. } x \in \Omega.$$

We derive from Lemma 3.2 (a) that $\xi(x) = \nabla u(x)$ a.e. on Ω, so

$$\nabla u_k(x) \to \nabla u(x) \text{ a.e. on } \Omega. \tag{3.5}$$

On the basis of the boundedness of $\{\nabla u_k(\cdot)\}_{k\geq 1} \subset L^p(\Omega, \mathbb{R}^N)$ and Hölder's inequality, we infer that $\{\|\nabla u_k(\cdot)\|^p\}_{k\geq 1} \subset L^1(\Omega)$ is uniformly integrable. This fact and (3.5) permit the use of Vitali's theorem (see, *e.g.*, [18, p. 901]), which implies $\|\nabla u_k\|_p^p \to \|\nabla u\|_p^p$. Recalling that $\nabla u_k \overset{w}{\to} \nabla u$ in $L^p(\Omega, \mathbb{R}^N)$ and $L^p(\Omega, \mathbb{R}^N)$ is uniformly convex, we deduce that $\nabla u_k \to \nabla u$ in $L^p(\Omega, \mathbb{R}^N)$. Thus V is of type $(S)_+$. $\square$

The next result relates local minimizers in $W_n^{1,p}(\Omega)$ with local minimizers in the smaller Banach space $C_n^1(\overline{\Omega})$. A result of this type was first proven for the Dirichlet Laplacian in [8], then extended to the Dirichlet p-Laplacian in [16, 20] (in the latter, for $p \geq 2$). Recently, the authors [29] established the result to the Neumann p-Laplacian ($1 < p < \infty$). Here we extend it to the Neumann nonlinear operators satisfying hypotheses H(a).

Let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with subcritical growth

$$|f_0(x, s)| \leq a(x) + c|s|^{r-1} \text{ for a.e. } x \in \Omega, \text{ all } s \in \mathbb{R},$$

where $a \in L^\infty(\Omega)_+$, $c > 0$ and

$$1 < r < p^* = \begin{cases} \dfrac{Np}{N-p} & \text{if } N > p \\ +\infty & \text{if } N \leq p\,. \end{cases}$$

We set $F_0(x, t) = \int_0^t f_0(x, s)\, ds$ and consider the continuously differentiable functional $\varphi_0 : W_n^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \int_\Omega G(x, \nabla u(x))\, dx - \int_\Omega F_0(x, u(x))\, dx \quad \text{for all } u \in W_n^{1,p}(\Omega).$$

Proposition 3.6. *If $u_0 \in W_n^{1,p}(\Omega)$ is a local $C_n^1(\overline{\Omega})$-minimizer of φ_0, i.e., there exists $r_1 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \text{ for all } h \in C_n^1(\overline{\Omega}),\ \|h\|_{C_n^1(\overline{\Omega})} \leq r_1\,,$$

then $u_0 \in C_n^1(\overline{\Omega})$ and it is a local $W_n^{1,p}(\Omega)$-minimizer of φ_0, i.e., there exists $r_2 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \text{ for all } h \in W_n^{1,p}(\Omega),\ \|h\| \leq r_2\,.$$

This result has been proven in [32]. The proof therein is done by arguing by contraction and relies on an appropriate application of Lagrange multiplier rule for the functional φ_0 together with the Moser iteration technique (see [21, Section 4.7]) and the regularity result in [22, Theorem 2]. Proposition 3.6 extends the corresponding result in [29], which treated the case of the Neumann p-Laplacian with any $1 < p < \infty$.

Remark 3.7. In this section, it is not used that the map $a(x, y)$ satisfies hypothesis $H(a)$(iv). This hypothesis will be used in Theorem 4.8.

4. Solutions of constant sign

In this section we produce solutions of constant sign for problem (P_λ) for certain range of the parameter $\lambda > 0$. To this end, we impose the following conditions on the nonlinearity $f(x, s)$:

H_1 $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(x, 0) = 0$ a.e. on Ω and

(i) f is Carathéodory (*i.e.*, for all $s \in \mathbb{R}$, $x \mapsto f(x, s)$ is measurable, and for a.a. $x \in \Omega$, $s \mapsto f(x, s)$ is continuous);

(ii) $f(x, s)s \geq 0$ for a.a. $x \in \Omega$ and all $s \in \mathbb{R}$;

(iii) for almost all $x \in \Omega$ and all $s \in \mathbb{R}$, we have

$$|f(x, s)| \leq a(x) + c|s|^{r-1} \quad \text{with } a \in L^\infty(\Omega)_+, c > 0, p < r < p^*;$$

(iv) $\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^p} = +\infty$ uniformly for a.a. $x \in \Omega$, where $F(x,t) = \int_0^t f(x,s)\, ds$;

(v) there exists $\theta \in ((r - p)\max\{1, \frac{N}{p}\}, r]$ with $\theta \geq 1$ such that

$$0 < \liminf_{|t|\to\infty} \frac{f(x, t)t - pF(x, t)}{|t|^\theta} \quad \text{uniformly for a.e. } x \in \Omega;$$

(vi) there exist $\delta \in (0, 1)$ and $\tau \in [1, p)$ such that

$$F(x, s) \geq \tilde{c}|s|^\tau \quad \text{for a.a. } x \in \Omega, \text{ all } |s| \leq \delta \text{ and some } \tilde{c} > 0;$$

(vii) for every $\ell > 0$ there exists $c_\ell > 0$ such that the function $s \mapsto f(x, s) + c_\ell|s|^{p-2}s$ is nondecreasing on $[-\ell, \ell]$ for a.a. $x \in \Omega$.

Remark 4.1. Hypothesis H_1(iv) implies that for almost all $x \in \Omega$, the nonlinearity $f(x, \cdot)$ exhibits a $(p - 1)$-superlinear growth near infinity. However, we do not assume that $f(x, \cdot)$ satisfies the usual in such cases Ambrosetti–Rabinowitz condition. Instead, we employ hypothesis H_1(v), which was first introduced in [13] and covers new situations. Hypothesis H_1(vii) is more general than $f(x, \cdot)$ be nondecreasing.

Example 4.2. For the sake of simplicity we drop the x-dependence. Consider the function f on $\mathbb{R}$ defined by

$$f(s) = \begin{cases} |s|^{\tau-2}s + s^+ & \text{if } |s| \le 1 \\ |s|^{\theta-2}s(\theta \ln|s| + 1) + s^+ - \sin((\pi(s^+)^\alpha) & \text{if } |s| > 1 \end{cases}$$

with constants $1 < \tau < p \le \theta < p^*$, $\theta > 2$, and $\alpha \ge 0$, satisfies hypotheses H_1. It is worth noting that f does not verify the Ambrosetti-Rabinowitz condition, it is not odd and generally, for $\alpha > 0$, it is not nondecreasing.

For the sake of clarity, we recall the notions of upper and lower solutions for problem (P_λ) which will play a major role subsequently.

Definition 4.3. (a) We say that $\overline{u} \in W^{1,p}(\Omega)$ is an upper solution for problem (P_λ) if

$$\int_\Omega (a(x, \nabla\overline{u}), \nabla v)_{\mathbb{R}^N}\, dx + \lambda \int_\Omega |\overline{u}|^{p-2}\overline{u} v\, dx \ge \int_\Omega f(x, \overline{u}) v\, dx$$

for all $v \in W^{1,p}(\Omega)$, $v \ge 0$. We say that $\overline{u}$ is a strict upper solution for problem (P_λ) if $\overline{u}$ is an upper solution but not a solution of (P_λ).

(b) Reversing the inequality in the above definition, we obtain the notions of lower solution and strict lower solution $\underline{u} \in W^{1,p}(\Omega)$ for problem (P_λ).

We proceed by introducing the set

$$S_+ = \{\lambda \in \mathbb{R}_+ : \text{ problem } (\mathrm{P}_\lambda) \text{ has a positive solution}\}.$$

Proposition 4.4. *$S_+ \ne \emptyset$, and if $\lambda \in S_+$ then $[\lambda, +\infty) \subset S_+$.*

Proof. In view of Proposition 3.5, the nonlinear operator $V + K_p$, where $K_p(u)(\cdot) = |u(\cdot)|^{p-2}u(\cdot)$, is maximal monotone, strictly monotone and coercive on $W_n^{1,p}(\Omega)$. So there is a unique $u \in W_n^{1,p}(\Omega)$ such that $V(u) + K_p(u) = 1$. Acting with $-u^- \in W_n^{1,p}(\Omega)$, Lemma 3.2 (c) yields $u \in C_+ \setminus \{0\}$ because

$$\frac{c_0}{p-1} \|\nabla u^-\|_p^p + \|u^-\|_p^p \le \langle V(u), -u^-\rangle + \|u^-\|_p^p \le 0.$$

By [24, Theorem 6] (see also [35, Theorem 1.2]), there exists $\theta_0 > 0$ such that $u(x) \ge \theta_0$ for all $x \in \overline{\Omega}$. Setting $\hat{\lambda} = 1 + \frac{\|f(\cdot,u(\cdot))\|_\infty}{\theta_0^{p-1}}$, we obtain that $u \in \operatorname{int} C_+$ is a strict upper solution of problem $(\mathrm{P}_{\hat{\lambda}})$. For

$$f_0(x,s) = \begin{cases} 0 & \text{if } s \le 0 \\ f(x,s) & \text{if } 0 \le s \le u(x) \\ f(x,u(x)) & \text{if } u(x) \le s, \end{cases}$$

we consider the auxiliary nonlinear Neumann problem:

$$\begin{cases} -\operatorname{div} a(x, \nabla y(x)) + \hat{\lambda}|y(x)|^{p-2}y(x) = f_0(x, y(x)) & \text{in } \Omega, \\ \dfrac{\partial y}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Let $\varphi_0 : W_n^{1,p}(\Omega) \to \mathbb{R}$ be the Euler functional for problem (4.1), that is

$$\varphi_0(z) = \int_\Omega G(x, \nabla z(x))\,dx + \frac{\hat{\lambda}}{p}\,\|z\|_p^p - \int_\Omega F_0(x, z(x))\,dx \quad \text{for all } z \in W_n^{1,p}(\Omega),$$

where $F_0(x,t) = \int_0^t f_0(x,s)\,ds$. Since φ_0 is coercive and sequentially weakly lower semicontinuous, there exists $y_0 \in W_n^{1,p}(\Omega)$ such that

$$\varphi_0(y_0) = \inf\{\varphi_0(y) :\ y \in W_n^{1,p}(\Omega)\}. \tag{4.2}$$

By H_1(vi) and using $\tau < p$, we infer that $\varphi_0(z) < 0$ provided $z \in \mathbb{R}$, $z > 0$ is sufficiently small. Hence, by (4.2), it is seen that $y_0 \neq 0$ and

$$V(y_0) + \hat{\lambda} K_p(y_0) = f_0(\cdot, y_0(\cdot)). \tag{4.3}$$

Acting on (4.3) with $-y_0^- \in W_n^{1,p}(\Omega)$ yields $y_0 \geq 0$. Knowing $V(u) + \hat{\lambda}K_p(u) \geq f(\cdot, u(\cdot)) = f_0(\cdot, u(\cdot))$ and (4.3), by acting with $(y_0 - u)^+ \in W_n^{1,p}(\Omega)$ we find that

$$\begin{aligned} &\int_{\{y_0>u\}} (a(x, \nabla u) - a(x, \nabla y_0), \nabla y_0 - \nabla u)_{\mathbb{R}^N}\,dx \\ &\quad + \hat{\lambda} \int_{\{y_0>u\}} (|u|^{p-2}u - |y_0|^{p-2}y_0)(y_0 - u)\,dx \\ &\geq \int_{\{y_0>u\}} (f_0(x,u) - f_0(x,y_0))(y_0 - u)\,dx = 0. \end{aligned}$$

We derive that $0 \leq y_0 \leq u$, $y_0 \neq 0$, and through (4.3), y_0 solves $(\mathrm{P}_{\hat{\lambda}})$. Taking H_1(ii) into account, by [24, Theorem 6], it is seen that $y_0 \in \operatorname{int} C_+$.

Let $\lambda \in S_+$, $u_\lambda \in \operatorname{int} C_+$ be a solution of (P_λ), and $\mu > \lambda$. It follows that u_λ is a strict upper solution for problem (P_μ). Then, as above, we get a solution $u_\mu \in \operatorname{int} C_+$ of (P_μ) with $u_\mu \leq u_\lambda$, thereby $\mu \in S_+$. □

Remark 4.5. The proof of Proposition 4.4 shows that if $\lambda_1 \in S_+$, $\lambda_1 < \lambda_2$, and $u_{\lambda_1} \in \operatorname{int} C_+$ is a solution of (P_{λ_1}), then there exists $u_{\lambda_2} \in \operatorname{int} C_+$ solution of (P_{λ_2}) such that $u_{\lambda_2} \leq u_{\lambda_1}$.

In fact, the following strong comparison principle holds.

Lemma 4.6. *If $\lambda_1 < \lambda_2$, $u_{\lambda_1} \in \operatorname{int} C_+$ is a solution of* (P_{λ_1})*, $u_{\lambda_2} \in \operatorname{int} C_+$ is a solution of* (P_{λ_2})*, and $u_{\lambda_2} \leq u_{\lambda_1}$, then $u_{\lambda_1} - u_{\lambda_2} \in \operatorname{int} C_+$.*

Proof. Fix $\gamma_1 > 0$ satisfying $\gamma_1 \le u_{\lambda_2}(x)$ for all $x \in \overline{\Omega}$. For some $\delta \in (0, \frac{\gamma_1}{2})$ we set $v_\delta(x) = u_{\lambda_1}(x) - \delta$, $x \in \overline{\Omega}$. Then $v_\delta \in \operatorname{int} C_+$ and, using the number $c_\ell > 0$ in hypothesis H_1(vii) corresponding to $\ell = \|u_{\lambda_1}\|_\infty$, it follows that

$$\begin{aligned}
&-\operatorname{div} a(x, \nabla v_\delta(x)) + (\lambda_2 + c_\ell) v_\delta(x)^{p-1} \\
&\ge -\operatorname{div} a(x, \nabla u_{\lambda_1}(x)) + (\lambda_2 + c_\ell) u_{\lambda_1}(x)^{p-1} - \varrho(\delta) \\
&\ge f(x, u_{\lambda_1}(x)) + (\lambda_2 - \lambda_1)\gamma_1^{p-1} + c_\ell u_{\lambda_1}(x)^{p-1} - \varrho(\delta) \quad \text{for a.a. } x \in \Omega,
\end{aligned}$$

with $\varrho(\delta) = \sup_{t \in u_{\lambda_1}(\overline{\Omega})}(t^{p-1} - (t-\delta)^{p-1})$. Note that $\varrho(\delta) \to 0$ as $\delta \downarrow 0$. For $\delta > 0$ small, we have $(\lambda_2 - \lambda_1)\gamma_1^{p-1} \ge \varrho(\delta)$, hence

$$\begin{aligned}
&-\operatorname{div} a(x, \nabla v_\delta(x)) + (\lambda_2 + c_\ell) v_\delta(x)^{p-1} \ge f(x, u_{\lambda_1}(x)) + c_\ell u_{\lambda_1}(x)^{p-1} \\
&\ge f(x, u_{\lambda_2}(x)) + c_\ell u_{\lambda_2}(x)^{p-1} = -\operatorname{div} a(x, \nabla u_{\lambda_2}(x)) + (\lambda_2 + c_\ell) u_{\lambda_2}(x)^{p-1}
\end{aligned}$$

for a.a. $x \in \Omega$ (see H_1(vii) and recall that $u_{\lambda_2} \le u_{\lambda_1}$). Acting with $(u_{\lambda_2} - v_\delta)^+ \in W_n^{1,p}(\Omega)$ and using the nonlinear Green's identity (see [11]) yield

$$\begin{aligned}
&\int_{\{u_{\lambda_2} > v_\delta\}} (a(x, \nabla v_\delta) - a(x, \nabla u_{\lambda_2}), \nabla u_{\lambda_2} - \nabla v_\delta)_{\mathbb{R}^N}\, dx \\
&\ge (\lambda_2 + c_\ell) \int_{\{u_{\lambda_2} > v_\delta\}} (u_{\lambda_2}^{p-1} - v_\delta^{p-1})(u_{\lambda_2} - v_\delta)\, dx.
\end{aligned}$$

It turns out that $u_{\lambda_2}(x) \le v_\delta(x)$ for all $x \in \overline{\Omega}$, thus $u_{\lambda_1} - u_{\lambda_2} \in \operatorname{int} C_+$. $\square$

Let $\varphi_\lambda : W_n^{1,p}(\Omega) \to \mathbb{R}$ be defined by

$$\varphi_\lambda(u) = \int_\Omega G(x, \nabla u(x))\, dx + \frac{\lambda}{p}\|u\|_p^p - \int_\Omega F(x, u(x))\, dx \quad \text{for all } u \in W_n^{1,p}(\Omega).$$

Hypotheses H_1(i) and (iii) ensure that $\varphi_\lambda \in C^1(W_n^{1,p}(\Omega))$.

Proposition 4.7. *Denote $\lambda_+ = \inf S_+$. Assume that $\lambda_+ < \lambda_1 < \mu < \lambda_2$, $u_{\lambda_1} \in \operatorname{int} C_+$ is a solution of* (P_{λ_1}), *$u_{\lambda_2} \in \operatorname{int} C_+$ is a solution of* (P_{λ_2}), *and $u_{\lambda_2} \le u_{\lambda_1}$. Then there exists $u_\mu \in \operatorname{int} C_+$ solution of* (P_μ) *such that $u_{\lambda_2} \le u_\mu \le u_{\lambda_1}$ and u_μ is a local minimizer of φ_μ.*

Proof. Since $\lambda_1 < \mu < \lambda_2$, we have that $u_{\lambda_1} \in \operatorname{int} C_+$ and $u_{\lambda_2} \in \operatorname{int} C_+$ are strict upper solution and strict lower solution for problem (P_μ), respectively. We truncate $f(x, \cdot)$ as follows:

$$\tilde{f}(x, s) = \begin{cases} f(x, u_{\lambda_2}(x)) & \text{if } s \le u_{\lambda_2}(x) \\ f(x, s) & \text{if } u_{\lambda_2}(x) \le s \le u_{\lambda_1}(x) \\ f(x, u_{\lambda_1}(x)) & \text{if } u_{\lambda_1}(x) \le s. \end{cases}$$

Setting $\tilde{F}(x,t) = \int_0^t \tilde{f}(x,s)\,ds$, the functional $\tilde{\varphi}_\mu : W_n^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tilde{\varphi}_\mu(u) = \int_\Omega G(x, \nabla u(x))\,dx + \frac{\mu}{p}\,\|u\|_p^p - \int_\Omega \tilde{F}(x, u(x))\,dx \quad \text{for all } u \in W_n^{1,p}(\Omega),$$

is continuously differentiable, coercive and sequentially weakly lower semicontinuous. Consequently, there exists $u_\mu \in W_n^{1,p}(\Omega)$ such that $\tilde{\varphi}_\mu(u_\mu) = \inf \tilde{\varphi}_\mu$, so

$$V(u_\mu) + \mu K_p(u_\mu) = \tilde{f}(\cdot, u_\mu(\cdot)).$$

Let us first act with $(u_\mu - u_{\lambda_1})^+ \in W_n^{1,p}(\Omega)$ and then with $(u_{\lambda_2} - u_\mu)^+ \in W_n^{1,p}(\Omega)$ to deduce $u_{\lambda_2} \le u_\mu \le u_{\lambda_1}$. Hence u_μ is a solution of (P_μ) and $u_\mu \in \operatorname{int} C_+$. Lemma 4.6 implies that $u_\mu - u_{\lambda_2} \in \operatorname{int} C_+$ and $u_{\lambda_1} - u_\mu \in \operatorname{int} C_+$. We infer that $u_\mu \in \operatorname{int} C_+$ is a local $C_n^1(\overline{\Omega})$-minimizer of φ_μ. Then Proposition 3.6 completes the proof. □

Now we state the first main result of this section.

Theorem 4.8. *If hypotheses* H(a), H_1 *hold and* $\lambda > \lambda_+$, *then problem* (P_λ) *has at least two* (*positive*) *solutions* $u_\lambda,\ v_\lambda \in \operatorname{int} C_+$ *with* $u_\lambda \le v_\lambda$, $u_\lambda \ne v_\lambda$.

Proof. Let $\lambda > \lambda_+$. Proposition 4.4 ensures the existence of one solution $u_\lambda \in \operatorname{int} C_+$ of problem (P_λ). Let $\lambda_+ < \lambda_1 < \lambda < \lambda_2$, $u_{\lambda_1} \in \operatorname{int} C_+$ a solution of (P_{λ_1}) and $u_{\lambda_2} \in \operatorname{int} C_+$ a solution of (P_{λ_2}) for which we can assume that $u_{\lambda_2} \le u_\lambda \le u_{\lambda_1}$ (see Remark 4.5). Consider the truncation

$$\hat{f}_\lambda^+(x,s) = \begin{cases} f(x, u_\lambda(x)) + c_\ell u_\lambda(x)^{p-1} & \text{if } s \le u_\lambda(x) \\ f(x,s) + c_\ell s^{p-1} & \text{if } s \ge u_\lambda(x), \end{cases}$$

with $c_\ell > 0$ in hypothesis H_1(vii) for $\ell = \|u_{\lambda_1}\|_\infty$. Using $\hat{F}_\lambda^+(x,t) = \int_0^t \hat{f}_\lambda^+(x,s)\,ds$, we define the functional $\hat{\varphi}_\lambda^+ \in C^1(W_n^{1,p}(\Omega))$ by

$$\hat{\varphi}_\lambda^+(u) = \int_\Omega G(x, \nabla u(x))\,dx + \frac{\lambda + c_\ell}{p}\,\|u\|_p^p - \int_\Omega \hat{F}_\lambda^+(x, u(x))\,dx.$$

Corresponding to $\mu > \lambda_+$, we formulate the auxiliary problem

$$(\mathcal{P}_\mu) \quad \begin{cases} -\operatorname{div} a(x, \nabla u(x)) + (\mu + c_\ell)|u(x)|^{p-2}u(x) = \hat{f}_\lambda^+(x, u(x)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Making use of hypothesis H_1(vii), we see that u_{λ_1} is an upper solution and u_{λ_2} is a lower solution for $(\mathcal{P}_\lambda)$. The reasoning in the proof of Proposition 4.7 applies to $\hat{f}_\lambda^+(x,s)$ truncated at $\{u_{\lambda_2}, u_{\lambda_1}\}$, thus we obtain a solution $\hat{u}_\lambda$ of $(\mathcal{P}_\lambda)$ which is also a local minimizer of $\hat{\varphi}_\lambda^+$ satisfying $\hat{u}_\lambda - u_{\lambda_2} \in \operatorname{int} C_+$ and $u_{\lambda_1} - \hat{u}_\lambda \in \operatorname{int} C_+$.

If $\hat{u}_\lambda \neq u_\lambda$, the conclusion is achieved because acting with $(u_\lambda - \hat{u}_\lambda)^+ \in W_n^{1,p}(\Omega)$ on

$$V(\hat{u}_\lambda) - V(u_\lambda) + \lambda(K_p(\hat{u}_\lambda) - K_p(u_\lambda)) + c_\ell K_p(\hat{u}_\lambda) = \hat{f}_\lambda^+(\cdot, \hat{u}_\lambda(\cdot)) - f(\cdot, u_\lambda(\cdot))$$

yields $u_\lambda \leq \hat{u}_\lambda$, so $\hat{u}_\lambda \in \operatorname{int} C_+$ is a second solution of (P_λ).

Now we suppose that $\hat{u}_\lambda = u_\lambda$. We can assume that u_λ is an isolated critical point and a local minimizer of $\hat{\varphi}_\lambda^+$. As in the proof of [25, Proposition 6], there is $r > 0$ such that

$$\hat{\varphi}_\lambda^+(u_\lambda) < \hat{\eta}_r =: \inf\{\hat{\varphi}_\lambda^+(v) : \|v - u_\lambda\| = r\}. \tag{4.4}$$

Claim 1: There exists $u \in W_n^{1,p}(\Omega)$ such that $\|u - u_\lambda\| > r$ and $\hat{\varphi}_\lambda^+(u) < \hat{\eta}_r$.

Hypotheses H_1(iii)-(iv) guarantee that, given $\varepsilon > 0$, there is a constant $c_\varepsilon > 0$ such that

$$\hat{F}_\lambda^+(x,s) \geq \frac{1}{p}\left(\frac{1}{\varepsilon} + c_\ell\right) s^p - c_\varepsilon \quad \text{for a.a. } x \in \Omega \text{ and all } s > 0.$$

For any constant $\xi > 0$, it follows that

$$\hat{\varphi}_\lambda^+(\xi) = \frac{\lambda + c_\ell}{p}\, \xi^p |\Omega|_N - \int_\Omega \hat{F}_\lambda^+(x,\xi)\, dx \leq \frac{\xi^p}{p}\left(\lambda - \frac{1}{\varepsilon}\right)|\Omega|_N + c_\varepsilon |\Omega|_N.$$

By choosing ε with $\varepsilon\lambda < 1$, Claim 1 is proven since $\varphi_\lambda^+(\xi) \to -\infty$ as $\xi \to +\infty$.

Claim 2: $\hat{\varphi}_\lambda^+$ satisfies the (C)-condition.

Let $\{u_k\}_{k\geq 1} \subset W_n^{1,p}(\Omega)$ be a sequence such that

$$|\hat{\varphi}_\lambda^+(u_k)| \leq M_1 \quad \text{for some } M_1 > 0, \text{ all } k \geq 1 \tag{4.5}$$

and

$$(1 + \|u_k\|)(\hat{\varphi}_\lambda^+)'(u_k) \to 0 \text{ in } W_n^{1,p}(\Omega)^* \text{ as } k \to \infty. \tag{4.6}$$

Writing (4.6) in the form

$$|\langle(\hat{\varphi}_\lambda^+)'(u_k), u\rangle| \leq \frac{\varepsilon_k}{1 + \|u_k\|}\|u\| \quad \text{for all } u \in W_n^{1,p}(\Omega), \text{ with } \varepsilon_k \downarrow 0, \tag{4.7}$$

and setting first $u = -u_k^-$ and then $u = u_k^+$ allow to get that

$$\{u_k^-\}_{k\geq 1} \text{ is bounded in } W_n^{1,p}(\Omega) \tag{4.8}$$

and

$$-\int_\Omega (a(x,\nabla u_k), \nabla u_k^+)_{\mathbb{R}^N}\, dx - (\lambda + c_\ell)\, \|u_k^+\|_p^p + \int_\Omega \hat{f}_\lambda^+(x,u_k) u_k^+\, dx \leq \varepsilon_k. \tag{4.9}$$

The inequalities (4.5) and (4.9) together with (4.8) imply that

$$\int_\Omega \big(p\, G(x, \nabla u_k^+) - (a(x, \nabla u_k^+), \nabla u_k^+)_{\mathbb{R}^N}\big)\, dx + \int_\Omega \Big(\hat{f}_\lambda^+(x, u_k^+) u_k^+ - p\, \hat{F}_\lambda^+(x, u_k^+)\Big)\, dx \le M_2\,,$$

for some $M_2 > 0$. In conjunction with hypothesis H(a) (iv), this gives

$$\int_\Omega (\hat{f}_\lambda^+(x, u_k^+) u_k^+ - p\, \hat{F}_\lambda^+(x, u_k^+))\, dx \le M_3 \ \text{ for all } k \ge 1, \tag{4.10}$$

with $M_3 > 0$. Hypothesis H_1(v) yields

$$\nu s^\theta - M_4 \le \hat{f}_\lambda^+(x, s)s - p\, \hat{F}_\lambda^+(x, s) \ \text{ for a.a. } x \in \Omega, \text{ all } s \ge 0,$$

with constants $\nu > 0$ and $M_4 > 0$. In view of (4.10), we infer that

$$\{u_k^+\}_{k\ge1} \text{ is bounded in } L^\theta(\Omega). \tag{4.11}$$

Assume first that $N > p$. Since $\theta \le r < p^*$, there is a unique $t \in [0, 1)$ such that $\frac{1}{r} = \frac{1-t}{\theta} + \frac{t}{p^*}$. Then the interpolation inequality $\|u_k^+\|_r \le \|u_k^+\|_\theta^{1-t}\, \|u_k^+\|_{p^*}^t$ (see, *e.g.*, [18, page 903]) and (4.11) lead to the estimate

$$\|u_k^+\|_r^r \le M_5 \|u_k^+\|^{tr}, \tag{4.12}$$

for some $M_5 > 0$. This also holds for $N \le p$ by applying the interpolation inequality with $\theta \le r < q$, for $q > \frac{p\theta}{\theta+p-r}$. Then, using Lemma 3.2 (c), (4.7), (4.8), H_1(iii) and (4.12), we see that

$$\begin{aligned} &\frac{c_0}{p-1}\|\nabla u_k^+\|_p^p + (\lambda + c_\ell)\|u_k^+\|_p^p \\ &\le \int_\Omega (a(x, \nabla u_k^+), \nabla u_k^+)_{\mathbb{R}^N}\, dx + (\lambda + c_\ell)\|u_k^+\|_p^p \\ &\le \varepsilon_k + \int_\Omega \hat{f}_\lambda^+(x, u_k^+) u_k^+\, dx \le \varepsilon_k + \bar{c}(\|u_k^+\| + \|u_k^+\|^{tr}), \end{aligned} \tag{4.13}$$

with a constant $\bar{c} > 0$. Notice that $tr < p$ because $\theta > (r - p)\max\{1, \frac{N}{p}\}$. Consequently, (4.13) renders that $\{u_k^+\}_{k\ge1}$ is bounded in $W_n^{1,p}(\Omega)$. Combining with (4.8), we conclude that $\{u_k\}_{k\ge1}$ is bounded in $W_n^{1,p}(\Omega)$. Passing to a relabelled subsequence, we may assume that $u_k \xrightarrow{\text{w}} u$ in $W_n^{1,p}(\Omega)$ and $u_k \to u$ in $L^r(\Omega)$. From (4.7) it turns out that $\lim_{k\to\infty}\langle V(u_k), u_k - u\rangle = 0$. By Proposition 3.5, Claim 2 is established.

Since (4.4) and Claims 1 and 2 hold true, we may apply Theorem 2.1. Thus there exists a critical point $v_\lambda \in W_n^{1,p}(\Omega)$ of $\hat{\varphi}_\lambda^+$ satisfying $\hat{\varphi}_\lambda^+(u_\lambda) < \hat{\eta}_r \leq \hat{\varphi}_\lambda^+(v_\lambda)$. It follows that $u_\lambda \neq v_\lambda$ and

$$V(v_\lambda) + (\lambda + c_\ell)K_p(v_\lambda) = \hat{f}_\lambda^+(\cdot, v_\lambda(\cdot)), \tag{4.14}$$

which implies

$$\int_{\{u_\lambda > v_\lambda\}} (a(x, \nabla v_\lambda) - a(x, \nabla u_\lambda), \nabla u_\lambda - \nabla v_\lambda)_{\mathbb{R}^N}\, dx$$
$$+ (\lambda + c_\ell) \int_{\{u_\lambda > v_\lambda\}} (|v_\lambda|^{p-2} v_\lambda - |u_\lambda|^{p-2} u_\lambda)(u_\lambda - v_\lambda)\, dx = 0,$$

so $u_\lambda \leq v_\lambda$. By means of (4.14) and nonlinear regularity theory, we infer that $v_\lambda \in \operatorname{int} C_+$ and it solves problem (P_λ). □

In the same way, denoting

$$S_- = \{\lambda \in \mathbb{R}_+ : \text{ problem } (\mathrm{P}_\lambda) \text{ has a negative solution}\}$$

and $\lambda_- = \inf S_-$, we can show that λ_- is finite and

Theorem 4.9. *If hypotheses* H(a), H_1 *hold and* $\lambda > \lambda_-$, *then problem* (P_λ) *has at least two* (*negative*) *solutions*

$$w_\lambda,\ y_\lambda \in -\operatorname{int} C_+,\ y_\lambda \leq w_\lambda\,,\ y_\lambda \neq w_\lambda\,.$$

Remark 4.10. Theorems 4.8 and 4.9 hold if H_1(ii), (iv)-(vii) are only imposed on the positive and negative half-lines, respectively.

Set $\lambda^* = \max\{\lambda_+, \lambda_-\}$. Theorems 4.8 and 4.9 ensure that:

Theorem 4.11. *If hypotheses* H(a), H_1 *hold and* $\lambda > \lambda^*$, *then problem* (P_λ) *has at least four nontrivial solutions of constant sign*

$$u_\lambda,\ v_\lambda \in \operatorname{int} C_+,\ u_\lambda \leq v_\lambda\,,\ u_\lambda \neq v_\lambda,$$
$$w_\lambda,\ y_\lambda \in -\operatorname{int} C_+,\ y_\lambda \leq w_\lambda\,,\ y_\lambda \neq w_\lambda.$$

5. Nodal solution

We now deal with the special case of problem (P_λ) driven by the p-Laplacian operator $\Delta_p u = \operatorname{div}(\|\nabla u\|^{p-2}\nabla u)$ (*i.e.*, $a(x, y) = \|y\|^{p-2}y$), where we produce in addition a nodal solution. The approach that we employ relies on the construction of a smallest positive solution u_+ and a biggest negative solution u_-, and then a nontrivial solution in the ordered interval $[u_-, u_+]$ which, thanks to the extremality of u_- and u_+, must be nodal (see [9, 15, 27] for related results).

We formulate the new hypotheses on $f(x, s)$:

H_2 $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $f(x, 0) = 0$ a.e. on Ω, hypotheses H_1(i), (iii)-(v), (vii) are satisfied, and (vi)$'$ there exist $\tau, \mu \in (1, p)$, $\xi > 0$ and $\delta_0 > 0$ such that

$$\begin{aligned} f(x,s) &\geq \xi s^{\tau-1} \text{ for a.a. } x \in \Omega, \text{ all } s > 0, \\ f(x,s) &\leq \xi |s|^{\tau-2} s \text{ for a.a. } x \in \Omega, \text{ all } s < 0, \\ \mu F(x,s) - f(x,s)s &\geq 0 \text{ for a.a. } x \in \Omega, |s| \leq \delta_0. \end{aligned}$$

Example 5.1. The function f in Example 4.2 satisfies hypotheses H_2. To see this, it suffices to choose $\tau < \mu < p$ for fulfilling hypothesis H_2(vi)$'$.

Our approach will be valid until certain point for the general problem (P_λ) before restricting to the case of the p-Laplacian.

Given a lower solution $\underline{u} \in W^{1,p}(\Omega)$ and an upper solution $\overline{u} \in W^{1,p}(\Omega)$ for (P_λ) (see Definition 4.3) with $0 \leq \underline{u} \leq \overline{u}$, we consider the ordered intervals

$$[\underline{u}, \overline{u}] = \{h \in W_n^{1,p}(\Omega) : \underline{u}(x) \leq h(x) \leq \overline{u}(x) \text{ a.e. on } \Omega\}$$

and

$$T^+(\underline{u}) = \{h \in W_n^{1,p}(\Omega) : \underline{u}(x) \leq h(x) \text{ a.e. on } \Omega\}.$$

Proposition 5.2. *If hypotheses* H(a) *and* H_1 *hold, and* $\lambda \in S_+$, *then problem* (P_λ) *admits a smallest solution* $\hat{x}_\lambda$ *in* $T^+(\underline{u})$ *and one has* $\hat{x}_\lambda \in [\underline{u}, \overline{u}]$.

Proof. For any $h \in L^{r'}(\Omega)$, where $\frac{1}{r} + \frac{1}{r'} = 1$, we formulate the auxiliary Neumann problem:

$$\begin{cases} -\operatorname{div} a(x, \nabla u(x)) + (\lambda + c_\ell)|u(x)|^{p-2}u(x) = h(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where c_ℓ corresponds by hypothesis H_1(vii) to $\ell = \|\overline{u}\|_\infty$. Taking into account that $V + (\lambda + c_\ell)K_p : W_n^{1,p}(\Omega) \to W_n^{1,p}(\Omega)^*$ is maximal monotone, strictly monotone and coercive, there is a unique $u =: S_{\lambda+c_\ell}(h) \in W_n^{1,p}(\Omega)$ which solves (5.1) and, by nonlinear regularity theory, $u \in C_n^1(\overline{\Omega})$.

Claim 1: $S_{\lambda+c_\ell} : L^{r'}(\Omega) \to W_n^{1,p}(\Omega)$ is completely continuous.

Let $h_k \xrightarrow{\text{w}} h$ in $L^{r'}(\Omega)$. Then $u_k = S_{\lambda+c_\ell}(h_k)$ satisfies

$$V(u_k) + (\lambda + c_\ell)K_p(u_k) = h_k\,. \tag{5.2}$$

Acting on (5.2) with $u_k \in W_n^{1,p}(\Omega)$ and using Lemma 3.2 (c) imply that $\{u_k\}_{k\geq1}$ is bounded in $W_n^{1,p}(\Omega)$. Along a relabelled subsequence, we have that $u_k \xrightarrow{\text{w}} u$ in $W_n^{1,p}(\Omega)$ and $u_k \to u$ in $L^r(\Omega)$. Let us now act on (5.2) with $u_k - u$ and then pass to the limit as $k \to \infty$. Through Proposition 3.5 we infer that $u_k \to u$ in $W_n^{1,p}(\Omega)$. In the limit, (5.2) results in $u = S_{\lambda+c_\ell}(h)$.

Claim 2: $h_1 \le h_2$ in $L^{r'}(\Omega) \implies S_{\lambda+c_\ell}(h_1) \le S_{\lambda+c_\ell}(h_2)$ in $W_n^{1,p}(\Omega)$.
Writing for $u_1 = S_{\lambda+c_\ell}(h_1)$ and $u_2 = S_{\lambda+c_\ell}(h_2)$ that

$$V(u_2) - V(u_1) + (\lambda + c_\ell)(K_p(u_2) - K_p(u_1)) = h_2 - h_1,$$

Claim 2 is readily verified by acting with $(u_1 - u_2)^+ \in W_n^{1,p}(\Omega)$.

Now let $\lambda \in S_+$. We introduce $R_\lambda = S_{\lambda+c_\ell} \circ N_\ell$, where $N_\ell(u) = c_\ell|u|^{p-2}u + f(\cdot, u)$. Then $u \in \operatorname{int} C_+$ is a solution of (P_λ) if and only if $u = R_\lambda(u)$. Moreover, since $\overline{u} \in W^{1,p}(\Omega)$ is an upper solution of (P_λ), by Claim 2 we derive $R_\lambda(\overline{u}) \le \overline{u}$ in $W^{1,p}(\Omega)$. Similarly, it is true that $\underline{u} \le R_\lambda(\underline{u})$ in $W^{1,p}(\Omega)$. In view of hypothesis H_1(vii), and by Claims 1 and 2, we have that the map $R_\lambda : [\underline{u}, \overline{u}] \to [\underline{u}, \overline{u}]$ is well defined, completely continuous and increasing. Consequently, Theorem 6.1 in [2] can be applied yielding the minimal fixed point $\hat{x}_\lambda \in [\underline{u}, \overline{u}]$ of R_λ in $[\underline{u}, \overline{u}]$. This is the smallest solution of (P_λ) in $T^+(\underline{u})$. □

Arguing as above leads to:

Proposition 5.3. *Let $\underline{v} \in W^{1,p}(\Omega)$ be a lower solution and let $\overline{v} \in W^{1,p}(\Omega)$ be an upper solution for problem* (P_λ) *with $\underline{v} \le \overline{v} \le 0$. If hypotheses* $\mathrm{H}(a)$ *and* H_1 *hold, and $\lambda \in S_-$, then problem* (P_λ) *has a biggest solution $\hat{v}_\lambda$ in*

$$T^-(\overline{v}) = \{h \in W_n^{1,p}(\Omega) \,:\, h(x) \le \overline{v}(x) \text{ a.e. on } \Omega\},$$

and one has $\hat{v}_\lambda \in [\underline{v}, \overline{v}]$.

We deal from now on with the following problem:

$$(\mathrm{P}'_\lambda) \qquad \begin{cases} -\Delta_p u(x) + \lambda |u(x)|^{p-2}u(x) = f(x, u(x)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Proposition 5.4. *If hypotheses* H_2 *hold and $\lambda > \lambda^*$, then problem* (P'_λ) *has a smallest positive solution $u_+ \in \operatorname{int} C_+$ and a biggest negative solution $u_- \in -\operatorname{int} C_+$.*

Proof. We only show the existence of the smallest positive solution because the proof of the existence of the biggest negative solution is similar. Let $\lambda > \lambda^*$ and $\hat{\theta} \in (0, (\frac{\xi}{\lambda})^{\tau-p})$. Hypothesis $\mathrm{H}_2(\mathrm{vi})'$ ensures that $\underline{u} := \hat{\theta}$ is a lower solution for (P'_λ) because

$$-\Delta_p \hat{\theta} + \lambda K_p(\hat{\theta}) = \lambda \hat{\theta}^{p-1} \le \xi \hat{\theta}^{\tau-1} \le f(\cdot, \hat{\theta}).$$

Fix $\mu \in (\lambda^*, \lambda)$. We note that if $u_\mu \in \operatorname{int} C_+$ is a solution of problem (P'_μ), then $\overline{u} := u_\mu$ is a (strict) upper solution of (P'_λ) which, for $\hat{\theta}$ sufficiently small, satisfies $\underline{u} \le \overline{u}$.

Corresponding to a sequence $\hat{\theta}_k \downarrow 0$ with $\hat{\theta}_1 = \hat{\theta}$, let $\underline{u}_k := \hat{\theta}_k$. According to Proposition 5.2, there is a smallest $u_*^k \in T^+(\underline{u}_k)$ with

$$-\Delta_p u_*^k + \lambda K_p(u_*^k) = f(\cdot, u_*^k(\cdot)) \text{ for all } k \geq 1, \tag{5.3}$$

and $u_*^k \in [\hat{\theta}_k, \overline{u}]$. It turns out that $\{u_*^k\}_{k\geq 1}$ is bounded in $W_n^{1,p}(\Omega)$. This gives rise to a relabeled subsequence with $u_*^k \overset{w}{\to} u_+$ in $W_n^{1,p}(\Omega)$ and $u_*^k \to u_+$ in $L^r(\Omega)$. Acting on (5.3) with $u_*^k - u_+$ and then letting $k \to \infty$, we obtain through Proposition 3.5 that

$$u_*^k \to u_+ \text{ in } W_n^{1,p}(\Omega). \tag{5.4}$$

Claim: $u_+ \neq 0$.

The auxiliary Neumann problem

$$\begin{cases} -\Delta_p u + \lambda |u|^{p-2}u = \xi(u^+)^{\tau-1} & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \tag{5.5}$$

possesses a nontrivial solution $u \in \operatorname{int} C_+$. This can be seen from the fact that the associated Euler functional ψ_λ defined by

$$\psi_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{\lambda}{p}\|u\|_p^p - \frac{\xi}{\tau}\|u^+\|_\tau^\tau \text{ for all } u \in W_n^{1,p}(\Omega)$$

is coercive and sequentially weakly lower semicontinuous, so there exists $u \in W_n^{1,p}(\Omega)$ such that $\psi_\lambda(u) = \inf \psi_\lambda$. Noting that if $\hat{\xi} > 0$ is small, since $p > \tau$, it follows that $\psi_\lambda(\hat{\xi}) < 0$, so $u \neq 0$. Furthermore, applying to (5.5) the strong maximum principle in [34], we get that $u \in \operatorname{int} C_+$.

Using that $u_*^k \in \operatorname{int} C_+$ allows to find $\theta_k > 0$ such that $\theta_k u \leq u_*^k$. Let us denote by θ_k the largest positive real for which this inequality holds. Suppose $\theta_k < 1$. By (5.3), $H_2(\text{vi})'$ and (5.5), we have

$$\begin{aligned} &-\Delta_p u_*^k + \lambda(u_*^k)^{p-1} = f(x, u_*^k) \geq \xi(u_*^k)^{\tau-1} \\ &\geq \xi(\theta_k u)^{\tau-1} = -\Delta_p(\theta_k u) + \lambda(\theta_k u)^{p-1} + \xi(\theta_k^{\tau-1} - \theta_k^{p-1})u^{\tau-1}. \end{aligned}$$

Since $\theta_k \in (0,1)$ and $\tau < p$, arguing as in the proof of Lemma 4.6, it is seen that $u_*^k - \theta_k u \in \operatorname{int} C_+$, which contradicts the maximality of θ_k. Thereby, $\theta_k \geq 1$, and so $u \leq u_*^k$, which proves the Claim taking into account (5.4).

The convergence (5.4) enables us to pass to the limit in (5.3) obtaining that u_+ is a nontrivial solution of (P'_λ). By the nonlinear regularity theory and since $u \leq u_+$ as shown above, we infer that $u_+ \in \operatorname{int} C_+$. In order to show that $u_+ \in \operatorname{int} C_+$ is the smallest positive solution of (P'_λ), let v be a positive solution of (P'_λ). Since $v \in \operatorname{int} C_+$, if k is sufficiently large we have $\underline{u}_k \leq v$. This forces $u_+ \leq v$. □

The following is the main result of this section supplying a nodal solution.

Theorem 5.5. *Assume hypotheses* H_2. *Then, for every* $\lambda > \lambda^*$, *problem* (P'_λ) *has at least five distinct nontrivial solutions*

$$u_\lambda,\ v_\lambda \in \operatorname{int} C_+,\ u_\lambda \le v_\lambda\,,\ u_\lambda \ne v_\lambda,$$
$$y_\lambda,\ w_\lambda \in -\operatorname{int} C_+,\ y_\lambda \le w_\lambda\,,\ y_\lambda \ne w_\lambda,$$

and

$$z_\lambda \in C^1_n(\overline{\Omega}) \text{ a nodal solution.}$$

Proof. Theorem 4.11 produces four constant sign solutions $u_\lambda,\ v_\lambda \in \operatorname{int} C_+$ and $w_\lambda,\ y_\lambda \in -\operatorname{int} C_+$. Let $u_+ \in \operatorname{int} C_+$ and $u_- \in -\operatorname{int} C_+$ be the two extremal constant sign solutions given by Proposition 5.4. We introduce the following three truncations of the nonlinearity $f(x,\cdot)$:

$$\hat{f}_+(x,s) = \begin{cases} 0 & \text{if } s \le 0 \\ f(x,s) & \text{if } 0 < s < u_+(x) \\ f(x,u_+(x)) & \text{if } u_+(x) \le s, \end{cases}$$

$$\hat{f}_-(x,s) = \begin{cases} f(x,u_-(x)) & \text{if } s \le u_-(x) \\ f(x,s) & \text{if } u_-(x) < s < 0 \\ 0 & \text{if } 0 \le s, \end{cases}$$

$$\hat{f}_0(x,s) = \begin{cases} f(x,u_-(x)) & \text{if } s \le u_-(x) \\ f(x,s) & \text{if } u_-(x) < s < u_+(x) \\ f(x,u_+(x)) & \text{if } u_+(x) \le s. \end{cases}$$

Corresponding to these truncations, we set

$$\hat{F}_\pm(x,t) = \int_0^t \hat{f}_\pm(x,s)\,ds, \quad \hat{F}_0(x,t) = \int_0^t \hat{f}_0(x,s)\,ds.$$

Then we define the C^1-functionals $\hat{\varphi}^\lambda_\pm,\ \hat{\varphi}^\lambda_0 : W^{1,p}_n(\Omega) \to \mathbb{R}$ by

$$\hat{\varphi}^\lambda_\pm(u) = \frac{1}{p}\,\|\nabla u\|_p^p + \frac{\lambda}{p}\,\|u\|_p^p - \int_\Omega \hat{F}_\pm(x,u(x))\,dx,$$
$$\hat{\varphi}^\lambda_0(u) = \frac{1}{p}\,\|\nabla u\|_p^p + \frac{\lambda}{p}\,\|u\|_p^p - \int_\Omega \hat{F}_0(x,u(x))\,dx \ \text{ for all } u \in W^{1,p}_n(\Omega),$$

and consider the order intervals in $W^{1,p}_n(\Omega)$: $T_+ = [0,u_+]$, $T_- = [u_-,0]$ and $T_0 = [u_-,u_+]$.

Claim 1: The critical points of $\hat{\varphi}^\lambda_\pm$ and $\hat{\varphi}^\lambda_0$ are in $T_\pm$ and T_0, respectively.

We prove the assertion only for $\hat{\varphi}_+^\lambda$ because the others are similar. Let $u \in W_n^{1,p}(\Omega)$ be a critical point of $\hat{\varphi}_+^\lambda$. This means

$$-\Delta_p u + \lambda K_p(u) = \hat{f}_+(\cdot, u(\cdot)). \tag{5.6}$$

Then a direct comparison with u_+ and 0 ensures $u \in T_+$.

Claim 2: The sets of critical points of $\hat{\varphi}_+^\lambda$ and $\hat{\varphi}_-^\lambda$ are $\{0, u_+\}$ and $\{u_-, 0\}$, respectively.

We only provide the proof for $\hat{\varphi}_+^\lambda$ since the other is analogous. If $u \in W_n^{1,p}(\Omega)$ is a critical point of $\hat{\varphi}_+^\lambda$, then Claim 1 implies that $u \in T_+$. By the nonlinear regularity theory and strong maximum principle [34] we find out that either $u = 0$ or $u \in \operatorname{int} C_+$. Then we can conclude because u_+ is the smallest positive solution of (P'_λ).

Claim 3: $u_+ \in \operatorname{int} C_+$ and $u_- \in -\operatorname{int} C_+$ are local minimizers of $\hat{\varphi}_0^\lambda$.

Taking into account that $\hat{\varphi}_+^\lambda$ is coercive and sequentially weakly lower semicontinuous, there exists $u \in W_n^{1,p}(\Omega)$ such that

$$\hat{\varphi}_+^\lambda(u) = \inf \hat{\varphi}_+^\lambda. \tag{5.7}$$

Claim 2 implies that $u \in \{0, u_+\}$. Set $\beta = \min_{\overline{\Omega}}\{u_+, -u_-\} > 0$. The first part of hypothesis $\mathrm{H}_2(\mathrm{vi})'$ yields that if $0 < s \le \beta$ is small enough, then

$$\hat{\varphi}_+^\lambda(s) \le \left(\frac{\lambda}{p} s^p - \frac{\xi}{\tau} s^\tau\right) |\Omega|_N < 0 = \hat{\varphi}_+^\lambda(0).$$

This guarantees that $u \neq 0$, which forces $u = u_+ \in \operatorname{int} C_+$. By (5.7) and since $\hat{\varphi}_+^\lambda|_{C_+} = \hat{\varphi}_0^\lambda|_{C_+}$, we derive that u_+ is a local $C_n^1(\overline{\Omega})$-minimizer of $\hat{\varphi}_0^\lambda$. Then Proposition 3.6 implies that u_+ is a local $W_n^{1,p}(\Omega)$-minimizer of $\hat{\varphi}_0^\lambda$. The proof for $u_- \in -\operatorname{int} C_+$ proceeds in the same way, which establishes Claim 3.

We may assume that $\hat{\varphi}_0^\lambda(u_-) \le \hat{\varphi}_0^\lambda(u_+)$ and there is $0 < \delta < \|u_- - u_+\|$ satisfying

$$\hat{\varphi}_0^\lambda(u_+) < \hat{\eta}_\rho^\lambda := \inf_{\partial B_\delta(u_+)} \hat{\varphi}_0^\lambda, \tag{5.8}$$

where $\partial B_\rho(u_+) = \{v \in W_n^{1,p}(\Omega) \,:\, \|v - u_+\| = \rho\}$. By Theorem 2.1 we find $z_\lambda \in W_n^{1,p}(\Omega)$ such that

$$(\hat{\varphi}_0^\lambda)'(z_\lambda) = 0, \tag{5.9}$$

$$\hat{\eta}_\rho^\lambda \le \hat{\varphi}_0^\lambda(z_\lambda) = \inf_{\gamma_0 \in \Gamma_0} \max_{t \in [-1,1]} \hat{\varphi}_0^\lambda(\gamma_0(t)), \tag{5.10}$$

where $\Gamma_0 = \{\gamma \in C([-1,1], W_n^{1,p}(\Omega)) \,:\, \gamma(-1) = u_-,\ \gamma(1) = u_+\}$. Combining (5.8) and (5.10) we see that $z_\lambda \notin \{u_-, u_+\}$, while (5.9) and Claim 1 show that z_λ is a solution of (P'_λ) that belongs to $C_n^1(\overline{\Omega})$ by the regularity theory.

The final step of the proof is to check that $z_\lambda \neq 0$ because then z_λ is necessarily nodal thanks to the extremality of u_- and u_+. Without any loss of generality we may suppose that z_λ and 0 are isolated critical points of $\hat{\varphi}_0^\lambda$. Property (5.8) implies that $H_1((\hat{\varphi}_0^\lambda)^b, (\hat{\varphi}_0^\lambda)^a) \neq 0$, for $a = \hat{\varphi}_0^\lambda(u_+)$ and any $b > \max_{t\in[0,1]} \hat{\varphi}_0^\lambda((1-t)u_- + tu_+)$ (see [12, pages 84-86]). According to [12, pages 89-90]), it is possible to choose z_λ so that

$$C_1(\hat{\varphi}_0^\lambda, z_\lambda) \neq 0. \tag{5.11}$$

On the other hand, we claim that

$$C_k(\hat{\varphi}_0^\lambda, 0) = 0 \ \text{ for every } k \geq 0. \tag{5.12}$$

The proof of (5.12) closely follows [30, Proposition 5]. For the sake of clarity we give it. First, by using $\mathrm{H}_2(\mathrm{vi})'$ and $\mathrm{H}_1(\mathrm{iii})$ we notice that, for each $v \in W_n^{1,p}(\Omega)$, $v \neq 0$, there exists $t^* = t^*(v) \in (0,1)$ fulfilling

$$\hat{\varphi}_0^\lambda(tv) < 0 \ \text{ for all } t \in (0, t^*). \tag{5.13}$$

By $\mathrm{H}_2(\mathrm{vi})'$, we see that

$$\mu \hat{F}_0(x,s) - \hat{f}_0(x,s)s = \mu F(x,s) - f(x,s)s \geq 0$$

for a.a. $x \in \Omega$ and all $|s| \leq \min\{\delta_0, \beta\}$. By the above estimate and taking into account hypothesis $\mathrm{H}_1(\mathrm{iii})$, and the boundedness of u_- and u_+, we derive

$$\mu \hat{F}_0(x,s) - \hat{f}_0(x,s)s \geq -c_2|s|^r \ \text{ for a.a. } x \in \Omega \text{ and all } s \in \mathbb{R},$$

for a constant $c_2 > 0$. Consequently, for $v \in W_n^{1,p}(\Omega)$ with $\hat{\varphi}_0^\lambda(v) = 0$, we infer that

$$\begin{aligned}
&\frac{d}{dt}\hat{\varphi}_0^\lambda(tv)\Big|_{t=1} = \langle (\hat{\varphi}_0^\lambda)'(v), v\rangle - \mu\hat{\varphi}_0^\lambda(v) \\
&= \left(1 - \frac{\mu}{p}\right)\left(\|\nabla v\|_p^p + \lambda\|v\|_p^p\right) + \int_\Omega (\mu \hat{F}_0(x, v(x)) - \hat{f}_0(x, v(x))v(x))\,dx \\
&\geq \left(1 - \frac{\mu}{p}\right)\left(\|\nabla v\|_p^p + \lambda\|v\|_p^p\right) - c_3\|v\|^r,
\end{aligned}$$

with $c_3 > 0$. Since $r \in (p, p^*)$ and $\mu < p$, there is $r_0 \in (0,1)$ such that

$$\frac{d}{dt}\hat{\varphi}_0^\lambda(tv)\Big|_{t=1} > 0 \ \text{ for all } v \in W_n^{1,p}(\Omega) \text{ with } 0 < \|v\| \leq r_0 \ \text{ and } \hat{\varphi}_0^\lambda(v) = 0. \tag{5.14}$$

For $r_0 > 0$ small enough we may admit that 0 is the unique critical point of $\hat{\varphi}_0^\lambda$ in $\overline{B}_{r_0} = \{v \in W_n^{1,p}(\Omega) : \|v\| \leq r_0\}$. We show that if $v \in W_n^{1,p}(\Omega)$ satisfies $0 < \|v\| \leq r_0$ and $\hat{\varphi}_0^\lambda(v) \leq 0$, then

$$\hat{\varphi}_0^\lambda(tv) \leq 0 \ \text{ for all } t \in [0,1]. \tag{5.15}$$

If the assertion were not true, there would exist $t_0 \in (0, 1)$ such that $\hat{\varphi}_0^\lambda(t_0 v) > 0$. Let t_1 stand for the maximal point in $(t_0, 1]$ with the property

$$\hat{\varphi}_0^\lambda(tv) > 0 \text{ for all } t \in [t_0, t_1). \tag{5.16}$$

Observing that $0 < \|t_1 v\| \leq r_0$ and $\hat{\varphi}_0^\lambda(t_1 v) = 0$, (5.14) yields

$$\frac{d}{dt}\hat{\varphi}_0^\lambda(tt_1 v)\Big|_{t=1} > 0.$$

We reach a contradiction in view of the fact that (5.16) implies

$$\frac{d}{dt}\hat{\varphi}_0^\lambda(tt_1 v)\Big|_{t=1} = t_1 \frac{d}{dt}\hat{\varphi}_0^\lambda(tv)\Big|_{t=t_1} = t_1 \lim_{t\uparrow t_1}\frac{\hat{\varphi}_0^\lambda(tv)}{t - t_1} \leq 0.$$

Therefore (5.15) holds true.

On the basis of (5.15) we can define $h : [0, 1] \times (\overline{B}_{r_0} \cap (\hat{\varphi}_0^\lambda)^0) \to \overline{B}_{r_0} \cap (\hat{\varphi}_0^\lambda)^0$ by $h(t, v) = (1 - t)v$. So $\overline{B}_{r_0} \cap (\hat{\varphi}_0^\lambda)^0$ is contractible in itself.

We now check that $(\overline{B}_{r_0} \cap (\hat{\varphi}_0^\lambda)^0) \setminus \{0\}$ is contractible in itself. To this end it suffices to construct a retraction $\omega : \overline{B}_{r_0} \setminus \{0\} \to (\overline{B}_{r_0} \cap (\hat{\varphi}_0^\lambda)^0) \setminus \{0\}$ and to invoke that $W_n^{1,p}(\Omega)$ is infinite dimensional since then $\overline{B}_{r_0} \setminus \{0\}$ is contractible in itself. Specifically, we set

$$\omega(v) = \eta(v)v \text{ for all } v \in \overline{B}_{r_0} \setminus \{0\},$$

with the function $\eta : \overline{B}_{r_0} \setminus \{0\} \to (0, 1]$ given by

$$\eta(v) = \begin{cases} 1 & \text{if } v \in \overline{B}_{r_0} \setminus \{0\},\ \hat{\varphi}_0^\lambda(v) \leq 0 \\ t(v) & \text{if } v \in \overline{B}_{r_0} \setminus \{0\},\ \hat{\varphi}_0^\lambda(v) > 0, \end{cases}$$

where, for $v \in \overline{B}_{r_0}$ with $\hat{\varphi}_0^\lambda(v) > 0$, $t(v) \in (0, 1)$ is the unique number that satisfies $\hat{\varphi}_0^\lambda(t(v)v) = 0$. The existence of $t(v)$ comes from (5.13). The uniqueness of $t(v)$ is a consequence of (5.15) and (5.14). Indeed, arguing by contradiction, suppose that

$$0 < t(v)_1 < t(v)_2 < 1 \text{ and } \hat{\varphi}_0^\lambda(t(v)_k\, v) = 0, \quad k = 1, 2.$$

By (5.15), it is known that $\hat{\varphi}_0^\lambda(t\, t(v)_2 v) \leq 0$ for all $t \in [0, 1]$. Hence $\frac{t(v)_1}{t(v)_2} \in (0, 1)$ is a maximizer of the function $t \mapsto \hat{\varphi}_0^\lambda(t\, t(v)_2\, v)$, $t \in [0, 1]$, so

$$\frac{d}{dt}\hat{\varphi}_0^\lambda(t\, t(v)_1\, v)\Big|_{t=1} = \frac{t(v)_1}{t(v)_2}\frac{d}{dt}\hat{\varphi}_0^\lambda(t\, t(v)_2\, v)\Big|_{t=\frac{t(v)_1}{t(v)_2}} = 0.$$

This contradicts (5.14). Therefore $t(v) \in (0, 1)$ is unique. Property (5.15) and the uniqueness of $t(v)$ lead to

$$\hat{\varphi}_0^\lambda(tv) < 0 \text{ if } t \in (0, t(v)) \text{ and } \hat{\varphi}_0^\lambda(tv) > 0 \text{ if } t \in (t(v), 1]. \tag{5.17}$$

We claim that η is continuous. We have to check the continuity of η at any v with $\hat{\varphi}_0^\lambda(v) = 0$, because otherwise it is a direct consequence of the implicit function theorem (due to $\hat{\varphi}_0^\lambda(t(v)v) = 0$ and (5.14)). Let $v_k \to v$ with $\hat{\varphi}_0^\lambda(v_k) > 0$. Arguing by contradiction and passing to a relabelled subsequence, suppose that there exists $\bar{t} \in (0, 1)$ such that $t(v_k) \leq \bar{t}$. By (5.17) we have that $\hat{\varphi}_0^\lambda(tv_k) > 0$ for all $t \in (\bar{t}, 1]$. Letting $k \to \infty$ we derive that $\hat{\varphi}_0^\lambda(tv) \geq 0$ for all $t \in (\bar{t}, 1]$. Combining with (5.15), it turns out $\hat{\varphi}_0^\lambda(tv) = 0$ for all $t \in (\bar{t}, 1]$, which implies that $\frac{d}{dt}\hat{\varphi}_0^\lambda(tv)\Big|_{t=1} = 0$. This contradicts (5.14). Therefore η is continuous. This implies that the map $[0, 1] \times \overline{B}_{r_0} \setminus \{0\} \to \overline{B}_{r_0} \setminus \{0\}$, $(t, v) \mapsto (t\eta(v) + 1 - t)v$ is a homotopy between ω and the identity, thus ω is indeed a retraction of $\overline{B}_{r_0} \setminus \{0\}$ onto $(\overline{B}_{r_0} \cap (\hat{\varphi}_0^\lambda)^0) \setminus \{0\}$. We have shown that $(\overline{B}_{r_0} \cap (\hat{\varphi}_0^\lambda)^0) \setminus \{0\}$ is contractible in itself. We obtain

$$C_k(\hat{\varphi}_0^\lambda, 0) = H_k(\overline{B}_{r_0} \cap (\hat{\varphi}_0^\lambda)^0, (\overline{B}_{r_0} \cap (\hat{\varphi}_0^\lambda)^0) \setminus \{0\}) = 0 \quad \text{for all } k \geq 0,$$

which establishes (5.12).

In view of (5.11) and (5.12), we conclude that $z_\lambda \neq 0$. □

6. Semilinear problem

In this section, we consider the following semilinear version of (P'_λ):

$$(\mathrm{P}''_\lambda) \qquad \begin{cases} -\Delta u(x) + \lambda u(x) = f(u(x)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

In what follows, by $\{\lambda_k\}_{k\geq 0}$ we denote the increasing sequence of distinct eigenvalues of $(-\Delta, H_n^1(\Omega))$. The hypotheses on the nonlinearity $f(s)$ are:

H_3 $f \in C^1(\mathbb{R})$, $f(0) = 0$, hypotheses H_1 (ii), (iv)-(v) and $\mathrm{H}_2(\mathrm{vi})'$ are satisfied with $p = 2$, and

(i) $|f'(s)| \leq c(1 + |s|^{r-2})$ for all $s \in \mathbb{R}$, some $c > 0$ and $2 < r < 2^*$;
(ii) there exists an integer $m \geq 1$ such that $f'(0) - \lambda_{m+1} \geq \lambda^*$.

Theorem 6.1. *If hypotheses* H_3 *hold and* $\lambda \in (f'(0) - \lambda_{m+1}, f'(0) - \lambda_m)$*, then problem* (P''_λ) *has at least six nontrivial solutions*

$$v_\lambda,\ u_\lambda \in C^2(\overline{\Omega}),\ v_\lambda - u_\lambda \in \operatorname{int} C_+,$$
$$w_\lambda,\ y_\lambda \in C^2(\overline{\Omega}),\ w_\lambda - y_\lambda \in \operatorname{int} C_+,$$

and

$$z_\lambda,\ \omega_\lambda \in C^2(\overline{\Omega}) \textit{ nodal solutions.}$$

Proof. Fix $\lambda \in (f'(0) - \lambda_{m+1}, f'(0) - \lambda_m)$. We notice that hypothesis H_1(vii) is fulfilled. This is valid because, as f being continuously differentiable, corresponding to the compact interval $[-\ell, \ell]$ we can choose a sufficiently large $c_\ell > 0$ such that $f'(s) + c_\ell > 0$ for all $s \in [-\ell, \ell]$. As all the other hypotheses are satisfied, Theorem 5.5 applies. We thus have the five solutions $u_\lambda, v_\lambda \in \operatorname{int} C_+$, $u_\lambda \le v_\lambda$, $u_\lambda \ne v_\lambda$, $y_\lambda, w_\lambda \in -\operatorname{int} C_+$, $y_\lambda \le w_\lambda$, $y_\lambda \ne w_\lambda$, and $z_\lambda \in C^1_n(\overline{\Omega})$ nodal. Since the solutions are bounded, there is a constant $c > 0$ such that $\Delta(v_\lambda - u_\lambda) \le (\lambda + c)(v_\lambda - u_\lambda)$, which yields $v_\lambda - u_\lambda \in \operatorname{int} C_+$ (see [34]). Similarly, we show that $w_\lambda - y_\lambda \in \operatorname{int} C_+$. Moreover, the regularity theory (see, *e.g.*, [33, page 219]) implies that $u_\lambda, v_\lambda, w_\lambda, y_\lambda, z_\lambda \in C^2(\overline{\Omega})$.

Let $\hat{\varphi}_0^\lambda$ be the functional in the proof of Theorem 5.5, where it was proven that u_-, u_+ are local minimizers of $\hat{\varphi}_0^\lambda$, that can be assumed to be isolated, and z_λ is a critical point of $\hat{\varphi}_0^\lambda$ of mountain pass type. Therefore, we have

$$C_k(\hat{\varphi}_0^\lambda, u_+) = C_k(\hat{\varphi}_0^\lambda, u_-) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \ge 0 \tag{6.1}$$

(see [12, page 33] and [23, page 175]) as well as

$$C_k(\hat{\varphi}_0^\lambda, z_\lambda) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \ge 0 \tag{6.2}$$

(by arguing as in [29, proof of Theorem 3.6] on the basis of [23, page 195]). Note that, if $u \in \ker(\hat{\varphi}_0^\lambda)''(0)$, then u satisfies

$$\begin{cases} -\Delta u(x) = (f'(0) - \lambda)u(x) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.3}$$

By hypothesis, we know that $\lambda_m < f'(0) - \lambda < \lambda_{m+1}$ which, in conjunction with (6.3), implies that $u = 0$. This ensures that 0 is a nondegenerate critical point of $\hat{\varphi}_0^\lambda$ and its Morse index is equal to $d_m = \dim \oplus_{k=0}^m E(\lambda_k)$ ($E(\lambda_k)$ being the eigenspace of $(-\Delta, H^1_n(\Omega))$ corresponding to the eigenvalue λ_k). Hence

$$C_k(\hat{\varphi}_0^\lambda, 0) = \delta_{k,d_m}\mathbb{Z} \quad \text{for all } k \ge 0 \tag{6.4}$$

(see [12, page 34] and [23, page 188]). Since the functional $\hat{\varphi}_0^\lambda$ is coercive, from the definition of critical groups at infinity, we have

$$C_k(\hat{\varphi}_0^\lambda, \infty) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \ge 0. \tag{6.5}$$

Suppose that $0, u_-, u_+, z_\lambda$ were the only critical points of $\hat{\varphi}_0^\lambda$. Then from (6.1), (6.2), (6.4), (6.5) and the Poincaré–Hopf formula (see (2.1)), we obtain $2(-1)^0 + (-1)^1 + (-1)^{d_m} = (-1)^0$, that is $(-1)^{d_m} = 0$, a contradiction. So, $\hat{\varphi}_0^\lambda$ has one more critical point $\omega_\lambda \in H^1_n(\Omega)$ distinct from $0, v_-, u_+, w_\lambda$. In addition, from Claim 1 in the proof of Theorem 5.5, we know that $\omega_\lambda \in [u_-, u_+]$. Hence ω_λ is a nontrivial solution of (P''_λ) which belongs to $C^2(\overline{\Omega})$ by the regularity theory. Taking into account that u_+ is the least positive solution and u_- is the greatest negative solution, this enables us to conclude that ω_λ is a second nodal solution of (P''_λ). □

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