

Regularity of the singular set for Mumford-Shah minimizers in \mathbb{R}^3 near a minimal cone

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Abstract. We prove that if (u, K) is a minimizer of the Mumford-Shah functional in an open set Ω of \mathbb{R}^3 , and if $x \in K$ and $r > 0$ are such that K is close enough to a minimal cone of type \mathbb{P} (a plane), \mathbb{Y} (three half planes meeting at x with 120° angles) or \mathbb{T} (cone over the 6 edges of a regular tetrahedron centered at x) in terms of Hausdorff distance in $B(x, r)$, then K is $C^{1,\alpha}$ equivalent to the minimal cone in $B(x, cr)$ where $c < 1$ is a universal constant.

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1. Introduction

The Mumford-Shah functional originally comes from an image-segmentation problem. If Ω is an open subset of \mathbb{R}^2 , for example a rectangle, and $g \in L^\infty(\Omega)$ is an image, D. Mumford and J. Shah [15] proposed to define

$$J(u, K) := \int_{\Omega \setminus K} |\nabla u|^2 dx + \int_{\Omega \setminus K} (u - g)^2 dx + \mathcal{H}^1(K) \quad (1.1)$$

and, to get a segmentation of the image g , to minimize the functional J over all the admissible pairs (u, K) where K is a closed one-dimensional set and u is regular outside K . More precisely (u, K) belongs to the set of admissible pairs \mathcal{A} defined by

$$\mathcal{A} := \left\{ (u, K); K \subset \Omega \text{ is closed, } u \in W_{\text{loc}}^{1,2}(\Omega \setminus K) \right\}. \quad (1.2)$$

For any solution (u, K) that minimizes J , the function u represents a “smoother” version of the image g and the set K stands for the edges of the image. One can easily be intuitively convinced that when being minimized, J tends to detect the singularities of g , which leads to solve the desired segmentation problem.

Of course there are no restrictions to define the functional in higher dimensions, simply replacing \mathcal{H}^1 by \mathcal{H}^{N-1} in (1.1) when $\Omega \subset \mathbb{R}^N$. Existence of minimizers is now a well-known result (but however nontrivial, see for instance [9]) using SBV theory. As far as the optimal regularity for the singular set K of a minimizer in \mathbb{R}^2 has been investigated, the following conjecture from D. Mumford and J. Shah is currently still open.

Conjecture 1.1 ([15, Mumford-Shah, 1989]). Let (u, K) be a reduced minimizer for the functional J . Then K is the finite union of C^1 arcs.

A pair $(u, K) \in \mathcal{A}$ is called “reduced”, when there exists no pair $(\tilde{u}, \tilde{K}) \in \mathcal{A}$ such that $\tilde{K} \subsetneq K$ and \tilde{u} is an extension of u in $W_{\text{loc}}^{1,2}(\Omega \setminus \tilde{K})$. Given a pair $(u, K) \in \mathcal{A}$, one can always find a reduced pair $(\tilde{u}, \tilde{K}) \in \mathcal{A}$ such that $\tilde{K} \subset K$ and \tilde{u} is an extension of u (see [6, Proposition 8.2]). We cannot expect any regularity result on non-reduced minimizers since we still get a minimizing pair from a minimizer (u, K) by adding some negligible set to K . In the sequel, we will always assume (u, K) to be reduced. Some partial results are true for this conjecture. For instance it is known that K is $C^{1,\alpha}$ almost everywhere (see [1, 3, 5]), which also leads to further regularity under more assumptions on g ([2, 11]).

Many results about the Mumford-Shah functional are stated in \mathbb{R}^2 . In dimension 3, lots of proprieties are unknown. The theorem of L. Ambrosio, N. Fusco and D. Pallara [1] about regularity of minimizers is one of the best regularity results valid in every dimension N . It says in particular that if K is flat enough in a ball B , and if the energy there is not too big, then K is a $C^{1,\alpha}$ hypersurface in a slightly smaller ball. The proof of L. Ambrosio, N. Fusco and D. Pallara relies on a “tilt-estimate” that it does not seem to be possible to generalize to get a similar perturbation result close to other geometric configurations different from a hyperplane.

We claim that dimension 3 is a natural step into optimal regularity results in higher dimension. Indeed, some works on minimal surfaces of soap bubbles-type in dimension 3 give some indications about what could be the singularities of a Mumford-Shah minimizer, at least when the energy is small. In particular in the famous paper of Jean Taylor [16] we can find the description of the three minimal cones in \mathbb{R}^3 . Jean Taylor also proves that any minimal surface is locally C^1 equivalent to one of these cones. So we can think that for the Mumford-Shah minimizers a similar description should hold. We prove in this paper that this is the case whenever the energy of u is small enough.

Our main theorem is a perturbation result near minimal cones in \mathbb{R}^3 . More precisely, assume that in a ball the singular set K of a Mumford-Shah minimizer is very close to a minimal cone in Hausdorff distance; then we prove that K is $C^{1,\alpha}$ equivalent to this cone in a slightly smaller ball. This is a generalization to the cones \mathbb{Y} and \mathbb{T} of what L. Ambrosio, N. Fusco et D. Pallara have done with hyperplanes in [1]. It is also a generalization in higher dimension of what G. David [6] did in \mathbb{R}^2 about the regularity near lines and propellers.

The key ingredient in the proof of our main result is a new way to construct some competitors using a stopping-time argument on the flatness of K (or closeness to minimal cones), together with a Whitney-type extension for the function u in the

region where the set K is geometrically bad. This technique was announced in the introduction of [14] and appears here like a powerful tool to get thin estimates on the energy of a minimizer, that lead to regularity.

Now, to be more precise, we start giving some definitions. Let Ω be an open subset of \mathbb{R}^N and let \mathcal{A} still denote the set of admissible pairs defined in (1.2).

Definition 1.2. Let $(u, K) \in \mathcal{A}$ and B be a ball such that $B \subset \Omega$. A competitor for the pair (u, K) in the ball B is a pair $(v, L) \in \mathcal{A}$ such that

$$\left. \begin{aligned} u &= v \\ K &= L \end{aligned} \right\} \text{ in } \Omega \setminus B$$

and in addition such that if x and y are two points in $\Omega \setminus (B \cup K)$ that are separated by K then they are also separated by L .

The expression “be separated by K ” means that x and y lie in different connected components of $\Omega \setminus K$.

Definition 1.3. A gauge function h is a non-negative and non-decreasing function on \mathbb{R}^+ such that $\lim_{t \rightarrow 0} h(t) = 0$.

Definition 1.4. Let Ω be an open subset of \mathbb{R}^N . A Mumford-Shah minimizer with gauge function h is a pair $(u, K) \in \mathcal{A}$ such that for every ball $B \subset \Omega$ and every competitor (v, L) in B we have

$$\int_{B \setminus K} |\nabla u|^2 dx + \mathcal{H}^{N-1}(K \cap B) \leq \int_{B \setminus L} |\nabla v|^2 dx + \mathcal{H}^{N-1}(L \cap B) + r^{N-1}h(r)$$

with r the radius of the ball B .

It is not difficult to prove that a minimizer for the functional J of the beginning of the introduction is a minimizer in the sense of Definition 1.4 with $h(r) = C_N \|g\|_\infty^2 r$ as gauge function, where C_N is a dimensional constant (see [6, Proposition 7.8, page 46]).

Definition 1.5. A global minimizer in \mathbb{R}^N is a Mumford-Shah minimizer in the sense of Definition 1.4 with $\Omega = \mathbb{R}^N$ and $h = 0$.

In this paper we will not work on global minimizers but they play an important role in the study of the Mumford-Shah functional, which is why we introduced the definition. In dimension 2, only three types of connected sets can give a global minimizer; K is a line and u is locally constant, K is a propeller (a union of three half-lines meeting with 120 degrees angles) and u is locally constant as well, and finally K is a half-line and u is a *cracktip*, namely $C\sqrt{r} \sin(\theta/2)$ with a proper constant C . Knowing whether there exist other global minimizers or not would give a positive answer to the Mumford-Shah conjecture. The main fact is that every blow-up limit of a Mumford-Shah minimizer is a global minimizer. We do not know much about global minimizers in dimension greater than 2. See [13] (or [12]) for a beginning of investigation in \mathbb{R}^3 .

Let us now define the minimal cones that will be used in the next sections. We define three types of cones. Cones of type 1 are planes in \mathbb{R}^3 , also called \mathbb{P} . Cones of types 2 and 3 and their spines are defined as in [8] and [14], in the following way.

Definition 1.6. Define $\text{Prop} \subset \mathbb{R}^2$ by

$$\begin{aligned} \text{Prop} = & \{(x_1, x_2) : x_1 \geq 0, x_2 = 0\} \\ & \cup \{(x_1, x_2) : x_1 \leq 0, x_2 = -\sqrt{3}x_1\} \\ & \cup \{(x_1, x_2) : x_1 \leq 0, x_2 = \sqrt{3}x_1\}. \end{aligned}$$

Then let $Y_0 = \text{Prop} \times \mathbb{R} \subset \mathbb{R}^3$. The spine of Y_0 is the line $L_0 = \{x_1 = x_2 = 0\}$. A cone of type 2 (or of type \mathbb{Y}) is a set $Y = R(Y_0)$ where R is the composition of a translation and a rotation. The spine of Y is then the line $R(L_0)$. We denote by \mathbb{Y} the set of all cones of type 2.

Definition 1.7. Let $A_1 = (1, 0, 0)$, $A_2 = (-\frac{1}{3}, \frac{2\sqrt{2}}{3}, 0)$, $A_3 = (-\frac{1}{3}, -\frac{\sqrt{2}}{3}, \frac{\sqrt{6}}{3})$, and $A_4 = (-\frac{1}{3}, -\frac{\sqrt{2}}{3}, -\frac{\sqrt{6}}{3})$ be the four vertices of a regular tetrahedron centered at 0. Let T_0 be the cone over the union of the 6 edges $[A_i, A_j]$ $i \neq j$. The spine of T_0 is the union of the four half lines $[0, A_i]$. A cone of type 3 (or of type \mathbb{T}) is a set $T = R(T_0)$ where R is the composition of a translation and a rotation. The spine of T is the image by R of the spine of T_0 . We denote by \mathbb{T} the set of all cones of type 3.

In the sequel we will also denote by $\text{type}(Z)$ the number 1, 2 or 3 corresponding to the type of the minimal cone Z .

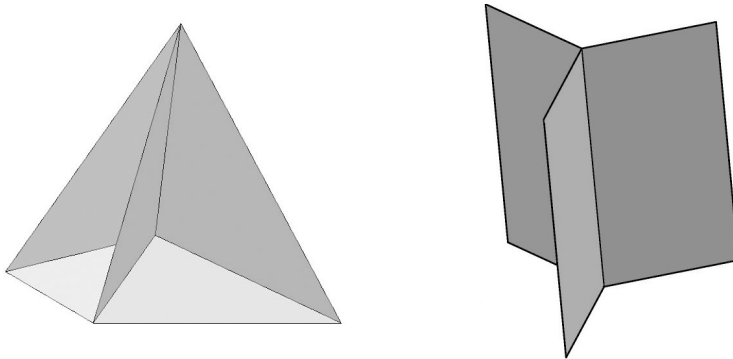


Figure 1.1. Cones of type \mathbb{Y} and \mathbb{T} .

We denote by $D_{x,r}$ the normalized Hausdorff distance between two closed sets E and F in $B(x, r)$ defined by

$$D_{x,r}(E, F) := \frac{1}{r} \left\{ \max \left\{ \sup_{y \in E \cap B(x,r)} d(y, F), \sup_{y \in F \cap B(x,r)} d(y, E) \right\} \right\}. \quad (1.3)$$

We now come to the main result of the paper.

Theorem 1.8. *For any constants $C > 0$ and $b \in (0, 1]$ we can find a positive constant $\varepsilon > 0$ such that the following holds. Let (u, K) be a reduced Mumford-Shah minimizer in $\Omega \subset \mathbb{R}^3$, with gauge function $h(r) = Cr^b$. Let $x \in K$ and $r > 0$ be such that $B(x, r) \subset \Omega$. Assume in addition that there is a minimal cone Z of type \mathbb{P} , \mathbb{Y} or \mathbb{T} centered at x such that*

$$D_{x,r}(K, Z) + h(r) \leq \varepsilon.$$

Then there is a diffeomorphism ϕ of class $C^{1,\alpha}$ from $B(x, cr)$ to its image such that $K \cap B(x, cr) = \phi(Z) \cap B(x, cr)$, where c is a universal constant.

When (u, K) is a Mumford-Shah minimizer in $\Omega \subset \mathbb{R}^N$ and $B(x, r)$ is a ball such that $B(x, r) \subset \Omega$, we denote by $\omega_2(x, r)$ the normalized energy of u in $B(x, r)$, defined by

$$\omega_2(x, r) := \frac{1}{r^{N-1}} \int_{B(x,r) \setminus K} |\nabla u|^2 dx. \quad (1.4)$$

Arguing with blow-up limits we also get a version of Theorem 1.8 with only a condition on the normalized energy instead of the geometric condition.

Theorem 1.9. *For any constants $C > 0$ and $b \in (0, 1]$ we can find a constant $\varepsilon > 0$ such that the following holds. Let (u, K) be a reduced Mumford-Shah minimizer in $\Omega \subset \mathbb{R}^3$, with gauge function $h(r) = Cr^b$. Let $x \in K$ and $r > 0$ be such that $B(x, r) \subset \Omega$ and*

$$\omega_2(x, r) + h(r) \leq \varepsilon.$$

Then there is a diffeomorphism ϕ of class $C^{1,\alpha}$ from $B(x, cr)$ to its image, and there is a minimal cone Z of type \mathbb{P} , \mathbb{Y} or \mathbb{T} such that $K \cap B(x, cr) = \phi(Z) \cap B(x, cr)$ where c is a universal constant.

In all the following we will work in \mathbb{R}^3 . However, the proof of Theorem 1.8 still works in higher dimension for the case of hyperplanes so that we could have a new proof of L. Ambrosio, N. Fusco, D. Pallara's entire result [1]. With the same proof we could also imagine to have other results in \mathbb{R}^N , but the analogue of Jean Taylor's Theorem in higher dimension is missing. Indeed, one of the ingredients to prove Theorem 1.8 is to use the description of the singularities for a minimal surface in \mathbb{R}^3 . In particular, we will use the recent work of G. David ([4, 7]) following J. Taylor [16], that is the analogue of Theorem 1.8 but for almost minimal sets which, are defined below.

Definition 1.10. An MS-competitor for the closed set E in $\Omega \subset \mathbb{R}^N$ is a closed set F such that there is a ball $B \subset \Omega$ of radius r with

$$F \setminus B = E \setminus B$$

and if $x, y \in \Omega \setminus (B \cup E)$ are separated by E then they are also separated by F .

Definition 1.11. A set $E \subset \Omega \subset \mathbb{R}^N$ is MS-almost minimal with gauge function h if

$$\mathcal{H}^{N-1}(E \cap B) \leq \mathcal{H}^{N-1}(F \cap B) + r^{N-1}h(r)$$

for all ball $B \subset \Omega$ and for all MS-competitor F for E in the ball B (of radius r).

If E is an MS-almost minimal set in \mathbb{R}^3 we set

$$\theta(x, r) = r^{-2}\mathcal{H}^2(E \cap B(x, r)).$$

By monotonicity results on the density of an almost minimal set (provided sufficient decay on h), the limit as r tends to 0 of θ exists at every point x (see [7, Proposition 5.24]). The limit is called “density” of E at the point x and will be denoted by $\theta(x)$. This quantity can only take three possible values, corresponding to the three minimal cones. Then we introduce the excess of density, defined by

$$f(x, r) = \theta(x, r) - \lim_{t \rightarrow 0} \theta(x, t) = \theta(x, r) - \theta(x).$$

[4, Corollary 12.25] says the following.

Theorem 1.12 ([4]). *For each choice of $b \in (0, 1]$, and $C_0 > 0$ we can find $\alpha > 0$ and $\varepsilon_1 > 0$ such that the following holds. Let E be a reduced MS-almost minimal set in $\Omega \subset \mathbb{R}^3$ with gauge function h . Suppose that $0 \in E$, $r_0 > 0$ is such that $B(0, 110r_0) \subset \Omega$ and h is satisfying*

$$h(r) \leq C_0 r^b \quad \text{for } 0 < r < 220r_0.$$

Assume in addition that

$$f(0, 110r_0) + C_0 r_0^b \leq \varepsilon_1 \tag{1.5}$$

and

$$D_{0,100r_0}(E, Z) \leq \varepsilon_1$$

where Z is a minimal cone centered at the origin such that

$$\mathcal{H}^2(Z \cap B(0, 1)) \leq d(0).$$

Then for $x \in E \cap B(0, r_0)$ and $0 < r \leq r_0$ there is a $C^{1,\alpha}$ diffeomorphism $\Phi : B(0, 2r) \rightarrow \Phi(B(x, 2r))$, such that $\Phi(0) = x$, $|\Phi(y) - y - x| \leq 10^{-2}r$ for $y \in B(0, 2r)$ and $E \cap B(x, r) = \Phi(Z) \cap B(x, r)$.

To apply Theorem 1.12 to a Mumford-Shah minimizer, the key point is to control the normalized energy of u (that is the quantity ω_2). A big part of this fact is already contained in a preliminary paper [14] where a decay estimate on the energy for energy minimizers is proved by a (technical) compactness argument. Actually we will be able to control the energy but with some rest that will be computed in term of what we call the “bad mass” $m(r)$, namely the Hausdorff measure of the

part where K is geometrically very bad. This quantity need also to be controlled, and for this we will use a second compactness argument.

The present paper is organized in three main sections. Section 1 describes some tools to construct new competitors, in particular using a stopping-time argument. In Section 2 we employ these tools in order to find some estimates on the normalized energy, the bad mass and minimality defect. Finally in the last section we prove that the above estimates imply the desired regularity result.

Let us go further and give more precisions about our approach and the key points to prove Theorem 1.8. The first section begins with the control of the normalized Jump. Although this section could seem somehow technical to the novice reader, it is actually not the most difficult part. Indeed, it consists in some easy generalizations in \mathbb{R}^3 of what G. David [5] already did in \mathbb{R}^2 . However, this preliminary work is crucial to avoid some topological and geometrical problems in all the sequel. Indeed, the inverse of the jump of u controls the size of the holes of K which allows us to work with a set F that is “separating” and which difference with the original set K has very small \mathcal{H}^2 measure. We also need a further property about the flatness of F at small scales, which will be called “Property \star ”. This is probably the only real difference with the 2-dimensional results about the jump contained in [5].

Then we give some new tools to construct some competitors. The method is based on a stopping time argument on the flatness, or more generally on the closeness to minimal cones. The stopping-time on the geometry of K is then combined with a Whitney extension for u . In particular, some preliminary work from [14] about the Whitney extension associated to a “geometric function” will be needed. We also introduce one of our main quantity, namely the “bad mass” denoted $m(r)$ which corresponds to the normalized total mass of “bad balls” for which the stopping-time stops.

We end the first section of the paper with a compactness Lemma that is needed later to control the bad mass. The rough idea to say that the stopping-time does not occur too often is as follows. It is proved in [4] that if E is an almost minimal set sufficiently close to a minimal cone in $B(r_0)$, then the closeness to minimal cones of $E \cap B(r)$ decays with the radius. By contradiction we deduce that if E is close enough to a minimal cone in a ball $B(r)$, and if E stops being close to a minimal cone in half of this ball, then E is not minimal. Consequently there is a set L that coincides with E at the boundary of the ball and satisfies $\mathcal{H}^2(E \cap B(r)) - \mathcal{H}^2(L \cap B(r)) > r\eta$. Applied to the singular set K of a Mumford-Shah minimizer, it will imply that the total mass of bad balls cannot be greater than $1/\eta$ times the lack of minimality of K . This is done in the next section.

Section 2 contains the “heart” of the proof. We will use the tools described in the previous section in order to get some estimates about the two main quantities that we want to control: the normalized energy $\omega_2(r)$ and the bad mass $m(r)$. The bad mass will be controlled using the compactness Lemma as it was described above. For the normalized energy, it also follows from a compactness argument that is almost contained in our preliminary paper [14]. Thank to the work in [14] we just need to compare an energy minimizer with a Mumford-Shah minimizer. At the end

of Section 2 we also prove that the minimality defect of K depends on those two main quantities.

Finally in the last section we prove that the estimates of Section 2 yields some decay on both $m(r)$ and $\omega_2(r)$, implying in particular that the stopping-time actually never happens if the singular set is close enough to a minimal cone and this leads to our regularity result. At the end we state a few different versions of the main theorem.

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2. Tools for the construction of competitors

Before describing some tools to construct new competitors, let us first recall some definitions and geometrical results from [14].

2.1. First definitions and notation

In the sequel the letter Z will always denote a generic minimal cone of type \mathbb{P} , \mathbb{Y} or \mathbb{T} . When Z is of type \mathbb{T} , the definition of its center is clear. When Z is of type \mathbb{Y} we will call *center* any point that belongs to the spine of Z , while in the case when Z is of type \mathbb{P} , any point $x \in Z$ is a center. We say that Z is centered at x_0 if x_0 is a center for Z . In the sequel we will use a notion of *almost centered* cones that is defined as follows.

Definition 2.1 (Almost centered). Let Z be a minimal cone and B a ball that meets Z . We say that Z is almost centered in B if the center of Z lies in $\frac{1}{10}B$.

The next lemma will be useful to deal with almost centered cones.

Lemma 2.2 ([14]). *Let Z be a minimal cone in \mathbb{R}^3 that contains 0 (but is not necessarily centered at 0). Then for any $r_0 > 0$ there exists a r_1 such that*

$$r_1 \in \{r_0, 10r_0, 100r_0\}$$

and such that we can find a cone Z' , containing 0 and centered in $B(0, \frac{1}{10}r_1)$ with $Z \cap B(0, r_1) = Z' \cap B(0, r_1)$.

Now in order to define the *jump* $J(x, r)$, a quantity that will be crucial in the sequel, we need to introduce some notations. Let (u, K) be a reduced Mumford-Shah minimizer in $\Omega \subset \mathbb{R}^3$. We denote by $\beta(x, r)$ the “generalized Peter Jones unilateral number” defined by

$$\beta(x, r) := \frac{1}{r} \inf_Z \left\{ \sup \{ \text{dist}(y, Z); y \in K \cap B(x, r) \} \right\} \tag{2.1}$$

where the infimum is taken over all the cones of type \mathbb{P} , \mathbb{Y} , or \mathbb{T} that contain x (but are not necessarily centered at x). Sometimes we will use the notation $\beta_K(x, r)$ to precise that the quantity is associated to the set K but it will not be a general rule.

Definition 2.3 (Associated cone). Let $K \subset \mathbb{R}^3$ be a closed set, $x \in K$ and $r > 0$. Then any minimal cone Z satisfying

$$\sup_{y \in K \cap B(x, r)} \text{dist}(y, Z) \leq 2r\beta_K(x, r)$$

will be called an associated cone in $B(x, r)$.

Notice that there always exists at least one associated minimal cone. We will choose one and denote this cone by $Z(x, r)$. In addition if $\beta_K(x, r) \leq 10^{-5}$ and if $Z(x, r)$ is almost centered in $B(x, r)$, we will denote $\kappa(x, r)$ the number of connected components of $B(x, r) \setminus Z(x, r)$ which is actually equal to $\text{type}(Z(x, r)) + 1$. Notice that from Lemma 2.2 we know in particular that if $Z(x, r)$ is not almost centered, then we can assume that $Z(x, r/10)$ or $Z(x, r/100)$ is.

Furthermore, for all $k \in \mathbb{N} \cap [1, \kappa(x, r)]$ we consider a ball $D_k(x, r)$ of radius $\frac{1}{10}r$ in such a way that each $D_k(x, r)$ is situated in one of the connected components of $B(x, r) \setminus Z(x, r)$, the farthest as possible from $Z(x, r)$. We also denote by $m_k(x, r)$ the mean value of u on $D_k(x, r)$. Then we introduce

$$\delta_{k,l}(x, r) = |m_k(x, r) - m_l(x, r)|$$

and finally, the normalized jump in $B(x, r)$ is defined by

$$J(x, r) := r^{-\frac{1}{2}} \min\{\delta_{k,l} : 1 < k, l < \kappa(x, r) \text{ and } k \neq l\}. \tag{2.2}$$

Now we need to define the jump in the case when $\beta(x, r) \leq 10^{-7}$ but with an associated minimal cone that is not necessarily almost centered. To do so, we remind that the recentering Lemma 2.2 insures that $B(x, r/10)$ or $B(x, r/100)$ has an associated cone which is almost centered. Moreover $\beta(x, r) \leq 10^{-7}$ implies that $\beta(x, r/10) \leq 10^{-5}$ and $\beta(x, r/100) \leq 10^{-5}$. Then we define the normalized jump $J(x, r)$ as being equal to the jump of the first ball between $B(x, r/10)$ or $B(x, r/100)$ for which the associated cone is almost centered.

All the parameters that define the jump (choice of cone $Z(x, r)$, constant 10 to have the almost centering property, diameter and position of the $D_k(x, r)$) are not so important since the difference is just multiplying the jump by a constant.

Finally we introduce a notion of separation.

Definition 2.4 (Separating). Let K be a closed set in \mathbb{R}^3 such that $\beta_K(x, r) \leq 10^{-5}$ and $Z(x, r)$ is almost centered in $B(x, r)$. We say that K is separating in $B(x, r)$ if each $D_k(x, r)$ lie in a different connected component of $B(x, r) \setminus K$.

Some notation:

- $\beta(x, r)$: defined in (2.1);
- Z : a generic minimal cone;
- $\text{type}(Z) \in \{1, 2, 3\}$: the type of the cone Z ;
- $Z(x, r)$: a minimal cone associated to $B(x, r)$;
- $\xi(x, r)$: the number of connected components of $B(x, r) \setminus Z(x, r)$ when $Z(x, r)$ is almost centered (equal to $\text{type}(Z(x, r) + 1)$);
- $J(x, r)$: the jump of u defined in (2.2) when $Z(x, r)$ is almost centered and defined in the paragraph after (2.2) in the general case;
- $D_k(x, r)$: the balls from the definition of the jump $J(x, r)$ when $Z(x, r)$ is almost centered;
- $m_k(x, r)$: the mean value of u on $D_k(x, r)$;
- C : a universal constant which value can change from line to line;
- dist : the euclidian distance in \mathbb{R}^3 ;
- $\omega_2(x, r)$: the normalized energy (defined in (1.4)).

2.2. Separation and control of the jump

It will be convenient to work with a set that is separating (in the sense of Definition 2.4). This is why in a first part we have to control the jump of function u , that will be useful to estimate the size of holes in K , and will allow us to replace K with a set F that contains K and is separating. The result is the same as [6, Proposition 1 page 303] but generalized to the case of \mathbb{Y} and \mathbb{T} . We also use the opportunity here to prove an additional fact about the set F (called Property \star) that will be used later. Recall that the normalized energy in the ball $B(x, r)$ is denoted by

$$\omega_2(x, r) := \frac{1}{r^2} \int_{B(x,r) \setminus K} |\nabla u|^2 dx.$$

Proposition 2.5. *Let (u, K) be a Mumford-Shah minimizer in $\Omega \subset \mathbb{R}^3$. Suppose that there is an $x \in K$, a $r > 0$ and a positive constant $\varepsilon < 10^{-10}$ such that $B := B(x, r) \subset \Omega$,*

$$\beta(x, r) \leq \varepsilon$$

and that the associated cone $Z(x, r)$ is almost centered. Moreover, assume that $J(x, r) \neq 0$,

$$\omega_2^{\frac{1}{2}}(x, r) J(x, r)^{-1} \leq \varepsilon \tag{2.3}$$

and that

$$\omega_2(x, r)^{\frac{1}{8}} \leq C J(x, r) \tag{2.4}$$

with C a positive universal constant given by the demonstration. Then there is a compact set $F(x, r) \subset B(x, r)$ such that

$$K \cap B(x, r) \subset F(x, r) \subset \{y \in B; \text{dist}(y, Z(x, r)) \leq Cr\sqrt{\varepsilon}\} \tag{2.5}$$

$$F(x, r) \text{ separates each } D_k(x, r) \text{ from } D_l(x, r) \text{ for } k \neq l \text{ in } B(x, r) \tag{2.6}$$

$$\mathcal{H}^2(F(x, r) \setminus K) \leq Cr^2\omega_2(x, r)^{\frac{1}{2}} J(x, r)^{-1}.$$

Moreover, $F(x, r)$ satisfies Property \star (defined below).

Property \star shows that we control the geometry of $F(x, r)$ at small scales when the geometry of K is controlled. This is the definition.

Definition 2.6 (Property \star). $F \supseteq K$ satisfies Property \star if for every $\varepsilon_0 < 10^{-5}$, $y \in K \cap B(x, r)$ and $s > 0$ such that

$$\inf\{t; \forall t' \geq t, \beta_K(y, t') \leq \varepsilon_0\} \leq s \leq d(y, \partial B(x, r))$$

we have

$$\beta_F(y, s) \leq \varepsilon_0.$$

Remark 2.7. Condition (2.4) allows us to have Property \star and Condition (2.3) is here to prove the last inclusion of (2.5). Proposition 2.5 is still true without Property \star and without Conditions (2.3) and (2.4). In this case, (2.5) is proved by use of a retraction as in [6, 44.1].

Proof. The first step is the same as [6, Proposition 1 page 303] but applied to \mathbb{Y} and \mathbb{T} as well. However we will write the entire proof here because it will be easier next to show Property \star .

For all λ we denote

$$S(\lambda) := \{y \in B(x, r); \text{dist}(y, Z(x, r)) \leq \lambda r\}$$

and denote by $A_k(\lambda)$ for any $k \in \mathbb{N} \cap [1, \kappa(x, r)]$ the connected component of $B(x, r) \setminus S(\lambda)$ that meets $D_k(x, r)$. Since x and r are fixed we will denote now D_k and m_k instead of $D_k(x, r)$ and $m_k(x, r)$ (recall that $m_k(x, r)$ is the mean value of u on $D_k(x, r)$).

We set $V = B(x, r) \setminus K$. Let us find a function v such that

$$v(y) = m_k \quad \text{for } y \in A_k(1/10) \tag{2.7}$$

and

$$\int_V |\nabla v| \leq C \int_V |\nabla u|. \tag{2.8}$$

To do this we consider for all k a function φ_k such that $0 \leq \varphi_k \leq 1$ and $\varphi_k = 1$ on $A_k(1/10)$, $\varphi_k = 0$ on $V \setminus A_k(1/100)$ and $|\nabla\varphi_k| \leq Cr^{-1}$. Then we set

$$\varphi = 1 - \sum_k \varphi_k$$

and

$$v = \varphi u + \sum_k \varphi_k m_k.$$

We have (2.7) trivially. Concerning (2.8) we have

$$\nabla v(y) = \varphi(y)\nabla u(y) - \sum_k 1_{A_k(1/100)}(y)\nabla\varphi_k(y)[u(y) - m_k]$$

and since $\varepsilon < 10^{-5}$, the $A_k(1/100)$ do not meet K and then applying Poincaré inequality in $A_k(1/100)$ gives

$$\begin{aligned} \int_{A_k(1/100)} |\nabla\varphi_k(y)||u(y) - m_k|dy &\leq Cr^{-1} \int_{A_k(1/100)} |u(y) - m_k|dy \\ &\leq C \int_{A_k(1/100)} |\nabla u(y)|dy \end{aligned}$$

thus (2.8) is verified.

Now we want to replace v with a smooth function w in V such that

$$w(y) = m_k \quad \text{for } y \in A_k(2/10) \tag{2.9}$$

and

$$\int_V |\nabla w| \leq C \int_V |\nabla u|. \tag{2.10}$$

For this purpose we use a Whitney extension. For all $z \in V$ we denote by $B(z)$ the ball $B(z, 10^{-2}d(z, \partial V))$, and we let $X \subset V$ be a maximal set such that for all $z \in X$, the $B(z)$ are disjoint. Note that by maximality, if $y \in V$, then $B(y)$ meets some $B(z)$ for a certain $z \in X$ hence $y \in 4B(z)$ thus the $4B(z)$ cover V .

For all $z \in X$ we choose a function φ_z which support is contained in $5B(z)$ such that $\varphi_z(y) = 1$ for all $y \in 4B(z)$, $0 \leq \varphi_z(y) \leq 1$ and $|\nabla\varphi_z(y)| \leq C \text{dist}(z, \partial V)^{-1}$ everywhere. Set $\Phi(y) = \sum_{z \in X} \varphi_z(y)$ on V . We have $\Phi(y) \geq 1$ because the $4B(z)$ cover V and the sum is locally finite (because all the $B(z)$ are disjoint and because the $5B(z)$ that contain a fixed point y have a radius equivalent to $d(y, \partial V)$). Then we set $\psi_z(y) = \varphi_z(y)/\Phi(y)$ such that $\sum_{z \in X} \psi_z(y) = 1$ on V . Finally, if $m_z(v)$ is the mean value of v on $B(z)$ we set for all $y \in V$

$$w(y) = \sum_{z \in X} m_z(v)\psi_z(y).$$

If $y \in A_k(2/10)$, $m_z(v) = m_k$ for all $z \in X$ such that $y \in B(z)$ thus (2.9) is verified. In addition,

$$\nabla w(y) = \sum_{z \in X} m_z(v) \nabla \psi_z(z) = \sum_{z \in X} [m_z(v) - m^y(v)] [\nabla \psi_z(y)]$$

where $m^y(v)$ is the mean value of v on $B(y) = B(y, 10^{-2} \text{dist}(y, \partial V))$. The sum at the point y has at most C terms, and all of these terms is less than

$$C \text{dist}(y, \partial V)^{-1} |m_z(v) - m^y(v)| \leq C \text{dist}(y, \partial V)^{-3} \int_{10B(y)} |\nabla v|$$

by Poincaré inequality and because all the $5B(z)$ that contain y are contained in $10B(y) \subset V$. Thus $|\nabla w(y)| \leq C \text{dist}(y, \partial V)^{-3} \int_{10B(y)} |\nabla v|$, and to obtain (2.10) it suffice to integrate on V , apply Fubini and use (2.8).

Then we apply the co-area formula (see [10, page 248], and also [6, Chapter 28]) to the function w on V . We obtain

$$\int_{\mathbb{R}} \mathcal{H}^2(\Gamma_t) dt = \int_V |\nabla w| \leq C \int_V |\nabla u|$$

where $\Gamma_t := \{y \in V; w(y) = t\}$ is the set of level t of the function w . Recall that

$$J(x, r) := r^{-\frac{1}{2}} \min\{\delta_{k,l}; k \neq l\}$$

and

$$\delta_{k,l} = |m_k - m_l|$$

where m_k is the mean value of u on D_k . For all $k_0 \neq k_1$ we know by definition that $\delta_{k_0,k_1} \geq \sqrt{r} J(x, r)$. Using the Tchebychev inequality we can choose $t_1 \in \mathbb{R}$ such that t_1 lies in $\frac{1}{10}[m_{k_0}, m_{k_1}]$ and such that

$$\begin{aligned} \mathcal{H}^2(\Gamma_{t_1}) &\leq C |m_{k_0} - m_{k_1}|^{-1} \int_V |\nabla u| \\ &\leq Cr^{-\frac{1}{2}} J(x, r)^{-1} \int_V |\nabla u| \\ &\leq Cr^2 J(x, r)^{-1} \omega_2(x, r)^{\frac{1}{2}}. \end{aligned} \tag{2.11}$$

For every pair $k_0 \neq k_1$ we do the same and choose t_2 etc., as many times as required by the number of connected components of $B(x, r) \setminus Z(x, r)$ (one if $Z(x, r)$ is a plane, three if $Z(x, r)$ is a \mathbb{Y} and four if $Z(x, r)$ is a \mathbb{T}). Then we set

$$F(x, r) = \bigcup_i \Gamma_{t_i} \cup [K \cap B(x, r)] \subset B(x, r).$$

The set $F(x, r)$, that we will also denote simply by F , is a closed set in $B(x, r)$ because each Γ_{t_i} is closed in $V = B(x, r) \setminus K$ and K is also a closed set. Since we have chosen some level sets, F separates the $A_k(2/10)$ to each other in $B(x, r)$. Indeed, if it is not the case then there is k, l and a continuous path γ that join $A_k(2/10)$ to $A_l(2/10)$ and that does not meet K (because $K \subset F$). Then $\gamma \subset V$, thus w is well defined and continuous on γ , it follows that there is a point $y \in \gamma$ such that $w(y) = t_i$. Then, $y \in F$, and this is a contradiction.

Now we want to prove the Property \star . Let $B(\bar{y}, s)$ be a ball centered on K such that $\beta(\bar{y}, 2^l s) \leq \varepsilon_0$ for all $0 \leq l \leq L$ where L is the first integer such that $B(\bar{y}, 2^{L+2} s)$ is not contained in $B(x, r)$. Set $B_l := B(\bar{y}, 2^l s)$ and possibly by extracting a subsequence we may suppose using Lemma 2.2 that in each B_l the minimal cone associated is almost centered. The radius of B_l is not as before exactly $2^l s$ but is equivalent with a factor 100. Thus the balls B_l forms a sequence of balls centered at \bar{y} such that $B_l \subset B_{l+1}$ and $B_0 = B(\bar{y}, s)$. Denote by Z_l the cone associated to B_l . We want to show that $F \cap B(\bar{y}, s) \subset Z_0(\varepsilon_0) := \{z; \text{dist}(z, Z_0) \leq \varepsilon_0 s\}$. By definition of F , it suffice to show that for all i

$$w(y) \neq t_i \text{ in } B(\bar{y}, s) \setminus Z_0(\varepsilon_0). \tag{2.12}$$

So let $y \in B(\bar{y}, s) \setminus Z_0(\varepsilon_0)$ and recall that

$$w(y) = \sum_{z \in X} m_z(v) \varphi_z(y).$$

Let $X(y) \subset X$ be the finite set of z such that $\varphi_z(y) \neq 0$. We claim that

$$\forall z \in X(y), \quad |m_z(v) - m_k| \leq Cr^{\frac{1}{2}} \omega_2(x, r)^{\frac{1}{8}} \tag{2.13}$$

where m_k is the mean value of u in an appropriate domain D_k (depending in which connected component lies y) and $m_z(v)$ is as before the mean value of v on $B_z := B(z, 10^{-2}d(z, \partial V))$. First of all, we can use the proof of [14, Lemma 16] to associate to each connected component of $B_l \setminus Z_l(\varepsilon_0)$, a component of $B_{l+1} \cap \{y; d(y, Z_{l+1}) \geq 10\varepsilon_0 r_l\}$, and by this way we can rely each component of $B_l \setminus Z_l(\varepsilon_0)$ to a certain A_k (that contain a D_k) (the argument is just to do an iteration on the scale since we know that the set K is close to a minimal cone at each scale that we look at). We denote by O_0 the component of $B_s \cap \{y; d(y, Z_0) \geq \varepsilon_0 s\}$ that contains y and by induction we denote by O_l the component of $B_l \setminus Z_l(\varepsilon)$ that is relied to O_0 .

With help of the particular geometric configuration in each B_l we can choose a domain G_l contained at the same time in O_l and in O_{l+1} , and of diameter equivalent to the diameter of B_l . We denote by $m_l(v)$ the mean value of v on G_l . We are now ready to estimate

$$\begin{aligned} |m_0(v) - m_L(v)| &\leq \sum_{l=0}^L |m_l(v) - m_{l+1}(v)| \leq \sum_{l=0}^L \frac{1}{|O_l|} \int_{O_l} |v - m_{l+1}(v)| \\ &\leq \sum_{l=0}^L C \frac{1}{(2^l s)^3} \int_{O_{l+1}} |v - m_{l+1}(v)| \leq \sum_{l=0}^L C (2^l s)^{-2} \int_{O_{l+1}} |\nabla v| \\ &\leq \sum_{l=0}^L C (2^l s)^{-\frac{1}{2}} \left(\int_{O_{l+1}} |\nabla v|^2 \right)^{\frac{1}{2}} \\ &\leq \sum_{l=0}^L C (2^l s)^{-\frac{1}{2}} \left(\int_{O_{l+1}} |\nabla v|^2 \right)^{\frac{3}{8}} \left(\int_{O_{l+1}} |\nabla v|^2 \right)^{\frac{1}{8}} \\ &\leq \sum_{l=0}^L C (2^l s)^{+\frac{1}{4}}. \end{aligned}$$

Next we use the classical estimate on the gradient of a Mumford-Shah minimizer that is

$$\int_{B(0,R) \setminus K} |\nabla u|^2 dx \leq C_N (1 + h(R)) R^{N-1} \tag{2.14}$$

obtained by comparing (u, K) and (v, K') where v is equal to 0 in $B(0, R)$ and $K' = (K \setminus B(0, R)) \cup \partial B(0, R)$. This yields

$$\begin{aligned} |m_0(v) - m_L(v)| &\leq \left(\int_{O_{l+1}} |\nabla v|^2 \right)^{\frac{1}{8}} \\ &\leq C \left(\int_V |\nabla v|^2 \right)^{\frac{1}{8}} \sum_{l=0}^L (2^l s)^{\frac{1}{4}} \leq C \left(\int_V |\nabla v|^2 \right)^{\frac{1}{8}} \sum_{l=0}^L (2^{-l} r)^{\frac{1}{4}} \\ &\leq C \left(\int_V |\nabla v|^2 \right)^{\frac{1}{8} + \infty} \sum_{l=0}^{\infty} (2^{-l} r)^{\frac{1}{4}} \leq Cr^{\frac{1}{4}} \left(\int_V |\nabla v|^2 \right)^{\frac{1}{8}} \\ &\leq Cr^{\frac{1}{4}} \left(\int_V |\nabla u|^2 \right)^{\frac{1}{8}} \end{aligned}$$

thus

$$|m_0(v) - m_L(v)| \leq Cr^{\frac{1}{2}} \omega_2(x, r)^{\frac{1}{8}}. \tag{2.15}$$

With a similar proof we also get

$$|m_L(v) - m_k| \leq Cr^{\frac{1}{2}} \omega_2(x, r)^{\frac{1}{8}}.$$

On the other hand, since $z \in X(y)$, then $\varphi_z(y)$ is not equal to zero. This implies that $\text{dist}(z, \partial V) \geq 2 \text{dist}(y, \partial V) \geq 2\varepsilon_0 s$ thus $B_z := B(z, 10^{-2} \text{dist}(z, \partial V)) \subset Z_0(\varepsilon_0)^c$. Since by hypothesis K does not meet this region, we can apply Poincaré inequality to prove that

$$|m_z(v) - m_0(v)| \leq Cr^{\frac{1}{2}} \omega_2(x, r)^{\frac{1}{8}}.$$

Finally

$$|m_z(v) - m_k| \leq |m_z(v) - m_0(v)| + |m_0(v) - m_L(v)| + |m_L(v) - m_k| \leq Cr^{\frac{1}{2}} \omega_2(x, r)^{\frac{1}{8}}$$

and this completes the proof of (2.13).

Now since $\sum_z \varphi_z(y) = 1$ we deduce that

$$|w(y) - m_k| = |w(y) - \sum_{z \in X(y)} \varphi_z(y) m_k| \leq \sum_{z \in X(y)} |m_z(v) - m_k| \leq Cr^{\frac{1}{2}} \omega_2(x, r)^{\frac{1}{8}}. \tag{2.16}$$

If we return to the choice of the t_i (see near (2.11)) we have taken $t_i \in \frac{1}{10} [m_{k_0}, m_{k_1}]$ for some k_0 and k_1 . So thank to (2.16), if $\omega_2(x, r)^{\frac{1}{8}}$ is small enough with respect to $J(x, r)$ (which is the case by assumption (2.3)) then for a suitable choice of t_i (that does not depend on ε_0) we are sure that $w(y) \neq t_i$ thus F does not meet the region $Z_s(\varepsilon_0)$, and Property \star is proved.

Finally we have to prove (2.5). With use of (2.3) and (2.11) we can find a cover of F by a family of balls B_j centered at $x_j \in K$, with radius equal to $C\sqrt{\varepsilon}r$ and such that $\frac{1}{2}B_j$ are disjoint. Otherwise we would have a hole in K of size greater than $C\varepsilon r^2$ which is a contradiction with (2.11). Now, for every $y \in F \cap B_j$ we have

$$\text{dist}(y, Z(x, r)) \leq \text{dist}(y, x_j) + \text{dist}(x_j, Z(x, r)) \leq C\sqrt{\varepsilon}r + \varepsilon r \leq C\sqrt{\varepsilon}r$$

and the conclusion follows. □

Lemma 7 of [6, on page 283] shows how to control the normalized jump in the flat case. Here we give a similar result for our definition of jump based on the cones of type \mathbb{P} , \mathbb{T} and \mathbb{Y} . The only substantial modification of the original proof consist in being careful with definition of the jump that depends on the existence of almost centered cones, but this is not much troublesome.

Lemma 2.8. *Let (u, K) be a Mumford-Shah minimizer in Ω . Let $x \in K$, r and r_1 being such that $B(x, r) \subset \Omega$ and $0 < r_1 \leq r \leq 2r_1$. Suppose in addition that $\beta(x, r) \leq 10^{-7}$. Then*

$$\left| \left(\frac{r_1}{r}\right)^{\frac{1}{2}} J(x, r_1) - J(x, r) \right| \leq C\omega_2(x, r)^{\frac{1}{2}} \leq C(1 + h(r))^{\frac{1}{2}} \tag{2.17}$$

with a constant C that depends only on N .

Proof. Since $\beta(x, r) \leq 10^{-7}$ we know that $\beta(x, t) \leq 10^{-5}$ for all $t \in [r_1/4, r]$. Assume first that $Z(x, r)$ and $Z(x, r_1)$ are almost centered. Then by use of Poincaré inequality and the fact that $\beta(x, t)$ stays small for $t \in [r_1, r]$ one can easily prove, after a suitable relabeling of $m_k(x, r_1)$ corresponding to the proper choice of connected component of $B(x, r) \setminus Z(x, r)$, that

$$\begin{aligned} |m_k(x, r) - m_k(x, r_1)| &\leq Cr^{-2} \int_{B(x,r) \setminus K} |\nabla u| \leq Cr^{\frac{1}{2}} \omega_2(x, r)^{\frac{1}{2}} \\ &\leq C(1 + h(r))^{\frac{1}{2}} r^{\frac{1}{2}}. \end{aligned} \tag{2.18}$$

Recall that the Jump is defined by

$$J(x, r) = r^{-\frac{1}{2}} \min\{\delta_{k,l}\}$$

where $\delta_{k,l} = |m_k(x, r) - m_l(x, r)|$. Thus (2.18) gives the estimate of $r^{\frac{1}{2}}J(x, r) - r_1^{\frac{1}{2}}J(x, r_1)$ that proves (2.17). Notice that $Z(x, r)$ could be of different type from $Z(x, r_1)$ but the estimate still hold in this case omitting one value m_k for the one of biggest type.

Now if $Z(x, r)$ or $Z(x, r_1)$ are not almost centered, then we argue by the same way with the proper radius $r_1/10$ or $r_1/100$ to get the same estimate with a slight modification of constant C and this ends the proof of the Lemma. \square

Lemma 2.9. *Let (u, K) be a Mumford-Shah minimizer in Ω . Then if $x \in K$ and r are such that $B(x, r) \subset \Omega$ and for all $r_1 < t < r$, $\beta(x, t) \leq 10^{-7}$, then*

$$J(x, r_1) \geq \left(\frac{r}{r_1}\right)^{\frac{1}{2}} [J(x, r) - C'] \tag{2.19}$$

where $C' := C(1 + h(r))^{\frac{1}{2}}$ and C depends only on N .

Proof. If $r_1 \leq r \leq 2r_1$ then (2.19) is a consequence of Lemma 2.8. Otherwise we use a sequence of radii r_k such that $r_k = 2r_{k-1}$ and we apply Lemma 2.8 sufficiently many times until r_k becomes greater than r . We obtain

$$\begin{aligned} J(x, r_1) &\geq 2^{\frac{k}{2}} J(x, 2^k r_1) - C2^{\frac{k}{2}} (1 + 2^{-\frac{1}{2}} + 2^{-\frac{2}{2}} + \dots) \\ &\geq 2^{\frac{k}{2}} \left[J(x, 2^k r_1) - \frac{C2^{\frac{1}{2}}}{1 - 2^{\frac{1}{2}}} \right] \end{aligned} \tag{2.20}$$

from which the conclusion follows. \square

2.3. Stopping-time balls, bad mass and standard assumptions

In all the sequel we will work under the following general assumptions.

Definition 2.10 (Standard Assumptions 1). We will say that we are under Standard Assumptions 1 in $B(x_0, r_0)$ when the situation is as follows. (u, K) is a Mumford-Shah minimizer in $\Omega \subset \mathbb{R}^3$, $x_0 \in K$, $B(x_0, 3r_0) \subset \Omega$,

$$\beta(x_0, 2r_0) \leq 10^{-5}\varepsilon, \tag{2.21}$$

$$J(x_0, r_0)^{-1} + \omega_2(x_0, r_0) \leq 10^{-5}\varepsilon \tag{2.22}$$

for some $\varepsilon < 10^{-8}$. Moreover we assume that the associated cone $Z(x_1, 2r_1)$ is almost centered. We also assume that ε is small enough so that (2.21) and (2.22) implies that the assumptions of Proposition 2.5 hold in $B(x_0, r_0)$. We denote $F(x_0, r_0)$ the corresponding separating set.

Remark 2.11. Under Standard Assumptions 1 and when no confusion is possible, we will sometimes denote only F instead of $F(x_0, r_0)$.

Our goal in this section is to construct a family of good balls \mathcal{G} by a stopping-time argument, with the condition that in all balls of \mathcal{G} , the singular set K will always look like a minimal cone.

We assume that we are under our Standard Assumptions 1 and we consider some constants ε_0 and ε'_0 such that $\varepsilon \leq \varepsilon'_0 \leq \varepsilon_0 < 10^{-8}$.

For all $x \in F(x_0, r_0)$ and $r \in (0, r_0)$, we say that $B(x, r)$ is a good ball (and then denote $B(x, r) \in \mathcal{G}$) if

$$K \cap B(x, r) \neq \emptyset, \mathcal{H}^2(F \cap B(x, r)) - \mathcal{H}^2(K \cap B(x, r)) \leq \varepsilon'_0 r^2, \tag{2.23}$$

and

$$\beta_K(x, r) \leq \varepsilon_0. \tag{2.24}$$

Observe that since by (2.21) we have $\beta(x_0, r_0) \leq 10^{-5}\varepsilon$, then the radii of balls that do not verify (2.24) is bounded by $10^{-5} \frac{\varepsilon}{\varepsilon_0} r_0$ and under our assumptions (in particular $\omega_2(x_0, 2r_0)^{\frac{1}{2}} J(x_0, 2r_0)^{-1} \leq C\varepsilon$), the radii of balls that do not verify (2.23) is bounded by $C\sqrt{\varepsilon}r_0$.

Now, for all $x \in F$ we define the stopping-time function

$$d(x) := \inf\{r; \forall t \in (r, r_0), B(x, t) \in \mathcal{G}\}$$

and the set

$$S := \bigcup_{x \in F} F \cap B(x, d(x)),$$

with the convention that a ball of radius 0 is the empty set. Then we introduce a new quantity called ‘‘Bad mass’’ defined for every $x \in B(x_0, r_0)$ and $r < r_0$ by

$$m(x, r) := \frac{1}{r^2} \mathcal{H}^2(S).$$

In the sequel, to estimate the quantity $m(x, r)$ it will be convenient to use a Vitali subfamily of balls. Indeed, with help of the Vitali covering lemma, from the collection of balls

$$\{B(x, Ad(x)); x \in F \text{ and } d(x) > 0\},$$

with A a constant that will be chosen later (in Remark 2.25), we get a disjoint subfamily $\{B_i\}_{i \in I}$ such that $\{5B_i\}_{i \in I}$ is covering. We will denote x_i and r_i the center and radius of B_i . Observe that

$$m(x_0, r_0) \simeq \frac{1}{r^2} \sum_{i \in I} r_i^2 \tag{2.25}$$

where the constant in the equivalence (2.25) depends on A .

Notice also that by definition $\beta_K(x, t) \leq \varepsilon_0$ for all $t \geq d(x)$, which implies $\beta_F(x, t) \leq C\varepsilon_0$ for all $t \geq d(x)$ using Property \star . Indeed, since $B(x, t) \in \mathcal{G}$ there exists a point in $z \in K \cap B(x, t)$ which allows us to apply Property \star to the balls centered at this point z .

2.4. Whitney extension

In order to construct some competitors, we explain how to extend the function u in the region where the geometry of K is bad, using a Whitney type extension. For this purpose, we recall some definitions and a result from [14].

Let E be a closed set in $B(x_0, r_0)$ such that $\mathcal{H}^2(E) < +\infty$. Suppose that there is a positive constant $\varepsilon_0 < 10^{-5}$ such that

$$\beta_E(x_0, r_0) \leq \varepsilon_0 \tag{2.26}$$

and that the associated cone $Z(x_0, r_0)$ is almost centered. Suppose in addition that E is separating in $B(x_0, r_0)$. Let $\rho \in [\frac{1}{2}r_0, \frac{3}{4}r_0]$ and assume that we have an application

$$\delta : B(x_0, \rho) \rightarrow \left[0, \frac{1}{4}r_0\right] \tag{2.27}$$

with the property that

$$\beta_E(x, r) \leq \varepsilon_0, \text{ for all } x \in K \cap B(x_0, \rho) \text{ and } r \text{ such that } \delta(x) \leq r \leq \frac{1}{4}r_0. \tag{2.28}$$

In addition we suppose that

$$\delta \text{ is } C_0 - \text{Lipschitz}. \tag{2.29}$$

The application δ will be called the ‘‘geometric function’’.

Definition 2.12 (Hypothesis H1). We will say that a closed set $E \subset B(x_0, r_0)$ with finite \mathcal{H}^2 measure is satisfying hypothesis $H1$ if all the above assumptions on E and δ are satisfied (i.e. (2.26), (2.27), (2.28) and (2.29) hold).

There exists a constant $U > 30C_0$, depending on C_0 and on a dimensional constant such that the following holds provided that ε_0 is small enough compared to U^{-1} . Define

$$\mathcal{V} := \bigcup_{x \in E \cap B(0, \rho)} B\left(x, \frac{10}{U} \delta(x)\right), \tag{2.30}$$

and set

$$\mathcal{V}_\rho := \bigcup_{x: B(x, \frac{10}{U} \delta(x)) \cap \partial B(x_0, \rho) \neq \emptyset} B\left(x, \frac{10}{U} \delta(x)\right). \tag{2.31}$$

With use of the same notations $\kappa(x_0, r_0)$ and D_k as in the section before, and recalling that by hypothesis, E is separating in $B(x_0, r_0)$, for all $k \in [1, \kappa(x_0, r_0)]$ we denote by $\Omega_k(x_0, r_0)$ the connected component of $B(x_0, r_0) \setminus E$ that contains D_k . We also define

$$\Delta_k := B(x_0, \rho) \cap (\Omega_k(x_0, r_0) \cup \mathcal{V}). \tag{2.32}$$

Now the following lemma is proved in [14] and will be needed in the sequel.

Lemma 2.13 ([14, Whitney Extension]). *Let E be a closed set in $B(x_0, r_0)$ satisfying Hypothesis H1 with a geometric function δ , a constant $\varepsilon_0 < 10^{-5}$ and a radius $\rho \in [\frac{1}{2}r_0, \frac{3}{4}r_0]$. Then for any function $u \in W^{1,2}(B(x_0, r_0) \setminus E)$, and for all $k \in [1, \kappa^{B(x_0, r_0)}]$, there is a function*

$$v_k \in W^{1,2}(\Delta_k \setminus \mathcal{V}_\rho)$$

such that

$$v_k = u \text{ in } B(x_0, \rho) \setminus \mathcal{V}$$

and

$$\int_{\Delta_k \setminus \mathcal{V}_\rho} |\nabla v_k|^2 dx \leq C \int_{B(x_0, r_0) \setminus E} |\nabla u|^2 dx \tag{2.33}$$

where C is a constant depending only on dimension and where \mathcal{V} , \mathcal{V}_ρ , and Δ_k are defined in (2.30), (2.31), and (2.32) with a certain constant $U > 30C_0$ given by the demonstration and depending on C_0 .

Remark 2.14. In the original statement of Lemma 2.13 (compare [14, Lemma 17]) one can find a slightly better estimate than (2.33) but in the sequel we only need the more simpler inequality (2.33).

We return now to our Mumford-Shah minimizer. From the bad balls $\{B_i\}_{i \in I}$ (constructed in Section 2.3), we want to apply Lemma 2.13 to get a good extension of u in each ball B_i . This extension will allow us replace in each bad ball the set K by a new set in order to get some estimates about the bad mass itself. This will be done in the next section.

We begin by introducing a geometric function associated to the balls $\{B_i\}_{i \in I}$. For this purpose we consider some functions

$$\psi_i = \begin{cases} r_i & \text{on } B_i \\ 0 & \text{in the complement of } 5B_i. \end{cases}$$

For instance let us take

$$\psi_i(x) := \min(r_i, \text{dist}(x, (5B_i)^c)). \tag{2.34}$$

Then for all x we define

$$\delta(x) := \sum_{i \in I} \psi_i(x). \tag{2.35}$$

The function $\delta(x)$ is a sort of regularized version of function $d(x)$.

Proposition 2.15. *Assume that $B(x_0, r_0)$ satisfies the Standard Assumptions 1. Then Application δ is a geometric function associated to F in $B(x_0, r_0)$ for all $\rho \in [\frac{1}{2}r_0, \frac{3}{4}r_0]$, with Lipschitz constant C_0 (a dimensional constant) and geometric constant ε_0 . In addition, we have Hypothesis H1 on F in $B(x_0, r_0)$ and*

$$\bigcup_{i \in I} \frac{10}{U} B_i \subset \mathcal{V}, \quad \mathcal{V}_\rho \subset \bigcup_{i: C_1 B_i \cap \partial B(x_0, \rho) \neq \emptyset} C_1 B_i \tag{2.36}$$

where \mathcal{V} and \mathcal{V}_ρ are defined in (2.30) and (2.31), and C_1 is a constant depending on dimension and U .

Proof. We have to verify (2.27), (2.28) and (2.29). Let $\rho \in [\frac{1}{2}r_0, \frac{3}{4}r_0]$. Recall that ε (the constant in the Standard Assumptions 1) is small as we want with respect to ε_0 (and A) and the radius of balls B_i are less than $C \frac{\varepsilon}{\varepsilon_0} A$ so that one can easily get $\delta(x) \leq \frac{1}{4}r_0$ thus (2.27) holds. Moreover it is clear by construction that ψ is C_0 -Lipschitz with C_0 the covering bound associated to the family $\{5B_i\}$ that depends only on dimension.

Now let $x \in F \cap B(x_0, \rho)$ and let r be a radius such that $\delta(x) \leq r \leq \frac{1}{4}r_0$. Firstly if x lie in the complement of all the $5B_i$ then $d(x) = 0$ so that $\beta_K(x, t) \leq \varepsilon_0$ for all $t > 0$ and this holds in particular for $r = t$ and replacing β_K by β_F using Property \star and the fact that $K \cap B(x, t)$ is not empty for every t . On the other hand if $x \in 5B_i$ for some ball B_i , then since the $\{5B_i\}$ are covering all the $\{B(y, Ad(y))\}$ we deduce that $Ad(x) \leq \text{dist}(x, (5B_i)^c)$ for some $5B_i$ containing x . Now by construction using the definition of ψ_i (see (2.34)),

$$\delta(x) := \sum_{i \in I} \psi_i(x) \geq \frac{1}{10} \min_{i: x \in 5B_i} \text{dist}(x, (5B_i)^c) \geq \frac{1}{10} Ad(x).$$

Therefore, provided $A > 10$ it comes that $r \geq \delta(x) \geq d(x)$. Next, we know by definition of $d(x)$ that $\beta_K(x, t) \leq \varepsilon_0$ for all $t > d(x)$. And since $K \cap B(x, r) \neq \emptyset$ Property \star implies $\beta_F(x, t) \leq \varepsilon_0$ for all $t \geq r$ and (2.28) is proved.

So we deduce that we have Hypothesis *H1* on F in $B(x_0, r_0)$ with application δ defined in (2.35).

Then, if $B_i = B(x_i, r_i)$ is a bad ball we have that $\delta(x_i) = r_i$ by definition of δ thus

$$\bigcup_{i \in I} \frac{10}{U} B_i \subset \mathcal{V} \tag{2.37}$$

and finally if $\delta(x) > 0$ then $x \in 5B_i$ for some $i \in I$ and assuming r_i to be the maximum among the radius of all possible balls $5B_i$ containing x one has $\frac{10}{U} \delta(x) \leq Cr_i \frac{10}{U}$ (where C is depending on the bounded cover constant of $5C_i$) thus

$$\mathcal{V} \subset \bigcup_{i \in I} \left(5 + \frac{10}{U}\right) B_i \tag{2.38}$$

which ends the proof of the proposition. □

Remark 2.16. Note that since the Lipschitz constant of δ depends only on dimension, then U is only a dimensional constant.

2.5. A compactness lemma for almost minimal sets

The purpose of this section is to show some geometrical results about almost minimal sets (see Definition 1.11). We want to give an argument which allows us to win something in each bad ball, in order to prove later that there are not so many. The main lemma says the following. If $B(x, r)$ is a ball such that $x \in K$ and $\beta_K(x, r) \leq \varepsilon_0$ but $\beta_K(x, r/100) > \varepsilon_0$, then there is a set that has smaller \mathcal{H}^2 -measure than K in $B(x, r)$. The argument will be by contradiction and compactness.

Recall that for any almost minimal set E in $B(x, r)$, we denote by $f(r)$ the excess of density

$$f(r) = \theta(x, r) - \lim_{t \rightarrow 0} \theta(x, t) \tag{2.39}$$

with

$$\theta(x, r) = r^{-2} \mathcal{H}^2(E \cap B(x, r)).$$

The limit in (2.39) exists because E is almost minimal (see [7, 2.3.]). For $x \in E$ we call $\theta(x)$ the density at x , that is $\theta(x) = \lim_{t \rightarrow 0} \theta(x, t)$. The function $\theta(x)$ can only take a finite number of values, more precisely $\theta(x) \in \{0, \pi, \frac{3\pi}{2}, d_+\}$ that are (excepted 0) densities of the three minimal cones in \mathbb{R}^3 .

Now [4, Proposition 12.28] gives the following result that will be needed. We give a statement only for almost minimal sets with vanishing gauge function (*i.e.* simply minimal sets) because it will be sufficient in the sequel. The same result holds for almost minimal sets under assumptions on $h(r)$ (see [4]).

Theorem 2.17 ([4]). *There exists $\varepsilon_1 > 0$ and $\alpha \in (0, 1)$ such that the following holds. Let E be a reduced MS-almost minimal set in $\Omega \subset \mathbb{R}^3$ with gauge function $h = 0$. Suppose that $0 \in E$, $r_0 > 0$ is such that $B(0, 110r_0) \subset \Omega$ and*

$$f(0, 110r_0) + D_{0,100r_0}(E, Z) \leq \varepsilon_1$$

where Z is a minimal cone centered at the origin satisfying

$$\mathcal{H}^2(Z \cap B(0, 1)) \leq d(0).$$

Then for all $x \in E$ and $r > 0$ such that $x \in E \cap B(0, 10r_0)$ and $0 < r < 10r_0$, we can find a minimal cone $Z(x, r)$, not necessarily centered at x or at the origin, such that

$$D_{x,r}(E, Z(x, r)) \leq \left(\frac{r}{r_0}\right)^\alpha.$$

See (1.3) for the definition of the normalized Hausdorff distance $D_{x,r}$. The constant α is a universal constant depending on dimension and other geometric facts.

For an almost minimal set E , the function $\theta(x, t)$ is nondecreasing in t thus the limit when t tends to 0 exists and that allows us to define the function $\theta(x)$. Unfortunately, if E is now the singular set of a Mumford-Shah minimizer, the monotonicity of θ is not known. So we have some difficulties to define the analogue of $f(r)$ for a Mumford-Shah minimizer. In order to use Theorem 2.17, we have to control $f(r)$ and that will be the role of the following lemmas. Our goal is to obtain a statement analogous to Theorem 2.17 but with only an hypothesis on $\beta(0, r_0)$ instead of $f(0, r_0)$.

First of all, an application of [7, Proposition 16.24] in $B(x, r10^{-3})$ with $\eta_1 = \varepsilon_2 10^3$, mixed with [7, Proposition 18.1] in $B(x, r10^{-5})$ and $\eta_1 = \varepsilon_7 10^{-5}$ (where ε_7 and ε_3 are defined in [7]) allows us to state the following lemma.

Lemma 2.18 ([7]). *There exists a $\eta_1 \geq 0$ such that if E is an almost minimal set in an open set $\Omega \in \mathbb{R}^3$, with gauge function $h(r) = 0$, if $x \in E$ and $r > 0$ are such that $B(x, r) \subset \Omega$, if there is Z , centered at x , of type \mathbb{P} , \mathbb{Y} or \mathbb{T} such that*

$$D_{x,r}(E, Z) \leq \eta_1$$

and if E is separating in $B(x, r)$, then there is a point $x \in E \cap B(x, r10^{-5})$, of the same type of Z .

We say that x has the same type as Z if $\theta(x)$ is equal to the density of the cone Z .

Remark 2.19. The hypothesis of separating are only useful for the case of \mathbb{T} and it is an open problem to know wether this assumption is necessary or not. See [7, Propositions 16.24 and 18.1] for more details.

Here is now the statement that will be useful for the next sections. The reader is invited to compare it with Theorem 2.17.

Lemma 2.20. *There exists $\eta_2 > 0$ and $\alpha \in (0, 1)$ such that the following holds. Let E be a reduced MS-almost minimal set in $\Omega \subset \mathbb{R}^3$ with gauge function $h = 0$. Suppose that $0 \in E$, $r_0 > 0$ is such that $B(0, 110r_0) \subset \Omega$ and*

$$D_{0,110r_0}(E, Z) \leq \eta_2$$

where Z is a minimal cone centered at the origin such that

$$\mathcal{H}^2(Z \cap B(0, 1)) \leq d(0)$$

and such that Z is separating in $B(0, 110r_0)$. Then for all $x \in E \cap B(0, 4r_0)$ and for all $0 < r < 5r_0$ there is a minimal cone $Z(x, r)$ such that

$$D_{x,r}(E, Z(x, r)) \leq \left(\frac{r}{r_0}\right)^\alpha.$$

Proof. We take $\eta_2 < \varepsilon_1$ (the constant of Theorem 2.17). In order to apply Theorem 2.17, all we have to prove is that

$$f(0, 110r_0) \leq \varepsilon_1.$$

We may assume that η_2 is smaller than η_1 so that we can apply Lemma 2.18 to E in $B(x, 110r_0)$ thus there is a point z in $B(x, 10^{-3}r_0)$ of same type of Z . In particular $\theta(z) = \mathcal{H}^2(Z \cap B(z, 1)) = \frac{1}{r^2} \mathcal{H}^2(Z \cap B(z, r))$ for all r . Hence we can compute the excess of density at z in $\bar{B}(z, 55r_0)$ by

$$f(z, 55r_0) = \frac{1}{(55r_0)^2} \left[\mathcal{H}^2(E \cap B(z, 55r_0)) - \mathcal{H}^2(Z \cap B(z, 55r_0)) \right].$$

Now define a competitor L by

$$L = \begin{cases} M \cup Z \cap B(z, 55r_0) & \text{in } \bar{B}(z, 55r_0) \\ E & \text{in } \Omega \setminus B(z, 55r_0) \end{cases}$$

where M is a little wall:

$$M := \{x \in \partial B(z, 55r_0); \text{dist}(x, Z) \leq 500\eta_2 r_0\}.$$

The set L is a MS-competitor for E thus

$$\begin{aligned} \mathcal{H}^2(E \cap B(z, 55r_0)) &\leq \mathcal{H}^2(L \cap B(z, 55r_0)) + (55r_0)^2 h(55r_0) \\ &\leq \mathcal{H}^2(M) + \mathcal{H}^2(Z \cap B(z, 55r_0)). \end{aligned}$$

Since $\mathcal{H}^2(M) \leq Cr_0^2 \eta_2$ we deduce

$$f(z, 55r_0) \leq C\eta_2.$$

Now if η_2 is small enough compared to ε_1 , we can apply Theorem 2.17 in $B(z, 55r_0)$ then for all $y \in E \cap B(z, 5r_0)$ and $0 < r < 10r_0$ we have

$$\beta(y, r) \leq \left(\frac{r}{r_0}\right)^\alpha. \tag{2.40}$$

In addition, since $\text{dist}(x, z) \leq 10^{-3}r_0$ we deduce that (2.40) is true for all $y \in B(x, 4r_0)$ and $0 < r < 5r_0$. □

Definition 2.21. By now we will denote by η_2 and α the constants given by Lemma 2.20 and we denote by $\bar{r}(\varepsilon_0) < 1$ the radius such that

$$(100\bar{r}(\varepsilon_0))^\alpha = \frac{1}{2}\varepsilon_0. \tag{2.41}$$

Technical Remark 2.22. Concerning the next lemma, it would be tempting to obtain a statement that one could apply directly on F instead of K . On the other hand to do this using the same technics as below, one would have to prove that F satisfies a uniform concentration property or something to make the Hausdorff measure lower semi-continuous. This is why we prefer arguing on K for which we already know that uniform concentration property holds. We will transfer the result on F later using Property \star . Notice that the uniform concentration property is verified in particular by the singular set of a Mumford-Shah minimizer. We do not recall here the definitions and result about uniform concentration and we refer to [6] (Section 35) and references therein for more details.

We are now ready to prove the main lemma of this section.

Lemma 2.23. *For every $\varepsilon_0 \in (0, \eta_2)$, and $r < \bar{r}(\varepsilon_0)$, there is a constant η_0 such that the following holds. Let E be a closed set of finite \mathcal{H}^2 measure in $B(0, 1) \subset \mathbb{R}^3$ that contains the origin, with the uniform concentration Property (with constant C_u), and such that*

$$\beta_E(0, 1) \leq \eta_2 \tag{2.42}$$

$$\beta_E(0, r) \geq \varepsilon_0. \tag{2.43}$$

Assume that the associated cone in $B(0, 1)$ is centered in $B(0, 10^{-5})$. In addition we assume that there is a set F that contains E , that is separating in $B(0, 1)$ and such that

$$\mathcal{H}^2(F) - \mathcal{H}^2(E) \leq \eta_0.$$

Then there is a MS-competitor L for E in $B(0, 1)$ such that

$$\mathcal{H}^2(E) - \mathcal{H}^2(L) \geq \eta_0.$$

Proof. The argument is by contradiction. If the lemma is not true, then there is an $\varepsilon_0 < \eta_2$ and a $r < \bar{r}(\varepsilon_0)$ such that for all $\eta > 0$ one can find a set E_η that verifies (2.42) and (2.43). In addition for all MS-competitor L_η for E_η we have

$$\mathcal{H}^2(E_\eta) - \mathcal{H}^2(L_\eta) \leq \eta. \tag{2.44}$$

Moreover for all η there is a set F_η that contain E_η , that is separating in $B(0, 1)$, and such that

$$\mathcal{H}^2(F_\eta) - \mathcal{H}^2(E_\eta) \leq \eta. \tag{2.45}$$

Now let η tend to 0. Passing if necessary to a subsequence, we may assume that the sequence of sets E_η converges to a certain E_0 in sense of Hausdorff distance. Passing to the limit, we deduce that this set E_0 still verifies (2.42) and (2.43).

We want to show that E_0 is a minimal set in $B(0, 1)$. Let L be a MS-competitor for E_0 in $B(0, R)$ with $R < 1$. Since E_η tends to E_0 for the Hausdorff distance (denoted D_H), we know that for any $\tau > 0$, $D_H(E_0, E_\eta) \leq \tau$ for η small enough. Thus denoting $T_\tau := \{x \in \partial B(0, R); \text{dist}(x, E_0) \leq \tau\}$, we have that $E_\eta \cap \partial B(0, R) \subset T_\tau$ for η small enough. Therefore, since $L = E_0$ on $\partial B(0, R)$, the set

$$L_\eta := [L \cap B(0, R)] \cup [E_\eta \setminus B(0, R)] \cup T_\tau$$

is a MS-competitor for E_η . Then applying (2.44) we obtain

$$\begin{aligned} \mathcal{H}^2(E_\eta \cap B(0, R)) &\leq \mathcal{H}^2(L_\eta \cap B(0, R)) + \eta \\ &\leq \mathcal{H}^2(L \cap B(0, R)) + \mathcal{H}^2(T_\tau) + \eta \\ &\leq \mathcal{H}^2(L \cap B(0, R)) + \eta + C\tau. \end{aligned}$$

Owing that by assumption the sets E_δ verify the uniform concentration property with same constant C_u , we are allowed to say that (see [6, Section 35])

$$\mathcal{H}^2(E_0) \leq \underline{\lim}_{\eta \rightarrow 0} \mathcal{H}^2(E_\eta).$$

Hence, letting η tend to 0 we obtain

$$\mathcal{H}^2(E_0 \cap B(0, R)) \leq \mathcal{H}^2(L \cap B(0, R)) + \tau$$

and letting τ tend to 0,

$$\mathcal{H}^2(E_0 \cap B(0, R)) \leq \mathcal{H}^2(L \cap B(0, R)).$$

This proves that E_0 is a minimal set.

On the other hand, E_0 is separating in $B(0, 1)$, because if it is not the case, we can find a continuous path γ joining D_{k_1} and D_{k_2} (two balls in different connected component of $B(0, 1) \setminus Z$) in $B(0, 1)$ and such that γ does not meet E_0 . Since E_η converge to E_0 for the Hausdorff distance, for all τ there is a η_τ such that all the E_η are τ close to E_0 for $\eta < \eta_\tau$. Let x be the point of γ that realizes the infimum of

$\text{dist}(x, E_0)$. Since γ is disjoint from E_0 , there is a ball centered at x with positive radius r that does not meet E_0 . Thus if we choose η smaller than r we get that all the E_η for $\eta < \eta_\tau$ contain a hole of size r , but this is not possible according to (2.45).

Thus finally E_0 is a minimal set in $B(0, 1)$, which is separating and verifies (2.42) and (2.43). We want now to apply Lemma 2.20 to obtain a contradiction. We know that

$$\beta_{E_0}(0, 1) \leq \eta_2$$

and that the associated cone is centered in $B(0, 10^{-5})$. We claim that

$$D_{0,1}(E_0, Z) \leq \eta_2$$

(i.e. with a bilateral definition of the distance). All we have to show is that for all $x \in Z \cap B(0, 1)$, $\text{dist}(x, E_0) \leq \eta_2$. If it is not the case, then we can find $x \in Z$ such that $B(x, \eta_2) \cap E_0 = \emptyset$. But then we can find a continuous path that join two different connected components of $B(0, 1) \setminus Z$ without meeting E , and that is not possible if E is separating. Thus (2.5) holds and then we can apply Lemma 2.20 in $B(0, 1)$ (i.e. $r_0 = \frac{1}{100}$), which implies that

$$\beta_{E_0}(0, r) \leq \frac{1}{2} \varepsilon_0$$

because of the definition of $\bar{r}(\varepsilon_0)$ (see (2.41)), and this yields a contradiction with (2.43) so the proof is now complete. \square

Applying Lemma 2.23 we deduce to following useful proposition.

Proposition 2.24. *Let $i \in I$ be an index such that $\frac{1}{A} B_i := B(x_i, d(x_i))$ does not verify (2.24). Then there is a MS-competitor L for K in*

$$\tilde{B}_i := B\left(x_i, \frac{M}{\bar{r}} d(x_i)\right)$$

such that

$$\mathcal{H}^2(K \cap \tilde{B}_i) - \mathcal{H}^2(L \cap \tilde{B}_i) \geq \eta_0 \tilde{r}_i^2$$

with $\tilde{r}_i := \frac{M}{\bar{r}} d(x_i)$ and M is a constant equal to 1, 10^5 or 10^{10} .

Proof. Since B_i do not verify (2.24), we know that

$$\beta(x_i, d(x_i)) \geq \varepsilon_0$$

and in addition

$$\beta\left(x_i, \frac{1}{\bar{r}} d(x_i)\right) \leq \varepsilon_0.$$

Multiplying if necessary the radius by 10^5 or 10^{10} , and using the proof of the re-centering Lemma 2.2, we can suppose that the center of the cone is in a ball of

radius 10^{-5} times smaller in $B(x_i, \frac{M}{\bar{r}}d(x_i))$ (M is the constant equal to 1, 10^5 or 10^{10}). Set

$$\tilde{r}_i := \frac{M}{\bar{r}}d(x_i).$$

Then if ε_0 is small enough compared to η_2 we have that

$$\beta(x_i, \tilde{r}_i) \leq \varepsilon_0 \leq \eta_2$$

with a cone centered in $B(x_i, 10^{-5}\tilde{r}_i)$. Moreover we have

$$\beta(x_i, \bar{r}\tilde{r}_i) \geq \frac{1}{M}\varepsilon_0.$$

We also have $F \cap B(x_i, \tilde{r}_i)$, that is a separating set in $B(x_i, \tilde{r}_i)$ and such that

$$\mathcal{H}^2(F \cap B(x_i, \tilde{r}_i)) - \mathcal{H}^2(K \cap B(x_i, \tilde{r}_i)) \leq \varepsilon'_0\tilde{r}_i.$$

Therefore, we can apply Lemma 2.23 in $B(x_i, \tilde{r}_i)$ with $\frac{1}{M}\varepsilon_0$ instead of ε_0 that we may suppose smaller than $C\varepsilon_1$. We can also take $\varepsilon'_0 \ll \eta_0$ and the proposition follows from Lemma 2.23. \square

Remark 2.25 (Choice of A). We can now fix our constant A , that will depend on ε_0 . We want that for every bad ball $B_i := B(x_i, Ad(x_i))$ with $i \in I$, the ball

$$B(x_i, \tilde{r}_i) := B\left(x_i, \frac{M}{\bar{r}}d(x_i)\right) \subset B\left(x_i, \frac{10A}{U}d(x_i)\right) \subset \mathcal{V}$$

in order to have that the extension of u given by Lemma 2.13 is well defined in each $B(x_i, \tilde{r}_i)$. Thus it suffices to take for instance

$$A = \frac{U10^{10}}{\bar{r}(\varepsilon_0)} > 1.$$

Comments about the constants The hierarchy between the constants

$$\varepsilon < \varepsilon'_0 < \varepsilon_0$$

has to be preserved. Notice that all the constants η_0 , η_1 , η_2 and ε_1 are coming from some independent results about minimal sets or almost minimal sets and the constants ε , ε'_0 and ε_0 can always be chosen small as we want with respect to them. The constants U and M are just some universal constants. On the other hand the constants \bar{r} and A are depending on ε_0 . The most important fact to keep in mind is that throughout all this paper, ε is always small as we want with respect to all the other constants.

3. Useful estimates

We are now ready to compute some estimates about the different quantities that will lead to regularity. Throughout this section we assume that $B(x_0, r_0)$ satisfies the Standard Assumptions 1 (see Definition 2.10). We start by finding a judicious radius ρ to begin the estimates.

3.1. Choice of the radius

The following lemma is very simple but it gives an estimate that will be crucial in the sequel.

Lemma 3.1. *Let $\{B_i\}_{i \in I}$ be the Vitali balls defined in Section 2.3 and set*

$$I(\rho) := \{i \in I; B_i \cap \partial B(x_0, \rho) \neq \emptyset\}. \tag{3.1}$$

Then there exists $\rho \in [\frac{r_0}{2}, \frac{3}{4}r_0]$ such that

$$\sum_{i \in I(\rho)} r_i^2 \leq C \sqrt{\varepsilon} r_0^2 m(x_0, r_0).$$

Proof. We select a $\rho \in T := [\frac{r_0}{2}, \frac{3}{4}r_0]$ such that the mass of the bad balls $\{B_i\}_{i \in I}$ that are meeting $\partial B(x_0, \rho)$ is less than average. By such a choice of ρ we have

$$\sum_{i \in I(\rho)} r_i^2 \leq \frac{1}{|T|} \int_T \sum_{i \in I(t)} r_i^2 dt \leq \frac{1}{|T|} \sum_{i \in I} \int_{t; i \in I(t)} r_i^2 \leq C \frac{1}{|T|} \sum_{i \in I} r_i^3.$$

Finally we have found a ρ that verifies

$$\sum_{i \in I(\rho)} r_i^2 \leq \frac{C}{r_0} \sum_{i \in I} r_i^3 \leq C \sup_i \{r_i\} \sum_{i \in I} r_i^2 \leq C \sqrt{\varepsilon} r_0^2 m(x_0, r_0). \tag{3.2}$$

□

3.2. Comparaison with an energy minimizing function

The next proposition will give the fundamental estimate that will be used to control the energy.

Proposition 3.2. *Assume that $B(x_0, r_0)$ satisfies the Standard Assumptions 1. Then for all $a < 1/2$ there exists $\varepsilon_2 := \varepsilon_2(a, \varepsilon_0)$ such that if $\varepsilon < \varepsilon_2$ then*

$$\begin{aligned} \omega_2(x_0, ar_0) &\leq 2\sqrt{a}\omega_2(x_0, r_0) + C \frac{1}{a^2} \omega_2(x_0, r_0)^{\frac{1}{2}} J(x_0, r_0)^{-1} \\ &\quad + C \frac{\sqrt{\varepsilon}}{a^2} m(x_0, r_0) + \frac{1}{a^2} h(r_0). \end{aligned} \tag{3.3}$$

Proof. We keep the same notation $I(\rho) := \{i \in I; B_i \cap \partial B(x_0, \rho) \neq \emptyset\}$ as before, and ρ is the radius given by Lemma 3.1. We are ready to compare u with an energy minimizing function and use the decay result of [14]. By construction of \mathcal{G} , the set F is $(\varepsilon_0, \sqrt{\varepsilon})$ -minimal in the sense of Definition 8 of [14]. In fact, we know by construction that F is ε_0 -minimal in the complement of the $\{B_i\}_{i \in I}$, and for all i , we have that $r_i \leq \sqrt{\varepsilon}r_0$. Set

$$G := F^\rho := \left(F \setminus \bigcup_{i \in I(\rho)} B_i \right) \cup \bigcup_{i \in I(\rho)} \partial B_i.$$

We may assume that ε is small enough with respect to the constant of [14] so that we can apply [14, Theorem 9]. Thus we know that the normalized energy decreases for all energy minimizer in $B(x_0, r_0) \setminus G$. In particular if w is the energy minimizer in $B(x_0, r_0) \setminus G$ that is equal to u on $\partial B(x_0, r_0) \setminus G = \partial B(x_0, r_0) \setminus F$ (for the existence of such a minimizer, one can see for example [6, page 97]), applying [14, Theorem 9] with $\gamma = \frac{1}{2} < 2(\sqrt{2} - 1)$, we have that for all $a < \frac{1}{2}$, there is a ε_2 (that depends on a and ε_0), such that if $\varepsilon < \varepsilon_2$ then

$$\frac{1}{(ar_0)^2} \int_{B(x_0, ar_0) \setminus G} |\nabla w|^2 \leq a^{\frac{1}{2}} \frac{1}{r_0^2} \int_{B(x_0, r_0) \setminus G} |\nabla w|^2. \tag{3.4}$$

The second useful fact is the following. Since (u, K) is a Mumford-Shah minimizer and (w, G) is a competitor we have

$$\begin{aligned} & \int_{B(x_0, r_0) \setminus K} |\nabla u|^2 + \mathcal{H}^2(K \cap B(x_0, r_0)) \\ & \leq \int_{B(x_0, r_0) \setminus G} |\nabla w|^2 + \mathcal{H}^2(G \cap B(x_0, r_0)) + r_0^2 h(r_0). \end{aligned}$$

Hence

$$\begin{aligned} & \int_{B(x_0, r_0) \setminus K} |\nabla u|^2 - \int_{B(x_0, r_0) \setminus G} |\nabla w|^2 \\ & \leq \mathcal{H}^2(G \cap B(x_0, r_0)) - \mathcal{H}^2(K \cap B(x_0, r_0)) + r_0^2 h(r_0) \\ & \leq Cr_0^2 \omega_2(x_0, r_0)^{\frac{1}{2}} J(x_0, r_0)^{-1} + C \sum_{i \in I(\rho)} r_j^2 + r_0^2 h(r_0) \\ & \leq Cr_0^2 \omega_2(x_0, r_0)^{\frac{1}{2}} J(x_0, r_0)^{-1} + C\sqrt{\varepsilon}r_0^2 m(x_0, r_0) + r_0^2 h(r_0). \end{aligned} \tag{3.5}$$

The third point is that ∇w and $\nabla(w - u)$ are orthogonal in $L^2(B(x_0, r_0))$. This comes from the fact that w is an energy minimizer in $B(x_0, r_0) \setminus G$ and u is a competitor for w . Thus

$$\int_{B(x_0, r_0) \setminus G} |\nabla u - \nabla w|^2 = \int_{B(x_0, r_0) \setminus G} |\nabla u|^2 - \int_{B(x_0, r_0) \setminus G} |\nabla w|^2.$$

We can now estimate the energy of u . Let $0 < a < \frac{1}{2}$, then

$$\begin{aligned} \int_{B(x_0, ar_0) \setminus G} |\nabla u|^2 &\leq 2 \int_{B(x_0, ar_0) \setminus G} |\nabla w|^2 + 2 \int_{B(x_0, ar_0) \setminus G} |\nabla w - \nabla u|^2 \\ &\leq 2a^{2+\frac{1}{2}} \int_{B(x_0, r_0) \setminus G} |\nabla w|^2 + 2 \int_{B(x_0, r_0) \setminus G} |\nabla w - \nabla u|^2 \\ &\leq 2a^{2+\frac{1}{2}} \int_{B(x_0, r_0) \setminus G} |\nabla u|^2 + 2 \int_{B(x_0, r_0) \setminus G} |\nabla u|^2 - 2 \int_{B(x_0, r_0) \setminus G} |\nabla w|^2. \end{aligned}$$

Hence, dividing by ar_0 and using (3.5) we get

$$\begin{aligned} \omega_2(x_0, ar_0) &\leq 2\sqrt{a}\omega_2(x_0, r_0) + C \frac{1}{a^2} \omega_2(x_0, r_0)^{\frac{1}{2}} J(x_0, r_0)^{-1} \\ &\quad + C \frac{\sqrt{\varepsilon}}{a^2} m(x_0, r_0) + \frac{1}{a^2} 2h(r_0). \end{aligned} \tag{3.6}$$

□

3.3. Control of the bad mass

We still assume that we are under our Standard Assumptions 1. The following proposition is an estimate about the bad mass $m(x_0, r)$. Let $\rho \in [\frac{r_0}{2}, \frac{3}{4}r_0]$ be the radius given by Lemma 3.1.

Proposition 3.3. *If $m(x_0, \frac{\rho}{2}) \geq \tau m(x_0, r_0)$ for some $\tau > 0$ and provided ε small enough with respect to τ and η_0 (the constant of Proposition 2.24), on has*

$$m\left(x_0, \frac{\rho}{2}\right) \leq \frac{C}{\eta_0} \left(\omega_2(x_0, r_0) + \omega_2(x_0, r_0)^{\frac{1}{2}} J(x_0, r_0)^{-1} + h(r_0) \right). \tag{3.7}$$

Proof. To prove Proposition 3.3, we will count the contribution of B_i for $i \in I$ and use Proposition 2.24 to say that there are not so many. Recall that the B_i are disjoint.

In order to estimate the bad mass we will construct a competitor for (u, K) in $B(x_0, r_0)$. We denote by I_1 the set of indices of bad balls B_i such that $B(x_i, d(x_i))$ doesn't verify (2.24) and $I_2 := I \setminus I_1$. In particular, balls of I_2 do not verify (2.23). Hence we know that if $i \in I_2$ we have

$$K \cap B(x_i, d(x_i)) = \emptyset \text{ or } d(x_i)^2 \leq \frac{1}{\varepsilon_0'} \left(\mathcal{H}^2(F \cap B(x_i, d(x_i))) - \mathcal{H}^2(K \cap B(x_i, d(x_i))) \right)$$

and since the B_i are disjoint we deduce that

$$\sum_{i \in I_2} r_i^2 \leq C \frac{1}{\varepsilon_0'} (\mathcal{H}^2(F(x_0, r_0)) - \mathcal{H}^2(K \cap B(x_0, r_0))) \leq Cr_0^2 \omega_2(x_0, r_0)^{\frac{1}{2}} J(x_0, r_0)^{-1}.$$

Now we have to count the contribution of I_1 . We will modify each B_i for $i \in I_1$ with the use of Proposition 2.24. Set

$$I'_1 := \{i \in I_1; B_i \cap B(x_0, \rho) \neq \emptyset \text{ and } B_i \cap \partial B(x_0, \rho) = \emptyset\}$$

and

$$I''_1 := \{i \in I_1; B_i \cap \partial B(x_0, \rho) \neq \emptyset\}.$$

Then define

$$\tilde{G} := \begin{cases} F(x_0, r_0) & \text{in } B(x_0, r_0) \setminus \bigcup_{i \in I'_1} B_i \\ L_i & \text{in } B_i \text{ for all } i \in I'_1 \end{cases}$$

where L_i is the set given by Proposition 2.24. Then set

$$G := \tilde{G} \cup \bigcup_{i \in I_\rho} \partial C_1 B_i$$

where C_1 is the constant in (2.36). For the function we use the extension of Proposition 2.13 which can be applied in $B(x_0, \rho)$ by Proposition 2.15. Thus we take

$$v = v^k \text{ in } \Omega^k \cap B(x_0, \rho),$$

and

$$v = u \text{ in } B(x_0, r_0) \setminus \left(B(x_0, \rho) \cup \bigcup_{i \in I(\rho)} C_1 B_i \right).$$

By choice of constant A we know that the function v is well defined in $B(x_0, r_0) \setminus G$. Notice that $m(x_0, \frac{\rho}{2}) \leq C \frac{1}{r_0} \sum_{i \in I'_1} r_i^2$ and $\sum_{i \in I''_1} r_i^2 \leq C r_0^2 \sqrt{\varepsilon} m(x_0, r_0)$ by Lemma 3.1. In addition G is a competitor (because of the same argument as [7, Remark 1.8.]). We apply now the fact that (u, K) is a Mumford-Shah minimizer and we obtain

$$\begin{aligned} & \int_{B(x_0, r_0) \setminus K} |\nabla u|^2 + \mathcal{H}^2(K \cap B(x_0, r_0)) \\ & \leq \int_{B(x_0, r_0) \setminus G} |\nabla v|^2 + \mathcal{H}^2(G \cap B(x_0, r_0)) + r_0^2 h(r_0) \\ & \leq C \int_{B(x_0, r_0) \setminus K} |\nabla u|^2 + \mathcal{H}^2(F(x_0, r_0)) - \eta_0 \sum_{i \in I'_1} r_i^2 + C \sum_{i \in I''_1} r_i^2 + C \sum_{i \in I_2} r_j^2 + r_0^2 h(r_0). \end{aligned}$$

Hence,

$$\begin{aligned} & \eta_0 C r_0 m \left(x_0, \frac{\rho}{2} \right) - C \sqrt{\varepsilon} r_0 m(x_0, r_0) \\ & \leq C \int_{B(x_0, r_0) \setminus K} |\nabla u|^2 + r_0^2 \omega_2(x_0, r_0)^{\frac{1}{2}} J(x_0, r_0)^{-1} + r_0^2 h(2r_0). \end{aligned}$$

Therefore, since $m(x_0, \frac{\rho}{2}) \geq \tau m(x_0, r_0)$ and if ε is small enough compared to η_0 and τ we deduce

$$m\left(x_0, \frac{\rho}{2}\right) \leq \frac{C}{\eta_0} \left(\omega_2(x_0, r_0) + \omega_2(x_0, r_0)^{\frac{1}{2}} J(x_0, r_0)^{-1} + h(2r_0)\right) \tag{3.8}$$

and the proposition follows. □

Now by the same sort of argument as in the Proposition before, we have this second estimate about m that will also be useful.

Proposition 3.4.

$$\begin{aligned} & m(x_0, r_0(1 - C\sqrt{\varepsilon})) \\ & \leq \frac{C}{\eta_0} \left(\omega_2(x_0, r_0) + \omega_2(x_0, r_0)^{\frac{1}{2}} J(x_0, r_0)^{-1} + \beta(x_0, r_0) + h(r_0)\right). \end{aligned} \tag{3.9}$$

Proof. The proof is very similar to Proposition 3.3. We keep the notation of I_1 and set

$$I'_1 := \{B_i; 5B_i \cap \partial B(x_0, r_0) \neq \emptyset\}.$$

Then we set

$$\tilde{G} := \begin{cases} F(x_0, r_0) & \text{in } B(x_0, r_0) \setminus \bigcup_{i \in I'_1} B_i \\ L_i & \text{in } B_i \text{ for all } i \in I'_1 \end{cases}$$

where the L_i are the sets given by Proposition 2.24. Our competitor is now

$$G := \tilde{G} \cup T_\beta$$

where T_β is a little wall of size $\beta := 10\beta(x_0, r_0)$

$$T_\beta := \{y \in \partial B(x_0, r_0); d(y, Z) \leq \beta r_0\}$$

with $Z = Z(x_0, r_0)$.

As before we have

$$\sum_{i \in I_2} r_i^2 \leq C \frac{1}{\varepsilon_0'} (\mathcal{H}^2(F(x_0, r_0)) - \mathcal{H}^2(K \cap B(x_0, r_0))) \leq Cr_0^2 \omega_2(x_0, r_0)^{\frac{1}{2}} J(x_0, r_0)^{-1}.$$

For the function we use the extension of Proposition 2.13 but applied in $B(x_0, 2r_0)$ and with $\rho = r_0$ and an analogous geometric function. We fix $v = v^k$ in $\Omega^k \cap B(x_0, \rho)$ and $v = u$ in $B(x_0, 2r_0) \setminus B(x_0, r_0)$. By choice of constant A we know that the function v is well defined in $B(x_0, r_0) \setminus G$ and since we added T_β there is no boundary problem.

We apply now the fact that (u, K) is a Mumford-Shah minimizer and we obtain,

$$\begin{aligned} & \int_{B(x_0, r_0) \setminus K} |\nabla u|^2 + \mathcal{H}^2(K \cap B(x_0, r_0)) \\ & \leq \int_{B(x_0, r_0) \setminus G} |\nabla v|^2 + \mathcal{H}^2(G \cap B(x_0, r_0)) + r_0^2 h(r_0) \\ & \leq C \int_{B(x_0, r_0) \setminus K} |\nabla u|^2 + \mathcal{H}^2(F(x_0, r_0)) - \eta_0 \sum_{i \in I'_1} r_i^2 + \mathcal{H}^2(T_\beta) + r_0^2 h(r_0). \end{aligned}$$

Hence,

$$\begin{aligned} \eta_0 m(x_0, r_0(1 - C\sqrt{\varepsilon})) & \leq C \int_{B(x_0, r_0) \setminus K} |\nabla u|^2 + r_0^2 \omega_2(x_0, r_0)^{\frac{1}{2}} J(x_0, r_0)^{-1} \\ & \quad + Cr_0^2 \beta(x_0, r_0) + r_0^2 h(2r_0) \end{aligned}$$

because all the B_i have a radius less than $C\sqrt{\varepsilon}r_0$ thus all the B_i for $i \in I_1$ such that $5B_i \cap \partial B(x_0, r_0) = \emptyset$ are contained in $B(x_0, r_0(1 - C\sqrt{\varepsilon}))$, and the proposition follows. \square

3.4. Control of the minimality defect

In this section we want to control the defect of minimality of K in terms of energy and bad mass. For some topological reasons we are not going to work directly on K , but we will use the set F to be sure that it is separating in B . We show in this section that F is an almost minimal set with gauge function depending on the energy of u and the bad mass.

Proposition 3.5. *Assume that $B(x_0, r_0)$ satisfies the Standard Assumptions 1. Then there is a positive constant $c_1 < 1$ such that for all MS-competitor L for the set F in the ball $B(x_0, c_1r_0)$, we have:*

$$\begin{aligned} & \frac{1}{r_0^2} [\mathcal{H}^2(F \cap B(x_0, c_1r_0)) - \mathcal{H}^2(L \cap B(x_0, c_1r_0))] \\ & \leq C [\omega_2(x_0, r_0) + \sqrt{\varepsilon}m(x_0, r_0) + h(r_0)]. \end{aligned}$$

Proof. Let $Z^0 := Z(x_0, r_0)$ be the associated cone in $B(x_0, r_0)$. We also denote Z_ε^0 the region

$$Z_\varepsilon^0 := \{x \in B(x_0, r_0); \text{dist}(x, Z^0) \leq \varepsilon\}. \tag{3.10}$$

We consider our usual Vitali balls $\{B_i\}_{i \in I}$ and we take the same functions as before

$$\psi_i := \begin{cases} r_i & \text{on } B_i \\ 0 & \text{in the complement of } 5B_i. \end{cases}$$

Then for all x we define as before

$$\delta(x) := \sum_{i \in I} \psi_i(x)$$

which is a geometric function, and finally for all $x \in B(x_0, \rho)$ (where ρ is the radius of Lemma 3.1) we set

$$\delta_1(x) := \max(\text{dist}(x, \partial B(x_0, \rho)), \delta(x)).$$

By the same way as for Proposition 2.15, one can easily prove that $\delta_1(x)$ is a geometric function associated to F in $B(x_0, r_0)$. Thus applying Lemma 2.13 we get some functions v^k such that $v^k \in W^{1,2}(\Omega^k \cup \mathcal{V} \setminus \mathcal{V}_\rho)$ and such that

$$\int_{\Omega^k \cup \mathcal{V} \setminus \mathcal{V}_\rho} |\nabla v^k|^2 \leq C \int_{B(x_0, \rho) \setminus F} |\nabla u|^2$$

in addition, v^k is equal to u on $\partial B(x_0, \rho) \cap \Omega^k \setminus \mathcal{V}$.

Moreover, since $\delta_1(x) \geq \text{dist}(x, \partial B(x, \rho))$, if ε is small enough we can easily deduce that there is a constant $c_1 < \frac{1}{2}$ depending on other constants like U and C_0 , such that $B(x_0, c_1 r_0) \subset \mathcal{V}$. Consider now

$$G' = \begin{cases} F & \text{in } B(x_0, r_0) \setminus B(x_0, c_1 r_0) \\ L & \text{in } B(x_0, c_1 r_0). \end{cases}$$

If L is a competitor for F in $B(x_0, c_1 r_0)$, we know that L is separating $B(x_0, c_1 r_0)$ into $\ell(x, \rho)$ big connected components (because F is separating and L is a topological competitor). Thus G' is separating in $B(x_0, \rho)$ and we denote by $(B(x_0, \rho) \setminus G)^k$ the big connected components.

Then set

$$G := G' \cup \bigcup_{i \in I(\rho)} \partial C_1 B_i$$

and

$$v := \begin{cases} u & \text{in } B(x_0, r_0) \setminus B(x_0, \rho) \\ v^k & \text{in } (B(x_0, \rho) \setminus G)^k \\ 0 & \text{in other components of } B(x_0, \rho) \setminus G. \end{cases}$$

Using that (u, K) is a Mumford-Shah minimizer and that (v, G) is a competitor we obtain

$$\int_{B(x_0, \rho) \setminus K} |\nabla u|^2 + \mathcal{H}^2(K) \leq \int_{B(x_0, \rho) \setminus G} |\nabla v|^2 + \mathcal{H}^2(G) + \rho^2 h(\rho)$$

thus

$$\begin{aligned} & \mathcal{H}^2(K \cap B(x_0, c_1 r_0)) - \mathcal{H}^2(L \cap B(x_0, c_1 r_0)) \\ & \leq C \left[\int_{B(x_0, r_0) \setminus K} |\nabla u|^2 + \sum_{i \in I''} r_i^2 + r_0^2 \omega(x_0, r)^{\frac{1}{2}} J(x_0, r)^{-1} + r_0^2 h(r_0) \right] \end{aligned}$$

and the proposition follows. □

4. Conclusion about regularity

Now we are ready to use all the preceding estimates in order to prove some regularity. We begin with this proposition about self-improving estimates.

Proposition 4.1. *There is an $\varepsilon > 0$, some $\tau_4 < \tau_3 < \tau_2 < \tau_1 < \varepsilon$ and $a < 1$ such that the following holds. Assume that $B(x, r)$ satisfies the Standard Assumptions 1 and that*

$$h(r) + J(x, r)^{-1} \leq \tau_4, \quad \omega_2(x, r) \leq \tau_3, \quad m(x, r) \leq \tau_2, \quad \beta(x, r) \leq \tau_1. \quad (4.1)$$

Then (4.1) is still true with ar instead of r and $10^{-5}\tau_i$ instead of τ_i for $i \in \{1, 2, 3, 4\}$.

Proof. We choose ε small enough such that all the results of the preceding sections holds. Then we fix $a < \frac{1}{4 \cdot 10^{10}}$ such that applying (3.3) to (u, K) gives

$$\omega_2(x, ar) \leq 10^{-6}\omega_2(x, r) + C_2\omega_2(x, r)^{\frac{1}{2}}J(x, r)^{-1} + C_2\sqrt{\varepsilon}m(x, r) + C_2h(r), \quad (4.2)$$

with a constant C_2 depending on a . Since a is chosen, we can fix τ_1 small enough such that for all $ar < t < r$ we have $\beta(x, t) \leq 10^{-6}$. Hence by Lemma 2.9

$$J(x, ar) \geq a^{-\frac{1}{2}}[J(x, r) - C'] \geq \frac{1}{2}a^{-\frac{1}{2}}J(x, r)$$

if τ_4 is small enough compared to C' . Then we deduce

$$J(x, ar)^{-1} \leq 2a^{\frac{1}{2}}J(x, r)^{-1} \leq 10^{-5}J(x, r)^{-1}$$

because $a < \frac{1}{4 \cdot 10^{10}}$. In addition if τ_4 is small enough compared to τ_3 , we have

$$C_2\tau_3^{\frac{1}{2}}\tau_4 \leq 10^{-6}\tau_3. \quad (4.3)$$

Therefore by (4.2),

$$\omega_2(x, ar) \leq 2 \cdot 10^{-6}\tau_3 + C_2\sqrt{\varepsilon}m(x, r) + C_2\tau_4 \leq 10^{-5}\tau_3$$

because τ_4 is small as we want with respect to τ_3 and under the further condition that for instance

$$C_2\sqrt{\varepsilon}\tau_2 < 10^{-6}\tau_3. \tag{4.4}$$

Now for $m(x, r)$ we have two cases. If $m(x, ar) \leq 10^{-5}m(x, r)$ then $m(x, ar) \leq 10^{-5}\tau_2$ and this is what we want. Otherwise, we have $m(x, ar) > 10^{-5}m(x, r)$ which implies $m(x, \frac{\rho}{2}) > a^2 10^{-5} \left(\frac{8}{3}\right)^2 m(x, r)$ and then we can apply Proposition 3.3 with $\tau = a^2 10^{-5} \left(\frac{8}{3}\right)^2$ to obtain

$$m(x, \rho/2) \leq \frac{C}{\eta_0}(\tau_3 + \tau_3^{\frac{1}{2}}\tau_4 + \tau_4)$$

which implies

$$\begin{aligned} m(x, ar) &\leq \frac{(\rho/2)^2}{(ar)^2} m\left(x, \frac{\rho}{2}\right) \\ &\leq C(a, \eta_0)(\tau_3 + \tau_3^{\frac{1}{2}}\tau_4 + \tau_4) \\ &\leq C(a, \eta_0)(2 + 10^{-6}/C_2)\tau_3 \leq 10^{-5}\tau_2, \end{aligned} \tag{4.5}$$

using (4.3) and provided that

$$10^5 C(a, \eta_0)(2 + 10^{-6}/C_2)\tau_3 \leq \tau_2. \tag{4.6}$$

So it suffice to choose ε small enough compared to η_0 and a in order to have the existence of $\tau_3 < \tau_2$ that verifies simultaneously (4.4) and (4.6). Hence, we control $\omega_2(x, ar)$ and $m(x, ar)$.

To finish we have to control $\beta(x, ar)$. For that we use the estimate in Proposition 3.5 and Lemma 2.23. Indeed, suppose by contradiction that $a \ll \bar{r}(10^{-5}\tau_1)c_1$ is such that

$$\beta(x, ar) \geq 10^{-5}\tau_1. \tag{4.7}$$

Then applying Lemma 2.23 with $\varepsilon_0 = 10^{-5}\tau_1$ gives a $\eta_0(\tau_1, a)$ and a competitor L for K in $B(x, c_1r)$ such that

$$\mathcal{H}^2(K) - \mathcal{H}^2(L) \geq \eta_0(\tau_1, a). \tag{4.8}$$

On the other hand, according to Proposition 3.5, if we choose τ_2 and τ_3 small enough compared to $\eta_0(\tau_1, a)$, the inequality (4.8) cannot hold. This shows that

$$\beta(x, ar) \leq 10^{-5}\tau_1$$

which achieves the proof of the proposition. □

We keep the constants a and τ_i given by the preceding proposition. Let b be the positive power such that $(a/200)^b = \frac{1}{2}$. Set

$$\tilde{h}_r(t) = \sup \left\{ \left(\frac{t}{s} \right)^b h(s); t \leq s \leq r \right\}$$

for $t < r$ and $\tilde{h}_r(t) = h(t)$ for $t > r$. According to [6, page 318], the function \tilde{h} is still a gauge function (*i.e.* monotone and with limit equal to 0 at 0). We also trivially have that $h(t) \leq \tilde{h}_r(t)$ and one can prove that

$$\tilde{h}_r(t) \geq \left(\frac{t}{t'} \right)^b \tilde{h}_r(t') \quad \text{for } 0 < t < t' \leq r. \tag{4.9}$$

Notice that since $(a/100)^b = \frac{1}{2}$, we have

$$\tilde{h}_r(at/100) \geq \frac{1}{2} \tilde{h}_r(t) \quad \text{for } 0 < t \leq r. \tag{4.10}$$

The purpose of Proposition 4.1 is just to have $\beta(x, r) \leq \tau_1$ at all scales in order to have more decay for the other quantities. Notice that at this step, an iteration of the last Proposition could prove that K is the bi-hölderian image of a minimal cone using the Reifenberg parametrization of G. David, T. De Pauw and T. Toro [8]. This will be done in Corollary 4.4 to prove that K is a separating set. Before that we will prove some more decay estimates.

Proposition 4.2. *We assume that we have the same hypothesis as in the proposition before. Then for all $0 < t < r$ we have*

$$\begin{aligned} J(x, t)^{-1} &\leq 2 \left(\frac{t}{r} \right)^b \tau_4 \\ \omega_2(x, t) &\leq C \left(\frac{t}{r} \right)^b \tau_3 + C \tilde{h}_r(t) \\ m(x, t) &\leq C \left(\frac{t}{r} \right)^b \tau_2 + C \tilde{h}_r(t). \end{aligned}$$

Proof. We begin by iterating Proposition 4.1. Let us do the first step. We start with some control of the different quantities ω_2, m, J, β in $B(x, r)$ and we obtain a similar control in $B(x, ar)$. The only problem that could occur to re-apply the proposition in $B(x, ar)$ is that the cone associated to $B(x, ar)$ might not be almost centered (which is required to have Standard Assumptions 1 in $B(x, ar/2)$). But in this case we know by the recentering Lemma that the cone associated to $B(x, ar/10)$ or $B(x, ar/100)$ is almost centered. Moreover since the conclusion of Proposition 4.1 yields a control of all the quantities by $10^{-5}\tau_i$ we are sure that in

$B(x, ar/10)$ or $B(x, ar/100)$ the quantities are still controlled at least by τ_i . Therefore we can apply the proposition in $B(x, ar/20)$ or $B(x, ar/200)$ and continue the iteration at all smaller scales. In conclusion we just constructed a sequence a_n converging to 0 such that $a_0 = 1, a_{n+1} \in \{aa_n, aa_n/20, aa_n/200\}$, such that each $B(x, a_n r)$ has an associated cone almost centered and

$$h(a_n r) + J(x, a_n r)^{-1} \leq \tau_4, \quad \omega_2(x, a_n r) \leq \tau_3, \quad m(x, a_n r) \leq \tau_2, \quad \beta(x, a_n r) \leq \tau_1$$

for all $n \in \mathbb{N}$.

Next we want to get some further decay. First we control the jump. Up to take a slightly smaller constant τ_1 (depending on a) we can assume that $\beta(x, t) \leq 10^{-7}$ for all $t < r$. Then by Lemma 2.9

$$J(x, t) \geq \left(\frac{r}{t}\right)^{-\frac{1}{2}} [J(x, r) - C'] \geq \frac{1}{2} \left(\frac{r}{t}\right)^{-\frac{1}{2}} J(x, r)$$

if τ_4 is small enough compared to C' . We deduce

$$J(x, t)^{-1} \leq 2 \left(\frac{t}{r}\right)^{\frac{1}{2}} J(x, r)^{-1}.$$

And since $a < \frac{1}{4}$ we have in particular

$$J(x, a_n r)^{-1} \leq 2 \left(\frac{1}{2}\right)^n \tau_4. \tag{4.11}$$

Now we want to show by induction that

$$\omega_2(x, a_n r) \leq 2^{-n} \tau_3 + C_3 \tilde{h}_r(a_n r) \quad \text{and} \quad m(x, a_n r) \leq 2^{-n} \tau_2 + C_3 \tilde{h}_r(a_n r) \tag{4.12}$$

for a suitable choice of constant C_3 .

If $n = 0$, (4.12) holds by our assumptions. Suppose by now that (4.12) is true for n . Then inequality (3.3) applied in $B(x, a_n r)$ yields

$$\begin{aligned} \omega_2(x, aa_n r) &\leq 10^{-6} \omega_2(x, a_n r) + C_2 \omega_2(x, a_n r)^{\frac{1}{2}} J(x, a_n r)^{-1} \\ &\quad + C_2 \sqrt{\varepsilon} m(x, a_n r) + C_2 h(a_n r). \end{aligned} \tag{4.13}$$

Now, using the inequality $2\alpha\beta \leq \alpha^2 + \beta^2$ we obtain

$$\omega_2(x, a_n r)^{\frac{1}{2}} \leq \frac{10^{-6}}{2C_2} \omega_2(x, a_n r) J(x, a_n r) + \frac{10^6 C_2}{2} J(x, a_n r)^{-1}. \tag{4.14}$$

Thus (4.13) becomes

$$\begin{aligned} \omega_2(x, aa_n r) &\leq \left(10^{-6} + \frac{10^{-6}}{2}\right) \omega_2(x, a_n r) + \frac{10^6}{2} C_2^2 J(x, a_n r)^{-2} \\ &\quad + C_2 \sqrt{\varepsilon} m(x, a_n r) + C_2 h(a_n r). \end{aligned}$$

Now using (4.11) and the induction hypothesis (4.12) we obtain

$$\begin{aligned}\omega_2(x, aa_nr) &\leq \frac{3}{2}10^{-6}2^{-n}\tau_3 + 10^6C_2^22^{-2n}\tau_2^2 + C_2\sqrt{\varepsilon}2^{-n}\tau_2 \\ &\quad + \left(\frac{3}{2}10^{-6}C_3 + C_2\sqrt{\varepsilon}C_3 + C_2\right)\tilde{h}_r(a_nr).\end{aligned}$$

Then, using that τ_4 is controlled by τ_3 , since ε is small as we want compared to C_2 , using also (4.4) and (4.10), and finally if we choose C_3 large enough with respect to C_2 we deduce that

$$\omega_2(x, aa_nr) \leq 3 \cdot 10^{-6}2^{-n}\tau_3 + 10^{-5}C_3\tilde{h}_r(a_nr)$$

which implies

$$\begin{aligned}\omega_2(x, a_{n+1}r) &\leq 200^2\omega_2(x, aa_nr) \leq 10^4[3 \cdot 10^{-6}2^{-n}\tau_3 + 10^{-5}C_3\tilde{h}_r(a_nr)] \\ &\leq 3 \cdot 10^{-2}2^{-n}\tau_3 + 10^{-1}C_3\tilde{h}_r(a_nr) \\ &\leq 2^{-(n+1)}\tau_3 + C_3\tilde{h}_r(a_{n+1}r).\end{aligned}\tag{4.15}$$

Concerning $m(x, r)$ it is a similar argument. We may assume that

$$m(x, aa_nr) > 10^{-5}m(x, a_nr)\tag{4.16}$$

otherwise

$$m(x, a_{n+1}r) \leq 10^4m(x, aa_nr) \leq 10^{-1}m(x, a_nr)$$

and we would conclude by the induction hypothesis.

Now if (4.16) holds, then the same argument employed to prove (4.5) implies that

$$m(x, aa_nr) \leq C(a, \eta_0)[\omega_2(x, a_nr) + \omega_2(x, a_nr)^{\frac{1}{2}}J(x, a_nr)^{-1} + h(a_nr)].$$

Then using again an inequality similar to (4.14) one gets

$$m(x, aa_nr) \leq C(a, \eta_0) \left[\frac{3}{2}\omega_2(x, a_nr) + \frac{1}{2}J(x, a_nr)^{-2} + h(a_nr) \right].$$

Setting $C_4 = C(a, \eta_0)$, using (4.11) and induction hypothesis we obtain

$$\begin{aligned}m(x, aa_nr) &\leq C_42^{-n}\tau_3 + C_42^{-2n+1}\tau_4^2 + C_4h(a_nr) \\ &\leq 10^{-4}2^{-n+1}\tau_2 + 10^{-4}C_3\tilde{h}_r(a_{n+1}r)\end{aligned}$$

because τ_3 and τ_4 are small as we want with respect to C_4 and τ_2 , because we can chose C_3 big enough with respect to C_4 and we also have used (4.10). Finally since $m(x, a_{n+1}r) \leq 10^4m(x, aa_nr)$ we conclude that

$$m(x, a_{n+1}r) \leq 2^{-n+1}\tau_2 + C_3\tilde{h}_r(a_{n+1}r).$$

To finish the proof let $0 < t < r$ and n such that $a_{n+1}r \leq t \leq a_n r$. In particular $(a/200)^{n+1}r \leq t$. Then we have

$$\begin{aligned} \omega_2(x, t) &= \frac{1}{t^2} \int_{B(x_0, t) \setminus K} |\nabla u|^2 \leq \left(\frac{a_n r}{t}\right)^2 \omega_2(x, a_n r) \\ &\leq \left(\frac{200}{a}\right)^2 \left(2^{-n} \tau_3 + C_3 \tilde{h}_r(a_n r)\right) \\ &\leq \left(\frac{200}{a}\right)^2 (a/200)^{bn} \tau_3 + C'_3 \tilde{h}_r(t) \\ &\leq C \left(\frac{t}{r}\right)^b \tau_3 + C'_3 \tilde{h}_r(t) \end{aligned}$$

and similarly,

$$\begin{aligned} m(x, t) &\leq \left(\frac{a_n r}{t}\right)^2 m(x, a_n r) \leq \left(\frac{200}{a}\right)^2 \left(2^{-n} \tau_2 + C_3 \tilde{h}_r(a_n r)\right) \\ &\leq \left(\frac{200}{a}\right)^2 (a/200)^{bn} \tau_2 + C'_3 \tilde{h}_r(t) \leq C \left(\frac{t}{r}\right)^b \tau_2 + C'_3 \tilde{h}_r(t), \end{aligned}$$

which ends the proof. \square

Proposition 4.3. *There is $\varepsilon > 0$ and $\tau'_4 < \tau'_3 < \tau'_2 < \tau'_1 < \varepsilon$ such that if the Standard Assumptions 1 are fulfilled in $B(x_0, r_0)$ and if*

$$h(r_0) + J(x_0, r_0)^{-1} \leq \tau'_4, \quad \omega_2(x_0, r_0) \leq \tau'_3, \quad m(x_0, r_0) \leq \tau'_2, \quad \beta(x_0, r_0) \leq \tau'_1,$$

then for all $x \in B(x_0, \frac{1}{10}r_0)$ and for all $0 < t < \frac{1}{200}r_0$ we have

$$\begin{aligned} J(x, t)^{-1} &\leq C \left(\frac{t}{r_0}\right)^b \\ \omega_2(x, t) &\leq C \left(\frac{t}{r_0}\right)^b + C \tilde{h}_r(t) \\ m(x, t) &\leq C \left(\frac{t}{r_0}\right)^b + C \tilde{h}_r(t) \\ \beta(x, t) &\leq \tau_1. \end{aligned}$$

Proof. It suffice to show that there are $\tau'_4 < \tau'_3 < \tau'_2 < \tau'_1 < \varepsilon$ such that if

$$h(r_0) + J(x_0, r_0)^{-1} \leq \tau'_4, \quad \omega_2(x_0, r_0) \leq \tau'_3, \quad m(x_0, r_0) \leq \tau'_2, \quad \beta(x_0, r_0) \leq \tau'_1$$

then for all $x \in B(x_0, \frac{1}{10}r_0)$ we have

$$\begin{aligned}
 h\left(\frac{1}{2}r_0\right) + J\left(x, \frac{1}{2}r_0\right)^{-1} &\leq \tau_4, & \omega_2\left(x, \frac{1}{2}r_0\right) &\leq \tau_3, \\
 m\left(x, \frac{1}{2}r_0\right) &\leq \tau_2, & \beta\left(x, \frac{1}{2}r_0\right) &\leq \tau_1
 \end{aligned}$$

hence we could apply all the work of preceding sections in $B(x, \frac{1}{10}r_0)$ and conclude.

Note that for all $x \in K \cap B(x_0, \frac{1}{10}r_0)$ we have

$$\begin{aligned}
 \omega_2(x, sr_0) &\leq \frac{1}{s}\omega_2(x_0, r_0) \\
 m(x, sr_0) &\leq 2\frac{1}{s}m(x_0, r_0) \\
 \beta(x, sr_0) &\leq \frac{1}{s}\beta(x_0, r_0)
 \end{aligned} \tag{4.17}$$

in addition if $\beta(x_0, r_0)$ is small enough then

$$J(x, sr_0)^{-1} \leq \frac{1}{s}J(x_0, r_0)^{-1}.$$

Finally, since h is non-decreasing

$$h(sr_0) \leq h(r_0).$$

We deduce that for $i \in [1, 4]$ we can set

$$\tau'_i := \frac{1}{400}\tau_i$$

so that all the assumptions of Proposition 4.2 are satisfied in $B(x, s)$ where $x \in B(x_0, r_0/10)$ and $s \in [r_0/200, r_0/2]$, except that the associated cone might not be almost centered. But for at least one radius $s \in [r_0/200, r_0/2]$ the associated cone is almost centered which allows us to apply Proposition 4.2 so that finally the decay holds for any $r < r_0/200$ as it is claim in the statement of the proposition. \square

Corollary 4.4. *In the same situation as in proposition before, if τ_1 is small enough we can choose*

$$F\left(x_0, \frac{1}{10}r_0\right) = K \cap B\left(x_0, \frac{1}{10}r_0\right).$$

Proof. The approach is to prove that K is separating in $B(x_0, \frac{1}{10}r_0)$, so that we can take $F = K$ in this ball. To show that K is separating it will be convenient

to apply [8, Theorem 1.1]. Actually one could probably prove it directly without using the whole result of [8], but this would make the present paper longer.

The main point is to show that for all $x \in B(x_0, \frac{1}{10}r_0)$ and for all r such that $B(x, r) \subset B(x_0, \frac{1}{200}r_0)$ there is a cone $Z(x, r)$ such that

$$D_{x,r}(K, Z(x, r)) \leq \varepsilon'$$

with ε' a certain constant given by [8, Theorem 1.1]. Recall that $D_{x,r}$ is the normalized Hausdorff distance

$$D_{x,r}(E, F) := \frac{1}{r} \max \left\{ \sup_{z \in E \cap B(x,r)} \{\text{dist}(z, F)\}, \sup_{z \in F \cap B(x,r)} \{\text{dist}(z, E)\} \right\}. \quad (4.18)$$

If we choose τ_1 small enough compared to ε' we know that for all x and for all r we have $\beta(x, r) \leq \varepsilon'$ by the preceding proposition. Hence we can find a cone $Z(x, r)$ that satisfy the first half of $D_{x,r}$. We have to show now that

$$\sup\{\text{dist}(z, K), z \in Z(x, r)\} \leq r\varepsilon'.$$

We know that $J(x, r)^{-1} \leq \tau_4$ and $\omega_2(x, r) \leq \tau_3$. Thus there is a set $F(x, r)$ that is separating in $B(x, r)$ and such that

$$\mathcal{H}^2(F(x, r) \cap K \cap B(x_0, r)) \leq C\omega_2(x, r)^{\frac{1}{2}}J(x, r)^{-1} \leq 10^{-6}\tau_3r^2$$

by (4.3). Then for all $z \in Z(x, r)$, we have

$$\text{dist}(z, K) \leq \text{dist}(z, y) + \text{dist}(y, K)$$

with y a point of $F(x, r)$ such that $\text{dist}(z, F(x, r)) = \text{dist}(z, y)$. If $\tau_1 \leq \frac{\varepsilon'}{2}$ we can suppose that $F(x, r) \subset \{y; \text{dist}(y, Z) \leq r\frac{\varepsilon'}{2}\}$. Thus $\text{dist}(z, y) \leq r\frac{\varepsilon'}{2}$. We claim that $\text{dist}(y, K) \leq r\frac{\varepsilon'}{2}$. The argument is by contradiction. If it is not true, then $K \cap B(y, r\frac{\varepsilon'}{2}) = \emptyset$. But $F(x, r)$ is contained in $T := \{y; \text{dist}(y, Z) \leq r\varepsilon'\}$. Let A^k be the connected components of $B(y, r\frac{\varepsilon'}{2}) \setminus T$. Then $F(x, r)$ separates the A^k in $B(y, r\frac{\varepsilon'}{2})$, and the minimal set that have this property is a cone of type \mathbb{P} , \mathbb{Y} or \mathbb{T} of area greater than $C\varepsilon'^2r^2$. On the other hand $\mathcal{H}^2(F(x, r) \setminus K) \leq \tau_3r^2$. Thus if τ_3 is small enough compared to ε' it is not possible, thus finally $\text{dist}(y, K) \leq \frac{\varepsilon'}{2}$ and

$$D_{x,r}(K, P) \leq \varepsilon'.$$

Now [8, Theorem 1.1] says that K is containing the image of a minimal cone by a homeomorphism from $B(x_0, \frac{1}{10}r_0)$ to $B(x_0, \frac{1}{5}r_0)$ and this proves in particular the desired separation property on K which implies that one can take $F = K$ as claimed. \square

Theorem 4.5. *There is some absolute positive constants ε and c such that the following is true. Let (u, K) be a Mumford-Shah minimizer in $\Omega \subset \mathbb{R}^3$ with gauge function h , let $x \in K$ and r be such that $B(x, r) \subset \Omega$ and*

$$\omega_2(x, r) + \beta(x, r) + J(x, r)^{-1} + h(r) \leq \varepsilon$$

where the associated cone in $B(x, r)$ denoted Z is of type \mathbb{P} , \mathbb{Y} or \mathbb{T} centered at x . Then there is a diffeomorphism ϕ of class $C^{1,\alpha}$ from $B(x, cr)$ to its image such that $K \cap B(x, cr) = \phi(Z) \cap B(x, cr)$.

Proof. To prove Theorem 4.5 it suffice to show that $K \cap B(x, cr)$ is an almost minimal set that verify the assumptions of Theorem 1.12. If ε is small enough, then $B(x, r/2)$ satisfies the Standard Assumptions 1 and provided ε small enough with respect to τ'_4 , all the quantities $\omega_2(x, r/2)$, $\beta(x, r/2)$, $J(x, r/2)^{-1}$ and $h(r/2)$ verify the hypothesis of Proposition 4.3. Furthermore, according to Proposition 3.4, $m(x, r/2)$ is also smaller that τ_2 . Therefore, all the assumptions of the last propositions are fulfilled.

By Corollary 4.4, we know that $F = K$ in $B(x, \frac{1}{20}r)$. So we can apply Proposition 3.5 directly on K (instead of F) and the decay result on ω_2 and m obtained in Proposition 4.3 shows that K is an almost minimal set in $B(x, \frac{1}{20}c_1r)$ with gauge function

$$\hat{h}(t) := C \left(\frac{t}{r}\right)^b + C\tilde{h}_r(t).$$

To conclude using Theorem 1.12 we have to verify (1.5). If ε and c are small enough we have that $\hat{h}(cr) \leq \varepsilon_1$ so we only have to control $f(x, r)$. To do this we can use the same argument as we used in Lemma 2.20. We use Lemma 2.18 to find a point x of same type of cone Z that define f , then we use the same competitor L as in the proof of 2.20 that is $Z \cup M$ where M is a small wall. We deduce a bound of f by β . Thus if the τ_i are small enough compared to ε_1 , (1.5) is verified hence the proof is achieved. □

Remark 4.6. Constant c in Theorem 4.5 is depending on c_1, U, α , and other constants. Thus, constant c is fairly small but one might give an explicit value by doing some long computations. On the other hand the constant ε cannot be explicitated since it rely on some infinitesimal coming from some compactness arguments.

Now we want to prove that the conditions on J and ω_2 can be removed in Theorem 4.5 if we suppose that c and ε are a bit smaller. To begin, we have to use this following lemma from [6].

Lemma 4.7 ([6]). *For each $\eta_1 \in (0, 1)$ there is some constants ε_3 and τ_1 such that the following holds. Let (u, K) is a Mumford-Shah minimizer in Ω , $x \in K$, $B(x, r) \subset \Omega$, and*

$$\omega_2(x, r) + h(r) + \beta(x, r) \leq \varepsilon_3.$$

In addition assume that K is not separating in $B(x, r)$. Then

$$J(x, \tau_1r) \geq \eta_1.$$

Proof. The proof is the same as [6, Lemma 5 page 333], generalized to the case of \mathbb{Y} and \mathbb{T} as well. This is not a serious issue so we omitted the proofs and refer to the aforementioned results. \square

About the normalized energy we also have this result that naturally comes from an argument with blow-up limits.

Lemma 4.8. *For each $\eta_2 > 0$ there is some constants ε_3 and a_0 with the following property. Let $\Omega \subset \mathbb{R}^3$ and let (u, K) be a Mumford-Shah minimizer in Ω with gauge function h . Let $x \in K$ and $r > 0$ be such that $B(x, r) \subset \Omega$. Suppose that $h(r) \leq \varepsilon_3$ and that we can find a cone Z of type \mathbb{P} , \mathbb{Y} or \mathbb{T} centered at x such that*

$$D_{x,r}(K, Z) \leq \varepsilon_3.$$

Then

$$\omega_2(x, a_0 r) \leq \eta_2.$$

Proof. The proof is the same as [6, Lemma 3 page 476], provided that we know who are the global minimizers in \mathbb{R}^3 with u locally constant. Thank to a result from G. David [4] we know that there are the cones of type \mathbb{P} , \mathbb{Y} and \mathbb{T} and this is exactly what we need to conclude. \square

Now we can state the main theorem.

Theorem 4.9. *For every $C > 0$ and $\alpha \in (0, 1)$, there is a $\varepsilon > 0$ such that the following holds. Let (u, K) be a Mumford-Shah minimizer in $\Omega \subset \mathbb{R}^3$ with gauge function $h(r) = Cr^\alpha$, let $x \in K$ and r be such that $B(x, r) \subset \Omega$ and $h(r) \leq \varepsilon$. Assume in addition that there is a cone Z of type \mathbb{P} , \mathbb{Y} or \mathbb{T} centered at x such that*

$$D_{x,r}(K, Z) \leq \varepsilon.$$

Then there is a diffeomorphism ϕ of class $C^{1,\alpha}$ from $B(x, cr)$ to its image, such that $K \cap B(x, cr) = \phi(Z)$.

Proof. We have to control the normalized jump and then apply Theorem 4.5. First assume that K is not separating. Then if ε is small enough compared to ε_3 we can use Lemma 4.8 and then Lemma 4.7 to obtain that

$$J(x, r')^{-1} \leq \bar{\varepsilon}$$

where $\bar{\varepsilon}$ is the constant of Theorem 4.5 and for a certain radius r' given by Lemma 4.7.

Now since ε is still small as we want, and possibly by taking a slightly smaller r' , we can assume that the associated cone in $\beta(x, r')$ is still centered near x and in addition

$$\beta(x, r') + J(x, r')^{-1} + \omega_2(x, r') + h(r') \leq \bar{\varepsilon}.$$

Then we apply Theorem 4.5 in $B(x, r')$ and the conclusion follows.

Finally if K is already separating in $B(x, r)$, then all the part of the proof corresponding to the construction of F and control of the jump is useless so that Theorem 4.5 is true with $J(x, r) = 0$ and this is enough to conclude. \square

This is an example of statement in terms of functional J .

Corollary 4.10. *Let $g \in L^\infty$ and $\Omega \subset \mathbb{R}^3$. Then there is some constants $\tilde{r} > 0$ and $\varepsilon > 0$, that depends only on $\|g\|_\infty$, such that for all pair $(u, K) \in \mathcal{A}$ that minimize the functional*

$$J(u, K) := \int_{\Omega \setminus K} |\nabla u|^2 dx + \int_{\Omega \setminus K} (u - g)^2 dx + \mathcal{H}^2(K),$$

for all $x \in K$ and $r < \tilde{r}$ such that there is a cone Z of type \mathbb{P} , \mathbb{Y} or \mathbb{T} centered at x with

$$D_{x,r}(K, Z) \leq \varepsilon$$

there is a diffeomorphism ϕ of class $C^{1,\alpha}$ from $B(x, cr)$ to $B(x, 10cr)$ such that $K \cap B(x, cr) = \phi(Z) \cap B(x, cr)$.

Proof. We know by [6, Proposition 7.8. page 46] that (u, K) is a Mumford-Shah minimizer with gauge function

$$h(r) = C_N \|g\|_\infty^2 r$$

where C_N depends only on dimension. The conclusion follows applying Theorem 4.9 in $B(x, r)$ if we choose

$$\tilde{r} = \frac{\tilde{\varepsilon}}{2C_N \|g\|_\infty^2}$$

where $\tilde{\varepsilon}$ is the constant of Theorem 4.9. □

Now we want a statement with only a condition about energy. We begin by this following lemma (D_H denotes the Hausdorff distance).

Lemma 4.11. *For every $\eta_4 > 0$ there exist a radius $R > 1$ and a $\eta_3 > 0$ such that for every Mumford-Shah minimizer (u, K) in $B(x, R) \subset \mathbb{R}^3$ such that $x \in K$ and*

$$\omega_2(x, R) + h(R) \leq \eta_3,$$

there is a minimal cone Z of type \mathbb{P} , \mathbb{Y} or \mathbb{T} that contains x and such that

$$D_H(K \cap B(x_0, 1), Z \cap B(x_0, 1)) \leq \eta_4.$$

Proof. The argument is by compactness. If it is not true, then we can find a $\eta_4 > 0$ such that for all $n > 0$, there is a Mumford-Shah minimizer (u_n, K_n) in $B(x, n)$ such that

$$\omega_2(x, n) + h(n) \leq \frac{1}{n^3} \tag{4.19}$$

and

$$\sup_Z D_H(K_n \cap B(x_0, 1), Z \cap B(x_0, 1)) \geq \eta_4 \tag{4.20}$$

where the supremum is taken over all minimal cones containing x . We let now tend n to infinity. Since (u_n, K_n) is a sequence of Mumford-Shah minimizers, with same gauge function $h_l(r) := \sup\{h(nr); n \geq l\}$, and such that

$$\int_{B(x,n)} |\nabla u|^2 \leq r \frac{1}{n} \leq C$$

by [6, Proposition 37.8] we can extract a subsequence such that (u_{n_k}, K_{n_k}) converges to (u, K) in \mathbb{R}^3 in the following sense: $D_H(K_{n_k} \cap A, K \cap A)$ tends to 0 for every compact set $A \subset \mathbb{R}^3$. Moreover for all connected component Ω of $\mathbb{R}^3 \setminus K$ and for all compact set A of Ω , there is a sequence a_k such that $\{u_{n_k} - a_k\}_{k \in \mathbb{N}}$ converges to u in $L^1(A)$. Then, using (4.19) and Proposition 37.18 of [6], we know that for every ball $B \subset \mathbb{R}^3$,

$$\int_{B \setminus K} |\nabla u|^2 \leq \liminf_{k \rightarrow +\infty} \int_{B \setminus K_n} |\nabla u_n|^2 \leq \lim_{k \rightarrow +\infty} r \frac{1}{n_k} = 0.$$

Thus $\nabla u = 0$ and u is locally constant. Finally, [6, Theorem 38.3] says that the limit (u, K) is a Mumford-Shah minimizer with gauge function $h_l(4r)$. Since it is true for all l , and that $\sup_l h_l = 0$, we can suppose that (u, K) is a Mumford-Shah minimizer with gauge function equals to zero, and u is locally constant. But in this case we know by [4] that K is a minimal cone of type \mathbb{P} , \mathbb{Y} or \mathbb{T} , and since for all n , K_n is containing x , it is still true for the limit K . In addition, there is a rank L such that for all $k \geq L$ we have $D_H(K \cap B(x_0, 1), K_{n_k} \cap B(x_0, 1)) \leq \frac{\eta_4}{2}$ which is in contradiction with (4.20) and achieves the proof. \square

Lemma 4.11 implies the following theorem.

Theorem 4.12. *For every $C > 0$ and $\alpha > 0$ there is some positive constants ε and $c < 1$ such that the following holds. Let (u, K) be a Mumford-Shah minimizer in $\Omega \subset \mathbb{R}^3$ with gauge function $h(r) = Cr^\alpha$, let $x \in K$ and r be such that $B(x, r) \subset \Omega$ and*

$$\omega_2(x, r) + h(r) \leq \varepsilon.$$

Then there is a diffeomorphism ϕ of class $C^{1,\alpha}$ from $B(x, cr)$ to its image, and there is a minimal cone Z such that $K \cap B(x, Cr) = \phi(Z) \cap B(x, cr)$.

Proof. Denote by $\bar{\varepsilon}$ the constant of Theorem 4.9. We apply Lemma 4.11 to (u, K) with $\eta_4 = \bar{\varepsilon}/100$. We know that there is a constant $c < 1$ and there is a cone Z that contains x such that

$$D_{x,cr}(Z, K) \leq \bar{\varepsilon}/100.$$

Dividing if necessary c by 100 we may assume that the center of the cone lies in $\frac{1}{10}B(x, cr)$. Thus

$$D_{x,cr}(Z, K) + \omega_2(x, cr) + h(r) \leq \bar{\varepsilon}$$

and then we can apply Theorem 4.9 in $B(x, cr)$, and the conclusion follows. \square

By the same way of Corollary 4.10, in terms of functional J we have the following statement.

Corollary 4.13. *Let $g \in L^\infty$ and $\Omega \subset \mathbb{R}^3$. Then there is some constants $\varepsilon > 0$, $\tilde{r} > 0$ and $c > 0$ depending only on $\|g\|_\infty$, such that for all pair $(u, K) \in \mathcal{A}$ that minimizes*

$$J(u, K) := \int_{\Omega \setminus K} |\nabla u|^2 dx + \int_{\Omega \setminus K} (u - g)^2 dx + \mathcal{H}^2(K),$$

for all $x \in K$ and $r < \tilde{r}$ such that

$$\omega_2(x, r) \leq \varepsilon$$

there is a diffeomorphism ϕ of class $C^{1,\alpha}$ from $B(x, cr)$ to its image such that $K \cap B(x, cr) = \phi(Z) \cap B(x, cr)$.

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