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Quantitative uniqueness for the power of the Laplacian with singular coefficients

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Abstract. In this paper we study the local behavior of a solution to the l-th power of the Laplacian with singular coefficients in lower order terms. We obtain a bound on the vanishing order of the nontrivial solution. Our proofs use Carleman estimates with carefully chosen weights. We will derive appropriate three-sphere inequalities and apply them to obtain doubling inequalities and the maximal vanishing order.

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1. Introduction

Assume that Ω is a connected open set containing 0 in $\mathbb{R}^n$ for $n \geq 2$. In this paper we are interested in the local behavior of u satisfying the following differential inequality:

$$|\Delta^l u| \leq K_0 \sum_{|\alpha| \leq l-1} |x|^{-2l+|\alpha|} |D^\alpha u| + K_0 \sum_{|\alpha|=l}^{[3l/2]} |x|^{-2l+|\alpha|+\epsilon} |D^\alpha u|, \tag{1.1}$$

where $0 < \epsilon < 1/2$ and $[h] = k \in \mathbb{Z}$ when $k \leq h < k+1$. For (1.1), a strong unique continuation was proved by the first author [9]. A similar result for the power of the Laplacian with lower derivatives up to l-th order can be found in [2]. On the other hand, a unique continuation property for the l-th power of the Laplacian with the same order of lower derivatives as in (1.1) was given in [12]. The results mentioned above concern only the qualitative behavior of the solution. In other words, they show that if u vanishes at 0 in infinite order or u vanishes in an open subset of Ω, then u must vanish identically in Ω.

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The aim of this paper is to study the strong unique continuation from a quantitative viewpoint. Namely, we are interested in the maximal vanishing order at 0 of any nontrivial solution to (1.1). It is worth mentioning that quantitative estimates of the strong unique continuation are useful in studying the nodal sets of eigenfunctions [3], or solutions of second-order elliptic equations [7, 11], or the inverse problem [1].

Perhaps, for the quantitative uniqueness problem, the most popular technique, introduced by Garofalo and Lin [4, 5], is to use the frequency function related to the solution. This method works quite efficiently for second-order strongly elliptic operators. However, this method cannot be applied to (1.1). Another method to derive quantitative estimates of the strong unique continuation is based on Carleman estimates, which was first initiated by Donnelly and Fefferman [3] where they studied the maximal vanishing order of the eigenfunction with respect to the corresponding eigenvalue on a compact smooth Riemannian manifold. Their method does not work for (1.1) either.

Recently, the first and third authors and Nakamura [10] introduced a method based on appropriate Carleman estimates to prove a quantitative uniqueness for second-order elliptic operators with sharp singular coefficients in lower order terms. A key strategy of our method is to derive three-sphere inequalities and then apply them to obtain doubling inequalities and the maximal vanishing order. Both steps require delicate choices of cut-off functions. Nevertheless, this method is quite versatile and can be adapted to treat many equations or even systems. The present work is an application of the ideas of [10] to the l-th power of the Laplacian with singular coefficients. The power $l = 2$ is the most interesting and useful case. It corresponds to the biharmonic operator with third-order derivatives. Our work provides a quantitative estimate of the strong unique continuation for this equation. To our best knowledge, this quantitative estimate has not been derived before.

Before stating the main results of the paper, we want to remark that if the right-hand side of (1.1) contains only l-th (or lower) order derivatives and has mild singular coefficients, then the Carleman estimate (3.1) alone is sufficient to derive doubling inequalities. However, if the highest order of the right-hand side of (1.1) is strictly larger than l, even with bounded coefficients, (3.1) is not enough to deduce doubling inequalities. The reason is that the constant in (3.1) behaves like $m^{2l-2|\alpha|}$ for $|\alpha| \le 2l$ and it decays to zero when $|\alpha| > l$. The trick to overcome this difficulty is to use three-sphere inequalities, which is another form of a quantitative uniqueness estimate.

We now state the main results of the paper. Assume that $B_{R_0'} \subset \Omega$ for some $R_0' > 0$.

Theorem 1.1. *There exists a positive number* $\tilde{R}_0 < e^{-1/2}$ *such that if* $0 < r_1 < r_2 < r_3 \le R_0' < 1$ *and* $r_1/r_3 < r_2/r_3 < \tilde{R}_0$, *then*

$$\int_{|x|<r_2} |u|^2 dx \le C \left(\int_{|x|<r_1} |u|^2 dx \right)^{\tau} \left(\int_{|x|<r_3} |u|^2 dx \right)^{1-\tau} \tag{1.2}$$

for $u \in H^{2l}(B_{R_0'})$ *satisfying* (1.1) *in* $B_{R_0'}$, *where* C *and* $0 < \tau < 1$ *depend on* r_1/r_3, r_2/r_3, n, l, *and* K_0.

Remark 1.2. From the proof, the constants C and τ can be explicitly written as $C = \max\{C_0(r_2/r_1)^n, \exp(B\beta_0)\}$ and $\tau = B/(A+B)$, where $C_0 > 1$ and β_0 are constants depending on n, l, K_0 and

$$A = A(r_1/r_3, r_2/r_3) = (\log(r_1/r_3) - 1)^2 - (\log(r_2/r_3))^2,$$
$$B = B(r_2/r_3) = -1 - 2\log(r_2/r_3).$$

The explicit forms of these constants are important in the proof of Theorem 1.3.

Theorem 1.3. *Let* $u \in H^{2l}_{\mathrm{loc}}(\Omega)$ *be a nonzero solution to* (1.1). *Then we can find a constant* R_2 (*depending on* n, l, ϵ, K_0) *and a constant* m_1 (*depending on* $n, l, \epsilon, K_0, \|u\|_{L^2(|x|<R_2^2)}/\|u\|_{L^2(|x|<R_2^4)}$) *such that if* $0 < r \le R_3$, *then*

$$C_3 r^{m_1} \le \int_{|x|<r} |u|^2 dx, \tag{1.3}$$

where R_3 *is a positive constants depending on* n, l, ϵ, K_0, m_1 *and* C_3 *is a positive constants depending on* $n, l, \epsilon, K_0, m_1, u$.

Theorem 1.4. *Let* $u \in H^{2l}_{\mathrm{loc}}(\Omega)$ *be a nonzero solution to* (1.1). *Then there exist positive constants* R_4 (*depending on* n, l, ϵ, K_0, m_1) *and* C_4 (*depending on* n, l, ϵ, K_0, m_1) *such that if* $0 < r \le R_4$, *then*

$$\int_{|x|\le 2r} |u|^2 dx \le C_4 \int_{|x|\le r} |u|^2 dx, \tag{1.4}$$

where m_1 *is the constant obtained in Theorem* 1.3.

The rest of the paper is devoted to the proofs of Theorems 1.1-1.4.

2. Three-sphere inequalities

In this section we will prove Theorem 1.1. To begin, we recall a Carleman estimate with weight $\varphi_\beta = \varphi_\beta(x) = \exp(\frac{\beta}{2}(\log|x|)^2)$ given in [9].

Lemma 2.1 ([9, Corollary 3.3]). *There exist a sufficiently large number $\beta_0 > 0$ and a sufficiently small number $r_0 > 0$, depending on n and l, such that for all $u \in U_{r_0}$ with $0 < r_0 < e^{-1}$, $\beta \geq \beta_0$, we have that*

$$\begin{aligned} &\sum_{|\alpha|\leq 2l} \beta^{3l-2|\alpha|} \int \varphi_\beta^2 |x|^{2|\alpha|-n} (\log|x|)^{2l-2|\alpha|} |D^\alpha u|^2 dx \\ &\leq \tilde{C}_0 \int \varphi_\beta^2 |x|^{4l-n} |\Delta^l u|^2 dx, \end{aligned} \tag{2.1}$$

where $U_{r_0} = \{u \in C_0^\infty(\mathbb{R}^n \setminus \{0\}) : \operatorname{supp}(u) \subset B_{r_0}\}$ and $\tilde{C}_0$ is a positive constant depending on n and l. Here $e = \exp(1)$.

Remark 2.2. The estimate (2.1) in Lemma 2.1 remains valid if we assume $u \in H^{2l}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ with compact support. This can be easily obtained by cutting off u for small $|x|$ and regularizing.

Proof. We first consider the case where $0 < r_1 < r_2 < R < 1/e$ and $B_R \subset \Omega$. The constant R will be chosen later. To use the estimate (2.1), we need to cut off u. So let $\xi(x) \in C_0^\infty(\mathbb{R}^n)$ satisfy $0 \leq \xi(x) \leq 1$ and

$$\xi(x) = \begin{cases} 0, & |x| \leq r_1/e, \\ 1, & r_1/2 < |x| < er_2, \\ 0, & |x| \geq 3r_2. \end{cases}$$

It is easy to check that for all multiindex α

$$\begin{cases} |D^\alpha \xi| = O(r_1^{-|\alpha|}) \text{ for all } r_1/e \leq |x| \leq r_1/2 \\ |D^\alpha \xi| = O(r_2^{-|\alpha|}) \text{ for all } er_2 \leq |x| \leq 3r_2. \end{cases} \tag{2.2}$$

On the other hand, repeating [8, proof of Corollary 17.1.4], we can show that

$$\int_{a_1 r<|x|<a_2 r} ||x|^{|\alpha|} D^\alpha u|^2 dx \leq C' \int_{a_3 r<|x|<a_4 r} |u|^2 dx, \quad |\alpha| \leq 2l, \tag{2.3}$$

for all $0 < a_3 < a_1 < a_2 < a_4$ such that $B_{a_4 r} \subset \Omega$, where the constant C' is independent of r and u.

Notice that the commutator $[\Delta^l, \xi]$ is a $2l-1$ order differential operator. Applying (2.1) to ξu and using (1.1), (2.2), (2.3) implies

$$
\begin{aligned}
&\sum_{|\alpha|\le 2l} \beta^{3l-2|\alpha|} \int_{r_1/2<|x|<er_2} \varphi_\beta^2 |x|^{2|\alpha|-n} (\log|x|)^{2l-2|\alpha|} |D^\alpha u|^2 dx \\
&\le \sum_{|\alpha|\le 2l} \beta^{3l-2|\alpha|} \int \varphi_\beta^2 |x|^{2|\alpha|-n} (\log|x|)^{2l-2|\alpha|} |D^\alpha (\xi u)|^2 dx \\
&\le \tilde{C}_0 \int \varphi_\beta^2 |x|^{4l-n} |\Delta^l (\xi u)|^2 dx \\
&\le 2\tilde{C}_0 \int \varphi_\beta^2 |x|^{4l-n} \xi^2 \Bigg(K_0 \sum_{|\alpha|\le l-1} |x|^{-2l+|\alpha|} |D^\alpha u| + K_0 \sum_{|\alpha|=l}^{[3l/2]} |x|^{-2l+|\alpha|+\epsilon} |D^\alpha u| \Bigg)^2 dx \\
&\quad + 2\tilde{C}_0 \int \varphi_\beta^2 |x|^{4l-n} \big| [\Delta^l, \xi] u \big|^2 dx \\
&\le \tilde{C}_1 \Bigg\{ \int_{r_1/2<|x|<er_2} \varphi_\beta^2 \Bigg(\sum_{|\alpha|\le l-1} |x|^{2|\alpha|-n} |D^\alpha u|^2 + \sum_{|\alpha|=l}^{[3l/2]} |x|^{2|\alpha|-n+2\epsilon} |D^\alpha u|^2 \Bigg) dx \\
&\quad + \int_{r_1/e<|x|<r_1/2} \varphi_\beta^2 \sum_{|\alpha|\le 2l-1} |x|^{2|\alpha|-n} |D^\alpha u|^2 dx \\
&\quad + \int_{er_2<|x|<3r_2} \varphi_\beta^2 \sum_{|\alpha|\le 2l-1} |x|^{2|\alpha|-n} |D^\alpha u|^2 dx \Bigg\} \\
&\le \tilde{C}_2 \Bigg\{ \int_{r_1/2<|x|<er_2} \varphi_\beta^2 \Bigg(\sum_{|\alpha|\le l-1} |x|^{2|\alpha|-n} |D^\alpha u|^2 + \sum_{|\alpha|=l}^{[3l/2]} |x|^{2|\alpha|-n+2\epsilon} |D^\alpha u|^2 \Bigg) dx \\
&\quad + r_1^{-n} \varphi_\beta^2 (r_1/e) \int_{r_1/e<|x|<r_1/2} \sum_{|\alpha|\le 2l-1} \big| |x|^{|\alpha|} D^\alpha u \big|^2 dx \\
&\quad + r_2^{-n} \varphi_\beta^2 (er_2) \int_{er_2<|x|<3r_2} \sum_{|\alpha|\le 2l-1} \big| |x|^{|\alpha|} D^\alpha u \big|^2 dx \Bigg\} \\
&\le \tilde{C}_3 \Bigg\{ \int_{r_1/2<|x|<er_2} \varphi_\beta^2 \Bigg(\sum_{|\alpha|\le l-1} |x|^{2|\alpha|-n} |D^\alpha u|^2 + \sum_{|\alpha|=l}^{[3l/2]} |x|^{2|\alpha|-n+2\epsilon} |D^\alpha u|^2 \Bigg) dx \\
&\quad + r_1^{-n} \varphi_\beta^2 (r_1/e) \int_{r_1/4<|x|<r_1} |u|^2 dx + r_2^{-n} \varphi_\beta^2 (er_2) \int_{2r_2<|x|<4r_2} |u|^2 dx \Bigg\},
\end{aligned}
\tag{2.4}
$$

where $\tilde{C}_1$, $\tilde{C}_2$, and $\tilde{C}_3$ are independent of r_1, r_2, and u.

We now choose $r_0 < e^{-\epsilon^{-1}([3l/2]-l)-1}$ small enough that

$$\begin{cases}(\log(er_0))^{-2} \le \dfrac{1}{2\tilde{C}_3}\\ (er_0)^{2\epsilon}(\log(er_0))^{2([3l/2]-l)} \le \dfrac{1}{2\tilde{C}_3}.\end{cases}$$

Letting $R \le r_0$ and $\beta \ge \beta_0 \ge \max\{2\tilde{C}_3, 1\}$, we can absorb the integral over $r_1/2 < |x| < er_2$ on the right-hand side of (2.4) into its left-hand side to obtain

$$\begin{aligned}&\int_{r_1/2<|x|<er_2} \varphi_\beta^2 \left(\sum_{|\alpha|\le l-1} |x|^{2|\alpha|-n}|D^\alpha u|^2 + \sum_{|\alpha|=l}^{[3l/2]} |x|^{2|\alpha|-n+2\epsilon}|D^\alpha u|^2\right) dx\\ &\le \tilde{C}_4\left\{r_1^{-n}\varphi_\beta^2(r_1/e)\int_{r_1/4<|x|<r_1} |u|^2 dx + r_2^{-n}\varphi_\beta^2(er_2)\int_{2r_2<|x|<4r_2} |u|^2 dx\right\},\end{aligned} \tag{2.5}$$

where $\tilde{C}_4 = 1/\tilde{C}_3$. Using (2.5) we have that

$$\begin{aligned}&r_2^{-n}\varphi_\beta^2(r_2)\int_{r_1/2<|x|<r_2} |u|^2 dx\\ &\le \int_{r_1/2<|x|<er_2} \varphi_\beta^2 |x|^{-n}|u|^2 dx\\ &\le \tilde{C}_4\left\{r_1^{-n}\varphi_\beta^2(r_1/e)\int_{r_1/4<|x|<r_1} |u|^2 dx + r_2^{-n}\varphi_\beta^2(er_2)\int_{2r_2<|x|<4r_2} |u|^2 dx\right\}.\end{aligned} \tag{2.6}$$

Dividing $r_2^{-n}\varphi_\beta^2(r_2)$ both sides of (2.6) we get

$$\begin{aligned}&\int_{r_1/2<|x|<r_2} |u|^2 dx\\ &\le \tilde{C}_4\left\{(r_2/r_1)^n[\varphi_\beta^2(r_1/e)/\varphi_\beta^2(r_2)]\int_{r_1/4<|x|<r_1} |u|^2 dx\right.\\ &\qquad\left. + [\varphi_\beta^2(er_2)/\varphi_\beta^2(r_2)]\int_{2r_2<|x|<4r_2} |u|^2 dx\right\}\\ &\le \tilde{C}_5\left\{(r_2/r_1)^n[\varphi_\beta^2(r_1/e)/\varphi_\beta^2(r_2)]\int_{|x|<r_1} |u|^2 dx\right.\\ &\qquad\left. + (r_2/r_1)^n[\varphi_\beta^2(er_2)/\varphi_\beta^2(r_2)]\int_{|x|<4r_2} |u|^2 dx\right\},\end{aligned} \tag{2.7}$$

where $\tilde{C}_5 = \max\{\tilde{C}_4, 1\}$. With such choice of $\tilde{C}_5$, we can see that

$$\tilde{C}_5(r_2/r_1)^n[\varphi_\beta^2(r_1/e)/\varphi_\beta^2(r_2)] > 1$$

for all $0 < r_1 < r_2$. Adding $\int_{|x|<r_1/2} |u|^2 dx$ to both sides of (2.7) and choosing $r_2 \le R = \min\{r_0, 1/4\}$, we get

$$\begin{aligned}\int_{|x|<r_2} |u|^2 dx \le 2\tilde{C}_5 (r_2/r_1)^n [\varphi_\beta^2(r_1/e)/\varphi_\beta^2(r_2)] \int_{|x|<r_1} |u|^2 dx \\ + 2\tilde{C}_5 (r_2/r_1)^n [\varphi_\beta^2(er_2)/\varphi_\beta^2(r_2)] \int_{|x|<1} |u|^2 dx.\end{aligned} \tag{2.8}$$

Setting

$$\begin{aligned}A &= \beta^{-1} \log[\varphi_\beta^2(r_1/e)/\varphi_\beta^2(r_2)] = (\log r_1 - 1)^2 - (\log r_2)^2 > 0,\\ B &= -\beta^{-1} \log[\varphi_\beta^2(er_2)/\varphi_\beta^2(r_2)] = -1 - 2\log r_2 > 0,\end{aligned}$$

inequality (2.8) becomes

$$\begin{aligned}&\int_{|x|<r_2} |u|^2 dx \\ &\le 2\tilde{C}_5 (r_2/r_1)^n \left\{ \exp(A\beta) \int_{|x|<r_1} |u|^2 dx + \exp(-B\beta) \int_{|x|<1} |u|^2 dx \right\}.\end{aligned} \tag{2.9}$$

To further simplify the terms on the right-hand side of (2.9), we consider two cases. If $\int_{|x|<r_1} |u|^2 dx \ne 0$ and

$$\exp(A\beta_0) \int_{|x|<r_1} |u|^2 dx < \exp(-B\beta_0) \int_{|x|<1} |u|^2 dx,$$

then we can pick $\beta > \beta_0$ such that

$$\exp(A\beta) \int_{|x|<r_1} |u|^2 dx = \exp(-B\beta) \int_{|x|<1} |u|^2 dx.$$

Using such a β, we obtain from (2.9) that

$$\begin{aligned}\int_{|x|<r_2} |u|^2 dx &\le 4\tilde{C}_5 (r_2/r_1)^n \exp(A\beta) \int_{|x|<r_1} |u|^2 dx \\ &= 4\tilde{C}_5 (r_2/r_1)^n \left(\int_{|x|<r_1} |u|^2 dx \right)^{\frac{B}{A+B}} \left(\int_{|x|<1} |u|^2 dx \right)^{\frac{A}{A+B}}.\end{aligned} \tag{2.10}$$

If $\int_{|x|<r_1} |u|^2 dx = 0$, then it follows from (2.9) that

$$\int_{|x|<r_2} |u|^2 dx = 0$$

since we can take β arbitrarily large. The three-sphere inequality obviously holds. On the other hand, if

$$\exp(-B\beta_0)\int_{|x|<1}|u|^2dx \le \exp(A\beta_0)\int_{|x|<r_1}|u|^2dx,$$

then we have

$$\begin{aligned}\int_{|x|<r_2}|u|^2dx &\le \left(\int_{|x|<1}|u|^2dx\right)^{\frac{B}{A+B}}\left(\int_{|x|<1}|u|^2dx\right)^{\frac{A}{A+B}}\\ &\le \exp(B\beta_0)\left(\int_{|x|<r_1}|u|^2dx\right)^{\frac{B}{A+B}}\left(\int_{|x|<1}|u|^2dx\right)^{\frac{A}{A+B}}.\end{aligned} \tag{2.11}$$

Putting together (2.10), (2.11), and setting $\tilde{C}_6 = \max\{4\tilde{C}_5(r_2/r_1)^n, \exp(B\beta_0)\}$, we arrive at

$$\int_{|x|<r_2}|u|^2dx \le \tilde{C}_6\left(\int_{|x|<r_1}|u|^2dx\right)^{\frac{B}{A+B}}\left(\int_{|x|<1}|u|^2dx\right)^{\frac{A}{A+B}}. \tag{2.12}$$

Now for the general case, we take $\tilde{R}_0 = R$ and consider $0 < r_1 < r_2 < r_3$ with $r_1/r_3 < r_2/r_3 \le \tilde{R}_0$. By scaling, *i.e.*, defining $\widehat{u}(y) := u(r_3y)$, we derive from (2.12) that

$$\int_{|y|<r_2/r_3}|\widehat{u}|^2dy \le C\left(\int_{|y|<r_1/r_3}|\widehat{u}|^2dy\right)^{\tau}\left(\int_{|y|<1}|\widehat{u}|^2dy\right)^{1-\tau}, \tag{2.13}$$

where $\tau = B/(A+B)$ with

$$\begin{aligned}A &= A(r_1/r_3, r_2/r_3) = (\log(r_1/r_3)-1)^2 - (\log(r_2/r_3))^2,\\ B &= B(r_2/r_3) = -1 - 2\log(r_2/r_3),\end{aligned}$$

and $C = \max\{4\tilde{C}_5(r_2/r_1)^n, \exp(B\beta_0)\}$. Note that $\tilde{C}_5$ can be chosen independent of the scaling factor r_3 provided $r_3 < 1$. Replacing the variable $y = x/r_3$ in (2.13) gives

$$\int_{|x|<r_2}|u|^2dx \le C\left(\int_{|x|<r_1}|u|^2dx\right)^{\tau}\left(\int_{|x|<r_3}|u|^2dx\right)^{1-\tau}.$$

This concludes the proof. □

3. Doubling inequalities and maximal vanishing order

In this section we prove Theorem 1.3 and Theorem 1.4. We begin with another Carleman estimate derived in [9, Lemma 2.1]: for any $u \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$ and for any $m \in \{k+1/2, k \in \mathbb{N}\}$, we have the following estimate

$$\sum_{|\alpha|\le 2l}\int m^{2l-2|\alpha|}|x|^{-2m+2|\alpha|-n}|D^\alpha u|^2dx \le C\int |x|^{-2m+4l-n}|\triangle^l u|^2dx, \tag{3.1}$$

where C depends only on the dimension n and the power l.

Remark 3.1. Using the cut-off function and regularization, the estimate (3.1) remains valid for any fixed m if $u \in H^{2l}_{\mathrm{loc}}(\mathbb{R}^n\backslash\{0\})$ with compact support.

Proof. In view of Remark 3.1, we can apply (3.1) to the function χu with $\chi(x) \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$. Thus, we define $\chi(x) \in C_0^\infty(\mathbb{R}^n\backslash\{0\})$ as

$$\chi(x) = \begin{cases} 0 & \text{if} \quad |x| \le \delta/3, \\ 1 & \text{if} \quad \delta/2 \le |x| \le (R_0+1)R_0R/4 = r_4R, \\ 0 & \text{if} \quad 2r_4R \le |x|, \end{cases}$$

where $\delta \le R_0^2R/4$, $R_0 > 0$ is a small number which will be chosen later and $R < 1$ is sufficiently small. Here the number R is not yet fixed and is given by $R = (\gamma m)^{-l/2\epsilon}$, where $\gamma > 0$ is a large constant which will be determined later. Using the estimate (3.1) and equation (1.1), we can derive that

$$\begin{aligned}
&\sum_{|\alpha|\le 2l}\int_{\delta/2\le|x|\le r_4R} m^{2l-2|\alpha|}|x|^{-2m+2|\alpha|-n}|D^\alpha u|^2dx \\
&\le \sum_{|\alpha|\le 2l}\int m^{2l-2|\alpha|}|x|^{-2m+2|\alpha|-n}|D^\alpha(\chi u)|^2dx \\
&\le C\int |x|^{-2m+4l-n}|\Delta^l(\chi u)|^2dx \\
&= C\int_{\delta/2\le|x|\le r_4R}|x|^{-2m+4l-n}|\Delta^l u|^2dx + C\int_{|x|>r_4R}|x|^{-2m+4l-n}|\Delta^l(\chi u)|^2dx \\
&\quad + C\int_{\delta/3\le|x|\le\delta/2}|x|^{-2m+4l-n}|\Delta^l(\chi u)|^2dx \\
&\le C'K_0^2\int_{\delta/2\le|x|\le r_4R}\left(\sum_{|\alpha|\le l-1}|x|^{2|\alpha|-n-2m}|D^\alpha u|^2 + \sum_{|\alpha|=l}^{[3l/2]}|x|^{2|\alpha|-n-2m+2\epsilon}|D^\alpha u|^2\right)dx \qquad (3.2) \\
&\quad + C\int_{|x|>r_4R}|x|^{-2m+4l-n}|\Delta^l(\chi u)|^2dx + C\int_{\delta/3\le|x|\le\delta/2}|x|^{-2m+4l-n}|\Delta^l(\chi u)|^2dx \\
&\le C'K_0^2(r_4R)^{2\epsilon}\int_{\delta/2\le|x|\le r_4R}\sum_{|\alpha|=l}^{[3l/2]}|x|^{2|\alpha|-n-2m}|D^\alpha u|^2dx \\
&\quad + C'K_0^2\int_{\delta/2\le|x|\le r_4R}\sum_{|\alpha|\le l-1}|x|^{2|\alpha|-n-2m}|D^\alpha u|^2dx \\
&\quad + C\int_{|x|>r_4R}|x|^{-2m+4l-n}|\Delta^l(\chi u)|^2dx + C\int_{\delta/3\le|x|\le\delta/2}|x|^{-2m+4l-n}|\Delta^l(\chi u)|^2dx,
\end{aligned}$$

where the constant C' depends on n and l.

By carefully checking the terms on both sides of (3.2), we now choose $\gamma \geq (2C'K_0^2)^{1/l}$ and thus

$$R^{2\epsilon} = (\gamma m)^{-l} \leq \frac{m^{-l}}{2C'K_0^2}.$$

Hence, choosing $R_0 < 1$ (which suffices to guarantee that $r_4^{2/\epsilon} = R_0^{2\epsilon}(R_0 + 1)^{2\epsilon}/4^{2\epsilon} < 1$) and m such that $m^2 > 2C'K_0^2$, we can remove the first two terms on the right-hand side of the last inequality in (3.2) and obtain

$$\begin{aligned} &\sum_{|\alpha|\leq 2l} \int_{\delta/2\leq|x|\leq r_4R} m^{2l-2|\alpha|}|x|^{-2m+2|\alpha|-n}|D^\alpha u|^2 dx \\ &\leq 2C\int_{\delta/3<|x|<\delta/2} |x|^{-2m+4l-n}|\Delta^l(\chi u)|^2 dx \\ &\quad + 2C\int_{r_4R<|x|<2r_4R} |x|^{-2m+4l-n}|\Delta^l(\chi u)|^2 dx. \end{aligned} \tag{3.3}$$

In view of the definition of χ, it is easy to see that for all multiindex α

$$\begin{cases} |D^\alpha \chi| = O(\delta^{-|\alpha|}) \text{ for all } \delta/3 < |x| < \delta/2, \\ |D^\alpha \chi| = O((r_4R)^{-|\alpha|}) \text{ for all } r_4R < |x| < 2r_4R. \end{cases} \tag{3.4}$$

Note that $R_0^2 \leq r_4$ provided $R_0 \leq 1/3$. Therefore, using (3.4) and (2.3) in (3.3), we derive

$$\begin{aligned} &m^2(2\delta)^{-2m-n}\int_{\delta/2<|x|\leq 2\delta}|u|^2dx + m^2(R_0^2R)^{-2m-n}\int_{2\delta<|x|\leq R_0^2R}|u|^2dx \\ &\leq \sum_{|\alpha|\leq 2l}\int_{\delta/2\leq|x|\leq r_4R} m^{2l-2|\alpha|}|x|^{-2m+2|\alpha|-n}|D^\alpha u|^2dx \\ &\leq C''\sum_{|\alpha|\leq 2l}\delta^{-4l+2|\alpha|}\int_{\delta/3<|x|<\delta/2}|x|^{-2m+4l-n}|D^\alpha u|^2dx \\ &\quad + C''\sum_{|\alpha|\leq 2l}(r_4R)^{-4l+2|\alpha|}\int_{r_4R<|x|<2r_4R}|x|^{-2m+4l-n}|D^\alpha u|^2dx \\ &\leq \tilde{C}'\delta^{-2m-n}\int_{|x|\leq\delta}|u|^2dx + C''(r_4R)^{-2m-n}\int_{|x|\leq R_0R}|u|^2dx, \end{aligned} \tag{3.5}$$

where $\tilde{C}' = C''3^{2m+n}$ and C'' is independent of R_0, R, and m.

We then add $m^2(2\delta)^{-2m-n}\int_{|x|\le\delta/2}|u|^2dx$ to both sides of (3.5) and obtain

$$\begin{aligned}
&\frac{1}{2}m^2(2\delta)^{-2m-n}\int_{|x|\le 2\delta}|u|^2dx+m^2(R_0^2R)^{-2m-n}\int_{|x|\le R_0^2R}|u|^2dx\\
&=\frac{1}{2}m^2(2\delta)^{-2m-n}\int_{|x|\le 2\delta}|u|^2dx+m^2(R_0^2R)^{-2m-n}\int_{|x|\le 2\delta}|u|^2dx\\
&\quad+m^2(R_0^2R)^{-2m-n}\int_{2\delta<|x|\le R_0^2R}|u|^2dx\\
&\le\frac{1}{2}m^2(2\delta)^{-2m-n}\int_{|x|\le 2\delta}|u|^2dx+\frac{1}{2}m^2(2\delta)^{-2m-n}\int_{|x|\le 2\delta}|u|^2dx\\
&\quad+m^2(R_0^2R)^{-2m-n}\int_{2\delta<|x|\le R_0^2R}|u|^2dx\\
&\le\tilde{C}''\delta^{-2m-n}\int_{|x|\le\delta}|u|^2dx+C''(r_4R)^{-2m-n}\int_{|x|\le R_0R}|u|^2dx\\
&=\tilde{C}''\delta^{-2m-n}\int_{|x|\le\delta}|u|^2dx\\
&\quad+m^2(R_0^2R)^{-2m-n}C''m^{-2}\left(\frac{R_0^2}{r_4}\right)^{2m+n}\int_{|x|\le R_0R}|u|^2dx
\end{aligned}\tag{3.6}$$

with $\tilde{C}''=\tilde{C}'+2^{2m+n}m^2$. We first observe that

$$\begin{aligned}
C''m^{-2}\left(\frac{R_0^2}{r_4}\right)^{2m+n}&=C''m^{-2}\left(\frac{4R_0}{R_0+1}\right)^{2m+n}\\
&\le C''m^{-2}(4R_0)^{2m+n}\le\exp(-2m)
\end{aligned}$$

for all $R_0\le 1/16$ and $m^2\ge C''$. Thus, we obtain

$$\begin{aligned}
&\frac{1}{2}m^2(2\delta)^{-2m-n}\int_{|x|\le 2\delta}|u|^2dx+m^2(R_0^2R)^{-2m-n}\int_{|x|\le R_0^2R}|u|^2dx\\
&\le\tilde{C}''\delta^{-2m-n}\int_{|x|\le\delta}|u|^2dx\\
&\quad+m^2(R_0^2R)^{-2m-n}\exp(-2m)\int_{|x|\le R_0R}|u|^2dx.
\end{aligned}\tag{3.7}$$

It should be noted that (3.7) is valid for all $m=j+\frac{1}{2}$ with $j\in\mathbb{N}$ and $j\ge j_0$, where j_0 depends on n, l, ϵ, and K_0. Setting $R_j=(\gamma(j+\frac{1}{2}))^{-l/2\epsilon}$ and using the relation $m = (\gamma)^{-1}(R)^{-2\epsilon/l}$, we get from (3.7) that

$$\begin{aligned}
&\frac{1}{2}m^2(2\delta)^{-2m-n}\int_{|x|\le 2\delta}|u|^2dx + m^2(R_0^2R_j)^{-2m-n}\int_{|x|\le R_0^2R_j}|u|^2dx \\
&\le \tilde{C}''\delta^{-2m-n}\int_{|x|\le\delta}|u|^2dx \\
&\quad + m^2(R_0^2R_j)^{-2m-n}\exp(-2cR_j^{-2\epsilon/l})\int_{|x|\le R_0R_j}|u|^2dx
\end{aligned}\tag{3.8}$$

for all $j \ge j_0$ and $c = \gamma^{-1}$. We now let j_0 be large enough that

$$R_{j+1} < R_j < 2R_{j+1} \quad \text{for all} \quad j \ge j_0.$$

Thus, if $R_{j+1} < R \le R_j$ for $j \ge j_0$, we can conclude that

$$\begin{cases}
\displaystyle\int_{|x|\le R_0^2R}|u|^2dx \le \int_{|x|\le R_0^2R_j}|u|^2dx, \\
\displaystyle\exp(-2cR_j^{-2\epsilon/l})\int_{|x|\le R_0R_j}|u|^2dx \le \exp(-cR^{-2\epsilon/l})\int_{|x|\le R}|u|^2dx,
\end{cases}\tag{3.9}$$

where we have used the inequality $R_0R_j \le R_j/16 < R_{j+1}$ to derive the second inequality above. Namely, we have from (3.8) and (3.9) that

$$\begin{aligned}
&\frac{1}{2}m^2(2\delta)^{-2m-n}\int_{|x|\le 2\delta}|u|^2dx + m^2(R_0^2R_j)^{-2m-n}\int_{|x|\le R_0^2R}|u|^2dx \\
&\le \tilde{C}''\delta^{-2m-n}\int_{|x|\le\delta}|u|^2dx \\
&\quad + m^2(R_0^2R_j)^{-2m-n}\exp(-cR^{-2\epsilon/l})\int_{|x|\le R}|u|^2dx.
\end{aligned}\tag{3.10}$$

If there exists $s \in \mathbb{N}$ such that

$$R_{j+1} < R_0^{2s} \le R_j \quad \text{for some} \quad j \ge j_0,\tag{3.11}$$

then replacing R by R_0^{2s} in (3.10) leads to

$$\begin{aligned}
&\frac{1}{2}m^2(2\delta)^{-2m-n}\int_{|x|\le 2\delta}|u|^2dx + m^2(R_0^2R_j)^{-2m-n}\int_{|x|\le R_0^{2s+2}}|u|^2dx \\
&\le \tilde{C}''\delta^{-2m-n}\int_{|x|\le\delta}|u|^2dx \\
&\quad + m^2(R_0^2R_j)^{-2m-n}\exp(-cR_0^{-4s\epsilon/l})\int_{|x|\le R_0^{2s}}|u|^2dx.
\end{aligned}\tag{3.12}$$

Here s and R_0 are yet to be determined. The trick now is to find suitable s and R_0 satisfying (3.11) such that the inequality

$$\exp(-cR_0^{-4s\epsilon/l})\int_{|x|\le R_0^{2s}}|u|^2dx\le\frac{1}{2}\int_{|x|\le R_0^{2s+2}}|u|^2dx \tag{3.13}$$

holds with such choices of s and R_0.

It is time to use the three-sphere inequality (1.2). To this end, we choose $r_1=R_0^{2k+2}$, $r_2=R_0^{2k}$ and $r_3=R_0^{2k-2}$ for $k\ge 1$. Note that $r_1/r_3<r_2/r_3\le R_0^2\le\tilde{R}_0$. Thus (1.2) implies

$$\begin{aligned}&\int_{|x|<R_0^{2k}}|u|^2dx\Big/\int_{|x|<R_0^{2k+2}}|u|^2dx\\&\le C^{1/\tau}\left(\int_{|x|<R_0^{2k-2}}|u|^2dx\Big/\int_{|x|<R_0^{2k}}|u|^2dx\right)^a,\end{aligned} \tag{3.14}$$

where

$$C=\max\{C_0R_0^{-2n},\exp(\beta_0(-1-4\log R_0))\}$$

and

$$\begin{aligned}a=\frac{1-\tau}{\tau}=\frac{A}{B}&=\frac{(\log(r_1/r_3)-1)^2-(\log(r_2/r_3))^2}{-1-2\log(r_2/r_3)}\\&=\frac{(4\log R_0-1)^2-(2\log R_0)^2}{-1-4\log R_0}.\end{aligned}$$

It is not hard to see that

$$\begin{cases}1<C\le C_0R_0^{-\beta_1},\\2<a\le -4\log R_0,\end{cases} \tag{3.15}$$

where $\beta_1=\max\{2n,4\beta_0\}$ and if R_0 is sufficiently small, *e.g.*, $R_0\le e^{-4}$. Combining (3.15) and using (3.14) recursively, we have

$$\begin{aligned}&\int_{|x|\le R_0^{2s}}|u|^2dx\Big/\int_{|x|\le R_0^{2s+2}}|u|^2dx\\&\le C^{1/\tau}\left(\int_{|x|<R_0^{2s-2}}|u|^2dx\Big/\int_{|x|<R_0^{2s}}|u|^2dx\right)^a\\&\le C^{\frac{a^{s-1}-1}{\tau(a-1)}}\left(\int_{|x|<R_0^{2}}|u|^2dx\Big/\int_{|x|<R_0^{4}}|u|^2dx\right)^{a^{s-1}}\end{aligned} \tag{3.16}$$

for all $s \geq 1$. Now from the definition of a, we have $\tau = 1/(a+1)$ and thus

$$\frac{a^{s-1}-1}{\tau(a-1)} = \frac{a+1}{a-1}(a^{s-1}-1) \leq 3a^{s-1}.$$

Then it follows from (3.16) that

$$\begin{aligned}
&\int_{|x|\leq R_0^{2s}} |u|^2dx \Big/ \int_{|x|\leq R_0^{2s+2}} |u|^2dx \\
&\leq C^{3(-4\log R_0)^{s-1}} \left(\int_{|x|<R_0^2} |u|^2dx \Big/ \int_{|x|<R_0^4} |u|^2dx\right)^{a^{s-1}} \qquad (3.17)\\
&\leq (C_0^3(R_0)^{-3\beta_1})^{(-4\log R_0)^{s-1}} \left(\int_{|x|<R_0^2} |u|^2dx \Big/ \int_{|x|<R_0^4} |u|^2dx\right)^{a^{s-1}}.
\end{aligned}$$

Thus, by (3.17), we can get that

$$\begin{aligned}
&\exp(-cR_0^{-4s\epsilon/l}) \int_{|x|\leq R_0^{2s}} |u|^2dx \\
&\leq \exp(-cR_0^{-4s\epsilon/l})(C_0^3(R_0)^{-3\beta_1})^{(-4\log R_0)^{s-1}} \qquad (3.18)\\
&\left(\int_{|x|<R_0^2} |u|^2dx \Big/ \int_{|x|<R_0^4} |u|^2dx\right)^{a^{s-1}} \int_{|x|\leq R_0^{2s+2}} |u|^2dx.
\end{aligned}$$

Let $\mu = -\log R_0$. Then if R_0 ($\leq \min\{e^{-4}, \sqrt{\tilde{R}_0}\}$) is sufficiently small, *i.e.*, μ is sufficiently large, we can see that

$$4t\epsilon\mu/l > (t-1)\log(4\mu) + \log(\log C_0^3 + 3\beta_1\mu) - \log(c/4),$$

for all $t \in \mathbb{N}$. In other words, for small R_0 we have that

$$(C_0^3 R_0^{-3\beta_1})^{(-4\log R_0)^{t-1}} < \exp(cR_0^{-4t\epsilon/l}/4) < (1/2)\exp(cR_0^{-4t\epsilon/l}/2), \qquad (3.19)$$

for all $t \in \mathbb{N}$. We now fix such an R_0 so that (3.19) holds and

$$-\frac{4\varepsilon}{l}\log R_0 - 2\log a > 0.$$

It is a key step in our proof that we can find a universal constant R_0. After fixing R_0, we then define a number t_0, depending on R_0 and u, as

$$\begin{aligned}
t_0 = &\left(\log 2 - \log(ac) + \log\log\left(\int_{|x|<R_0^2} |u|^2dx \Big/ \int_{|x|<R_0^4} |u|^2dx\right)\right)\\
&\times\left(-\frac{4\varepsilon}{l}\log R_0 - \log a\right)^{-1}.
\end{aligned}$$

With this choice of t_0, we can see that

$$\left(\int_{|x|<R_0^2} |u|^2 dx \Big/ \int_{|x|<R_0^4} |u|^2 dx\right)^{a^{t-1}} \leq \exp(cR_0^{-4t\epsilon/l}/2) \tag{3.20}$$

for all $t \geq t_0$. Let s_1 be the smallest positive integer such that $s_1 \geq t_0$. If

$$R_0^{2s_1} \leq R_{j_0} = (\gamma(j_0 + 1/2))^{-l/2\epsilon}, \tag{3.21}$$

then we can find $j_1 \in \mathbb{N}$ with $j_1 \geq j_0$ such that (3.11) holds, *i.e.*,

$$R_{j_1+1} < R_0^{2s_1} \leq R_{j_1}.$$

On the other hand, if

$$R_0^{2s_1} > R_{j_0}, \tag{3.22}$$

then we pick the smallest positive integer $s_2 > s_1$ such that $R_0^{2s_2} \leq R_{j_0}$ and thus we can also find $j_1 \in \mathbb{N}$ with $j_1 \geq j_0$ for which (3.11) holds. We now define

$$s = \begin{cases} s_1 & \text{if} \quad (3.21) \quad \text{holds,} \\ s_2 & \text{if} \quad (3.22) \quad \text{holds.} \end{cases}$$

It is important to note that with such an s, (3.11) is satisfied for some j_1 and (3.19), (3.20) hold. Therefore, we set $m_1 = n + 2(j_1 + 1/2)$ and $m = (m_1 - n)/2$. Combining (3.18), (3.19) and (3.20) yields

$$\begin{aligned}
&\exp(-cR_0^{-4s\epsilon/l}) \int_{|x|\leq R_0^{2s}} |u|^2 dx \\
&\leq \exp(-cR_0^{-4s\epsilon/l})(C_0^3(R_0)^{-3\beta_1})^{(-3\log R_0)^{s-1}} \\
&\quad \left(\int_{|x|<R_0^2} |u|^2 dx \Big/ \int_{|x|<R_0^4} |u|^2 dx\right)^{a^{(s-1)}} \int_{|x|\leq R_0^{2s+2}} |u|^2 dx \\
&\leq \frac{1}{2} \int_{|x|\leq R_0^{2s+2}} |u|^2 dx
\end{aligned}$$

which is (3.13). Using (3.13) in (3.12), we have

$$\begin{aligned}&\frac{1}{2}m^2(2\delta)^{-2m-n}\int_{|x|\le 2\delta}|u|^2dx+\frac{1}{2}m^2(R_0^2R_{j_1})^{-2m-n}\int_{|x|\le R_0^{2s+2}}|u|^2dx\\&\le \tilde{C}''\delta^{-2m-n}\int_{|x|\le\delta}|u|^2dx.\end{aligned}\tag{3.23}$$

From (3.23), we get

$$\frac{(m_1-n)^2}{8\tilde{C}''}(R_0^2R_{j_1})^{-m_1}\int_{|x|\le R_0^{2s+2}}|u|^2dx\le\delta^{-m_1}\int_{|x|\le\delta}|u|^2dx \tag{3.24}$$

and

$$\frac{1}{2}m^2(2\delta)^{-2m-n}\int_{|x|\le 2\delta}|u|^2dx\le\tilde{C}''\delta^{-2m-n}\int_{|x|\le\delta}|u|^2dx$$

which implies

$$\int_{|x|\le 2\delta}|u|^2dx\le\frac{8\tilde{C}''}{(m_1-n)^2}2^{m_1}\int_{|x|\le\delta}|u|^2dx. \tag{3.25}$$

The estimates (3.24) and (3.25) are valid for all $\delta\le R_0^{2s+2}/4$. Therefore, (1.3) holds with $R_2=R_0$, $R_3=R_0^{2s+2}/4$ and $C_3=\frac{(m_1-n)^2}{8\tilde{C}''}(R_0^2R_{j_1})^{-m_1}\int_{|x|\le R_0^{2s+2}}|u|^2dx$. Moreover (1.4) holds with $R_4=R_0^{2s+2}/8$ and $C_4=\frac{8\tilde{C}''}{(m_1-n)^2}2^{m_1}$ and the proof is now complete. Here we have proved (1.3) and (1.4) together. In fact, (1.3) can be seen as a corollary of (1.4) (see, for example, [6, page 135]). □

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