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Maximizers for the Strichartz norm for small solutions of mass-critical NLS

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Abstract. Consider the mass-critical nonlinear Schrödinger equations in both focusing and defocusing cases for initial data in L^2 in space dimension N. By Strichartz inequality, solutions to the corresponding linear problem belong to a global L^p space in the time and space variables, where $p = 2 + \frac{4}{N}$. In $1D$ and $2D$, the best constant for the Strichartz inequality was computed by D. Foschi who has also shown that the maximizers are the solutions with Gaussian initial data.

Solutions to the nonlinear problem with small initial data in L^2 are globally defined and belong to the same global L^p space. In this work we show that the maximum of the L^p norm is attained for a given small mass. In addition, in $1D$ and $2D$, we show that the maximizer is unique and obtain a precise estimate of the maximum. In order to prove this we show that the maximum for the linear problem in $1D$ and $2D$ is nondegenerated.

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1. Introduction

We study the L^2-critical nonlinear Schrödinger (NLS) equation in space dimension $N \geq 1$:

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u + \gamma\, |u|^{\frac{4}{N}} u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^N, \\ u_{\restriction t=0} = f \in L^2(\mathbb{R}^N). \end{cases} \tag{1.1}$$

We will consider both focusing ($\gamma = +1$) and defocusing ($\gamma = -1$) equations.

Let us first recall some properties of the linear problem:

$$i\partial_t u + \frac{1}{2}\Delta u = 0, \quad u_{\restriction t=0} = f. \tag{1.2}$$

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Denote by $u = e^{i\frac{t}{2}\Delta}f$ the solution to (1.2). The mass $\|u(t)\|^2_{L^2}$ of the solution is conserved. Solutions to the linear problem satisfy the Strichartz inequality (see [24]):

$$\forall f \in L^2, \quad \left\| e^{i\frac{t}{2}\Delta} f \right\|_{L^{\frac{4}{N}+2}_{t,x}} \leq C\|f\|_{L^2}, \tag{1.3}$$

where

$$\|u\|_{L^{\frac{4}{N}+2}_{t,x}} = \left(\iint_{\mathbb{R}\times\mathbb{R}^N} |u(t,x)|^{\frac{4}{N}+2}\, dt\, dx \right)^{\frac{1}{\frac{4}{N}+2}}.$$

By standard profile decomposition arguments, one can easily show that the maximum for the Strichartz inequality is attained. The best constant and maximizers for the Strichartz estimates were computed by D. Foschi [11] (see also [13] for another proof) for $N = 1, 2$. Before stating this result, we first recall some symmetries of the equations (1.1) and (1.2).

The following group of transformations leaves the solutions invariant under the nonlinear and linear Schrödinger evolution. If $\{\theta_0, \rho_0, t_0, \xi_0, x_0\} \in \mathbb{R}\times(0,+\infty)\times\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$, then if u is a solution to (1.1) (respectively (1.2)), so is

$$e^{i\theta_0}\rho_0^{\frac{N}{2}} e^{ix\cdot\xi_0} e^{-i\frac{t}{2}|\xi_0|^2} u\left(\rho_0^2 t + t_0, \rho_0\left(x - \frac{t}{2}\xi_0 \right) + x_0 \right). \tag{1.4}$$

This includes phase invariance, scaling, time-translation, Galilean transformation and space-translation. Another transformation of (1.1) and (1.2) is the pseudo-conformal inversion (see [25]):

$$\frac{1}{t^{N/2}} \exp\left(\frac{i|x|^2}{2t} \right) u\left(-\frac{1}{t}, \frac{x}{t} \right). \tag{1.5}$$

Note that all the preceding transformations leave the mass and the $L^{\frac{4}{N}+2}_{t,x}$ norm of the solutions invariant. The linear equation is of course also invariant under the multiplication by a scalar: if $u(t,x)$ is a solution, so is $c_0\, u(t,x)$, $c_0 \in \mathbb{R}$.

Consider the following normalized Gaussian:

$$G_0(x) = \frac{1}{\pi^{N/4}} e^{-\frac{|x|^2}{2}}, \quad \text{thus,} \quad \int_{\mathbb{R}^N} |G_0|^2 dx = 1,$$

and its linear evolution:

$$G(t,x) = e^{\frac{1}{2}it\Delta} G_0 = \frac{1}{\pi^{N/4}} \frac{1}{(1+it)^{N/2}} e^{-\frac{|x|^2}{2(1+it)}}. \tag{1.6}$$

Theorem A (Foschi). *For all $f \in L^2(\mathbb{R}^N)$, $N = 1, 2$,*

$$\left\| e^{i\frac{t}{2}\Delta} f \right\|^{\frac{4}{N}+2}_{L^{\frac{4}{N}+2}_{t,x}} \leq C_S\, \|f\|^{\frac{4}{N}+2}_{L^2(\mathbb{R}^N)}, \qquad C_S = \begin{cases} \frac{1}{\sqrt{3}}, & N = 1 \\ \frac{1}{2}, & N = 2. \end{cases}$$

Furthermore, the equality holds if and only if $e^{i\frac{t}{2}\Delta}f$ is, up to the symmetries (1.4) *of the equation, one of the solutions $c_0 G$, $c_0 \in \mathbb{C}$.*

Let us mention that the effect of the pseudo-conformal transformation (1.5) on G may be expressed only with the invariances (1.4) and we can omit it from consideration in Theorem A.

The Strichartz estimate (1.3) is the key ingredient to prove that the Cauchy problem (1.1) is locally wellposed in L^2 (see [8]). For small data, the solution is also globally wellposed and the global $L_{t,x}^{\frac{4}{N}+2}$ norm is finite, which implies that the solution scatters in L^2. This was extended to large radial data in the defocusing case $\gamma = -1$, in [30] for $N \geq 3$ and in [16] for $N = 2$ (in this last work, the focusing case $\gamma = 1$ below the mass of the ground-state is also treated). The proofs are mainly based on technics developed for the energy-critical NLS (see *e.g.* [1,2,26,29] and [15]).

In all these studies, a global Strichartz norm (in the mass-critical case, the $L^{\frac{4}{N}+2}$ norm) appears as the relevant norm to control. In this work we consider

$$I(\delta) = \sup_{\|f\|_{L^2(\mathbb{R}^N)}=\delta} \iint_{\mathbb{R}\times\mathbb{R}^N} |u(t,x)|^{\frac{4}{N}+2}\, dt\, dx,$$

where $\delta > 0$ is small and u is the solution to (1.1). The results cited above imply that $I(\delta)$ is finite for small δ, and, in the defocusing case with $N \geq 2$, for large δ if we restrict the maximum to radial solutions. A natural extension to Theorem A would be to show that this maximum is achieved by a unique solution (up to symmetries) of (1.1) and give a precise estimate of $I(\delta)$.

Our main result is the following:

Theorem 1.1. *Fix $\gamma \in \{-1,+1\}$. There exists a $\delta_0 > 0$ such that for all δ in $(0,\delta_0)$, the maximum $I(\delta)$ is attained: there exists a solution u_δ of* (1.1) *with initial condition f_δ such that*

$$\|f_\delta\|_{L^2} = \delta, \quad I(\delta) = \iint_{\mathbb{R}\times\mathbb{R}^N} |u_\delta(t,x)|^{\frac{4}{N}+2}\, dt\, dx.$$

If $N = 1$ or $N = 2$, the maximizer u_δ is unique up to the transformations (1.4), (1.5) *of the equation. Furthermore, as $\delta \to 0$,*

$$I(\delta) = C_S \delta^{\frac{4}{N}+2} + \gamma D_N \delta^{\frac{8}{N}+2} + \mathcal{O}\left(\delta^{\frac{12}{N}+2}\right), \tag{1.7}$$

where $D_1 = \dfrac{1}{\pi}\displaystyle\sum_{k\geq 1} \frac{(2k)!}{k\, 9^k\, (k!)^2} \approx 0.0867$ *and* $D_2 = \dfrac{1}{2\pi}\ln\dfrac{4}{3} \approx 0.0458.$

Remark 1.2. In particular, in the focusing case in $1D$ and $2D$, the maximum of the Strichartz norm is, for small data, higher than in the linear case. In the defocusing case, the effect of the nonlinearity is to lower this maximum.

Remark 1.3. The constant D_N may be expressed as

$$D_N = -\left(2+\frac{4}{N}\right) \operatorname{Im} \iint |G(t)|^{\frac{4}{N}} \overline{G}(t) \int_0^t e^{i\frac{(t-s)}{2}\Delta} \left(|G(s)|^{\frac{4}{N}} G(s)\right) ds\, dt\, dx. \quad (1.8)$$

Remark 1.4. The proof also shows that in $1D$ and $2D$, the initial condition of any maximizer with small mass δ is (after transformations) close to δG_0, where G_0 is the normalized Gaussian. See Proposition 3.3 and Remark 3.4 for a precise statement.

Estimates of Strichartz norms for critical nonlinear problems are only known in a few cases. Super-exponential bounds were obtained by T. Tao for radial defocusing energy-critical equations: Schrödinger equation in space dimension higher than 3 [26], and wave equation in $3D$ [27]. An equivalent of the maximizum is given in [9] for the energy-critical focusing Schrödinger and wave equations (in space dimensions 3, 4 and 5), close to the energy threshold given by the stationary solution.

The fact that the maximum of the Strichartz norm is attained is new for a nonlinear equation. The proof of this result is based on time-dependent adaptation to concentration-compactness arguments (see *e.g.* [18]) and on a super-additivity property of $I(\delta)$ which we show by general estimates on small solutions of (1.1). As stated in Proposition 2.12, the proof would extend to larger data provided the scattering of all solutions and the super-additivity properties are shown for those data also. This proof is flexible and should also easily adapt to other equations, *e.g.* the energy-critical NLS and wave equations for small data and (together with the methods of [9]) close to the energy threshold.

On the other hand, the proof of the uniqueness of the maximizer and of the estimate (1.7) is specific to the mass-critical problem, and strongly relies on the results of [11] and [13]. A key element is the nondegeneracy of the Gaussian for the nonlinear problem, in the orthogonal space of the null directions related to the invariances of the equation:

Theorem 1.5. *Assume $N = 1, 2$. There exists $c > 0$ such that if $\varphi \in L^2$ satisfies the following orthogonality properties ($x \in \mathbb{R}^N$)*

$$\int \varphi\, G_0 = \int \varphi\, |x|^2 G_0 = 0, \qquad \int \varphi\, x G_0 = 0_{\mathbb{R}^N}, \quad (1.9)$$

then

$$Q(\varphi) \geq c\|\varphi\|_{L^2}^2,$$

where Q is the quadratic form associated to the second derivative of the mapping

$$f \mapsto C_S \left(\int |f|^2\, dx\right)^{1+\frac{2}{N}} - \iint \left|e^{i\frac{t}{2}\Delta} f\right|^{2+\frac{4}{N}} dt\, dx$$

from L^2 to $[0, \infty)$, at the critical point $f = G_0$.

We refer to (3.3) for an expression of Q. This result is an analogue, for the Strichartz estimate, to the non-degeneracy of the maximizer $\frac{1}{(1+|x|^2)^{\frac{N-2}{2}}}$ for the Sobolev imbedding $\dot{H}^1(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ (see [21]).

To show Theorem 1.5, we apply a lens transform ([4, 20, 22]), related to the pseudo-conformal inversion, to the solutions of (1.1), which turns the Laplace operator into the harmonic operator $-\Delta + |x|^2$. The result then follows from explicit computations and a formula of Wei-Min Wang [33] on products of eigenfunctions for the harmonic oscillator.

The outline of the paper is as follows. In Section 2 we show that the maximizer is attained and in Section 3 we prove the estimate on $I(\delta)$. In Section 4 we show the uniqueness of the maximizer. Section 5 is devoted to the proof of Theorem 1.5.

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2. Existence of a maximizer

In this section, where there is no restriction on the dimension $N \geq 1$, we show the first part of Theorem 1.1:

Proposition 2.1. *There exists* $\delta_0 > 0$ *such that if* $\delta \in (0, \delta_0)$*, then there exists a solution* u_δ *of* (1.1)*, with initial condition* f_δ *such that*

$$\|f_\delta\|_{L^2(\mathbb{R}^N)} = \delta \quad \textit{and} \quad \iint_{\mathbb{R}\times\mathbb{R}^N} |u_\delta|^{\frac{4}{N}+2}\, dt\, dx = I(\delta). \tag{2.1}$$

After some preliminaries (Section 2.1) we show in Section 2.2 a crucial super-additivity property of $I(\delta)$, which relies on rough estimates of $I(\delta)$ and its growth rate. In Section 2.3 we use this property to prove Proposition 2.1 by concentration-compactness arguments.

2.1. Profile decomposition

We recall here from [19] a profile decomposition adapted to the Strichartz estimate for the linear equation (1.2). We start with a long time perturbation result for the equation (1.1).

Lemma 2.2 (Long time perturbation). *Let* $A > 0$*. There exists* $C = C(A) > 0$ *and a small* $\delta_0 = \delta_0(A) > 0$ *such that the following holds: Let* $u \in C^0(\mathbb{R}, L^2_x)$ *and solves*

$$i\partial_t u + \frac{1}{2}\Delta u + \gamma|u|^{\frac{4}{N}}u = 0.$$

Let $\tilde{u} = \tilde{u}(x,t) \in C^0(\mathbb{R}, L^2_x)$ *and define*

$$e = i\partial_t \tilde{u} + \frac{1}{2}\Delta\tilde{u} + \gamma|\tilde{u}|^{\frac{4}{N}}\tilde{u}.$$

Assume $\|\tilde{u}\|_{L^{\frac{4}{N}+2}_{t,x}} \leq A$, *and for some* $\varepsilon < \delta_0$

$$\|e\|_{L^{\frac{2(N+2)}{N+4}}_{t,x}} \leq \varepsilon \quad \text{and} \quad \left\| e^{i\frac{(t-t_0)}{2}\Delta}(u(t_0) - \tilde{u}(t_0)) \right\|_{L^{\frac{4}{N}+2}_{t,x}} \leq \varepsilon,$$

then

$$\|u - \tilde{u}\|_{L^{\frac{4}{N}+2}_{t,x}} \leq C\,\varepsilon.$$

We skip the proof of Lemma 2.2. We refer to [2, 6, 15, 29] for similar result for the energy-critical case, [12] for a subcritical case and [31, Lemma 3.1] for a statement close to Lemma 2.2 in the mass-critical case.

We next turn to the profile decomposition. If $\Gamma_0 = \{\rho_0, t_0, \xi_0, x_0\} \in (0, +\infty) \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$, and u is a function of space and time, we will denote by $\Gamma_0(u)$ the function

$$\Gamma_0(u) = \rho_0^{\frac{N}{2}} e^{ix\cdot\xi_0} e^{-i\frac{t}{2}|\xi_0|^2} u\left(\rho_0^2 t + t_0, \rho_0\left(x - \frac{t}{2}\xi_0\right) + x_0\right). \tag{2.2}$$

As we have seen in the introduction, if u is a solution to the linear equation (1.2) (respectively, to the nonlinear equation (1.1)), then $\Gamma_0(u)$ is also a solution to (1.2) (respectively, to (1.1)). We say that two sequences of transformations $\{\Gamma^1_n\}_n$ and $\{\Gamma^2_n\}_n$ are orthogonal when

$$\lim_{n\to\infty} \frac{\rho^1_n}{\rho^2_n} + \frac{\rho^2_n}{\rho^1_n} + \frac{|\xi^1_n - \xi^2_n|}{\rho^1_n} + \left|t^1_n - t^2_n\right| + \left|\frac{t^1_n}{2}\,\frac{\xi^1_n - \xi^2_n}{\rho^1_n} + x^1_n - x^2_n\right| = +\infty. \tag{2.3}$$

We recall from [19, Theorem 2] (see [14] in space dimension 1, [3] for general space dimension) , the following profile decomposition result:

Lemma 2.3. *Let* $\{f_n\}$ *be a bounded sequence in* $L^2(\mathbb{R}^N)$. *Then there exists a subsequence of* $\{f_n\}$ *(still denoted by* $\{f_n\}$*), a family* $\{U^j\}_{j\geq 1}$ *of solutions to* (1.2)*, and sequences of parameters* $\{\Gamma^j_n\}_n$*, such that if* $j \neq k$, $\left\{\Gamma^j_n\right\}_n$ *is orthogonal to* $\{\Gamma^k_n\}_n$ *and for all* J,

$$f_n(x) = \sum_{j=1}^{J} \Gamma^j_n\left(U^j\right)(0, x) + h^J_n(x), \tag{2.4}$$

where

$$\lim_{J\to+\infty} \limsup_{n\to\infty} \left\| e^{i\frac{t}{2}\Delta} h^J_n \right\|_{L^{\frac{4}{N}+2}_{t,x}} = 0.$$

Remark 2.4. As a consequence of the orthogonality of the transformations Γ_n^j, the following Pythagorean expansions hold for all $J \geq 1$:

$$\|f_n\|_{L^2}^2 - \sum_{j=1}^{J} \left\|U^j(0)\right\|_{L^2}^2 - \left\|h_n^J\right\|_{L^2}^2 \underset{n\to+\infty}{\longrightarrow} 0, \tag{2.5}$$

$$\left\|e^{i\frac{t}{2}\Delta} f_n\right\|_{L_{t,x}^{\frac{4}{N}+2}}^{\frac{4}{N}+2} - \sum_{j=1}^{J} \left\|U^j\right\|_{L_{t,x}^{\frac{4}{N}+2}}^{\frac{4}{N}+2} - \left\|e^{i\frac{t}{2}\Delta} h_n^J\right\|_{L_{t,x}^{\frac{4}{N}+2}}^{\frac{4}{N}+2} \underset{n\to+\infty}{\longrightarrow} 0. \tag{2.6}$$

Let $\{f_n\}_n$ be a sequence in L^2 and assume that the corresponding solution to (1.1) is globally defined and satisfies $\|f_n\|_{L_{t,x}^{\frac{4}{N}+2}} < \infty$. Consider the profile decomposition given by Lemma 2.3. Let V^j be the nonlinear profile associated to $\{U^j, t_n^j\}_n$, that is the unique solution of (1.1) such that

$$\lim_{n\to\infty} \left\|U^j\left(t_n^j\right) - V^j\left(t_n^j\right)\right\|_{L^2} = 0.$$

Assume also that the V^j's are globally defined and such that $\|V^j\|_{L^{2+\frac{4}{N}}}$ is finite for all j. Combining Lemmas 2.2 and 2.3, one gets a nonlinear version of the decomposition (2.4):

Corollary 2.5. *Let $\{f_n\}_n$ is as above and $\{u_n\}_n$ be the sequence of solutions to* (1.1) *with initial conditions $\{f_n\}_n$. Then*

$$u_n(t,x) = \sum_{j=1}^{J} \Gamma_n^j\left(V^j\right)(t,x) + h_n^J(t,x) + r_n^J(t,x) \tag{2.7}$$

with

$$\lim_{J\to+\infty} \lim_{n\to+\infty} \left(\left\|r_n^J\right\|_{L_{t,x}^{\frac{4}{N}+2}} + \sup_{t\in\mathbb{R}} \left\|r_n^J(t)\right\|_{L^2} \right) = 0.$$

Remark 2.6. Using the orthogonality of the sequences of transformations $\{\Gamma_n^j\}_n$, it is easy to check that

$$\lim_{J\to\infty} \limsup_{n\to\infty} \left| \|u_n\|_{L_{t,x}^{\frac{4}{N}+2}}^{\frac{4}{N}+2} - \sum_{j=1}^{J} \left\|V^j\right\|_{L_{t,x}^{\frac{4}{N}+2}}^{\frac{4}{N}+2} \right| = 0. \tag{2.8}$$

2.2. A superadditivity property of the maximum

In this paragraph we give various estimates on $I(\delta)$. The main result is the following proposition, which is one of the steps (along with a concentration-compactness argument) in showing that the maximizer is attained:

Proposition 2.7. *There exists* $\delta_0 > 0$ *such that if* $0 < \sqrt{\alpha^2+\beta^2} < \delta_0$, *then*

$$I(\alpha) + I(\beta) < I\left(\sqrt{\alpha^2+\beta^2}\right).$$

Remark 2.8. Superadditivity (or subadditivity for minimizers) conditions are classical in this context (see [17, Subsection I.2]).

The proof of Proposition 2.7 relies on two estimates on $I(\delta)$ that we treat in Lemmas 2.9 and 2.11 below.

Lemma 2.9. *There exists a constant* $C_0 > 0$ *such that for small* $\delta > 0$,

$$\left| I(\delta) - C_S \delta^{\frac{4}{N}+2} \right| \le C_0 \delta^{\frac{8}{N}+2}, \tag{2.9}$$

where C_S *is the best constant for the Strichartz inequality*

$$\iint \left| e^{i\frac{t}{2}\Delta} f \right|^{\frac{4}{N}+2} dt\, dx \le C_S \|f\|_{L^2}^{\frac{4}{N}+2}. \tag{2.10}$$

Before proving this lemma, we start by a straightforward consequence of the small data well-posedness theory for equation (1.1) (see [8]).

Claim 2.10. There exists a constant $C > 0$ such that if $\|f\|_{L^2}$ is small, then

$$\left\| e^{i\frac{t}{2}\Delta} f - u \right\|_{L_{t,x}^{\frac{4}{N}+2}} \le C \|f\|_{L^2}^{\frac{4}{N}+1},$$

where u is the solution of (1.1) with initial condition f.

Sketch of proof. The Cauchy problem theory for (1.1) implies that for small initial data

$$\|u\|_{L_{t,x}^{\frac{4}{N}+2}} \le 2\|f\|_{L^2}.$$

Since

$$u(t) = e^{i\frac{t}{2}\Delta} f + i\gamma \int_0^t e^{\frac{i}{2}(t-s)\Delta} |u(s)|^{\frac{4}{N}} u(s) ds,$$

the claim follows from Theorem A and the Strichartz estimate

$$\left\| \int_0^t e^{\frac{i}{2}(t-s)\Delta} \varphi(s) ds \right\|_{L_{t,x}^{\frac{4}{N}+2}} \le C \|\varphi\|_{L_{t,x}^{\frac{2(N+2)}{N+4}}}. \qquad \square$$

Proof of Lemma 2.9. Let u be a solution of (1.1) with initial condition f such that $\|f\|_{L^2(\mathbb{R}^N)} = \delta$. Then

$$\begin{aligned}
&\left| \iint \left| e^{i\frac{t}{2}\Delta} f(x) \right|^{\frac{4}{N}+2} dt\, dx - \iint |u(t,x)|^{\frac{4}{N}+2}\, dt\, dx \right| \\
&\le C \left| \left\| e^{i\frac{t}{2}\Delta} f \right\|_{L_{t,x}^{\frac{4}{N}+2}} - \|u\|_{L_{t,x}^{\frac{4}{N}+2}} \right| \left(\left\| e^{i\frac{t}{2}\Delta} f \right\|_{L_{t,x}^{\frac{4}{N}+2}}^{\frac{4}{N}+1} + \|u\|_{L_{t,x}^{\frac{4}{N}+2}}^{\frac{4}{N}+1} \right) \\
&\le C \left\| e^{i\frac{t}{2}\Delta} f - u \right\|_{L_{t,x}^{\frac{4}{N}+2}} \delta^{\frac{4}{N}+1} \le C \delta^{\frac{8}{N}+2},
\end{aligned}$$

where the last line follows from the triangle inequality and then from Claim 2.10. Applying the previous inequality to the initial data $f = \delta F_0$, where F_0 is the initial condition of a maximizer for Strichartz estimate (2.10), and then to a sequence $\{f_n\}_n$ such that $\|f_n\|_{L^2} = \delta$ and $\iint |u_n|^{\frac{4}{N}+2} \to I(\delta)$, we obtain (2.9). □

We next estimate the rate of growth of $I(\delta)$.

Lemma 2.11. *If δ is small and $\varepsilon \le \frac{1}{2}\delta$, then*

$$I(\delta) + c_1 \delta^{\frac{4}{N}+1} \varepsilon \le I(\delta+\varepsilon) \le I(\delta) + C_1 \delta^{\frac{4}{N}+1} \varepsilon, \tag{2.11}$$

where $c_1 = \frac{4}{N} C_S$ and $C_1 = 2\left(\frac{4}{N}+2\right) C_S$.

Proof. Step 1. We first show that there exist $C_2, \epsilon_0 > 0$ such that if $f \in L^2$ with $\|f\|_{L^2} + \epsilon \le \epsilon_0$, u is the solution of (1.1) with the initial condition f, and v_ϵ is the solution of (1.1) with the initial condition $(1+\epsilon)f$, then

$$\left| (1+\epsilon)^{\frac{4}{N}+2} \iint \left| u \right|^{\frac{4}{N}+2} - \iint \left| v_\epsilon \right|^{\frac{4}{N}+2} \right| \le C_2\, \epsilon\, \|f\|_{L^2}^{\frac{8}{N}+2}.$$

First, observe that $u_\epsilon = (1+\epsilon)u$ is a solution to the equation

$$i\partial_t u_\epsilon + \frac{1}{2}\Delta u_\epsilon + \frac{1}{(1+\epsilon)^{\frac{4}{N}}} |u_\epsilon|^{\frac{4}{N}} u_\epsilon = 0, \quad u_\epsilon\restriction_{t=0} = (1+\epsilon) f.$$

We rewrite the above equation as

$$i\partial_t u_\epsilon + \frac{1}{2}\Delta u_\epsilon + |u_\epsilon|^{\frac{4}{N}} u_\epsilon = \left(1 - \frac{1}{(1+\epsilon)^{\frac{4}{N}}}\right) |u_\epsilon|^{\frac{4}{N}} u_\epsilon,$$

noting that for small ϵ, Strichartz estimate implies

$$\begin{aligned}
\left\| \left(1 - \frac{1}{(1+\epsilon)^{\frac{4}{N}}}\right) |u_\epsilon|^{\frac{4}{N}} u_\epsilon \right\|_{L_{t,x}^{\frac{2(N+2)}{N+4}}} &\le C\,\epsilon \left\| |u_\epsilon|^{1+\frac{4}{N}} \right\|_{L_{t,x}^{\frac{2(N+2)}{N+4}}} \\
&= C\,\epsilon \|u_\epsilon\|_{L_{t,x}^{\frac{4}{N}+2}}^{1+\frac{4}{N}} \le C\epsilon \|f\|_{L^2}^{1+\frac{4}{N}}.
\end{aligned}$$

Since v_ϵ is a solution of

$$i\partial_t v_\epsilon + \frac{1}{2}\Delta v_\epsilon + |v_\epsilon|^{\frac{4}{N}} v_\epsilon = 0, \quad v_{\epsilon\restriction t=0} = (1+\epsilon)f,$$

by the long time perturbation Lemma 2.2, we get

$$\|u_\epsilon - v_\epsilon\|_{L^{\frac{4}{N}+2}_{t,x}} \leq C\epsilon \|f\|_{L^2}^{\frac{4}{N}+1}.$$

Hence,

$$\begin{aligned}\left|\iint |u_\epsilon|^{\frac{4}{N}+2}\,dt\,dx - \iint |v_\epsilon|^{\frac{4}{N}+2}\,dt\,dx\right| &\leq C\,\|u_\epsilon - v_\epsilon\|_{L^{\frac{4}{N}+2}_{t,x}}\,\|f\|_{L^2}^{\frac{4}{N}+1}\\ &\leq C\epsilon\|f\|_{L^2}^{\frac{8}{N}+2},\end{aligned}$$

which concludes Step 1.

Step 2. Let $\varepsilon, \delta > 0$. First, we show the lower bound of $I(\delta+\varepsilon)$. Let $f \in L^2(\mathbb{R}^N)$ be such that

$$\|f\|_{L^2} = \delta \quad \text{and} \quad \iint |u(t,x)|^{\frac{4}{N}+2}\,dt\,dx \geq I(\delta) - \delta^{\frac{8}{N}+1}\varepsilon, \tag{2.12}$$

where u is the corresponding solution of (1.1) and we used the supremum property of $I(\delta)$. Let u_ε be the solution of (1.1) with the initial condition $\left(1+\frac{\varepsilon}{\delta}\right)f$. Then $\|u_\varepsilon(0)\|_{L^2} = \delta + \varepsilon$. By Step 1,

$$\begin{aligned}I(\delta+\varepsilon) &\geq \iint |u_\varepsilon(t,x)|^{\frac{4}{N}+2}\,dt\,dx\\ &\geq \left(1+\frac{\varepsilon}{\delta}\right)^{\frac{4}{N}+2}\iint |u(t,x)|^{\frac{4}{N}+2} - C_2\frac{\varepsilon}{\delta}\,\delta^{\frac{8}{N}+2}.\end{aligned}$$

By (2.12), we get

$$I(\delta+\varepsilon) \geq \left[1+\left(\frac{4}{N}+2\right)\frac{\varepsilon}{\delta}\right]\left(I(\delta) - \delta^{\frac{8}{N}+1}\varepsilon\right) - C_2\delta^{\frac{8}{N}+1}\varepsilon.$$

Lemma 2.9 implies $I(\delta) \geq C_S\delta^{\frac{4}{N}+2} - C_0\delta^{\frac{8}{N}+2}$, hence,

$$\begin{aligned}I(\delta+\varepsilon) \geq I(\delta) &+ C_S\left(\frac{4}{N}+2\right)\delta^{\frac{4}{N}+1}\varepsilon\\ &- \left[\left(\frac{4}{N}+2\right)C_0 + \left(1+\left(\frac{4}{N}+2\right)\frac{\varepsilon}{\delta}\right) + C_2\right]\delta^{\frac{8}{N}+1}\varepsilon.\end{aligned}$$

Now if $\varepsilon < \frac{1}{2}\delta$ and

$$\delta < \left(\frac{C_S}{4+6C_0+C_2}\right)^{N/4},$$

the last term in the expression above will be less than $2C_S\delta^{\frac{4}{N}+1}\varepsilon$, and thus, the right side in (2.11) follows with $c_1 = \frac{4}{N}C_S$.

The upper bound on $I(\delta+\varepsilon)$ follows similarly from Step 1 and Lemma 2.9, obtaining the left side in (2.11) with $C_1 = 2C_S\left(\frac{4}{N}+2\right)$. □

We next prove Proposition 2.7.

Proof. Without loss of generality, we can assume $0 < \alpha \le \beta$.

Step 1. We first show that there exists a large constant $C_3 > 0$ such that the conclusion of the proposition holds if

$$C_3\beta^{\frac{2}{N}+1} \le \alpha \le \beta. \tag{2.13}$$

By Lemma 2.9,

$$I(\alpha)+I(\beta) \le C_S\alpha^{\frac{4}{N}+2} + C_S\beta^{\frac{4}{N}+2} + 2C_0\beta^{\frac{8}{N}+2},$$

$$\text{and}\quad C_S\left(\alpha^2+\beta^2\right)^{\frac{2}{N}+1} \le I\left(\sqrt{\alpha^2+\beta^2}\right) + 2C_0\beta^{\frac{8}{N}+2}.$$

There is a constant $\kappa_N > 0$ such that $1+x^{\frac{2}{N}+1}+\kappa_N x \le (1+x)^{\frac{2}{N}+1}$ for $x \in [0,1]$. As a consequence, $\alpha^{\frac{4}{N}+2}+\beta^{\frac{4}{N}+2}+\kappa_N\,\beta^{\frac{4}{N}}\alpha^2 \le \left(\alpha^2+\beta^2\right)^{\frac{2}{N}+1}$. Combining with the previous estimates, we get

$$I(\alpha)+I(\beta)+C_S\,\kappa_N\,\beta^{\frac{4}{N}}\alpha^2 - 4C_0\beta^{\frac{8}{N}+2} \le I\left(\sqrt{\alpha^2+\beta^2}\right),$$

which yields the announced result if C_3 is chosen large in (2.13).

Step 2. We next show that the conclusion of the Proposition still holds if

$$0 < \alpha < C_3\beta^{\frac{2}{N}+1}, \tag{2.14}$$

where C_3 is the constant defined in Step 1. Choosing δ_0 small enough, $\beta \le \delta_0$ and (2.14) imply

$$\frac{\alpha^2}{4\beta} \le \sqrt{\alpha^2+\beta^2} - \beta \le \frac{\beta}{2}.$$

By Lemma 2.11, with $\delta = \beta$ and $\varepsilon = \sqrt{\alpha^2+\beta^2} - \beta$,

$$I(\beta) \le I\left(\sqrt{\alpha^2+\beta^2}\right) - c_1\beta^{\frac{4}{N}+1}\left(\sqrt{\alpha^2+\beta^2}-\beta\right) \le I\left(\sqrt{\alpha^2+\beta^2}\right) - \frac{c_1}{4}\beta^{\frac{4}{N}}\alpha^2.$$

Combining with Lemma 2.9 we get, taking a smaller δ_0 if necessary,

$$\begin{aligned} I(\alpha)+I(\beta) &\leq I\left(\sqrt{\alpha^2+\beta^2}\right)-\frac{c_1}{4}\beta^{\frac{4}{N}}\alpha^2+2C_S\alpha^{\frac{4}{N}+2} \\ &\leq I\left(\sqrt{\alpha^2+\beta^2}\right)+\alpha^2\beta^{\frac{4}{N}}\left(2C_SC_3^{\frac{4}{N}}\beta^{\frac{8}{N^2}}-\frac{c_1}{4}\right), \end{aligned}$$

which shows that the conclusion of the proposition holds also in this case, provided $\delta_0>0$ is small enough. □

2.3. Proof of the existence of the maximizer

Let us show Proposition 2.1. We will prove the following more general result:

Proposition 2.12. *Assume that there exists a constant $A>0$ such that*

i. *Scattering: for all $f\in L^2$ such that $\|f\|_{L^2}\leq A$, the solution u of* (1.1) *with initial condition f is globally defined and*

$$\delta\leq A\Longrightarrow I(\delta)<\infty.$$

ii. *Superadditivity: if $0<\sqrt{\alpha^2+\beta^2}=A$, and $\alpha,\beta>0$, then*

$$I(\alpha)+I(\beta)<I(A).$$

Then there exists a solution u_A of (1.1) *with initial condition $f_A\in L^2$ such that*

$$\|f_A\|_{L^2}=A,\qquad \iint|u_A|^{2+\frac{4}{N}}=I(A).$$

In view of the small data global well-posedness theory and Proposition 2.7, Proposition 2.12 implies Proposition 2.1. Let us prove Proposition 2.12.

Let $\{u_n\}_n$ be a sequence of solutions to (1.1) with initial data f_n such that

$$\|f_n\|_{L^2(\mathbb{R}^N)}=A,\qquad \lim_{n\to\infty}\iint_{\mathbb{R}^N}|u_n|^{\frac{4}{N}+2}=I(A).$$

We will show that there exist a subsequence of $\{u_n\}_n$ and a sequence $\{\Gamma_n\}_n$ of transformations such that $\{\Gamma_n(u_n)\}_n$ converges strongly in L^2. Consider, after extraction, a profile decomposition of the sequence $\{f_n\}_n$:

$$f_n=\sum_{j=1}^{J}\Gamma_n^j\left(U^j\right)_{\restriction t=0}+h_n^J. \tag{2.15}$$

It is sufficient to show that $U^j=0$ except for one j and that $\lim_{n\to\infty}\|h_n^J\|_{L^2}=0$, which we will do in two steps.

Step 1: *no dichotomy*. First assume that there are at least two nonzero profiles, say $U^1 \neq 0$ and $U^2 \neq 0$. Let V^1 be the nonlinear profiles associated to $\{U^1, t_n^1\}$ and V_n the solution of (1.1) given by

$$V_n = \Gamma_n^1(V^1).$$

Let W_n be the sequence of solutions to (1.1) with initial condition

$$W_n(0) = f_n - V_n(0).$$

Let $r_n = u_n - V_n - W_n$. By assumption (2.12), all the nonlinear profiles V^j scatter. Thus, one can use Corollary 2.5, showing

$$\lim_{n\to\infty} \sup_{t\in\mathbb{R}} \|r_n(t)\|_{L^2} = 0.$$

Furthermore, (see (2.5) and Remark 2.6)

$$\int |f_n|^2 = \int |V_n(0)|^2 + \int |W_n(0)|^2 + o_n(1) \tag{2.16}$$

$$\iint |u_n|^{\frac{4}{N}+2} = \iint |V_n|^{\frac{4}{N}+2} + \iint |W_n|^{\frac{4}{N}+2} + o_n(1). \tag{2.17}$$

Let $\varepsilon = \|U^1(0)\|_{L^2}$. Then for all n, $\varepsilon = \|V_n(0)\|_{L^2}$. By (2.16),

$$\|W_n(0)\|_{L^2}^2 = A^2 - \varepsilon^2 + o_n(1).$$

By our assumptions, $\varepsilon > 0$ (otherwise, U^1 would be zero) and $A^2 - \varepsilon^2 > 0$ (otherwise, U^2 would be zero). Using that $\iint |u_n|^{\frac{4}{N}+2}$ tends to $I(A)$ as $n \to \infty$, and that by Lemma 2.2, $\limsup_n \int |W_n|^{\frac{4}{N}+2} \leq I(\sqrt{A^2-\varepsilon^2})$, we get by (2.17)

$$I(A) \leq I(\varepsilon) + I\left(\sqrt{A^2-\varepsilon^2}\right). \tag{2.18}$$

This contradicts assumption (2.12), concluding Step 1.

Step 2: *non vanishing and the end of the proof*. There must be one nonzero profile in (2.15). If not, then

$$\lim_{n\to\infty} \iint |u_n|^{\frac{4}{N}+2} = 0,$$

showing that $I(A) = 0$, a contradiction. It remains to show that the remainder $h_n = h_n^J$ in (2.15) tends to 0 in L^2. Denote by

$$\varepsilon = \lim_{n\to\infty} \|h_n\|_{L^2},$$

then, using again Lemma 2.2, we get $I(A) \leq I(\sqrt{A^2-\varepsilon^2})$, which shows by assumption (2.12) that $\varepsilon = 0$.

Denoting by U^1 the only nonzero profile in (2.15), we have shown that $(\Gamma_n^1)^{-1}(u_n)$ tends to U^1 in L^2, and therefore,

$$\|U^1\|_{L^2} = A, \qquad \iint |U^1|^{\frac{4}{N}+2} = I(A),$$

concluding the proof of the proposition. □

3. Estimate of the maximum of the Strichartz norm

In the remainder of the paper, we restrict ourselves to $1D$ and $2D$. In this section we prove the second part of Theorem 1.1:

Proposition 3.1. *Assume that $N = 1$ or $N = 2$. Then as $\delta \to 0$,*

$$I(\delta) = \iint |u_\delta|^{2+\frac{4}{N}} = C_S\delta^{2+\frac{4}{N}} + \gamma D_N \delta^{2+\frac{8}{N}} + \mathcal{O}\left(\delta^{2+\frac{12}{N}}\right),$$

where $D_1 = \dfrac{1}{\pi}\displaystyle\sum_{k\geq 1} \frac{(2k)!}{k\, 9^k\, (k!)^2} \approx 0.0867$ *and* $D_2 = \dfrac{1}{2\pi}\ln\dfrac{4}{3} \approx 0.0458.$

Before proving Proposition 3.1, we define the quadratic form associated to the maximum of the Strichartz estimate that appears in Theorem 1.5. By Theorem A, if G is the Gaussian solution defined by (1.6) and $\varphi \in L^2$, then

$$C_S\left(\int |G_0+\varphi|^2\right)^{1+\frac{2}{N}} - \iint \left|G + e^{i\frac{t}{2}\Delta}\varphi\right|^{2+\frac{4}{N}} \geq 0.$$

Expanding the above inequality and using that G is a maximizer, we obtain that the linear part vanishes, *i.e.*,

$$\forall\varphi \in L^2, \quad C_S \operatorname{Re}\int G_0\varphi = \operatorname{Re}\iint |G|^{\frac{4}{N}}\overline{G}\, e^{i\frac{t}{2}\Delta}\varphi. \tag{3.1}$$

The expansion at second order in φ yields

$$C_S\left(\int |G_0+\varphi|^2\right)^{1+\frac{2}{N}} - \iint \left|G + e^{i\frac{t}{2}\Delta}\varphi\right|^{2+\frac{4}{N}} = Q(\varphi) + \mathcal{O}\left(\|\varphi\|_{L^2}^3\right), \tag{3.2}$$

where Q is a (real) nonnegative symmetric quadratic form on L^2 defined by

$$\begin{aligned} Q(\varphi) = C_S&\left[\frac{N+2}{N}\int|\varphi|^2 + \frac{4(N+2)}{N^2}\left(\operatorname{Re}\int G_0\varphi\right)^2\right] \\ &- \frac{(N+2)^2}{N^2}\iint |G|^{\frac{4}{N}}\left|e^{i\frac{t}{2}\Delta}\varphi\right|^2 \\ &- \frac{2(N+2)}{N^2}\operatorname{Re}\iint |G|^{\frac{4}{N}-2}\overline{G}^2\left(e^{i\frac{t}{2}\Delta}\varphi\right)^2. \end{aligned} \tag{3.3}$$

By the transformations of the linear equation (respectively, multiplication by a real number, phase shift, space translation, Galilean invariance, scaling and time translation), we have

$$Q(G_0) = Q(iG_0) = Q(xG_0) = Q(ixG_0) = Q(x^2G_0) = Q(ix^2G_0) = 0, \quad (3.4)$$

if $N = 1$ and

$$Q(G_0)=Q(iG_0)=Q(x_jG_0)=Q(ix_jG_0)=Q(|x|^2G_0)=Q(i|x|^2G_0)=0, \quad (3.5)$$

(where $j = 1, 2$) if $N = 2$. Theorem 1.5, which will be proved in Section 5 states that Q is positive definite in the subspace of functions in L^2 that are orthogonal to the directions in (3.4) or (3.5). This non-degeneracy property is crucial in the proof of Proposition 3.1, which is divided in two parts.

3.1. Choice of the maximizer

We first give a corollary to the linear profile decomposition that will be needed in the proof. Recall from (1.6) the definition of the normalized Gaussian G .

Lemma 3.2. *Let $\{f_n\}_n$ be a sequence in $L^2(\mathbb{R}^N)$ such that*

$$\lim_{n\to\infty} \|f_n\|_{L^2} = 1, \quad (3.6)$$

and

$$\lim_{n\to\infty} \iint \left|e^{i\frac{t}{2}\Delta} f_n\right|^{\frac{4}{N}+2} dt\, dx = C_S. \quad (3.7)$$

Then there exist a subsequence of $\{f_n\}_n$ (still denoted by $\{f_n\}_n$), a phase θ_0 and a sequence $\{\Gamma_n\}_n$ of transformations of the form (2.2) *such that*

$$\lim_{n\to\infty} \left\|f_n - e^{i\theta_0}\Gamma_n(G)\right\|_{L^2} = 0, \quad (3.8)$$

where G is the normalized Gaussian solution defined in (1.6).

Proof. This is an application of Lemma 2.3 and the uniqueness result of Foschi [11].

After extraction of a subsequence, the sequence $\{f_n\}_n$ admits a profile decomposition of the form (2.4). At least one of the profiles is nonzero. Indeed, if it was not the case, $\left\|e^{i\frac{t}{2}\Delta} f_n\right\|_{L^{\frac{4}{N}+2}}$ would tend to 0, a contradiction with (3.7). Reordering the profiles, we may assume that $U^1 \neq 0$. By the Pythagorean expansion (2.6) and by (3.7)

$$C_S + o_n(1) = \iint \left|e^{i\frac{t}{2}\Delta} f_n\right|^{\frac{4}{N}+2} dt\, dx \le C_S \left(\left\|U^1\right\|_{L^2}^{\frac{4}{N}+2} + \left\|w_n^1\right\|_{L^2}^{\frac{4}{N}+2}\right) + o_n(1).$$

Using that by (2.5), $\left\|w_n^1\right\|_{L^2}^2 = 1 - \left\|U^1\right\|_{L^2}^2 + o_n(1)$, we obtain from the previous expression that

$$1 \leq \left(\left\|U^1\right\|_{L^2}^2\right)^{\frac{4}{N}+2} + \left(1 - \left\|U^1\right\|_{L^2}^2\right)^{\frac{4}{N}+2},$$

which shows that $\|U^1\|_{L^2} = 1$ (we already excluded the case $\|U^1\|_{L^2} = 0$), and by (2.5) again

$$\lim_{n\to\infty} \|f_n - \Gamma_n^1(U^1)(0)\|_{L^2} = 0.$$

By our assumptions on f_n we obtain, passing to the limit, that $\|U^1(0)\|_{L^2} = 1$ and $\|U^1\|_{L^{\frac{4}{N}+2}}^{\frac{4}{N}+2} = C_S$, which shows by Theorem A that $U^1(0) = G_0$ up to the symmetries of the equation (*i.e.*, the transformations of the form (2.2) and the multiplication by a phase $e^{i\theta_0}$), which completes the proof. □

Proposition 3.3. *There exists* $\delta_0 > 0$ *such that if* $\left\{u_\delta^*\right\}_{0<\delta<\delta_0}$ *is a family of maximizers, i.e.* u_δ^* *satisfies* (2.1)*, then for all* $\delta \in (0, \delta_0)$ *there exists a transformation* u_δ *of* u_δ^* *such that* $f_\delta = u_\delta(0, x)$ *satisfies:*

$$f_\delta = \alpha_\delta G_0 + \varphi_\delta, \quad \lim_{\delta\to 0^+} \frac{\alpha_\delta}{\delta} = 1,$$

with φ_δ *satisfying the orthogonality properties* (1.9) *and*

$$\forall \delta \in (0, \delta_0), \quad \|\varphi_\delta\|_{L^2} \leq C\delta^{1+\frac{2}{N}}. \tag{3.9}$$

By "transformation" we mean a symmetry of (1.1) which is a combination of transformations of the form (1.4) and (1.5).

Remark 3.4. We will later improve the estimates on φ_δ and α_δ and obtain (see (3.22), (3.24)):

$$\forall \delta \in (0, \delta_0), \quad \|\varphi_\delta\|_{L^2} \leq C\delta^{1+\frac{4}{N}} \text{ and } |\alpha_\delta - \delta| \leq C\delta^{1+\frac{4}{N}}.$$

Proof. The proof is divided into three steps.

Step 1. *Closeness to* G_0. In this step we show that if δ is small enough, there exists a transformation v_δ of u_δ^* which satisfies the maximizer equations (2.1) and

$$\lim_{\delta\to 0} \delta^{-1}\|g_\delta - \delta G_0\|_{L^2} = 0, \quad \text{where} \quad g_\delta(x) = v_\delta(0, x). \tag{3.10}$$

Arguing by contradiction, we see that it is sufficient to show that for any sequence $\delta_n \to 0$ there exists (after extraction of a subsequence) a sequence of solutions $\left\{v_{\delta_n}\right\}_n$ that are obtained as transformations of $u_{\delta_n}^*$ and satisfy (3.10).

By Claim 2.10 and Lemma 2.9, there exists a constant $C > 0$ such that

$$\left| \iint |e^{i\frac{t}{2}\Delta} f_{\delta_n}^*|^{2+\frac{4}{N}}\, dt\, dx - C_S \delta_n^{2+\frac{4}{N}} \right| \leq C \delta_n^{2+\frac{8}{N}}.$$

By Lemma 3.2, we obtain after extraction of subsequences that there exist $\theta_0 \in \mathbb{R}$ and a sequence of transformations $\{\Gamma_n\}$ such that

$$\lim_{n\to\infty} \delta_n^{-1} \left\| f_{\delta_n}^* - \delta_n e^{i\theta_0} \Gamma_n(G)_{\restriction t=0} \right\|_{L^2} = 0. \tag{3.11}$$

Note that, by (1.6),

$$\Gamma_n(G)_{\restriction t=0} = \rho_n^{\frac{N}{2}} e^{ix\cdot\xi_n} G(t_n, \rho_n x + x_n) = \frac{\rho_n^{\frac{N}{2}} e^{ix\cdot\xi_n}}{\pi^{N/4}(1+it_n)^{N/2}} e^{-\frac{|\rho_n x + x_n|^2}{2(1+it_n)}}.$$

And thus, by the change of variable $y = \frac{\rho_n x + x_n}{\sqrt{1+t_n^2}}$,

$$\begin{aligned} &\left\| f_{\delta_n}^*(x) - \delta_n e^{i\theta_0} \Gamma_n(G)_{\restriction t=0} \right\|_{L^2}^2 \\ &= \int \left| e^{i\tau_n + i\frac{t_n|y|^2}{2} + i\frac{\sqrt{1+t_n^2}y - x_n}{\rho_n}\cdot\xi_n} \frac{\left(1+t_n^2\right)^{\frac{N}{4}}}{\rho_n^{\frac{N}{2}}} \overline{f}_{\delta_n}^* \left(\frac{\sqrt{1+t_n^2}y - x_n}{\rho_n} \right) - \frac{\delta_n e^{-\frac{|y|^2}{2}}}{\pi^{\frac{N}{4}}} \right|^2 dy, \end{aligned}$$

where $e^{i\tau_n} = \left(\frac{\sqrt{1+t_n^2}}{1+it_n} \right)^{\frac{N}{2}}$. Consider the solution w_{δ_n} of (1.1) with initial condition

$$h_{\delta_n}(x) = e^{i\tau_n + i\frac{\sqrt{1+t_n^2}y - x_n}{\rho_n}\cdot\xi_n} \frac{\left(1+t_n^2\right)^{\frac{N}{4}}}{\rho_n^{\frac{N}{2}}} \overline{f}_{\delta_n}^* \left(\frac{\sqrt{1+t_n^2}y - x_n}{\rho_n} \right),$$

and the solution v_{δ_n} of (1.1) with initial condition $g_{\delta_n} = e^{i\frac{t_n|y|^2}{2}} h_{\delta_n}$. Then w_{δ_n} is an image of $u_{\delta_n}^*$ by phase, scaling, space translation and Galilean transformation (see (1.4)). Furthermore, v_{δ_n} is obtained from w_{δ_n} with a combination of pseudo-conformal transformation and time translation. Namely:

$$v_{\delta_n}(t,x) = \frac{t_n^{N/2}}{(t_n^2 t + t_n)^{N/2}} e^{\frac{it_n|x|^2}{2(t_n t+1)}} w_{\delta_n}\left(\frac{t}{1+t_n t}, \frac{t_n x}{t_n^2 t + t_n} \right).$$

All these transformations preserve the L^2 norm and the global space-time $L^{2+\frac{4}{N}}$ norm, which shows that

$$\left\| g_{\delta_n} \right\|_{L^2} = \delta_n, \qquad \iint \left| v_{\delta_n} \right|^{\frac{4}{N}+2} = I(\delta_n).$$

By (3.11),

$$\lim_{n\to\infty} \frac{1}{\delta_n} \left\| g_{\delta_n} - \delta_n G_0 \right\|_{L^2} = 0,$$

concluding the first step.

Step 2. *Orthogonality conditions.* We next show that the statement of the proposition holds if (3.9) is replaced by the weaker condition

$$\lim_{\delta\to 0} \delta^{-1} \|\varphi_\delta\|_{L^2} = 0. \tag{3.12}$$

For this we must show that there exists a transformation u_δ of v_δ such that φ_δ satisfies the orthogonality conditions (1.9). Consider the unit ball

$$B_{L^2}(G_0, 1) = \left\{ f \in L^2,\ \|f - G_0\|_{L^2} < 1 \right\},$$

and define, for small $\delta > 0$, a differentiable mapping

$$\Phi_\delta : \mathbb{R} \times (0, +\infty) \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \times B_{L^2}(G_0, 1) \longrightarrow \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$$

as follows. If $\theta_0 \in \mathbb{R}$, $\Gamma_0 \in (0, +\infty) \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$, $f \in B_{L^2}(G_0, 1)$, $\tilde{u}_\delta$ is the solution of (1.1) with initial condition δf and

$$U_\delta(x) = \delta G_0 - e^{i\theta_0}\Gamma_0\left(\tilde{u}_\delta\right)_{\restriction t=0} = \delta G_0 - e^{i\theta_0}\rho_0^{\frac{N}{2}} e^{ix\cdot\xi_0}\tilde{u}_\delta\left(t_0, \rho_0 x + x_0\right),$$

then $\Phi_\delta(\theta_0, \Gamma_0, f) = (\Phi_\delta^1, \Phi_\delta^2, \Phi_\delta^3, \Phi_\delta^4, \Phi_\delta^5)$ is defined by

$$\Phi_\delta^1 = \frac{1}{\delta}\,\mathrm{Im}\int U_\delta\, G_0, \quad \Phi_\delta^2 = \frac{1}{\delta}\,\mathrm{Re}\int U_\delta\left(|x|^2 - \frac{N}{2}\right) G_0, \quad \Phi_\delta^3 = \frac{1}{\delta}\,\mathrm{Im}\int U_\delta x\, G_0,$$
$$\Phi_\delta^4 = \frac{1}{\delta}\,\mathrm{Re}\int U_\delta x\, G_0, \quad \Phi_\delta^5 = \frac{1}{\delta}\,\mathrm{Im}\int U_\delta\left(|x|^2 - \frac{N}{2}\right) G_0.$$

Denote by $\Gamma_{id} = (1,0,0,0)$ the identical transformation. Note that $\Phi_\delta(0, \Gamma_{id}, G_0) = 0$. Then:

Claim 3.5. For small δ, there exist (θ, Γ) close to $(0, \Gamma_{id})$ such that

$$\Phi_\delta\left(\theta_\delta, \Gamma_\delta, \frac{1}{\delta} g_\delta\right) = 0,$$

where g_δ is the initial condition of the maximizer v_δ defined in Step 1.

We refer to Appendix A for the proof of Claim 3.5 which is based on a standard application of the implicit function theorem.

Let u_δ be the solution of (1.1) with initial condition

$$f_\delta = e^{i\theta_\delta}\Gamma_\delta\left(v_\delta\right)_{\restriction t=0}.$$

Then by (3.10),

$$\lim_{\delta\to\infty} \delta^{-1} \|f_\delta - \delta G_0\|_{L^2} = 0. \tag{3.13}$$

Furthermore, from the invariance of the L^2 and $L_{t,x}^{2+\frac{4}{N}}$ norms by the transformations of the equation, u_δ satisfies the maximizer equations (2.1).

The fact that $\Phi_\delta\left(\theta_\delta, \Gamma_\delta, \delta^{-1} g_\delta\right) = 0$ means that f_δ satisfies the orthogonality conditions

$$\operatorname{Im}\int (f_\delta - \delta G_0)\ G_0 = 0, \qquad \int (f_\delta - \delta G_0)\, x\, G_0 = 0, \tag{3.14}$$

$$\int (f_\delta - \delta G_0)\left(|x|^2 - \frac{N}{2}\right) G_0 = 0. \tag{3.15}$$

Let $\alpha_\delta = \operatorname{Re}\int f_\delta G_0$ and $\varphi_\delta = f_\delta - \alpha_\delta G_0$, so that $\operatorname{Re}\int \varphi_\delta G_0 = 0$. By (3.14) and (3.15), φ_δ satisfies the orthogonality conditions (1.9). By (3.13), $\lim_{\delta\to 0} \alpha_\delta/\delta = 1$, which concludes Step 2.

Step 3. *Proof of the estimate* (3.9). In this step we conclude the proof of Proposition 3.3 using the coercivity of Q (Theorem 1.5). To simplify notations, we will omit the index δ and write u, f, φ and α instead of u_δ, f_δ, φ_δ and α_δ. All estimates stated hold for small $\delta > 0$.

By Claim 2.10,

$$\left| \iint |u|^{2+\frac{4}{N}}\, dt\, dx - \iint \left| e^{i\frac{t}{2}\Delta} f\, dt\, dx \right|^{2+\frac{4}{N}} \right| \\ \leq C\delta^{1+\frac{4}{N}} \left\| u - e^{i\frac{t}{2}\Delta} f \right\|_{L_{t,x}^{2+\frac{4}{N}}} \leq C\delta^{2+\frac{8}{N}}.$$

Recalling that $\frac{1}{\alpha} f = G_0(x) + \frac{1}{\alpha}\varphi$ and using the expansion of the Strichartz norm, we obtain

$$\begin{aligned} \iint |u|^{2+\frac{4}{N}}\ dt\, dx &= \alpha^{2+\frac{4}{N}} \iint \left| e^{i\frac{t}{2}\Delta} \frac{1}{\alpha} f \right|^{2+\frac{4}{N}} dt\, dx + \mathcal{O}\left(\delta^{2+\frac{8}{N}}\right) \\ &= C_S \left(\int |f|^2\ dx \right)^{1+\frac{2}{N}} - \alpha^{2+\frac{4}{N}} Q\left(\frac{1}{\alpha}\varphi\right) \\ &\quad + \alpha^{2+\frac{4}{N}} \mathcal{O}\left(\frac{1}{\alpha^3} \|\varphi\|_{L^2}^3 \right) + \mathcal{O}\left(\delta^{2+\frac{8}{N}}\right) \\ &= C_S \delta^{2+\frac{4}{N}} - \alpha^{\frac{4}{N}} Q(\varphi) + \mathcal{O}\left(\alpha^{\frac{4}{N}-1} \|\varphi\|_{L^2}^3 \right) + \mathcal{O}\left(\delta^{2+\frac{8}{N}}\right). \end{aligned}$$

Using that u satisfies (2.1), we get

$$\iint |u|^{2+\frac{4}{N}}\ dt\, dx = I(\delta) = C_S \delta^{2+\frac{4}{N}} + \mathcal{O}\left(\delta^{2+\frac{8}{N}}\right),$$

and thus,

$$\alpha^{\frac{4}{N}} Q(\varphi) = \mathcal{O}\left(\alpha^{\frac{4}{N}-1}\|\varphi\|_{L^2}^3\right) + \mathcal{O}\left(\delta^{2+\frac{8}{N}}\right) = \mathcal{O}\left(\alpha^{\frac{4}{N}}\|\varphi\|_{L^2}^2 \frac{\|\varphi\|_{L^2}}{\alpha}\right) + \mathcal{O}\left(\delta^{2+\frac{8}{N}}\right).$$

By Theorem 1.5, $\|\varphi\|_{L^2}^2 \lesssim Q(\varphi)$, and thus, using that $\frac{1}{\alpha}\|\varphi\|_{L^2} \to 0$ as $\delta \to 0$,

$$\alpha^{\frac{4}{N}}\|\varphi\|_{L^2}^2 = \mathcal{O}\left(\delta^{2+\frac{8}{N}}\right),$$

which shows (3.9). □

3.2. Proof of the estimate on the maximum

The idea of the proof of Proposition 3.1 is to compare $I(\delta)$ with the $L^{2+\frac{4}{N}}$ norm of H_δ, the solution to the nonlinear equation (1.1) with the Gaussian initial data δG_0. We have

$$\iint |u_\delta|^{2+\frac{4}{N}}\, dt\, dx = I(\delta) \geq \iint |H_\delta|^{2+\frac{4}{N}}\, dt\, dx.$$

The global $L^{2+\frac{4}{N}}$ of H_δ may be estimated as follows:

Lemma 3.6. *Let*

$$D_N = -\left(2+\frac{4}{N}\right) \operatorname{Im} \iint |G(t)|^{\frac{4}{N}} \overline{G}(t) \int_0^t e^{i\frac{(t-s)}{2}\Delta}\left(|G(s)|^{\frac{4}{N}} G(s)\right) ds\, dt\, dx. \quad (3.16)$$

Then for small $\delta > 0$,

$$\iint |H_\delta|^{2+\frac{4}{N}}\, dt\, dx = \delta^{2+\frac{4}{N}} \iint |G|^{2+\frac{4}{N}}\, dt\, dx + \gamma D_N \delta^{2+\frac{8}{N}} + \mathcal{O}\left(\delta^{2+\frac{12}{N}}\right). \quad (3.17)$$

The exact value of the constant D_N will be computed in Appendix B (dimension 1) and Appendix C (dimension 2).

Proof of Lemma 3.6. Since G is the linear evolution of G_0, we have

$$H_\delta = \delta G + i\gamma \int_0^t e^{i\frac{(t-s)}{2}\Delta} |H_\delta(s)|^{\frac{4}{N}} H_\delta(s) ds.$$

We approximate H_δ by v_δ:

$$v_\delta(t,x) = \delta\left(G(t,x) + \gamma \delta^{\frac{4}{N}} r(t,x)\right),$$

where

$$r(t,x) = i\int_0^t e^{i\frac{(t-s)}{2}\Delta} |G(s)|^{\frac{4}{N}} G(s)\, ds, \quad (3.18)$$

in other words, v_δ solves

$$i\partial_t v_\delta + \frac{1}{2}\Delta v_\delta + \gamma\delta^{\frac{4}{N}+1}|G|^{\frac{4}{N}}G = 0, \quad v_\delta(0,x) = \delta G_0(x),$$

and r solves

$$i\partial_t r + \frac{1}{2}\Delta r + |G|^{\frac{4}{N}}G = 0, \quad r(0,x) = 0.$$

Since by Claim 2.10

$$\begin{aligned}\left\| |H_\delta|^{\frac{4}{N}}H_\delta - \delta^{\frac{4}{N}+1}|G|^{\frac{4}{N}}G\right\|_{L_{t,x}^{\frac{2(N+2)}{N+4}}} \\ \le C\,\|H_\delta - \delta G\|_{L_{t,x}^{2+\frac{4}{N}}}\left(\|H_\delta\|_{L_{t,x}^{2+\frac{4}{N}}}^{\frac{4}{N}} + \|\delta G\|_{L_{t,x}^{2+\frac{4}{N}}}^{\frac{4}{N}}\right) \le C\delta^{\frac{8}{N}+1},\end{aligned}$$

by Strichartz estimates, we have

$$\|H_\delta - v_\delta\|_{L_{t,x}^{2+\frac{4}{N}}} \le C\delta^{1+\frac{8}{N}},$$

and thus,

$$\left|\iint |H_\delta|^{2+\frac{4}{N}}\,dt\,dx - \iint |v_\delta|^{2+\frac{4}{N}}\,dt\,dx\right| \lesssim \|H_\delta - v_\delta\|_{L_{t,x}^{2+\frac{4}{N}}}\|\delta G_0\|_{L^2}^{1+\frac{4}{N}} \lesssim \delta^{2+\frac{12}{N}},$$

which is exactly the power of higher order terms in (3.17). It remains to estimate $\iint |v_\delta|^{2+\frac{4}{N}}$. Note that if A and B are functions of space and time,

$$\begin{aligned}&\iint |A+B|^{2+\frac{4}{N}} \\ &\quad = \iint |A|^{2+\frac{4}{N}} + \left(2+\frac{4}{N}\right)\mathrm{Re}\iint |A|^{\frac{4}{N}}A\overline{B} + \mathcal{O}\left(\iint |A|^{\frac{4}{N}}|B|^2 + |B|^{2+\frac{4}{N}}\right).\end{aligned} \tag{3.19}$$

By (3.19) and the definition of v_δ we get,

$$\begin{aligned}\iint |v_\delta|^{2+\frac{4}{N}}\,dt\,dx &= \delta^{2+\frac{4}{N}}\iint |G|^{2+\frac{4}{N}}\,dt\,dx \\ &\quad + \delta^{2+\frac{8}{N}}\left(2+\frac{4}{N}\right)\mathrm{Re}\iint |G|^{\frac{4}{N}}\overline{G}r\,dt\,dx + \mathcal{O}\left(\delta^{2+\frac{12}{N}}\right),\end{aligned}$$

which concludes the proof of Lemma 3.6 in view of the definition (3.16) of D_N. □

We next prove Proposition 3.1. Let u_δ, f_δ, φ_δ and α_δ be as in Proposition 3.3. We have

$$u_\delta = \underbrace{e^{i\frac{t}{2}\Delta}\left(\alpha_\delta G_0 + \varphi_\delta\right)}_{A} + \underbrace{i\gamma \int_0^t e^{i\frac{(t-s)}{2}\Delta}\left(|u_\delta(s)|^{\frac{4}{N}} u_\delta(s)\right) ds}_{B}. \tag{3.20}$$

By (3.9) and Strichartz estimate (1.3),

$$\left\| e^{i\frac{t}{2}\Delta}\varphi_\delta \right\|_{L^{2+\frac{4}{N}}} \leq C\|\varphi_\delta\|_{L^2} \leq C\delta^{1+\frac{2}{N}}.$$

Expanding the B term in (3.20) and applying Strichartz estimates again to bound the terms in φ_δ, we get (the $\mathcal{O}$'s are estimated in the space $L^{2+\frac{4}{N}}_{t,x}$).

$$\begin{aligned} B &= i\gamma \int_0^t e^{i\frac{(t-s)}{2}\Delta}\left(|u_\delta(s)|^{\frac{4}{N}} u_\delta(s)\right) ds \\ &= i\gamma \int_0^t e^{i\frac{(t-s)}{2}\Delta}\left[\left|\alpha_\delta G(s) + e^{i\frac{s}{2}\Delta}\varphi_\delta\right|^{\frac{4}{N}}\left(\alpha_\delta G(s) + e^{i\frac{s}{2}\Delta}\varphi_\delta\right)\right] ds + \mathcal{O}\left(\delta^{1+\frac{8}{N}}\right) \\ &= i\gamma\alpha_\delta^{1+\frac{4}{N}} \int_0^t e^{i\frac{(t-s)}{2}\Delta}\left(|G(s)|^{\frac{4}{N}} G(s)\right) ds + \mathcal{O}\left(\delta^{\frac{4}{N}}\|\varphi_\delta\|_{L^2} + \delta^{1+\frac{8}{N}}\right). \end{aligned}$$

And thus, by (3.19) and (3.20),

$$\begin{aligned} &\iint |u_\delta|^{2+\frac{4}{N}}\, dt\, dx = \iint \left|\alpha_\delta G + e^{i\frac{t}{2}\Delta}\varphi_\delta\right|^{2+\frac{4}{N}} dt\, dx \\ &-\left(2+\frac{4}{N}\right)\gamma\alpha_\delta^{2+\frac{8}{N}}\operatorname{Im}\iint |G(t)|^{\frac{4}{N}}\overline{G}(t)\int_0^t e^{i\frac{(t-s)}{2}\Delta}\left(|G(s)|^{\frac{4}{N}} G(s)\right) ds\, dt\, dx \\ &+\mathcal{O}\left(\delta^{1+\frac{8}{N}}\|\varphi_\delta\|_{L^2}\right) + \mathcal{O}\left(\delta^{2+\frac{12}{N}}\right). \end{aligned} \tag{3.21}$$

By the equation (3.2)

$$\begin{aligned} &\iint \left|\alpha_\delta G + e^{i\frac{t}{2}\Delta}\varphi_\delta\right|^{2+\frac{4}{N}} dt\, dx \\ &\qquad = \alpha_\delta^{2+\frac{4}{N}}\left[C_S\left\|G_0 + \frac{1}{\alpha_\delta}\varphi_\delta\right\|_{L^2}^{2+\frac{4}{N}} - Q\left(\frac{1}{\alpha_\delta}\varphi_\delta\right) + \mathcal{O}\left(\frac{1}{\alpha_\delta^3}\|\varphi_\delta\|_{L^2}^3\right)\right]. \end{aligned}$$

By (3.9) and (3.21), using that

$$\|\alpha_\delta G_0 + \varphi_\delta\|_{L^2}^2 = \delta^2 = \alpha_\delta^2 + \|\varphi_\delta\|_{L^2}^2 = \alpha_\delta^2 + \mathcal{O}\left(\delta^{2+\frac{4}{N}}\right), \tag{3.22}$$

we get, in view of the definition (3.16) of D_N,

$$\begin{aligned}&\iint |u_\delta|^{2+\frac{4}{N}}\,dt\,dx\\&=C_S\delta^{2+\frac{4}{N}}-\alpha_\delta^{\frac{4}{N}}Q(\varphi_\delta)+\gamma D_N\alpha_\delta^{2+\frac{8}{N}}+\mathcal{O}\left(\delta^{1+\frac{8}{N}}\|\varphi_\delta\|_{L^2}\right)+\mathcal{O}\left(\delta^{2+\frac{12}{N}}\right).\end{aligned}\tag{3.23}$$

By Lemma 3.6,

$$\iint |u_\delta|^{2+\frac{4}{N}}\,dt\,dx \geq C_S\delta^{2+\frac{4}{N}}+\gamma D_N\delta^{2+\frac{8}{N}}+\mathcal{O}\left(\delta^{2+\frac{12}{N}}\right).$$

Combining with (3.23), we get

$$\begin{aligned}C_S\delta^{2+\frac{4}{N}}-\alpha_\delta^{\frac{4}{N}}Q(\varphi_\delta)+\gamma D_N\alpha_\delta^{2+\frac{8}{N}}+\mathcal{O}(\delta^{2+\frac{12}{N}})+\mathcal{O}\left(\delta^{1+\frac{8}{N}}\|\varphi_\delta\|_{L^2}\right)\\ \geq C_S\delta^{2+\frac{4}{N}}+\gamma D_N\delta^{2+\frac{8}{N}}.\end{aligned}$$

Using that by (3.22)

$$\left|\delta^{2+\frac{8}{N}}-\alpha_\delta^{2+\frac{8}{N}}\right|=\mathcal{O}\left(\delta^{2+\frac{12}{N}}\right),$$

this simplifies to

$$\alpha_\delta^{\frac{4}{N}}Q(\varphi_\delta)=\mathcal{O}\left(\delta^{2+\frac{12}{N}}\right)+\mathcal{O}\left(\delta^{1+\frac{8}{N}}\|\varphi_\delta\|_{L^2}\right).$$

Let $X=\|\varphi_\delta\|_{L^2}\delta^{-1-\frac{4}{N}}$. By the preceding estimate and Theorem 1.5, there exists a constant $C>0$ independent of δ such that $X^2\leq C(1+X)$. This implies that X is bounded independently of δ, *i.e.*

$$\|\varphi_\delta\|_{L^2}=\mathcal{O}\left(\delta^{1+\frac{4}{N}}\right).\tag{3.24}$$

By (3.23) again,

$$I(\delta)=\iint |u_\delta|^{2+\frac{4}{N}}=C_S\delta^{2+\frac{4}{N}}+\gamma D_N\delta^{2+\frac{8}{N}}+\mathcal{O}\left(\delta^{2+\frac{12}{N}}\right).\tag{3.25}$$

The proof is complete, except for the computation of D_N which is given in appendices B and C. Note that as announced in Remark 3.4, the estimate (3.24) improves the preceding estimate (3.9) on φ_δ. □

4. Uniqueness

In this section we show the uniqueness part of Theorem 1.1. We assume again

$$N \in \{1, 2\}.$$

By Proposition 2.1, there exists, for small $\delta > 0$, a maximizer for $I(\delta)$, *i.e.* a solution u_δ of (1.1) such that

$$\|f_\delta\|_{L^2} = \delta, \quad \iint |u_\delta|^{2+\frac{4}{N}} = I(\delta) \tag{4.1}$$

(as usual $f_\delta(x) = u_\delta(0, x)$). By Proposition 3.3 and Remark 3.4, assuming again that δ is small, any maximizer for $I(\delta)$ satisfies, after transformation, the following properties:

$$f_\delta = \alpha_\delta G_0 + \varphi_\delta, \tag{4.2}$$

where $\varphi_\delta \in L^2(\mathbb{R}^N)$ and $\alpha_\delta > 0$ are such that

$$\int \varphi G_0 = \int \varphi |x|^2 G_0 = 0, \quad \int \varphi x G_0 = 0_{\mathbb{R}^N}, \tag{4.3}$$

$$\|\varphi_\delta\|_{L^2} \le C\delta^{1+\frac{4}{N}}, \quad \alpha_\delta > 0 \quad \text{and} \quad |\alpha_\delta - \delta| \le C\delta^{1+\frac{4}{N}}. \tag{4.4}$$

We must show that if $C > 0$, there exists $\delta_0 > 0$ such that if $\delta \in (0, \delta_0)$, there is at most one solution u_δ of (1.1) satisfying (4.1), (4.2), (4.3) and (4.4).

Let us fix a small $\delta > 0$ and a maximizer u_δ satisfying (4.1), (4.2), (4.3) and (4.4). The strategy of the proof is to expand $\int |v|^{2+\frac{4}{n}}$, where v is a solution of (1.1) which is close to u_δ. In Section 4.1 we expand v and $\int |v|^{2+\frac{4}{n}}$ at first order, in Section 4.2 we obtain a second order expansion involving the quadratic form Q. Assuming that v is another maximizer, the conclusion will follow from Theorem 1.5.

4.1. Linearization

Lemma 4.1. *There exists a linear operator* $L_\delta : L_{t,x}^{2+\frac{4}{N}} \to L_{t,x}^{2+\frac{4}{N}}$ *such that*

$$\forall h \in L_{t,x}^{2+\frac{4}{N}}, \quad \|(L_\delta - 1)h\|_{L_{t,x}^{2+\frac{4}{N}}} \le C\delta^{\frac{4}{N}} \|h\|_{L_{t,x}^{2+\frac{4}{N}}}, \tag{4.5}$$

with the following property: if v is a solution of (1.1) *with the initial condition $f_\delta + \psi$, where*

$$\|\psi\|_{L^2} \le \delta, \tag{4.6}$$

then

$$\left\| v - u_\delta - L_\delta\left(e^{i\frac{t}{2}\Delta}\psi\right)\right\|_{L_{t,x}^{2+\frac{4}{N}}} \le C\delta^{\frac{4}{N}-1}\|\psi\|_{L^2}^2. \tag{4.7}$$

Proof. Let $w = v - u_\delta$. Then by Lemma 2.2,

$$\|w\|_{L_{t,x}^{2+\frac{4}{N}}} \le C\|\psi\|_{L^2}. \tag{4.8}$$

Writing Duhamel's formula for u_δ and $v = u_\delta + w$, we get

$$w = e^{i\frac{t}{2}\Delta}\psi + i\gamma\int_0^t e^{i\frac{(t-s)}{2}\Delta}\left(|u_\delta(s)+w(s)|^{\frac{4}{N}}(u_\delta(s)+w(s)) - |u_\delta(s)|^{\frac{4}{N}}u_\delta(s)\right)ds.$$

Expanding $|u_\delta(s)+w(s)|^{\frac{4}{N}}(u_\delta(s)+w(s))$, one can write the preceding equality as

$$w = e^{i\frac{t}{2}\Delta}\psi + \widetilde{L_\delta}w + \widetilde{R_\delta}(w), \tag{4.9}$$

where the linear operator $\widetilde{L_\delta} : L_{t,x}^{2+\frac{4}{N}} \to L_{t,x}^{2+\frac{4}{N}}$ satisfies

$$\left\|\widetilde{L_\delta}w\right\|_{L_{t,x}^{2+\frac{4}{N}}} \le C\delta^{\frac{4}{N}}\|w\|_{L_{t,x}^{2+\frac{4}{N}}}, \tag{4.10}$$

and $\widetilde{R_\delta}$ satisfies

$$\left\|\widetilde{R_\delta}(w)\right\| \le C\left(\delta^{\frac{4}{N}-1}\|w\|^2_{L_{t,x}^{2+\frac{4}{N}}} + \|w\|^{1+\frac{4}{N}}_{L_{t,x}^{2+\frac{4}{N}}}\right). \tag{4.11}$$

Letting for small δ

$$L_\delta = \left(1 - \widetilde{L_\delta}\right)^{-1},$$

we obtain by (4.10) that L_δ satisfies (4.5). The estimate (4.7) follows from (4.6), (4.8), (4.9) and (4.11). □

Lemma 4.2. *Let L_δ be as in Lemma* 4.1. *Then for small $\delta > 0$,*

$$\mathrm{Re}\iint |u_\delta|^{\frac{4}{N}}\overline{u_\delta}L_\delta\left(e^{i\frac{t}{2}\Delta}\psi\right) = \mu_\delta\,\mathrm{Re}\int\overline{f_\delta}\psi, \tag{4.12}$$

where $\mu_\delta > 0$, which depends only on u_δ, satisfies

$$\left|\mu_\delta - C_S\delta^{\frac{4}{N}}\right| \le C\delta^{\frac{8}{N}}. \tag{4.13}$$

Proof. Indeed, by definition

$$I(\delta) = \max\iint |v|^{2+\frac{4}{N}}, \tag{4.14}$$

where the maximum is taken over all solutions v of (1.1) with initial condition $f_\delta + \psi$, such that $\int |f_\delta + \psi|^2 = \delta^2$. For such a solution v, write, as in the proof of Lemma 4.1, $v = u_\delta + w$. Then

$$\begin{aligned}\iint |v|^{2+\frac{4}{N}} &= \iint |u_\delta + w|^{2+\frac{4}{N}}\\ &= \iint |u_\delta|^{2+\frac{4}{N}} + \left(2+\frac{4}{N}\right)\operatorname{Re}\iint |u_\delta|^{\frac{4}{N}}\overline{u_\delta}w + \mathcal{O}\left(\delta^{\frac{4}{N}}\|\psi\|_{L^2}^2\right)\\ &= \iint |u_\delta|^{2+\frac{4}{N}} + \left(2+\frac{4}{N}\right)\operatorname{Re}\iint |u_\delta|^{\frac{4}{N}}\overline{u_\delta}L_\delta\left(e^{i\frac{t}{2}\Delta}\psi\right)\\ &\quad + \mathcal{O}\left(\delta^{\frac{4}{N}}\|\psi\|_{L^2}^2\right).\end{aligned}$$

The existence of μ_δ then follows from the Lagrange multiplier equation for the maximizing problem (4.14).

We next estimate μ_δ. By (4.2) and (4.4)

$$f_\delta = \delta G_0 + \mathcal{O}\left(\delta^{1+\frac{4}{N}}\right) \text{ in } L^2.$$

Thus by Claim 2.10,

$$u_\delta = \delta G + \mathcal{O}\left(\delta^{1+\frac{4}{N}}\right) \text{ in } L^{2+\frac{4}{N}}_{t,x}. \tag{4.15}$$

As a consequence, we obtain (assuming $\|\psi\|_{L^2} \le \delta$)

$$\begin{aligned}\operatorname{Re}\iint |u_\delta|^{\frac{4}{N}}\overline{u}_\delta L_\delta\left(e^{i\frac{t}{2}\Delta}\psi\right) &= \operatorname{Re}\iint |u_\delta|^{\frac{4}{N}}\overline{u}_\delta\, e^{i\frac{t}{2}\Delta}\psi + \mathcal{O}\left(\delta^{1+\frac{8}{N}}\|\psi\|_{L^2}\right)\\ &= \delta^{1+\frac{4}{N}}\operatorname{Re}\iint |G|^{\frac{4}{N}}\overline{G}\, e^{i\frac{t}{2}\Delta}\psi + \mathcal{O}\left(\delta^{1+\frac{8}{N}}\|\psi\|_{L^2}\right).\end{aligned}$$

On the other hand,

$$\operatorname{Re}\int \overline{f}_\delta\psi = \operatorname{Re}\int \alpha_\delta G_0\psi + \mathcal{O}\left(\delta^{1+\frac{4}{N}}\|\psi\|_{L^2}\right) = \delta\operatorname{Re}\int G_0\psi + \mathcal{O}\left(\delta^{1+\frac{4}{N}}\|\psi\|_{L^2}\right).$$

Combining with (4.12), we get

$$\delta^{1+\frac{4}{N}}\operatorname{Re}\iint |G|^{\frac{4}{N}}\overline{G}e^{i\frac{t}{2}\Delta}\psi = \delta\mu_\delta\operatorname{Re}\int G_0\psi + \mathcal{O}\left(\delta^{1+\frac{8}{N}}\|\psi\|_{L^2} + \mu_\delta\delta^{1+\frac{4}{N}}\|\psi\|_{L^2}\right).$$

By (3.1),

$$C_S\delta^{1+\frac{4}{N}}\operatorname{Re}\int G_0\psi = \delta\mu_\delta\operatorname{Re}\int G_0\psi + \mathcal{O}\left(\mu_\delta\delta^{1+\frac{4}{N}}\|\psi\|_{L^2} + \delta^{1+\frac{8}{N}}\|\psi\|_{L^2}\right).$$

This holds for all small $\psi \in L^2$, yielding (4.13). □

4.2. Second order expansion

Lemma 4.3. *Let v be a solution of* (1.1) *with initial condition $f_\delta + \psi$, and assume*

$$\int |f_\delta + \psi|^2 = \delta^2.$$

Then

$$\iint |v|^{2+\frac{4}{N}} = I(\delta) - \delta^{\frac{4}{N}} Q(\psi) + \mathcal{O}\left(\delta^{\frac{8}{N}} \|\psi\|_{L^2}^2 + \delta^{\frac{4}{N}-1} \|\psi\|_{L^2}^3 + \|\psi\|_{L^2}^{2+\frac{4}{N}}\right). \quad (4.16)$$

Proof. Using that $\int |f_\delta|^2 = \delta^2$, we get

$$\int |\psi|^2 = -2\,\mathrm{Re} \int \overline{f_\delta} \psi, \quad (4.17)$$

and thus by (4.2) and (4.4),

$$\delta^2 \left|\mathrm{Re} \int G_0 \psi\right|^2 \leq C\left(\delta^{\frac{8}{N}+2} \|\psi\|_{L^2}^2 + \|\psi\|_{L^2}^4\right). \quad (4.18)$$

Expanding $|u_\delta + w|^{2+\frac{4}{N}}$ at second order in w, we obtain

$$\begin{aligned}
\iint |v|^{2+\frac{4}{N}} &= \iint |u_\delta + w|^{2+\frac{4}{N}} \\
&= \iint |u_\delta|^{2+\frac{4}{N}} + \left(2 + \frac{4}{N}\right) \mathrm{Re} \iint |u_\delta|^{\frac{4}{N}} \overline{u}_\delta w \\
&\quad + \left(1 + \frac{2}{N}\right)^2 \iint |u_\delta|^{\frac{4}{N}} |w|^2 + \frac{2}{N}\left(1 + \frac{2}{N}\right) \mathrm{Re} \iint |u_\delta|^{\frac{4}{N}-2} \overline{u}_\delta^2 w^2 \\
&\quad + \mathcal{O}\left(\delta^{\frac{4}{N}-1} \|\psi\|_{L^2}^3\right) + \mathcal{O}\left(\|\psi\|_{L^2}^{2+\frac{4}{N}}\right).
\end{aligned}$$

By Lemma 4.1, Lemma 4.2 and (4.17),

$$\begin{aligned}
\mathrm{Re} \iint |u_\delta|^{\frac{4}{N}} \overline{u}_\delta w &= \mathrm{Re} \iint |u_\delta|^{\frac{4}{N}} \overline{u}_\delta L_\delta\left(e^{i\frac{t}{2}\Delta} \psi\right) + \mathcal{O}\left(\delta^{\frac{8}{N}} \|\psi\|_{L^2}^2\right) \\
&= \mu_\delta \int \overline{f}_\delta \psi + \mathcal{O}\left(\delta^{\frac{8}{N}} \|\psi\|_{L^2}^2\right) \\
&= -\frac{\mu_\delta}{2} \int |\psi|^2 + \mathcal{O}\left(\delta^{\frac{8}{N}} \|\psi\|_{L^2}^2\right) \\
&= -\frac{C_S}{2} \delta^{\frac{4}{N}} \int |\psi|^2 + \mathcal{O}\left(\delta^{\frac{8}{N}} \|\psi\|_{L^2}^2\right).
\end{aligned}$$

By (4.15), then Lemma 4.1,

$$\begin{aligned}\iint |u_\delta|^{\frac{4}{N}}|w|^2 &= \delta^{\frac{4}{N}} \iint |G|^{\frac{4}{N}}|w|^2 + \mathcal{O}\left(\delta^{\frac{8}{N}}\|\psi\|_{L^2}^2\right) \\ &= \delta^{\frac{4}{N}} \iint |G|^{\frac{4}{N}}\left|e^{i\frac{t}{2}\Delta}\psi\right|^2 + \mathcal{O}\left(\delta^{\frac{8}{N}}\|\psi\|_{L^2}^2\right),\end{aligned}$$

and similarly

$$\operatorname{Re}\iint |u_\delta|^{\frac{4}{N}-2}\overline{u}_\delta^2 w^2 = \delta^{\frac{4}{N}}\operatorname{Re}\iint |G|^{\frac{4}{N}-2}\overline{G}^2\left(e^{i\frac{t}{2}\Delta}\psi\right)^2 + \mathcal{O}\left(\delta^{\frac{8}{N}}\|\psi\|_{L^2}^2\right).$$

Combining the preceding estimates, we obtain

$$\begin{aligned}\iint |v|^{2+\frac{4}{N}} = \iint |u_\delta|^{2+\frac{4}{N}} &- C_S\left(\frac{N+2}{N}\right)\delta^{\frac{4}{N}}\int |\psi|^2 \\ &+ \left(1+\frac{2}{N}\right)^2\delta^{\frac{4}{N}}\iint |G|^{\frac{4}{N}}\left|e^{i\frac{t}{2}\Delta}\psi\right|^2 \\ &+ \frac{2}{N}\left(1+\frac{2}{N}\right)\delta^{\frac{4}{N}}\operatorname{Re}\iint |G|^{\frac{4}{N}-2}\overline{G}^2\left(e^{i\frac{t}{2}\Delta}\psi\right)^2 \\ &+ \mathcal{O}\left(\delta^{\frac{8}{N}}\|\psi\|_{L^2}^2 + \delta^{\frac{4}{N}-1}\|\psi\|_{L^2}^3 + \|\psi\|_{L^2}^{2+\frac{4}{N}}\right),\end{aligned}$$

which yields (4.16) in view of (4.18) and the definition (3.3) of Q. □

We can now conclude the proof of the uniqueness of the maximizer. Assume that $\delta > 0$ is small and consider a solution $\tilde{u}_\delta$ of (1.1) with initial condition $\tilde{f}_\delta = \tilde{\alpha}_\delta G_0 + \tilde{\varphi}_\delta$. Assume that $\tilde{u}_\delta$, $\tilde{f}_\delta$, $\tilde{\varphi}_\delta$ and $\tilde{\alpha}_\delta$ also satisfy (4.1), (4.2), (4.3) and (4.4). We must show that $\tilde{u}_\delta = u_\delta$. Let

$$\psi = (\tilde{\alpha}_\delta - \alpha_\delta)G_0 + \tilde{\varphi}_\delta - \varphi_\delta.$$

By (4.4), $\|\psi\|_{L^2} \le C\delta^{\frac{4}{N}+1}$. By Lemma 4.3 with $v = \tilde{u}_\delta$,

$$I(\delta) = \iint |v|^{2+\frac{4}{N}} = I(\delta) - \delta^{\frac{4}{N}}Q(\psi) + \mathcal{O}\left(\delta^{\frac{8}{N}}\|\psi\|_{L^2}^2 + \delta^{\frac{4}{N}-1}\|\psi\|_{L^2}^3 + \|\psi\|_{L^2}^{2+\frac{4}{N}}\right),$$

and thus,

$$\delta^{\frac{4}{N}}Q(\psi) \le C\left(\delta^{\frac{8}{N}}\|\psi\|_{L^2}^2 + \delta^{\frac{4}{N}-1}\|\psi\|_{L^2}^3 + \|\psi\|_{L^2}^{2+\frac{4}{N}}\right) \le C\delta^{\frac{8}{N}}\|\psi\|_{L^2}^2. \quad (4.19)$$

Since G_0 is in the kernel of Q, $Q(\psi) = Q(\tilde{\varphi}_\delta - \varphi_\delta)$. Using that φ_δ and $\tilde{\varphi}_\delta$ satisfy the orthogonality conditions (1.9), we deduce from Theorem 1.5:

$$c\|\tilde{\varphi}_\delta - \varphi_\delta\|^2 \le Q(\psi). \quad (4.20)$$

Using that

$$\alpha_\delta^2 + \int |\varphi_\delta|^2 = \delta^2 = \tilde{\alpha}_\delta^2 + \int |\tilde{\varphi}_\delta|^2,$$

we obtain, in view of (4.4),

$$|\tilde{\alpha}_\delta - \alpha_\delta| = \left| \frac{\tilde{\alpha}_\delta^2 - \alpha_\delta^2}{\alpha_\delta + \tilde{\alpha}_\delta} \right| \leq \frac{1}{\delta} \left| \int |\varphi_\delta|^2 - \int |\tilde{\varphi}_\delta|^2 \right| \leq C\delta^{\frac{4}{N}} \|\varphi_\delta - \tilde{\varphi}_\delta\|_{L^2},$$

and thus for small δ,

$$\|\psi\|_{L^2}^2 = (\tilde{\alpha}_\delta - \alpha_\delta)^2 + \|\varphi_\delta - \tilde{\varphi}_\delta\|_{L^2}^2 \leq 2\|\varphi_\delta - \tilde{\varphi}_\delta\|_{L^2}^2. \tag{4.21}$$

Combining (4.19), (4.20) and (4.21), we get

$$\delta^{\frac{4}{N}} \|\psi\|_{L^2}^2 \leq C\delta^{\frac{8}{N}} \|\psi\|_{L^2}^2,$$

a contradiction if $\delta > 0$ is small and $\psi \neq 0$. Thus, $\psi = 0$ and $u_\delta = \tilde{u}_\delta$, which completes the proof.

5. Coercivity of the quadratic form

In this section we show Theorem 1.5.

Let F be the $N+2$-dimensional space of the null directions for Q that are generated by the continuous symmetries of the linear Schrödinger equation:

$$F = \operatorname{span}_{\mathbb{C}}\{G_0, x_j G_0, |x|^2 G_0\}$$

($j = 1$ or $j = 1, 2$ in dimension 1 and 2 respectively).

We must show that there exists a constant $c > 0$ such that

$$\varphi \in F^\perp \Longrightarrow Q(\varphi) \geq c\|\varphi\|_{L^2}^2.$$

It turns out that F is generated by eigenfunctions for the harmonic oscillator defined in Section 5.1.1. Indeed, in dimension 1, F is spanned by h_0, h_1 and h_2 and in dimension 2 by h_{00}, h_{10}, h_{01} and $h_{20} + h_{02}$.

The outline of this section is as follows. In Section 5.1 we recall some properties of the harmonic oscillator $\mathcal{H} = -\Delta + |x|^2$ and of a lens transform that will be used in the proof. In Section 5.2 we show that the proof of Theorem 1.5 reduces to the proof that $Q(\varphi) > 0$ for any eigenfunction φ of the harmonic oscillator $\mathcal{H}$ that is orthogonal to F. In Section 5.3 and Section 5.4 we treat the reduced problem in $1D$ and $2D$ respectively by estimating the values taken by the quadratic form on the eigenfunctions of $\mathcal{H}$.

5.1. Preliminaries

5.1.1. *Harmonic oscillator*

Consider the linear Schrödinger equation with the harmonic potential:

$$i\partial_\tau u - \frac{1}{2}\mathcal{H}u = 0, \quad (\tau, y) \in \mathbb{R} \times \mathbb{R}^N, \quad \text{where } \mathcal{H} = -\Delta + |y|^2. \tag{5.1}$$

In what follows we briefly recall spectral property of $\mathcal{H}$. We refer to [5] and references therein for more details.

We first review the spectral properties of $\mathcal{H}$ in one space dimension. The spectrum of $\mathcal{H}$ consists of positive eigenvalues $\lambda_n = 2n + 1$, $n = 0, 1, \ldots$, and the corresponding eigenfunctions are

$$h_n(y) = (-1)^n\, c_n\, e^{y^2/2}\, \partial_y^n(e^{-y^2}), \quad c_n = \frac{1}{\sqrt{n!}\, 2^{n/2}}, \tag{5.2}$$

here the coefficients c_n are chosen so that $\|h_n\|^2_{L^2(\mathbb{R})} = \sqrt{\pi}$. Equivalently, these are the Hermite functions

$$h_n(y) = \frac{H_n(y)}{\sqrt{2^n\, n!}}\, e^{-y^2/2}, \tag{5.3}$$

with $H_n(y)$ being the n^{th} Hermite polynomial:

$$H_n(y) = (-1)^n\, e^{y^2}\, \partial_y^n(e^{-y^2}).$$

Thus, $H_0(y) = 1$, $H_1(y) = 2y$, $H_2(y) = 4y^2 - 2$, $H_3(y) = 8y^3 - 12y$, $H_4(y) = 16y^4 - 48y^2 + 12$, etc. These eigenfunctions are orthogonal

$$\int_{\mathbb{R}} h_j(y)\, h_k(y)\, dy = \frac{1}{\sqrt{2^j\, j!}\sqrt{2^k\, k!}} \int_{\mathbb{R}} H_j(y)\, H_k(y)\, e^{-y^2}\, dy = \sqrt{\pi}\, \delta_{jk}, \tag{5.4}$$

and they span $L^2(\mathbb{R})$.

In the $2D$ set up, $y = (y_1, y_2) \in \mathbb{R}^2$, the spectrum of $\mathcal{H}$ consists as well of a discrete set of positive eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ and, for $n \in \mathbb{N}$, one has

$$\lambda_n = 2n + 2.$$

To each eigenvalue λ_n there corresponds a set of eigenfunctions $h_{jk}(y)$ with the property that $j + k = n$ and $h_{jk}(y) = h_j(y_1)h_k(y_2)$, where the h_n's are the one-dimensional eigenfunctions. For example, $h_{00}(y) = e^{-|y|^2}$ is the only eigenfunction corresponding to the smallest eigenvalue $\lambda_0 = 2$. For $\lambda_1 = 4$, the eigenfunctions are

$$h_{10}(y) = \sqrt{2}\, y_1\, e^{-|y|^2/2} \quad \text{and} \quad h_{01}(y) = \sqrt{2}\, y_2\, e^{-|y|^2/2},$$

for $\lambda_2 = 6$, they are

$$h_{20}(y) = 2^{-1/2}(2y_1^2 - 1)e^{-|y|^2/2}, \quad h_{02}(y) = 2^{-1/2}(2y_2^2 - 1)e^{-|y|^2/2}$$

$$\text{and} \quad h_{11}(y) = 2y_1y_2e^{-|y|^2/2}.$$

5.1.2. *The Lens transform*

For a function $u(t,x) : I \times \mathbb{R}^N \to \mathbb{C}$, define the *lens transform*[1] $\text{Ł}u$ of u by

$$\text{Ł}u(\tau, y) = \frac{1}{\cos^{N/2}\tau}\, u\left(\tan\tau, \frac{y}{\cos\tau}\right) e^{-i|y|^2\frac{\tan\tau}{2}}.$$

The new variables (τ, y) are defined by $t = \tan\tau$ and $x = \frac{y}{\cos\tau}$, $\tau \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and thus, $\text{Ł}u : \tan^{-1}(I) \cap (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^N \to \mathbb{C}$. If $I = \mathbb{R}$, then $\text{Ł}u : (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}^N \to \mathbb{C}$: the lens transform compactifies the time. For more details see for example [4,28] and reference therein.

If $u(t,x)$ solves (1.1) (for some $\gamma \in \mathbb{R}$), then $v = \text{Ł}u(\tau, y)$ solves

$$i\partial_\tau v - \frac{1}{2}\mathcal{H}v = -\gamma|v|^{\frac{4}{N}}v, \tag{5.5}$$

and vice versa.

The lens transform preserves the initial data $(\text{Ł}u)(0) = u(0)$, and thus, the mass of the solution:

$$\|(\text{Ł}u)(0)\|_{L^2} = \|u(0)\|_{L^2}.$$

Furthermore, all Strichartz norms are also preserved, in particular:

$$\|\text{Ł}u\|_{L_{t,x}^{\frac{4}{N}+2}((-\pi/2,\pi/2)\times\mathbb{R}^N)} = \|u\|_{L_{t,x}^{\frac{4}{N}+2}(\mathbb{R}\times\mathbb{R}^N)}.$$

Example 5.1. Let $G_0 = \frac{1}{\pi^{N/4}}e^{-|x|^2/2}$. The solution to the linear Schrödinger equation (1.2) is given by (1.6). The definition of Ł shows that the solution $e^{-i\frac{\tau}{2}\mathcal{H}}G_0$ of (5.1) is given by

$$\widetilde{G}(\tau, y) = \frac{1}{\pi^{N/4}}\, e^{-i\frac{N}{2}\tau}\, e^{-|y|^2/2} = (\text{Ł}G)(\tau, y),$$

which is consistent with the fact that G_0 is an eigenfunction for the eigenvalue $\lambda_0 = N$ of $\mathcal{H}$ (in dimension $N = 1, 2$).

[1] We use the name 'lens transform' as in [28] but it should not be confused with the pseudo-conformal inversion (1.5) of Talanov which is sometimes also called the lens transform.

For later use we note that using the invariance of the initial condition and the $L^{\frac{4}{N}+2}$ norm by the lens transform Ł, we can rewrite the definition (3.3) of the quadratic form as

$$\begin{aligned} Q(\varphi) = C_S \left[\frac{N+2}{N} \int |\varphi|^2 + \frac{4(N+2)}{N^2} \left(\operatorname{Re} \int G_0 \varphi \right)^2 \right] \\ - \frac{(N+2)^2}{N^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}^N} G_0^{\frac{4}{N}} \left| e^{-i\frac{\tau}{2}\mathcal{H}} \varphi \right|^2 \\ - \frac{2(N+2)}{N^2} \operatorname{Re} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{\mathbb{R}^N} G_0^{\frac{4}{N}} e^{iN\tau} \left(e^{-i\frac{\tau}{2}\mathcal{H}} \varphi \right)^2 . \end{aligned} \tag{5.6}$$

5.2. Reduction of the problem

We prove here the following proposition:

Proposition 5.2. *Assume that the conclusion of Theorem* 1.5 *does not hold. Then there exists an eigenfunction φ of $\mathcal{H}$, satisfying the orthogonality relations* (1.9) *and such that $Q(\varphi) = 0$.*

We define
$$E = \{\varphi \in L^2, \quad Q(\varphi) = 0\}.$$
Since Q is a real positive quadratic form, we know that E is a real vector space. Before proving Proposition 5.2, we need a few preliminary results.

Lemma 5.3. *Let $\{\varphi_n\}$ be a* bounded *sequence in L^2 such that*
$$\lim_{n\to\infty} Q(\varphi_n) = 0. \tag{5.7}$$
Then there exists a subsequence of $\{\varphi_n\}$ that converges strongly in L^2 to an element of E.

Proof. Assume after extraction,
$$\varphi_n \rightharpoonup \varphi \text{ weakly in } L^2 \text{ as } n \to \infty.$$
Write
$$Q(\varphi) = c_Q \int |\varphi|^2 + B(\varphi, \varphi), \tag{5.8}$$
where $c_Q = C_S \frac{N+2}{N}$ and the symmetric bilinear form B is defined by

$$\begin{aligned} B(\varphi, \psi) = C_S \frac{4(N+2)}{N^2} \left(\operatorname{Re} \int G_0 \varphi \right) \left(\operatorname{Re} \int G_0 \psi \right) \\ - \frac{(N+2)^2}{N^2} \operatorname{Re} \int_{\mathbb{R}} \int_{\mathbb{R}^N} |G|^{\frac{4}{N}} \left(e^{i\frac{t}{2}\Delta} \varphi \right) \left(e^{-i\frac{t}{2}\Delta} \overline{\psi} \right) \\ - \frac{2(N+2)}{N^2} \operatorname{Re} \int_{\mathbb{R}} \int_{\mathbb{R}^N} G^{\frac{4}{N}} \left(e^{i\frac{t}{2}\Delta} \varphi \right) \left(e^{i\frac{t}{2}\Delta} \psi \right) . \end{aligned}$$

We will use the following standard property of the Schrödinger linear flow:

Claim 5.4.

$$\psi_n \rightharpoonup 0 \text{ weakly in } L^2 \Longrightarrow e^{i\frac{t}{2}\Delta}\psi_n \to 0 \text{ strongly in } L^2_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^N).$$

Indeed, by the local smoothing effect [7, 23, 32], $e^{i\frac{t}{2}\Delta}$ defines a continuous map from $L^2(\mathbb{R}^N)$ to $L^2\left(\mathbb{R}, H^{1/2}_{\mathrm{loc}}(\mathbb{R}^N)\right)$. Using the equation (1.2), we see that it also defines a continuous map from $L^2(\mathbb{R}^N)$ to $H^{1/4}_{\mathrm{loc}}\left(\mathbb{R}^{N+1}\right)$. The claim follows from the local compactness of the embedding of $H^{1/4}$ in L^2.

Combining Claim 5.4 with the decay of G at infinity, we get

$$\psi_n \rightharpoonup 0 \text{ weakly in } L^2 \Longrightarrow B(\psi_n,\psi_n) \to 0. \tag{5.9}$$

We will show by contradiction that $\{\varphi_n\}$ is a Cauchy sequence in L^2. If not, there exist sequences of integer $\{j_n\}$, $\{k_n\}$ that go to ∞ such that

$$\forall n, \quad \|\varphi_{k_n} - \varphi_{j_n}\|_{L^2} \geq \varepsilon_0 > 0. \tag{5.10}$$

The weak convergence of $\{\varphi_n\}$ in L^2 implies

$$\varphi_{k_n} - \varphi_{j_n} \rightharpoonup 0 \text{ weakly in } L^2. \tag{5.11}$$

Furthermore, (5.7) and Cauchy-Schwarz inequality (Q is positive) implies

$$0 \leq Q(\varphi_{j_n} - \varphi_{k_n}) \leq 2\left(Q(\varphi_{j_n}) + Q(\varphi_{k_n})\right) \longrightarrow 0 \text{ as } n\to\infty.$$

Combining with (5.9) and (5.11) one gets

$$\lim_{n\to\infty} \|\varphi_{j_n} - \varphi_{k_n}\|_{L^2} = 0,$$

contradicting (5.10). The proof is complete. □

Lemma 5.5. *The space E is a finite dimensional vector space over $\mathbb{C}$.*

Proof. The space E is a vector space over $\mathbb{R}$. To show that it is a vector space over $\mathbb{C}$, it is sufficient to show that it is stable by multiplication by i. Let $\varphi \in E$. Write $\varphi = \alpha G_0 + \tilde{\varphi}$, with $\alpha = \int \varphi G_0$, so that

$$\int \tilde{\varphi} G_0 = 0. \tag{5.12}$$

The function $i\alpha G$ is in E and E is stable by addition. To show that $i\varphi \in E$ we must show that $i\tilde{\varphi} \in E$. By (5.6),

$$\begin{aligned} Q(i\tilde{\varphi}) = Q(\tilde{\varphi}) - \frac{8(N+2)}{N^2} C_S \left(\operatorname{Re}\int G_0\tilde{\varphi}\right)^2 \\ + \frac{4(N+2)}{N}\operatorname{Re}\int_{-\pi/2}^{\pi/2}\int_{\mathbb{R}^N} G_0^{\frac{4}{N}} e^{iN\tau}\left(e^{-i\frac{\tau}{2}\mathcal{H}}\tilde{\varphi}\right)^2. \end{aligned}$$

We know that $\tilde{\varphi} \in E$, so $Q(\tilde{\varphi}) = 0$ and it suffices to show:

$$\left(\mathrm{Re} \int G_0 \tilde{\varphi}\right)^2 = 0 \tag{5.13}$$

$$\mathrm{Re} \int_{-\pi/2}^{\pi/2} \int_{\mathbb{R}^N} G_0^{\frac{4}{N}} e^{iN\tau} \left(e^{-i\frac{\tau}{2}\mathcal{H}} \tilde{\varphi}\right)^2 = 0. \tag{5.14}$$

The first equality follows immediately from (5.12). Let us show the second equality in the case $N = 2$. By (5.12), $\tilde{\varphi}$ is orthogonal to the first eigenfunction h_{00} of $\mathcal{H}$. Thus, $e^{-i\frac{\tau}{2}\mathcal{H}} \tilde{\varphi}$ is of the form

$$e^{-i\frac{\tau}{2}\mathcal{H}} \tilde{\varphi} = \sum_{\substack{(n_1,n_2)\in\mathbb{N}^2 \\ n_1+n_2\geq 1}} \alpha_{n_1 n_2} e^{-i\tau(n_1+n_2+1)} h_{n_1 n_2}(y),$$

where by definition $\alpha_{n_1 n_2} = \int_{\mathbb{R}^2} \tilde{\varphi}(y) h_{n_1 n_2}(y) dy$. It follows from the definition of $h_{n_1 n_2}$ that it is even if $n_1 + n_2$ is even and odd if $n_1 + n_2$ is odd. Expanding $\left(e^{-i\frac{\tau}{2}\mathcal{H}} \tilde{\varphi}\right)^2$, we can write

$$\mathrm{Re} \int_{-\pi/2}^{\pi/2} \int_{\mathbb{R}^2} G_0^2 e^{2i\tau} \left(e^{-i\frac{\tau}{2}\mathcal{H}} \tilde{\varphi}\right)^2 = \mathrm{Re} \int_{-\pi/2}^{\pi/2} \int_{\mathbb{R}^2} G_0^2 e^{2i\tau} \sum_{m\geq 4} e^{-i\tau m} g_m(y)\, dy\, d\tau,$$

where $m \geq 4$ and $g_m \in C^\infty\left(\mathbb{R}^N\right)$ is exponentially decaying. Again, g_m is even if m is even and odd if m is odd. Then (5.14) will follow from

$$\mathrm{Re} \int_{-\pi/2}^{\pi/2} \int_{\mathbb{R}^2} G_0^2(x) e^{i2\tau} e^{-i\tau m} g_m(y)\, dy\, d\tau = 0. \tag{5.15}$$

We distinguish two cases. If m is odd, then $\int_{\mathbb{R}^2} G_0(y)^2 g_m(y)\, dy = 0$ (it is the integral of an odd function on $\mathbb{R}^2$), and (5.15) follows. If m is even, using that $m \geq 4$, we get that $\int_{-\pi/2}^{\pi/2} e^{2i\tau - i\tau m}\, d\tau = 0$, which implies also (5.15). This completes the proof of (5.14) in the case $N = 2$. To prove (5.14) in the case $N = 1$ write

$$e^{-i\frac{\tau}{2}\mathcal{H}} \tilde{\varphi}_0 = \sum_{n\geq 1} \alpha_n e^{-i\tau\left(n+\frac{1}{2}\right)} h_n(y),$$

and argue as above. We leave the details to the reader.

It follows immediately from Lemma 5.3 that the unit ball of $\left(E, \|\cdot\|_{L^2}\right)$ is compact, concluding the proof of Lemma 5.5. □

We next prove Proposition 5.2. Let $\widetilde{E} = F^\perp \cap E$. By definition, $\widetilde{E}$ is the subspace of functions $\varphi \in L^2$ satisfying $Q(\varphi) = 0$ and the orthogonality relations (1.9). By Lemma 5.5 it is a complex, finite dimensional vector space.

We argue by contradiction, assuming that the conclusion of Theorem 1.5 does not hold.

Step 1. *Existence of a nontrivial null-space for Q.* In this step we show that the negation of Theorem 1.5 implies that $\widetilde{E}$ is not reduced to $\{0\}$. Indeed, in this case, there exists a sequence φ_n in L^2 such that

$$\forall n, \quad \varphi_n \in F^{\perp} \text{ and } n Q(\varphi_n) < \|\varphi_n\|_{L^2} = 1. \tag{5.16}$$

By Lemma 5.3, a subsequence of $\{\varphi_n\}_n$ converges strongly in L^2 to some $\psi \in E$. The condition $\|\varphi_n\|_{L^2} = 1$ implies that $\|\psi\|_{L^2} = 1$ and, in particular, that $\psi \neq 0$. Furthermore, $\varphi_n \in F^{\perp}$ for all n and $F^{\perp}$ is closed, thus, $\psi \in F^{\perp}$, which shows as announced that $\dim \widetilde{E} \geq 1$.

Step 2. *Stability by the harmonic evolution.* In this step we show that $\widetilde{E}$ is invariant by $e^{-i\frac{\tau_0}{2}\mathcal{H}}$ for any $\tau_0 \in \mathbb{R}$. As $\widetilde{E}$ is a complex vector space, it is equivalent to show that $\widetilde{E}$ is invariant by $S(t_0) = e^{-i\frac{\mathcal{H}-N}{2}\tau_0}$. The space F admits a basis of eigenfunctions of $\mathcal{H}$, thus $F^{\perp}$ is stable by $S(\tau_0)$. To prove that E is stable by $S(\tau_0)$, we rewrite the equation (3.2) using the lens transform of Section 5.1.2

$$\begin{aligned} C_S \left(\int_{\mathbb{R}^N} |G_0 + \varphi|^2 \right)^{1+\frac{2}{N}} - \int_{-\pi/2}^{\pi/2} \int_{\mathbb{R}^N} \left| e^{-i\frac{N\tau}{2}} G_0 + e^{-i\frac{\tau}{2}\mathcal{H}} \varphi \right|^{2+\frac{4}{N}} dy\, d\tau \\ = Q(\varphi) + \mathcal{O}\left(\|\varphi\|_{L^2}^3 \right). \end{aligned} \tag{5.17}$$

We will show that the two terms in the first line of (5.17) do not change when replacing φ by $S(\tau_0)\varphi$, which will imply that

$$Q(S(\tau_0)\varphi) = Q(\varphi), \tag{5.18}$$

and thus, that E and $\widetilde{E} = E \cap F$ are stable by $S(\tau_0)$).

By mass conservation

$$\begin{aligned} \int_{\mathbb{R}^N} |G_0 + S(\tau_0)\varphi|^2 &= \int_{\mathbb{R}^N} \left| e^{-i\frac{N\tau_0}{2}} G_0 + e^{-i\frac{\tau_0}{2}\mathcal{H}} \varphi \right|^2 \\ &= \int \left| e^{-i\frac{\tau_0}{2}\mathcal{H}} (G_0 + \varphi) \right|^2 = \int |G_0 + \varphi|^2 . \end{aligned} \tag{5.19}$$

Similarly,

$$\begin{aligned} &\int_{-\pi/2}^{\pi/2} \int_{\mathbb{R}^N} \left| e^{-i\frac{N\tau}{2}} G_0 + e^{-i\frac{\tau}{2}\mathcal{H}} S(\tau_0) \varphi \right|^{2+\frac{4}{N}} \\ &= \int_{-\pi/2+\tau_0}^{\pi/2+\tau_0} \int_{\mathbb{R}^N} \left| e^{-i\frac{\tau}{2}\mathcal{H}} (G_0 + \varphi) \right|^{2+\frac{4}{N}} \\ &= \int_{-\pi/2}^{\pi/2} \int_{\mathbb{R}^N} \left| e^{-i\frac{\tau}{2}\mathcal{H}} (G_0 + \varphi) \right|^{2+\frac{4}{N}} . \end{aligned}$$

The last equality is consequence of the following known identity (see *e.g.* equality (2.5) in [5]), which can be easily checked by expanding φ in the Hilbert basis of L^2 given by the eigenfunctions of $\mathcal{H}$:

$$e^{-i\frac{\pi+\tau}{2}\mathcal{H}}\varphi(y) = e^{-iN\frac{\pi}{2}}e^{-i\frac{\tau}{2}\mathcal{H}}\varphi(-y).$$

This concludes the proof of (5.18).

Step 3. *End of the proof.* We have shown that $e^{-i\frac{\tau}{2}\mathcal{H}}$ is a strongly continuous group of operators on the finite dimensional vector space $\widetilde{E}$. As a consequence, $e^{-i\frac{\tau}{2}\mathcal{H}} = e^{\tau A}$ for some $A \in \mathcal{L}(\widetilde{E})$ (see for example [10, Theorem 2.9, page 11]).

Let $f \in \widetilde{E}$. Then

$$\lim_{\tau\to 0}\frac{e^{-i\frac{\tau}{2}\mathcal{H}}f - f}{\tau} = \lim_{\tau\to 0}\frac{e^{\tau A}f - f}{\tau} = Af.$$

This shows that f is in the domain of $\mathcal{H}$ and that $Af = -\frac{i}{2}\mathcal{H}f$. As a consequence, $\mathcal{H} = 2iA$ is a continuous linear operator on $\widetilde{E}$. Using that $\widetilde{E}$ is finite dimensional, we deduce that $\mathcal{H}$ admits an eigenfunction in $\widetilde{E}$, concluding the proof of Proposition 5.2. □

From now on we treat each dimension separately.

5.3. 1D case

In this case, the quadratic form is

$$\begin{aligned} Q(\varphi) = \sqrt{3}\int|\varphi|^2\,dy &+ \frac{4\sqrt{3}}{\sqrt{\pi}}\left(\operatorname{Re}\int e^{-y^2/2}\varphi(y)\,dy\right)^2 \\ &- \frac{9}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int e^{-2y^2}\left|e^{-i\frac{\tau}{2}\mathcal{H}}\varphi\right|^2\,dy\,d\tau \\ &- \frac{6}{\pi}\operatorname{Re}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int e^{-2y^2}e^{i\tau}\left(e^{-i\frac{\tau}{2}\mathcal{H}}\varphi\right)^2\,dy\,d\tau. \end{aligned} \tag{5.20}$$

Recall that h_0 is the 0th Hermite function (the eigenfunction corresponding to $\lambda_0 = 1$), and $e^{-i\frac{\tau}{2}\mathcal{H}}h_0 = e^{-i\frac{\tau}{2}}\,e^{-y^2/2}$. Similarly,

$$h_1(y) = \sqrt{2}\,ye^{-y^2/2} \quad \rightsquigarrow \quad e^{-i\frac{\tau}{2}\mathcal{H}}h_1(y) = \sqrt{2}\,e^{-\frac{3}{2}i\tau}\,y\,e^{-y^2/2},$$

and

$$h_2(y) = \frac{1}{\sqrt{2}}(2y^2-1)e^{-y^2/2} \quad \rightsquigarrow \quad e^{-i\frac{\tau}{2}\mathcal{H}}h_2(y) = \frac{1}{\sqrt{2}}e^{-\frac{5}{2}i\tau}\,(2y^2-1)\,e^{-y^2/2},$$

then it is easy to check that

$$Q(h_0) = Q(ih_0) = Q(h_1) = Q(ih_1) = Q(h_2) = Q(ih_2) = 0.$$

Note that for the rest of h_j, $j \geq 3$, we have $e^{-i\frac{\tau}{2}\mathcal{H}}h_j = e^{-i(2j+1)\frac{\tau}{2}}h_j$, and when computing the quadratic form $Q(h_j)$, we obtain that by orthogonality of $\{h_j\}$ the second term in (5.20) is zero. Integration in t over the full circle makes the fourth term vanish, therefore producing

$$Q(h_j) = \sqrt{3}\int |h_j(y)|^2\,dy - 9\int e^{-2y^2}|h_j(y)|^2\,dy.$$

Since e^{-2y^2} is dominated by e^{-y^2}, we estimate the second term by

$$\int e^{-y^2}|h_j(y)|^2\,dy = \frac{(2j)!}{2^{2j}(j!)^2}\sqrt{\frac{\pi}{2}},$$

(see [33, Lemma 2.1]). Then, using the following estimate for the central binomial coefficient

$$\binom{2m}{m} \leq \frac{4^m}{\sqrt{3m+1}}, \quad m \geq 1, \tag{5.21}$$

we obtain

$$\begin{aligned} Q(h_j) &\geq \sqrt{3\pi}\left(1 - 3\sqrt{\frac{3}{2}}\frac{(2j)!}{2^{2j}(j!)^2}\right) \\ &\geq \sqrt{3\pi}\left(1 - \frac{3\sqrt{3}}{\sqrt{2}\sqrt{3j+1}}\right) > 0, \end{aligned}$$

for $j > 4$. Explicit computation shows that

$$Q(h_3) = \frac{2\sqrt{\pi}}{3\sqrt{3}} \quad \text{for} \quad h_3(y) = \frac{1}{\sqrt{3}}(2y^3 - 3y)e^{-y^2/2}$$

and

$$Q(h_4) = \frac{8\sqrt{\pi}}{9\sqrt{3}} \quad \text{for} \quad h_4(y) = \frac{1}{2\sqrt{6}}(4y^4 - 12y^2 + 3)e^{-y^2/2},$$

concluding the proof that $Q(h_j) > 0$ for all $j \geq 3$.

5.4. 2D case

Recall from Section 5.1.1 the definitions of the basis h_{jk} of eigenfunctions of $\mathcal{H}$. By definition $h_{jk}(y) = h_j(y_1)h_k(y_2)$, where $\{h_j\}_{j\geq 0}$ is the orthogonal system in $L^2(\mathbb{R})$ of eigenfunctions of the $1D$ harmonic oscillator. The function h_{jk} corresponds to the eigenvalue λ_m with $m = j+k$, and $\lambda_m = 2m+2 = 2(j+k)+2$. For a fixed m there are $m+1$ independent eigenfunctions $h_{jk} \equiv h_{j,m-j}$, $0 \leq j \leq m$, corresponding to λ_m. The space F is exactly

$$F = \operatorname{span}_{\mathbb{C}} \left\{h_{00}, h_{01}, h_{10}, h_{02} + h_{20}\right\}.$$

By Proposition 5.2, the proof of Theorem 1.5 in $2D$ is reduced to the following:

Proposition 5.6. *Assume that $N = 2$. Then*

$$\textit{If } \alpha \neq \beta \textit{ or } \gamma \neq 0, \quad Q\left(\alpha h_{02} + \beta h_{20} + \gamma h_{11}\right) > 0. \tag{5.22}$$

$$\textit{If } m \geq 3 \textit{ and } \sum_{j=0}^{m} |\alpha_j|^2 \neq 0, \textit{ then } Q\left(\sum_{j=0}^{m} \alpha_j h_{j,m-j}\right) > 0. \tag{5.23}$$

Proof. Let $\varphi \in L^2$. By (5.6) with $N = 2$, we have

$$\begin{aligned} Q(\varphi) = &\int_{\mathbb{R}^2} |\varphi|^2 + 2\left(\operatorname{Re}\int G_0\varphi\right)^2 \\ &- 4\int_{-\pi/2}^{\pi/2}\int_{\mathbb{R}^2} G_0^2 \left|e^{-i\frac{\tau}{2}\mathcal{H}}\varphi\right|^2 \\ &- 2\operatorname{Re}\int_{-\pi/2}^{\pi/2}\int_{\mathbb{R}^2} G_0^2 e^{i2\tau}\left(e^{-i\frac{\tau}{2}\mathcal{H}}\varphi\right)^2. \end{aligned} \tag{5.24}$$

It is easy to check that $Q(h_{00}) = 0$.

Let $m \geq 1$. Any eigenfunction of $\mathcal{H}$ for the eigenvalue $2m+2$ is of the form

$$\varphi = \sum_{j=0}^{m} \alpha_j h_{j,m-j}. \tag{5.25}$$

If φ is of this form, then the second integral in $Q(\varphi)$ vanishes because of the orthogonality of the h_{jk}'s and so does the last term, since $\int_{-\pi/2}^{\pi/2} e^{i2mt}\,dt = 0$ as $m \in \mathbb{N} \setminus \{0\}$.

Recall that the first eigenfunction for $\mathcal{H}$ is $h_{00}(y) = e^{-\frac{1}{2}|y|^2}$. Using that $G_0 = \frac{1}{\sqrt{\pi}} e^{-\frac{|y|^2}{2}}$, we obtain

$$Q(\varphi) = \mathcal{B}(\varphi,\varphi), \quad \mathcal{B}(\varphi,\psi) = \operatorname{Re}\int_{\mathbb{R}^2} \varphi\overline{\psi} - 4\operatorname{Re}\int_{\mathbb{R}^2} h_{00}^2\varphi\overline{\psi}. \tag{5.26}$$

In particular, if $j + k \geq 1$,

$$\begin{aligned} Q(h_{jk}) &= \left(\int h_j^2(y_1)\,dy_1\right)\left(\int h_k^2(y_2)\,dy_2\right) \\ &\quad - 4\left(\int e^{-y_1^2} h_j^2(y_1)\,dy_1\right)\left(\int e^{-y_2^2} h_k^2(y_2)\,dy_2\right) \\ &= \pi\left(1 - \frac{(2j)!(2k)!}{2^{2(j+k)-1}\,(j!)^2\,(k!)^2}\right), \end{aligned}$$

where in the last line we used the product of Hermite functions from [33, Lemma 2.1]. As expected we get $Q(h_{01}) = Q(h_{10}) = 0$.

Define

$$G(j,k) = \begin{cases} \dfrac{(j+k)!}{2^{(j+k)-1/2}\sqrt{j!}\sqrt{k!}\left(\frac{j+k}{2}\right)!} & \text{for} \quad j+k-\text{even}, \\ 0 & \text{for} \quad j+k-\text{odd}. \end{cases}$$

For a product of two G functions, write

$$F(m, j, k) = G(j, k)G(m - j, m - k).$$

Observe that F is symmetric, *i.e.*,

$$F(m, j, k) = F(m, k, j) = F(m, m - j, m - k) = F(m, m - k, m - j).$$

Note as well that

$$Q(h_{j,m-j}) = \pi\,(1 - F(m, j, j)), \qquad j \neq k \Longrightarrow \mathcal{B}\left(h_{j,m-j}, h_{k,m-k}\right) = \pi F(m, j, k),$$

and that for $\alpha, \beta, \gamma \in \mathbb{C}$

$$\frac{1}{\pi} Q(\alpha h_{02} + \beta h_{20} + \gamma h_{11}) = \frac{1}{4}|\alpha - \beta|^2 + \frac{1}{2}|\gamma|^2,$$

which is equal to zero if and only if $\alpha = \beta$ and $\gamma = 0$. This shows (5.22).

Let us show (5.23).

We have

$$\begin{aligned}
&\frac{1}{\pi}\, Q\left(\sum_{j=0}^{m} \alpha_j h_{j,m-j}\right) \\
&= \sum_{j=0}^{m} |\alpha_j|^2 \,(1 - F(m, j, j)) - 2\left(\operatorname{Re} \sum_{\substack{j<k,\\ j+k-\text{even}}} \alpha_j \overline{\alpha}_k F(m, j, k)\right) \\
&\geq \sum_{j=0}^{m} |\alpha_j|^2 - \left(\sum_{j=0}^{m} |\alpha_j|^2\, F(m, j, j) + \sum_{\substack{j<k,\\ j+k-\text{even}}} (|\alpha_j|^2 + |\alpha_k|^2) F(m, j, k)\right) \\
&\geq \sum_{j=0}^{m} |\alpha_j|^2 \left(1 - \sum_{\substack{k\in[0,m],\\ j+k-\text{even}}} F(m, j, k)\right),
\end{aligned}$$

where we used the symmetry of F in the last line. By Cauchy-Schwarz, for any $j \in [0, m]$ we obtain

$$
\begin{aligned}
\mathcal{F}(m,j) &:= \sum_{\substack{k\in[0,m],\\ j+k-\text{even}}} F(m,j,k)\\
&= \frac{2}{2^{2m}} \sum_{\substack{k\in[0,m],\\ j+k-\text{even}}} \frac{(j+k)!(2m-(j+k))!}{\sqrt{j!k!(m-j)!(m-k)!}\left(\frac{j+k}{2}\right)!\left(m-\frac{j+k}{2}\right)!}\\
&\leq \frac{2}{2^{2m}} \left(\sum_{\substack{k\in[0,m],\\ j+k-\text{even}}} \binom{j+k}{k}\binom{2m-(j+k)}{m-k} \right)^{1/2}\\
&\qquad \times \left(\sum_{\substack{k\in[0,m],\\ j+k-\text{even}}} \binom{j+k}{\frac{j+k}{2}}\binom{2m-(j+k)}{m-\frac{j+k}{2}} \right)^{1/2}\\
&\leq \frac{2}{4^m}\, \mathrm{I} \times \mathrm{II}.
\end{aligned}
$$

By elementary combinatorial arguments (see Appendix D) and (5.21), we estimate the term I

$$
\begin{aligned}
\mathrm{I}^2 &\leq \frac{1}{2}\left[\binom{2m+1}{m+1} + \binom{2m}{m}\right] = \frac{1}{2}\left[\frac{m+1}{2m+2}\binom{2m+2}{m+1} + \binom{2m}{m}\right]\\
&< \frac{1}{2}\left[\frac{1}{2}\frac{4^{m+1}}{\sqrt{3(m+1)+1}} + \frac{4^m}{\sqrt{3m+1}}\right]\\
&= 4^m \left(\frac{1}{\sqrt{3m+4}} + \frac{1}{2\sqrt{3m+1}}\right).
\end{aligned}
$$

For the term II we use (5.21), then decompose into fractions:

$$
\begin{aligned}
\mathrm{II}^2 &\leq 4^m \sum_{\substack{k\in[0,m],\\ j+k-\text{even}}} \frac{1}{\sqrt{3\,(\frac{j+k}{2})+1}}\, \frac{1}{\sqrt{3\,(m-\frac{j+k}{2})+1}}\\
&= \frac{4^m}{\sqrt{3m+2}} \sum_{\substack{k\in[0,m],\\ j+k-\text{even}}} \left(\frac{1}{3\,(\frac{j+k}{2})+1} + \frac{1}{3\,(m-\frac{j+k}{2})+1}\right)^{1/2}.
\end{aligned}
$$

Using the inequality $\sqrt{a+b} \leq \sqrt{a}+\sqrt{b}$, reindexing the summation and estimating the sum we obtain

$$\begin{aligned}
\mathrm{II}^2 &\le \frac{4^m}{\sqrt{3m+2}} \sum_{l=0}^{[m/2]} \left(\frac{1}{\sqrt{3l+1}} + \frac{1}{\sqrt{3(m-l)+1}} \right) \\
&= \frac{4^m}{\sqrt{3m+2}} \left(\sum_{l=0}^{m} \frac{1}{\sqrt{3l+1}} + \frac{1}{\sqrt{3\frac{m}{2}+1}}\, \chi_{\{m-\text{even}\}} \right) \\
&\le \frac{4^m}{\sqrt{3m+2}} \left(\frac{2}{3} (\sqrt{3m+1} - 1) + 1 + \frac{1}{\sqrt{1.5m+1}}\, \chi_{\{m-\text{even}\}} \right),
\end{aligned}$$

where $\chi_{\{m-\text{even}\}} = 1$ if m is even, 0 if m is odd. Hence,

$$\begin{aligned}
\mathcal{F}(m, j) \le 2 \Bigg[&\left(\frac{1}{\sqrt{3m+4}} + \frac{1}{2\sqrt{3m+1}} \right) \\
&\times \frac{1}{\sqrt{3m+2}} \left(\frac{2\sqrt{3m+1}+1}{3} + \frac{1}{\sqrt{1.5m+1}}\, \chi_{\{m-\text{even}\}} \right) \Bigg]^{1/2},
\end{aligned}$$

which is less than 1 for $m \ge 7$. For $m = 3, 4, 5, 6$ we provide the values of $\mathcal{F}(m, j)$ in Table 5.1 (which are all smaller than 1). □

$m = 3$	$\mathcal{F}(3,0)$	$\mathcal{F}(3,1)$	$\mathcal{F}(3,2)$	$\mathcal{F}(3,3)$			
	0.841	0.591	0.591	0.841			
$m = 4$	$\mathcal{F}(4,0)$	$\mathcal{F}(4,1)$	$\mathcal{F}(4,2)$	$\mathcal{F}(4,3)$	$\mathcal{F}(4,4)$		
	0.785	0.5	0.664	0.5	0.785		
$m = 5$	$\mathcal{F}(5,0)$	$\mathcal{F}(5,1)$	$\mathcal{F}(5,2)$	$\mathcal{F}(5,3)$	$\mathcal{F}(5,4)$	$\mathcal{F}(5,5)$	
	0.718	0.492	0.573	0.573	0.492	0.718	
$m = 6$	$\mathcal{F}(6,0)$	$\mathcal{F}(6,1)$	$\mathcal{F}(6,2)$	$\mathcal{F}(6,3)$	$\mathcal{F}(6,4)$	$\mathcal{F}(6,5)$	$\mathcal{F}(6,6)$
	0.673	0.454	0.563	0.495	0.563	0.454	0.673

Table 5.1. Values of $\mathcal{F}(m, j)$ for $3 \le m \le 6$.

Appendix

A. Implicit function theorem and orthogonality conditions

In this appendix we prove Claim 3.5. By explicit computation,

$$\nabla G_0 = -x G_0, \quad \Delta G_0 = (|x|^2 - N) G_0. \tag{A.1}$$

The preceding identities imply that at the point $(0, \Gamma_{id}, G_0)$:

$$\frac{\partial U_\delta}{\partial \theta_0} = -i\delta G_0, \qquad \frac{\partial U_\delta}{\partial \rho_0} = -\frac{N}{2}\delta G_0 - \delta x \cdot \nabla G_0 = -\frac{N}{2}\delta G_0 + \delta |x|^2 G_0,$$
$$\frac{\partial U_\delta}{\partial \xi_0} = -i\delta x G_0, \qquad \frac{\partial U_\delta}{\partial x_0} = -\nabla G_0 = \delta x G_0\,,$$
$$\frac{\partial U_\delta}{\partial t_0} = -\frac{i}{2}\delta \Delta G_0 - i\gamma \delta^{\frac{4}{N}+1}|G_0|^{\frac{4}{N}} G_0 = \frac{i}{2}\delta(N - |x|^2)G_0 - i\gamma\delta^{\frac{4}{N}+1}|G_0|^{\frac{4}{N}}G_0.$$

Using the equalities

$$\int G_0^2 = 1, \quad \int |x|^2 G_0^2 = \frac{N}{2}, \quad \int |x|^4 G_0^2 = \frac{N(N+2)}{4}, \tag{A.2}$$

which follow from the normalization of G_0 and (A.1), we get that the Jacobian $\left(\frac{\partial \Phi_\delta^k}{\partial \theta_0}, \frac{\partial \Phi_\delta^k}{\partial \rho_0}, \frac{\partial \Phi_\delta^k}{\partial \xi_0}, \frac{\partial \Phi_\delta^k}{\partial x_0}, \frac{\partial \Phi_\delta^k}{\partial t_0}\right)_{k=1\dots 5}$ of Φ_δ with respect to the variables $(\theta_0, \rho_0, \xi_0, x_0, t_0)$ at the point $(0, \Gamma_{id}, G_0)$ is of the form

$$\begin{pmatrix} -1 & 0 & 0 & 0 & \frac{1}{4} + \mathcal{O}(\delta^4) \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} + \mathcal{O}(\delta^4) \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} + \mathcal{O}(\delta^2) \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} + \mathcal{O}(\delta^2) \end{pmatrix}$$

in dimensions $N = 1$ or 2 respectively. Using that these matrices are invertible, and that their inverses may be estimated uniformly with respect to $\delta \in (0, \delta_0)$ (δ_0 small), we deduce from the implicit functions theorem that there exists $\varepsilon > 0$ and a constant $C > 0$ such that for small δ, if $\|f - G_0\|_{L^2} < \varepsilon$, there exists $(\theta_\delta, \rho_\delta, \xi_\delta, x_\delta, t_\delta) = (\theta_\delta, \Gamma_\delta)$ such that

$$|\theta_\delta| + |\rho_\delta - 1| + |\xi_\delta| + |x_\delta| + |t_\delta| \le C\|f - G_0\|_{L^2} \text{ and } \Phi_\delta(\theta_\delta, \Gamma_\delta, f) = 0.$$

Applying this to the family $\left\{\delta^{-1} g_\delta\right\}_\delta$ of Step 1 in the proof of Proposition 3.3, we get as announced that there exists $(\theta_\delta, \Gamma_\delta) = (\theta_\delta, \rho_\delta, \xi_\delta, x_\delta, t_\delta)$ such that

$$\lim_{\delta \to \infty} |\theta_\delta| + |\rho_\delta - 1| + |\xi_\delta| + |x_\delta| + |t_\delta| = 0 \text{ and } \Phi_\delta\left(\theta_\delta, \Gamma_\delta, \delta^{-1} g_\delta\right) = 0,$$

concluding the proof.

B. Constant in 1D and the generating function trick

By (3.16),

$$D_1 = 6\,\mathrm{Re} \iint |G(t)|^4 \overline{G}(t) r(t)\, dt\, dx,$$

where r is the solution to

$$i\partial_t r + \frac{1}{2}\Delta r + |G|^4 G = 0, \quad r(0,x) = 0.$$

Let Ł be the lens transform defined in Section 5.1.2. By the change of variable $t = \tan\tau$, $x = \frac{y}{\cos\tau}$, $\tau \in (-\pi/2, \pi/2)$, we get

$$D_1 = 6\,\mathrm{Re} \int_{\mathbb{R}} \int_{-\pi/2}^{\pi/2} |ŁG|^4\, \overline{ŁG}\, Łr\, d\tau\, dy.$$

By the example at the end of Section 5.1.2, $ŁG = \frac{1}{\pi^{1/4}} e^{-i\tau/2} e^{-y^2/2}$, and thus,

$$D_1 = \frac{6}{\pi^{5/4}} \mathrm{Re} \int_{\mathbb{R}} \int_{-\pi/2}^{\pi/2} e^{-5y^2/2} e^{i\tau/2} Łr\, d\tau\, dy. \tag{B.1}$$

Denote $\tilde{r} = Łr$. An explicit computation shows that $\tilde{r}$ solves

$$i\partial_\tau \tilde{r} - \frac{1}{2}\mathcal{H}\tilde{r} + \frac{1}{\pi^{\frac{5}{4}}} e^{-\frac{i}{2}\tau} e^{-\frac{5}{2}y^2} = 0, \quad \tilde{r}(0,y) = 0.$$

By Duhamel's formula

$$\tilde{r}(\tau, y) = \frac{i}{\pi^{5/4}} e^{-\frac{i\tau}{2}\mathcal{H}} \int_0^\tau e^{-\frac{i\sigma}{2}} e^{\frac{i\sigma}{2}\mathcal{H}} \left(e^{-\frac{5}{2}y^2}\right) d\sigma. \tag{B.2}$$

Decompose

$$e^{-\frac{5}{2}y^2} = \sum_{k\geq 0} \alpha_k h_k$$

with $\{h_k\}$'s as in (5.2) or (5.3), and

$$\mathcal{H} h_k = \lambda_k h_k \equiv (2k+1)\, h_k.$$

Then the coefficients α_k's are given by

$$\alpha_k = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-\frac{5}{2}y^2}\, h_k(y)\, dy. \tag{B.3}$$

Note that for k odd, the eigenfunction h_k is odd, and thus, the corresponding coefficient $\alpha_k = 0$. In the end of this appendix we compute the rest of (even) coefficients using a generating function trick of Wang [33] and obtain

$$\alpha_{2j} = (-1)^j \frac{\sqrt{(2j)!}}{3^j\sqrt{3}\, j!}. \tag{B.4}$$

Since

$$e^{\frac{1}{2}i\sigma\mathcal{H}}\left(e^{-\frac{5}{2}y^2}\right) = \alpha_0 e^{\frac{1}{2}i\sigma} h_0 + \sum_{k\geq 1} \alpha_k e^{i(k+\frac{1}{2})\sigma} h_k,$$

by (B.2) we have

$$\tilde{r}(\tau, y) = \frac{i}{\pi^{5/4}}\, e^{-i\frac{\tau}{2}} \left(\tau\alpha_0\, h_0(y) - i\sum_{k\geq 1}\frac{\alpha_k}{k}(1 - e^{-ik\tau})h_k(y)\right). \tag{B.5}$$

Substituting $\tilde{r}$ back into (B.1), we obtain that the zeroth term from (B.5) vanishes when integrating in τ, and thus,

$$\begin{aligned} D_1 &= \frac{6}{\pi^{\frac{5}{2}}}\operatorname{Re}\int\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-\frac{5}{2}y^2}\sum_{k\geq 1}\frac{\alpha_k}{k}(1 - e^{-ik\tau})h_k(y)\, d\tau\, dy \\ &= \frac{6}{\pi^{\frac{5}{2}}}\sum_{k\geq 1}\frac{\alpha_k}{k}\operatorname{Re}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(1 - e^{-ik\tau}\right) d\tau \int e^{-\frac{5}{2}y^2}\, h_k(y)\, dy \\ &= \frac{6}{\pi^{\frac{5}{2}}}\sum_{k\geq 1}\frac{\alpha_k}{k}\cdot\pi\cdot\sqrt{\pi}\alpha_k, \end{aligned}$$

where we have used $\int_{-\pi/2}^{\pi/2} e^{-ik\tau}\, d\tau = 0$ if k is even, and $\alpha_k = 0$ if k is odd. By (B.4) and keeping only even terms ($k = 2j$), we have

$$D_1 = \frac{6}{\pi}\sum_{j\geq 1}\frac{(\alpha_{2j})^2}{2j} = \frac{1}{\pi}\sum_{j\geq 1}\frac{(2j)!}{j\, 3^{2j}\, (j!)^2}, \tag{B.6}$$

and since $\sum_{j\geq 1}\frac{(2j)!}{j\, 9^j\, (j!)^2} \approx 0.2724$, we get

$$D_1 = \frac{1}{\pi}\sum_{k\geq 1}\frac{(2k)!}{k\, 9^k\, (k!)^2} \approx \frac{1}{\pi}0.2724 \approx 0.0867.$$

Proof of (B.4). Here we compute coefficients of decomposition of $e^{-\frac{5}{2}x^2}$ in Hermite basis, adapting a method from [33]. Recall the k-th Hermite polynomial H_k

$$h_k(x) = \frac{H_k(x)}{\sqrt{2^k\, k!}}\, e^{-\frac{x^2}{2}}.$$

We have

$$\alpha_k = \frac{1}{\sqrt{2^k\, k!\, \pi}} \int_{-\infty}^{+\infty} H_k(x)\, e^{-3x^2} dx. \tag{B.7}$$

Using the generating function representation

$$e^{2tx-t^2} = \sum_{n=0}^{+\infty} \frac{t^n}{n!} H_n(x), \tag{B.8}$$

we observe that it is equivalent to

$$e^{2\frac{t}{\sqrt{3}}\sqrt{3}x - \left(\frac{t}{\sqrt{3}}\right)^2} \times e^{-\frac{2}{3}t^2} = \sum_{n=0}^{+\infty} \frac{t^n}{n!} H_n(x),$$

on the other hand, using (B.8) again on the left side

$$\sum_{j=0}^{+\infty} \frac{1}{j!} \left(\frac{t}{\sqrt{3}}\right)^j H_j\left(\sqrt{3}x\right) \times \sum_{k=0}^{+\infty} \frac{1}{k!} \left(-\frac{2}{3}\right)^k t^{2k} = \sum_{n=0}^{+\infty} \frac{t^n}{n!} H_n(x).$$

Expanding the product on the left-hand side and identifying the powers of t, we get

$$\begin{aligned} \frac{1}{n!} H_n(x) &= \sum_{j+2k=n} \left(-\frac{2}{3}\right)^k \frac{1}{j!\, k!\, (\sqrt{3})^j} H_j\left(\sqrt{3}x\right) \\ &= \frac{1}{(\sqrt{3})^n} \sum_{j+2k=n} \frac{(-2)^k}{j!\, k!} H_j\left(\sqrt{3}x\right) \end{aligned}$$

Integrating both sides against e^{-3x^2}, we obtain

$$\begin{aligned} \frac{1}{n!} \int H_n(x)\, e^{-3x^2}\, dx &= \frac{1}{(\sqrt{3})^n} \sum_{j+2k=n} \frac{(-2)^k}{j!\, k!} \int_{\mathbb{R}} H_j\left(\sqrt{3}x\right)\, e^{-3x^2}\, dx \\ &= \frac{1}{(\sqrt{3})^n} \sum_{j+2k=n} \frac{(-2)^k}{j!\, k!} \frac{1}{\sqrt{3}} \int_{\mathbb{R}} H_0(y)\, H_j(y)\, e^{-y^2}\, dy. \end{aligned}$$

Thus by (5.4)

$$\begin{aligned} \frac{1}{n!} \int H_n(x)\, e^{-3x^2}\, dx &= \frac{1}{(\sqrt{3})^{n+1}} \sum_{j+2k=n} \frac{(-2)^k}{j!\, k!} \sqrt{2^j\, j!}\, \delta_{0j} \sqrt{\pi} \\ &= \begin{cases} 0 & \text{if } n \text{ is odd;} \\ \dfrac{(-2)^k \sqrt{\pi}}{(\sqrt{3})^{2k+1}\, k!} & \text{if } n \text{ is even, } n = 2k. \end{cases} \end{aligned}$$

Thus,

$$\int_{-\infty}^{\infty} H_{2k}(x)\, e^{-3x^2}\, dx = \frac{(2k)!}{k!}\, \frac{(-2)^k \sqrt{\pi}}{(\sqrt{3})^{2k+1}},$$

which by (B.7) implies that

$$\alpha_{2k} = (-1)^k \frac{\sqrt{(2k)!}}{3^k \sqrt{3}\, k!}. \qquad \square$$

C. Constant in 2D

Claim C.1.

$$D_2 = \frac{1}{2\pi} \ln \frac{4}{3}.$$

Proof. Recall from (3.18) the definition of r. By (3.16) we must show

$$\operatorname{Re} \iint |G|^2\, \overline{G}\, r\, dt\, dx = \frac{1}{8\pi} \ln \frac{4}{3}.$$

We will prove this result by direct computation of the integral, which is essentially an integral of a Gaussian function (in x) and rational functions (in s and t).

By (1.6),

$$r(t,x) = \frac{i}{\pi^{3/2}} \int_0^t \frac{1}{(1+is)(1+s^2)}\, e^{i\frac{(t-s)}{2}\Delta} \left(e^{-\frac{|x|^2}{2}\frac{(3-is)}{(1+s^2)}} \right) ds.$$

Noting that

$$e^{i\frac{t}{2}\Delta}\left(e^{-\alpha|x|^2}\right) = \frac{1}{(1+2\alpha i t)^{N/2}}\, e^{-\frac{\alpha|x|^2}{1+2\alpha i t}}, \quad \operatorname{Re}\alpha > 0,$$

we get

$$r(t,x) = \frac{i}{\pi^{3/2}} \int_0^t \frac{1}{(1+is)(1+s^2+(s+3i)(t-s))}\, e^{-\frac{|x|^2}{2}\frac{3-is}{1+s^2+(s+3i)(t-s)}}\, ds.$$

Let

$$A = 1+s^2+(s+3i)(t-s) = 1+st+3i(t-s),$$

$$B = \frac{1}{2}\left(\frac{2}{1+t^2} + \frac{1}{1-it} + \frac{3-is}{A} \right).$$

Thus

$$|G|^2\, \overline{G}\, r = \frac{i}{\pi^3} \int_0^t \frac{1}{(1+t^2)(1-it)(1+is)A} e^{-|x|^2 B}\, ds.$$

Integrating in space we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} |G|^2\,\overline{G}\,r\,dx &= \frac{i}{\pi^2}\int_0^t \frac{1}{(1+t^2)(1-it)(1+is)\,A\,B}\,ds \\ &= \frac{1}{\pi^2}\int_0^t \frac{i}{(1-it)(1+is)(3(1+st)+5i(t-s))}\,ds.\end{aligned} \tag{C.1}$$

By fraction decomposition with respect to the variable s,

$$\begin{aligned}&\operatorname{Re}\left[\frac{i}{(1-it)(1+is)(3(1+st)+5i(t-s))}\right] \\ &\quad= \operatorname{Re}\left[\frac{1}{8(1+t^2)}\left(\frac{i}{1+is}+\frac{5i-3t}{3+5it+(3t-5i)s}\right)\right] \\ &\quad= \frac{1}{8(1+t^2)}\left(\frac{s}{1+s^2}+\frac{25(t-s)-9t(1+ts)}{9(1+ts)^2+25(t-s)^2}\right).\end{aligned}$$

Integrating with respect to the variable s and coming back to (C.1) we get:

$$\operatorname{Re}\left(\int_{\mathbb{R}^2}|G|^2\,\overline{G}\,r\,dx\right) = -\frac{1}{16\pi^2}\frac{\ln(1+t^2)+2\ln 3-\ln(9+25t^2)}{1+t^2}.$$

Finally, we compute the space-time norm:

$$\begin{aligned}&\int_{-\infty}^{\infty}\operatorname{Re}\left(\int_{\mathbb{R}^2}|G|^2\,\overline{G}\,r\,dx\right)dt \\ &= -\frac{1}{16\pi^2}\left(\int_{-\infty}^{\infty}\frac{\ln(1+t^2)}{1+t^2}\,dt + 2\ln 3\int_{-\infty}^{\infty}\frac{dt}{1+t^2} - \int_{-\infty}^{\infty}\frac{\ln(9+25t^2)}{(1+t^2)}\,dt\right).\end{aligned}$$

We have

$$\int_{-\infty}^{\infty}\frac{1}{\left(1+t^2\right)}\,dt = \pi.$$

By the change of variable $t=\tan\tau$, $\tau\in(-\pi/2,\pi/2)$ and the classical formulas

$$\begin{aligned}&\int_0^{\frac{\pi}{2}}\ln(\cos\tau)\,d\tau = -\frac{\pi}{2}\ln 2, \\ &\int_0^{\pi}\ln(a+b\cos\tau)\,d\tau = \pi\ln\left(\frac{a+\sqrt{a^2-b^2}}{2}\right),\ \ a>|b|,\end{aligned}$$

one gets

$$\int_{-\infty}^{\infty}\frac{\ln(9+25t^2)}{(1+t^2)} = 6\pi\ln 2, \qquad \int_{-\infty}^{\infty}\frac{\ln\left(1+t^2\right)}{\left(1+t^2\right)}dt = 2\pi\ln 2.$$

We leave the details of the computations to the reader. Combining the preceding equalities, we obtain as announced

$$\int_{-\infty}^{\infty} \operatorname{Re}\left(\int_{\mathbb{R}^2} |G|^2\, \overline{G}\, r\, dx\right) dt = \frac{1}{8\pi}(\ln 4 - \ln 3). \qquad \square$$

D. Bound of a sum of binomial coefficients

Let $m \geq 1$ and $j \in \{0, \dots, m\}$. In this appendix we sketch the proof of the following inequality

$$\sum_{\substack{k\in\{0,\dots,m\}\\ j+k \text{ even}}} \binom{j+k}{j}\binom{2m-(j+k)}{m-j} \leq \frac{1}{2}\binom{2m+1}{m+1} + \frac{1}{2}\binom{2m}{m}. \tag{D.1}$$

For $n \in \mathbb{N}^*$, let $I_n = \{1, \dots, n\}$. Let $\mathcal{P}(I_{2m+1})$ be the set of all subsets of I_{2m+1}. Define $O_{m,j} \subset \mathcal{P}(I_{2m+1})$ and $E_{m,j} \subset \mathcal{P}(I_{2m+1})$ as follows: a subset of I_{2m+1} is in $O_{m,j}$ (respectively, $E_{m,j}$) if it has $m+1$ elements $a_1 < a_2 < \dots < a_{m+1}$ and if a_{j+1} is odd (respectively, even). Then for fixed $j \in \{0, \dots, m\}$,

$$\left|O_{m,j}\right| = \sum_{\substack{k\in\{0,\dots,m\}\\ j+k \text{ even}}} \binom{j+k}{j}\binom{2m-(j+k)}{m-j}, \quad \binom{2m+1}{m+1} = \left|O_{m,j}\right| + \left|E_{m,j}\right|.$$

Let us construct a one-to-one map Φ_j from $O_{m,j}$ to the disjoint union of $E_{m,j}$ and the set of m-elements subsets of I_{2m}. Let S be a set which is in $O_{m,j}$, and $a_1 < a_2 < \dots < a_{m+1}$ its $m+1$ elements. Then if $j \geq 1$ and $a_j < a_{j+1} - 1$, or $j = 0$ and $a_1 > 1$, we denote by $\Phi_j(S)$ the element of $E_{m,j}$ $\{a_1, \dots, a_j, a_{j+1} - 1, a_{j+2}, \dots, a_{m+1}\}$ (i.e obtained from S by shifting only the element a_{j+1} to the left). If $a_j = a_{j+1} - 1$, or $j = 0$ and $a_1 = 1$, we denote by $\Phi_j(S)$ the subset $\{a_1, \dots, a_j, a_{j+2}, \dots, a_m\}$ of I_{2m}. The mapping Φ_j is clearly one-to-one: in the first case one can recover S by shifting the $j+1$ element of $\Phi_j(S)$ to the right. In the second case, by adding to the set $\Phi_j(S)$ the element $b_j + 1$ (1 if $j = 0$), where b_j is the jth element of $\Phi_j(S)$. Finally we obtain:

$$|O_{m,j}| \leq |E_{m,j}| + \binom{2m}{m} \leq \binom{2m+1}{m+1} - |O_{m,j}| + \binom{2m}{m},$$

which yields (D.1).

References

[1] J. BOURGAIN. *Refinements of Strichartz' inequality and applications to 2D-NLS with critical nonlinearity*, Internat. Math. Res. Notices (1998), 253–283.

[2] J. BOURGAIN, *Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case*, J. Amer. Math. Soc. **12** (1999), 145–171.

[3] P. BÉGOUT and A. VARGAS, *Mass concentration phenomena for the L^2-critical nonlinear Schrödinger equation*, Trans. Amer. Math. Soc. **359** (2007), 5257–5282.

[4] R. CARLES, *Critical nonlinear Schrödinger equations with and without harmonic potential*, Math. Models Methods Appl. Sci. **12** (2002), 1513–1523.

[5] R. CARLES, *Rotating points for the conformal nls scattering operator*, Dyn. Partial Differ. Equ. **6** (2009), 35–51.

[6] J. COLLIANDER, M. KEEL, G. STAFFILANI, H. TAKAOKA and T. TAO, *Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$*, Ann. of Math. **167** (2008), 767–865.

[7] P. CONSTANTIN and J.-C. SAUT, *Local smoothing properties of Schrödinger equations*, Indiana Univ. Math. J. **38** (1989), 791–810.

[8] T. CAZENAVE and F. B. WEISSLER, *The Cauchy problem for the critical nonlinear Schrödinger equation in H^s*, Nonlinear Anal. **14** (1990), 807–836.

[9] T. DUYCKAERTS and F. MERLE, *Scattering norm estimate near the threshold for energy-critical focusing semilinear wave equation*, Indiana Univ. Math. J. **58** (2009), 1971–2002.

[10] K.-J. ENGEL and R. NAGEL, "One-parameter Semigroups for Linear Evolution Equations", Vol. 194 of *Graduate Texts in Mathematics*, Springer-Verlag, New York, 2000, with contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[11] D. FOSCHI, *Maximizers for the Strichartz inequality*, J. Eur. Math. Soc. (JEMS) **9** (2007), 739–774.

[12] J. HOLMER and S. ROUDENKO, *A sharp condition for scattering of the radial 3D cubic nonlinear Schrödinger equation*, Comm. Math. Phys. **282** (2008), 435–467.

[13] D. HUNDERTMARK and V. ZHARNITSKY, *On sharp Strichartz inequalities in low dimensions*, Int. Math. Res. Not. 2006, Art. ID 3408, 18 pages.

[14] S. KERAANI, *On the blow up phenomenon of the critical nonlinear Schrödinger equation*, J. Funct. Anal. **235** (2006), 171–192.

[15] C. E. KENIG and F. MERLE, *Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case*, Invent. Math. **166** (2006), 645–675.

[16] R. KILLIP, T. TAO and M. VISAN, *The cubic nonlinear schrödinger equation in two dimensions with radial data*, J. Eur. Math. Soc. (JEMS) **11** (2009), 1203 –1258.

[17] P.-L. LIONS, *The concentration-compactness principle in the calculus of variations. The locally compact case*, I, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), 109–145.

[18] P.-L. LIONS, *The concentration-compactness principle in the calculus of variations. The limit case*, I, Rev. Mat. Iberoamericana **1** (1985), 145–201.

[19] F. MERLE and L. VEGA, *Compactness at blow-up time for L^2 solutions of the critical nonlinear Schrödinger equation in 2D*, Internat. Math. Res. Notices (1998), 399–425.

[20] U. NIEDERER, *The maximal kinematical invariance groups of Schrödinger equations with arbitrary potentials*, Helv. Phys. Acta **47** (1974), 167–172.

[21] O. REY, *The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent*, J. Funct. Anal. **89** (1990), 1–52.

[22] A. V. RYBIN, G. G. VARZUGIN, M. LINDBERG, J. TIMONEN and R. K. BULLOUGH, *Similarity solutions and collapse in the attractive gross-pitaevskii equation*, Phys. Rev. E **62** (2000), 6224–6228.

[23] P. SJÖLIN, *Convergence properties for the Schrödinger equation*, Rend. Sem. Mat. Fis. Milano **57** (1989), 293–297.

[24] R. S. STRICHARTZ, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. **44** (1977), 705–714.

[25] V.I. TALANOV, *Focusing of light in cubic media*, JETP Lett. **11**, 199–201.

[26] T. TAO, *Global well-posedness and scattering for the higher-dimensional energy-critical nonlinear Schrödinger equation for radial data*, New York J. Math. **11** (2005), 57–80 (electronic).

[27] T. TAO, *Spacetime bounds for the energy-critical nonlinear wave equation in three spatial dimensions*, Dyn. Partial Differ. Equ. **3** (2006), 93–110.

[28] T. TAO, *A pseudoconformal compactification of the nonlinear Schrödinger equation and applications*, New York J. Math. **15** (2009), 265–282.

[29] T. TAO and M. VISAN, *Stability of energy-critical nonlinear Schrödinger equations in high dimensions*, Electron. J. Differential Equations (2005), pages No. 118, 28 pp. (electronic).

[30] T. TAO, M. VISAN and X. ZHANG, *Global well-posedness and scattering for the defocusing mass-critical nonlinear Schrödinger equation for radial data in high dimensions*, Duke Math. J. **140** (2007), 165–202.

[31] T. TAO, M. VISAN and X. ZHANG, *Minimal-mass blowup solutions of the mass-critical NLS*, Forum Math. **20** (2008), 881–919.

[32] L. VEGA, *Schrödinger equations: pointwise convergence to the initial data*, Proc. Amer. Math. Soc. **102** (1988), 874–878.

[33] W.-M. WANG, *Pure point spectrum of the Floquet Hamiltonian for the quantum harmonic oscillator under time quasi-periodic perturbations*, Comm. Math. Phys. **277** (2008), 459–496.

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