Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)
Vol. X (2011), 269-315

BV solutions of rate independent variational inequalities

VINCENZO RECUPERO

Abstract. We prove a theorem providing a geometric characterization of BV continuous vector rate independent operators. We apply this theorem to rate independent evolution variational inequalities and deduce new continuity properties of their solution operator: the vectorial play operator.

Mathematics Subject Classification (2010): 49J40 (primary); 47J35, 74C15, 34C55, 26A45 (secondary).

1. Introduction

In several mathematical models of elastoplasticity, the nonlinear dependence between deformation and stress tensors is described by means of the following evolution variational inequality. Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and $\mathcal{Z} \subseteq \mathcal{H}$ be a closed convex subset containing 0. Given $T > 0$ and $u : [0, T] \longrightarrow \mathcal{H}$, find $y : [0, T] \longrightarrow \mathcal{H}$ such that

$$\langle u(t) - y(t) - z, y'(t)\rangle \geq 0 \qquad \forall z \in \mathcal{Z}, \quad t \in [0, T], \tag{1.1}$$

where y' denotes the time derivative of y. The references [14,15,20] contain surveys of the physical models described by (1.1). The special one dimensional case $\mathcal{H} = \mathbb{R}$ has been deeply studied by several authors: we refer to the monographs [6, 12, 19, 27].

Inequality (1.1) can be solved by using classical tools from the theory of evolution equations governed by maximal monotone operators. In particular it is well known that if $u \in W^{1,1}(0, T; \mathcal{H})$ then there exists a unique $y \in W^{1,1}(0, T; \mathcal{H})$ satisfying (1.1) and the initial condition

$$u(0) - y(0) = z_0, \tag{1.2}$$

where $z_0 \in \mathcal{Z}$ is fixed. The resulting solution operator $\mathsf{P} : W^{1,1}(0, T; \mathcal{H}) \longrightarrow W^{1,1}(0, T; \mathcal{H})$ is usually called (*vector*) *play operator*. The suggestive terms *input* and *output* are used for u and v respectively. Regarding problem (1.1)-(1.2) there are two important issues to be considered. First of all the continuity of the solution

Received November 14, 2009; accepted in revised form March 4, 2010.

operator $\mathsf{P} : u \longmapsto y$ with respect to different topologies. Secondly the extension of such operator to classes of functions more general than $W^{1,1}(0, T; \mathcal{H})$. Both questions have an applicative relevance since the continuity of the extension ensures robustness of the model and applicability of mathematical tools including numerical simulation.

It is well-known that the operator P is continuous on $W^{1,1}(0, T; \mathcal{H})$ endowed with its natural topology: this was proved in [13, Proposition 3.1] in the finite dimensional case, whereas for general $\mathcal{H}$ it is proved in [14, Theorem 3.12, page 34]. The continuity with respect to the topology of uniform convergence is proved in [14, Corollary 3.8, page 32]. As far as the extension of P is concerned, it seems that the first answer to this question in the infinite dimensional case can be found in [14]. In that book the play operator is extended to the space $BV(0, T; \mathcal{H}) \cap C(0, T; \mathcal{H})$. In order to do this, the variational inequality (1.1) is replaced by the integral inequality

$$\int_0^T \big\langle u(t) - y(t) - z(t), \mathrm{d}y(t) \big\rangle \geq 0 \qquad \forall z \in C([0, T]; \mathcal{Z}). \tag{1.3}$$

Here the integral is meant in the sense of Riemann-Stieltjes. In [14] the problem (1.3)–(1.2) is first solved for step functions, then a solution for continuous BV inputs is found by an *a priori estimates-limit procedure*. In [14] it is also proved the continuity with respect to the topology of uniform convergence and it is shown that this extension of P is continuous in $BV(0, T; \mathcal{H}) \cap C(0, T; \mathcal{H})$ endowed with the strict metric, provided $\mathcal{Z}$ is bounded and its boundary satisfies suitable smoothness conditions, the general case being left as an open problem. Let us recall that the *strict metric* is defined by

$$d_s(u, v) := \|u - v\|_{L^1} + |\operatorname{V}(u, [0, T]) - \operatorname{V}(v, [0, T])|, \tag{1.4}$$

where $\operatorname{V}(u, [0, T])$ is the variation of u on $[0, T]$. This is a natural metric on BV because every function u of bounded variation admits a sequence of smooth functions u_n such that $d_s(u_n, u) \to 0$ as n goes to infinity.

In the paper [16] the play operator is further extended to the space of possibly discontinuous functions of bounded variation. In that paper the integral in (1.3) is understood in the Young sense and the continuity with respect to the topology of uniform convergence is proved. The continuity with respect to the strict convergence of BV is left as an open question.

In the present paper we address the issue of BV-continuity by studying the problem of the extension of a general *rate independent operator*: indeed the play operator P is *rate independent*, *i.e.*

$$\mathsf{P}(u \circ \phi) = \mathsf{P}(u) \circ \phi \tag{1.5}$$

whenever $u \in W^{1,p}(0, T; \mathcal{H})$ and $\phi : [0, T] \longrightarrow [0, T]$ is an increasing surjective Lipschitz reparametrization. Thus we study when a general rate independent operator R, acting on the space of Lipschitz mappings, can be continuously extended to all $BV(0,T;\mathcal{H})$. In our main theorem we prove that such extension exists if and only if R is *locally isotone*, *i.e.*

$$\mathrm{V}(u,[c,d]) = \|u(d)-u(c)\|_{\mathcal{H}} \Longrightarrow \mathrm{V}(\mathsf{R}(u),[c,d]) = \|\mathsf{R}(u)(d)-\mathsf{R}(u)(c)\|_{\mathcal{H}} \quad (1.6)$$

whenever u is Lipschitz and $[c,d]$ is a subinterval of $[0,T]$. Moreover this extension is unique if we identify functions which are equal almost everywhere. Condition (1.6) has the advantage of a clear geometrical meaning that can be easily applied to the play operator and translated in terms of the convex set $\mathcal{Z}$. It turns out that P can be continuously extended to $BV(0,T;\mathcal{H})$ if and only if either $\mathcal{Z}$ is a vector subspace or

$$\mathcal{Z} = \{x \in \mathcal{H} \ : \ -\alpha \leq \langle f, x\rangle \leq \beta\} \quad (1.7)$$

for some $f \in \mathcal{H} \setminus \{0\}$ and $\alpha, \beta \in [0,\infty]$. Therefore in many simple cases (*e.g.* $\mathcal{Z}$ is a cylinder or a ball for $\dim(\mathcal{H}) > 1$) the operator P cannot be continuously extended to BV. However, as a by-product of the proof, we obtain that P can always be continuously extended to $BV(0,T;\mathcal{H}) \cap C(0,T;\mathcal{H})$ for every $\mathcal{Z}$. Therefore we extend the result of [14] where the continuity is proved only for smooth $\mathcal{Z}$.

The scalar case was dealt in the papers [21, 22] where we proved that a BV-continuous scalar rate independent operator $\mathsf{R} : W^{1,\infty}(0,T;\mathbb{R}) \longrightarrow W^{1,\infty}(0,T;\mathbb{R})$ can be continuously extended to $BV(0,T;\mathbb{R})$ (in a unique manner) if and only if it is locally isotone. When $\mathcal{H} = \mathbb{R}$, local isotonicity is a very natural generalization of *local monotonicity*, well-known in hysteresis: a scalar operator R is called locally monotone if

$$u \text{ increasing (resp. decreasing) on } [c,d] \Longrightarrow \mathsf{R}(u) \text{increasing (resp. decreasing) on } [c,d]$$

whenever $[c,d]$ is a subinterval of $[0,T]$. In this special case every convex set $\mathcal{Z}$ is an interval and P is locally monotone, therefore P can always be continuously extended to $BV(0,T;\mathbb{R})$. In applications, locally monotonicity is verified in many particular cases, therefore the result applies to a wide class of concrete rate independent operators (*cf.* [21, Section 5]).

Let us also observe that our procedure will yield a representation formula for the extension $\overline{\mathsf{P}}$ of P (and in general for a rate independent operator R). Indeed we prove that

$$\overline{\mathsf{P}}(u) = \mathsf{P}(\widetilde{u}) \circ \ell_u, \quad (1.8)$$

where

$$\ell_u(t) = \frac{T}{\mathrm{V}(u,[0,T])}\mathrm{V}(u,[0,t]) \quad (1.9)$$

and $\widetilde{u}$ is a Lipshitz map such that

$$u = \widetilde{u} \circ \ell_u. \quad (1.10)$$

Even if P cannot be continuosly extended to all of $BV(0,T;\mathcal{H})$, we prove that $\overline{\mathsf{P}}$ has a good continuity property, namely $\overline{\mathsf{P}}(u_n) \to \overline{\mathsf{P}}(u)$ in $L^1(0,T;\mathcal{H})$ whenever

u_n converges strictly to u. This property suggests that $\overline{\mathsf{P}}(u)$ can be defined to be a generalized solution of (1.1)-(1.2) when u is of bounded variation. We show that this notion of solution does not coincide with the one proposed in [16], indeed it solves a variational inequality similar to (1.3), but containing an extra term due to the jumps of u. A comparison between this two notions of solution is given.

Now we give a brief plan of the paper. In the next section we recall the main definitions and notations about vector valued functions of bounded variation. In Sections 3 and 4 we state precisely the main results and we present their proofs. In Section 5 we apply the abstract results to rate independent variational inequalities. Finally in the Appendix we prove some technical results about $\mathcal{H}$-valued BV maps and convex sets in a Hilbert space.

ACKNOWLEDGEMENTS. The author is grateful to P. Krejčí and G. Savaré for stimulating discussions and useful suggestions.

2. Preliminaries

2.1. List of notation

In the paper we will use the following notation.

- B^A, set of functions defined on a set A with values in a set B.
- $\mathscr{P}(S)$, power set of a set S.
- $\mathbb{N}$, set of strictly positive integer numbers $\{1, 2, \ldots\}$.
- χ_S, characteristic function of a set S: $\chi_S(t) = 1$ if $t \in S$ and $\chi_S(t) = 0$ if $t \notin S$.
- $\overline{S}$, closure of a subset $S \subseteq T$, with T topological space.
- $\mathring{S}$, interior of a subset $S \subseteq T$, with T topological space.
- $f(t-) := \lim_{s \nearrow t} f(s)$, $f(t+) := \lim_{s \searrow t} f(s)$, with $f \in T^S$, T topological space, $S \subseteq \mathbb{R}$.
- $\mathrm{Cont}(f)$, continuity set of $f \in T^S$, with S, T topological spaces.
- $\mathrm{Discont}(f) = S \smallsetminus \mathrm{Cont}(f)$, with $f \in T^S$ and S, T topological spaces.
- $\|f\|_\infty := \sup\{\|f(s)\|_{\mathcal{X}} \ : \ s \in S\}$, with $f \in \mathcal{X}^S$, S set, $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ Banach space.
- $Lip(I; \mathcal{X})$, $\mathcal{X}$-valued Lipschitz continuous functions defined on I, with $I \subseteq \mathbb{R}$ interval, $\mathcal{X}$ Banach space.
- $\mathrm{Lip}(f)$, Lipschitz constant of $f \in \mathcal{X}^I$, with $I \subseteq \mathbb{R}$ interval, $\mathcal{X}$ Banach space.
- $\mathcal{X}'$, topological dual space of a Banach space $\mathcal{X}$.
- $x_n \rightharpoonup x$, weak convergence: $f(x_n) \to f(x)$ for all $f \in \mathcal{X}'$, with $x_n, x \in \mathcal{X}$, $\mathcal{X}$ Banach space.

- $f_n \overset{*}{\rightharpoonup} f$, weak* convergence: $f_n(x) \to f(x)$ for all $x \in \mathcal{X}$, with $f_n, f \in \mathcal{X}'$, $\mathcal{X}$ Banach space.
- $\mathrm{seg}[x, y] := \{(1-\lambda)x + \lambda y \ : \ \lambda \in [0, 1]\}$, segment joining x and y in a Hilbert space.
- $\mathrm{Proj}_{\mathcal{Z}}$, projection operator on a closed convex set $\mathcal{Z}$ in a Hilbert space.
- $\mathscr{B}(T)$, family of Borel sets of T, T topological space.
- $L^1(\mu, T; \mathcal{X}) = L^1(\mu; \mathcal{X})$, space of μ-integrable $\mathcal{X}$-valued maps, with μ positive measure on T.
- $\mathcal{L}^1$, one dimensional Lebesgue measure, $L^1(I; \mathcal{X}) := L^1(\mathcal{L}^1, I; \mathcal{X})$, $I \subseteq \mathbb{R}$ interval.

Let us emphasize that we do not identify two functions defined on the real line which are equal $\mathcal{L}^1$-almost everywhere ($\mathcal{L}^1$-a.e.). Moreover throughout the paper we assume that

$$I := \left]a, b\right[\,, \quad -\infty \le a < b \le \infty, \tag{2.1}$$

the open interval in $\mathbb{R}$ with endpoints a, b, and

$$\begin{cases} \mathcal{H} \text{ is a real Hilbert space with inner product } (x, y) \longmapsto \langle x, y\rangle \\ \|x\|_{\mathcal{H}} := \langle x, x\rangle^{1/2}. \end{cases} \tag{2.2}$$

2.2. Pointwise and essential variations

In this subsection $\mathcal{X}$ denotes a Banach space with norm $\|\cdot\|_{\mathcal{X}}$. We collect the main definitions and results concerning Banach valued functions with bounded pointwise variation. All the results are standard in the real case, however we give proofs whenever we are not able to provide a reference for the vector case.

Definition 2.1. If J is a subinterval of I, the symbol $St(J; \mathcal{X})$ denotes the set of *$\mathcal{X}$-valued step maps on J*, *i.e.* maps $f : J \longrightarrow \mathcal{X}$ such that J can be partitioned into a finite number of (possibly degenerate) intervals $J_1, \ldots, J_m$ and f is constant on each J_j for $j = 1, \ldots, m$. A function $f : J \longrightarrow \mathcal{X}$ is called *regulated on J* if at each point $t \in J$ there exist $f(t-)$ and $f(t+)$ in $\mathcal{X}$, with the convention that $f(t-) := f(t)$ (respectively $f(t+) := f(t)$) if $t \in J$ and t is the right (resectively the left) endpoint of J. We denote by $Reg(J; \mathcal{X})$ the set of regulated maps on J.

Every $f \in Reg(J; \mathcal{X})$ is locally the uniform limit of a sequence $f_n \in St(J; \mathcal{X})$ (*cf.*, *e.g.*, [3, Theorem 3, Section 2.1]), hence f is $\mathcal{L}^1$-measurable, the set $\{t \in J \ : f(t-) \ne f(t+)\}$ is at most countable, and if J is compact then f is bounded. Of course every monotone real function is regulated. In this regard we warn the reader about the terminology: by an *increasing function* on J, we mean a function $f : J \longrightarrow \mathbb{R}$ such that $(f(t_1) - f(t_2))(t_1 - t_2) \ge 0$ for every $t_1, t_2 \in J$. Same convention is adopted for the term *decreasing*. Finally f is *monotone* if it is increasing or if it is decreasing.

Definition 2.2. We recall that a *subdivision* of a nondegenerate subinterval $J \subseteq I$ is a family $(s_j)_{j=0}^m$, $m \in \mathbb{N}$, with the property that $s_0 < \cdots < s_m$ and $s_j \in J$ for $j = 0, \dots, m$. The set of all subdivisions of J is indicated by $\mathfrak{S}(J)$. If $f \in \mathcal{X}^I$ and $\mathfrak{s} = (s_j)_{j=0}^m \in \mathfrak{S}(J)$, the *variation of f with respect to* $\mathfrak{s}$ is defined by

$$\mathrm{V}(f, \mathfrak{s}) := \sum_{j=1}^{m} \|f(s_j) - f(s_{j-1})\|_{\mathcal{X}}.$$

If J is nondegenerate the *pointwise variation of u on J* is defined by

$$\mathrm{V}_p(f, J) := \sup\left\{ \mathrm{V}(f, \mathfrak{s}) \ : \ \mathfrak{s} \in \mathfrak{S}(J)\right\},$$

otherwise we set $\mathrm{V}_p(f,J)=0$. We define $BV_p(I; \mathcal{X}) := \{f \in \mathcal{X}^I \ : \ \mathrm{V}_p(f,I) < \infty\}$.

If $f \in \mathcal{X}^I$, $\mathrm{V}_p(f, I) < \infty$, and $t_0 \in I$, the inequality $\|f(t)\|_{\mathcal{X}} \leq \mathrm{V}_p(f, I) + \|f(t_0)\|_{\mathcal{X}}$ yields the boundedness of f. Moreover it is well known there exist (in $\mathcal{X}$) $\lim_{t \to \inf I+} f(t)$, $\lim_{t \to \sup I-} f(t)$, $f(t+)$, and $f(t-)$ for every $t \in I$. In particular f is regulated, $\mathcal{L}^1$-measurable, and $\mathrm{Discont}(f)$ is at most countable. We can define the maps $f_-, f_+ \in \mathcal{X}^I$ by setting

$$f_-(t) := f(t-), \quad f_+(t) := f(t+), \qquad t \in I. \tag{2.3}$$

It is easy to check that $\mathrm{V}_p(f_+, I) = \mathrm{V}_p(f_-, I)$. Let us observe that if $g_1, g_2 \in \mathcal{X}^I$ are two functions in the same $\mathcal{L}^1$-equivalence class and $\mathrm{V}_p(g_j, I) < \infty$, $j = 1, 2$, then every $t \in I$ is a left Lebesgue point of g_j, hence

$$(g_1)_-(t) = \lim_{h \nearrow 0} \frac{1}{h} \int_{t-h}^{t} g_1(s)\,\mathrm{d}s = \lim_{h \nearrow 0} \frac{1}{h} \int_{t-h}^{t} g_2(s)\,\mathrm{d}s = (g_2)_-(t).$$

In the same manner we see that $(g_1)_+ = (g_2)_+$. This remark allows us to formulate the following:

Definition 2.3. Let $f \in \mathcal{X}^I$ be given. If there is no $\mathcal{L}^1$-representative g of f such that $\mathrm{V}_p(g, I) < \infty$, we set $\mathrm{V}_e(f, I) := \infty$. Otherwise if $g \in \mathcal{X}^I$ is such that $f = g$ $\mathcal{L}^1$-a.e. in I and $\mathrm{V}_p(g, I) < \infty$, we set

$$\mathrm{V}_e(f, I) := \mathrm{V}_p(g_-, I) \ (= \mathrm{V}_p(g_+, I)), \tag{2.4}$$

where g_- is defined in (2.3). The real extended number $\mathrm{V}_e(f, I)$ is called *essential variation of f*.

Now let $f \in \mathcal{X}^I$ be left-continuous, then $\mathrm{Discont}(f)$ is at most countable (proof: to every $t \in \mathrm{Discont}(f)$ associate a triple $(p, q, r) \in \mathbb{Q}^3$ such that $0 < p < \limsup_{\tau \to t} \|f(\tau) - f(t)\|_{\mathcal{X}}$, $\|f(s) - f(t)\|_{\mathcal{X}} < p$ whenever $q < s < t$, and $\limsup_{\tau \to s} \|f(\tau) - f(s)\|_{\mathcal{X}} > p$ whenever $t < s < r$; from the left continuity it follows that the correspondence $t \longmapsto (p, q, r)$ is one-to-one). Therefore if $\mathfrak{s} = (s_j)_{j=1}^m$ is a subdivision of I and f is left-continuous, then for every $\varepsilon > 0$ we can find another subdivision $\mathfrak{t} = (t_j)_{j=1}^m$ such that $t_j \in \text{Cont}(f)$, $t_j < s_j$, and $\|f(t_j) - f(s_j)\|_{\mathcal{X}} < \varepsilon/(4m)$ for $j = 1, \ldots, m$. Hence $\text{V}(f, \mathfrak{s}) + \varepsilon/2 \leq \text{V}(f, \mathfrak{t}) + \varepsilon$, thus we have proved the following:

Lemma 2.4. *If $f : I \longrightarrow \mathcal{X}$ is left-continuous, then*

$$\text{V}_p(f, I) = \sup\Big\{\text{V}(f, \mathfrak{s}) \;:\; \mathfrak{s} = (s_j) \in \mathfrak{S}(I),\ s_j \in \text{Cont}(f)\Big\}.$$

Let us notice that if $f, f_n \in \mathcal{X}^I$ and $f_n(t) \to f(t)$ for every $t \in \text{Cont}(f)$, then $\text{V}(f_n, \mathfrak{s}) \to \text{V}(f, \mathfrak{s})$ for every $\mathfrak{s} \in \mathfrak{S}(I)$. Hence thanks to Lemma 2.4 we have the following:

Corollary 2.5. *Assume that $f, f_n \in \mathcal{X}^I$ and $f_n(t) \to f(t)$ for every $t \in \text{Cont}(f)$. Then $\text{V}_e(f, I) \leq \liminf_{n\to\infty} \text{V}_e(f_n, I)$.*

2.3. Vector Stieltjes measures

Now we recall the connection between functions with bounded variation and *Borel vector measures* on the real line, *i.e.* maps $\mu : \mathscr{B}(I) \longrightarrow \mathcal{X}$ such that $\mu(\bigcup_{n=1}^\infty B_n) = \sum_{n=1}^\infty \mu(B_n)$ whenever (B_n) is a sequence of mutually disjoint Borel subsets of I. Let us also recall that if $\mu : \mathscr{B}(I) \longrightarrow \mathcal{X}$ is a vector measure, then $\left\bracevert\mu\right\bracevert : \mathscr{B}(I) \longrightarrow [0, \infty]$ is defined by

$$\left\bracevert\mu\right\bracevert(B) := \sup\left\{\sum_{n=1}^\infty \|\mu(B_n)\|_{\mathcal{X}} : B = \bigcup_{n=1}^\infty B_n,\ B_n \in \mathscr{B}(I),\ B_h \cap B_k = \varnothing \text{ if } h \neq k\right\}.$$

The map $\left\bracevert\mu\right\bracevert$ is a positive measure which is called *total variation of* μ and we set

$$A(\mu) := \{t \in I \;:\; \left\bracevert\mu\right\bracevert(\{t\}) \neq 0\}, \tag{2.5}$$

the *set of atoms of* μ. The vector measure μ is called *with bounded variation* if $\left\bracevert\mu\right\bracevert(I) < \infty$ (see, *e.g.*, [8, Chapter I, Section 3.]). In this case the equality $\|\mu\| := \left\bracevert\mu\right\bracevert(I)$ defines a norm on the space of measures with bounded variation. Let us recall the following proposition whose proof can be found in [8, Theorem 1, section III.17.2, page 358]:

Theorem 2.6. *If $f \in \mathcal{X}^I$ and $\text{V}_p(f, I) < \infty$ then there exists a unique vector measure $\mu_f : \mathscr{B}(I) \longrightarrow \mathcal{X}$ such that for every $c, d \in I$ with $c < d$ we have*

$$\mu_f(]c, d[) = f(d-) - f(c+), \qquad \mu_f([c, d]) = f(d+) - f(c-), \tag{2.6}$$

$$\mu_f([c, d[) = f(d-) - f(c-), \qquad \mu_f(]c, d]) = f(d+) - f(c+). \tag{2.7}$$

Moreover μ_f is with bounded variation and if $f_- : I \longrightarrow \mathcal{X}$ is defined by (2.3), *then $\mu_f = \mu_{f_-}$. Vice versa if $\mu : \mathscr{B}(I) \longrightarrow \mathcal{X}$ is a vector measure with bounded variation, then the map $f_\mu : I \longrightarrow \mathcal{X}$ defined by $f_\mu(t) := \mu(]a, t[)$ is such that $\text{V}_p(f_\mu, I) < \infty$ and $\mu_{f_\mu} = \mu$.*

Usually μ_f is called *the Lebesgue-Stieltjes measure associated with* f. Observe that from Theorem 2.6 it follows that $\mu_f(I) = \lim_{t\to\sup I-} f(t) - \lim_{t\to\inf I+} f(t)$ and that $\mu_f(\{t\}) = f(t+) - f(t-)$ for every $t \in I$. Now we recall the characterization of the total variation of μ_f (see [8, Remark 5, Section III.17.2, page 362]):

Proposition 2.7. *Let* $f : I \longrightarrow \mathcal{X}$ *be such that* $\mathrm{V}_p(f, I) < \infty$ *and let* $f_- : I \longrightarrow \mathcal{X}$ *be defined by* (2.3). *Define* $V_f : I \longrightarrow]0, \infty]$ *by* $V_f(t) := \mathrm{V}_p(f_-,]a, t[)$, *that is the pointwise variation of* f_- *on* $]a, t[$. *Then* $|\mu_f| = \mu_{V_f}$

It follows that if $f \in \mathcal{X}^I$, $\mathrm{V}_p(f, I)$ and J is an open subinterval of I, then

$$|\mu_f|(J) = \mathrm{V}_p(f_-, J) = \mathrm{V}_e(f, J). \tag{2.8}$$

2.4. Integrals with respect to vector measures

Let $\mathcal{X}_j$, $j = 1, 2, 3$, be Banach spaces with norms $\|\cdot\|_{\mathcal{X}_j}$ and let $\mathcal{X}_1 \times \mathcal{X}_2 \longrightarrow \mathcal{X}_3 : (x_1, x_2) \longmapsto x_1 \bullet x_2$ be a bilinear form such that $\|x_1 \bullet x_2\|_{\mathcal{X}_3} \le \|x_1\|_{\mathcal{X}_1} \|x_2\|_{\mathcal{X}_2}$ for every $x_j \in \mathcal{X}_j$, $j = 1, 2$. Assume that $\mu : \mathscr{B}(I) \longrightarrow \mathcal{X}_2$ is a vector measure with bounded variation. Let $f : \mathcal{X}_1^I$ be a *step map with respect to* μ, *i.e.* there exist $f_1, \dots, f_m \in \mathcal{X}_1$ and $A_1, \dots, A_m \in \mathscr{B}(I)$ mutually disjoint such that $|\mu|(A_j) < \infty$ for every j and

$$f = \sum_{j=1}^{m} \chi_{A_j} f_j.$$

The set of step maps with respect to μ is denoted by $St(|\mu|; \mathcal{X}_1)$ and the integral of f is the vector defined by

$$\int_I f \bullet \mathrm{d}\mu := \sum_{j=1}^{m} f_j \bullet \mu(A_j) \in \mathcal{X}_3.$$

It can be proved that the map $St(|\mu|; \mathcal{X}_1) \longrightarrow \mathcal{X}_3$ associating to every f the integral $\int_I f \bullet \mathrm{d}\mu$ is linear and continuous when $St(|\mu|; \mathcal{X}_1)$ is endowed with the L^1-semimetric $\|f - g\|_{L^1(|\mu|;\mathcal{X}_1)} := \int_I \|f - g\|_{\mathcal{X}_1} \mathrm{d}|\mu|$. Therefore it admits a unique continuous extension $I_\mu : L^1(|\mu|; \mathcal{X}_1) \longrightarrow \mathcal{X}_3$ and we set

$$\int_I f \bullet \mathrm{d}\mu := I_\mu(f), \qquad f \in L^1(|\mu|; \mathcal{X}_1).$$

The following fundamental inequality holds:

$$\left\| \int_I f \bullet \mathrm{d}\mu \right\|_{\mathcal{X}_3} \le \int_I \|f\|_{\mathcal{X}_1} \mathrm{d}|\mu|. \tag{2.9}$$

We will use the previous integral in two particular cases, namely when

a) $\mathcal{X}_1 = \mathbb{R}$, $\mathcal{X}_2 = \mathcal{X}_3 = \mathcal{H}$, $\lambda \bullet x := \lambda x$ ($\int_I f \bullet \mathrm{d}\mu = \int_I f \,\mathrm{d}\mu$, integral of a real function with respect to a vector measure);
b) $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{H}$, $\mathcal{X}_3 = \mathbb{R}$, $x_1 \bullet x_2 := \langle x_1, x_2\rangle$ ($\int_I f \bullet \mathrm{d}\mu = \int_I \langle f, \mathrm{d}\mu\rangle$, integral of a vector function with respect to a vector measure).

2.5. Maps whose derivative is a measure

Now we are going to present a brief summary of facts about functions of bounded variation with values in $\mathcal{H}$. We adopt the notations of [2], which is our main reference for the finite dimensional case.

Definition 2.8. A map $u \in L^1(I; \mathcal{H})$ is called *of bounded variation* (*on* I) if its distributional derivative is a measure with bounded variation, *i.e.* if there exists a measure $\mathrm{D}\,u : \mathscr{B}(I) \longrightarrow \mathcal{H}$ such that $\|\mathrm{D}\,u\|\,(I) < \infty$ and

$$-\int_I \varphi'(t)u(t)\,\mathrm{d}t = \int_I \varphi\,\mathrm{d}\mathrm{D}\,u \qquad \forall \varphi \in C^1_c(I; \mathbb{R}).$$

We set $A(u) := A(\mathrm{D}\,u)$ and the space of maps of bounded variation on I is denoted by $BV(I; \mathcal{H})$.

Proposition 2.9. *Assume that* $u \in BV(I; \mathcal{H})$ *and define* $v \in \mathcal{H}^I$ *by* $v(t) := \mathrm{D}\,u(]a, t[)$. *Then* v *is left-continuous,* $\mathrm{V}_p(v, I) < \infty$, *and* $\mathrm{D}\,u = \mu_v = \mathrm{D}\,v$. *Moreover there exists a unique* $u_a \in \mathcal{H}$ *such that*

$$u(t) = u_a + \mathrm{D}\,u(]a, t[) \qquad \textit{for } \mathcal{L}^1\textit{-a.e. } t \in I. \tag{2.10}$$

We have

$$\mathrm{V}_e(u, I) = \mathrm{V}_p(v, I) < \infty.$$

Proof. The left continuity of v is a straightforward consequence of the continuity of measures. It is easy to check that $\mathrm{V}_p(v, I) \le \|\mathrm{D}\,u\|\,(I) < \infty$. The last part of Theorem 2.6 yields $\mu_v = \mathrm{D}\,u$. Now take $\varphi \in C^1_c(I; \mathbb{R})$. Thanks to Lemma A.1 of Section A.1 in the Appendix we have

$$\begin{aligned} -\int_I \varphi'(t)v(t)\,\mathrm{d}t &= -\int_I \varphi'(t)\int_{]a,t[}\mathrm{d}\mathrm{D}\,u\,\mathrm{d}t = -\int_I \varphi'(t)\int_I \chi_{]a,t[}(s)\,\mathrm{d}\mathrm{D}\,u(s)\,\mathrm{d}t \\ &= -\int_I \varphi'(t)\int_I \chi_{]a,t[}(s)\,\mathrm{d}t\,\mathrm{d}\mathrm{D}\,u(s) = -\int_I\int_s^b \varphi'(t)\,\mathrm{d}t\,\mathrm{d}\mathrm{D}\,u(s) \\ &= \int_I \varphi(s)\,\mathrm{d}\mathrm{D}\,u(s). \end{aligned}$$

Hence we have proved that $\mathrm{D}\,v = \mathrm{D}\,u$. Therefore $u - v$ is $\mathcal{L}^1$-a.e. equal to a constant $u_a \in \mathcal{H}$ thus $\mathrm{V}_e(u, I) = \mathrm{V}_p(u_a + v, I) = \mathrm{V}_p(v, I) < \infty$. □

In the same way we can prove that setting $w(t) := \mathrm{D}\,u(]a,t])$, $t \in I$, then w is right-continuous and $u(t) = u_a + \mathrm{D}\,u(]a,t])$ for $\mathcal{L}^1$-a.e. $t \in I$. Therefore we infer the following:

Corollary 2.10. *Assume that $u \in L^1(I;\mathcal{H})$. Then $u \in BV(I;\mathcal{H})$ if and only if $\mathrm{V}_e(u,I) < \infty$. In this case, if $u_a \in \mathcal{H}$ is the unique vector such that* (2.10) *holds, the functions $u^l, u^r \in \mathcal{H}^I$ defined by*

$$u^l(t) := u_a + \mathrm{D}\,u(]a,t[), \quad u^r(t) := u_a + \mathrm{D}\,u(]a,t]), \qquad t \in I,$$

are respectively the left-continuous and the right-continuous representatives of u (with respect to $\mathcal{L}^1$). We have $u_a = u^l(a+) = u^r(a+)$ and we have $\mathrm{V}_p(u^l,I) = \mathrm{V}_p(u^r,I) = \mathrm{V}_e(u,I) = \|\mathrm{D}\,u\|$.

If not otherwise specified, we understand that a mapping $f \in BV(I;\mathcal{H})$ is extended to $\overline{I}$ by setting $f(a) := f^r(a+)$ and $f(b) := f^r(b-)$ (if a and/or b are finite).

Corollary 2.11. *If $u, u_n \in BV(I;\mathcal{H})$ are such that $u_n \to u$ in $L^1(I;\mathcal{H})$, then $\|\mathrm{D}\,u\| \le \liminf_{n\to\infty} \|\mathrm{D}\,u_n\|$.*

Proof. Since $\|\mathrm{D}\,v\| = \mathrm{V}_e(v,I) = \mathrm{V}_e(v^l,I)$ it is not restrictive to assume that u and u_n are the left-continuous representatives. Let us consider a subsequence of $\|\mathrm{D}\,u_n\|$ which is convergent to $\lambda \in \mathbb{R}$ and that we do not relabel. There exists a further subsequence n_k such that $u_{n_k} \to u$ $\mathcal{L}^1$-a.e. in I. Redefining every u_{n_k} on a suitable $\mathcal{L}^1$-null set of $\mathrm{Cont}(u)$ we obtain that $u_{n_k}(t) \to u(t)$ for every $t \in \mathrm{Cont}(u)$, therefore by Corollary 2.5 $\mathrm{V}_e(u,I) \le \lambda$. The thesis follows. ☐

The *strict semimetric* on $BV(I;\mathcal{H})$ is defined as follows:

$$d_s(u,v) := \|u-v\|_{L^1(I;\mathcal{H})} - |\|\mathrm{D}\,u\| - \|\mathrm{D}\,v\||, \qquad u,v \in BV(I;\mathcal{H}). \quad (2.11)$$

If $d_s(u_n,u) \to 0$ we also say that $u_n \to u$ *strictly on I*. The strict metric induces a natural topology on $BV(I;\mathcal{H})$, indeed we have the following:

Proposition 2.12. *If $u \in BV(I;\mathcal{H})$ then there exists a sequence (u_n) in $C^\infty(\overline{I};\mathcal{H})$ such that $u_n \to u$ strictly on I.*

The previous proposition is classical if $\mathcal{H}$ is finite dimensional. In the Appendix we provide a proof for the general case (see Proposition A.2).

Let us also mention the fact that d_s is not complete, this is important if we consider the problem of extending a $BV(I;\mathcal{H})$-valued operator in a continuous manner.

Let us recall that if $\mathrm{D}\,u = v\mathcal{L}^1$ and $v \in L^p(I;\mathcal{H})$, $p \in [1,\infty]$, then the distributional derivative u' equals $\mathcal{L}^1$-a.e. the pointwise derivative and $u' = v$ $\mathcal{L}^1$-a.e. in I. For $k \in \mathbb{N}$ we define $W^{k,p}(I;\mathcal{H}) := \{u \in L^p(I;\mathcal{H}) \ : \ u^{(k)} \in L^p(I;\mathcal{H})\}$. It is well known that $\mathrm{V}_p(u,I) = \int_I \|u'(t)\|_{\mathcal{H}}\,\mathrm{d}t$ whenever $u \in W^{1,1}(I;\mathcal{H})$, therefore

$W^{1,1}(I;\mathcal{H}) \subseteq BV(I;\mathcal{H})$. Moreover $f \in W^{1,\infty}(I;\mathcal{H})$ if and only if its continuous representative belong to $Lip(I;\mathcal{H}) \cap L^1(I;\mathcal{H})$. The standard semimetric on $W^{1,p}(I;\mathcal{H})$ is

$$\|u\|_{W^{1,p}(I;\mathcal{H})} := \|u\|_{L^p(I;\mathcal{H})} + \|u'\|_{L^p(I;\mathcal{H})}, \qquad u \in W^{1,p}(I;\mathcal{H})$$

(see the appendix of [4] for details).

2.6. Rate independent operators

Now we recall the notion of (vector) rate independent operator. In the last decades, operators of this kind have been extensively studied in the scalar case in several research articles and in the monographs [6, 12, 14, 19, 27]. The vector case has been object of fewer investigations than the scalar case: see *e.g.* [12] for the finite dimensional case and [14] for the Hilbert case.

Definition 2.13. Assume that $F \subseteq BV(I;\mathcal{H})$. We say that $\mathsf{R} : F \longrightarrow BV(I;\mathcal{H})$ is a *rate independent operator* if

$$\mathsf{R}(u \circ \phi) = \mathsf{R}(u) \circ \phi \tag{2.12}$$

for every $u \in F$ and every $\phi : \overline{I} \longrightarrow \overline{I}$ increasing and surjective such that $u \circ \phi \in F$.

Notice that in defining ϕ from $\overline{I}$ into itself, we allow, *e.g.*, time rescalings that are equal to $b \in \mathbb{R}$ on an interval $]t_0, b[$ for a certain time $t_0 \in\ ]a, b[$. Of course the definition makes sense if we extend any $u \in BV(I;\mathcal{H})$ to $\overline{I}$, by setting $f(a) := f^r(a+)$, $f(b) := f^r(b-)$, for a and/or b finite.

Definition 2.14. Assume that $F \subseteq BV(I;\mathcal{H})$, $F \neq \varnothing$. We say that $\mathsf{R} : F \longrightarrow BV(I;\mathcal{H})$ is *locally isotone* if for every $c, d \in \overline{I}$, $c < d$,

$$\begin{aligned} \mathrm{V}_e(u,]c, d[) = \|u(d) - u(c)\|_{\mathcal{H}} \Longrightarrow \mathrm{V}_e(\mathsf{R}(u),]c, d[) \\ = \|\mathsf{R}(u)(d) - \mathsf{R}(u)(c)\|_{\mathcal{H}}. \end{aligned} \tag{2.13}$$

The notion of locally isotone rate independent operator was introduced in [21, Remark 4.6] and it is a natural generalization of the notion of local monotonicity, well known in hysteresis. In the scalar case the local monotonicity of R means that if $\mathsf{R}(u)$ is monotone increasing (respectively decreasing) on $[c, d]$ than u is monotone increasing (respectively decreasing) on the same interval. Instead condition (2.13) simply means that $\mathsf{R}(u)$ is monotone on $]c, d[$ whenever u is monotone on $[c, d]$, hence the term 'isotone'. Since we will use Definition 2.14 only for $F \subseteq C(J;\mathcal{H})$, the essential variation can be replaced by the pointwise variation on $[c, d]$. In this case and when the dimension of $\mathcal{H}$ is greater than one, condition (2.13) means that if u is an injective parametrization of a segment on $[c, d]$, then $\mathsf{R}(u)$ is also an injective parametrization of another segment on $[c, d]$.

3. Main results

In this section we state the main results of this paper. To this aim we first need some properties on reparametrizations. We set $I =]a, b[$ with $a, b \in \mathbb{R}$, $a < b$.

3.1. Reparametrizations

We follow [11, Section 2.5.16], with some slight differences, due to the fact that we assign the same arc length to two functions which are equal $\mathcal{L}^1$-a.e. Moreover we need a normalization factor. Set $I :=]a, b[$ with $a, b \in \mathbb{R}$, $a < b$. If $u \in BV(I; \mathcal{H})$, define $\ell_u : [a, b] \longrightarrow [a, b]$ by

$$\ell_u(t) := \begin{cases} a + \frac{b-a}{\|\mathrm{D}\, u\|} \left|\mathrm{D}\, u\right| (]a, t[) & \text{if } \|\mathrm{D}\, u\| \neq 0 \\ a & \text{if } \|\mathrm{D}\, u\| = 0 \end{cases}, \qquad t \in \overline{I}. \tag{3.1}$$

The function ℓ_u is increasing and left-continuous. Moreover $\mathrm{Discont}(\ell_u) = A(u)$ and

$$\ell_u(\overline{I}) = \overline{I} \setminus \bigcup_{t \in A(u)}]\ell_u(t), \ell_u(t+)] .$$

If $t_1 < t_2$ we have $\|u^l(t_1) - u^l(t_2)\|_{\mathcal{H}} \leq \left|\mathrm{D}\, u\right| ([t_1, t_2[) = \left|\mathrm{D}\, u\right| (]a, t_2[) - \left|\mathrm{D}\, u\right| (]a, t_1[)$ therefore

$$\|u^l(t_1) - u^l(t_2)\|_{\mathcal{H}} \leq \frac{\|\mathrm{D}\, u\|}{b-a} |\ell_u(t_1) - \ell_u(t_2)| \qquad \forall t_1, t_2 \in I. \tag{3.2}$$

This inequality yields that $u^l(\ell_u^{-1}(\sigma))$ is a singleton for every $\sigma \in \ell_u(I)$, therefore there is a unique function $U : \ell_u(I) \longrightarrow \mathcal{H}$ such that $U \circ \ell = u^l$. From (3.2) it also follows that U is the unique Lipschitz function such that $U \circ \ell_u = u$ $\mathcal{L}^1$-a.e. and its Lipschitz constant satisfies $\mathrm{Lip}(U) \leq \|\mathrm{D}\, u\|/(b-a)$. In order to extend U to all of I we define $\widetilde{u} : I \longrightarrow \mathcal{H}$ by setting

$$\widetilde{u}(\sigma) := (1-\lambda)u^l(t) + \lambda u^l(t+) \text{ if } \sigma = (1-\lambda)\ell_u(t) + \lambda \ell_u(t+), \ t \in I, \ \lambda \in [0, 1].$$

It is clear that $\widetilde{u}$ extends U and that $\mathrm{Lip}(\widetilde{u}) = \mathrm{Lip}(U)$. The function $\widetilde{u}$ may be regarded as a kind of reparametrization of u by the normalized arc length. We summarize the previous discussions in the following proposition.

Proposition 3.1. *Assume a, b are finite and let $u \in BV(I; \mathcal{H})$. Let $\ell_u : \overline{I} \longrightarrow \overline{I}$ be its "normalized" arc length defined by* (3.1). *Then there exists a unique function $\widetilde{u} \in Lip(I; \mathcal{H})$ such that*

$$u = \widetilde{u} \circ \ell_u \quad \mathcal{L}^1\text{-a.e. in } I, \tag{3.3}$$

$$\widetilde{u} \text{ is affine on } [\ell_u(t), \ell_u(t+)] \qquad \forall t \in A(u). \tag{3.4}$$

3.2. Main abstract results

Here is our main result.

Theorem 3.2. *Let I be bounded. Assume that* $\mathsf{R} : Lip(I;\mathcal{H}) \longrightarrow BV(I;\mathcal{H}) \cap C(I;\mathcal{H})$ *is a rate independent operator which is continuous when* $Lip(I;\mathcal{H})$ *and* $BV(I;\mathcal{H}) \cap C(I;\mathcal{H})$ *are endowed with the strict topology. Then* R *admits a unique continuous extension to* $BV(I;\mathcal{H}) \cap C(I;\mathcal{H})$*. Moreover* R *can be continuosly extended to all of* $BV(I;\mathcal{H})$ *if and only if* R *is locally isotone. This extension is unique if we identify functions which are* $\mathcal{L}^1$*-a.e. equal in I and it is given by* $\overline{\mathsf{R}} : BV(I;\mathcal{H}) \longrightarrow BV(I;\mathcal{H})$

$$\overline{\mathsf{R}}(u) := \mathsf{R}(\widetilde{u}) \circ \ell_u, \qquad u \in BV(I;\mathcal{H}), \tag{3.5}$$

where $\widetilde{u}$ *and* ℓ_u *are defined by Proposition* 3.1*. The operator* $\overline{\mathsf{R}}$ *is rate independent.*

Even if R is not locally isotone we have the following continuity property

Proposition 3.3. *Let I be bounded. Assume that* $\mathsf{R} : Lip(I;\mathcal{H}) \longrightarrow BV(I;\mathcal{H}) \cap C(I;\mathcal{H})$ *is a rate independent operator which is continuous when* $Lip(I;\mathcal{H})$ *and* $BV(I;\mathcal{H}) \cap C(I;\mathcal{H})$ *are endowed with the strict topology. Let* $\overline{\mathsf{R}} : BV(I;\mathcal{H}) \longrightarrow BV(I;\mathcal{H})$ *be defined by formula* (3.5)*. Then* $\|\overline{\mathsf{R}}(u_n) - \overline{\mathsf{R}}(u)\|_{L^1(I;\mathcal{H})} \to 0$ *whenever* $u_n \to u$ *strictly on I,* $u, u_n \in BV(I;\mathcal{H})$*.*

Finally we present the following theorem that will allows us to infer new continuity properties of the vector play operator (defined in Section 3.3) also in the classical framework of absolutely continuous inputs.

Theorem 3.4. *Let I be bounded. Let F be such that* $Lip(I;\mathcal{H}) \subseteq F \subseteq BV(I;\mathcal{H}) \cap C(I;\mathcal{H})$*. Assume that* $\mathsf{R} : F \longrightarrow BV(I;\mathcal{H}) \cap C(I;\mathcal{H})$ *is rate independent and has the following continuity property:*

$$v, v_n \in Lip(I;\mathcal{H}),\ \|v_n - v\|_{W^{1,1}(I;\mathcal{H})} \to 0 \Longrightarrow \mathsf{R}(v_n) \to \mathsf{R}(v) \text{ strictly on } I \tag{3.6}$$

as $n \to \infty$. Then R *is continuous with respect to the strict topology, i.e.*

$$u_n \to u \quad \text{strictly on } I \qquad \Longrightarrow \qquad \mathsf{R}(u_n) \to \mathsf{R}(u) \quad \text{strictly on } I \tag{3.7}$$

as $n \to \infty$.

The previous theorem implies in particular that Theorem 3.2 holds if we replace the strict continuity by the condition (3.6), which is well-known in many particular concrete cases.

Remark 3.5. We point out that we proved a particular case of Theorem 3.2 in [21, 22]: namely the case $\mathcal{H} = \mathbb{R}$, even if in those papers we did not observe that the existence of the continuous extension to $BV(I;\mathbb{R}) \cap C(I;\mathbb{R})$ is granted even if R is not locally isotone. The scalar version of Theorem 3.4 is proved in [24].

The vectorial case is not a rephrasing of the scalar case, but different proofs are needed. Moreover in the vector case the condition of local isotonicity has a clear geometrical meaning. This kind of geodesic condition allows to infer new continuity properties of the vector play operator that are very different form the scalar case. This analysis is performed in Section 5.

3.3. Main applications

In this section we state the main applications of the abstract theorems to rate independent variational inequalities. We assume that

$$\mathcal{Z} \text{ is a closed convex subset of } \mathcal{H}, \quad 0 \in \mathcal{Z}, \tag{3.8}$$

$$z_0 \in \mathcal{Z}, \tag{3.9}$$

$$0 < T < \infty. \tag{3.10}$$

In order to define the play operator we need to recall the following result.

Proposition 3.6. *For every $u \in W^{1,\infty}(]0,T[\,;\mathcal{H})$ there exists $y \in W^{1,\infty}(]0,T[\,;\mathcal{H})$ such that*

$$u(t) - y(t) \in \mathcal{Z} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in \,]0,T[, \tag{3.11}$$

$$\langle u(t) - y(t) - z, y'(t)\rangle \geq 0 \qquad \forall z \in \mathcal{Z}, \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \,]0,T[, \tag{3.12}$$

$$u(0) - y(0) = z_0. \tag{3.13}$$

There is a unique $y \in C([0,T]\,;\mathcal{H})$ which satisfies (3.11)-(3.13) *(equivalently such solution is unique if we identify functions agreeing outside a set having zero Lebesgue measure).*

The previous result is well-known, anyway we will need to outline its proof in Section 5.1. If $u \in W^{1,1}(]0,T[\,;\mathcal{H})$ and $\mathsf{P}(u) := y$, where y is the unique continuous solution of (3.11) – (3.13), we define an operator

$$\mathsf{P} : W^{1,1}(]0,T[\,;\mathcal{H}) \longrightarrow W^{1,1}(]0,T[\,;\mathcal{H})$$

which is usually called (*vector*) *play operator*. It is well known that P is rate independent. The main application of the abstract results is the following

Theorem 3.7. *The play operator is continuous with respect to the strict topology and it admits a unique continuous extension to $BV(]0,T[\,;\mathcal{H}) \cap C([0,T]\,;\mathcal{H})$. Moreover it can be continuously extended to $BV(]0,T[\,;\mathcal{H})$ if and only if $\mathcal{Z}$ is a vector subspace or if*

$$\mathcal{Z} = \{x \in \mathcal{H} \,:\, -\alpha \leq \langle f, x\rangle \leq \beta\}$$

for some $f \in \mathcal{H} \smallsetminus \{0\}$ and $\alpha, \beta \in [0,\infty]$. In both cases such extension is given by

$$\overline{\mathsf{P}}(u) = \mathsf{P}(\widetilde{u}) \circ \ell_u,$$

where $\widetilde{u} \in Lip([0,T]\,;\mathcal{H})$ and ℓ_u are defined by Proposition 3.1 *with $a = 0$, $b = T$.*

Let us observe that the Theorem 3.7 improves a previous result in [14, Proposition 4.11], where the continuity of P in $BV(]0,T[\,;\mathcal{H})\cap C([0,T]\,;\mathcal{H})$ was proved for separable $\mathcal{H}$ and for $\mathcal{Z}$ having suitable regularity properties, *i.e.* such that at every point $x \in \partial\mathcal{Z}$ there exists a unique outward normal $n(x)$ and the resulting mapping n is continuous (see the Appendix A.5 for the notion of normal vectors).

Moreover we answer in a complete manner to the open question about the continuous extendibility of the play operator to $BV(]0,T[\,;\mathcal{H})$.

Remark 3.8. Let us remark that in [14] the strict metric is defined by $\tilde{d}_s(u,v) := \|u-v\|_\infty + |\operatorname{V}_p(u,[0,T]) - \operatorname{V}_p(u,[0,T])|$ for $u, v \in BV(]0,T[\,;\mathcal{H})\cap C([0,T]\,;\mathcal{H})$ continuous of bounded variation. But in the continuous case this turns out to be topologically equivalent to the definition adopted in our paper, by virtue of Corollary 4.8 of Section 4.2 below.

4. Proof of abstract results

Let us recall that $I :=]a,b[$, with $a, b \in [-\infty,\infty]$, $a < b$.

4.1. Properties of reparametrizions

Lemma 4.1. *Let $v : I \longrightarrow \mathcal{H}$ be such that $\operatorname{V}_p(v,I) < \infty$ and let $\beta : \overline{I} \longrightarrow \overline{I}$ be an increasing function satisfying $\beta(a) = a$, $\beta(b) = b$, and $\operatorname{Discont}(v)\cap\operatorname{Discont}(\beta) = \varnothing$. Moreover assume that*

$$\operatorname{V}_p(v,[\beta(t-),\beta(t+)]) = \|v(\beta(t+)) - v(\beta(t-))\|_{\mathcal{H}} \qquad \forall t \in \operatorname{Discont}(\beta). \quad (4.1)$$

Then $\operatorname{V}_p(v\circ\beta, I) = \operatorname{V}_p(v,I)$.

Proof. We prove the lemma when β is left-continuous and $\beta(a) = \beta(a+)$ (for a finite), the other cases being similar (however we need only this case). The inequality $\operatorname{V}_p(v\circ\beta, I) \le \operatorname{V}_p(v,I)$ is obvious, hence $\operatorname{V}_p(v,I)$ is an upper bound for $\{\sum_{j=1}^n \|v(\beta(t_j)) - v(\beta(t_{j-1}))\|_{\mathcal{H}} \,:\, n\in\mathbb{N},\ a < t_0 \le \cdots \le t_n < b\}$. Let $\varepsilon > 0$ be arbitrarily fixed. There exists a subdivision $(t_j)_{j=0}^n$ of I such that

$$\operatorname{V}_p(v,I) < \sum_{j=1}^n \|v(t_j) - v(t_{j-1})\|_{\mathcal{H}} + \varepsilon/2. \quad (4.2)$$

For every $\sigma \in \operatorname{Discont}(\beta)$ there is a possibly empty subset $E_\sigma \subseteq \{t_j\}$ contained in $[\beta(\sigma-),\beta(\sigma+)[$. Adding the points $\beta(\sigma-) = \beta(\sigma)$, $\beta(\sigma+)$ to E_σ, the sum in (4.2) can only increase. Moreover, thanks to the assumption (4.1) we can also replace E_σ by $\{\beta(\sigma),\beta(\sigma+)\}$ without affecting such a sum. Therefore we can assume that (4.2) holds for a subdivision (t_j) such that

$$\{t_j\}_{j=0}^n = \{s_0^1,\dots,s_{k_1-1}^1\}\cup\{\beta(\sigma_1),\beta(\sigma_1+)\}\cup\{s_0^2,\dots,s_{k_2-1}^2\}\cup\{\beta(\sigma_2),\beta(\sigma_2+)\}\cup \cdots\cdots\cup\{s_0^m,\dots,s_{k_m-1}^m\}\cup\{\beta(\sigma_m),\beta(\sigma_m+)\}\cup\{s_0^{m+1},\dots,s_{k_{m+1}}^{m+1}\}$$

where

$$\sigma_i \in \mathrm{Discont}(\beta), \qquad s^i_{k_m} := \beta(\sigma_i) \qquad \forall i = 1, \dots, m;$$
$$\{s^i_0, \dots, s^i_{k_i}\} \subseteq \beta(\overline{I}) \qquad \forall i = 1, \dots, m+1.$$

Hence, setting

$$\tau^i_h := \beta^{-1}(s^i_h) \qquad i = 1, \dots, m+1, \quad j = 0, \dots, k_{m+1},$$

we can write (β is left-continuous)

$$\begin{aligned} &\sum_{j=1}^{n} \|v(t_j) - v(t_{j-1})\|_{\mathcal{H}} \\ &= \sum_{i=1}^{m} \Bigg(\sum_{h=1}^{k_i} \|v(\beta(\tau^i_h)) - v(\beta(\tau^i_{h-1}))\|_{\mathcal{H}} + \|v(\beta(\sigma_i+)) - v(\beta(\sigma_i))\|_{\mathcal{H}} \\ &\qquad + \|v(\beta(\tau^{i+1}_0)) - v(\beta(\sigma_i+))\|_{\mathcal{H}} \Bigg) + \sum_{h=1}^{k_{m+1}} \|v(\beta(\tau^{m+1}_h)) - v(\beta(\tau^{m+1}_{h-1}))\|_{\mathcal{H}} \end{aligned}$$

The fact that $\mathrm{Discont}(v) \cap \mathrm{Discont}(\beta) = \varnothing$ yields that for every $i = 1, \dots, m$ there exists $\tilde{\sigma}_i$ very near σ_i, such that $\sigma_i < \tilde{\sigma}_i$ and $\|v(\beta(\sigma_i+)) - v(\beta(\tilde{\sigma}_i))\|_{\mathcal{H}} < \varepsilon/(2m)$, so that

$$\begin{aligned} &\sum_{j=1}^{n} \|v(t_j) - v(t_{j-1})\|_{\mathcal{H}} \\ &\le \sum_{i=1}^{m} \Bigg(\sum_{h=1}^{k_i} \|v(\beta(\tau^i_h))) - v(\beta(\tau^i_{h-1})\|_{\mathcal{H}} + \|v(\beta(\tilde{\sigma}_i)) - v(\beta(\sigma_i))\|_{\mathcal{H}} \\ &\qquad + \|v(\beta(\tau^{i+1}_0)) - v(\beta(\tilde{\sigma}_i))\|_{\mathcal{H}} + \varepsilon/m \Bigg) \\ &\quad + \sum_{h=1}^{k_{m+1}} \|v(\beta(\tau^{m+1}_h)) - v(\beta(\tau^{m+1}_{h-1}))\|_{\mathcal{H}}. \end{aligned}$$

That is, we have found a subdivision $(\theta_j)_{j=0}^{r}$ such that $\mathrm{V}_p(v, I) < \sum_{j=1}^{n} \|v(\beta(\theta_j)) - v(\beta(\theta_{j-1}))\|_{\mathcal{H}} + \varepsilon$, and the lemma is proved. □

Lemma 4.2. *Assume that I is bounded and that $u \in BV(I; \mathcal{H})$. Let $\widetilde{u}$ and ℓ_u be the maps provided by Proposition* 3.1. *Let $\phi : \overline{I} \longrightarrow \overline{I}$ be increasing and surjective, and set $v := u \circ \phi$. Then $\ell_v = \ell_u \circ \phi$ and $\widetilde{v} = \widetilde{u} \circ \ell_v$, or in other terms $\widetilde{u \circ \phi} = \widetilde{u}$.*

Proof. The assumptions on ϕ implies that $\mathrm{V}_e(v,]0,t[) = \mathrm{V}_p(u^l \circ \phi,]0,t[) = \mathrm{V}_p(u^l,]0,\phi(t)[) = \mathrm{V}_e(u,]0,\phi(t)[)$ for every $t \in \overline{I}$, therefore

$$\ell_v(t) = \frac{b-a}{\|\mathrm{D}\,v\|}\,\mathrm{V}_e(v,]0,t[) = \frac{b-a}{\|\mathrm{D}\,v\|}\,\mathrm{V}_e(u,]0,\phi(t)[) = (\ell_u \circ \phi)(t) \qquad \forall t \in I.$$

Thus we have $\widetilde{v} \circ \ell_v = v = u \circ \phi = \widetilde{u} \circ \ell_u \circ \phi = \widetilde{u} \circ \ell_v$ and the thesis follows from the uniqueness of $\widetilde{v}$. □

Lemma 4.3. *Assume that $u \in BV(I;\mathcal{H})$ and let $\widetilde{u}$ be its reparametrization defined by Proposition* 3.1. *Then we have that*

$$\|\widetilde{u}'(\sigma)\|_{\mathcal{H}} = \frac{\|\mathrm{D}\,u\|}{b-a} \qquad \text{for } \mathcal{L}^1\text{-a.e. } \sigma \in I. \tag{4.3}$$

In particular $\|\mathrm{D}\,u\| = \|\mathrm{D}\,\widetilde{u}\|$.

Proof. Observe that by (3.4) and Lemma 4.1 we have that

$$\mathrm{V}_p(\widetilde{u},]a,\ell_u(t)[) = \mathrm{V}_p(\widetilde{u} \circ \ell_u,]a,t[) = \mathrm{V}_e(\widetilde{u} \circ \ell_u,]a,t[) \qquad \forall t \in \overline{I},$$

last equality holding because $\widetilde{u} \circ \ell_u$ is left-continuous. But $u = \widetilde{u} \circ \ell_u$ $\mathcal{L}^1$-a.e., therefore, by (3.1),

$$\mathrm{V}_p(\widetilde{u},]a,\ell_u(t)[) = \mathrm{V}_e(u,]a,t[) = \frac{\|\mathrm{D}\,u\|}{b-a}\,(\ell_u(t) - a) \qquad \forall t \in \overline{I}.$$

In particular, for $t = b$, this yields the equality $\|\mathrm{D}\,u\| = \|\mathrm{D}\,\widetilde{u}\|$. More generally if $\sigma \in \ell_u(\overline{I})$, *i.e.* $\sigma = \ell_u(t)$ for some $t \in \overline{I}$, then $\mathrm{V}_p(\widetilde{u},]a,\sigma[) = \frac{\|\mathrm{D}\,u\|}{b-a}\,(\sigma - a)$. But $\sigma \longmapsto \mathrm{V}_p(\widetilde{u},]a,\sigma[)$ is continuous on I and affine on $I \setminus \ell_u(I)$, hence we get that

$$\mathrm{V}_p(\widetilde{u},]a,\sigma[) = \frac{\|\mathrm{D}\,u\|}{b-a}\,(\sigma - a) \qquad \forall \sigma \in I.$$

Therefore, as $\widetilde{u}$ is Lipschitz continuous, we have

$$\frac{\|\mathrm{D}\,u\|}{b-a}\,(\sigma - a) = \mathrm{V}_p(\widetilde{u},]a,\sigma[) = \int_a^\sigma \|\widetilde{u}'(\tau)\|_{\mathcal{H}}\,\mathrm{d}\tau \qquad \forall \sigma \in \overline{I},$$

thus differentiating we infer that $\|\widetilde{u}'(\sigma)\|_{\mathcal{H}} = \|\mathrm{D}\,u\|/(b-a)$ for $\mathcal{L}^1$-a.e. $\sigma \in I$. □

4.2. Properties of strict convergence

Let us start by recalling the following:

Lemma 4.4. *Let $v_n : I \longrightarrow \mathbb{R}$ be a sequence of increasing functions which is pointwise converging to a continuous function $v : I \longrightarrow \mathbb{R}$. Assume that the sequences $v_n(a+)$ and $v_n(b-)$ have a finite limit. Then v_n converges uniformly to v.*

Proof. If I is bounded, a proof can be found in [9, Theorem 10, page 166]. If I is unbounded, the lemma can be easily inferred by the bounded case, *e.g.*, letting ψ be a homeomorphism from $[a,b]$ to $[0,1]$ and defining $w := v \circ \psi^{-1}$, $w_n(s) := v_n \circ \psi^{-1}$, $s \in [0,1]$, for every $n \in \mathbb{N}$. Then the assumptions on $v_n(a+)$ and $v_n(b-)$ allow to apply [9, Theorem 10, page 166] to w_n, and this yields the result for v_n. □

Lemma 4.5. *If $u, u_n \in BV(I;\mathcal{H})$ and $u_n \to u$ strictly on I, then $\|\mathrm{D}\,u_n\|(]c,d[) \to \|\mathrm{D}\,u\|(]c,d[)$ for every $c, d \in \overline{I} \smallsetminus A(u)$, $c < d$.*

Proof. As $c, d \notin A(u)$ we have, thanks to Corollary 2.11 and formula (2.8)

$$\begin{aligned}
&\limsup_{n\to\infty} \|\mathrm{D}\,u_n\|(]c,d[)\\
&\le \limsup_{n\to\infty} (\|\mathrm{D}\,u_n\|(I) - \|\mathrm{D}\,u_n\|(]a,c[) - \|\mathrm{D}\,u_n\|(]c,d[))\\
&\le \|\mathrm{D}\,u\|(]a,b[) - \liminf_{n\to\infty}\|\mathrm{D}\,u_n\|(]a,c[) - \liminf_{n\to\infty}\|\mathrm{D}\,u_n\|(]c,d[)\\
&\le \|\mathrm{D}\,u\|(]a,b[) - \|\mathrm{D}\,u\|(]a,c[) - \|\mathrm{D}\,u\|(]c,d[) = \|\mathrm{D}\,u\|(]c,d[).
\end{aligned}$$

On the other hand by Corollary 2.11 we know that

$$\|\mathrm{D}\,u\|(]c,d[) \le \liminf_{n\to\infty} \|\mathrm{D}\,u_n\|(]c,d[)$$

and we are done. □

Lemma 4.6. *Assume $u, u_n \in BV(I;\mathcal{H})$ are left-continuous and $u_n \to u$ strictly as $n \to \infty$. Then $u_n(t) \to u(t)$ for every $t \in I \smallsetminus A(u)$. Moreover $u_n(a+) \to u(a+)$ and $u_n(b-) \to u(b-)$.*

Proof. Take $t \in I \smallsetminus A(u) = \mathrm{Cont}(u)$ and $\varepsilon > 0$. By elementary properties of the pointwise variation we have that $\lim_{s\nearrow t} \mathrm{V}_p(u,[s,t]) = \|u(t) - u(t-)\|_{\mathcal{H}} = 0$. Moreover the set $\mathrm{Cont}(u)$ is at most countable and $\|u - u_n\|_{L^1(I;\mathcal{H})} \to 0$, hence, possibly extracting a subsequence which we do not relabel, there exists $t_0 \in \mathrm{Cont}(u)$ such that $t_0 < t$, $u_n(t_0) \to u(t_0)$, and $\|\mathrm{D}\,u\|(]t_0,t[) = \mathrm{V}_p(u,[t_0,t]) < \varepsilon/2$. Then

$$\begin{aligned}
\|u_n(t) - u(t)\|_{\mathcal{H}} &\le \|u_n(t) - u_n(t_0)\|_{\mathcal{H}} + \|u_n(t_0) - u(t_0)\|_{\mathcal{H}} + \|u(t_0) - u(t)\|_{\mathcal{H}}\\
&\le \|\mathrm{D}\,u_n\|(]t_0,t[) + \|u_n(t_0) - u(t_0)\|_{\mathcal{H}} + \|\mathrm{D}\,u\|(]t_0,t[).
\end{aligned}$$

Therefore taking the upper limit for $n \to \infty$ and using Lemma 4.5 we get $\limsup_{n\to\infty} \|u_n(t) - u(t)\|_{\mathcal{H}} \le \varepsilon$, which proves the first part of the Lemma because of the arbitrariness of ε and the uniqueness of the limit. A similar argument proves the convergence of $u_n(a+)$ and $u_n(b-)$. □

A straightforward consequence of the previous lemma is the following:

Corollary 4.7. *Assume $u, u_n \in BV(I;\mathcal{H})$ and $u_n \to u$ strictly as $n \to \infty$. Then $u_n(t) \to u(t)$ for $\mathcal{L}^1$-a.e. $t \in I$.*

Corollary 4.8. *Assume $u, u_n \in BV(I;\mathcal{H}) \cap C(I;\mathcal{H})$ and $u_n \to u$ strictly as $n \to \infty$. Then $u_n \to u$ uniformly on compact subsets of $\overline{I}$.*

Proof. Thank to Lemma 4.6 we have that $u_n(t) \to u(t)$ for every $t \in \overline{I}$. Therefore in order to prove the uniform convergence it is enough to prove equicontinuity. First of all let us observe that Lemma 4.5 yields the pointwise convergence of the sequence $V_n(t) := \|\mathrm{D}\, u_n\|\,(]a,t[)$ to the function $V(t) := \|\mathrm{D}\, u\|\,(]a,t[)$. For every $n \in \mathbb{N}$ the function V_n is increasing and V is continuous, due to the continuity of u. Therefore we can apply Lemma 4.4 and deduce that V_n is uniformly convergent to V. Hence the sequence (V_n) is equicontinuous, and this implies that for $\varepsilon > 0$ arbitrarily given, there exists $\delta > 0$ such that for every $c, d \in I$ the following implication holds:

$$0 \le d - c < \delta \implies \sup_{n\in\mathbb{N}} \|\mathrm{D}\, u_n\|\,(]c,d[) < \varepsilon.$$

Now we can infer the equicontinuity of (u_n), indeed if $0 \le t - s < \delta$ we get

$$\|u_n(t) - u_n(s)\|_{\mathcal{H}} = \|\,\mathrm{D}\, u_n(]s,t[)\|_{\mathcal{H}} \le \|\mathrm{D}\, u_n\|\,(]s,t[) < \varepsilon \quad \forall n \in \mathbb{N}. \qquad \square$$

Lemma 4.9. *Let $c, d \in \mathbb{R}$ be such that $c < d$ and assume $x, y \in \mathcal{H}$. Then the affine map $w : [c,d] \longrightarrow \mathcal{H}$ defined by $w(t) := x + t(y-x)/(d-c)$ is the only minimizer of the functional $v \longmapsto \|v'\|^2_{L^2(]c,d[;\mathcal{H})}$ in the set $\{v \in Lip([c,d];\mathcal{H}) : v(c) = x,\ v(d) = y\}$.*

Proof. Let us consider $z \in Lip([c,d];\mathcal{H})$ such that $z(c) = x$ and $z(d) = y$. We first consider the case when $z([c,d]) \ne w([c,d])$. Hence there exists $t_0 \in I$ such that $z(t_0)$ does not belong to the segment with endpoints x and y. We have that $\|z(t_0) - x\|_{\mathcal{H}} + \|y - z(t_0)\|_{\mathcal{H}} > \|y - x\|_{\mathcal{H}}$, therefore $\mathrm{V}_p(z,[c,d]) > \|y-x\|_{\mathcal{H}} = \mathrm{V}_p(w,[c,d])$. Hence using Schwarz inequality we have

$$\begin{aligned}
\|z'\|_{L^2(]c,d[;\mathcal{H})} &= \left(\int_c^d \|z'(t)\|^2_{\mathcal{H}}\,\mathrm{d}t\right)^{1/2} \ge \frac{1}{(d-c)^{1/2}}\int_c^d \|z'(t)\|_{\mathcal{H}}\,\mathrm{d}t \\
&> \frac{1}{(d-c)^{1/2}}\|y-x\|_{\mathcal{H}} = \left(\int_c^d \frac{\|y-x\|^2_{\mathcal{H}}}{(d-c)^2}\,\mathrm{d}t\right)^{1/2} \\
&= \|w'\|_{L^2(]c,d[;\mathcal{H})}.
\end{aligned}$$

If instead $z([c,d])$ is the segment $w([c,d])$, then it easily seen that we can reduce to a one dimensional problem, and the affine functions are the only minimizers of the given functional with Dirichlet boundary conditions. $\square$

Proposition 4.10. *Assume I is bounded, $u, u_n \in BV(I;\mathcal{H})$ for every $n \in \mathbb{N}$ and $u_n \to u$ strictly on I. Let ℓ and ℓ_n be the "normalized" arc length functions of*

u and u_n defined as in (3.1)*, and let $\widetilde{u}$ and $\widetilde{u}_n$ be the unique Lipschitz functions satisfying* (3.3)-(3.4) *with $u, \widetilde{u}, \ell_u$ replaced respectively by $u, \widetilde{u}, \ell$ and $u_n, \widetilde{u}_n, \ell_n$, as given by Proposition 3.1. Then*

$$\ell_n(t) \to \ell(t) \qquad \forall t \in I \smallsetminus A(u), \tag{4.4}$$

$$\widetilde{u}_n \to \widetilde{u} \qquad \text{in } W^{1,p}(I;\mathcal{H}) \quad \forall p \in [1, +\infty[\,. \tag{4.5}$$

Proof. Formula (4.4) is a consequence of (3.1) and Lemma 4.5. Now we prove (4.5). From the strict convergence of u_n and Lemma 4.3 we obtain the convergence

$$\|\mathrm{D}\,\widetilde{u}_n\| \to \|\mathrm{D}\,\widetilde{u}\| \qquad \text{as } n \to \infty. \tag{4.6}$$

Now observe that $\widetilde{u}(a+) = u(a+)$ and $\widetilde{u}_n(a+) = u_n(a+)$, therefore by Lemma 4.6 we get

$$\widetilde{u}_n(a+) \to \widetilde{u}(a+) \qquad \text{in } \mathcal{H} \tag{4.7}$$

as $n \to \infty$. We also have

$$\|\widetilde{u}_n(\sigma)\|_{\mathcal{H}} \le \|\widetilde{u}_n(a+)\|_{\mathcal{H}} + \|\mathrm{D}\,\widetilde{u}_n\| \qquad \forall \sigma \in I \tag{4.8}$$

and, by (4.3)

$$\|\widetilde{u}_n'\|_{L^\infty(I;\mathcal{H})} = \frac{\|\mathrm{D}\,u_n\|}{b-a}. \tag{4.9}$$

Hence (4.6)-(4.9) let us infer that $(\widetilde{u}_n)$ is bounded in $W^{1,p}(I;\mathcal{H})$ for every $p \in [1,\infty]$. Hence there exists $\widehat{u} \in Lip(I;\mathcal{H})$ such that, at least for a subsequence which we do not relabel,

$$\widetilde{u}_n \overset{*}{\rightharpoonup} \widehat{u} \qquad \text{in } W^{1,p}(I;\mathcal{H}), \quad p \in [1,\infty]. \tag{4.10}$$

This convergence, together with (4.7) implies that $\widetilde{u}_n(a+) \to \widehat{u}(a+) = \widetilde{u}(a+)$ in $\mathcal{H}$, from which we infer that

$$\widetilde{u}_n(\sigma) \rightharpoonup \widehat{u}(\sigma) \qquad \forall \sigma \in I, \tag{4.11}$$

indeed for every $x \in \mathcal{H}$

$$\langle \widetilde{u}_n(\sigma) - \widehat{u}(\sigma), x\rangle = \langle \widetilde{u}_n(a+) - \widehat{u}(a+), x\rangle + \int_0^\sigma \langle \widetilde{u}_n'(\tau) - \widehat{u}'(\tau), x\rangle \,\mathrm{d}\tau \to 0.$$

Now for every $x \in \mathcal{H}$ and for every $n \in \mathbb{N}$ define $f_n^x : I \longrightarrow \mathbb{R}$ by $f_n^x(\sigma) := \langle \widetilde{u}_n(\sigma), x\rangle$ and $f^x : I \longrightarrow \mathbb{R}$ by $f^x(\sigma) := \langle \widehat{u}(\sigma), x\rangle$. We have seen that $f_n^x \to f^x$ pointwise in I. Estimate (4.8) and (4.6) imply that $\|f_n^x\|_\infty$ is bounded, and for every pair $\sigma, \tau \in I$ we have, thanks to (4.9), that

$$\begin{aligned} |f_n^x(\sigma) - f_n^x(\tau)| &\le \|x\|_{\mathcal{H}} \|\widetilde{u}_n(\sigma) - \widetilde{u}_n(\tau)\|_{\mathcal{H}} \le \|x\|_{\mathcal{H}} \operatorname{Lip}(\widetilde{u}_n)|\tau - \sigma| \\ &\le \|x\|_{\mathcal{H}} \frac{\|\mathrm{D}\,u_n\|}{b-a}|\sigma - \tau|, \end{aligned}$$

thus $(f_n^x)_n$ is equicontinuous and $f_n^x \to f^x$ uniformly on I for every $x \in \mathcal{H}$. But $\ell_n(t) \to \ell(t)$ for $\mathcal{L}^1$-a.e. $t \in I$, hence for every $x \in \mathcal{H}$ we have that $f_n^x(\ell_n(t)) \to f^x(\ell(t))$ for $\mathcal{L}^1$-a.e. $t \in I$, *i.e.*

$$\widetilde{u}_n(\ell_n(t)) \rightharpoonup \widehat{u}(\ell(t)) \quad \text{in } \mathcal{H}, \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in I. \tag{4.12}$$

On the other hand by Corollary 4.7 we know that $\widetilde{u}_n(\ell_n(t)) = u_n(t) \to u(t)$ for $\mathcal{L}^1$-a.e. $t \in I$, hence, by construction of $\widetilde{u}$ and by the continuity of $\widetilde{u}$ and $\widehat{u}$, we get that $\widehat{u} = \widetilde{u}$ on $\ell(I)$. Observe now that $\widetilde{u}_n' \rightharpoonup \widehat{u}'$ in $L^2(I;\mathcal{H})$, therefore

$$\begin{aligned}
\|\widehat{u}'\|^2_{L^2(I;\mathcal{H})} &\le \liminf_{n\to\infty} \|\widetilde{u}_n'\|^2_{L^2(I;\mathcal{H})} = \liminf_{n\to\infty} \int_a^b \left(\frac{\|\mathrm{D}\, u_n\|}{b-a}\right)^2 \mathrm{d}\sigma \\
&= \int_a^b \left(\frac{\|\mathrm{D}\, u\|}{b-a}\right)^2 \mathrm{d}\sigma = \int_a^b \|\widetilde{u}'(\sigma)\|^2_{\mathcal{H}}\, \mathrm{d}\sigma = \|\widetilde{u}'\|^2_{L^2(I;\mathcal{H})}.
\end{aligned}$$

Thus by Lemma 4.9 and by (3.4), we infer that $\widetilde{u} = \widehat{u}$ on I, so that

$$\widetilde{u}_n \rightharpoonup \widetilde{u} \qquad \text{in } W^{1,p}(I;\mathcal{H}). \tag{4.13}$$

Now we prove that $\widetilde{u}_n' \to \widetilde{u}'$ in $W^{1,p}(I;\mathcal{H})$ for every $p \in [1,+\infty[$. For every $n \in \mathbb{N}$ we have that

$$\|\widetilde{u}_n'\|^p_{L^p(I;\mathcal{H})} = \int_a^b \|\widetilde{u}_n'(\sigma)\|^p_{\mathcal{H}}\, \mathrm{d}\sigma = \int_a^b \left(\frac{\|\mathrm{D}\, u_n\|}{b-a}\right)^p \mathrm{d}\sigma.$$

Hence, since $\|\mathrm{D}\, u_n\| \to \|\mathrm{D}\, u\|$ as $n \to \infty$, we get that

$$\lim_{n\to\infty} \|\widetilde{u}_n'\|^p_{L^p(I;\mathcal{H})} = \int_a^b \left(\frac{\|\mathrm{D}\, u\|}{b-a}\right)^p \mathrm{d}\sigma = \int_a^b \|\widetilde{u}'(\sigma)\|^p_{\mathcal{H}}\, \mathrm{d}\sigma.$$

Therefore we have shown that

$$\|\widetilde{u}_n'\|_{L^p(I;\mathcal{H})} \to \|\widetilde{u}'\|_{L^p(I;\mathcal{H})} \qquad \text{as } n \to \infty, \tag{4.14}$$

But we also have

$$\widetilde{u}_n' \rightharpoonup \widetilde{u}' \qquad \text{in } L^p(I;\mathcal{H}) \tag{4.15}$$

as $n \to \infty$. Hence, as $L^p(I;\mathcal{H})$ is uniformly convex for $p \in]1,+\infty[$, we have that (4.14)–(4.15) imply that

$$\widetilde{u}_n' \to \widetilde{u}' \qquad \text{in } L^p(I;\mathcal{H}) \tag{4.16}$$

for every $p \in]1,+\infty[$ as $n \to \infty$ (*cf. e.g.* [5, Proposition III.30]). Since I is bounded we get that (4.16) holds also for $p = 1$ and we are done. From (4.16) follows that $\widetilde{u}_n \to \widetilde{u}$ in $L^p(I;\mathcal{H})$ for every $p \in [1,\infty[$, and we are done. □

Remark 4.11. If u_n is a strictly convergent sequence, then in general $\widetilde{u}_n$ does not converge in $W^{1,\infty}(I;\mathcal{H})$. A counterexample in the scalar case is given in [24, Remark 4.1].

We conclude this section with a lemma which is useful to study rate independent operators that are not locally isotone. If $x, y \in \mathcal{H}$, we use the notation $\operatorname{seg}[x, y]$ to denote the segment $\{(1-\lambda)x + \lambda y \ : \ \lambda \in [0,1]\}$.

Lemma 4.12. *Assume that I is bounded and that $u \in BV(I;\mathcal{H})$ is left-continuous and that there exist $c, d \in I$ such that $c < d$ and $\mathrm{V}_p(u,[c,d]) = \|u(d)-u(c)\|_{\mathcal{H}}$. Then $u([c,d]) \subseteq \operatorname{seg}[u(c),u(d)]$ and $\widetilde{u}$ is affine on $[\ell_u(c),\ell_u(d)]$. Moreover if u is continuous then $u([c,d]) = \operatorname{seg}[u(c),u(d)]$.*

Proof. The inclusion $u([c,d]) \subseteq \operatorname{seg}[u(c),u(d)]$ is an easy consequence of the euclidean structure of $\mathcal{H}$ even if u is not left-continuous. If u is continuous it is clear that equality holds. Concerning the last property, observe that $\widetilde{u}(\ell_u(c)) = u(c)$ and $\widetilde{u}(\ell_u(d)) = u(d)$, hence using (4.3), (3.1) and the left continuity of u

$$\begin{aligned}\mathrm{V}_p(\widetilde{u},[\ell_u(c),\ell_u(d)]) &= \int_{\ell_u(c)}^{\ell_u(d)} \frac{\|\mathrm{D}u\|}{b-a}\,\mathrm{d}\sigma = \frac{\|\mathrm{D}u\|}{b-a}(\ell_u(d)-\ell_u(c)) \\ &= |\mathrm{D}u|\,([c,d[) = \mathrm{V}_p(u,\,]c,d[) = \mathrm{V}_p(u,[c,d]).\end{aligned}$$

Hence by the first part of the lemma we infer that $\widetilde{u}([\ell_u(c),\ell_u(d)]) = \operatorname{seg}[u(c),u(d)]$ and it is not difficult to see that

$$\widetilde{u}(\sigma) = u(c) + \frac{\mathrm{V}_p(\widetilde{u},[\ell_u(c),\sigma])}{\|u(d)-u(c)\|_{\mathcal{H}}}[u(d)-u(c)] \qquad \forall \sigma \in [\ell_u(c),\ell_u(d)]\,. \tag{4.17}$$

But $\mathrm{V}_p(\widetilde{u},[\ell_u(c),\sigma]) = \int_{\ell_u(c)}^{\sigma} \|\mathrm{D}u\|/(b-a)\,\mathrm{d}\sigma = (\sigma-\ell_u(c))\|\mathrm{D}u\|/(b-a)$ which together with (4.17) yields that $\widetilde{u}$ is affine. ☐

4.3. Proof of main theorems

We start with the

Proof of Proposition 3.3. Let us recall that $\overline{\mathsf{R}} : BV(I;\mathcal{H}) \longrightarrow \mathcal{H}^I$ is defined by

$$\overline{\mathsf{R}}(u) := \mathsf{R}(\widetilde{u}) \circ \ell_u, \qquad u \in BV(I;\mathcal{H}),$$

where $\widetilde{u}$ and ℓ_u are defined by Proposition 3.1. The rate independence of R implies that $\overline{\mathsf{R}}$ extends R, indeed if $u \in Lip(I;\mathcal{H})$, then $\ell_u \in Lip(I)$ and $\mathsf{R}(u) = \mathsf{R}(\widetilde{u} \circ \ell_u) = \mathsf{R}(\widetilde{u}) \circ \ell_u = \overline{\mathsf{R}}(u)$. It is clear that $\overline{\mathsf{R}}(u) \in BV(I;\mathcal{H})$ for every $u \in BV(I;\mathcal{H})$. In order to prove the proposition let us take a sequence (u_n) which strictly converges to u and let us denote the normalized arc length function ℓ_{u_n} simply by ℓ_n. We have to show that

$$\overline{\mathsf{R}}(u_n) \to \overline{\mathsf{R}}(u) \qquad \text{in } L^1(I;\mathcal{H}) \tag{4.18}$$

as $n \to \infty$. From Proposition 4.10 we infer that $\widetilde{u}_n \to \widetilde{u}$ strictly on I, hence, as R is continuous, we have that $\mathsf{R}(\widetilde{u}_n) \to \mathsf{R}(\widetilde{u})$ strictly on I. Moreover $\mathsf{R}(\widetilde{u}_n)$ and $\mathsf{R}(\widetilde{u})$ are continuous maps, hence by Corollary 4.8 $\mathsf{R}(\widetilde{u}_n) \to \mathsf{R}(\widetilde{u})$ uniformly on I. Therefore by (4.4) we have that $\ell_n \to \ell_u$ $\mathcal{L}^1$-a.e. in I and

$$\overline{\mathsf{R}}(u_n)(t) = \mathsf{R}(\widetilde{u}_n)(\ell_n(t)) \to \mathsf{R}(\widetilde{u})(\ell_u(t)) = \overline{\mathsf{R}}(u)(t) \qquad \forall t \in I \setminus A(u).$$

Observe also that by the uniform convergence we get

$$\sup_{n\in\mathbb{N}} \|\overline{\mathsf{R}}(u_n)\|_\infty = \sup_{n\in\mathbb{N}} \|\mathsf{R}(\widetilde{u}_n) \circ \ell_n\|_\infty \leq \sup_{n\in\mathbb{N}} \|\mathsf{R}(\widetilde{u}_n)\|_\infty < \infty,$$

thus in order to obtain (4.18) it suffices to apply the dominated convergence theorem. □

For the sake of clarity let us explicitly state the following elementary fact:

Lemma 4.13. *If $c, d \in \overline{I}$, $c < d$, $u, v \in BV(I; \mathcal{H})$, and if $\phi : [c, d] \longrightarrow \overline{I}$ is a continuous increasing nonconstant function such that $u = v \circ \phi$, then*

$$\mathrm{V}_p(u, [c, d]) = \|u(d) - u(c)\|_{\mathcal{H}} \iff \mathrm{V}_p(v, [\phi(c), \phi(d)]) = \|v(\phi(d)) - v(\phi(c))\|_{\mathcal{H}}.$$

Lemma 4.14. *Let $F \subseteq \mathcal{H}^I$ be such that $Lip(I; \mathcal{H}) \subseteq F \subseteq BV(I; \mathcal{H}) \cap C(I; \mathcal{H})$. Assume that $\mathsf{R} : F \longrightarrow BV(I; \mathcal{H}) \cap C(I; \mathcal{H})$ is rate independent. If $c, d \in \overline{I}$, $c < d$, and if $u \in BV(I; \mathcal{H}) \cap C(I; \mathcal{H})$ is constant on $[c, d]$, then $\mathsf{R}(u)$ is constant on $[c, d]$.*

Proof. Since u is equal to a constant on $[c, d]$, we have that $\ell_u(t) = \ell_u(c)$ for every $t \in [c, d]$. Moreover, since u is continuous, then ℓ_u is also continuous, therefore by rate independence we have $\mathsf{R}(u)(t) = \mathsf{R}(\widetilde{u})(\ell_u(t)) = \mathsf{R}(\widetilde{u})(\ell_u(c))$ for every $t \in [c, d]$, and the lemma is proved. □

Lemma 4.15. *Let $F \subseteq \mathcal{H}^I$ be such that $Lip(I; \mathcal{H}) \subseteq F \subseteq BV(I; \mathcal{H}) \cap C(I; \mathcal{H})$. Assume that $\mathsf{R} : F \longrightarrow BV(I; \mathcal{H}) \cap C(I; \mathcal{H})$ is rate independent. If R is not locally isotone then there exist $u \in Lip(I; \mathcal{H})$ and $c, d \in \overline{I}$ such that $c < d$, u is affine nonconstant on $[c, d]$ and $\mathrm{V}_e(\mathsf{R}(u), [c, d]) > \|\mathsf{R}(u)(d) - \mathsf{R}(u)(c)\|_{\mathcal{H}}$.*

Proof. By assumption there exists $v \in F$ and $s, t \in I$, $s < t$, such that $\mathrm{V}_p(v, [s, t]) = \|v(s) - v(t)\|_{\mathcal{H}}$ and $\mathrm{V}_p(\mathsf{R}(v), [s, t]) > \|\mathsf{R}(v)(t) - \mathsf{R}(v)(s)\|_{\mathcal{H}}$. Now let us consider the reparametrized map $\widetilde{v} \in Lip(I; \mathcal{H})$. By Lemma 4.12 we have that $\widetilde{v}$ is affine on $[\ell_v(t), \ell_v(s)]$. Moreover as ℓ_v is continuous and $\mathsf{R}(v) = \mathsf{R}(\widetilde{v}) \circ \ell_v$ we infer that $\mathrm{V}_p(\mathsf{R}(\widetilde{v}), [\ell_v(s), \ell_v(t)]) = \mathrm{V}_p(\mathsf{R}(v), [s, t]) > \|\mathsf{R}(v)(t) - \mathsf{R}(v)(s)\|_{\mathcal{H}} = \|\mathsf{R}(\widetilde{u})(\ell_u(t)) - \mathsf{R}(\widetilde{u})(\ell_u(s))\|_{\mathcal{H}}$. Hence the lemma follows with $u = \widetilde{v}$, $c = \ell_v(s)$, $d = \ell_v(t)$, and observing that $\widetilde{v}$ is non constant on $[c, d]$ by Lemma 4.14. □

The following proposition proves part of Theorem 3.2.

Proposition 4.16. *Let $F \subseteq \mathcal{H}^I$ be such that $Lip(I;\mathcal{H}) \subseteq F \subseteq BV(I;\mathcal{H}) \cap C(I;\mathcal{H})$. Assume that $\mathsf{R} : F \longrightarrow BV(I;\mathcal{H})$ is rate independent and continuous when F and $BV(I;\mathcal{H})$ are endowed with the strict metric. If R is not locally isotone then R cannot be continuosly extended to $BV(I;\mathcal{H})$.*

Proof. By previous Lemma there exists $u \in Lip(I;\mathcal{H})$ and $c, d \in I$, $c < d$, such that u is non constant on $[c,d]$, $\mathrm{V}_p(u,[c,d]) = \|u(d)-u(c)\|_{\mathcal{H}}$, and $\mathrm{V}_p(\mathsf{R}(u),[c,d]) > \|\mathsf{R}(u)(d) - \mathsf{R}(u)(c)\|_{\mathcal{H}}$. Observe that as $u = \widetilde{u} \circ \ell_u$ we have that $\mathsf{R}(u) = \mathsf{R}(\widetilde{u}) \circ \ell_u$ so that

$$\mathrm{V}_p(\mathsf{R}(\widetilde{u}),[\ell_u(c),\ell_u(d)]) > \|\mathsf{R}(\widetilde{u})(\ell_u(d)) - \mathsf{R}(\widetilde{u})(\ell_u(c))\|_{\mathcal{H}}. \tag{4.19}$$

Now let us define $z \in \mathcal{H}^I$ by

$$z(t) := \begin{cases} u(t) & \text{if } t \notin \,]c,d[\\ u(c) & \text{if } t \in \,]c,d[\end{cases}$$

Since u is non constant we have that $\mathrm{Discont}(z) = \mathrm{Discont}(\ell_u) = \{d\}$, thus $z \in BV(I;\mathcal{H}) \setminus C(I;\mathcal{H})$, $A(z) = \{d\}$, and

$$\ell_z(t) = \ell_u(t) \text{ if } t \notin \,]c,d[, \qquad \ell_z(t) = \ell_u(c) \text{ if } t \in \,]c,d[, \tag{4.20}$$

since u is affine on $[c,d]$. From the equalities $u = \widetilde{u} \circ \ell_u$, $z = \widetilde{z} \circ \ell_z$, from (4.20), and from the uniqueness properties of reparametrizations stated in Proposition 3.1 it follows that

$$\widetilde{z}(\sigma) = \widetilde{u}(\sigma) \quad \text{if } \sigma \notin \,]\ell_u(d),\ell_u(d+)[, \qquad \widetilde{z} \text{ is affine on }]\ell_u(d),\ell_u(d+)[.$$

But $\widetilde{u}$ is affine on $]\ell_u(d),\ell_u(d+)[$, thus $\widetilde{z} = \widetilde{u}$ and

$$\mathrm{V}_p(\mathsf{R}(\widetilde{z}),[\ell_u(c),\ell_u(d)]) > \|\mathsf{R}(\widetilde{z})(\ell_u(d)) - \mathsf{R}(\widetilde{z})(\ell_u(c))\|_{\mathcal{H}}. \tag{4.21}$$

Now let $z_n \in Lip(I;\mathcal{H})$ be such that $z_n \to z$ strictly in $BV(I;\mathcal{H})$. Let us denote the functions ℓ_{z_n} simply by ℓ_n. By Proposition 3.3 we have that

$$\mathsf{R}(z_n) \to \mathsf{R}(\widetilde{z}) \circ \ell_z \qquad \text{in } L^1(I;\mathcal{H}) \tag{4.22}$$

as $n \to \infty$. Now let us compute the limit of $\|\mathrm{D}\,\mathsf{R}(z_n)\|$. Thanks to the continuity of $\mathsf{R}(\widetilde{z}_n)$ and ℓ_n we have

$$\|\mathrm{D}\,(\mathsf{R}(z_n))\| = \|\mathrm{D}\,(\mathsf{R}(\widetilde{z}_n \circ \ell_n))\| = \mathrm{V}_p(\mathsf{R}(\widetilde{z}_n) \circ \ell_n, I) = \mathrm{V}_p(\mathsf{R}(\widetilde{z}_n), I) = \|\mathrm{D}\,(\mathsf{R}(\widetilde{z}_n))\|.$$

Now, R is continuous on $Lip(I;\mathcal{H})$, hence $\|\mathrm{D}\,(\mathsf{R}(\widetilde{z}_n))\| \to \|\mathrm{D}\,(\mathsf{R}(\widetilde{z}))\|$, therefore

$$\|\mathrm{D}\,(\mathsf{R}(z_n))\| \to \|\mathrm{D}\,(\mathsf{R}(\widetilde{z}))\|. \tag{4.23}$$

Let us compute $\| \mathrm{D}\,(\mathsf{R}(\widetilde{z}) \circ \ell_z)\|$. Since $\mathsf{R}(\widetilde{z})$ is continuous, we have that $\mathsf{R}(\widetilde{z}) \circ \ell_z$ is left-continuous, therefore, using elementary properties of the pointwise variation,

$$\begin{aligned}\| \mathrm{D}\,(\mathsf{R}(\widetilde{z}) \circ \ell_z)\| &= \mathrm{V}_p(\mathsf{R}(\widetilde{z}) \circ \ell_z, I)\\ &= \mathrm{V}_p(\mathsf{R}(\widetilde{z}) \circ \ell_z,]a,d]) + \mathrm{V}_p(\mathsf{R}(\widetilde{z}) \circ \ell_z, [d,b[)\\ &= \mathrm{V}_p(\mathsf{R}(\widetilde{z}),]a, \ell_z(d)[) + \|\mathsf{R}(\widetilde{z})(\ell_z(d+)) - \mathsf{R}(\widetilde{z})(\ell_z(d-))\|_{\mathcal{H}}\\ &\quad + \mathrm{V}_p(\mathsf{R}(\widetilde{z}),]\ell_z(d+), b[).\end{aligned}$$

Thus by (4.21) we infer that

$$\begin{aligned}\| \mathrm{D}\,(\mathsf{R}(\widetilde{z}) \circ \ell_z)\| &< \mathrm{V}_p(\mathsf{R}(\widetilde{z}),]a, \ell_z(d)]) + \mathrm{V}_p(\mathsf{R}(\widetilde{z}), [\ell_z(d+), \ell_z(d)])\\ &\quad + \mathrm{V}_p(\mathsf{R}(\widetilde{z}), [\ell_z(d+), b[)\\ &= \mathrm{V}_p(\mathsf{R}(\widetilde{z}), I) = \| \mathrm{D}\,(\mathsf{R}(\widetilde{z}))\|\end{aligned}$$

hence we deduce that $\| \mathrm{D}\,(\mathsf{R}(\widetilde{z}))\| \neq \| \mathrm{D}\,(\mathsf{R}(\widetilde{z}) \circ \ell_z)\|$, that together with (4.22) and (4.23) implies that $\mathsf{R}(z_n)$ does not have a limit in $BV(I;\mathcal{H})$ with the strict topology and this concludes the proof. ☐

Now we can address the proof of the main theorem.

Proof of Theorem 3.2. We have already shown in the proof of Proposition 3.3 that $\overline{\mathsf{R}}$ extend R and maps $BV(I;\mathcal{H})$ into itself. Moreover if u is continuous then $\overline{\mathsf{R}}(u)$ is also continuous. In order to prove continuity let us take a sequence (u_n) which strictly converges to u and let us denote the normalized arc length functions ℓ_u and ℓ_{u_n} simply by ℓ and ℓ_n respectively. Since $\overline{\mathsf{R}}(u_n) \to \overline{\mathsf{R}}(u)$ by Proposition 3.3, it remains to study the convergence of the variations. Since $\mathsf{R}(\widetilde{u})$ is continuous, $\mathsf{R}(\widetilde{u}) \circ \ell_u$ is left-continuous, therefore

$$\| \mathrm{D}\,(\overline{\mathsf{R}}(u))\| = \| \mathrm{D}\,(\mathsf{R}(\widetilde{u}) \circ \ell))\| = \mathrm{V}_p(\mathsf{R}(\widetilde{u}) \circ \ell, I).$$

For the same reason

$$\| \mathrm{D}\,(\overline{\mathsf{R}}(u_n))\| = \| \mathrm{D}\,(\mathsf{R}(\widetilde{u}) \circ \ell_n)\| = \mathrm{V}_p(\mathsf{R}(\widetilde{u}) \circ \ell_n, I) \qquad \forall n \in \mathbb{N}.$$

If $u_n, u \in BV(I;\mathcal{H}) \cap C(I;\mathcal{H})$ for every $n \in \mathbb{N}$, then ℓ_n and ℓ are continuous, therefore

$$\mathrm{V}_p(\mathsf{R}(\widetilde{u}) \circ \ell, I) = \mathrm{V}_p(\mathsf{R}(\widetilde{u}), I),\ \ \mathrm{V}_p(\mathsf{R}(\widetilde{u}_n) \circ \ell_n, I) = \mathrm{V}_p(\mathsf{R}(\widetilde{u}_n), I) \quad \forall n \in \mathbb{N}. \tag{4.24}$$

Hence we obtain that

$$\lim_{n\to\infty} \| \mathrm{D}\,\overline{\mathsf{R}}(u_n)\| = \lim_{n\to\infty} \| \mathrm{D}\,\mathsf{R}(\widetilde{u}_n)\| = \| \mathrm{D}\,\mathsf{R}(\widetilde{u})\| = \| \mathrm{D}\,\overline{\mathsf{R}}(u)\|$$

so that $\overline{\mathsf{R}}$ is continuous from $BV(I;\mathcal{H}) \cap C(I;\mathcal{H})$ into itself endowed with the strict semimetric. Now let us assume that R is locally isotone and that u and u_n are not necessarily continuous. By construction $\widetilde{u}$ is affine on the interval $[\ell(t-), \ell(t+)]$ for every $t \in \mathrm{Discont}(\ell)$, therefore $\mathsf{R}(\widetilde{u})$ is also affine on these intervals, because R is locally isotone. Therefore Lemma 4.1 implies that $\mathrm{V}_p(\mathsf{R}(\widetilde{u})\circ\ell, I) = \mathrm{V}_p(\mathsf{R}(\widetilde{u}), I)$ and we can deduce the equality

$$\| \mathrm{D}\,(\overline{\mathsf{R}}(u))\| = \| \mathrm{D}\,(\mathsf{R}(\widetilde{u}))\|.$$

The same argument shows that

$$\| \mathrm{D}\,(\overline{\mathsf{R}}(u_n))\| = \| \mathrm{D}\,(\mathsf{R}(\widetilde{u}_n))\|.$$

Therefore also in this case we obtain that $\| \mathrm{D}\,\overline{\mathsf{R}}(u_n)\| \to \| \mathrm{D}\,\overline{\mathsf{R}}(u)\|$ and we have that $\overline{\mathsf{R}}$ is continuous form $BV(I;\mathcal{H})$ into itself proveided that $\overline{\mathsf{R}}$ is locally isotone. If we consider $BV(I;\mathcal{H})$ as a space of $\mathcal{L}^1$-classes of equivalence, then the strict metric induces a Hausdorff topology, therefore the uniqueness of the extension is a consequence of the density of $Lip(I;\mathcal{H})$ in $BV(I;\mathcal{H})$. Now let $\phi : \overline{I} \longrightarrow \overline{I}$ be increasing and surjective and set $v := u \circ \phi$, where $u \in BV(I;\mathcal{H})$. From Lemma 4.2 we infer that

$$\overline{\mathsf{R}}(u \circ \phi) = \mathsf{R}(\widetilde{u \circ \phi}) \circ \ell_v = \mathsf{R}(\widetilde{u}) \circ \ell_u \circ \phi = \overline{\mathsf{R}}(u) \circ \phi,$$

hence $\overline{\mathsf{R}}$ is rate independent. □

We conclude with the

Proof of Theorem 3.4. Assume that $u_n, u \in F$ and $u_n \to u$ strictly on I. For simplicity we set $\ell := \ell_u$ and $\ell_n := \ell_{u_n}$ for every $n \in \mathbb{N}$, where ℓ_u and ℓ_{u_n} are the "normalized" arc length functions of u and u_n defined as in (3.1), and $\widetilde{u}$ and $\widetilde{u}_n$ are the reparametrizations satisfying (3.3)-(3.4) with $u, \widetilde{u}, \ell_u$ replaced respectively by $u, \widetilde{u}, \ell$ and $u_n, \widetilde{u}_n, \ell_n$, as given by Proposition 3.1. Rate independence implies that

$$\mathsf{R}(u_n) = \mathsf{R}(\widetilde{u}_n \circ \ell_n) = \mathsf{R}(\widetilde{u}_n) \circ \ell_n \qquad \forall\, n \in \mathbb{N}. \tag{4.25}$$

The continuity of u and Proposition 4.10 let us infer that

$$\ell_n(t) \to \ell(t) \qquad \forall\, t \in \overline{I}, \tag{4.26}$$

$$\widetilde{u}_n \to \widetilde{u} \qquad \text{in } W^{1,1}(I;\mathcal{H}) \tag{4.27}$$

as $n \to \infty$. Hence by the assumption (3.6) we have that

$$\mathsf{R}(\widetilde{u}_n) \to \mathsf{R}(\widetilde{u}) \qquad \text{strictly on } I \tag{4.28}$$

for $n \to \infty$. From this convergence, the continuity of $\mathsf{R}(\widetilde{u}_n)$ and $\mathsf{R}(\widetilde{u})$, and Corollary 4.8 we get that $\mathsf{R}(u_n) \to \mathsf{R}(u)$ uniformly on I, therefore $\mathsf{R}(\widetilde{u}_n) \circ \ell_n \to \mathsf{R}(\widetilde{u}) \circ \ell$ pointwise in I. Finally, since $\|\mathsf{R}(\widetilde{u}_n) \circ \ell_n\|_\infty \le \|\mathsf{R}(\widetilde{u}_n)\|_\infty < +\infty$, by (4.25) and the dominated convergence theorem we infer that $\mathsf{R}(u_n) \to \mathsf{R}(\widetilde{u}) \circ \ell$

in $L^1(I;\mathcal{H})$. Now, by the continuity of u and by rate independence, we have $\mathsf{R}(\widetilde{u})\circ\ell = \mathsf{R}(\widetilde{u}\circ\ell) = \mathsf{R}(u)$, therefore we have proved that

$$\mathsf{R}(u_n) \to \mathsf{R}(u) \qquad \text{in } L^1(I;\mathcal{H}) \tag{4.29}$$

as $n \to \infty$. It is left to prove the convergence of the variations. By (4.25), the continuity of ℓ_u, and by Lemma 4.1 we have that

$$\|\,\mathrm{D}(\mathsf{R}(u))\| = \|\,\mathrm{D}(\mathsf{R}(\widetilde{u}))\|, \qquad \|\,\mathrm{D}(\mathsf{R}(u_n))\| = \|\,\mathrm{D}(\mathsf{R}(\widetilde{u}_n))\|$$

for every $n \in \mathbb{N}$. Since by convergence (4.27) we have that $\|\,\mathrm{D}(\mathsf{R}(\widetilde{u}_n))\| \to \|\,\mathrm{D}(\mathsf{R}(\widetilde{u}))\|$, we infer that $\|\,\mathrm{D}(\mathsf{R}(u_n))\| \to \|\,\mathrm{D}(\mathsf{R}(u))\|$ and we are done. □

4.4. Reduction to the case of open intervals

We have proved so far theorems for operators acting on spaces of functions defined on an open interval $]a,b[$. However one may be interested to the case of a compact interval, say $[0,T]$, $T > 0$, and a source like $\mu = \mathrm{D}u = \delta_0 x$, where $x \in \mathcal{H}$ and δ_0 is the unit mass concentrated in $t = 0$. In this section we show that rate independence allows to reduce the case of compact intervals to the case of open ones by means of the following procedure.

Assume that $a, b \in \mathbb{R}$ with $a < b$. As in [16] we consider the following set

$$D := \{u \in \mathcal{H}^{[a,b]} \ : \ u|_{]a,b[} \in BV(]a,b[\,;\mathcal{H})\}. \tag{4.30}$$

The essential variation is modified accordingly :

$$\mathrm{V}_e^D(u,[a,b]) := \mathrm{V}_e(u,]a,b[) + \|u^l(a+) - u(a)\|_{\mathcal{H}} + \|u(b) - u^l(b-)\|_{\mathcal{H}}, \ \ u \in D, \tag{4.31}$$

and the strict semimetric on D is defined by $d_s^D(u,v) := \|u - v\|_{L^1(]a,b[;\mathcal{H})} + |\,\mathrm{V}_e^D(u,[a,b]) - \mathrm{V}_e^D(v,[a,b])|$, $u, v \in D$. The notion of rate independence does not change: $\mathsf{Q} : D \longrightarrow D$ is called rate independent, if $\mathsf{Q}(u\circ\phi) = \mathsf{Q}(u)\circ\phi$ for every $\phi : [a,b] \longrightarrow [a,b]$ increasing and surjective.

Fix $\delta \in \,]0,(b-a)/2[$ and let $\gamma : [a+\delta, b-\delta] \longrightarrow [a,b]$ be the affine function such that $\gamma(a+\delta) = a$, $\gamma(b-\delta) = b$. Define $\alpha : [a,b] \longrightarrow [a,b]$ and $\beta : [a,b] \longrightarrow [a,b]$ by

$$\alpha(t) := \begin{cases} a & \text{if } t = a \\ \gamma(t) & \text{if } t \in \,]a+\delta, b-\delta[\\ b & \text{if } t = b \end{cases}, \qquad \beta(t) := \begin{cases} a & \text{if } t = a \\ \gamma^{-1}(t) & \text{if } t \in \,]a,b[\\ b & \text{if } t = b \end{cases}$$

(α is increasing and β is the right inverse of α). If $u_n, u \in D$ for every $n \in \mathbb{N}$, then $u_n \to u$ strictly in D if and only if $u_n\circ\alpha \to u\circ\alpha$ strictly in $BV(]a,b[\,;\mathcal{H})$.

Let $\mathsf{R} : Lip([a,b]\,;\mathcal{H}) \longrightarrow D \cap C([a,b]\,;\mathcal{H})$ be a rate independent operator which is continuous with respect to the strict metric. Let $\overline{\mathsf{R}} : BV(]a,b[\,;\mathcal{H}) \longrightarrow$

$BV(]a,b[\,;\mathcal{H})$ be the extension of R defined by Theorem 3.2. Now we define $\overline{\mathsf{Q}} : D \longrightarrow D$ by setting $\overline{\mathsf{Q}}(u) := \overline{\mathsf{R}}(u \circ \alpha) \circ \beta$ for every $u \in D$. Using rate independence it is easily seen that $\overline{\mathsf{Q}}(u_n) \to \overline{\mathsf{Q}}(u)$ in $L^1(]a,b[\,;\mathcal{H})$ whenever $u_n \to u$ strictly in D. Moreover by Theorem 3.2 we infer that Q is continuous with respect to d_s^D if and only if R is locally isotone.

Another way to reduce to open intervals consists in artificially extending any $u \in D$ to the interval $]a-1, b+1[$ by setting $u(t) = u(a)$ and $u(t) = u(b)$ for $t < a$ and $t > b$ respectively. This procedure is used in [25] for the scalar play operator.

5. Application to variational inequalities

In this section we assume that (3.8)–(3.10) hold.

5.1. Review of classical stop and play operators

Problem 5.1 (P). Assume $p \in [1, \infty]$ and (3.8)–(3.10) hold.
Given $u \in W^{1,p}(]0,T[\,;\mathcal{H})$ find $y \in W^{1,p}(]0,T[\,;\mathcal{H})$ such that

$$u(t) - y(t) \in \mathcal{Z} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in\,]0,T[, \tag{5.1}$$

$$\big\langle u(t) - y(t) - z, y'(t) \big\rangle \geq 0 \qquad \forall z \in \mathcal{Z},\ \text{for } \mathcal{L}^1\text{-a.e. } t \in\,]0,T[, \tag{5.2}$$

$$u(0) - y(0) = z_0. \tag{5.3}$$

Strictly related to the previous problem is the following:

Problem 5.2 (S). Assume $p \in [1, \infty]$ and (3.8)–(3.10) hold.
Given $u \in W^{1,p}(]0,T[\,;\mathcal{H})$ find $x \in W^{1,p}(]0,T[\,;\mathcal{H})$ such that

$$x(t) \in \mathcal{Z} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in\,]0,T[, \tag{5.4}$$

$$\big\langle x(t) - z, u'(t) - x'(t) \big\rangle \geq 0 \qquad \forall z \in \mathcal{Z},\ \text{for } \mathcal{L}^1\text{-a.e. } t \in\,]0,T[, \tag{5.5}$$

$$x(0) = z_0. \tag{5.6}$$

Let us remark that in (5.3) and (5.6), $y(0)$ and $x(0)$ denote the traces of y and x, *i.e.* the values in $t = 0$ of the continuous representatives of y and x. Observe that the two problems are related by the formula $u = x + y$, indeed if y is a solution of problem (**P**), then $x := u - y$ is a solution of problem (**S**). Vice versa given a solution x of problem (**S**), then a solution of the problem (**P**) is given by $y := u - x$.

Let $I_{\mathcal{Z}} : \mathcal{H} \longrightarrow [0, \infty]$ be the *indicator function of* $\mathcal{Z}$, defined by

$$I_{\mathcal{Z}}(x) := \begin{cases} 0 & \text{if } x \in \mathcal{Z} \\ \infty & \text{if } x \notin \mathcal{Z}. \end{cases} \tag{5.7}$$

Since $I_{\mathcal{Z}}$ is convex and lower semicontinuous and $I_{\mathcal{Z}} \not\equiv \infty$, it makes sense to consider its *subdifferential* $\partial I_{\mathcal{Z}} : \mathcal{H} \longrightarrow \mathscr{P}(H)$ which is defined by $\partial I_{\mathcal{Z}}(x) :=$

$\{y \in \mathcal{H} : \langle y, z - x\rangle \le 0\ \forall z \in \mathcal{Z}\}$ if $x \in \mathcal{Z}$ and by $\partial I_{\mathcal{Z}}(x) := \varnothing$ if $x \notin \mathcal{Z}$. It is well known that $\partial I_{\mathcal{Z}}$ is a *(multivalued) monotone operator*, *i.e.* $\langle y_1 - y_2, x_1 - x_2\rangle \ge 0$ for every $x_j \in \mathcal{Z}$, $y_j \in \partial I_{\mathcal{Z}}(x_j)$, $j = 1, 2$. Moreover $\partial I_{\mathcal{Z}}$ is *maximal monotone*, *i.e.* it is monotone and its *graph* $\{(x, y) \in \mathcal{H} \times \mathcal{H} : y \in \partial I_{\mathcal{Z}}(x)\}$ is not contained in the graph of another monotone operator. Let us also note that $\mathcal{Z} = D(\partial I_{\mathcal{Z}}) := \{x \in \mathcal{H} : \partial I_{\mathcal{Z}}(x) \neq \varnothing\}$, the *domain* of $\partial I_{\mathcal{Z}}$. For the theory of maximal monotone operators we refer to [4, Chapter II]. Let us remark that in our case $\partial I_{\mathcal{Z}}(x) = N_{\mathcal{Z}}(x)$, the normal cone to $\mathcal{Z}$ at x (*cf.* Section A.5 of the Appendix for the definition of normal cone; see also [4, Example 2.8.2, Chapter 2, page 46]).

Problem (**S**) can be solved using the classical theory of evolution equations governed by maximal monotone operators, indeed by [4, Proposition 3.4, Remark 3.7] we infer that for every $u \in W^{1,p}(]0, T[\,; \mathcal{H})$ and $z_0 \in \mathcal{Z}$, there exists $x \in W^{1,p}(]0, T[\,; \mathcal{H})$ such that $x(t) \in \mathcal{Z}$ for $\mathcal{L}^1$-a.e. $t \in\,]0, T[$ and

$$x'(t) + \partial I_{\mathcal{Z}}(x(t)) \ni u'(t) \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in\,]0, T[\tag{5.8}$$

$$x(0) = z_0. \tag{5.9}$$

Moreover for any right Lebesgue point $t \in [0, T[$ of u' there exists the right derivative $x'_+(t)$ of x and

$$x'_+(t) + \mathrm{Proj}_{N_{\mathcal{Z}}(x(t))}(u'(t+)) = u'(t+)\ \forall t \in [0, T[\text{ right Lebesgue point of } u', \tag{5.10}$$

where in this case we set $u'(t+) := \lim_{h \searrow 0} h^{-1} \int_t^{t+h} u'(s)\,\mathrm{d}s$.

Observe that this solution is unique if consider $\mathcal{L}^1$-classes of equivalence (or if we require x to be continuous). Hence from the definition of subdifferential we immediately obtain the following:

Proposition 5.3. *Both problems* (**P**) *and* (**S**) *admit a solution for every* $p \in [1, \infty]$. *These solutions are unique if we consider* $\mathcal{L}^1$*-classes of equivalence* (*or if we require* x *to be continuous*).

Identifying mappings that differ on a set of zero Lebesgue measure, the previous theorem allows us to define two solution operators

$$\mathsf{P} : W^{1,p}(]0, T[\,; \mathcal{H}) \longrightarrow W^{1,p}(]0, T[\,; \mathcal{H}),$$
$$\mathsf{S} : W^{1,p}(]0, T[\,; \mathcal{H}) \longrightarrow W^{1,p}(]0, T[\,; \mathcal{H})$$

associating with every $u \in W^{1,p}(]0, T[\,; \mathcal{H})$ the solutions y and x of Problems (**P**) and (**S**) respectively. The operators P and S are usually called *play operator* and *stop operator* and have an important role in many physical applications. We have seen that the play and stop operators are related by the formula

$$\mathsf{P}(u) + \mathsf{S}(u) = u \qquad \forall u \in W^{1,p}(]0, T[\,; \mathcal{H}),$$

which is generally known as ‘stop-play duality’. Let us also stress that if $x := \mathsf{S}(u)$ then $x'(t) \in T_{\mathcal{Z}}(x(t))$ for $\mathcal{L}^1$-a.e. t, where $T_{\mathcal{Z}}(x)$ is the tangent cone to $\mathcal{Z}$ at x (see

Section A.5). Moreover $u' = x' + y'$ is the unique orthogonal decomposition of u' into the tangential and normal component.

It is well known (and easy to check) that P and S are rate independent operators. The convex set $\mathcal{Z}$ is often called *characteristic* of P.

It is worth noting that the play operator has a simple geometric interpretation (*cf.* [12, section 16.1, page 151]). In fact the inclusion (5.8), $y'(t) \in \partial I_Z(x(t))$, means that $\langle y'(t), z + y(t) - u(t)\rangle \le 0$ for every $z \in \mathcal{Z}$, *i.e.* $\langle y'(t), z - u(t)\rangle \le 0$ for every $z \in \mathcal{Z} + y(t)$; hence if $y(t) = \mathsf{P}(u)(t)$, then

$$y'(t) \in N_{y(t)+\mathcal{Z}}(u(t)) \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in \,]0, T[, \tag{5.11}$$

$$y(0) = y_0 := u(0) - z_0. \tag{5.12}$$

Let us interpret the point $u(t)$ as the motion of a pivot in the space $\mathcal{H}$. If $y_0 := u_0 - z_0$ let us imagine that at the initial time $t = 0$ the pivot is inserted in the convex set $y_0 + \mathcal{Z}$. Then the pivot starts to move according the law of motion $u(t)$. If initially $u(0)$ lies in the interior of $y(0) + \mathcal{Z}$ then $N_{y_0+\mathcal{Z}}(u(t)) = \{0\}$, thus $y(t) \equiv y_0$ solves the inclusion as long as $u(t)$ does not touch the boundary of $y_0 + \mathcal{Z}$. When $u(t)$ touches the boundary of $y_0 + \mathcal{Z}$ then $y(t)$ starts to move in such a way that $y'(t) \in N_{y(t)+\mathcal{Z}}(u(t))$. We interpret this solution by saying that the convex set $y_0 + \mathcal{Z}$ moves in the direction of the outward normal in $u(t) = y(t)+x(t) \in y(t)+\mathcal{Z}$. This interpretation is maybe easier to visualize if we assume that $z_0 = u(0) = 0 \in \mathcal{Z}$.

5.2. The play operator on functions of bounded variation

Now we state a fundamental continuity property of the operator S. This property is well-known under the assumption that $\mathcal{H}$ is separable and it is proved in [14] or [15]. In those references, the existence Theorem 5.3 is deduced from a generalized formulation for BV mappings, thus we give here a proof using only the formulation in the regular case.

Proposition 5.4. *The operator* $\mathsf{S} : W^{1,\infty}(]0, T[\,; \mathcal{H}) \longrightarrow W^{1,\infty}(]0, T[\,; \mathcal{H})$ *is continuous when* $W^{1,\infty}(]0, T[\,; \mathcal{H})$ *is endowed with the topology induced by the norm* $\|\cdot\|_{W^{1,1}(]0,T[;\mathcal{H})}$.

Proof. Assume that $u, u_n \in W^{1,\infty}(]0, T[\,; \mathcal{H})$ and $\|u - u_n\|_{W^{1,1}(]0,T[;\mathcal{H})} \to 0$. If $x = \mathsf{S}(u)$ and $x_n = \mathsf{S}(u_n)$ then there exist $\xi, \xi_n \in \mathcal{H}^{]0,T[}$ $\mathcal{L}^1$-measurable such that $\xi(t) \in \partial I_{\mathcal{Z}}(x(t))$, $\xi_n(t) \in \partial I_{\mathcal{Z}}(x_n(t))$ for $\mathcal{L}^1$-a.e. t, and

$$x_n'(t)+\xi_n(t) = u_n'(t), \qquad x'(t)+\xi(t) = u'(t) \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in \,]0, T[. \tag{5.13}$$

Multiplying the difference of these two equations by $x_n(t) - x(t)$, using the monotonicity of $\partial I_{\mathcal{Z}}$, and integrating in time, we infer that

$$\frac{1}{2}\|x_n(t) - x(t)\|^2_{\mathcal{H}} \le \int_0^t \|u_n'(s) - u'(s)\|_{\mathcal{H}}\|x_n(s) - x(s)\|_{\mathcal{H}}\,\mathrm{d}s, \qquad \forall t \in [0, T],$$

hence by Gronwall Lemma [4, Lemma A.5, page 157]

$$\|x_n(t) - x(t)\|_{\mathcal{H}} \leq \int_0^t \|u_n'(s) - u'(s)\|_{\mathcal{H}}\, \mathrm{d}s \qquad \forall t \in [0, T] .$$

It follows that $x_n \to x$ uniformly on $[0, T]$. On the other hand multiplying the equations (5.13) respectively by $x_n'(t)$ and $x'(t)$, $t \in [0, T]$, and using [4, Lemma 3.3, page 73] we get

$$\|x_n'(t)\|_{\mathcal{H}}^2 = \langle u_n'(t), x_n'(t)\rangle, \qquad \|x'(t)\|_{\mathcal{H}}^2 = \langle u'(t), x'(t)\rangle . \tag{5.14}$$

The first equation and Schwarz inequality implies that $\|x_n'(t)\|_{\mathcal{H}} \leq \|u_n'(t)\|_{\mathcal{H}}$ for every t, therefore, at least for a subsequence which we do not relabel, we have that $x_n \rightharpoonup x$ in $W^{1,2}(]0, T[\,; \mathcal{H})$. It follows that

$$\begin{aligned}\lim_{n\to\infty} \|x_n'\|^2_{L^2(]0,T[;\mathcal{H})} &= \lim_{n\to\infty} \int_0^T \langle u_n'(s), x_n'(s)\rangle\, \mathrm{d}s \\ &= \int_0^T \langle u'(s), x'(s)\rangle\, \mathrm{d}s = \|x'\|^2_{L^2(]0,T[;\mathcal{H})}.\end{aligned}$$

We infer that $x_n \to x$ in $W^{1,2}(]0, T[\,; \mathcal{H})$, hence in $W^{1,1}(]0, T[\,; \mathcal{H})$. □

Applying the previous continuity property and the general Theorems 3.3 and 3.2 we can infer the following result, proving part of Theorem 3.7.

Theorem 5.5. *The play operator* $\mathsf{P} : W^{1,1}(]0, T[\,; \mathcal{H}) \longrightarrow W^{1,1}(]0, T[\,; \mathcal{H})$ *is continuous with respect to the stricty convergence and admits a unique continuous extension* $\overline{\mathsf{P}} : BV(]0, T[\,; \mathcal{H}) \cap C(]0, T[\,; \mathcal{H}) \longrightarrow BV(]0, T[\,; \mathcal{H}) \cap C(]0, T[\,; \mathcal{H})$. *We have* $\overline{\mathsf{P}}(u) = \mathsf{P}(\widetilde{u}) \circ \ell_u$ *for every* $u \in BV(]0, T[\,; \mathcal{H}) \cap C(]0, T[\,; \mathcal{H})$, *where* $\widetilde{u} \in Lip([0, T]\,; \mathcal{H})$ *and* ℓ_u *are defined by Proposition* 3.1 *with* $a = 0$, $b = T$.

In the next theorem we show that P is not locally isotone when the dimension of $\mathcal{H}$ is strictly greater than one.

Theorem 5.6. *The operator* P *is locally isotone isotone if and only if* $\mathcal{Z}$ *is a vector subspace or*

$$\mathcal{Z} = \{x \in \mathcal{H} \ :\ -\alpha \leq \langle f, x\rangle \leq \beta\} \tag{5.15}$$

for some $f \in \mathcal{H} \setminus \{0\}$ *and* $\alpha, \beta \in [0, \infty]$.

Proof. The "if" part of the theorem is clear. Let us prove the "only if" direction assuming by contradiction that $\mathcal{Z}$ is not of the form (5.15). By Proposition A.13 of the Appendix A.5 there exist $z_1, z_2 \in \partial\mathcal{Z}$ such that $z_1 \neq z_2$, $N_{\mathcal{Z}}(z_j) \neq \{0\}$, $j = 1, 2$, and there exists $v \in N_{\mathcal{Z}}(z_2) \setminus \{0\}$ such that $v \notin N_{\mathcal{Z}}(z_1) \cup T_{\mathcal{Z}}(z_1)$. We first deal with the case when $z_0 = z_1$.

Define $u(t) := z_1 + tv$, $t \geq 0$. By Proposition 5.3 and equations (5.8)–(5.9) there exists a unique $x \in C([0, \infty[\,; \mathcal{H})$, absolutely continuous on compact sets such that $x(t) \in \mathcal{Z}$ for every $t \geq 0$, $x(0) = z_1$, and $x'(t) + \partial I_{\mathcal{Z}}(x(t)) \ni u'(t)$ for $\mathcal{L}^1$-a.e. $t \geq 0$. Since $u'(t) = v$ for every $t \geq 0$, thanks to [4, Proposition 3.5, page 69] the right derivative $x'_+(t)$ exists for every $t \geq 0$ and by (5.10)

$$x'_+(t) + \mathrm{Proj}_{N_{\mathcal{Z}}(x(t))}(v) = v \qquad \forall t \in [0, \infty[\,. \tag{5.16}$$

We can also apply [4, Theorem 3.10, page 89] and infer that

$$\|x'_+(t)\|_{\mathcal{H}} \to 0 \qquad \text{as } t \to \infty. \tag{5.17}$$

Now we set $y := u - x$ and observe that $y(0) = 0$. We show that there exists $\bar{t} > 0$ such that

$$\mathrm{V}_p(y, [0, \bar{t}]) > \|y(\bar{t}) - y(0)\|_{\mathcal{H}}. \tag{5.18}$$

To this aim we assume by contradiction that there exists a locally Lipschitz continuous map $\psi : [0, \infty[\longrightarrow \mathbb{R}$ such that $\psi(0) = 0$, ψ is increasing and $y(t) = \psi(t)w$ for $t \in [0, \infty[$, for some $w \in \mathcal{H}$, $\|w\|_{\mathcal{H}} = 1$. Observe that $y'_+(0) = \mathrm{Proj}_{N_{\mathcal{Z}}(z_1)}(v)$ and $v \notin N_Z(z_1) \cup T_{\mathcal{Z}}(z_1)$, hence $y'_+(0) = \psi'_+(0)w \neq 0$, thus $\psi'_+(0) > 0$ and $v \notin \mathbb{R}w$. We have

$$x'_+(t) = v - \psi'_+(t)w \qquad \forall t \geq 0,$$

therefore from (5.17) we infer that for every $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that $\|v - \psi'_+(t_\varepsilon)w\|_{\mathcal{H}} < \varepsilon$. This is impossible because $\|v - \psi'_+(t_\varepsilon)w\|_{\mathcal{H}} \geq \|v - \mathrm{Proj}_{\mathbb{R}w}(v)\|_{\mathcal{H}} > 0$, thus (5.18) is proved. Now let $\phi : [0, T] \longrightarrow \left[0, \bar{t}\right]$ be a strictly increasing continuously differentiable map and define $u_T := u \circ \phi \in C^1([0, T]\,; \mathcal{H})$. It is easily seen, namely by rate independence, that $y_T := \mathsf{P}(u_T) = y \circ \phi$, and we have $\mathrm{V}_p(u_T, [0, T]) = \|u_T(T) - u_T(0)\|_{\mathcal{H}}$, whereas $\mathrm{V}_p(y_T, [0, T]) > \|y_T(T) - y_T(0)\|_{\mathcal{H}}$, thus P is not locally isotone when $z_0 = z_1$. The general case is reduced to the previous one by considering a map $u : [0, \infty[\longrightarrow \mathcal{H}$ and $t_1 > 0$ such that $u|_{[0,t_1]}$ is an injective parametrization of the segment joining z_0 and z_1. Then $\mathsf{S}(u) = u$ and $\mathsf{P}(u) = 0$ on $[0, t_1]$, so that for $t > t_1$ we can argue as above. □

The previous theorem together with Theorem 3.2 allow us to deduce Theorem 3.7, thereby giving a complete characterization of BV-continuous play operators in terms of $\mathcal{Z}$. If $\mathcal{H} = \mathbb{R}$ then every closed convex set is of the form (5.15) (a closed interval), therefore P admits a conitnuous extension to all of $BV(]0, T[\,; \mathbb{R})$. Now we prove that if $u \in BV(]0, T[\,; \mathcal{H})$, then $y := \overline{\mathsf{P}}(u)$ solves a suitable generalized variational inequality.

Theorem 5.7. *If $u \in BV(]0, T[\,; \mathcal{H})$ and $y := \overline{\mathsf{P}}(u) = \mathsf{P}(\widetilde{u}) \circ \ell_u$, then*

$$u(t) - y(t) \in \mathcal{Z} \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in \,]0, T[, \tag{5.19}$$

$$\int_{]0,t[} \langle u^r - y^r - z, \mathrm{d}\mathrm{D}\, y\rangle \geq \sum_{s \in]0,t[} \langle u^r(s) - y^r(s) - z(s), y^r(s) - y^r(s-)\rangle$$
$$\forall z \in L^1(]0, T[\,; \mathcal{H}),\ z(]0, T[) \subseteq \mathcal{Z},\ \forall t \in [0, T], \tag{5.20}$$

$$u(0) - y(0) = z_0. \tag{5.21}$$

Proof. Conditions (5.19) and (5.21) are easily checked. If $y := \overline{\mathsf{P}}(u)$, then by Corollary A.8 we have

$$\begin{aligned}
&\int_{]0,t[} \langle u^r - y^r - z, \mathrm{dD}\, y\rangle \\
&= \int_{]0,t[} \langle \widetilde{u}(\ell_u(s+)) - \mathsf{P}(\widetilde{u})(\ell_u(s+)) - z(s), \mathrm{dD}(\mathsf{P}(\widetilde{u}) \circ \ell_u)(s)\rangle \\
&= \int_{I\setminus A(u)} \langle \widetilde{u}(\ell_u(s)) - \mathsf{P}(\widetilde{u})(\ell_u(s)) - z(s), (\mathsf{P}(\widetilde{u}))'(\ell_u(s))\rangle \,\mathrm{dD}\,\ell_u(s) \\
&\quad + \sum_{s\in]0,t[\cap A(u)} \langle \widetilde{u}(\ell_u(s+)) - \mathsf{P}(\widetilde{u})(\ell_u(s+)) \\
&\quad - z(s), \mathsf{P}(\widetilde{u})(\ell_u(s+)) - \mathsf{P}(\widetilde{u})(\ell_u(s))\rangle.
\end{aligned} \tag{5.22}$$

Now set

$$F = \left\{\sigma \in [0, T] \ : \ \langle \widetilde{u}(\sigma) - \mathsf{P}(\widetilde{u})(\sigma) - z, (\mathsf{P}(\widetilde{u}))'(\sigma)\rangle \geq 0 \ \forall z \in [-r, r]\right\}.$$

Thanks to Proposition 3.6, formula (3.12), we know that $\mathcal{L}^1([0, T]\setminus F) = 0$. Let us set $E := \{s \in [0, T]\setminus A(u) \ : \ \ell_u(s) \in [0, T]\setminus F\}$. Since $A(u) = \mathrm{Discont}(\ell_u)$, in view of Proposition A.5 we get that $\mathrm{D}\,\ell_u(E) = 0$, therefore

$$\begin{aligned}
&\mathrm{D}\,\ell_u([0, T]) \\
&= \mathrm{D}\,\ell_u([0, T]\setminus E) \leq \mathrm{D}\,\ell_u(\{s \in [0, T] \ : \ \ell_u(s) \in F\}) \\
&= \mathrm{D}\,\ell_u\left(\left\{s\in[0, T] : \langle \widetilde{u}(\ell_u(s)) - \mathsf{P}(\widetilde{u})(\ell_u(s)) - z(s), (\mathsf{P}(\widetilde{u}))'(\ell_u(s))\rangle \geq 0\right\}\right).
\end{aligned}$$

This implies that $\langle \widetilde{u}(\ell_u(s)) - \mathsf{P}(\widetilde{u})(\ell_u(s)) - z(s), (\mathsf{P}(\widetilde{u}))'(\ell_u(s))\rangle \geq 0$ for $\mathrm{D}\,\ell_u$-a.e. $s \in [0, t]$, therefore

$$\int_{[0,t]\setminus A(u)} \langle \widetilde{u}(\ell_u(s)) - \mathsf{P}(\ell_u(s)) - z(s), \mathsf{P}(\widetilde{u}))'(\ell_u(s))\rangle \,\mathrm{dD}\,\ell_u(s) \geq 0. \tag{5.23}$$

The thesis follows. ☐

As a corollary we obtain the result proved in [14, Theorem 3.1].

Corollary 5.8. *If $u \in BV(]0, T[\,; \mathcal{H}) \cap C(]0, T[\,; \mathcal{H})$, then $y := \overline{\mathsf{P}}(u) = \mathsf{P}(\widetilde{u}) \circ \ell_u$ is the unique map such that*

$$u(t) - y(t) \in \mathcal{Z} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in]0, T[, \tag{5.24}$$

$$\int_{]0,t[} \langle u(s) - x(s) - z(s), \mathrm{dD}\, y(s)\rangle \geq 0 \quad \forall z\in C([0, T]; \mathcal{Z}),\ \forall t\in[0, T], \tag{5.25}$$

$$u(0) - y(0) = z_0. \tag{5.26}$$

Finally if $p \in [1, \infty]$ and $u \in W^{1,p}(]0, T[\,; \mathcal{H})$, then $\overline{\mathsf{P}}(u)$ is the unique continuous solution of Problem (**P**).

Proof. When u and y are continuous, al the terms of the sum in (5.20) vanish. Uniqueness is standard (see [23, Theorem 3.1] or [14, Theorem 3.1]). □

Observe that our procedure provides a formula for the solution of (5.24)-(5.26) and we obtain this solution as a direct consequence of the existence of the classical problem (**P**).

5.3. Final remarks on BV-solutions

Now we compare the extension $\overline{\mathsf{P}}$ with the notion of play operator on BV maps given in [16]. In that paper the evolution is studied on the closed interval $[0, T]$. For simplicity we limit ourselves to the case of the open interval $]0, T[$, indeed we have shown in subsection 4.4 that this is not a restriction. In [16, Theorem 2.3] it is proved that if $u \in BV(]0, T[\,; \mathcal{H})$ then there exists a map $\xi \in BV(]0, T[\,; \mathcal{H})$ such that

$$u(t) - \xi(t) \in \mathcal{Z} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in\,]0, T[, \tag{5.27}$$

$$\int_{]0,t[} \langle u^r(s) - \xi^r(s) - z(s), \mathrm{d}\xi(s)\rangle \geq 0$$
$$\forall z \in Reg([0, T]\,; \mathcal{H}),\ z([0, T]) \subseteq \mathcal{Z}, \forall t \in [0, T], \tag{5.28}$$

$$u(0) - \xi(0) = z_0. \tag{5.29}$$

The integral in (5.28) is meant in the sense of Young and the solution is unique if we identify maps differing only in a $\mathcal{L}^1$-null set. In the Appendix A.4 we recall the definition of Young integral and we prove that the Young integral in (5.28) coincides with $\int_{]0,t[} \langle u^r(s) - \xi^r(s) - z(s), \mathrm{d}\mathrm{D}\xi(s)\rangle$. Therefore when $u \in BV(]0, T[\,; \mathcal{H}) \cap C(]0, T[\,; \mathcal{H})$, we infer from Corollary 5.8 that the solution ξ found in [16] is exactly $\overline{\mathsf{P}}(u) = \mathsf{P}(\widetilde{u}) \circ \ell_u$. One can conjecture that $\overline{\mathsf{P}}(u)$ is the solution of (5.27)-(5.29) even if u is not continuous. The following simple example shows that this conjecture is false. Assume that $T = 1$, $\mathcal{H} = \mathbb{R}^2$, $z_0 = (0, 0)$, and $\mathcal{Z} = \overline{B_1((-1, 0))}$. Let $u \in BV(]0, 1[\,; \mathbb{R}^2)$ be defined by $u(t) := \chi_{]1/2,1[}(t)(0, 1)$. By [16, Proposition 4.3] the solution ξ of (5.27)-(5.29) is

$$\xi(t) := \begin{cases} (0, 0) & \text{if } 0 < t \leq 1/2 \\ (1 - 1/\sqrt{2}, 1 - 1/\sqrt{2}) & \text{if } 1/2 < t < 1 \end{cases} \tag{5.30}$$

Now observe that the normalized arc-length of u is $\ell_u = \chi_{]1/2,1[}$ and the reparametrization $\widetilde{u}$ is given by $\widetilde{u}(t) = (0, t)$, $t \in\,]0, 1[$. Therefore $\mathsf{P}(\widetilde{u})(t) = (-1 + (1 - \tanh^2(t))^{1/2}, \tanh(t))$ and we infer that $\overline{\mathsf{P}}(u) = \mathsf{P}(\widetilde{u}) \circ \ell_u$ is given by

$$\overline{\mathsf{P}}(u)(t) := \begin{cases} (0, 0) & \text{if } 0 < t \leq 1/2 \\ (-1 + (1 - \tanh^2(1))^{1/2}, \tanh(1)) & \text{if } 1/2 < t < 1 \end{cases} \tag{5.31}$$

It follows that $\xi \neq \overline{\mathsf{P}}(u)$ and therefore we could consider ξ and $\overline{\mathsf{P}}(u)$ as two different notions of solutions, indeed it is very natural to approximate any u by a strictly convergent sequence $u_n \in Lip(]0, T[\,; \mathcal{H})$ and we have proved that $\mathsf{P}(u_n) \to \overline{\mathsf{P}}(u)$ in L^1. Therefore it seems important to perform a careful comparison of the two solutions from the modelling point of view. In this regard in the paper [17] it is shown that the solution of (5.27)-(5.29) is the vanishing-viscosity limit of suitable viscous regularized problems. Anyway the problem of defining weak solutions of nonsmooth rate independent processes is object of an intensive research: see, *e.g.*, the recent paper [26] and the references therein.

Let us remark again that the one dimensional case is different: for $\mathcal{H} = \mathbb{R}$ the two notions of solutions are the same: indeed if v is monotone on an interval $[c, d]$ and $y = \mathsf{P}(v)$, then $(v(d) - y(d) - z)(y(d) - y(c)) \geq 0$ for every $z \in \mathcal{Z}$ and the sum in (5.22) is positive (see the details in [25]).

A. Appendix

In this appendix we assume that (2.2) holds.

A.1. Iterated integrals with respect to vector measures

Lemma A.1. *Let $\mu : \mathscr{B}(\mathbb{R}) \longrightarrow \mathcal{H}$ be a vector measure. If $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $f \in L^1(\mathcal{L}^1 \times \|\mu\|\,; \mathbb{R})$, then*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(t, s)\, \mathrm{d}t\, \mathrm{d}\mu(s) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(t, s)\, \mathrm{d}\mu(s)\, \mathrm{d}t. \tag{A.1}$$

Proof. Since $f \in L^1(\mathcal{L}^1 \times \|\mu\|\,; \mathbb{R})$ there exists a sequence of integers k_n and there exist

$$f_j^n \in \mathbb{R}, \quad A_j^n, B_j^n \in \mathscr{B}(\mathbb{R}) \qquad \forall n \in \mathbb{N},\ 1 \leq j \leq k_n$$

such that the sequence of step functions f_n defined by

$$f_n = \sum_{j=1}^{j_n} f_j^n \chi_{A_j^n \times B_j^n}$$

satisfies the following property

$$f_n \to f \qquad \text{in } L^1(\mathcal{L}^1 \times \|\mu\|\,; \mathbb{R}) \tag{A.2}$$

(see, *e.g.*, [18, Theorem 6.3, page 150]). Thanks to Fubini Theorem we can define the functions $\phi, \psi, \phi_n, \psi_n \in \mathbb{R}^{\mathbb{R}}$ by setting

$$\phi_n(s) := \int_{\mathbb{R}} f_n(t, s)\, \mathrm{d}t, \qquad \phi(s) := \int_{\mathbb{R}} f(t, s)\, \mathrm{d}t, \qquad s \in \mathbb{R},$$

$$\psi_n(t) := \int_{\mathbb{R}} f_n(t, s)\, \mathrm{d}\mu(s), \quad \psi(t) := \int_{\mathbb{R}} f(t, s)\, \mathrm{d}\mu(s), \qquad t \in \mathbb{R}.$$

We have

$$\|\phi - \phi_n\|_{L^1(|\mu|;\mathbb{R})} = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} (f(t,s) - f_n(t,s))\,\mathrm{d}t \right| \mathrm{d}\,|\mu|\,(s)$$
$$\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(t,s) - f_n(t,s)|\,\mathrm{d}t\,\mathrm{d}\,|\mu|\,(s) \tag{A.3}$$

and

$$\|\psi - \psi_n\|_{L^1(\mathcal{L}^1;\mathbb{R})} = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} (f(t,s) - f_n(t,s))\,\mathrm{d}\mu(s) \right| \mathrm{d}t$$
$$\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(t,s) - f_n(t,s)|\,\mathrm{d}\,|\mu|\,(s)\,\mathrm{d}t \tag{A.4}$$

On the other hand by Fubini theorem

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(t,s) - f_n(t,s)|\,\mathrm{d}t\,\mathrm{d}\,|\mu|\,(s) = \int_{\mathbb{R}} \int_{\mathbb{R}} |f(t,s) - f_n(t,s)|\,\mathrm{d}\,|\mu|\,(s)\,\mathrm{d}t$$
$$= \int_{\mathbb{R}^2} |f(t,s) - f_n(t,s)|\,\mathrm{d}(\mathcal{L}^1 \times |\mu|)(t,s),$$

therefore, thanks to (A.2), (A.3)–(A.4), we obtain that

$$\lim_{n\to\infty} \|\phi - \phi_n\|_{L^1(|\mu|;\mathbb{R})} = \lim_{n\to\infty} \|\psi - \psi_n\|_{L^1(\mathcal{L}^1;\mathbb{R})} = 0. \tag{A.5}$$

Equations (A.5) and (2.9) yield

$$\lim_{n\to\infty} \int_{\mathbb{R}} \phi_n(s)\,\mathrm{d}\mu(s) = \int_{\mathbb{R}} \psi_n(s)\,\mathrm{d}\mu(s) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(t,s)\,\mathrm{d}t\,\mathrm{d}\mu(s),$$
$$\lim_{n\to\infty} \int_{\mathbb{R}} \psi_n(t)\,\mathrm{d}t = \int_{\mathbb{R}} \psi_n(t)\,\mathrm{d}t = \int_{\mathbb{R}} \int_{\mathbb{R}} f(t,s)\,\mathrm{d}\mu(s)\,\mathrm{d}t,$$

Therefore we can deduce equation (A.1) by observing that

$$\int_{\mathbb{R}} \phi_n(s)\,\mathrm{d}\mu(s) = \sum_{j=1}^{k_n} \mathcal{L}^1(A_j^n) f_j^n \mu(B_j) = \int_{\mathbb{R}} \psi_n(t)\,\mathrm{d}\mu(t). \tag{A.6}$$

□

A.2. *BV* approximation by smooth vector functions

Proposition A.2. *If $u \in \mathcal{H}^I$ is such that $\mathrm{V}_e(u, I) < \infty$, then there exists a sequence (u_n) in $u_n \in C^\infty(\overline{I}; \mathcal{H})$ such that $u_n \to u$ in $\mathrm{L}^1_{\mathrm{loc}}(I; \mathcal{H})$ and $\mathrm{V}_e(u_n, I) \to \mathrm{V}_e(u, I)$ as $n \to \infty$. If $u \in L^1(I; \mathcal{H})$ then the sequence can be chosen in such a way that $u_n \in L^1(I; \mathcal{H})$ and $u_n \to u$ in $L^1(I; \mathcal{H})$ as $n \to \infty$.*

Proof. Without loss of generality we can assume that $\mathrm{V}_p(u, I) < \infty$ and that u is left-continuous, thus $\mathrm{V}_e(u, I) = \mathrm{V}_p(u, I)$. If $I =]a, b[\neq \mathbb{R}$, we extend u to $\mathbb{R}$ by means of the map $\overline{u} : \mathbb{R} \longrightarrow \mathcal{H}$ defined by $\overline{u}(t) := u(t)$ if $t \in]a, b[$, $\overline{u}(t) := u(a+)$ if $t \leq a$, $\overline{u}(t) := u(b-)$ if $t \geq b$. In this way $\mathrm{V}_e(u, I) = \mathrm{V}_e(\overline{u}, I) = \mathrm{V}_p(\overline{u}, I)$. Now let $\rho_n \in C_c^\infty(I; \mathbb{R})$ be a sequence of symmetric mollifiers, *i.e.* $\rho_n \geq 0$, $\rho_n(t) = 0$ iff $|t| \geq 1/n$, $\int_{\mathbb{R}} \rho_n = 1$ and $\rho_n(t) = \rho_n(-t)$ for every $t \in \mathbb{R}$. Then we can define the convolution $u_n \in \mathcal{H}^{\mathbb{R}}$ by $u_n(t) := \int_{\mathbb{R}} \rho_n(t - s)\overline{u}(s)\,\mathrm{d}s$, $t \in \mathbb{R}$. By [10, Theorem 10, page 219] (applied on bounded intervals) we have that $u_n \in C^\infty(\mathbb{R}; \mathcal{H})$ and $u_n(t) \to \overline{u}(t)$ for every Lebesgue point t of $\overline{u}$, hence for every $t \in \mathrm{Cont}(u)$. Therefore by Corollary 2.5 we have that $\mathrm{V}_e(u, I) \leq \liminf_{n\to\infty} \mathrm{V}_e(u_n, I)$. On the other hand it is easy to check that $\mathrm{V}_e(u_n, I) = \mathrm{V}_p(u_n, I) \leq \mathrm{V}_p(u, I)$, indeed for every $(t_j)_{j=1}^m \in \mathfrak{S}(I)$ we have

$$\begin{aligned}\sum_{j=1}^{m} \|u_n(t_j) - u(t_j)\| &= \sum_{j=1}^{m} \left\| \int_{\mathbb{R}} \rho_n(s)(\overline{u}(t_j - s) - \overline{u}(t_{j-1} - s))\,\mathrm{d}s \right\| \\ &\leq \mathrm{V}_p(\overline{u}, I) = \mathrm{V}_p(u, I).\end{aligned}$$

Therefore $\limsup_{n\to\infty} \mathrm{V}_e(u_n, I) \leq \mathrm{V}_e(u, I)$ and the convergence of the variations is proved. The L^1_{loc}-convergence follows from the dominated convergence theorem. The remaining assertion on the L^1-convergence can be proved first approximating u by a step function u_ε having compact support, and then approximating u_ε by convolution. □

A.3. A chain rule in $BV(I; \mathcal{H})$

In this subsection we are going to prove a chain rule for $\mathcal{H}$-valued functions of bounded variation (Theorem A.7). In the finite dimensional case this chain rule has been proved in the appendix of [7]. Since in that paper compactness theorems for measures are exploited, we need to employ a different technique.

Lemma A.3. *Let I be an interval, $h : I \longrightarrow \mathbb{R}$ be increasing and let $C_h := \{\alpha \in \mathbb{R} : \mathrm{card}(h^{-1}(\alpha)) > 1\}$. Then C_h is at most countable and* $\mathrm{D}\,h(h^{-1}(C_h)) = 0$.

Proof. Assume first that I is bounded. Since h is increasing, for every $\alpha \in \mathbb{R}$ we have that $h^{-1}(\alpha)$ is an interval contained in I, and $h^{-1}(\alpha) \cap h^{-1}(\beta) = \varnothing$ whenever $\alpha \neq \beta$. Therefore $\sum_{\alpha\in\mathbb{R}} \mathcal{L}^1(h^{-1}(\alpha)) \leq \mathcal{L}^1(I) < \infty$. Moreover $h^{-1}(\alpha)$ is nondegenerate if and only if $\mathrm{card}(h^{-1}(\alpha)) > 1$, therefore $\mathcal{L}^1(h^{-1}(\alpha)) > 0$ if and only if $\alpha \in C_h$. It follows that C_h is discrete if I is bounded. If I is unbounded it is enough to partition it in a countable sequence of bounded intervals, and apply the result to each interval separately. We have that $h^{-1}(C_h) = \bigcup_{\alpha\in C_h} h^{-1}(\alpha)$ and this union is disjoint. Since C_h is at most countable we get that $\mathrm{D}\,h(h^{-1}(C_h)) = \sum_{\alpha\in C_h} \mathrm{D}\,h(h^{-1}(\alpha)) = 0$ because each $h^{-1}(\alpha)$ is an interval where h is constant, so $\mathrm{D}\,h(h^{-1}(\alpha)) = 0$. □

Let us recall the following lemma which can be proved first for step functions and then by approximation (see, *e.g.* [10, Lemma 8, Section III.10, page 182]).

Lemma A.4. *Let T, S be two sets in $\mathbb{R}$ and $\mu : \mathscr{B}(T) \longrightarrow [-\infty, \infty]$ be a measure which is finite on compact sets. Let $\psi : T \longrightarrow S$ and $\nu : \mathscr{B}(S) \longrightarrow [-\infty, \infty]$ be a measure such that $\nu(B) = \mu(\psi^{-1}(B))$ for every $B \in \mathscr{B}(S)$. Then*

$$\int_A f \,\mathrm{d}\nu = \int_{\psi^{-1}(A)} f \circ \psi \,\mathrm{d}\mu \qquad \forall A \in \mathscr{B}(S).$$

for every $f \in L^1(\nu, S; \mathcal{H})$.

Lemma A.5. *Let $I \subseteq \mathbb{R}$ be an open interval and assume that $h : I \longrightarrow \mathbb{R}$ is increasing, bounded, and $h(t) \in \,]h(t-), h(t+)[$ for every $t \in \mathrm{Discont}(h)$. Then $\mathrm{D}\,h(h^{-1}(B)) = \mathcal{L}^1(B)$ for every $B \in \mathscr{B}(h(\mathrm{Cont}(h)))$.*

Proof. Assume that $I = \,]a, b[$ and set $J := \,]h(a+), h(b-)[$, $X := \mathrm{Cont}(h)$, $Y := h(\mathrm{Cont}(h))$. Thus

$$Y = J \setminus \bigcup_{t \in \mathrm{Discont}(h)}]h(t-), h(t+)[\,. \tag{A.7}$$

If V is an open set in Y then there exists an open set A in J and a sequence of mutually disjoint intervals $[c_n, d_n[\, \subseteq J$ such that

$$A = \bigcup_{n=1}^{\infty} [c_n, d_n[\, \subseteq J, \qquad V = Y \cap A. \tag{A.8}$$

It is not restrictive to assume that

$$c_n \notin \,]h(s-), h(s+)]\,, \quad d_n \notin [h(t-), h(t+)[\ \ \forall s,t \in \mathrm{Discont}(h),\ s \le t,\ \forall n \in \mathbb{N}, \tag{A.9}$$

indeed if (A.9) does not hold for some n, then we can replace $[c_n, d_n[$ by $[h(s-), h(t+)[$, so that V differs by $Y \cap A$ by a set of $\mathcal{L}^1$-measure zero and $h^{-1}(V)$ differs from $h^{-1}(Y \cap A)$ at most by a set where h is constant, hence where $\mathrm{D}\,h$ is zero (namely $h^{-1}(h(s-))$). From (A.7)–(A.9) we infer that

$$\mathrm{D}\,h(h^{-1}([c_n, d_n[)) = d_n - c_n$$

and

$$\begin{aligned} V &= \Big(J \setminus \bigcup_{t \in \mathrm{Discont}(h)}]h(t-), h(t+)[\Big) \cap A \\ &= A \setminus \bigcup_{\substack{t \in \mathrm{Discont}(h) \\ h(t) \in A}}]h(t-), h(t+)[\,. \end{aligned} \tag{A.10}$$

Observe that in this way the last difference in (A.10) is proper, hence

$$\mathcal{L}^1(V) = \sum_{n=1}^{\infty}(d_n - c_n) - \sum_{\substack{t\in \mathrm{Discont}(h)\\ h(t)\in A}} (h(t+) - h(t-)).$$

On the other hand we have

$$\begin{aligned} h^{-1}(V) &= h^{-1}\Big(A \setminus \bigcup_{\substack{t\in \mathrm{Discont}(h)\\ h(t)\in A}}]h(t-), h(t+)[\,\Big) \\ &= h^{-1}(A) \setminus \bigcup_{\substack{t\in \mathrm{Discont}(h)\\ h(t)\in A}} h^{-1}(]h(t-), h(t+)[) \end{aligned}$$

and this difference is proper, hence

$$\begin{aligned} \mathrm{D}\,h(h^{-1}(V)) &= \mathrm{D}\,h(h^{-1}(A)) - \mathrm{D}\,h\Big(\bigcup_{\substack{t\in \mathrm{Discont}(h)\\ h(t)\in A}} h^{-1}(]h(t-), h(t+)[)\Big) \\ &= \mathrm{D}\,h(h^{-1}(A)) - \sum_{\substack{t\in \mathrm{Discont}(h)\\ h(t)\in A}} (h(t+) - h(t-)). \end{aligned}$$

Let us compute $\mathrm{D}\,h(h^{-1}(A))$. We have

$$\begin{aligned} \mathrm{D}\,h(h^{-1}(A)) &= \mathrm{D}\,h\Big(h^{-1}\Big(\bigcup_{n=1}^{\infty} [c_n, d_n[\,\Big)\Big) = \mathrm{D}\,h\Big(\bigcup_{n=1}^{\infty} h^{-1}([c_n, d_n[)\Big) \\ &= \sum_{n=1}^{\infty} \mathrm{D}\,h(h^{-1}([c_n, d_n[)) = \sum_{n=1}^{\infty}(d_n - c_n) = \mathcal{L}^1(A). \end{aligned}$$

Therefore $\mathrm{D}\,h(h^{-1}(V)) = \mathcal{L}^1(V)$ for every open set V. By the coincidence criterion for measures (see *e.g.* [2, Proposition 1.8, page 5]) we get the thesis. □

Corollary A.6. *Let $I \subseteq \mathbb{R}$ be an open interval and assume that $h : I \longrightarrow \mathbb{R}$ is increasing, bounded, and $h(t) \in\,]h(t-), h(t+)[$ for every $t \in \mathrm{Discont}(h)$. Then*

$$\int_{h(B)} f \,\mathrm{d}\mathcal{L}^1 = \int_B f \circ h \,\mathrm{d}\mathrm{D}\,h \qquad \forall B \in \mathscr{B}(I),\ B \subseteq \mathrm{Cont}(h)$$

for every $f \in \mathrm{L}^1(\mathcal{L}^1; \mathcal{H})$.

Proof. Take $A := h(B)$ and set $Z := h^{-1}(h(B)) \setminus B$. We have that $Z \subseteq \{\alpha \in \mathbb{R} : \operatorname{card}(h^{-1}(\alpha)) > 1\}$, therefore by Lemma A.3 $\mathrm{D}\,h(Z) = 0$. Hence by Lemmas A.4 and A.5 we get that

$$\begin{aligned}\int_{h(B)} f \,\mathrm{d}\mathcal{L}^1 &= \int_{h^{-1}(h(B))} f \circ h \,\mathrm{d}\mathrm{D}\,h \\ &= \int_B f \circ h \,\mathrm{d}\mathrm{D}\,h + \int_Z f \circ h \,\mathrm{d}\mathrm{D}\,h = \int_B f \circ h \,\mathrm{d}\mathrm{D}\,h. \qquad \square\end{aligned}$$

Theorem A.7. *Let $I, J \subseteq \mathbb{R}$ be open intervals with J bounded. Assume that $h : I \longrightarrow J$ is increasing and that $f \in W^{1,\infty}(J; \mathcal{H})$. Define $g : I \longrightarrow \mathcal{H}$ by*

$$g(t) := \begin{cases} f'(h(t)) & \text{if } t \in \operatorname{Cont}(h) \\ \dfrac{f(h(t+)) - f(h(t-))}{h(t+) - h(t-)} & \text{if } t \in \operatorname{Discont}(h), \end{cases} \tag{A.11}$$

where f' is any representive in the $\mathcal{L}^1$-class of f'. Then $\mathrm{D}(f \circ h) = g\,\mathrm{D}\,h$.

Proof. We may assume that $h(t) \in \,]h(t-), h(t+)[$ whenever $t \in \operatorname{Discont}(h)$. It is enough to check that the measures $\mathrm{D}(f \circ h)$ and $g\,\mathrm{D}\,h$ coincide on intervals of type $[c, d[$, with $c, d \in I$, $c < d$. We have

$$\mathrm{D}(f \circ h)([c, d[) = f(h(d-)) - f(h(c-)). \tag{A.12}$$

If $\operatorname{Discont}(h) = \{t_k : k \in \mathbb{N}\}$ then we have

$$\begin{aligned}\int_{[c,d[} g(t) \,\mathrm{d}\mathrm{D}\,h(t) &= \int_{\operatorname{Cont}(h)\cap[c,d[} g(t) \,\mathrm{d}\mathrm{D}\,h(t) + \int_{\operatorname{Discont}(h)\cap[c,d[} g(t) \,\mathrm{d}\mathrm{D}\,h(t) \\ &= \int_{\operatorname{Cont}(h)\cap[c,d[} f'(h(t)) \,\mathrm{d}\mathrm{D}\,h(t) + \sum_{k\in\mathbb{N}, t_k\in[c,d[} (f(h(t_k+)) - f(h(t_k-))).\end{aligned}$$

Now

$$\begin{aligned}&\int_{\operatorname{Cont}(h)\cap[c,d[} f'(h(t)) \,\mathrm{d}\mathrm{D}\,h(t) \\ &= \int_{h(\operatorname{Cont}(h)\cap[c,d[)} f'(\sigma) \,\mathrm{d}\sigma \\ &= \int_{[h(c-),h(d-)[\setminus \bigcup_{t_k\in[c,d[}[h(t_k-),h(t_k+)]} f'(\sigma) \,\mathrm{d}\sigma \\ &= \int_{[h(c-),h(d-)[} f'(\sigma) \,\mathrm{d}\sigma - \int_{\bigcup_{t_k\in[c,d[}]h(t_k-),h(t_k+)]} f'(\sigma) \,\mathrm{d}\sigma \\ &= \int_{[h(c-),h(d-)[} f'(\sigma) \,\mathrm{d}\sigma - \sum_{t_k\in[c,d[} \int_{]h(t_k-),h(t_k+)]} f'(\sigma) \,\mathrm{d}\sigma \\ &= f(h(d-)) - f(h(c-)) - \sum_{t_k\in[c,d[} (f(h(t_k+)) - f(h(t_k-)))\end{aligned} \tag{A.13}$$

Therefore

$$(g\,\mathrm{D}\,h)([c,d[) = \int_{[c,d[} g(t)\,\mathrm{d}\mathrm{D}\,h(t) = f(h(d-)) - f(h(c-))$$

and the theorem is proved. □

Corollary A.8. *Let $I, J \subseteq \mathbb{R}$ be open intervals with J bounded. Assume that $h : I \longrightarrow J$ is increasing and left-continuous, $g \in Lip(J;\mathcal{H})$, and $f \in L^1(I;\mathcal{H})$. Then*

$$\int_I \langle f, \mathrm{dD}(g \circ h)\rangle = \int_{\mathrm{Cont}(h)} \langle f(t), g'(h(t))\rangle\,\mathrm{dD}\,h(t) + \sum_{t\in\mathrm{Discont}(h)} \langle f(t), g(h(t+)) - g(h(t))\rangle.$$

A.4. The Young and Lebesgue integrals

In this section we show that the Young integral with respect to a function g of bounded variation coincides with the ordinary Lebesgue integral with respect to the measure $\mathrm{D}\,g$, the distributional derivative of g. Let us now recall the definition of Young integral given in [16]. Assume that $I =]a,b[\subseteq \mathbb{R}$ is an open interval and $J \subseteq I$ is a bounded subinterval. Let us consider $f : I \longrightarrow \mathcal{H}$ and let $g \in Reg(I;\mathcal{H})$ be bounded. Let $\mathfrak{s} = \{t_0, \ldots, t_m\} \in \mathfrak{S}(J)$ be a subdivision of J and let $\mathfrak{c} = (c_j)_{j=1}^m$ be a family of numbers that is *consistent with* $\mathfrak{s}$, *i.e.* $t_{j-1} < c_j < t_j$ for every $j = 1, \ldots, m$. The *Young integral sum* is defined by

$$\begin{aligned} S_Y(f, g, \mathfrak{s}, \mathfrak{c}) := &\sum_{j=1}^m \langle f(c_j), g(t_j-) - g(t_{j-1}+)\rangle \\ &+ \sum_{j=0}^m \langle f(t_j), g(t_j+) - g(t_j-)\rangle. \end{aligned} \tag{A.14}$$

We say that *f is Young integrable with respect to g on J* if there exists $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\mathfrak{s}_\varepsilon \in \mathfrak{S}(J)$ which satisfies the inequality

$$|L - S_Y(f, g, \mathfrak{s}, \mathfrak{c})| < \varepsilon$$

whenever $\mathfrak{s}_\varepsilon \subseteq \mathfrak{s}$ and $\mathfrak{c}$ is consistent with $\mathfrak{s}$. The number L is uniquely determined and is called *Young integral of f with respect to g on J* and is denoted by one of the symbols

$$\int_J \langle f, \mathrm{d}g\rangle, \qquad \int_J \langle f(t), \mathrm{d}g(t)\rangle.$$

Now we can compare the Young and Lebesgue integrals.

Lemma A.9. *If $f \in Reg(I;\mathcal{H})$ is bounded and $g \in BV(I;\mathcal{H}) \cap Reg(I;\mathcal{H})$, then f is Young integrable with respect to g on J, $f \in L^1(\|\mathrm{D}\, g\|, J;\mathcal{H})$, and $\int_J \langle f, \mathrm{d}g\rangle = \int_J \langle f, \mathrm{d}\mathrm{D}\, g\rangle$.*

Proof. For simplicity we assume that J is closed. First of all we have that $f \in L^1(\|\mathrm{D}\, g\|, J;\mathcal{H})$, because it is bounded on J and $\mathrm{D}\, g$-measurable. Let us neglect the trivial case when $\mathrm{D}\, g$ is zero. Then there is a step map f_ε with respect to intervals such that $\|f - f_\varepsilon\|_\infty < \varepsilon/(2\|\mathrm{D}\, g\|)$. We may assume that there are a subdivision $(t_j)_{j=0}^m$ and vectors $x_1, \ldots, x_m$ such that $t_0 = \inf J$, $t_m = \sup J$, $f_\varepsilon = \sum_{j=1}^m \chi_{]t_{j-1},t_j[}x_j + \sum_{j=0}^m \chi_{\{t_j\}} f(t_j)$. Observe that $\sup_{t\in]t_{j-1},t_j[} |f(t) - x_j| < \varepsilon/(2\|\mathrm{D}\, g\|)$ for every j. Therefore if we take *e.g.* $c_j := (t_{j-1} + t_j)/2$, then

$$
\begin{aligned}
&\left|\sum_{j=1}^m \langle f(c_j), g(t_j-) - g(t_{j-1}+)\rangle + \sum_{j=1}^m \langle f(t_j), g(t_j+) - g(t_j-)\rangle - \int_J \langle f, \mathrm{d}\mathrm{D}\, g\rangle\right| \\
&\le \left|\sum_{j=1}^m \langle f(c_j), g(t_j-) - g(t_{j-1}+)\rangle - \sum_{j=1}^m \langle x_j, g(t_j-) - g(t_{j-1}+)\rangle\right| \\
&\quad + \left|\sum_{j=1}^m \langle x_j, g(t_j-) - g(t_{j-1}+)\rangle + \sum_{j=1}^m \langle f(t_j), g(t_j+) - g(t_j-)\rangle - \int_J \langle f, \mathrm{d}\mathrm{D}\, g\rangle\right| \\
&= \left|\sum_{j=1}^m \langle x_j - f(c_j), g(t_j-) - g(t_{j-1}+)\rangle\right| + \left|\int_J \langle f_\varepsilon, \mathrm{d}\mathrm{D}\, g\rangle - \int_J \langle f, \mathrm{d}\mathrm{D}\, g\rangle\right| \\
&\le \sum_{j=1}^m \|x_j - f(c_j)\|_{\mathcal{H}} \|g(t_j-) - g(t_{j-1}+)\|_{\mathcal{H}} + \int_J \|f_\varepsilon(t) - f(t)\|_{\mathcal{H}} \,\mathrm{d}\|\mathrm{D}\, g\|(t) \\
&\le \sum_{j=1}^m \frac{\varepsilon}{2\|\mathrm{D}\, g\|} \|\mathrm{D}\, g\|(]t_{j-1}, t_j[) + \frac{\varepsilon}{2\|\mathrm{D}\, g\|}\|\mathrm{D}\, g\|(J) < \varepsilon.
\end{aligned}
$$

On the other hand by [16, Corollary 3.10] we have that f is Young integrable with respect to g, hence $\int_J \langle f, \mathrm{d}g\rangle = \int_J \langle f, \mathrm{d}\mathrm{D}\, g\rangle$. □

A.5. Convex sets and normal cones

In this subsection we assume that

$$\mathcal{Z} \subseteq \mathcal{H} \text{ is a closed convex subset and } 0 \in \mathcal{Z}, \tag{A.15}$$

and the projection mapping to $\mathcal{Z}$ is denoted by $\mathrm{Proj}_{\mathcal{Z}}$. If $x \in \mathcal{Z}$ then $N_{\mathcal{Z}}(x)$, the *normal cone to $\mathcal{Z}$ at x* and $T_{\mathcal{Z}}(x)$, the *tangent cone to $\mathcal{Z}$ at x*, are defined by

$$N_{\mathcal{Z}}(x) := \{y \in \mathcal{H} \ :\ \langle y, z - x\rangle \le 0 \ \forall z \in \mathcal{Z}\}, \tag{A.16}$$

$$T_{\mathcal{Z}}(x) := \{w \in \mathcal{H} \ :\ \langle w, y\rangle \le 0 \ \forall y \in N_{\mathcal{Z}}(x)\}. \tag{A.17}$$

Let us recall that $x \in \mathcal{Z}$ is a *support point for* $\mathcal{Z}$ if there exist $f \in \mathcal{H} \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that $\mathcal{Z} \subseteq \{y : \langle f, y\rangle \leq \alpha\}$ and $\langle f, x\rangle = \alpha$. The set $\{y : \langle f, y\rangle = \alpha\}$ is called *supporting hyperplane to* $\mathcal{Z}$ *at* x. The set of support points is denoted by $\mathrm{Supp}(\mathcal{Z})$ and has the following two properties (see *e.g.* [1, Lemma 7.37, Section 7.8, page 280]):

$$x \in \mathrm{Supp}(\mathcal{Z}) \iff N_{\mathcal{Z}}(x) \neq \varnothing \iff \mathrm{Proj}_{\mathcal{Z}}^{-1}(x) \setminus \mathcal{Z} \neq \varnothing, \tag{A.18}$$

$$\mathrm{Supp}(\mathcal{Z}) \text{ is dense in } \partial\mathcal{Z}. \tag{A.19}$$

In this subsection for $f \in \mathcal{H}\setminus\{0\}$ and $\gamma \in \mathbb{R}$ we will use the notation $\{f = \gamma\} := \{x \in \mathcal{H} : \langle f, x\rangle = \gamma\}$. Two hyperplanes $\{f = \alpha\}$, $\{g = \beta\}$ are *parallel* if f and g are parallel.

Lemma A.10. *Let us suppose that* (A.15) *holds. Assume* $f \in \mathcal{H}\setminus\{0\}$ *and that for every* $x \in \mathrm{Supp}(\mathcal{Z})$ *and for every supporting hyperplane* Π_x *at* x*,* Π_x *is parallel to* $\{f = 0\}$*. Then there exist* $\alpha, \beta \in \mathbb{R}$ *(possibly* $\alpha = \beta$*) such that*

$$\mathrm{Supp}(\mathcal{Z}) = \partial\mathcal{Z} = \{f = \alpha\} \cup \{f = \beta\}. \tag{A.20}$$

Proof. For convenience we set $P_\gamma := \{f = \gamma\}$, $\gamma \in \mathbb{R}$. It is easy to check that there exist $\alpha, \beta \in \mathbb{R}$ (possibly $\alpha = \beta$) such that

$$\mathrm{Supp}(\mathcal{Z}) \subseteq P_\alpha \cup P_\beta. \tag{A.21}$$

Now we claim that

$$\mathrm{Supp}(\mathcal{Z}) = \partial\mathcal{Z}. \tag{A.22}$$

Indeed, if we assume that there exists $x \in \partial\mathcal{Z} \setminus \mathrm{Supp}(\mathcal{Z})$ and we select a sequence $x_n \in \mathrm{Supp}(\mathcal{Z})$ such that $x_n \to x$, then there exists $y_n \notin \mathcal{Z}$ such that $x_n = \mathrm{Proj}_{\mathcal{Z}}(y_n)$ for every $n \in \mathbb{N}$. By the assumptions we can take y_n such that $y_n - x_n = \lambda f$ for some $\lambda \neq 0$, indeed by (A.21) we can assume that every x_n belongs to only one supporting hyperplane P_γ where $\gamma \in \{\alpha, \beta\}$ is fixed. We deduce that $y_n = \lambda f - x_n \to \lambda f - x$ as $n \to \infty$. Observe that $y_n \in \lambda f - P_\gamma$ for every n and that $\lambda f - P_\gamma$ is closed and disjoint from $\mathcal{Z}$: it follows that $y := \lambda f - x \in \lambda f - P_\gamma$ and $y \notin \mathcal{Z}$. By the continuity of the projection, $\mathrm{Proj}_{\mathcal{Z}}(y_n) \to \mathrm{Proj}_{\mathcal{Z}}(y) = x$. This means that $x \in \mathrm{Supp}(\mathcal{Z})$ and (A.22) is proved.

Now we can prove (A.20). Assume by contradiction that there is $\gamma \in \{\alpha, \beta\}$ such that P_γ contains points both in $\partial\mathcal{Z}$ and in the complement of $\partial\mathcal{Z}$. Take $p \in P_\gamma \setminus \partial\mathcal{Z}$. Since P_γ is a supporting hyperplane, p is not an interior point of $\mathcal{Z}$, hence $p \notin \mathcal{Z}$. Thus there is a ball $B_\rho(p)$ such that $B_\rho(p) \cap \mathcal{Z} = \varnothing$ and $\partial B_\rho(p) \cap \mathcal{Z} \neq \varnothing$. By convexity we can take $z \in \partial B_\rho(p) \cap \partial\mathcal{Z} \cap P_\gamma$. It follows that $p - z$ is a normal vector to $\mathcal{Z}$ which is not parallel to f, a contradiction which proves the lemma. □

Lemma A.11. *Assume that* (A.15) *holds and* $\mathcal{Z}$ *is properly contained in a closed vector subspace* $\mathcal{V} \neq \mathcal{H}$*. Then there exist two hyperplanes* Π_1, Π_2 *supporting* $\mathcal{Z}$ *respectively at* x_1 *and* x_2 *such that* Π_1 *and* Π_2 *are not parallel and* $x_1 \notin \Pi_1 \cap \Pi_2$.

Proof. We can assume that $\mathcal{V}$ is the smallest closed vector subspace containing $\mathcal{Z}$ (it suffices to replace $\mathcal{V}$ by the intersection of all closed subspaces containing $\mathcal{Z}$, which is still a closed subspace).

Take $y_2 \in \mathcal{V} \smallsetminus \mathcal{Z}$ and set $x_2 := \mathrm{Proj}_{\mathcal{Z}}(y_2)$. Then if $f_2 := y_2 - x_2$ and $\alpha_2 = \langle f_2, x_2 \rangle$ the hyperplane $\Pi_2 := \{f_2 = \alpha_2\}$ supports $\mathcal{Z}$ at x_2 because $\mathcal{Z} \subseteq \mathcal{V}$.

Now observe that $\mathcal{W} := \Pi_2 \cap \mathcal{V}$ is a proper subset of $\mathcal{V}$ because $y_2 \in \mathcal{V} \smallsetminus \Pi_1$. We have that $\mathcal{Z}$ is not contained in $\mathcal{W}$, because otherwise $0 \in \mathcal{W}$, thus $\mathcal{V}$ would not be the smallest closed vector subspace containing $\mathcal{Z}$. Hence there is $x_1 \in \mathcal{V} \smallsetminus \mathcal{W}$. Now take $f_1 \in \mathcal{H} \smallsetminus \mathcal{V}$ such that $\langle f_1, v \rangle = 0$ for every $v \in \mathcal{V}$. It follows that $\mathrm{Proj}_{\mathcal{Z}}(x_1 + f_1) = x_1$, hence $x_1 \in \mathrm{Supp}(\mathcal{Z})$ and $\Pi_1 := \{f_1 = 0\}$ is a supporting hyperplane for $\mathcal{Z}$ in x_1. Since $f_2 \in \mathcal{V}$ we have $\langle f_1, f_2 \rangle = 0$, thus the two hyperplanes are not parallel. Moreover $x_1 \notin \Pi_1 \cap \Pi_2$. □

Lemma A.12. *Assume that* (A.15) *holds and that $\mathcal{Z}$ is not a closed vector subspace and that $\mathcal{Z}$ is not of the form*

$$\mathcal{Z} = \{x \in \mathcal{H} \ : \ -\alpha \le \langle f, x \rangle \le \beta\} \tag{A.23}$$

for some $f \in \mathcal{H} \smallsetminus \{0\}$, $\alpha, \beta \in [0, \infty]$. Then there exist $x_1, x_2 \in \mathrm{Supp}(\mathcal{Z})$ and two supporting hyperplanes Π_1, Π_2 of $\mathcal{Z}$ respectively at x_1 and x_2, such that Π_1 and Π_2 are not parallel and $x_1 \notin \Pi_1 \cap \Pi_2$.

Proof. By Lemma A.10 we infer that there exist two supporting hyperplanes Π_1 and Π_2 which are not parallel. Let $x_1, x_2 \in \mathrm{Supp}(\mathcal{Z})$ be such that Π_j supports $\mathcal{Z}$ at x_j, $j = 1, 2$. Of course we can take $x_1 \neq x_2$. We can also assume that $x_1 \notin \Pi_1 \cap \Pi_2$, indeed we have the two following possibilities.

(i) If $\mathcal{Z}$ is not contained in $\Pi_1 \cap \Pi_2$, then we consider another support point $x_3 \notin \Pi_1 \cap \Pi_2$ with supporting hyperplane Π_3: if Π_3 is parallel to, say, Π_2, then we replace Π_2 by Π_3.

(ii) If $\mathcal{Z}$ is contained in $\Pi_1 \cap \Pi_2$, then $\Pi_1 \cap \Pi_2$ is a vector subspace and we can apply Lemma A.11. □

In the proof of the following proposition we use the argument of the Bishop-Phelps theorem (see [1, Theorem 7.43, Section, 7.9, page 284]).

Proposition A.13. *Assume that* (A.15) *holds, that $\mathcal{Z}$ is not a vector subspace and that $\mathcal{Z}$ is not of the type*

$$\mathcal{Z} = \{x \in \mathcal{H} \ : \ -\alpha \le \langle f, x \rangle \le \beta\} \tag{A.24}$$

for some $f \in \mathcal{H} \smallsetminus \{0\}$, $\alpha, \beta \in [0, \infty]$. Then there exist $z_j \in \partial\mathcal{Z}$, $j = 1, 2$, such that $z_1 \neq z_2$, $N_{\mathcal{Z}}(z_1) \neq \{0\}$ and there exists $v \in N_{\mathcal{Z}}(z_2) \smallsetminus \{0\}$ such that $v \notin N_{\mathcal{Z}}(z_1) \cup T_{\mathcal{Z}}(z_2)$.

Proof. By Lemma A.12 there exist $x_j \in \mathrm{Supp}(\mathcal{Z})$, $j = 1, 2$, and two supporting hyperplanes $\Pi_j = \{f_j = \alpha_j\}$ of $\mathcal{Z}$ at x_j, such that Π_1 and Π_2 are not parallel and $x_1 \notin \Pi_1 \cap \Pi_2$. Observe that $\langle f_2, x_2 - x_1\rangle \neq 0$, because otherwise we would have that $x_1 \in \Pi_2$. Hence, as $f_2 \in N_{\mathcal{Z}}(x_2)$, we have that $\langle f_2, x_1 - x_2\rangle < 0$, which implies that $f_2 \notin N_{\mathcal{Z}}(x_1)$. Now let us set

$$v_\lambda := \sqrt{1-\lambda^2} f_1 + \lambda f_2, \qquad 0 \le \lambda \le 1,$$

and

$$\Lambda := \{\lambda \in [0, 1] \ : \ v_\lambda \in N_{\mathcal{Z}}(x_1)\}.$$

It is an easy exercise to show that Λ is a closed interval containing 0, but not 1. It is not restrictive to assume that

$$\{\lambda f_1 + \mu(\overline{x} - x_1) \ : \ \lambda, \mu > 0\} \cap N_{\mathcal{Z}}(x_1) = \varnothing$$

where $\overline{x} = \mathrm{Proj}_{\Pi_1 \cap \Pi_2}(x_1)$, indeed it suffices to replace f_1 by v_{λ_1}, with $\lambda_1 = \max \Lambda$. Therefore if we take

$$f \in \{\lambda f_1 + \mu(\overline{x} - x_1) : \ \lambda, \mu > 0\}. \tag{A.25}$$

we have that $f \notin N_{\mathcal{Z}}(x_1)$. We can take $\|f\|_{\mathcal{H}} = 1$, thus $f = \lambda_1 f_1 + \lambda_2 f_2$ for some $\lambda_1, \lambda_2 \in \,]0, 1[$. Since $N_{\mathcal{Z}}(x_1) \cup T_{\mathcal{Z}}(x_1)$ is a closed cone, we infer that there exists $\varepsilon \in \,]0, 1[$ such that $B_\varepsilon(f)$ is contained in the complement of $N_{\mathcal{Z}}(x_1) \cup T_{\mathcal{Z}}(x_1)$.

Now consider the convex cone

$$K := \{x \in \mathcal{H} \ : \ \langle f, x\rangle \ge \varepsilon \|x\|_{\mathcal{H}}/(4+\varepsilon)\}$$

and define the partial order "$\preceq$" in $\mathcal{Z}$ by setting $x \preceq y$ iff $y - x \in K$. Let $\mathcal{C}$ be a totally ordered subset of $\mathcal{Z}$. Using the set of indexes $\Gamma = \mathcal{C}$ and setting $x_\gamma := \gamma$ for each $\gamma \in \mathcal{C}$, we can consider $\mathcal{C} = (x_\gamma)$ as an increasing net. For every γ we have

$$\langle f, x_\gamma\rangle = \lambda_1\langle f_1, x_\gamma\rangle + \lambda_2\langle f_2, x_\gamma\rangle \le \alpha_1 + \alpha_2. \tag{A.26}$$

Thus $\langle f, x_\gamma\rangle$ is an increasing net of real numbers that is bounded above, hence it is a Cauchy net. Observe that $\varepsilon\|x_{\gamma_1} - x_{\gamma_2}\|_{\mathcal{H}} \le (4+\varepsilon)|\langle f, x_{\gamma_1} - x_{\gamma_2}\rangle|$ hence (x_γ) is a Cauchy net and it converges to some $x_\infty \in \mathcal{Z}$. It follows that x_∞ is an upper bound of $\mathcal{C}$. Hence by Zorn's Lemma there is a maximal element $x_m \in \mathcal{Z}$ with respect to $\preceq$. This is equivalent to the equality $\mathcal{Z} \cap [x_m + K] = \{x_m\}$. Hence $\mathcal{Z} \cap [x_m + \mathring{K}] = \varnothing$, thus there exists $f_\varepsilon \in \mathcal{H} \setminus \{0\}$ such that $\|f_\varepsilon\|_{\mathcal{H}} = 1$ and $\langle f, z\rangle \le \langle f, x_m + v\rangle$ for every $z \in \mathcal{Z}$ and $v \in K$. It follows that $x_m \in \partial\mathcal{Z}$ and $\{f_\varepsilon = \langle f_\varepsilon, x_m\rangle\}$ is a supporting hyperplane for $\mathcal{Z}$ at x_m. Moreover $\langle f_\varepsilon, v\rangle \ge 0$ for every $v \in K$. Using elementary geometry it is not hard to show that $\|f - f_\varepsilon\|_{\mathcal{H}} \le \varepsilon$ (see [1, Lemma 7.41, Section, 7.9, page 282] for a Banach space proof). Therefore we infer that $f_\varepsilon \in N_{\mathcal{Z}}(x_m)$ and $f_\varepsilon \notin N_{\mathcal{Z}}(x_1) \cup T_{\mathcal{Z}}(x_1)$ because the ball $B_\varepsilon(f)$ is contained in the complement of $N_{\mathcal{Z}}(x_1) \cup T_{\mathcal{Z}}(x_1)$. The proposition is proved with $z_1 = x_1$, $z_2 = x_m$ and $v = f_\varepsilon$. □

References

[1] C. D. ALIPRANTIS and K. C. BORDER, “Infinite Dimensional Analysis” (Third Edition), Springer, Berlin, Heidelberg, 2006.

[2] L. AMBROSIO, N. FUSCO and D. PALLARA, “Functions of Bounded Variation and Free Discontinuity Problems”, Oxford Mathematical Monographs, Clarendon Press, Oxford, 2000.

[3] N. BOURBAKI, “Fonctions d’une variable rèelle”, Hermann, Paris, 1958.

[4] H. BREZIS, “Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert”, North-Holland Mathematical Studies, Vol. 5, North-Holland Publishing Company, Amsterdam, 1973.

[5] H. BREZIS, “Analyse fonctionelle - Théorie et applications”, Masson, Paris, 1983.

[6] M. BROKATE and J. SPREKELS, “Hysteresis and Phase Transitions”, Applied Mathematical Sciences, Vol. 121, Springer-Verlag, New York, 1996.

[7] G. DAL MASO, P. LE FLOCH and F. MURAT, *Definition and weak stability of nonconservative products*, J. Math. Pures Appl. **74** (1995), 483–548.

[8] N. DINCULEANU, “Vector Measures”, International Series of Monographs in Pure and Applied Mathematics, Vol. 95, Pergamon Press, Berlin, 1967.

[9] J. L. DOOB, “Measure Theory”, Springer-Verlag, New York, 1994.

[10] N. DUNFORD and J. SCHWARTZ, “Linear Operators, Part 1”, Wiley Interscience, New York, 1958.

[11] H. FEDERER, “Geometric Measure Theory”, Springer-Verlag, Berlin-Heidelberg, 1969.

[12] M. A. KRASNOSEL’SKIĬ and A. V. POKROVSKIĬ, “Systems with Hysteresis”, Springer-Verlag, Berlin Heidelberg, 1989.

[13] P. KREJČÍ, *Vector hysteresis models*, European J. Appl. Math. **2** (1991), 281–292.

[14] P. KREJČÍ, “Hysteresis, Convexity and Dissipation in Hyperbolic Equations”, Gakuto International Series Mathematical Sciences and Applications, Vol. 8, Gakkōtosho, Tokyo, 1996.

[15] P. KREJČÍ, *Evolution variational inequalities and multidimensional hysteresis operators*, In: “Nonlinear Differential Equations” (Chvalatice, 1998), Vol. 404, Chapman & Hall/CRC Res. Notes Math., 1999, 47–110.

[16] P. KREJČÍ and P. LAURENÇOT, *Generalized variational inequalities*, J. Convex Anal. **9** (2002), 159–183.

[17] P. KREJČÍ and M. LIERO, *Rate independent Kurzweil processes*, Appl. Math. **54** (2009), 117–145.

[18] S. LANG, “Real and Functional Analysis” (Third Edition), Graduate Text in Mathematics, Vol. 142, Springer Verlag, New York, 1993.

[19] I. D. MAYERGOYZ, “Mathematical Models of Hysteresis”, Springer-Verlag, New York, 1991.

[20] A. MIELKE, *Evolution in rate-independent systems*, In: “Handbook of Differential Equations, Evolutionary Equations”, Vol. 2, C. Dafermos and E. Fereisl (eds.), Elsevier, 2005, 461–559.

[21] V. RECUPERO, *BV-extension of rate independent operators*, Math. Nachr. **282** (2009), 86–98.

[22] V. RECUPERO, *On locally isotone rate independent operators*, Appl. Math. Letters. **20** (2007), 1156–1160.

[23] V. RECUPERO, *The play operator on the rectifiable curves in a Hilbert space*, Math. Methods Appl. Sci. **31** (2008), 1283–1295.

[24] V. RECUPERO, *Sobolev and strict continuity of general hysteresis operators*, Math. Methods Appl. Sci. **32** (2009), 2003–2018.

[25] V. RECUPERO, *On a class of scalar variational inequalities with measure data*, Appl. Anal. **88** (2009), 1739–1753.

[26] U. STEFANELLI, *A variational characterization of rate independent evolution*, Math. Nachr. **282** (2009), 1492–1512.

[27] A. VISINTIN, “Differential Models of Hysteresis”, Applied Mathematical Sciences, Vol. 111, Springer-Verlag, Berlin Heidelberg, 1994.

Dipartimento di Matematica
Politecnico di Torino
Corso Duca degli Abruzzi, 24
10129 Torino, Italia
vincenzo.recupero@polito.it