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Least energy nodal solution of a singular perturbed problem with jumping nonlinearity

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Abstract. In this paper we study the asymptotic behavior of the least energy nodal solution of a problem with a jumping nonlinearity.

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1. Introduction

There has been a considerable interest to understand the asymptotic behavior of positive solutions of the elliptic problem

$$\begin{cases} \varepsilon^2 \Delta u - u + f(u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a parameter, f is a superlinear function, Ω is a smooth bounded domain in $\mathbb{R}^N$. Let $F(u) = \int_0^u f(t)dt$. In this paper, we consider the problem

$$\begin{cases} \varepsilon^2 \Delta u - \lambda_1 u^+ + \lambda_2 u^- + f(u) = 0 & \text{in} \Omega \\ u^{\pm} \neq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

where $\lambda_1 > 0, \lambda_2 > 0$ with $\lambda_1 \neq \lambda_2$, and $u^{\pm} = \max\{\pm u, 0\}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function satisfying:

(f1) $f(t) = o(t)$ as $t \to 0$;

(f2) $f(t) = O(|t|^p)$ as $t \to +\infty$ for some $p \in (1, \frac{N+2}{N-2})$ if $N \geq 3$ and $p > 1$ if $N = 1, 2$;

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(f3) there exists a constant $\theta > 2$ such that $\theta F(t) \leq tf(t)$ where

$$F(t) = \int_0^t f(s)ds;$$

(f4) $|t|f'(t) > f(t)(sgn\ t)$ for all $t \neq 0$.

Condition (f4) implies that $\frac{1}{2}f(t)t - F(t)$ is strictly increasing in $(0, +\infty)$. Problem (1.1) arises in various applications, such as chemotaxis, population genetic, chemical reactor theory. Problem (1.2) arises in the study of population dynamics with jumping nonlinearity [9]. It can also be considered as the limiting problem of the following elliptic system

$$\begin{cases} \varepsilon^2 \Delta u - \lambda_1 u + \mu_1 u^3 + \beta u v^2 = 0 & \text{in } \Omega \\ \varepsilon^2 \Delta v - \lambda_2 v + \mu_2 v^3 + \beta v u^2 = 0 & \text{in } \Omega \\ u, v > 0 & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

The system (1.3) arises in the Bose-Einstein condenstates and nonlinear optics. An important phenomena of (1.3) is the so-called *phase separation*. As $\beta \to -\infty$, the components u, v separates and the difference function $u - v$ approaches a solution of (1.2) with $f(u) = \mu_1 u_+^3 - \mu_2 u_-^3$. This has been proved for the least energy solution of (1.3) in [5, 7] and for radial solutions on two dimensional balls in [20]. We refer to [1, 2, 4, 5, 8, 10, 14, 19, 20] and the references therein.

Existence and concentration of positive solution of this type of problems were extensively studied by Ni-Takagi [16, 17], Ni-Wei [18], del Pino- Felmer [11].

Define

$$I_{\lambda_1}(W) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla W|^2 + \frac{\lambda_1}{2}\int_{\mathbb{R}^N} W^2 - \int_{\mathbb{R}^N} F(W)$$

and

$$I_{\lambda_2}(W) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla W|^2 + \frac{\lambda_2}{2}\int_{\mathbb{R}^N} W^2 - \int_{\mathbb{R}^N} F(W).$$

Let W_{λ_1} be a least energy positive solution of

$$\begin{cases} -\Delta u + \lambda_1 u = f(u) & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{1.4}$$

and W_{λ_2} be a least positive solution of

$$\begin{cases} -\Delta u + \lambda_2 u = f(u) & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u \in H^1(\mathbb{R}^N). \end{cases} \tag{1.5}$$

By Gidas, Ni and Nirenberg [13], it is well known that W_{λ_i} is radially decreasing and decays as

$$W_{\lambda_i}(|x|) \sim e^{-\sqrt{\lambda_i}|x|}|x|^{\frac{1-N}{2}} \text{ as } |x| \to +\infty$$

for $i = 1, 2$. Throughout the course of the paper we will call W_{λ_i} an entire solution or a ground state.

In this paper, we prove the existence of a least energy nodal solution and show that for ε sufficiently small, the solution has a exactly one positive spike and one negative spike and the spikes concentrate at two distinct points of Ω, in other words they repel each other. We define a function $\varphi : \Omega \times \Omega \to \mathbb{R}$ by

$$\varphi(x, y) = \min\left\{\sqrt{\lambda_1}d(x, \partial\Omega), \sqrt{\lambda_2}d(y, \partial\Omega)), \frac{1}{2}\frac{\sqrt{\lambda_1}\sqrt{\lambda_2}}{\sqrt{\lambda_1}+\sqrt{\lambda_2}}|x-y|\right\}.$$

Theorem 1.1. *There exists* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon < \varepsilon_0$*, the least energy nodal solution* $u_\varepsilon \in H_0^1(\Omega)$ *of* (1.2) *having exactly one positive local maximum* (*hence a global maximum*) *point* P_ε^1 *and one negative local minimum* (*hence a global minimum*) *point* P_ε^2 *and*

$$\lim_{\varepsilon\to 0}\varphi(P_\varepsilon^1, P_\varepsilon^2) = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}}\varphi(x, y),$$

with $u_\varepsilon(P_\varepsilon^i) \to (-1)^{i-1}W_{\lambda_i}(0)$ *and* $u_\varepsilon \to 0$ *in* $C^1_{\text{loc}}(\Omega \setminus \{P_\varepsilon^1, P_\varepsilon^2\})$.

Note that for sufficiently small $\varepsilon > 0$, the least energy positive solution to the problem (1.1) has a unique maxima P_ε; u_ε decays exponentially away from P_ε and $d(P_\varepsilon, \partial\Omega) \to \max_{P\in\Omega} d(P, \partial\Omega)$ as $\varepsilon \to 0$, which implies that the solution concentrates at an interior point furthest from the boundary of Ω. This was studied by Ni–Wei [15]. For the least energy nodal solution, the problem was studied by Noussair–Wei [18] when $\lambda_1 = \lambda_2 = 1$ and $f(u) = u^p$. They obtain the same results as in Theorem 1.1. In addition, they prove that $u_\varepsilon(x) = W(\frac{x-P_\varepsilon^1}{\varepsilon}) - W(\frac{x-P_\varepsilon^2}{\varepsilon}) + v_\varepsilon$, where $\|v_\varepsilon\|_{L^\infty(\Omega)} \to 0$ as $\varepsilon \to 0$ and W is the unique solution of the limiting problem. The study of asymptotic behavior involves the uniqueness and non-degeneracy of solution of the limiting problem. Then using the expansion, an asymptotic expansion of the energy is obtained. This approach does not work here since u_+ and u_- are not differentiable. Neither we have uniqueness nor nondegeneracy of the ground state. There is another approach by del Pino and Felmer [11] where they used variational characterizations of positive solutions and symmetrization technique. However their approach works well for positive solutions but does not work for sign-changing solutions. We shall modify the approach of del Pino and Felmer. The problem here is more complicated since the solution is sign-changing and we have to estimate the interaction of the positive and negative components.

2. Preliminaries

Without loss of generality, we consider $0 < \lambda_1 < \lambda_2$. The associated functional to the problem (1.2) is

$$E_\varepsilon(u) = \int_\Omega \left(\frac{\varepsilon^2}{2}|\nabla u|^2 + \frac{\lambda_1}{2}(u^+)^2 + \frac{\lambda_2}{2}(u^-)^2 - F(u) \right) dx.$$

Note that from (f_2), $E_\varepsilon \in \mathcal{C}^1(H_0^1(\Omega), \mathbb{R})$. Moreover, if $u_\varepsilon \in H_0^1(\Omega)$ is a critical point of E_ε, then $u_\varepsilon \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ and hence u_ε is a classical solution of (1.2). Note that $E_\varepsilon(u) = E_{\varepsilon,\lambda_1}(u) + E_{\varepsilon,\lambda_2}(u)$ where

$$E_{\varepsilon,\lambda_1}(u) = \int_\Omega \left(\frac{\varepsilon^2}{2}|\nabla u^+|^2 + \frac{\lambda_1}{2}(u^+)^2 - F(u^+) \right) dx,$$

$$E_{\varepsilon,\lambda_2}(u) = \int_\Omega \left(\frac{\varepsilon^2}{2}|\nabla u^-|^2 + \frac{\lambda_2}{2}(u^-)^2 - F(u^-) \right) dx.$$

Define the Nehari set as

$$\mathcal{N}_\varepsilon = \Big\{ u \in H_0^1(\Omega) : u^\pm \not\equiv 0, \varepsilon^2 \int_\Omega |\nabla u^+|^2 + \lambda_1 \int_\Omega (u^+)^2 = \int_\Omega f(u^+)u^+; \\ \varepsilon^2 \int_\Omega |\nabla u^-|^2 + \lambda_2 \int_\Omega (u^-)^2 = \int_\Omega f(u^-)u^- \Big\}. \quad (2.1)$$

Define the positive and negative Nehari set as

$$\mathcal{N}_\varepsilon^+ = \{u \in H_0^1(\Omega) : \langle E'_{\varepsilon,\lambda_1}(u), u\rangle = 0; u \not\equiv 0 \text{ and } u \geq 0\} \quad (2.2)$$

and

$$\mathcal{N}_\varepsilon^- = \{u \in H_0^1(\Omega) : \langle E'_{\varepsilon,\lambda_2}(u), u\rangle = 0; u \not\equiv 0 \text{ and } -u \geq 0\} \quad (2.3)$$

respectively. Note that any u belonging to $\mathcal{N}_\varepsilon$ is sign-changing. Moreover, all the sign-changing solutions of (1.2) are contained in $\mathcal{N}_\varepsilon$. Also note that $\mathcal{N}_\varepsilon^+ \cap \mathcal{N}_\varepsilon^- = \emptyset$. Let

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} E_\varepsilon(u). \quad (2.4)$$

Remark 2.1. The set $\mathcal{N}_\varepsilon$ is not a manifold in $H_0^1(\Omega)$ due to the lack of differentiability of the map $u \mapsto u^\pm$. In fact, $\mathcal{N}_\varepsilon \cap H^2(\Omega)$ is a $\mathcal{C}^1$ manifold of codimension 2 in $H^2(\Omega)$, see [1]. Hence it is not clear whether a minimizer of E_ε on $\mathcal{N}_\varepsilon$ is indeed a solution of (1.2).

Remark 2.2. Define $h^\pm(t) = E_\varepsilon(tu_\varepsilon^\pm)$. Note that $h^\pm$ is strictly increasing for $t \in (0, 1)$ and strictly decreasing in $t \in (1, +\infty)$. This implies that $\max_{0<t<+\infty} h^\pm(t)$ exists and occurs at $t = 1$.

We will show that there exists $u_\varepsilon \in \mathcal{N}_\varepsilon$ such that $c_\varepsilon = E_\varepsilon(u_\varepsilon)$, and that u_ε is a least energy sign-changing solution. We state some elementary lemmas,

Lemma 2.3. *For all $\varepsilon > 0$, $\mathcal{N}_\varepsilon^+$ and $\mathcal{N}_\varepsilon^-$ are closed subsets of $H_0^1(\Omega)$.*

$$0 < c_\varepsilon^+ = \inf_{u\in\mathcal{N}_\varepsilon^+} E_{\varepsilon,\lambda_1}(u) = \inf_{u\in H_0^1(\Omega),u\neq 0} \max_{t\geq 0} E_{\varepsilon,\lambda_1}(tu)$$

and

$$0 < c_\varepsilon^- = \inf_{u\in\mathcal{N}_\varepsilon^-} E_{\varepsilon,\lambda_2}(u) = \inf_{u\in H_0^1(\Omega),u\neq 0} \max_{t\geq 0} E_{\varepsilon,\lambda_2}(tu).$$

Moreover, $\mathcal{N}_\varepsilon^\pm$ is a $\mathcal{C}^1$ manifold of codimension 1 and every minimizer u of E_ε on $\mathcal{N}_\varepsilon^\pm$ is positive.

Proof. This follows trivially by using (f_4) and Sobolev embedding theorem. See [15]. $\mathcal{N}_\varepsilon^\pm$ is a $\mathcal{C}^1$ manifold of codimension 1 follows from [3]. □

Lemma 2.4. *There exists some $u_\varepsilon \in \mathcal{N}_\varepsilon$ such that c_ε is achieved. Moreover, u_ε is a weak solution and hence a classical nodal solution of* (1.2).

Proof. Let $\varepsilon > 0$ be fixed. We use the argument by Bartsch, Weth and Willem [2]. Since $c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} E_\varepsilon(u)$, there exists a minimizing sequence $u_{\varepsilon,n} \in \mathcal{N}_\varepsilon$ such that $E_\varepsilon(u_{\varepsilon,n}) \to c_\varepsilon$ as $n \to +\infty$. Note that by $(f3)$, E_ε is coercive on $\mathcal{N}_\varepsilon$, as

$$E_\varepsilon(u_{\varepsilon,n}) \geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \int_\Omega \left\{\varepsilon^2|\nabla u_{\varepsilon,n}|^2 + \lambda_1(u_{\varepsilon,n}^+)^2 + \lambda_2(u_{\varepsilon,n}^-)^2\right\}. \tag{2.5}$$

and hence there exist $b(\varepsilon) > 0, d(\varepsilon) > 0$ independent of n such that $b(\varepsilon) \leq \|u_{\varepsilon,n}^\pm\|_{H_0^1(\Omega)} \leq d(\varepsilon)$. Therefore there exist $u_\varepsilon^\pm \in H_0^1(\Omega)$ such that $u_{\varepsilon,n}^\pm \rightharpoonup u_\varepsilon^\pm$ as $n \to +\infty$ and by the Rellich Lemma $u_{\varepsilon,n}^\pm \to u_\varepsilon^\pm$ in $L^q(\Omega)$ for $q \in (1, \frac{2N}{N-2})$. This implies that $u_\varepsilon^\pm \geq 0$ and $u_\varepsilon^+.u_\varepsilon^- = 0$ since $u_{\varepsilon,n}^+.u_{\varepsilon,n}^- = 0$. Thus $u_\varepsilon^\pm$ are indeed the positive and negative part of $u_\varepsilon = u_\varepsilon^+ - u_\varepsilon^-$. From the fact that (2.2) and (2.3) we have $\|u_{\varepsilon,n}^\pm\|_{L^q(\Omega)}$ has a positive lower bound and this implies $u_\varepsilon^\pm \not\equiv 0$. But also we have

$$\lim_{n\to\infty} \int_\Omega f(u_{\varepsilon,n}^\pm)u_{\varepsilon,n}^\pm = \int_\Omega f(u_\varepsilon^\pm)u_\varepsilon^\pm \tag{2.6}$$

and

$$\lim_{n\to\infty} \int_\Omega F(u_{\varepsilon,n}^\pm) = \int_\Omega F(u_\varepsilon^\pm). \tag{2.7}$$

From (2.6) using Fatou's lemma we have

$$\|u_\varepsilon^\pm\|_{H_0^1(\Omega)}^2 \leq \int_\Omega f(u_\varepsilon^\pm)u_\varepsilon^\pm.$$

By a variant Remark 2.2 there exist $s, t \in (0, 1]$ such that

$$\|tu_\varepsilon^+\|^2_{H_0^1(\Omega)} = \int_\Omega f(tu_\varepsilon^+)tu_\varepsilon^+$$

and

$$\|su_\varepsilon^-\|^2_{H_0^1(\Omega)} = \int_\Omega f(su_\varepsilon^-)su_\varepsilon^-.$$

This implies $tu_\varepsilon^+ - su_\varepsilon^- \in \mathcal{N}_\varepsilon$ and hence

$$\begin{aligned} E_\varepsilon(tu_\varepsilon^+ - su_\varepsilon^-) &= E_{\varepsilon,\lambda_1}(tu_\varepsilon^+) + E_{\varepsilon,\lambda_2}(su_\varepsilon^-) \\ &\le \lim_{n\to\infty} E_{\varepsilon,\lambda_1}(u_{\varepsilon,n}^+) + \lim_{n\to\infty} E_{\varepsilon,\lambda_2}(u_{\varepsilon,n}^-) = c_\varepsilon. \end{aligned} \tag{2.8}$$

Note that we have used the fact (f4), (2.6), (2.7) to obtain

$$E_{\varepsilon,\lambda_1}(tu_\varepsilon^+) \le \lim_{n\to\infty} E_{\varepsilon,\lambda_1}(u_\varepsilon^+) \text{ and } E_{\varepsilon,\lambda_2}(su_\varepsilon^-) \le \lim_{n\to\infty} E_{\varepsilon,\lambda_2}(u_\varepsilon^-).$$

Hence we have $c_\varepsilon \le E_\varepsilon(tu_\varepsilon^+ - su_\varepsilon^-) \le c_\varepsilon$ and indeed $tu_\varepsilon^+ - su_\varepsilon^-$ is a minimizer in $\mathcal{N}_\varepsilon$.

By Remark 2.1 we want to show that $v_\varepsilon := tu_\varepsilon^+ - su_\varepsilon^-$ is a critical point of E_ε. If possible, let $E'_\varepsilon(v_\varepsilon) \neq 0$ and then there exist $\delta > 0$ and $\lambda > 0$ such that

$$\|E'_\varepsilon(w)\| \ge \lambda \text{ whenever } \|v_\varepsilon - w\| \le \delta. \tag{2.9}$$

Define a square $S = (\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{2}, \frac{3}{2})$ and for any $(m, n) \in S$

$$\psi(m, n) = mv_\varepsilon^+ - nv_\varepsilon^-.$$

Then from (2.8) we have

$$\tilde{c}_\varepsilon = \max_{\partial S} E_\varepsilon(\psi) < c_\varepsilon\,. \tag{2.10}$$

Indeed our earlier comments, $E_\varepsilon(\psi) < c_\varepsilon$ on S except at $(1, 1)$. Choose $\tau = \min\{\frac{c_\varepsilon - \tilde{c}_\varepsilon}{2}, \frac{\lambda\delta}{8}\}$ and $B(v_\varepsilon, \delta)$ be ball centered at v_ε. Then by Willem [21, Lemma 2.3, page 38], there exist a deformation $\eta \in \mathcal{C}([0, 1] \times H_0^1(\Omega); H_0^1(\Omega))$ such that
(a) $\eta(t, w) = w$ if $t = 0$ or if $w \in E_\varepsilon^{-1}(c_\varepsilon - 2\tau, c_\varepsilon + 2\tau)$,
(b) $\eta(1, E_\varepsilon^{c_\varepsilon+\tau} \cap B(v_\varepsilon, \delta)) \subset E_\varepsilon^{c_\varepsilon-\tau}$,
(c) $E_\varepsilon(\eta(1, w)) \le E_\varepsilon(w), \forall w \in H_0^1(\Omega)$. Moreover, by our remarks and results in [21], we have

$$\max_{(m,n)\in\bar{S}} E_\varepsilon(\eta(1, \psi(m, n)) < c_\varepsilon. \tag{2.11}$$

The idea of the proof is to obtain a contradiction. To this end we claim that $\eta(1, \psi(S)) \cap \mathcal{N}_\varepsilon \neq \emptyset$. Define $h(m, n) = \eta(1, \psi(m, n))$ and

$$\Pi_1(m, n) = \left(E'_\varepsilon(m v_\varepsilon^+) v_\varepsilon^+, \, E'_\varepsilon(n v_\varepsilon^-) v_\varepsilon^- \right)$$

$$\Pi_2(m, n) = \left(\frac{1}{m} E'_\varepsilon(h^+(m, n)) h^+(m, n), \, \frac{1}{n} E'_\varepsilon(h^-(m, n)) h^-(m, n) \right).$$

Note that the first component of $\Pi_1(m, n)$ is positive if $m < 1$ and is negative if $m > 1$ with an analogous property for the second component. Hence by the product rule for degree theory we have $\deg(\Pi_1, S, 0) = 1$. Moreover, as $\psi = h$ on ∂S (by our choice of τ and the property (a) of the deformation) we must have $\deg(\Pi_1, S, 0) = \deg(\Pi_2, S, 0)$. Hence there exists a tuple $(m_0, n_0) \in S$ such that $\Pi_2(m_0, n_0) = 0$ which implies $h(m_0, n_0) = \eta(1, \psi(m_0, n_0)) \in \mathcal{N}_\varepsilon$. □

Lemma 2.5. *Let $\omega_{\varepsilon,\lambda_1}$ and $\omega_{\varepsilon,\lambda_2}$ be the least energy solutions of*

$$\begin{cases} -\varepsilon^2 \Delta u + \lambda_1 u = f(u) & \text{in } B_r(0) \\ u > 0 & \text{in } B_r(0) \\ u = 0 & \text{on } \partial B_r(0) \end{cases} \tag{2.12}$$

$$\begin{cases} -\varepsilon^2 \Delta u + \lambda_2 u = f(u) & \text{in } B_r(0) \\ u > 0 & \text{in } B_r(0) \\ u = 0 & \text{on } \partial B_r(0) \end{cases} \tag{2.13}$$

respectively. Then for sufficiently small $\varepsilon > 0$, we have

$$E_{\varepsilon,\lambda_1}(\omega_{\varepsilon,\lambda_1}) = \varepsilon^N \left\{ I_{\lambda_1}(W_{\lambda_1}) + e^{-\frac{2\sqrt{\lambda_1} r(1+o(1))}{\varepsilon}} \right\}$$

$$E_{\varepsilon,\lambda_2}(\omega_{\varepsilon,\lambda_2}) = \varepsilon^N \left\{ I_{\lambda_2}(W_{\lambda_2}) + e^{-\frac{2\sqrt{\lambda_2} r(1+o(1))}{\varepsilon}} \right\}$$

where $o(1) \to 0$ as $\varepsilon \to 0$.

Proof. For the proof see [11]. □

Let $\Lambda = \{x \in \Omega : \sqrt{\lambda_1}|x - P_1| = \sqrt{\lambda_2}|x - P_2|\}$.

Lemma 2.6. *We have for $\varepsilon > 0$ sufficiently small*

$$c_\varepsilon \leq \varepsilon^N \left\{ I_{\lambda_1}(W_{\lambda_1}) + I_{\lambda_2}(W_{\lambda_2}) + e^{-\frac{2\varphi(P_1, P_2)}{\varepsilon}} + o(e^{-\frac{2\varphi(P_1, P_2)}{\varepsilon}}) \right\}. \tag{2.14}$$

Proof. Let v_ε be a positive solution of

$$\begin{cases} -\varepsilon^2\Delta u + \lambda_1 u = f(u) & \text{in } B_{r_1}(P_1) \\ u > 0 & \text{in } B_{r_1}(P_1) \\ u = 0 & \text{on } B_{r_1}(P_1) \end{cases} \tag{2.15}$$

where $r_1 = \min\{d(P_1, \partial\Omega), d(P_1, \Lambda)\}$. Let w_ε be a positive solution of

$$\begin{cases} -\varepsilon^2\Delta u + \lambda_2 u = f(u) & \text{in } B_{r_2}(P_2) \\ u > 0 & \text{in } B_{r_2}(P_2) \\ u = 0 & \text{on } B_{r_2}(P_2) \end{cases} \tag{2.16}$$

where $r_2 = \min\{d(P_2, \partial\Omega), d(P_2, \Lambda)\}$. Note that supp $v_\varepsilon \cap$ supp $w_\varepsilon = \emptyset$ and $v_\varepsilon \in \mathcal{N}_\varepsilon^+$ and $w_\varepsilon \in \mathcal{N}_\varepsilon^-$. Then we have $v_\varepsilon - w_\varepsilon \in \mathcal{N}_\varepsilon$ and hence we have from (2.15) and (2.16),

$$\begin{aligned} c_\varepsilon &\le E_\varepsilon(v_\varepsilon - w_\varepsilon) \\ &\le E_{\varepsilon,\lambda_1}(v_\varepsilon) + E_{\varepsilon,\lambda_2}(w_\varepsilon) \\ &\le \varepsilon^N \Big\{ I_{\lambda_1}(W_{\lambda_1}) + e^{-\frac{2r_1}{\varepsilon}} + I_{\lambda_2}(W_{\lambda_2}) + e^{-\frac{2r_2}{\varepsilon}} + o(e^{-\frac{2r_1}{\varepsilon}}) + o(e^{-\frac{2r_2}{\varepsilon}}) \Big\}. \end{aligned}$$

Hence we have,

$$\begin{aligned} c_\varepsilon &\le \varepsilon^N \Big\{ I_{\lambda_1}(W_{\lambda_1}) + e^{-\frac{2\min\{r_1,r_2\}}{\varepsilon}} + I_{\lambda_2}(W_{\lambda_2}) + o(e^{-\frac{2\min\{r_1,r_2\}}{\varepsilon}}) \Big\} \\ &\le \varepsilon^N \Big\{ I_{\lambda_1}(W_{\lambda_1}) + I_{\lambda_2}(W_{\lambda_2}) + e^{-\frac{2\varphi(P_1,P_2)}{\varepsilon}} + o(e^{-\frac{2\varphi(P_1,P_2)}{\varepsilon}}) \Big\}. \end{aligned} \tag{2.17}$$

□

Corollary 2.7. *We also have* $c_\varepsilon \ge \varepsilon^N \Big\{ I_{\lambda_1}(W_{\lambda_1}) + I_{\lambda_2}(W_{\lambda_2}) + o(1) \Big\}$.

Proof.

$$c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} \{E_{\varepsilon,\lambda_1}(u) + E_{\varepsilon,\lambda_2}(u)\} \ge \inf_{u\in\mathcal{N}_\varepsilon^+} E_{\varepsilon,\lambda_1}(u) + \inf_{u\in\mathcal{N}_\varepsilon^-} E_{\varepsilon,\lambda_2}(u)$$

this implies the result. □

Lemma 2.8. *As* $\varepsilon \to 0$,

$$\frac{d(P_\varepsilon^1, \partial\Omega)}{\varepsilon} \to +\infty, \ \frac{d(P_\varepsilon^2, \partial\Omega)}{\varepsilon} \to +\infty, \ \frac{|P_\varepsilon^1 - P_\varepsilon^2|}{\varepsilon} \to +\infty.$$

Proof. As $\varepsilon^2 \Delta u_\varepsilon(P^1_\varepsilon) \leq 0$ it implies that $f(u_\varepsilon(P^1_\varepsilon)) \geq \lambda_1 u_\varepsilon(P^1_\varepsilon)$ which implies that $Cu_\varepsilon^{p-1}(P^1_\varepsilon) \geq \lambda_1$, hence there exists a positive constant β such that $u_\varepsilon(P^1_\varepsilon) \geq \beta$ and similarly we obtain that $u_\varepsilon(P^2_\varepsilon) \leq -\beta$. Also by Lemma 2.6,

$$\varepsilon^2 \int_\Omega |\nabla u_\varepsilon|^2 + \lambda_1 \int_\Omega (u_\varepsilon^+)^2 + \lambda_2 \int_\Omega (u_\varepsilon^-)^2 \leq C\varepsilon^N$$

and hence by Moser iteration we obtain $\|u_\varepsilon\|_{L^\infty(\Omega)} \leq C$.

Suppose that $\lim_{\varepsilon\to 0} \frac{d(P^1_\varepsilon, \partial\Omega)}{\varepsilon} \leq C$. By scaling $v_\varepsilon(x) = u_\varepsilon(\varepsilon x + P^1_\varepsilon)$, then (1.2) reduces to,

$$\begin{cases} \Delta v_\varepsilon - \lambda_1 v_\varepsilon + \lambda_2 v_\varepsilon^- + f(v_\varepsilon) = 0 & \text{in} \Omega_\varepsilon \\ v_\varepsilon^\pm \neq 0 & \text{in } \Omega_\varepsilon \\ v_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon \end{cases} \tag{2.18}$$

where $\Omega_\varepsilon = \frac{x - P^1_\varepsilon}{\varepsilon}$. Note that from (2.6), $\|v_\varepsilon\|_{H^1_0(\Omega_\varepsilon)} \leq C$; there exists $W \in H^1(\mathbb{R}^N)$ we have $v_\varepsilon \rightharpoonup W$ in $H^1(\mathbb{R}^N)$ and by the Sobolev embedding theorem we have $v_\varepsilon \to W$ in $L^p_{\text{loc}}(\mathbb{R}^N)$. Hence $v_\varepsilon \to W$ point-wise almost everywhere in $\mathbb{R}^N$. Also by Schauder estimates, it follows that there exists $C > 0$ such that $\|v_\varepsilon\|_{C^{2,\beta}_{\text{loc}}(\mathbb{R}^N)} \leq C$ for some $0 < \beta \leq 1$. Hence by the Ascoli-Arzela's theorem there exists $W \neq 0$ such that

$$\|v_\varepsilon - W\|_{C^2_{\text{loc}}(\mathbb{R}^N)} \to 0 \text{ as } \varepsilon \to 0$$

where W is a nontrivial solution satisfying

$$\begin{cases} \Delta W - \lambda_1 W + f(W) = 0 & \text{in } \mathbb{R}^N_+ \\ \sup W \geq \beta,\, W \in H^1 & \\ W = 0 & \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{2.19}$$

where $\mathbb{R}^N_+ = \{y : y_n > -a\}$. Then by a result in [12] we obtain $W \equiv 0$, a contradiction. Similarly $\lim_{\varepsilon\to 0} \frac{d(P^2_\varepsilon, \partial\Omega)}{\varepsilon} = +\infty$. Now we prove that $\lim_{\varepsilon\to 0} \frac{|P^1_\varepsilon - P^2_\varepsilon|}{\varepsilon} = +\infty$. By applying the Schauder estimates we obtain a $C > 0$ such that $\|\varepsilon D u_\varepsilon\|_{L^\infty} \leq C$. If possible let $\lim_{\varepsilon\to 0} \frac{|P^1_\varepsilon - P^2_\varepsilon|}{\varepsilon} = \delta < +\infty$. Then it easily follows that $u_\varepsilon(P^1_\varepsilon) \geq \beta$ and $u_\varepsilon(P^2_\varepsilon) \leq -\beta$ which implies that $u_\varepsilon(P^1_\varepsilon) - u_\varepsilon(P^2_\varepsilon) \geq 2\beta$. Then

$$2\beta \leq |u_\varepsilon(P^1_\varepsilon) - u_\varepsilon(P^2_\varepsilon)| \leq \varepsilon \|D u_\varepsilon\|_\infty \frac{|P^1_\varepsilon - P^2_\varepsilon|}{\varepsilon}.$$

Suppose $P_\varepsilon = \frac{P_\varepsilon^1 - P_\varepsilon^2}{\varepsilon}$. Then along a subsequence $|P_\varepsilon| \to \delta \in (0, +\infty)$. Define $v_\varepsilon = u_\varepsilon(\varepsilon y + P_\varepsilon^1)$. Then $v_\varepsilon \to W$ in $\mathcal{C}^2_{\mathrm{loc}}(\mathbb{R}^N)$ and W satisfies

$$\begin{cases} -\Delta W + \lambda_1 W^+ - \lambda_2 W^- = f(W) & \text{in } \mathbb{R}^N \\ W(0) \geq \beta, \quad W(P) \leq -\beta \\ W \in H^1(\mathbb{R}^N) \end{cases} \tag{2.20}$$

where $P = \lim_{\varepsilon\to 0} \frac{P_\varepsilon^1 - P_\varepsilon^2}{\varepsilon}$ which implies that W is a nodal solution of (2.20) and hence a critical point of the functional

$$I_\infty(u) = \int_{\mathbb{R}^N} \left(\frac{1}{2}|\nabla u|^2 + \frac{\lambda_1}{2}(u^+)^2 + \frac{\lambda_2}{2}(u^-)^2 - F(u)\right)dx$$

and in particular we have $\langle I'_\infty(W), W^\pm\rangle = 0$ and $W \in \mathcal{N}_\infty$ where

$$\mathcal{N}_\infty = \Big\{u \in H^1(\mathbb{R}^N) : u^\pm \not\equiv 0, \int_{\mathbb{R}^N} |\nabla u^+|^2 + \lambda_1 \int_{\mathbb{R}^N} (u^+)^2 = \int_{\mathbb{R}^N} f(u^+)u^+; \\ \int_{\mathbb{R}^N} |\nabla u^-|^2 + \lambda_2 \int_{\mathbb{R}^N} (u^-)^2 = \int_{\mathbb{R}^N} f(u^-)u^-\Big\}.$$

But by (2.1) we know that $\varepsilon^N(I_{\lambda_1}(W_{\lambda_1}) + I_{\lambda_2}(W_{\lambda_2}) + o(1)) \geq \varepsilon^N(I_\infty(W^+) + I_\infty(W^-) + o(1))$. This implies

$$I_\infty(W^+) + I_\infty(W^-) \leq I_{\lambda_1}(W_{\lambda_1}) + I_{\lambda_2}(W_{\lambda_2}) = c_{\lambda_1} + c_{\lambda_2}$$

where c_{λ_i} is a mountain pass critical value with respect to the functional I_{λ_i}, i.e.

$$c_{\lambda_i} = \inf_{u\in H^1(\mathbb{R}^N), u\not\equiv 0, \int_{\mathbb{R}^N} |\nabla u|^2 + \lambda_i \int_{\mathbb{R}^N} u^2 = \int_{\mathbb{R}^N} f(u)u} I_{\lambda_i}(u). \tag{2.21}$$

Also it easily follows that $I_\infty(W^+) = I_{\lambda_1}(W^+) \geq c_{\lambda_1}$, $I_\infty(W^-) = I_{\lambda_2}(W^-) \geq c_{\lambda_2}$. Since any minimizer c_{λ_i} is a weak solution, we have $c_{\lambda_1} = I_{\lambda_1}(W^+), c_{\lambda_2} = I_{\lambda_2}(W^-)$. Thus $W^+ = W_{\lambda_1}(x - R)$ and $W^- = W_{\lambda_2}(x - S)$ for some R, S in $\mathbb{R}^N$. The first equality implies $W^+ > 0$ on $\mathbb{R}^N$ which contradicts that W changes sign. □

Lemma 2.9. *For sufficiently small $\varepsilon > 0$, u_ε has exactly one positive local maximum and one negative local minimum.*

Proof. Note that from Lemma 2.6, we obtain that $c_\varepsilon \leq \varepsilon^N(I_{\lambda_1}(W_{\lambda_1}) + I_{\lambda_2}(W_{\lambda_2}) + o(1))$. Suppose it has two positive local maxima as P_ε and Q_ε and a negative local minimum R_ε. Then it follows similarly as in the proof of Lemma 2.8 one can show that $\frac{|P_\varepsilon - Q_\varepsilon|}{\varepsilon} \to +\infty$, $\frac{|Q_\varepsilon - R_\varepsilon|}{\varepsilon} \to +\infty$ and $\frac{|P_\varepsilon - R_\varepsilon|}{\varepsilon} \to +\infty$ as $\varepsilon \to 0$. Also note that $\frac{1}{2} f(u_\varepsilon)u_\varepsilon - F(u_\varepsilon) \geq 0$ by assumption (f4), and thus

$$\begin{aligned} c_\varepsilon = E_\varepsilon(u_\varepsilon) &= \int_\Omega \left(\frac{1}{2} f(u_\varepsilon)u_\varepsilon - F(u_\varepsilon) \right) dx \\ &\geq \int_{B_{\varepsilon R}(P_\varepsilon)} \left(\frac{1}{2} f(u_\varepsilon)u_\varepsilon - F(u_\varepsilon) \right) + \int_{B_{\varepsilon R}(Q_\varepsilon)} \left(\frac{1}{2} f(u_\varepsilon)u_\varepsilon - F(u_\varepsilon) \right) \\ &\quad + \int_{B_{\varepsilon R}(R_\varepsilon)} \left(\frac{1}{2} f(u_\varepsilon)u_\varepsilon - F(u_\varepsilon) \right) \\ &\geq \varepsilon^N \left(2I_{\lambda_1}(W_{\lambda_1}) + I_{\lambda_2}(W_{\lambda_2}) + o(1) \right) \end{aligned} \tag{2.22}$$

a contradiction to Lemma 2.6. Hence u_ε has exactly one positive maximum and one negative minimum. □

Now let us define

$$d_\varepsilon = \min\left\{ \sqrt{\lambda}_1 d(P_\varepsilon^1, \partial\Omega), \sqrt{\lambda}_2 d(P_\varepsilon^2, \partial\Omega), \frac{\sqrt{\lambda_1}\sqrt{\lambda_2}}{\sqrt{\lambda_1} + \sqrt{\lambda_2}} |P_\varepsilon^1 - P_\varepsilon^2| \right\}.$$

Then by the above lemma $\frac{d_\varepsilon}{\varepsilon} \to +\infty$ as $\varepsilon \to 0$. Now let us re-scale the problem by $\overline{\varepsilon} = \frac{\varepsilon}{d_\varepsilon}$ and $\overline{x} = d_\varepsilon \overline{x}$. Then we have

$$\Delta u - \lambda_1 u^+ + \lambda_2 u^- + f(u) = 0 \text{ in } \overline{\Omega}_{d_\varepsilon} = \frac{\Omega}{d_\varepsilon}. \tag{2.23}$$

Lemma 2.10. *For any* $0 < \delta' < 1$, *there exists a constant* $C > 0$ *independent of* δ' *such that*

$$u_\varepsilon^+ \leq Ce^{-\frac{\sqrt{\lambda_1}(1-\delta')|x-P_\varepsilon^1|}{\varepsilon}} \text{ and } u_\varepsilon^- \leq Ce^{-\frac{\sqrt{\lambda_2}(1-\delta')|x-P_\varepsilon^2|}{\varepsilon}} \quad \forall x \in \Omega.$$

Proof. Let $v_\varepsilon^i(y) = u_\varepsilon(\varepsilon y + P_\varepsilon^i)$. Then $v_\varepsilon^1 \to W_{\lambda_1}$ in $C^2_{\text{loc}}(\mathbb{R}^N)$. Also we have $W_{\lambda_1}(r) \leq Ce^{-\sqrt{\lambda_1}r}$ for all r. Let $R = \ln \frac{C}{\zeta}$ such that $\zeta = Ce^{-R}$. Then there exist an $\varepsilon_0 > 0$ such that $v_\varepsilon^+(y) \leq W_{\lambda_1}(y) + \zeta \leq 2\zeta$. Let us consider the domain $\Omega^1 = \Omega \setminus B_{\varepsilon R}(P_\varepsilon^1)$ where $R > 0$ is large. Hence we can choose a $\zeta > 0$, independent of ε such that $v_\varepsilon^+ \leq C$ on $\partial B_R(0)$. This implies that $u_\varepsilon^+ \leq 2\zeta$ on $\partial B_{\varepsilon R}(P_\varepsilon^1)$. For any $0 < \delta' < 1$, choose ζ in such a way that

$$\frac{f(u_\varepsilon)}{\lambda_1 u_\varepsilon^+} < \delta',$$

consider the equation with $u_\varepsilon > 0$

$$-\varepsilon^2 \Delta u_\varepsilon + \lambda_1 u_\varepsilon = \frac{f(u_\varepsilon)}{u_\varepsilon} u_\varepsilon \ \text{ in } \Omega^1.$$

Then we obtain,

$$\begin{cases} -\varepsilon^2\Delta u_\varepsilon + (1-\delta')\lambda_1 u_\varepsilon \le 0 & \text{in } \Omega^1 \\ u_\varepsilon > 0 & \text{in } \Omega^1 \\ u_\varepsilon \le 2\zeta & \text{in } \partial B_{\varepsilon R}(P_\varepsilon^1) \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.24}$$

Using a comparison argument we obtain $u_\varepsilon^+ \le Ce^{-\frac{\sqrt{\lambda_1}(1-\delta')|x-P_\varepsilon^1|}{\varepsilon}}$. We obtain the other estimate similarly. □

3. Lower bound of the energy expansion

In order to obtain the greatest lower bound of the energy E_ε we consider three cases.

Case 1. Suppose that

$$\frac{d_\varepsilon}{\sqrt{\lambda_1}d(P_\varepsilon^1, \partial\Omega)} \to 1 \ \text{ as } \varepsilon \to 0.$$

Note that

$$c_\varepsilon \ge \inf_{u\in\mathcal{N}_\varepsilon^+} E_{\varepsilon,\lambda_1}(u) + \inf_{u\in\mathcal{N}_\varepsilon^-} E_{\varepsilon,\lambda_2}(u).$$

We use del Pino-Felmer's symmetrization technique in [11] to conclude that

$$E_{\varepsilon,\lambda_1}(u_\varepsilon^+) \ge \varepsilon^N\left\{I_{\lambda_1}(W_{\lambda_1}) + \frac{1}{2}e^{-2\frac{\sqrt{\lambda_1}(d(P_\varepsilon^1,\partial\Omega)+o(1))}{\varepsilon}}\right\}.$$

We also deduce that

$$E_{\varepsilon,\lambda_2}(u_\varepsilon^-) \ge \varepsilon^N\left\{I_{\lambda_2}(W_{\lambda_2}) + \frac{1}{2}e^{-2\frac{(d_\varepsilon+o(1))}{\varepsilon}}\right\}$$

and as $d_\varepsilon = \sqrt{\lambda_1}d(P_\varepsilon^1, \partial\Omega) + o(1)$, we have

$$c_\varepsilon \ge \varepsilon^N\left(I_{\lambda_1}(W_{\lambda_1}) + I_{\lambda_2}(W_{\lambda_2}) + e^{-\frac{2(d_\varepsilon+o(1))}{\varepsilon}}\right). \tag{3.1}$$

Case 2. Suppose that

$$\frac{d_\varepsilon}{\sqrt{\lambda_2}d(P_\varepsilon^2, \partial\Omega)} \to 1 \ \text{ as } \varepsilon \to 0.$$

Then we argue as in Case 1.

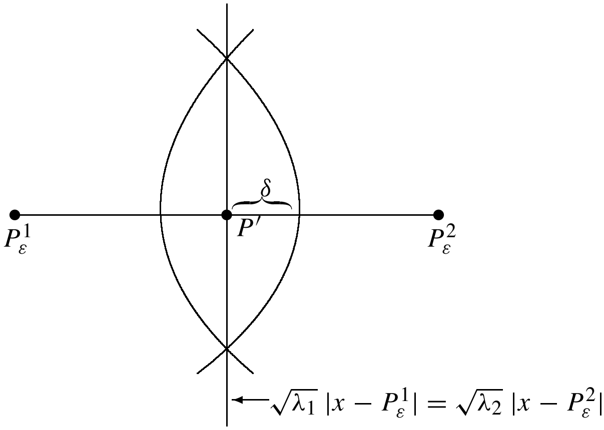

Figure 3.1. The region of intersection.

Case 3.
Suppose that

$$d_\varepsilon = \frac{\sqrt{\lambda_1}\sqrt{\lambda_2}}{\sqrt{\lambda_1}+\sqrt{\lambda_2}}|P_\varepsilon^1 - P_\varepsilon^2|.$$

Then we can choose $\delta > 0$ such that $d_\varepsilon \geq (1+5\delta)\sqrt{\lambda_1}d(P_\varepsilon^1, \partial\Omega)$, $d_\varepsilon \geq (1+5\delta)\sqrt{\lambda_2}d(P_\varepsilon^2, \partial\Omega)$. Furthermore, we define $|P' - P_\varepsilon^1| = \frac{\sqrt{\lambda_2}}{\sqrt{\lambda_1}+\sqrt{\lambda_2}}|P_\varepsilon^1 - P_\varepsilon^2| = d_{\varepsilon,1}$. Then we have

$$|P' - P_\varepsilon^2| = \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1}+\sqrt{\lambda_2}}|P_\varepsilon^1 - P_\varepsilon^2| = d_{\varepsilon,2}.$$

We consider balls $B_{d_{\varepsilon,1}+\delta}(P_\varepsilon^1)$ and $B_{d_{\varepsilon,2}+\delta_2}(P_\varepsilon^2)$, where $0 < \delta \ll d_{\varepsilon,1}$ is small and $\delta_2 \sim \frac{\sqrt{\lambda_2}}{\sqrt{\lambda_1}}\delta$ is defined by

$$(d_{\varepsilon,1}+\delta)^2 - d_{\varepsilon,1}^2 = (d_{\varepsilon,2}+\delta_2)^2 - d_{\varepsilon,2}^2. \tag{3.2}$$

Define the intersection $\Gamma_\varepsilon = B_{d_{\varepsilon,1}+\delta}(P_\varepsilon^1) \cap B_{d_{\varepsilon,2}+\delta}(P_\varepsilon^2)$. Then the total volume of $\Gamma_\varepsilon \approx \delta O(\delta^{\frac{N-1}{2}})$. Since $\Gamma_\varepsilon = (\Gamma_\varepsilon \cap \{u_\varepsilon \geq 0\}) \cup (\Gamma_\varepsilon \cap \{u_\varepsilon \leq 0\})$, we either have $|\Gamma_\varepsilon \cap \{u_\varepsilon \geq 0\}| \leq \frac{1}{2}|\Gamma_\varepsilon|$ or $|\Gamma_\varepsilon \cap \{u_\varepsilon \leq 0\}| \leq \frac{1}{2}|\Gamma_\varepsilon|$.

Without loss of generality, let

$$|\Gamma_\varepsilon \cap \{u_\varepsilon \geq 0\}| \leq \frac{1}{2}|\Gamma_\varepsilon|\,.$$

Thus

$$|B_{d_{\varepsilon,1}+\delta}(P_\varepsilon^1) \cap \{u_\varepsilon > 0\}| \leq |B_{d_{\varepsilon,1}+\delta}(P_\varepsilon^1)| - \frac{1}{2}|\Gamma_\varepsilon| = |B_{r_\varepsilon}(0)|$$

where $r_\varepsilon = (d_{1,\varepsilon} + \delta)(1 - \eta)$ for some $0 < \eta < 1$, where $\eta \sim \delta^{\frac{N+1}{2}}$. We define a smooth function

$$\chi(x) = \begin{cases} 1 & \text{if } |x - P_\varepsilon^1| \le (d_{\varepsilon,1} + \delta)(1 - \eta) \\ 0 & \text{if } |x - P_\varepsilon^1| \ge (d_{\varepsilon,1} + \delta) \end{cases} \tag{3.3}$$

and $0 \le \chi \le 1$ and $|\nabla \chi| \le \frac{C}{(d_{\varepsilon,1}+\delta)\eta}$. Then the support of $u_\varepsilon^+ \chi^2$ is contained in $B_{d_{\varepsilon,1}+\delta}(P_\varepsilon^1)$. Multiplying (1.2) by $u_\varepsilon^+ \chi^2$ we obtain

$$\int_\Omega \varepsilon^2 \nabla u_\varepsilon \nabla(u_\varepsilon^+ \chi^2) + \lambda_1 (u_\varepsilon^+)^2 \chi^2 = \int_\Omega f(u_\varepsilon) u_\varepsilon^+ \chi^2 . \tag{3.4}$$

Now let us compute

$$\begin{aligned}
\int_\Omega \varepsilon^2 \nabla u_\varepsilon \nabla(u_\varepsilon^+ \chi^2) &= \int_\Omega \varepsilon^2 \nabla u_\varepsilon^+ \nabla(u_\varepsilon^+ \chi^2) \\
&= \int_\Omega \varepsilon^2 \nabla u_\varepsilon^+ \Big\{ \chi \nabla(u_\varepsilon^+ \chi) + u_\varepsilon^+ \chi \nabla \chi \Big\} \\
&= \int_\Omega \varepsilon^2 \Big\{ (\nabla(u_\varepsilon^+ \chi) - u_\varepsilon^+ \nabla \chi) \nabla(u_\varepsilon^+ \chi) + u_\varepsilon^+ \chi \nabla \chi \nabla u_\varepsilon^+ \Big\} \\
&= \int_\Omega \varepsilon^2 \Big\{ |\nabla(u_\varepsilon^+ \chi)|^2 - u_\varepsilon^+ \nabla \chi \nabla(u_\varepsilon^+ \chi) + u_\varepsilon^+ \chi \nabla \chi \nabla u_\varepsilon^+ \Big\} \qquad (3.5) \\
&= \int_\Omega \varepsilon^2 \Big\{ |\nabla(u_\varepsilon^+ \chi)|^2 - u_\varepsilon^+ \chi \nabla \chi \nabla u_\varepsilon^+ - (u_\varepsilon^+)^2 |\nabla \chi|^2 \\
&\qquad\qquad + u_\varepsilon^+ \chi \nabla \chi \nabla u_\varepsilon^+ \Big\} \\
&= \varepsilon^2 \int_\Omega |\nabla(u_\varepsilon^+ \chi)|^2 - \varepsilon^2 \int_\Omega (u_\varepsilon^+)^2 |\nabla \chi|^2
\end{aligned}$$

where

$$\varepsilon^2 \int_\Omega (u_\varepsilon^+)^2 |\nabla \chi|^2 \le C \varepsilon^N e^{-\sqrt{\lambda_1} \frac{2(1-\frac{\eta}{2})(d_{\varepsilon,1}+\delta)}{\varepsilon}} . \tag{3.6}$$

On the other hand

$$\begin{aligned}
\int_\Omega f(u_\varepsilon) u_\varepsilon^+ \chi^2 &= \int_\Omega f(u_\varepsilon^+ \chi) u_\varepsilon^+ \chi + \int_\Omega \{ f(u_\varepsilon^+ \chi) - f(u_\varepsilon) \chi \} u_\varepsilon^+ \chi \\
&= \int_\Omega f(u_\varepsilon^+ \chi) u_\varepsilon^+ \chi + O\left(\varepsilon^N e^{-\frac{(p+1)\sqrt{\lambda_1}(d_{\varepsilon,1}+\delta)(1-\frac{\eta}{2})}{\varepsilon}} \right). \qquad (3.7)
\end{aligned}$$

Note that in order to derive (3.6), we use the assumption (f_2), Lemma 2.10, (3.3)

$$u_\varepsilon^+ \le C e^{-\frac{\sqrt{\lambda_1}(1-\delta')|x-P_\varepsilon^1|}{\varepsilon}}, \qquad \delta' = \frac{\eta}{2(1-\eta)},$$

and $|\nabla\chi| \neq 0$ if $|x - P_{\varepsilon_1}| \geq (d_{\varepsilon,1} + \delta)(1 - \eta)$. Moreover, note that $\{f(u_\varepsilon^+\chi) - f(u_\varepsilon)\chi\}u_\varepsilon^+\chi = 0$ if $\chi = 1$. When $(d_{\varepsilon,1} + \delta)(1 - \eta) \leq |x - P_\varepsilon^1| \leq (d_{\varepsilon,1} + \delta)$ using (f2) we obtain

$$\left\{f(u_\varepsilon^+\chi) - f(u_\varepsilon)\chi\right\} u_\varepsilon^+\chi \leq Ce^{-(p+1)\frac{\sqrt{\lambda_1}(1-\delta')|x-P_\varepsilon^1|}{\varepsilon}}$$

and hence

$$\int_\Omega \left\{f(u_\varepsilon^+\chi) - f(u_\varepsilon)\chi\right\} u_\varepsilon^+\chi \leq C\varepsilon^N e^{-\frac{\sqrt{\lambda_1}(p+1)(d_{\varepsilon,1}+\delta)(1-\delta')}{\varepsilon}}$$
$$\leq C\varepsilon^N e^{-\frac{\sqrt{\lambda_1}(p+1)(d_{\varepsilon,1}+\delta)(1-\frac{\eta}{2})}{\varepsilon}}.$$

Hence combining (3.4), (3.5) and (3.7) we have

$$\begin{aligned}\varepsilon^2 \int_\Omega |\nabla(u_\varepsilon^+\chi)|^2 + \lambda_1 \int_\Omega (u_\varepsilon^+\chi)^2 \\ = \int_\Omega f(u_\varepsilon^+\chi)u_\varepsilon^+\chi + O\left(\varepsilon^N e^{-\frac{2\sqrt{\lambda_1}(d_{\varepsilon,1}+\delta)(1-\frac{\eta}{2})}{\varepsilon}}\right).\end{aligned} \tag{3.8}$$

Let $v_\varepsilon = t_\varepsilon u_\varepsilon^+\chi$ where t_ε is such that

$$\varepsilon^2 \int_\Omega |\nabla v_\varepsilon|^2 + \lambda_1 \int_\Omega v_\varepsilon^2 = \int_\Omega f(v_\varepsilon)v_\varepsilon.$$

Now we claim that

$$t_\varepsilon = 1 + O\left(e^{-\frac{2\sqrt{\lambda_1}(1-\frac{\eta}{2})(d_{\varepsilon,1}+\delta)}{\varepsilon}}\right).$$

Define $\tilde{\sigma} : [0, +\infty) \times [0, \beta^\star) \to \mathbb{R}$ such that

$$\tilde{\sigma}(t, \beta) = \int_\Omega f(tu_\varepsilon^+\chi)u_\varepsilon^+\chi - \int_\Omega f(u_\varepsilon^+\chi)u_\varepsilon^+\chi - \beta \int_\Omega f'(u_\varepsilon^+\chi)(u_\varepsilon^+\chi)^2$$

for some $\beta^\star > 0$. Then $\tilde{\sigma} \in C^1$. Note that $\tilde{\sigma}(1, 0) = 0$ and

$$\tilde{\sigma}_t(1, 0) = \int_\Omega f'(u_\varepsilon^+\chi)(u_\varepsilon^+\chi)^2 \neq 0.$$

Hence by implicit function theorem, there exists a C^1 function $\beta \mapsto t(\beta)$ such that $\tilde{\sigma}(t(\beta), \beta) = 0$, for small β and $t(0) = 1$. Letting $t_\varepsilon = 1 + \beta$, we have from (3.8)

$$\beta \sim \frac{\varepsilon^2 \int_\Omega |\nabla u_\varepsilon^+\chi|^2 + \lambda_1 \int_\Omega (u_\varepsilon^+\chi)^2 - \int_\Omega f(u_\varepsilon^+\chi)u_\varepsilon^+\chi}{\varepsilon^2 \int_\Omega |\nabla u_\varepsilon^+\chi|^2 + \lambda_1 \int_\Omega (u_\varepsilon^+\chi)^2 - \int_\Omega f'(u_\varepsilon^+)(u_\varepsilon^+\chi)^2}.$$

Hence

$$\beta \sim \frac{O\left(\varepsilon^N e^{-\frac{2\sqrt{\lambda_1}(d_{\varepsilon,1}+\delta)(1-\frac{\eta}{2})}{\varepsilon}}\right)}{\int_\Omega f(u_\varepsilon^+\chi)u_\varepsilon^+\chi - \int_\Omega f'(u_\varepsilon^+\chi)(u_\varepsilon^+\chi)^2}$$

which implies $\beta = O(e^{-\frac{2\sqrt{\lambda_1}(1-\frac{\eta}{2})(d_{\varepsilon,1}+\delta)}{\varepsilon}})$. Then we obtain,

$$\frac{\varepsilon^2}{2}\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} |\nabla v_\varepsilon|^2 = \frac{\varepsilon^2}{2}\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} |\nabla(u_\varepsilon^+\chi)|^2 + \varepsilon^2\beta\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} |\nabla u_\varepsilon^+\chi|^2 + O(\beta^2\varepsilon^N),$$

$$\frac{\lambda_1}{2}\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} v_\varepsilon^2 = \frac{\lambda_1}{2}\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} (u_\varepsilon^+\chi)^2 + \lambda_1\beta\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} (u_\varepsilon^+\chi)^2 + O(\beta^2\varepsilon^N),$$

and

$$\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} F(v_\varepsilon) = \int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} F(u_\varepsilon^+\chi) + \beta\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} f(u_\varepsilon^+\chi)u_\varepsilon^+\chi + O(\beta^2\varepsilon^N).$$

Also we have

$$\varepsilon^2\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} |\nabla u_\varepsilon^+\chi|^2 + \lambda_1\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} (u_\varepsilon^+\chi)^2 - \int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} f(u_\varepsilon^+\chi)u_\varepsilon^+\chi = O(\beta\varepsilon^N).$$

Using the above facts we have,

$$\begin{aligned}
&\frac{\varepsilon^2}{2}\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} |\nabla v_\varepsilon|^2 + \frac{\lambda_1}{2}\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} v_\varepsilon^2 - \int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} F(v_\varepsilon) \\
&= \frac{\varepsilon^2}{2}\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} |\nabla u_\varepsilon^+\chi|^2 + \frac{\lambda_1}{2}\int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} (u_\varepsilon^+\chi)^2 \\
&\quad - \int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} F(u_\varepsilon^+\chi) + O(\varepsilon^N|t_\varepsilon - 1|^2) \\
&= \int_{B_{d_{1,\varepsilon}+\delta}(P_\varepsilon^1)} \left(\frac{1}{2}f(u_\varepsilon^+\chi)u_\varepsilon^+\chi - F(u_\varepsilon^+\chi)\right) + O(\varepsilon^N|t_\varepsilon - 1|^2) \qquad (3.9) \\
&= \int_\Omega \left(\frac{1}{2}f(u_\varepsilon^+)u_\varepsilon^+ - F(u_\varepsilon^+)\right) \\
&\quad + O\left(\varepsilon^N|t_\varepsilon - 1|^2 + e^{-\frac{\sqrt{\lambda_1}(p+1)(1-\frac{\eta}{2})(d_{\varepsilon,1}+\delta)}{\varepsilon}}\right) \\
&= E_{\varepsilon,\lambda_1}(u_\varepsilon^+) + \varepsilon^N O\left(e^{-\frac{\sqrt{\lambda_1}(2+\sigma)(d_{\varepsilon,1}+\delta)}{\varepsilon}}\right)
\end{aligned}$$

for some $\sigma \in (0, \min(1, p-1))$. Thus we have

$$\begin{aligned} E_{\varepsilon,\lambda_1}(u_\varepsilon^+) &\geq \inf_{\mathcal{N}_\varepsilon^+} E_{\varepsilon,\lambda_1,B_{d_\varepsilon+\delta}(P_\varepsilon^1)}(v) - C\varepsilon^N e^{-\frac{\sqrt{\lambda_1}(2+\sigma)(d_{\varepsilon,1}+\delta)}{\varepsilon}} \\ &\geq \varepsilon^N \left\{ I_{\lambda_1}(W_{\lambda_1}) + e^{-\frac{2\sqrt{\lambda_1}(1-\frac{\eta}{2})(d_{\varepsilon,1}+\delta)}{\varepsilon}} \right\} - C\varepsilon^N e^{-\frac{\sqrt{\lambda_1}(2+\sigma)(d_{\varepsilon,1}+\delta)}{\varepsilon}} \\ &\geq \varepsilon^N \left\{ I_{\lambda_1}(W_{\lambda_1}) + \frac{1}{2} e^{-\frac{2\sqrt{\lambda_1}(1-\frac{\eta}{2})(d_{\varepsilon,1}+\delta)}{\varepsilon}} \right\} \\ &\geq \varepsilon^N \left\{ I_{\lambda_1}(W_{\lambda_1}) + \frac{1}{2} e^{-\frac{2(1-\frac{\eta}{2})(d_\varepsilon+\delta)}{\varepsilon}} \right\}. \end{aligned}$$

Similarly we obtain the estimate for $E_{\varepsilon,\lambda_2}(u_\varepsilon^-)$. This proves the result.

Proof of Theorem 1.1. This follows from Lemma 2.6 and Section 3. □

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