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Weighted estimates for nonhomogeneous quasilinear equations with discontinuous coefficients

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Abstract. We obtain local and global $W^{1,q}$ estimates on weighted Lebesgue spaces with certain Muckenhoupt weights for solutions to a nonhomogeneous p-Laplace type equation with VMO coefficients in a $\mathcal{C}^1$ domain. These estimates can be viewed as weighted norm inequalities for certain nonlinear singular operators (without any explicit kernel) arising from the p-Laplacian, and are applicable to a quasilinear Riccati type equation.

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1. Introduction

In this paper we are concerned with local and global estimates for gradients of solutions to a nonhomogeneous quasilinear equation on certain weighted Lebesgue spaces. Given $1 < p < +\infty$ and a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, the equation under our consideration takes the form:

$$\begin{cases} \mathrm{div}[(A\nabla u \cdot \nabla u)^{\frac{p-2}{2}} A\nabla u] = \mathrm{div}\vec{F} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $A = \{A_{ij}(x)\}_{n\times n}$ is a symmetric matrix with measurable coefficients satisfying the ellipticity condition

$$\Lambda^{-1}|\xi|^2 \leq A(x)\xi \cdot \xi \leq \Lambda|\xi|^2$$

for some constant $\Lambda > 0$, almost every $x \in \Omega$, and all $\xi \in \mathbb{R}^n$.

All solutions u to (1.1) are understood in the weak sense, *i.e.*, $u \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega (A\nabla u \cdot \nabla u)^{\frac{p-2}{2}} A\nabla u \cdot \nabla\varphi dx = \int_\Omega \vec{F} \cdot \nabla\varphi dx$$

for all test functions $\varphi \in C_0^\infty(\Omega)$.

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In the case A is the identity matrix equation (1.1) is a non-homogeneous p-Laplace equation that naturally arises in different contexts of mathematics. Regularity estimates for gradients of solutions to this equation have been considered by several authors. The reader is referred to the pioneer work of Iwaniec [9] in the case A is the identity matrix, to [11, 12] in the case the A has VMO coefficients, and to [1,2] in the case A has small BMO coefficients. Basically, the results in [1,2,12] say that if $\vec{F} \in L^{\frac{q}{p-1}}(\Omega)$ for some $q > p$, then under certain mild assumptions on Ω one has the regularity estimate

$$\int_\Omega |\nabla u|^q dx \leq C \int_\Omega |\vec{F}|^{\frac{q}{p-1}} dx \tag{1.2}$$

for a constant $C > 0$ independent of $\vec{F}$ and u. We mention here that Ω is assumed to be of class $\mathcal{C}^{1,\alpha}$, $0 < \alpha < 1$, in [12], whereas it is assumed to be Lipschitz (with small Lipschitz constant) in [1] and Reifenberg flat (with a certain smallness condition) in [2].

In this paper we obtain a weighted version of (1.2) assuming that Ω is of class $\mathcal{C}^1$ and that each component of A belongs to $VMO(\Omega)$, the space of functions of vanishing mean oscillation in Ω. In fact, our approach could be easily modified to cover the case where Ω is Lipschitz with small Lipschitz constant and A has small BMO coefficients as in [1] (see Remarks 3.2 and 3.4 below).

Our approach in the present paper is similar to that in [11, 12] and [9], which makes use of the Fefferman-Stein sharp maximal function and $C^{1,\alpha}$ regularity estimates obtained earlier for homogeneous p-Laplace type equations. In particular, we obtain a weighted version of Fefferman-Stein inequality for a local dyadic sharp maximal function which enables us to apply some of the available results in [11,12] to the weighted situation.

The class of weights considered in the paper is the well-known Muckenhoupt A_s weights for certain $1 \leq s < \infty$. Weighted $W^{1,q}$ estimates obtained in this paper are motivated by our work in [19] on quasilinear Riccati type equations with super-critical growth in the gradient. In fact, Theorem 1.1 below is employed in an indispensable way to derive a *capacitary inequality*, which is essential for the treatment of the quasilinear Riccati type equation

$$-\mathrm{div}[(A\nabla u \cdot \nabla u)^{\frac{p-2}{2}} A\nabla u] = |\nabla u|^q + \omega, \qquad q > p,$$

with measure data ω (see [19]). Our weighted estimates are also motivated by the successful use of Muckenhoupt weights to obtain certain capacitary inequalities in [14, Section 2] and in the recent papers [15–17]. On the other hand, estimates obtained in this paper can be viewed, in a sense, as weighted norm inequalities for certain *nonlinear singular operators* (without any explicit kernel) arising from the p-Laplacian (see [3] and [6,9]).

We now recall that a function $f \in L^1(\Omega)$ is an element of $VMO(\Omega)$ if the integral average

$$I(x,r) = \frac{1}{|B(x,r) \cap \Omega|} \int_{B(x,r)\cap\Omega} |f(y) - f_{B(x,r)\cap\Omega}| dy$$

is uniformly bounded in $x \in \Omega$, $0 < r \leq \operatorname{diam}(\Omega)$, and in addition $I(x,r)$ tends uniformly in $x \in \Omega$ to zero as r tends to zero; see [20]. Here and in what follows we use the notation

$$f_E = \frac{1}{|E|}\int_E f(y)dy$$

to denote the integral average of f over a measurable set $E \subset \mathbb{R}^n$ of positive Lebesgue measure.

For a Muckenhoupt A_s weight w we write $[w]_{A_s}$ to denote the A_s constant of w (see Section 2). One of our main results reads as follows.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\mathcal{C}^1$-boundary. Suppose that $\vec{F} \in L^{\frac{q}{p-1}}(\Omega, \mathbb{R}^n)$ and that w is an $A_{q/p}$ weight where $q > p > 1$. Then there exists a unique solution $u \in W_0^{1,q}(\Omega)$ to the equation*

$$\begin{cases} \operatorname{div}[(A\nabla u \cdot \nabla u)^{\frac{p-2}{2}} A\nabla u] = \operatorname{div}\vec{F} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Moreover, one has the estimate

$$\int_\Omega |\nabla u|^q w dx \leq C \int_\Omega \left(|\vec{F}|^{\frac{q}{p-1}} + |u|^q\right) w dx,$$

where C is a constant depending only on n, p, q, Λ, $[w]_{A_{q/p}}$, Ω, and the VMO data of A.

Theorem 1.1 follows from an existence result obtained in [12, Theorem 1.6] and the local as well as boundary weighted estimates obtained in Theorems 3.1 and 3.3 below.

2. Preliminaries on weighted norm inequalities

For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ the *Hardy-Littlewood maximal function* of f is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy, \tag{2.1}$$

and the *Fefferman-Stein sharp maximal function* of f is defined by

$$\mathcal{M}^\# f(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y) - f_{B(x,r)}|dy. \tag{2.2}$$

If the suprema in (2.1) and (2.2) are restricted to $0 < r \leq \rho$ for some $\rho > 0$, then we have by definition the corresponding *truncated maximal functions* $\mathcal{M}_\rho f$ and $\mathcal{M}^\#_\rho f$.

It follows directly from these definitions that

$$f \leq \mathcal{M}_\rho f \leq \mathcal{M} f \qquad \text{and} \qquad \mathcal{M}^{\#}_\rho f \leq \mathcal{M}^{\#} f \leq 2\mathcal{M} f. \tag{2.3}$$

Recall that a nonnegative function $w \in L^1_{\rm loc}(\mathbb{R}^n)$ is an A_1 weight if the there exists a constant $C > 0$ such that

$$\mathcal{M} w(x) \leq C\, w(x)$$

for a.e. $x \in \mathbb{R}^n$. In this case the smallest constant C in the preceding inequality will be denoted by $[w]_{A_1}$ and is called the A_1 constant of w. On the other hand, for $1 < s < +\infty$ a nonnegative function $w \in L^1_{\rm loc}(\mathbb{R}^n)$ is called an A_s weight if the quantity

$$[w]_{A_s} = \sup \left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{\frac{-1}{s-1}}dx\right)^{s-1} < +\infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. This quantity is then called the A_s constant of w.

It is easy to see from Hölder's inequality that one has the inclusion $A_s \subset A_r$ with $[w]_{A_r} \leq [w]_{A_s}$ whenever $1 \leq s \leq r < \infty$. A nontrivial result on A_s weights is the following "open-end property" (see *e.g.*, [7, Corollary 9.2.6]).

Lemma 2.1. *If w is an A_s weight, $1 < s < \infty$, then there exists $\epsilon_0 = \epsilon_0(n, s, [w]_{A_s})$, $0 < \epsilon_0 < s - 1$ such that w is an $A_{s-\epsilon_0}$ weight with $[w]_{A_{s-\epsilon_0}} \leq C[w]_{A_s}$.*

A broader class of weights is the A_∞ weights, which by definition are the union of A_s weights for $1 \leq s < +\infty$. We will employ the following characterization of A_∞ weights (see *e.g.*, [7, Theorem 9.3.3]).

Lemma 2.2. *A weight w is an A_∞ weight if and only if there are constants $C, \delta > 0$ such that for every cube $Q \subset \mathbb{R}^n$ and every measurable subset $E \subset Q$ one has*

$$w(E) \leq C\left(\frac{|E|}{|Q|}\right)^{\delta} w(Q), \tag{2.4}$$

where we denote by $w(E)$ the integral $\int_E w(x)dx$. Moreover, if $w \in A_s$ for some $s \geq 1$ then it satisfies (2.4) with constants C and δ depending only on n, s and the A_s constant $[w]_{A_s}$ of w.

We will refer to the constants C and δ in (2.4) as the A_∞ constants of w.

In the next two lemmas we recall well-known and striking results on weighted norm inequalities for maximal functions, especially when one compares them to the pointwise estimates in (2.3).

Lemma 2.3 (Muckenhoupt [18]). *Let w be an A_p weight. Then there exists a constant $C = C(n, p, [w]_{A_p}) > 0$ such that*

$$\|\mathcal{M} f\|_{L^p(\mathbb{R}^n, w)} \leq C\, \|f\|_{L^p(\mathbb{R}^n, w)} \tag{2.5}$$

for all $f \in L^p(\mathbb{R}^n, w)$. Conversely, if (2.5) holds for all $f \in L^p(\mathbb{R}^n, w)$ then w must be an A_p weight.

Lemma 2.4 (Fefferman and Stein [10]). *Let w be an A_∞ weight and let $0 < p_0 < \infty$. Then for each $p_0 \le p < \infty$ there exists a constant $C > 0$ depending only on n, p and the A_∞ constants of w such that*

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^n,w)} \le C \left\|\mathcal{M}^{\#} f\right\|_{L^p(\mathbb{R}^n,w)}$$

for all locally integrable functions f for which $\mathcal{M}f \in L^{p_0}(\mathbb{R}^n, w)$.

The proof of this lemma was first given in [10] in the unweighted case, *i.e.*, $w \equiv 1$, and it could be adapted to the weighted case as stated above, (see [7, page 715]).

We next describe a local dyadic version of Lemma 2.4 that will be needed later. Let Q_0 be a cube in $\mathbb{R}^n$. We define by induction the families $\mathcal{D}_k$, $k = 0, 1, 2, \ldots$ of open subcubes of the cube Q_0: $\mathcal{D}_0 = \{Q_0\}$. Suppose that the family $\mathcal{D}_k$ is given for some $k \ge 0$. Then the family $\mathcal{D}_{k+1}$ consists of all cubes obtained by dividing dyadically every cube of $\mathcal{D}_k$ into 2^n cubes of equal side-lengths. The cubes of $\mathcal{D}_k$ are disjoint and have side-length $2^{-k}|Q_0|^{1/n}$. Moreover, every two cubes from the union $\mathcal{D}^{Q_0} = \cup_k \mathcal{D}_k$ are either disjoint or one includes another.

For an integrable function f on Q_0, we define the following *local dyadic maximal functions* of Hardy-Littlewood and Fefferman-Stein associated to the cube Q_0:

$$\mathcal{M}^{dy} f(x) = \mathcal{M}^{dy}_{Q_0} f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy,$$

$$\mathcal{M}^{\#,\,dy} f(x) = \mathcal{M}^{\#,\,dy}_{Q_0} f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where the suprema are taken over all cubes Q in $\mathcal{D}^{Q_0}$ that contain x.

We first prove a *good λ distributional inequality* for $\mathcal{M}^{dy}$ and $\mathcal{M}^{\#,\,dy}$.

Lemma 2.5. *Let w be an A_∞ weight in $\mathbb{R}^n$. Then there exist constants C, $\delta > 0$ depending only on n and the A_∞ constants of w such that for all $f \in L^1(Q_0)$, all $\epsilon > 0$, and all $\lambda \ge |f|_{Q_0}$ we have the estimate*

$$\begin{aligned} &w(\{x \in Q_0 : \mathcal{M}^{dy} f(x) > 2\lambda, \mathcal{M}^{\#,\,dy} f(x) \le \epsilon\lambda\}) \\ &\le C\epsilon^\delta w(\{x \in Q_0 : \mathcal{M}^{\#,\,dy} f(x) > \lambda\}). \end{aligned}$$

Proof. For any $\lambda \ge |f|_{Q_0}$, we let $\Omega_\lambda = \{x \in Q_0 : \mathcal{M}^{dy} f(x) > \lambda\}$. Then for each $x \in \Omega_\lambda$ there is a maximal cube Q^x in $\mathcal{D}^{Q_0}$ containing x such that

$$\frac{1}{|Q^x|} \int_{Q^x} |f(y)| dy > \lambda. \tag{2.6}$$

Note that $Q^x \subsetneqq Q_0$ since $|f|_{Q_0} \le \lambda$. Let $\{Q_j\}_{j=1}^\infty = \{Q^x : x \in \Omega_\lambda\}$. Then any two different cubes in the collection $\{Q_j\}_{j=1}^\infty$ are disjoint by maximality, and moreover $\Omega_\lambda = \cup_j Q_j$.

We first prove that for any $\epsilon > 0$ and for all Q_j one has the estimate

$$|\{x \in Q_j : \mathcal{M}^{dy} f(x) > 2\lambda, \mathcal{M}^{\#, dy} f(x) \leq \epsilon\lambda\}| \leq 2^n \epsilon |Q_j|. \tag{2.7}$$

To this end we let $x \in Q_j$ for which $\mathcal{M}^{dy} f(x) > 2\lambda$. Then we find from (2.6) and the maximality of Q_j that

$$\mathcal{M}^{dy}(f\chi_{Q_j})(x) = \mathcal{M}^{dy} f(x) > 2\lambda.$$

We next denote by Q_j^* the unique cube in $\mathcal{D}^{Q_0}$ of twice side-length of Q_j (recalling that $Q_j \neq Q_0$). By the maximality of Q_j we have $|f_{Q_j^*}| \leq |f|_{Q_j^*} \leq \lambda$. Therefore, for $x \in Q_j$ for which $\mathcal{M}^{dy} f(x) > 2\lambda$ we can estimate

$$\mathcal{M}^{dy}((f - f_{Q_j^*})\chi_{Q_j})(x) \geq \mathcal{M}^{dy}(f\chi_{Q_j})(x) - |f_{Q_j^*}| > 2\lambda - \lambda = \lambda.$$

This implies that

$$\left|\{x \in Q_j : \mathcal{M}^{dy} f(x) > 2\lambda\}\right| \leq \left|\{x \in Q_j : \mathcal{M}^{dy}((f - f_{Q_j^*})\chi_{Q_j})(x) > \lambda\}\right|. \tag{2.8}$$

We now use the weak type (1, 1) estimate with constant 1:

$$\left\|\mathcal{M}^{dy} g\right\|_{L^{1,\infty}(Q_0)} \leq \|g\|_{L^1(Q_0)}, \qquad \forall g \in L^1(Q_0),$$

to deduce from (2.8) that

$$\begin{aligned} |\{x \in Q_j : \mathcal{M}^{dy} f(x) > 2\lambda\}| &\leq \frac{1}{\lambda}\int_{Q_j} |f(y) - f_{Q_j^*}| dy \\ &\leq \frac{2^n |Q_j|}{\lambda} \mathcal{M}^{\#, dy} f(z) \end{aligned} \tag{2.9}$$

for all $z \in Q_j$. At this point, to prove (2.7) we may assume that $\mathcal{M}^{\#} f(z) \leq \epsilon\lambda$ for some $z \in Q_j$. With this z and inequality (2.9) we obtain the estimate (2.7).

Finally, we use Lemma 2.2 to deduce from (2.7) that

$$w(\{x \in Q_j : \mathcal{M}^{dy} f(x) > 2\lambda, \mathcal{M}^{\#, dy} f(x) \leq \epsilon\lambda\}) \leq C(2^n\epsilon)^{\delta} w(Q_j)$$

for constants $C, \delta > 0$ depending only on the A_∞ constants of w. Since $\Omega_\lambda = \cup_j Q_j$ and since the cubes in $\{Q_j\}$ are pairwise disjoint we conclude from the last inequality that

$$w(\{x \in Q_0 : \mathcal{M}^{dy} f(x) > 2\lambda, \mathcal{M}^{\#, dy} f(x) \leq \epsilon\lambda\}) \leq C(2^n\epsilon)^{\delta} w(\Omega_\lambda).$$

This completes the proof of the lemma. □

We are now ready to prove a local dyadic version of Lemma 2.4. In the case the weight $w \equiv 1$, a similar result was established in [8, Lemma 4].

Theorem 2.6. *Let w be an A_∞ weight in $\mathbb{R}^n$ and let $1 < s < \infty$. Then there exists a constant $C > 0$ depending only n, s and the A_∞ constants of w such that for all $f \in L^s(Q_0)$ one has the estimate*

$$\int_{Q_0} (\mathcal{M}^{dy} f)^s w(x)dx \le C \int_{Q_0} (\mathcal{M}^{\#,\,dy} f)^s w(x)dx + 2^{s+1}\, w(Q_0)(|f|_{Q_0})^s.$$

Proof. We first employ the well-known formula

$$\int_{Q_0} |g|^s w(x)dx = s\int_0^\infty \lambda^{s-1} w(\{x \in Q_0 : |g(x)| > \lambda\})d\lambda, \tag{2.10}$$

which holds for all measurable functions g on Q_0, to write

$$\begin{aligned}
&\int_{Q_0} (\mathcal{M}^{dy} f)^s w(x)dx \\
&= s\int_0^\infty \lambda^{s-1} w(\{x \in Q_0 : \mathcal{M}^{dy} f(x) > \lambda\})d\lambda \\
&= s2^s\int_0^\infty \lambda^{s-1} w(\{x \in Q_0 : \mathcal{M}^{dy} f(x) > 2\lambda\})d\lambda.
\end{aligned} \tag{2.11}$$

It is obvious that

$$\begin{aligned}
&s2^s\int_0^{|f|_{Q_0}} \lambda^{s-1} w(\{x \in Q_0 : \mathcal{M}^{dy} f(x) > 2\lambda\})d\lambda \\
&\le s2^s\int_0^{|f|_{Q_0}} \lambda^{s-1} w(Q_0)d\lambda = 2^s\, w(Q_0)(|f|_{Q_0})^s.
\end{aligned} \tag{2.12}$$

On the other hand, for any $M > |f|_{Q_0}$ and any $\epsilon > 0$ one has

$$\begin{aligned}
&s2^s\int_{|f|_{Q_0}}^M \lambda^{s-1} w(\{x \in Q_0 : \mathcal{M}^{dy} f(x) > 2\lambda\})d\lambda \\
&\le s2^s\int_{|f|_{Q_0}}^M \lambda^{s-1} w(\{x \in Q_0 : \mathcal{M}^{dy} f(x) > 2\lambda, \mathcal{M}^{\#,\,dy} f(x) \le \epsilon\lambda\})d\lambda \\
&\quad + s2^s\int_{|f|_{Q_0}}^M \lambda^{s-1} w(\{x \in Q_0 : \mathcal{M}^{\#,\,dy} f(x) > \epsilon\lambda\})d\lambda.
\end{aligned}$$

It thus follows from Lemma 2.5 that

$$\begin{aligned} s2^s \int_{|f|_{Q_0}}^{M} &\lambda^{s-1} w(\{x \in Q_0 : \mathcal{M}^{dy} f(x) > 2\lambda\}) d\lambda \\ &\leq C\epsilon^{\delta} s2^s \int_0^M \lambda^{s-1} w(\{x \in Q_0 : \mathcal{M}^{dy} f(x) > \lambda\}) d\lambda \\ &\quad + s2^s \int_0^M \lambda^{s-1} w(\{x \in Q_0 : \mathcal{M}^{\#, dy} f(x) > \epsilon\lambda\}) d\lambda \end{aligned} \tag{2.13}$$

for constants $C, \ \delta > 0$ independent of ϵ.

Now combining estimates (2.12), (2.13), and a change of variable we obtain

$$\begin{aligned} s2^s \int_0^M &\lambda^{s-1} w(\{x \in Q_0 : \mathcal{M}^{dy} f(x) > 2\lambda\}) d\lambda \\ &\leq C\epsilon^{\delta} s2^{2s} \int_0^{\frac{M}{2}} \lambda^{s-1} w(\{x \in Q_0 : \mathcal{M}^{dy} f(x) > 2\lambda\}) d\lambda \\ &\quad + s2^s \int_0^M \lambda^{s-1} w(\{x \in Q_0 : \mathcal{M}^{\#, dy} f(x) > \epsilon\lambda\}) d\lambda \\ &\quad + 2^s\, w(Q_0)(|f|_{Q_0})^s. \end{aligned} \tag{2.14}$$

Thus in (2.14) if we choose $\epsilon > 0$ so that $C\epsilon^{\delta} s2^{2s} = \frac{s2^s}{2}$ we deduce

$$\begin{aligned} \frac{s2^s}{2} \int_0^M &\lambda^{s-1} w(\{x \in Q_0 : \mathcal{M}^{dy} f(x) > 2\lambda\}) d\lambda \\ &\leq s2^s \int_0^M \lambda^{s-1} w(\{x \in Q_0 : \mathcal{M}^{\#, dy} f(x) > \epsilon\lambda\}) d\lambda \\ &\quad + 2^s\, w(Q_0)(|f|_{Q_0})^s. \end{aligned} \tag{2.15}$$

Finally, in view of (2.10), (2.11), and (2.15) we find

$$\int_{Q_0} (\mathcal{M}^{dy} f)^s w(x) dx \leq C \int_{Q_0} (\mathcal{M}^{\#, dy} f)^s w(x) dx + 2^{s+1} w(Q_0)(|f|_{Q_0})^s,$$

which is the desired inequality. □

The following consequence of Theorem 2.6 will play a crucial role in our approach to gradient estimates below. In the case the weight $w \equiv 1$, this result was obtained in [11, Lemma 2.4] by a different method that does not seem to be applicable in our situation.

Corollary 2.7. *Let w be an A_s weight in $\mathbb{R}^n$ with $1 < s < \infty$. Then there exist constants $\kappa = \kappa(n, s, [w]_{A_s}) > \sqrt{n}$ and $C = C(n, s, [w]_{A_s}) > 0$ such that for all $f \in L^s(\mathbb{R}^n)$ with* $\mathrm{supp}(f) \subset B(x_0, R)$, *$R > 0$, we have the estimate*

$$\int_{B(x_0, R)} |f(x)|^s w(x) dx \leq C \int_{B(x_0, \kappa R)} (\mathcal{M}^{\#}_{\kappa R} f(x))^s w(x) dx.$$

Proof. Let Q_0 be a cube centered at x_0 with side-length $\frac{2\kappa R}{\sqrt{n}}$, where $\kappa > \sqrt{n}$ to be determined later. We have $B(x_0, R) \subset Q_0 \subset B(x_0, \kappa R)$. Thus by Theorem 2.6 we can estimate

$$\begin{aligned}\int_{B(x_0,R)} |f(x)|^s w(x)dx &\le \int_{Q_0} (\mathcal{M}^{dy} f)^s w(x)dx \\ &\le C \int_{Q_0} (\mathcal{M}^{\#,dy} f)^s w(x)dx \\ &\quad + 2^{s+1} w(Q_0)(|f|_{Q_0})^s.\end{aligned} \tag{2.16}$$

Given $x \in Q_0$ and a cube Q in $\mathcal{D}^{Q_0}$ that contains x, we let B be the smallest ball centered at x that contains Q. Then the radius of B is $\sqrt{n}/2$ times the side-length of Q. It is evident that

$$|f - f_Q| \le |f - a| + |a - f_Q| \le |f - a| + \frac{1}{|Q|}\int_Q |f(z) - a|dz$$

for all $a \in \mathbb{R}$, which in particular gives

$$\frac{1}{|Q|}\int_Q |f(y) - f_Q|dy \le \frac{2}{|Q|}\int_Q |f(y) - f_B|dy \le \frac{c(n)}{|B|}\int_B |f(y) - f_B|dy.$$

Hence,

$$\mathcal{M}^{\#,dy} f(x) \le c(n)\, \mathcal{M}^{\#}_{\kappa R} f(x). \tag{2.17}$$

On the other hand, by Hölder's inequality and the A_s condition on w

$$\begin{aligned}&w(Q_0)\left(\frac{1}{|Q_0|}\int_{Q_0} |f(y)|dy\right)^s \\ &\quad = w(Q_0)\left(\frac{1}{|Q_0|}\int_{B(x_0,R)} |f(y)|w(y)^{\frac{1}{s}} w(y)^{\frac{-1}{s}} dy\right)^s \\ &\quad \le C\,\frac{w(Q_0)}{w(B(x_0,R))}\left(\frac{|B(x_0,R)|}{|Q_0|}\right)^s \int_{B(x_0,R)} f(y)^s w(y)dy.\end{aligned} \tag{2.18}$$

We next employ the open-end property of A_s weights, Lemma 2.1, to find $\epsilon_0 = \epsilon_0(n, s, [w]_{A_s})$, $\epsilon_0 \in (0, s-1)$ such that w is an $A_{s-\epsilon_0}$ weight with $[w]_{A_{s-\epsilon_0}} \le C[w]_{A_s}$. With this ϵ_0 we can now estimate using Hölder's inequality

$$\begin{aligned}|B(x_0,R)|^{s-\epsilon_0} &= \left(\int_{Q_0} \chi_{B(x_0,R)}(y) w(y)^{\frac{1}{s-\epsilon_0}} w(y)^{\frac{-1}{s-\epsilon_0}} dy\right)^{s-\epsilon_0} \\ &\le w(B(x_0,R))\left(\int_{Q_0} w(y)^{\frac{-1}{s-\epsilon_0-1}} dy\right)^{s-\epsilon_0-1} \\ &\le \frac{w(B(x_0,R))}{w(Q_0)} |Q_0|^{s-\epsilon_0}[w]_{p-\epsilon_0}.\end{aligned}$$

Thus combining this and inequality (2.18) we deduce that

$$\begin{aligned} w(Q_0)\left(\frac{1}{|Q_0|}\int_{Q_0}|f(y)|dy\right)^s &\\ \leq C\left(\frac{|B(x_0,R)|}{|Q_0|}\right)^{\epsilon_0}\int_{B(x_0,R)} f(y)^s w(y)dy & \qquad (2.19)\\ \leq C\kappa^{-\epsilon_0}\int_{B(x_0,R)} f(y)^s w(y)dy, & \end{aligned}$$

where the constants $C = C(n, s, [w]_{A_s})$.

Finally, we combine (2.16), (2.17), and (2.19) to obtain

$$\begin{aligned} &\int_{B(x_0,R)}|f(x)|^s w(x)dx\\ &\leq C\int_{Q_0}(\mathcal{M}^{\#}_{\kappa R}f)^s w(x)dx + C\kappa^{-\epsilon_0}\int_{B(x_0,R)}|f(x)|^s w(x)dx, \end{aligned}$$

which will give the desired estimate if we choose κ so that $C\kappa^{-\epsilon_0} = \frac{1}{2}$. ☐

Remark 2.8. In the case $w \equiv 1$, we can estimate the last term on the right-hand side of (2.16) by Hölder's inequality as follows:

$$\begin{aligned} 2^{s+1}|Q_0|(|f|_{Q_0})^s &\leq \frac{2^{s+1}}{|Q_0|^{s-1}}\left(\int_{B(x_0,R)} f(y)dy\right)^s\\ &\leq 2^{s+1}\frac{|B(x_0,R)|^{s-1}}{|Q_0|^{s-1}}\int_{B(x_0,R)} f(y)^s dy\\ &\leq 2^{s+1}\left(\frac{\sqrt{n}}{\kappa}\right)^{n(s-1)}\int_{B(x_0,R)} f(y)^s dy. \end{aligned}$$

Thus in this case one can take $\kappa = \sqrt{n}2^{\frac{s+2}{n(s-1)}}$.

3. Weighted $W^{1,q}$ estimates for quasilinear equations

In this section we obtain the main results of the paper. Our first result concerns with a local interior gradient estimate on weighted spaces that extends the unweighted case considered previously in [1,5,9,11], and [2].

Theorem 3.1. *Let $1 < p < q < \infty$ and let w be an $A_{q/p}$ weight in $\mathbb{R}^n$. Suppose that $\vec{F} \in L^{\frac{q}{p-1}}_{\rm loc}(\Omega, \mathbb{R}^n)$ and that $u \in W^{1,q}_{\rm loc}(\Omega)$ is a weak solution to*

$$\operatorname{div}[(A\nabla u\cdot\nabla u)^{\frac{p-2}{2}}A\nabla u] = \operatorname{div}\vec{F} \qquad in\ \Omega.$$

Then for every $x_0 \in \Omega$ *there exist* $d > 0$ *and* $C > 0$ *such that* $B(x_0, 6d) \subset \Omega$ *and*

$$\int_{B(x_0,d)} |\nabla u|^q w dx \le C \int_{B(x_0,6d)} (|\vec{F}|^{\frac{q}{p-1}} + |u|^q) w dx.$$

Here d *and* C *depend only on* n, p, q, Λ, $[w]_{A_{q/p}}$, $\operatorname{dist}(x_0, \partial\Omega)$, *and the VMO data of* A.

Proof. Let $\kappa = \kappa(n, s, [w]_{A_s}) > \sqrt{n}$ be as in Corollary 2.7, where $s = q/p > 1$. Fix $x_0 \in \Omega$ and let $h \ge 2$, $d > 0$ to be determined appropriately later so that $B(x_0, 8h\kappa d) \subset \Omega$. We set

$$\overline{u} = u\zeta^{\frac{p}{p-1}},$$

where $\zeta \in C_0^\infty(B(x_0, 2d))$, $0 \le \zeta \le 1$, is a cut-off function such that $\zeta \equiv 1$ in $B(x_0, d)$, and $|\nabla\zeta| \le c/d$. It is clear that $\overline{u} \in W^{1,q}(B(x, R))$ for any $0 < R < \operatorname{dist}(x, \partial\Omega)$.

For each $x \in B(x_0, 2\kappa d)$ and R, $0 < R < 2h\kappa d$ we consider the unique solution $v \in W^{1,p}(B(x, R))$ to the problem

$$\begin{cases} \operatorname{div}[(A_B\nabla v \cdot \nabla v)^{\frac{p-2}{2}} A_B\nabla v] = 0 & \text{in } B(x, R), \\ v - \overline{u} \in W_0^{1,p}(B(x, R)). \end{cases} \tag{3.1}$$

Here the matrix $A_B = A_{B(x,R)}$ is the constant matrix whose entries are the integral averages of the corresponding entries of the matrix A over the ball $B = B(x, R)$. Note that $B(x, 3R) \subset \Omega$ since $h \ge 2$ and $B(x_0, 8h\kappa d) \subset \Omega$.

We now recall the following basic estimate for the gradient of v obtained *e.g.*, in [13, 21], and [4]: *There exist constants* $C = C(n, p, \Lambda) > 0$ *and* $\alpha = \alpha(n, p, \Lambda) \in (0, 1)$ *such that for every* $\rho \in (0, R/2]$ *one has*

$$\begin{aligned} &\frac{1}{|B(x,\rho)|} \int_{B(x,\rho)} |\nabla v - (\nabla v)_{B(x,\rho)}| dy \\ &\qquad \le C \left(\frac{\rho}{R}\right)^\alpha \left(\frac{1}{|B(x,R)|} \int_{B(x,R)} |\nabla \overline{u}|^p dy\right)^{\frac{1}{p}}. \end{aligned} \tag{3.2}$$

Next, for brevity we set $G = (|\vec{F}|^{\frac{p}{p-1}} + |u|^p)\chi_{B(x_0,6d)}$ and

$$\|A\|_{*,R} = \sup \frac{1}{|B(x,r)|} \int_{B(x,r)} |A(y) - A_{B(x,r)}| dy,$$

where the supremum is taken over all balls $B(x, r) \subset \Omega$ for which $r \le R$. Here for an $n \times n$ matrix $B = \{B_{ij}\}$, $|B|$ denotes its norm, *i.e.*,

$$|B| = \sqrt{\sum_{i,j=1}^n B_{ij}^2}.$$

Since $w \in A_{q/p}$, by Lemma 2.1 there exists ϵ_0, $0 < \epsilon_0 < q/p - 1$ such that $w \in A_{q/p-\epsilon_0}$ and $[w]_{A_{q/p-\epsilon_0}} \leq C[w]_{A_{q/p}}$. We now choose $\overline{p} \in (p, q)$ so that $q/\overline{p} = q/p - \epsilon_0$, *i.e.*, $\overline{p} = pq/(q - p\epsilon_0)$ and let $R = h\rho$ with $0 < \rho \leq 2\kappa d$. Then thanks to Lemma 3.7 in [11] we have the following estimate: *For every* $\epsilon \in (0, 1)$, *there are constants* $C(\epsilon) > 0$ *and* $C(\epsilon, h, d) > 0$ *such that*

$$\begin{aligned}
&\frac{1}{|B(x,R)|}\int_{B(x,R)} |\nabla\overline{u} - \nabla v|^p dy \\
&\qquad \leq C(\epsilon)\, \|A\|_{*,R}^{1-p/\overline{p}} \left(\frac{1}{|B(x,R)|}\int_{B(x,R)} |\nabla\overline{u}|^{\overline{p}} dy\right)^{p/\overline{p}} \\
&\qquad\quad + \frac{\epsilon}{|B(x,R)|}\int_{B(x,R)} |\nabla\overline{u}|^p dy + \frac{C(\epsilon,h,d)}{|B(x,3R)|}\int_{B(x,3R)} G dy.
\end{aligned} \tag{3.3}$$

On the other hand, by triangle and Hölder's inequalities we can estimate

$$\begin{aligned}
&\frac{1}{|B(x,\rho)|}\int_{B(x,\rho)} \big||\nabla\overline{u}| - (|\nabla\overline{u}|)_{B(x,\rho)}\big| dy \\
&\qquad \leq \frac{2}{|B(x,\rho)|}\int_{B(x,\rho)} \big||\nabla\overline{u}| - |(\nabla v)_{B(x,\rho)}|\big| dy \\
&\qquad \leq \frac{2}{|B(x,\rho)|}\int_{B(x,\rho)} \big(|\nabla\overline{u} - \nabla v| + |\nabla v - (\nabla v)_{B(x,\rho)}|\big) dy \\
&\qquad \leq 2\left(\frac{1}{|B(x,\rho)|}\int_{B(x,\rho)} |\nabla\overline{u} - \nabla v|^p dy\right)^{\frac{1}{p}} \\
&\qquad\quad + \frac{2}{|B(x,\rho)|}\int_{B(x,\rho)} |\nabla v - (\nabla v)_{B(x,\rho)}| dy.
\end{aligned} \tag{3.4}$$

We now take $\epsilon = h^{-p(n+\alpha)}$ in (3.3) where α is the exponent in (3.2). Then it follows from (3.4), (3.3), and estimate (3.2) that

$$\begin{aligned}
&\frac{1}{|B(x,\rho)|}\int_{B(x,\rho)} \big||\nabla\overline{u}| - (|\nabla\overline{u}|)_{B(x,\rho)}\big| dy \\
&\leq C(h)\, \|A\|_{*,R}^{1/p-1/\overline{p}} \left(\frac{1}{|B(x,R)|}\int_{B(x,R)} |\nabla\overline{u}|^{\overline{p}} dy\right)^{1/\overline{p}} \\
&\quad + Ch^{-\alpha}\left(\frac{1}{|B(x,R)|}\int_{B(x,R)} |\nabla\overline{u}|^p dy\right)^{1/p} \\
&\quad + C(h,d)\left(\frac{1}{|B(x,3R)|}\int_{B(x,3R)} G dy\right)^{1/p}.
\end{aligned}$$

This holds for every $x \in B(x_0, 2\kappa d)$ and every $R = h\rho$ with $h \geq 2$, $0 < \rho \leq 2\kappa d$. Thus we can take the supremum over $\rho \in (0, 2\kappa d]$ in the above inequality to derive the pointwise estimate

$$\begin{aligned}\mathcal{M}^{\#}_{2\kappa d}(|\nabla\overline{u}|)(x) \le C(h)\,\|A\|^{1/p-1/\overline{p}}_{*,\,2h\kappa d}\left[\mathcal{M}(|\nabla\overline{u}|^{\overline{p}})(x)\right]^{1/\overline{p}} \\ + Ch^{-\alpha}\left[\mathcal{M}(|\nabla\overline{u}|^{p})(x)\right]^{1/p} + C(h,d)\left[\mathcal{M}G(x)\right]^{1/p}\end{aligned}$$

for all $x \in B(x_0, 2\kappa d)$. At this point we apply Corollary 2.7 with $s = q/p > 1$ and $f = |\nabla\overline{u}|^p$, which is compactly supported in $B(x_0, 2d)$, to deduce

$$\begin{aligned}\int_{B(x_0,2d)} |\nabla\overline{u}|^q w\,dx \le C(h,[w]_{A_s})\,\|A\|^{q/p-q/\overline{p}}_{*,\,2h\kappa d}\int_{\mathbb{R}^n}[\mathcal{M}(|\nabla\overline{u}|^{\overline{p}})]^{q/\overline{p}} w\,dx \\ + C([w]_{A_s})h^{-\alpha q}\int_{\mathbb{R}^n}[\mathcal{M}(|\nabla\overline{u}|^{p})]^{q/p} w\,dx \\ + C(h,d,[w]_{A_s})\int_{\mathbb{R}^n}(\mathcal{M}G)^{q/p} w\,dx,\end{aligned}$$

where the constants C may depend also on n, p, q, and Λ.

Since $s = q/p > 1$, $q/\overline{p} > 1$ and $w \in A_{p/q} \cap A_{q/\overline{p}}$, we can now use the weighted version of Hardy-Littlewood maximal function estimate, Lemma 2.3, to obtain from the above inequality

$$\begin{aligned}\int_{B(x_0,2d)} |\nabla\overline{u}|^q w\,dx \le C(h,[w]_{A_s})\,\|A\|^{q/p-q/\overline{p}}_{*,\,2h\kappa d}\int_{B(x_0,2d)}|\nabla\overline{u}|^q w\,dx \\ + C([w]_{A_s})h^{-\alpha q}\int_{B(x_0,2d)}|\nabla\overline{u}|^q w\,dx \\ + C(h,d,[w]_{A_s})\int_{B(x_0,6d)}(|\vec{F}|^{\frac{p}{p-1}} + |u|^p)^{q/p} w\,dx.\end{aligned}$$

Finally, in the last inequality we first choose h large enough so that

$$C([w]_{A_s})h^{-\alpha q} \le \frac{1}{4}$$

and then choose d small enough so that $B(x_0, 8h\kappa d) \subset \Omega$ and

$$C(h,[w]_{A_s})\,\|A\|^{q/p-q/\overline{p}}_{*,\,2h\kappa d} \le \frac{1}{4}, \tag{3.5}$$

we can absorb the first two terms on the right-hand side to the left-hand side obtaining

$$\int_{B(x_0,2d)} |\nabla\overline{u}|^q w\,dx \le C(h,d,[w]_{A_s})\int_{B(x_0,6d)}(|\vec{F}|^{\frac{p}{p-1}} + |u|^p)^{q/p} w\,dx.$$

This inequality gives the desired estimate and hence completes the proof of the theorem. □

Remark 3.2. We observe that the only assumption on A needed in the proof of Theorem 3.1 is condition (3.5). Thus it is enough to assume that A has small BMO coefficients as in [1], where the smallness condition now of course depends also on the weight w.

The proof of Theorem 3.1 can now be adapted to obtain the corresponding boundary estimate. The corresponding result in the unweighted case can be found in [1,2,12].

Theorem 3.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $\mathcal{C}^1$-boundary, and let $x_0 \in \partial\Omega$ and $R_0 > 0$. Suppose that $\vec{F} \in L^{\frac{q}{p-1}}(B(x_0, R_0) \cap \Omega)$ for some $q > p > 1$, and that w is an $A_{q/p}$ weight in $\mathbb{R}^n$. Then there exist $a > 1$, $d \in (0, R_0/a)$, and $C > 0$ such that for any weak solution $u \in W^{1,q}(B(x_0, R_0) \cap \Omega)$ to the problem*

$$\begin{cases} \mathrm{div}[(A\nabla u \cdot \nabla u)^{\frac{p-2}{2}} A\nabla u] = \mathrm{div}\vec{F} & \text{in } B(x_0, R_0) \cap \Omega, \\ u = 0 & \text{on } \partial\Omega \cap B(x_0, R_0) \end{cases}$$

one has the estimate

$$\int_{B(x_0,d)\cap\Omega} |\nabla u|^q w dx \leq C \int_{B(x_0,ad)\cap\Omega} (|\vec{F}|^{\frac{q}{p-1}} + |u|^q) w dx.$$

Here d and C depend only on $n, p, q, \Lambda, [w]_{A_{q/p}}, R_0$, and the VMO data of A, whereas the number a depends only on $\partial\Omega$.

Proof. Let $x_0 \in \partial\Omega$ and $R_0 > 0$ be as in the lemma. For $x \in \mathbb{R}^n$ we write $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Since $\partial\Omega$ is of class $\mathcal{C}^1$ we may assume that there is a $\mathcal{C}^1$ function $h : \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$\Omega \cap B(x_0, r) = \{x = (x', x_n) \in B(x_0, r) : x_n > h(x')\}$$

for all $r \leq R_0/M$, where $M > 1$ is a constant depending only on $\partial\Omega$.

Now let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^1$-diffeomorphism defined by

$$y = \Phi(x) = \Phi((x', x_n)) = (x', x_n - h(x'))$$

and set

$$x = \Phi^{-1}(y) = \Psi(y).$$

Next, for a fixed r, $0 < r < R_0/M$ we choose $s > 0$ so small that $B^+(\Phi(x_0), s) \subset \Phi(\Omega \cap B(x_0, r))$ and define

$$u_1(y) = u(\Psi(y))$$

for all $y \in B^+(\Phi(x_0), s)$. Here $B^+(\Phi(x_0), s) = B(\Phi(x_0), s) \cap \mathbb{R}^n_+$. Then we see that u_1 is a weak solution to

$$\begin{cases} \mathrm{div}[(A_1\nabla u_1 \cdot \nabla u_1)^{\frac{p-2}{2}} A_1\nabla u_1] = \mathrm{div}\vec{F}_1 & \text{in } B^+(\Phi(x_0), s), \\ u_1 = 0 & \text{on } B(\Phi(x_0), s) \cap \partial\mathbb{R}^n_+, \end{cases}$$

where

$$A_1(y) = [\nabla\Phi(\Psi(y))]^T A(\Psi(y))[\nabla\Phi(\Psi(y))] \tag{3.6}$$

and

$$\vec{F}_1(y) = [\nabla\Phi(\Psi(y))]^T \vec{F}(\Psi(y)); \tag{3.7}$$

see [12, pages 479–480].

Sine $\nabla\Phi(x)$ is continuous and $A(x) \in VMO$, we see that $A_1(y) \in VMO$ as well. Moreover, it is easy to see that if $w(x)$ is an A_s weight, $s \geq 1$, then $w_1(y) = w(\Psi(y))$ is also an A_s weight. These observations imply that the boundary of Ω could be locally flattened by a $\mathcal{C}^1$ diffeomorphism and Theorem 3.3 can be reduced to the case where $\partial\Omega$ is locally $\partial\mathbb{R}^n_+$.

Therefore, we may assume that $x_0 \in \partial\mathbb{R}^n_+$, $\vec{F} \in L^{\frac{q}{p-1}}(B^+(x_0, R_0))$, and $u \in W^{1,q}(B^+(x_0, R_0))$ is a weak solution to

$$\begin{cases} \operatorname{div}[(A\nabla u \cdot \nabla u)^{\frac{p-2}{2}} A\nabla u] = \operatorname{div}\vec{F} & \text{in } B^+(x_0, R_0), \\ u = 0 & \text{on } B(x_0, R_0) \cap \partial\mathbb{R}^n_+. \end{cases}$$

Our goal is to show that

$$\int_{B^+(x_0,d)} |\nabla u|^q w dx \leq C \int_{B^+(x_0,6d)} (|\vec{F}|^{\frac{q}{p-1}} + |u|^q) w dx. \tag{3.8}$$

for some $d > 0$ to be determined appropriately later so that $B^+(x_0, 8h\kappa d) \subset B^+(x_0, R_0)$. Here $h \geq 2$ is also to be determined, and $\kappa = \kappa(n, q/p, [w]_{A_{q/p}})$ is as in Corollary 2.7.

As in the proof of Theorem 3.1 we set

$$\overline{u} = u\zeta^{\frac{p}{p-1}},$$

where $\zeta \in C_0^\infty(B(x_0, 2d))$, $0 \leq \zeta \leq 1$, is a cut-off function such that $\zeta \equiv 1$ in $B(x_0, d)$, and $|\nabla\zeta| \leq c/d$. It is clear that $\overline{u} \in W^{1,q}(B(x, R) \cap \mathbb{R}^n_+)$, $\overline{u} = 0$ on $B(x, R) \cap \partial\mathbb{R}^n_+$ for every $x \in B^+(x_0, 2\kappa d)$ and $0 < R < 2h\kappa d$.

For every $x \in B^+(x_0, 2\kappa d)$ and every R, $0 < R < 2h\kappa d$ we consider the unique solution $v \in W^{1,p}(B(x, R) \cap \mathbb{R}^n_+)$ to the problem

$$\begin{cases} \operatorname{div}[(A_B\nabla v \cdot \nabla v)^{\frac{p-2}{2}} A_B\nabla v] = 0 & \text{in } B(x, R) \cap \mathbb{R}^n_+, \\ v - \overline{u} \in W_0^{1,p}(B(x, R) \cap \mathbb{R}^n_+). \end{cases}$$

Here the matrix $A_B = A_{B(x,R)\cap\mathbb{R}^n_+}$ is the constant matrix whose entries are the integral averages of the corresponding entries of the matrix A over the set $B = B(x, R) \cap \mathbb{R}^n_+$. Note that $B(x, 3R) \cap \mathbb{R}^n_+ \subset B^+(x_0, R_0)$ since $h \geq 2$ and $B^+(x_0, 8h\kappa d) \subset B^+(x_0, R_0)$.

By Lemmas 3.7 and 3.11 in [12], inequalities similar to (3.2) and (3.3) hold with $B(x, \rho)\cap\mathbb{R}^n_+$, $B(x, R)\cap\mathbb{R}^n_+$, and $B(x, 3R)\cap\mathbb{R}^n_+$ in place of $B(x, \rho)$, $B(x, R)$, and $B(x, 3R)$, respectively. In this setting $\|A\|_{*,\, R}$ should be understood as

$$\|A\|_{*,\, R} = \sup \frac{1}{|B(x,r)\cap\mathbb{R}^n_+|}\int_{B(x,r)\cap\mathbb{R}^n_+} |A(y) - A_{B(x,r)\cap\mathbb{R}^n_+}|dy,$$

where the supremum is taken over all balls $B(x, r)$ with $x \in \mathbb{R}^n_+$ and $0 < r \leq R$.

Thus to obtain (3.8) we can proceed as in the proof of Theorem 3.1. However, to be able to utilize Corollary 2.7 at some point we need to employ a certain extension result obtained in [12, Lemma 2.3]. We omit the details here and the reader is referred to [12, pages 484–485], for a similar situation. □

Remark 3.4. By Remark 3.2 and in view of (3.6) and (3.7) we see that the proof of Theorem 3.3 can be adapted to the case where Ω is only Lipschitz with small Lipschitz constant and A has small BMO coefficients as in [1].

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