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Duality of multiparameter Hardy spaces H^p on spaces of homogeneous type

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Dedicated to Professor Shanzhen Lu on the occasion of his 70th birthday with appreciation and friendship

Abstract. In this paper, we introduce the Carleson measure space CMO^p on product spaces of homogeneous type in the sense of Coifman and Weiss [4], and prove that it is the dual space of the product Hardy space H^p of two homogeneous spaces defined in [15]. Our results thus extend the duality theory of Chang and R. Fefferman [2, 3] on $H^1(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ with $\mathrm{BMO}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ which was established using bi-Hilbert transform. Our method is to use discrete Littlewood-Paley analysis in product spaces recently developed in [13] and [14].

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1. Introduction

The Hardy and BMO spaces play an important role in modern harmonic analysis and applications in partial differential equations. In [7], C. Fefferman and Stein showed that the space of functions of bounded mean oscillation on $\mathbb{R}^n$, $\mathrm{BMO}(\mathbb{R}^n)$, is the dual space of the Hardy space $H^1(\mathbb{R}^n)$. They also obtained a characterization of the BMO space in terms of the Carleson measure. For the multi-parameter product case, S.-Y. Chang and R. Fefferman in [3] proved using bi-Hilbert transform the following:

Theorem 1.1. *The dual space of* $H^1(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ *is* $\mathrm{BMO}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$.

Recently, using a new version of Journé covering lemma, Ferguson and Lacey in [10] gave a new characterization of the product $\mathrm{BMO}(\mathbb{R} \times \mathbb{R})$ by bicommutator of Hilbert transforms (see also Lacey and Terwilleger [21]). Furthermore, Lacey, Petermichl, Pipher and Wick established in [20] such a characterization of product $\mathrm{BMO}(\mathbb{R}^n \times \mathbb{R}^m)$ using multiparameter commutators of Riesz transforms.

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However, the characterization of the dual space of $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ when $0 < p < 1$ appears to be an open question. One of the main purposes of this paper is to establish such duality theory for $p \leq 1$. In fact, we will achieve this by proving a more general theorem. Namely, we will establish the dual space of Hardy spaces on the product of two homogeneous spaces which includes the dual space of $H^p(\mathbb{R}^n \times \mathbb{R}^m)$ for all $0 < p < 1$ as a special case.

Spaces of homogeneous type were first introduced by R. Coifman and G. Weiss [4] in the 1970's in order to extend the theory of Calderón-Zygmund singular integrals to a more general setting. There are, however, no translations or dilations, and no analogue of the Fourier transform or convolution operation on such spaces.

In 1985, using Coifman's idea on decomposition of the identity operator, G. David, J. L. Journé and S. Semmes [5] developed the Littlewood-Paley analysis on spaces of homogeneous type and used it to give a proof of the $T1$ theorem on this general setting.

Recently, the first and third authors of this paper, established in [13] theory of the multi-parameter Hardy space $H^p_F(\mathbb{R}^n \times \mathbb{R}^m)$ associated with the flag singular integrals, where the L^p theory has been developed by Muller-Ricci-Stein [24] and Nage-Ricci-Stein [25], and the Hardy space $H^p_Z(\mathbb{R}^3)$ associated with the non-classical Zygmund dilation in [14] using the discrete Littlewood-Paley analysis and proved that the singular integral operators introduced by Ricci-Stein [27] are bounded on such Hardy spaces. In these two papers [13] and [14], the Carleson measure spaces $\mathrm{CMO}^p_F(\mathbb{R}^n \times \mathbb{R}^m)$ and $\mathrm{CMO}^p_Z(\mathbb{R}^3)$ are introduced for all $0 < p \leq 1$, and the duality of between $H^p_F(\mathbb{R}^n \times \mathbb{R}^m)$ with $\mathrm{CMO}^p_F(\mathbb{R}^n \times \mathbb{R}^m)$, $H^p_Z(\mathbb{R}^3)$ and $\mathrm{CMO}^p_Z(\mathbb{R}^3)$ are established. Such CMO^p spaces when $p = 1$ play the same role as BMO space. Moreover, the authors of [13] and [14] established the multiparameter Hardy space theory using discrete Littlewood-Paley analysis and proved boundeness of singular integral operators on Hardy spaces H^p and from H^p to L^p by bypassing the deep Journe's geometric covering lemma. As a consequence, they provided an alternative approach of proving boundedness of singular integral operators on product Hardy spaces using rectangle atoms discovered by R. Fefferman in [6] (see also Pipher [26]).

For the multi-parameter product spaces of homogeneous type, denoted by $\mathcal{X} \times \mathcal{X}$, the Hardy space $H^p(\mathcal{X} \times \mathcal{X})$ for $p_0 < p \leq 1$ with some p_0 close to 1 was established in [15], see more details in the next section. The boundedness of singular integral operators on $H^p(\mathcal{X} \times \mathcal{X})$ were proved in [17] without using Journe's covering lemma. Subsequently, the boundedness from $H^p(\mathcal{X} \times \mathcal{X})$ to $L^p(\mathcal{X} \times \mathcal{X})$ was established in [12] by proving that the density result of $L^q(\mathcal{X} \times \mathcal{X}) \cap H^p(\mathcal{X} \times \mathcal{X})$ in $H^p(\mathcal{X} \times \mathcal{X})$ for $1 < q < \infty$ and $0 < p \leq 1$ close to 1 and that $||f||_{L^p(\mathcal{X}\times\mathcal{X})} \leq C||f||_{H^p(\mathcal{X}\times\mathcal{X})}$ for $f \in L^q(\mathcal{X} \times \mathcal{X}) \cap H^p(\mathcal{X} \times \mathcal{X})$. However, the BMO space and the duality theory of Hardy space $H^p(\mathcal{X} \times \mathcal{X})$ remain an open question on $\mathcal{X} \times \mathcal{X}$. The main purpose of this paper is to establish such a duality theory. We will achieve this goal by introducing the Carleson measure space CMO^p for $p \leq 1$ and sufficiently close to 1, on product spaces $\mathcal{X} \times \mathcal{X}$ of homogeneous type and prove that it is the dual of the product Hardy space $H^p(\mathcal{X} \times \mathcal{X})$ mentioned above.

To be more specific, let $p_0 < 1$ and $p_0 < p \leq 1$. We will give the precise value of p_0 in Section 5. Then the main result of this paper is as follows.

Theorem 1.2. *For $p_0 < p \leq 1$, $(H^p(\mathcal{X} \times \mathcal{X}))' = \mathrm{CMO}^p(\mathcal{X} \times \mathcal{X})$. Namely, the dual space for $H^p(\mathcal{X} \times \mathcal{X})$ is $\mathrm{CMO}^p(\mathcal{X} \times \mathcal{X})$.*

In particular, when $p = 1$ we obtain the duality of H^1 with BMO on product spaces of homogeneous type.

Since the key tool that Chang and R. Fefferman used in establishing the dual space $\mathrm{BMO}(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ of Hardy space $H^1(\mathbb{R}^2_+ \times \mathbb{R}^2_+)$ is the bi-Hilbert transform, it is extremely difficult to apply their method to work on $\mathcal{X} \times \mathcal{X}$. Therefore, we will follow the ideas recently developed by the first and third authors in [13] and [14]. The basic scheme is as follows.

First, the discrete Calderón reproducing formula on the product of two homogeneous spaces will play a role in defining the Hardy spaces $H^p(\mathcal{X} \times \mathcal{X})$. The continuous version of such a formula for space of homogeneous type was given by the first author with E. Sawyer in [18], see [16] for the discrete Calderón's identity. One of the essential parts of our paper is to verify the dual spaces $\mathrm{CMO}^p(\mathcal{X} \times \mathcal{X})$ is well defined. This is accomplished by using the Min-Max comparison principle involving the CMO^p norms. We will establish the Min-Max comparison principle by using the discrete Calderón reproducing formula on $\mathcal{X} \times \mathcal{X}$ (See Theorem 3.2 and its proof in Section 3).

Second, we introduce the product sequence spaces s^p and c^p and prove that the dual of s^p is c^p, *i.e.*, $(s^p)' = c^p$. Spaces s^p and c^p in one parameter case of $\mathbb{R}^n$ were introduced and studied by Frazier and Jawerth in [9]. Since the main tools they used are the Fourier transform and the estimates on distribution functions, it seems difficult to carry out their methods to product sequence spaces. We will give a constructive proof which applies to the product sequence spaces of two homogeneous spaces.

Third, we prove that $H^p(\mathcal{X} \times \mathcal{X})$ can be lifted to s^p and s^p can be projected to $H^p(\mathcal{X} \times \mathcal{X})$ and that the combination of the lifting and projection operators equals the identity operator on $H^p(\mathcal{X} \times \mathcal{X})$. Similar results hold for $\mathrm{CMO}^p(\mathcal{X} \times \mathcal{X})$ and c^p.

Then, as a consequence, Theorem 1.2 follows from the second and third steps.

Since spaces of homogeneous type include compact Lie groups, C^∞ manifolds with doubling volume measures for geodesic balls, Carnot-Caratheodory spaces, nilpotent Lie groups such as the Heisenberg group, the d-sets in $\mathbb{R}^n$, and many other cases, so our result includes the duality theory of Hardy spaces in these cases.

A brief description of the contents of this paper is as follows. In Section 2, we provide some preliminaries on spaces of homogeneous type and recall the product Hardy space $H^p(\mathcal{X} \times \mathcal{X})$. In Section 3, we give the precise definition of $\mathrm{CMO}^p(\mathcal{X} \times \mathcal{X})$ and establish the Min-Max comparison principle for such spaces. In section 4, we develop the product sequence spaces s^p and c^p and obtain the duality of s^p with c^p by a constructive proof. Finally, Theorem 1.2 will be showed in Section 5.

2. Preliminaries

We begin by recalling some necessary definitions and notation on spaces of homogeneous type.

A quasi-metric ρ on a set $\mathcal{X}$ is a function $\rho : \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ satisfying that:

1. $\rho(x, y) = 0$ if and only if $x = y$;
2. $\rho(x, y) = \rho(y, x)$ for all $x, y \in \mathcal{X}$;
3. there exists a constant $A \in [1, \infty)$ such that for all x, y and $z \in \mathcal{X}$,

$$\rho(x, y) \leq A[\rho(x, z) + \rho(z, y)].$$

Any quasi-metric defines a topology, for which the balls $B(x, r) = \{y \in \mathcal{X} : \rho(x, y) < r\}$ form a base. However, the balls themselves need not be open when $A > 1$.

The following spaces of homogeneous type are variants of those introduced by Coifman and Weiss in [4].

Definition 2.1. Let $\theta \in (0, 1]$. A space of homogeneous type, $(\mathcal{X}, \rho, \mu)_\theta$, is a set $\mathcal{X}$ together with a quasi-metric ρ and a nonnegative Borel regular measure μ on $\mathcal{X}$ and there exists a constant $C_0 > 0$ such that for all $0 < r < \operatorname{diam}\mathcal{X}$ and all $x, x^{'}, y \in \mathcal{X}$,

$$\mu(B(x, r)) \sim r, \tag{2.1}$$

$$| \rho(x, y) - \rho(x^{'}, y) | \leq C_0 \rho(x, x^{'})^\theta [\rho(x, y) + \rho(x^{'}, y)]^{1-\theta} . \tag{2.2}$$

Through out the paper, we assume that $\mu(\mathcal{X}) = \infty$.

We first recall the following construction given independently by Christ in [1] and by Sawyer-Wheeden in [28], which provides an analogue of the grid of Euclidean dyadic cubes on spaces of homogeneous type. We will follow the statement given in [1].

Lemma 2.2. *Let $(\mathcal{X}, \rho, \mu)$ be a space of homogeneous type, then, there exists a collection $\{Q_\alpha^k \subset \mathcal{X} : k \in \mathbb{Z}, \alpha \in I_k\}$ of open subsets, where I_k is some index set, and constant $\delta = 1/2$, and $C_1, C_2 > 0$, such that*

(i) $\mu(\mathcal{X} \setminus \bigcup_\alpha Q_\alpha^k) = 0$ *for each fixed k and $Q_\alpha^k \bigcap Q_\beta^k = \emptyset$ if $\alpha \neq \beta$;*
(ii) *for any α, β, k, l with $l \geq k$, either $Q_\beta^l \subset Q_\alpha^k$ or $Q_\beta^l \bigcap Q_\alpha^k = \emptyset$;*
(iii) *for each (k, α) and each $l < k$ there is a unique β such that $Q_\alpha^k \subset Q_\beta^l$;*
(iv) $\operatorname{diam}(Q_\alpha^k) \leq C_1 \left(\frac{1}{2}\right)^k$*;*
(v) *each Q_α^k contains some ball $B\left(z_\alpha^k, C_2 \left(\frac{1}{2}\right)^k\right)$, where $z_\alpha^k \in \mathcal{X}$.*

In fact, we can think of Q_α^k as being a dyadic cube with diameter roughly $(\frac{1}{2})^k$ centered at z_α^k. As a result, we consider CQ_α^k to be the dyadic cube with the same center as Q_α^k and diameter $C\mathrm{diam}(Q_\alpha^k)$. In the following, for $k \in \mathbb{Z}$ and $\tau \in I_k$, we will denote by $Q_\tau^{k,v}$, $v = 1, 2, \cdots, N(k, \tau)$, the set of all cubes $Q_\tau^{k+J} \subset Q_\tau^k$, where J is a fixed large positive integer, and denote by $y_\tau^{k,v}$ a point in $Q_\tau^{k,v}$.

Now we introduce the approximation to identity on $\mathcal{X}$.

Definition 2.3 ([15]). Let $\theta > 0$ be given in Definition 2.1. A sequence $\{S_k\}_{k\in\mathbb{Z}}$ of operators is said to be an approximation to identity of order $\varepsilon \in (0, \theta]$, if there exists a constant $C > 0$ such that for all $k \in \mathbb{Z}$ and all $x, x', y, y' \in \mathcal{X}$, $S_k(x, y)$, the kernel of S_k, is a function from $\mathcal{X} \times \mathcal{X}$ into $\mathbb{C}$ satisfying

$$|S_k(x, y)| \le C\frac{2^{-k\varepsilon}}{(2^{-k} + \rho(x, y))^{1+\varepsilon}}; \tag{2.3}$$

$$|S_k(x, y) - S_k(x', y)| \le C\left(\frac{\rho(x, x')}{2^{-k} + \rho(x, y)}\right)^{\varepsilon}\frac{2^{-k\varepsilon}}{(2^{-k} + \rho(x, y))^{1+\varepsilon}} \quad \text{for } \rho(x, x') \le \frac{1}{2A}(2^{-k} + \rho(x, y)); \tag{2.4}$$

$$|S_k(x, y) - S_k(x, y')| \le C\left(\frac{\rho(y, y')}{2^{-k} + \rho(x, y)}\right)^{\varepsilon}\frac{2^{-k\varepsilon}}{(2^{-k} + \rho(x, y))^{1+\varepsilon}} \quad \text{for } \rho(y, y') \le \frac{1}{2A}(2^{-k} + \rho(x, y)); \tag{2.5}$$

$$\begin{aligned}|S_k(x, y) - S_k(x, y') - S_k(x', y) + S_k(x', y')| &\le C\left(\frac{\rho(x, x')}{2^{-k} + \rho(x, y)}\right)^{\varepsilon}\\ &\quad\times\left(\frac{\rho(y, y')}{2^{-k} + \rho(x, y)}\right)^{\varepsilon}\frac{2^{-k\varepsilon}}{(2^{-k} + \rho(x, y))^{1+\varepsilon}}\\ &\quad\text{for } \rho(x, x'),\ \rho(y, y') \le \frac{1}{2A}(2^{-k} + \rho(x, y));\end{aligned} \tag{2.6}$$

$$\int_{\mathcal{X}} S_k(x, y)d\mu(y) = \int_{\mathcal{X}} S_k(x, y)d\mu(x) = 1. \tag{2.7}$$

We remark that by a construction of Coifman, in what follows, we will use an approximation to the identity of order ε with $\varepsilon = \theta$.

To recall the definition of $H^p(\mathcal{X} \times \mathcal{X})$, we need to introduce the space of test functions on $\mathcal{X} \times \mathcal{X}$. Before we do this, we shall introduce the space of test functions on $\mathcal{X}$.

Definition 2.4. Fix $\beta > 0, \gamma > 0$. A function f defined on $\mathcal{X}$ is said to be a test function of type (β, γ) centered at x_0 with width $r > 0$ if f satisfies the following conditions:

(i) $|f(x)| \le C\dfrac{r^{\gamma_1}}{(r + \rho(x, x_0))^{1+\gamma}}$;

(ii) $|f(x) - f(x')| \le C\left(\frac{\rho(x,x')}{r+\rho(x,x_0)}\right)^{\beta} \cdot \frac{r^{\gamma}}{(r+\rho(x,x_0))^{1+\gamma}}$ for $\rho(x,x') \le \frac{1}{2A}(r+\rho(x,x_0))$;

(iii) $\int_{\mathcal{X}} f(x)\mathrm{d}\mu(x) = 0$.

If f is a test function of type (β, γ) centered at $x_0 \in \mathcal{X}$ with width $r > 0$, then we write $f \in \mathcal{G}(x_0, r, \beta, \gamma)$ and define the norm of f by

$$||f||_{\mathcal{G}(x_0,r,\beta,\gamma)} = \inf\{C : \text{(i), (ii) hold}\}.$$

We now introduce the space of test functions on $\mathcal{X} \times \mathcal{X}$.

Definition 2.5 ([17]). For $i = 1, 2$, fix $\gamma_i > 0$ and $\beta_i > 0$. A function f defined on $\mathcal{X} \times \mathcal{X}$ is said to be a test function of type $(\beta_1, \beta_2; \gamma_1, \gamma_2)$ centered at $(x_0, y_0) \in \mathcal{X} \times \mathcal{X}$ with width $r_1, r_2 > 0$ if for any fixed $y \in \mathcal{X}$, $f(x, y)$, as a function of x, is a test function of type (β_1, γ_1) centered at $x_0 \in \mathcal{X}$ with width $r_1 > 0$ and satisfies the following conditions:

$$(1)\quad ||f(\cdot,y)||_{\mathcal{G}(x_0,r_1,\beta_1,\gamma_1)} \le C\frac{r_2^{\gamma_2}}{(r_2+\rho(y,y_0))^{1+\gamma_2}};$$

$$(2)\quad ||f(\cdot,y)-f(\cdot,y')||_{\mathcal{G}(x_0,r_1,\beta_1,\gamma_1)} \le C\left(\frac{\rho(y,y')}{r_2+\rho(y,y_0)}\right)^{\beta_2}\frac{r_2^{\gamma_2}}{(r_2+\rho(y,y_0))^{1+\gamma_2}}$$

$$\text{for}\ \ \rho(y,y') \le \frac{1}{2A}[r_2+\rho(y,y_0)].$$

Similarly, for any fixed $x \in \mathcal{X}$, $f(x, y)$, as a function of y, is a test function of type (β_2, γ_2) centered at $y_0 \in \mathcal{X}$ with width $r_2 > 0$ and satisfies the same conditions of (1) and (2) above by interchanging the role of x and y, namely,

$$(3)\quad ||f(x,\cdot)||_{\mathcal{G}(y_0,r_2,\beta_2,\gamma_2)} \le C\frac{r_1^{\gamma_1}}{(r_1+\rho(x,x_0))^{1+\gamma_1}};$$

$$(4)\quad ||f(x,\cdot)-f(x',\cdot)||_{\mathcal{G}(y_0,r_2,\beta_2,\gamma_2)} \le C\left(\frac{\rho(x,x')}{r_1+\rho(x,x_0)}\right)^{\beta_1}\frac{r_1^{\gamma_1}}{(r_1+\rho(x,x_0))^{1+\gamma_1}}$$

$$\text{for}\ \ \rho(x,x') \le \frac{1}{2A}[r_1+\rho(x,x_0)].$$

If f is a test function of type $(\beta_1, \beta_2; \gamma_1, \gamma_2)$ centered at $(x_0, y_0) \in \mathcal{X} \times \mathcal{X}$ with width $r_1, r_2 > 0$, then we write $f \in \mathcal{G}(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ and define the norm of f by $\|f\|_{\mathcal{G}(x_0,y_0;r_1,r_2;\beta_1,\beta_2;\gamma_1,\gamma_2)} = \inf\{C : \text{(1), (2), (3) and (4) hold}\}$.

We denote by $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ the class of $\mathcal{G}(x_0, y_0; r_1, r_2; \beta_1, \beta_2; \gamma_1, \gamma_2)$ with $r_1 = r_2 = 1$ for fixed $(x_0, y_0) \in \mathcal{X} \times \mathcal{X}$. It is easy to see that $\mathcal{G}(x_1, y_1; r_1, r_2;$

$\beta_1, \beta_2; \gamma_1, \gamma_2) = \mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ with an equivalent norm for all $(x_1, y_1) \in \mathcal{X} \times \mathcal{X}$. We can easily check that the space $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ is a Banach space with respect to the norm in $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$.

For any $0 < \beta_1, \beta_2, \gamma_1, \gamma_2 < \theta$, the space $\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ is defined to be the completion of the space $\mathcal{G}(\theta, \theta; \theta, \theta)$ in $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ when $0 < \beta_1, \gamma_1 \leq \varepsilon_1$ and $0 < \beta_2, \gamma_2 \leq \varepsilon_2$. We define $\|f\|_{\overset{\circ}{\mathcal{G}}(\beta_1,\beta_2;\gamma_1,\gamma_2)} = \|f\|_{\mathcal{G}(\beta_1,\beta_2;\gamma_1,\gamma_2)}$. Then, obviously, $\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ is a Banach space. Hence we can define the dual space $(\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$ to be the set of all linear functionals $\mathcal{L}$ from $\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ to $\mathbb{C}$ with the property that there exists $C \geq 0$ such that for all $f \in \overset{\circ}{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)$,

$$|\mathcal{L}(f)| \leq C\|f\|_{\overset{\circ}{\mathcal{G}}(\beta_1,\beta_2;\gamma_1,\gamma_2)}.$$

In [15], to define the product Hardy space $H^p(\mathcal{X} \times \mathcal{X})$, they first introduced the Littlewood-Paley-Stein square function on $\mathcal{X} \times \mathcal{X}$ by

$$g(f)(x_1, x_2) = \left\{ \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} |D_{k_1} D_{k_2}(f)(x_1, x_2)|^2 \right\}^{1/2},$$

where $D_{k_i} = S_{k_i} - S_{k_i-1}$ with S_{k_i} being an approximation to the identity for $i = 1, 2$, and proved that $\|g(f)\|_p \approx \|f\|_p$ for $1 < p < \infty$. Then $H^p(\mathcal{X} \times \mathcal{X})$ is defined as follows.

Definition 2.6. Let $\{S_{k_i}\}_{k_i \in \mathbb{Z}}$ be an approximation to the identity of order $\theta, i = 1, 2$. Set $D_{k_i} = S_{k_i} - S_{k_{i-1}}$ for all $k_i \in \mathbb{Z}$. For $\frac{1}{1+\theta} < p \leq 1$ and $\frac{1}{p} - 1 < \beta_i, \gamma_i < \theta$, the Hardy space $H^p(\mathcal{X} \times \mathcal{X})$ is defined to be the set of all $f \in \left(\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2, \gamma_1, \gamma_2)\right)'$ such that $\|g(f)\|_{L^p(\mathcal{X}\times\mathcal{X})} < \infty$, and we define

$$\|f\|_{H^p(\mathcal{X}\times\mathcal{X})} = \|g(f)\|_{L^p(\mathcal{X}\times\mathcal{X})}.$$

In order to verify that the definition of $H^p(\mathcal{X} \times \mathcal{X})$ is independent of the choice of approximations to the identity, the following Min-Max comparison principle for $H^p(\mathcal{X} \times \mathcal{X})$ was proved in [15].

Lemma 2.7. *Let all the notation be the same as in Definition* 2.6. *Moreover, for* $i = 1, 2$, *let* $\{P_{k_i}\}_{k_i \in \mathbb{Z}}$ *be another approximation to the identity of order* θ *and* $E_{k_i} = P_{k_i} - P_{k_i-1}$ *for all* $k_i \in \mathbb{Z}$. *And let* $\{Q_{\tau_i}^{k_i,v_i} : k_i \in \mathbb{Z}, \tau_i \in I_{k_i}, v_i = 1, \cdots, N(k_i, \tau_i)\}$ *and* $\{Q_{\tau_i'}^{k_i',v_i'} : k_i' \in \mathbb{Z}, \tau_i' \in I_{k_i'}, v_i' = 1, \cdots, N(k_i', \tau_i')\}$ *be sets of dyadic cubes of* $\mathcal{X}$ *as mentioned in Lemma* 2.2. *Then for* $\frac{1}{1+\theta} < p < \infty$ *there is a constant* $C > 0$

such that for all $f \in \left(\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2, \gamma_1, \gamma_2)\right)'$ *with* $\frac{1}{r}p - 1 < \beta_i, \gamma_i < \theta$,

$$\Bigg\|\Bigg\{\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}\sum_{v_2=1}^{N(k_2,\tau_2)}\sup_{z_1\in Q_{\tau_1}^{k_1,v_1}, z_2\in Q_{\tau_2}^{k_2,v_2}}|D_{k_1}D_{k_2}(f)(z_1,z_2)|^2$$

$$\times\chi_{Q_{\tau_1}^{k_1,v_1}}(\cdot)\chi_{Q_{\tau_2}^{k_2,v_2}}(\cdot)\Bigg\}^{1/2}\Bigg\|_{L^p(\mathcal{X}\times\mathcal{X})}$$

$$\le C\Bigg\|\Bigg\{\sum_{k_1'=-\infty}^{\infty}\sum_{k_2'=-\infty}^{\infty}\sum_{\tau_1'\in I_{k_1'}}\sum_{\tau_2'\in I_{k_2'}}\sum_{v_1'=1}^{N(k_1',\tau_1')}\sum_{v_2'=1}^{N(k_2',\tau_2')}\inf_{z_1\in Q_{\tau_1'}^{k_1',v_1'}, z_2\in Q_{\tau_2'}^{k_2',v_2'}}|E_{k_1'}E_{k_2'}(f)(z_1,z_2)|^2$$

$$\times\chi_{Q_{\tau_1'}^{k_1',v_1'}}(\cdot)\chi_{Q_{\tau_2'}^{k_2',v_2'}}(\cdot)\Bigg\}^{1/2}\Bigg\|_{L^p(\mathcal{X}\times\mathcal{X})}.$$

To prove Lemma 2.7, in [15] they established the following discrete Calderón reproducing formula on $\mathcal{X}\times\mathcal{X}$.

Lemma 2.8. *Let all the notation be the same as in Definition* 2.6. *Then there are families of linear operators* $\{\tilde{D}_{k_i}\}_{k_i\in\mathbb{Z}}$ *and* $\{\bar{D}_{k_i}\}_{k_i\in\mathbb{Z}}$ *such that for all* $f \in \overset{\circ}{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ *with* $\beta_i, \gamma_i \in (0, \theta)$,

$$\begin{aligned} f(x_1,x_2) &= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}\sum_{v_2=1}^{N(k_2,\tau_2)}\mu(Q_{\tau_1}^{k_1,v_1})\mu(Q_{\tau_2}^{k_2,v_2}) \\ &\quad\times\tilde{D}_{k_1}\tilde{D}_{k_2}(x_1,x_2,y_1,y_2)D_{k_1}D_{k_2}(f)(y_1,y_2) \\ &= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}\sum_{v_2=1}^{N(k_2,\tau_2)}\mu(Q_{\tau_1}^{k_1,v_1})\mu(Q_{\tau_2}^{k_2,v_2}) \\ &\quad\times D_{k_1}D_{k_2}(x_1,x_2,y_1,y_2)\bar{D}_{k_1}\bar{D}_{k_2}(f)(y_1,y_2), \end{aligned} \tag{2.8}$$

where y_i *is any point in* $Q_{\tau_i}^{k_i,v_i}$ *for* $i = 1, 2$ *and the series converges in both the norm of* $\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ *and the norm of* $L^p(\mathcal{X}\times\mathcal{X})$ *with* $1 < p < \infty$. *Moreover,* $\tilde{D}_{k_i}(x, y)$, *the kernel of* $\tilde{D}_{k_i}$ *satisfies the conditions* (2.3) *and* (2.4) *of Definition* 2.3 *with* θ *replaced by any* $\theta' < \theta$ *and*

$$\int_{\mathcal{X}}\tilde{D}_{k_i}(x,y)\mathrm{d}\mu(y) = \int_{\mathcal{X}}\tilde{D}_{k_i}(x,y)\mathrm{d}\mu(x) = 0; \tag{2.9}$$

similarly, $\bar{D}_{k_i}(x, y)$*, the kernel of* $\bar{D}_{k_i}$ *satisfies the conditions* (2.3) *and* (2.5) *of Definition* 2.3 *with* θ *replaced by any* $\theta' < \theta$ *and* (2.9)*, for all* $k_i \in \mathbb{Z}$ *with* $i = 1, 2$.

For any $f \in (\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$, (2.8) *also holds in* $(\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$.

In this paper, we use notation $a \sim b$ and $b \lesssim c$ for $a, b, c \geq 0$ to mean that there exists $C > 0$, so that $a/C \leq b \leq C \cdot a$ and $b \leq C \cdot c$, respectively. The value of C varies from one usage to the next, but it depends only on constants quantified in the relevant preceding hypotheses. We use $a \vee b$ and $a \wedge b$ to mean $\max(a, b)$ and $\min(a, b)$ for any $a, b \in \mathbb{R}$, respectively.

3. $\mathrm{CMO}^p(\mathcal{X} \times \mathcal{X})$ and the Min-Max comparison principle

To characterize the dual space of $H^p(\mathcal{X} \times \mathcal{X})$, we introduce the Carleson measure space CMO^p on $\mathcal{X} \times \mathcal{X}$, which is motivated by ideas of Chang and R. Fefferman [2].

Definition 3.1. Let $i = 1, 2,\ 0 < \beta_i,\ \gamma_i < \theta$, $\{S_{k_i}\}_{k_i \in \mathbb{Z}}$ be an approximation to the identity of order θ. Set $D_{k_i} = S_{k_i} - S_{k_i - 1}$ for all $k_i \in \mathbb{Z}$. The Carleson measure space $\mathrm{CMO}^p(\mathcal{X} \times \mathcal{X})$ is defined to be the set of all $f \in (\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2, \gamma_1, \gamma_2))'$ such that

$$\|f\|_{\mathrm{CMO}^p(\mathcal{X}\times\mathcal{X})} = \sup_{\Omega} \Bigg(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \int_{\Omega} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{v_1=1}^{N(k_1,\tau_1)} \sum_{v_2=1}^{N(k_2,\tau_2)} \times \chi_{\{Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2} \subset \Omega\}}(k_1, k_2, \tau_1, \tau_2, v_1, v_2) \big| D_{k_1} D_{k_2}(f)(x_1, x_2) \big|^2 \times \chi_{Q_{\tau_1}^{k_1,v_1}}(x_1) \chi_{Q_{\tau_2}^{k_2,v_2}}(x_2) \mathrm{d}\mu(x_1) \mathrm{d}\mu(x_2) \Bigg)^{1/2} < \infty, \tag{3.1}$$

where the sup is taken over all open sets Ω in $\mathcal{X} \times \mathcal{X}$ with finite measures.

In order to verify that the definition of $\mathrm{CMO}^p(\mathcal{X} \times \mathcal{X})$ is independent of the choice of the approximations to identity, we establish the Min-Max comparison principle involving the CMO^p norm. To this end and for the sake of simplicity, we first give some notation as follows.

We write $R = Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2}$, $R' = Q_{\tau_1'}^{k_1',v_1'} \times Q_{\tau_2'}^{k_2',v_2'}$;

$$\sum_{R \subseteq \Omega} = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{v_1=1}^{N(k_1,\tau_1)} \sum_{v_2=1}^{N(k_2,\tau_2)} \chi_{\{Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2} \subset \Omega\}}(k_1, k_2, \tau_1, \tau_2, v_1, v_2);$$

$$\sum_{R' \subseteq \Omega} = \sum_{k_1'=-\infty}^{\infty} \sum_{k_2'=-\infty}^{\infty} \sum_{\tau_1' \in I_{k_1'}} \sum_{\tau_2' \in I_{k_2'}} \sum_{v_1'=1}^{N(k_1',\tau_1')} \sum_{v_2'=1}^{N(k_2',\tau_2')} \chi_{\{Q_{\tau_1'}^{k_1',v_1'} \times Q_{\tau_2'}^{k_2',v_2'} \subset \Omega\}}(k_1', k_2', \tau_1', \tau_2', v_1', v_2');$$

$$\sum_{R'} = \sum_{k_1'=-\infty}^{\infty} \sum_{k_2'=-\infty}^{\infty} \sum_{\tau_1' \in I_{k_1'}} \sum_{\tau_2' \in I_{k_2'}} \sum_{v_1'=1}^{N(k_1',\tau_1')} \sum_{v_2'=1}^{N(k_2',\tau_2')};$$

$$\mu(R) = \mu(Q_{\tau_1}^{k_1,v_1})\mu(Q_{\tau_2}^{k_2,v_2}); \qquad \mu(R') = \mu(Q_{\tau_1'}^{k_1',v_1'})\mu(Q_{\tau_2'}^{k_2',v_2'});$$

$$r(R,R') = \left(\frac{\mu(Q_{\tau_1}^{k_1,v_1})}{\mu(Q_{\tau_1'}^{k_1',v_1'})} \wedge \frac{\mu(Q_{\tau_1'}^{k_1',v_1'})}{\mu(Q_{\tau_1}^{k_1,v_1})} \right)^{1+\varepsilon'} \left(\frac{\mu(Q_{\tau_2}^{k_2,v_2})}{\mu(Q_{\tau_2'}^{k_2',v_2'})} \wedge \frac{\mu(Q_{\tau_2'}^{k_2',v_2'})}{\mu(Q_{\tau_2}^{k_2,v_2})} \right)^{1+\varepsilon'};$$

$$v(R,R') = \left(\mu(Q_{\tau_1}^{k_1,v_1}) \vee \mu(Q_{\tau_1'}^{k_1',v_1'}) \right) \left(\mu(Q_{\tau_2}^{k_2,v_2}) \vee \mu(Q_{\tau_2'}^{k_2',v_2'}) \right);$$

$$P(R,R') = \left(1 + \frac{\mathrm{dist}(Q_{\tau_1}^{k_1,v_1}, Q_{\tau_1'}^{k_1',v_1'})}{\mu(Q_{\tau_1}^{k_1,v_1}) \vee \mu(Q_{\tau_1'}^{k_1',v_1'})} \right)^{-(1+\theta')} \cdot \left(1 + \frac{\mathrm{dist}(Q_{\tau_2}^{k_2,v_2}, Q_{\tau_2'}^{k_2',v_2'})}{\mu(Q_{\tau_2}^{k_2,v_2}) \vee \mu(Q_{\tau_2'}^{k_2',v_2'})} \right)^{-(1+\theta')};$$

$$S_R = \sup_{x_1 \in Q_{\tau_1}^{k_1,v_1}, x_2 \in Q_{\tau_2}^{k_2,v_2}} |D_{k_1} D_{k_2}(f)(x_1,x_2)|^2;$$

$$T_{R'} = \inf_{y_1' \in Q_{\tau_1'}^{k_1',v_1'}, y_2' \in Q_{\tau_2'}^{k_2',v_2'}} \left| D_{k_1'} D_{k_2'}(f)(y_1',y_2') \right|^2.$$

Now we state the main theorem of this section as follows.

Theorem 3.2. *Let all the notations be the same as above. For* $\frac{2}{2+\theta} < p \le 1$ *all* $f \in \mathrm{CMO}^p(\mathcal{X} \times \mathcal{X})$*,*

$$\sup_{\Omega} \left(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R \subseteq \Omega} \mu(R) S_R \right)^{1/2} \lesssim \sup_{\Omega} \left(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R' \subseteq \Omega} \mu(R') T_{R'} \right)^{1/2}, \quad (3.2)$$

where Ω *ranges over the open sets in* $\mathcal{X} \times \mathcal{X}$ *with finite measures.*

Proof. First, for each p satisfying $\frac{2}{2+\theta} < p \le 1$, we choose $\epsilon \in (0,\theta)$ such that $\frac{2}{2+\theta} < \frac{2}{2+\epsilon} < p \le 1$.

Then, for any $f \in \mathrm{CMO}^p(\mathcal{X} \times \mathcal{X})$, it is easy to see that the right-hand side of (3.2) is finite and can be controlled by $C\|f\|_{\mathrm{CMO}^p(\mathcal{X}\times\mathcal{X})}$.

To prove (3.2), we need to show that for any open set $\Omega \in \mathcal{X} \times \mathcal{X}$ with finite measure, the following inequality holds.

$$\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R \subseteq \Omega} \mu(R) S_R \lesssim \sup_{\overline{\Omega}} \frac{1}{\mu(\overline{\Omega})^{\frac{2}{p}-1}} \sum_{R' \subset \overline{\Omega}} \mu(R') T_{R'}, \tag{3.3}$$

where $\overline{\Omega}$ ranges over all open sets in $\mathcal{X} \times \mathcal{X}$ with finite measures.

To begin with, for each fix Ω, we first consider the estimate of the term S_R in the left-hand side of (3.3) for every $R = Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2} \subset \Omega$. To estimate this, we recall the almost orthogonal property of $D_{k_i} \tilde{D}_{k_i'}$ for $i = 1, 2$, namely, for any $0 < \varepsilon' < \theta'$

$$|D_{k_i} \tilde{D}_{k_i'}(x, y)| \le C 2^{-|k_i - k_i'|\varepsilon'} \frac{2^{-(k_i \wedge k_i')\theta'}}{(2^{-(k_i \wedge k_i')} + \rho(x, y))^{1+\theta'}}$$

(see [18] for more details).

Now for any $(x_1, x_2) \in R$, using the discrete Calderón reproducing formula (2.8), the above almost orthogonal property and the Hölder inequality, we can obtain that

$$\begin{aligned}
|D_{k_1} D_{k_2}(f)(x_1,x_2)|^2 \lesssim & \Bigg| \sum_{k_1'=-\infty}^{\infty} \sum_{k_2'=-\infty}^{\infty} \sum_{\tau_1' \in I_{k_1'}} \sum_{\tau_2' \in I_{k_2'}} \sum_{v_1'=1}^{N(k_1',\tau_1')} \sum_{v_2'=1}^{N(k_2',\tau_2')} \mu(Q_{\tau_1'}^{k_1',v_1'}) \mu(Q_{\tau_2'}^{k_2',v_2'}) \\
& \times D_{k_1} D_{k_2} \widetilde{D}_{k_1} \widetilde{D}_{k_2}(x_1, x_2, y_1', y_2') D_{k_1'} D_{k_2'}(f)(y_1', y_2') \Bigg|^2 \\
\lesssim & \sum_{k_1'=-\infty}^{\infty} \sum_{k_2'=-\infty}^{\infty} \sum_{\tau_1' \in I_{k_1'}} \sum_{\tau_2' \in I_{k_2'}} \sum_{v_1'=1}^{N(k_1',\tau_1')} \\
& \cdot \sum_{v_2'=1}^{N(k_2',\tau_2')} 2^{-|k_1 - k_1'|\varepsilon'} 2^{-|k_2 - k_2'|\varepsilon'} \mu(Q_{\tau_1'}^{k_1',v_1'}) \mu(Q_{\tau_2'}^{k_2',v_2'}) \\
& \times \frac{2^{-(k_1 \wedge k_1')\theta'}}{(2^{-(k_1 \wedge k_1')} + \rho(y_1, y_1'))^{1+\theta'}} \\
& \cdot \frac{2^{-(k_2 \wedge k_2')\theta'}}{(2^{-(k_2 \wedge k_2')} + \rho(y_2, y_2'))^{1+\theta'}} |D_{k_1'} D_{k_2'}(f)(y_1', y_2')|^2,
\end{aligned} \tag{3.4}$$

where ε' is chosen to satisfy $\epsilon < \varepsilon' < \theta' < \theta$, and for $i = 1, 2$, y_i is the center of $Q_{\tau_i}^{k_i,v_i}$ and y_i' is any point in $Q_{\tau_i'}^{k_i',v_i'}$, respectively.

From (2.1) and Lemma 2.2, we know that each dyadic cube Q^k_α satisfies that

$$\mu(Q^k_\alpha)\sim 2^{-k},\text{ which yields } 2^{-|k_i-k_i'|}\sim \frac{\mu(Q^{k_i,v_i}_{\tau_i})}{\mu(Q^{k_i',v_i'}_{\tau_i'})}\wedge\frac{\mu(Q^{k_i',v_i'}_{\tau_i'})}{\mu(Q^{k_i,v_i}_{\tau_i})}$$

$$\text{and } 2^{-(k_i\wedge k_i')}\sim(\mu(Q^{k_i,v_i}_{\tau_i})\vee\mu(Q^{k_i',v_i'}_{\tau_i'}))\text{ for } i=1,2.$$

Also note that $\rho(y_i,y_i')\geq \operatorname{dist}(Q^{k_i,v_i}_{\tau_i},Q^{k_i',v_i'}_{\tau_i'})$. Since the last inequality of (3.4) is independent of (x_1,x_2), then combining the above estimates, it follows that

$$\begin{aligned}
S_R\lesssim&\sum_{k_1'=-\infty}^{\infty}\sum_{k_2'=-\infty}^{\infty}\sum_{\tau_1'\in I_{k_1'}}\sum_{\tau_2'\in I_{k_2'}}\sum_{v_1'=1}^{N(k_1',\tau_1')}\sum_{v_2'=1}^{N(k_2',\tau_2')}\mu(Q^{k_1',v_1'}_{\tau_1'})\mu(Q^{k_2',v_2'}_{\tau_2'})\\
&\times\left(\frac{\mu(Q^{k_1,v_1}_{\tau_1})}{\mu(Q^{k_1',v_1'}_{\tau_1'})}\wedge\frac{\mu(Q^{k_1',v_1'}_{\tau_1'})}{\mu(Q^{k_1,v_1}_{\tau_1})}\right)^{\varepsilon'}\\
&\cdot\frac{(\mu(Q^{k_1,v_1}_{\tau_1})\vee\mu(Q^{k_1',v_1'}_{\tau_1'}))^{\theta'}}{(\mu(Q^{k_1,v_1}_{\tau_1})\vee\mu(Q^{k_1',v_1'}_{\tau_1'})+\operatorname{dist}(Q^{k_1,v_1}_{\tau_1},Q^{k_1',v_1'}_{\tau_1'}))^{1+\theta'}}\\
&\times\left(\frac{\mu(Q^{k_2,v_2}_{\tau_2})}{\mu(Q^{k_2',v_2'}_{\tau_2'})}\wedge\frac{\mu(Q^{k_2',v_2'}_{\tau_2'})}{\mu(Q^{k_2,v_2}_{\tau_2})}\right)^{\varepsilon'}\\
&\cdot\frac{(\mu(Q^{k_2,v_2}_{\tau_2})\vee\mu(Q^{k_2',v_2'}_{\tau_2'}))^{\theta'}}{(\mu(Q^{k_2,v_2}_{\tau_2})\vee\mu(Q^{k_2',v_2'}_{\tau_2'})+\operatorname{dist}(Q^{k_2,v_2}_{\tau_2},Q^{k_2',v_2'}_{\tau_2'}))^{1+\theta'}}\cdot T_{R'}.
\end{aligned}\tag{3.5}$$

Now combining (3.5) and the following equality

$$\prod_{i=1}^{2}\mu(Q^{k_i,v_i}_{\tau_i})\mu(Q^{k_i',v_i'}_{\tau_i'})=\prod_{i=1}^{2}\Big(\mu(Q^{k_i,v_i}_{\tau_i})\vee\mu(Q^{k_i',v_i'}_{\tau_i'})\Big)^2\left(\frac{\mu(Q^{k_i,v_i}_{\tau_i})}{\mu(Q^{k_i',v_i'}_{\tau_i'})}\wedge\frac{\mu(Q^{k_i',v_i'}_{\tau_i'})}{\mu(Q^{k_i,v_i}_{\tau_i})}\right),$$

we obtain that the left-hand side of (3.3), namely, $\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R\subset\Omega} \mu(R)S_R$, is bounded by

$$\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R\subset\Omega} \sum_{R'} v(R, R')r(R, R')P(R, R')T_{R'}. \tag{3.6}$$

Thus, to finish the proof of the theorem, we need to prove that (3.6) can be controlled by

$$\sup_{\bar{\Omega}} \frac{1}{\mu(\bar{\Omega})^{\frac{2}{p}-1}} \sum_{R'\subset\bar{\Omega}} \mu(R')T_{R'}, \tag{3.7}$$

where $\bar{\Omega}$ ranges over the open sets in $\mathcal{X} \times \mathcal{X}$ with finite measures.

We first point out that the terms $v(R, R')$ and $P(R, R')$ characterize the geometrical properties between R and R'. Namely, when the difference of the sizes of R and R' grows bigger, $v(R, R')$ becomes smaller; when the distance between R and R' grows bigger, $P(R, R')$ becomes smaller. Hence, what we should do next is to that, for each R, decompose the set of all dyadic rectangles $\{R'\}$ into annuli according to the distance between R and R'. Next, for each annuli, we give a precise estimate by considering the difference of the sizes of R and R'. Finally, we add up all the estimates on each annuli and then finish our proof.

Now let's go into the details. For the sake of simplicity, we denote $Q_{\tau_i}^{k_i,v_i}$, $Q_{\tau_i'}^{k_i',v_i'}$ by Q_i, Q_i', respectively, for $i = 1, 2$. Define

$$\Omega^0 =: \bigcup_{R=Q_1\times Q_2\subset\Omega} 3A^2(Q_1 \times Q_2).$$

And for each R, let

$$\begin{aligned}
A_{0,0}(R) &= \{R' :\ 3A^2R' \cap 3A^2R \neq \emptyset\};\\
A_{j,0}(R) &= \{R' :\ 3A^2R \cap 3A^2(2^jQ_1' \times Q_2') \neq \emptyset\ ;\\
&\qquad 3A^2R \cap 3A^2(2^{j-1}Q_1' \times Q_2') = \emptyset\};\\
A_{0,k}(R) &= \{R' :\ 3A^2R \cap 3A^2(Q_1' \times 2^kQ_2') \neq \emptyset\ ;\\
&\qquad 3A^2R \cap 3A^2(Q_1' \times 2^{k-1}Q_2') = \emptyset\};\\
A_{j,k}(R) &= \{R' :\ 3A^2R \cap 3A^2(2^jQ_1' \times 2^kQ_2') \neq \emptyset\ ;\\
&\qquad 3A^2R \cap 3A^2(2^{j-1}Q_1' \times 2^kQ_2') = \emptyset;\\
&\qquad 3A^2R \cap 3A^2(2^jQ_1' \times 2^{k-1}Q_2') = \emptyset;\\
&\qquad 3A^2R \cap 3A^2(2^{j-1}Q_1' \times 2^{k-1}Q_2') = \emptyset\},
\end{aligned}$$

where $j, k \geq 1$.

Since for each $R^{'}=Q_1^{'}\times Q_2^{'}$, $\lim_{j,k\to\infty}3A^2(2^jQ_1^{'}\times 2^kQ_2^{'})=\mathcal{X}\times\mathcal{X}$, we can see that for any $R\subset\Omega$, there must be some j and k such that $R'\in A_{j,k}(R)$. This implies that for each $R\subset\Omega$, $\{R'\}\subseteq\cup_{j,k\geq 0}A_{j,k}(R)$.

Then, we have

$$\begin{aligned}(3.6)\ \leq\ & \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\sum_{R\subset\Omega}\sum_{R'\in A_{0,0}(R)}v(R,R^{'})r(R,R^{'})P(R,R^{'})T_{R'}\\ &+\sum_{j\geq 1}\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\sum_{R\subset\Omega}\sum_{R'\in A_{j,0}(R)}v(R,R^{'})r(R,R^{'})P(R,R^{'})T_{R'}\\ &+\sum_{k\geq 1}\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\sum_{R\subset\Omega}\sum_{R'\in A_{0,k}(R)}v(R,R^{'})r(R,R^{'})P(R,R^{'})T_{R'}\\ &+\sum_{j,k\geq 1}\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\sum_{R\subset\Omega}\sum_{R'\in A_{j,k}(R)}v(R,R^{'})r(R,R^{'})P(R,R^{'})T_{R'}\\ =:\ & I+I\!I+I\!I\!I+IV.\end{aligned}$$

We first estimate term I. Define

$$B_{0,0}=\{R':\ 3A^2R'\cap\Omega^0\neq\emptyset\}.$$

Then we claim that

$$I\leq\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\sum_{R'\in B_{0,0}}\sum_{\{R:\ R\subset\Omega,R'\in A_{0,0}(R)\}}v(R,R^{'})r(R,R^{'})P(R,R^{'})T_{R'}.\quad(3.8)$$

To show this claim, we only need to point out that for any $R'\notin B_{0,0}$, we have $3A^2R'\cap\Omega^0=\emptyset$. Thus, for any $R\subset\Omega$, we can see that $3A^2R'\cap 3A^2R=\emptyset$, which implies that $R'\notin A_{0,0}(R)$. Hence, we can obtain that $\cup_{R\subset\Omega}A_{0,0}(R)\subset B_{0,0}$. This yields that the claim (3.8) holds.

Now we continue to decompose $B_{0,0}$. Let $\mathcal{F}_h^{0,0}=\{R^{'}:\mu(3A^2R^{'}\cap\Omega^0)>\frac{1}{2^h}\mu(3A^2R^{'})\}$, $\mathcal{D}_h^{0,0}=\mathcal{F}_h^{0,0}\setminus\mathcal{F}_{h-1}^{0,0}$, $h\geq 1$, $\mathcal{F}_0^{0,0}=\emptyset$, and $\Omega_h^{0,0}=\cup_{R'\in\mathcal{D}_h^{0,0}}R^{'}$, $h\geq 1$. From these definitions, we can see that

$$B_{0,0}=\bigcup_{h\geq 1}\mathcal{D}_h^{0,0}.$$

Then (3.8) can be rewritten as

$$I\leq\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\sum_{h\geq 1}\sum_{R^{'}\in\mathcal{D}_h^{0,0}}\sum_{\{R:\ R\subset\Omega,R'\in A_{0,0}(R)\}}v(R,R^{'})r(R,R^{'})P(R,R^{'})T_{R'}.\quad(3.9)$$

To estimate the right-hand side of (3.9), we only need to consider

$$\sum_{\{R:\ R\subset\Omega,R'\in A_{0,0}(R)\}}v(R,R^{'})r(R,R^{'})\qquad(3.10)$$

since $P(R, R') \le 1$ for any $R' \in \mathcal{D}_h^{0,0}$ and R satisfying $R' \in A_{0,0}(R)$. In what follows, we use a simple geometrical argument, which is a generalization of Chang and R. Fefferman's idea, see more details in [2].

Since $3A^2R \cap 3A^2R' \neq \emptyset$, we can split (3.10) into four cases:

Case 1: $\mu(Q_1') \ge \mu(Q_1)$, $\mu(Q_2') \le \mu(Q_2)$.

First, it is easy to see that $\mu(Q_1 \times 3AQ_2') \lesssim \mu(3A^2R \cap 3A^2R')$. So we have

$$\frac{\mu(Q_1)}{\mu(3AQ_1')}\mu(3A^2R') \lesssim \mu(3A^2R \cap 3A^2R') \le \mu(3A^2R' \cap \Omega^0) \le \frac{1}{2^{h-1}}\mu(3A^2R'),$$

which yields that $2^{h-1}\mu(Q_1) \lesssim \mu(3AQ_1') \lesssim \mu(Q_1')$. Since all the Q_i and Q_i' ($i = 1, 2$) are dyadic cubes with measures equivalent to 2^{-a} for some $a \in \mathbb{Z}$, then we have $\mu(Q_1') \sim 2^{h-1+n}\mu(Q_1)$, for some $n \ge 0$. For each fixed n, the numbers of such Q_1's must be $\lesssim \frac{C_1}{C_2} \cdot 5A \cdot 2^n$.

As for Q_2, $\mu(Q_2) \sim 2^m\mu(Q_2')$ for some $m \ge 0$. For each fixed m, the number of such Q_2's is $\lesssim \frac{C_1}{C_2} \cdot 5A$ since $3AQ_2 \cap 3AQ_2' \neq \emptyset$. Thus

$$\begin{aligned}\sum_{\text{Case1}} r(R, R')v(R, R') &= \sum_{\text{case1}} \left(\frac{\mu(Q_1)}{\mu(Q_1')}\right)^{1+\varepsilon'} \left(\frac{\mu(Q_2')}{\mu(Q_2)}\right)^{1+\varepsilon'} \mu(Q_1')\mu(Q_2) \\ &\lesssim \sum_{n,m\ge 0} 2^{-(h-1+n)(1+\varepsilon')}2^n\mu(Q_1')2^{-m(1+\epsilon')}2^m\mu(Q_2') \\ &\lesssim 2^{-h(1+\varepsilon')}\mu(R').\end{aligned}$$

Case 2: $\mu(Q_1') \le \mu(Q_1)$, $\mu(Q_2') \ge \mu(Q_2)$.

This can be handled in a similar way to that of case 1.

Case 3: $\mu(Q_1') \ge \mu(Q_1)$, $\mu(Q_2') \ge \mu(Q_2)$.

Since

$$\mu(R) \lesssim \mu(3A^2R' \cap 3A^2R) \le \mu(3A^2R' \cap \Omega^0) \le \frac{1}{2^{h-1}}\mu(3A^2R'),$$

we have $2^{h-1}\mu(R) \lesssim \mu(R')$. Using the same idea as in Case 1, we can obtain that $\mu(R') \sim 2^{h-1+n}\mu(R)$ for some $n \ge 0$, and that for each fixed n, the number of such R's is $\lesssim \frac{C_1^2}{C_2^2}5^2A^2 \cdot 2^n$. Combining these results, we can get

$$\begin{aligned}\sum_{\text{Case3}} r(R, R')v(R, R') &= \sum_{\text{case3}} \left(\frac{\mu(R)}{\mu(R')}\right)^{1+\varepsilon'} \mu(Q_1')\mu(Q_2') \\ &\lesssim \sum_{n\ge 0} 2^{-(h-1+n)(1+\varepsilon')}2^n\mu(R') \\ &\lesssim 2^{-h(1+\varepsilon')}\mu(R').\end{aligned}$$

Case 4: $\mu(Q_1^{'}) \le \mu(Q_1), \mu(Q_2^{'}) \le \mu(Q_2)$.

From

$$\mu(R^{'}) \lesssim \mu(3A^2R^{'} \cap 3A^2R) \le \mu(3A^2R^{'} \cap \Omega^0) \le \frac{1}{2^{h-1}}\mu(3A^2R^{'}),$$

we have that $\mu(R^{'}) \le C\frac{1}{2^{h-1}}\mu(R^{'})$, where C is a constant depending only on A, C_1 and C_2. This yields that $h \le h_0 = \left[\log_2(2C)\right] + 1$. Thus we can see that in this case, there are at most h_0 terms in (3.9) is nonzero.

Since $\mu(R) \ge \mu(R')$, we obtain that $\mu(R) \sim 2^n\mu(R^{'})$ for some $n \ge 0$. For each fixed n, the number of such R's is $\lesssim 5^2A^2\frac{C_1^2}{C_2^2}$. Therefore

$$\begin{aligned}
\sum_{\text{Case4}} r(R, R^{'})v(R, R^{'}) &= \sum_{\text{case4}} \left(\frac{\mu(R^{'})}{\mu(R)}\right)^{1+\varepsilon^{'}} \mu(R) \\
&\lesssim \sum_{n\ge 0} 2^{-n(1+\varepsilon^{'})}2^n\mu(R^{'}) \\
&\lesssim \mu(R^{'}).
\end{aligned}$$

Now we have finished the estimate of (3.10). Then from (3.9), we have

$$\begin{aligned}
I &\le \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h\ge 1}\sum_{R'\in\mathcal{D}_h^{0,0}} \left(\sum_{R\in \text{ Case1}} + \sum_{R\in \text{ Case2}} + \right. \\
&\quad \left. + \sum_{R\in \text{ Case3}} + \sum_{R\in \text{ Case4}} \right) v(R, R^{'})r(R, R^{'})T_{R'} \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We first consider the terms I_1, I_2 and I_3. Noting that we have chosen ϵ and ε' satisfying that $\frac{2}{2+\theta} < \frac{2}{2+\theta'} < \frac{2}{2+\varepsilon'} < \frac{2}{2+\epsilon} < p \le 1$ and combining with the fact that $\mu(\Omega_h^{0,0}) \lesssim h2^h\mu(\Omega)$ for $h \ge 1$, we have

$$\begin{aligned}
I_1, I_2, I_3 &\lesssim \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h\ge 1} 2^{-h(1+\varepsilon^{'})}\mu(\Omega_h^{0,0})^{\frac{2}{p}-1} \frac{1}{\mu(\Omega_h^{0,0})^{\frac{2}{p}-1}} \sum_{R'\subset\Omega_h^{0,0}} \mu(R^{'})T_{R'} \\
&\lesssim \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h\ge 1} 2^{-h(1+\varepsilon^{'})}(h2^h)^{(\frac{2}{p}-1)}\mu(\Omega)^{\frac{2}{p}-1} \\
&\quad \times \sup_{\bar\Omega} \frac{1}{\mu(\bar\Omega)^{\frac{2}{p}-1}} \sum_{R'\subset\bar\Omega} \mu(R^{'})T_{R'} \\
&\lesssim \sup_{\bar\Omega} \frac{1}{\mu(\bar\Omega)^{\frac{2}{p}-1}} \sum_{R'\subset\bar\Omega} \mu(R^{'})T_{R'}.
\end{aligned}$$

As to I_4, from the estimate in Case 4 we can see that

$$I_4 \le \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h=1}^{h_0} \sum_{R' \in \mathcal{D}_h^{0,0}} \sum_{R \in \, case 4} r(R, R^{'}) v(R, R^{'}) T_{R'}.$$

Thus, we have

$$\begin{aligned} I_4 &\lesssim \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h=1}^{h_0} \mu(\Omega_h^{0,0})^{\frac{2}{p}-1} \frac{1}{\mu(\Omega_h^{0,0})^{\frac{2}{p}-1}} \sum_{R' \subset \Omega_h^{0,0}} \mu(R^{'}) T_{R'} \\ &\lesssim \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h=1}^{h_0} (h2^h)^{(\frac{2}{p}-1)} \mu(\Omega)^{\frac{2}{p}-1} \times \sup_{\bar{\Omega}} \frac{1}{\mu(\bar{\Omega})^{\frac{2}{p}-1}} \sum_{R' \subset \bar{\Omega}} \mu(R^{'}) T_{R'} \\ &\lesssim \sup_{\bar{\Omega}} \frac{1}{\mu(\bar{\Omega})^{\frac{2}{p}-1}} \sum_{R' \subset \bar{\Omega}} \mu(R^{'}) T_{R'}. \end{aligned}$$

Combining the estimates from I_1 to I_4, we can get

$$I \lesssim \sup_{\bar{\Omega}} \frac{1}{\mu(\bar{\Omega})^{\frac{2}{p}-1}} \sum_{R' \subset \bar{\Omega}} \mu(R^{'}) T_{R'}.$$

We now only need to estimate IV because the estimates of II and III can be derived from the same skills as in I and IV. First consider each term in IV as follows:

$$a_{j,k} =: \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R \subset \Omega} \sum_{R' \in A_{j,k}(R)} v(R, R^{'}) r(R, R^{'}) P(R, R^{'}) T_{R'} \tag{3.11}$$

for some $j, k \ge 1$.

Define

$$B_{j,k} = \{R' : \ 3A^2(2^j Q_1^{'} \times 2^k Q_2^{'}) \cap \Omega^0 \neq \emptyset\}$$

for the above j, k.

Then we claim that

$$a_{j,k} \le \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{R' \in B_{j,k}} \sum_{\{R: \ R \subset \Omega, R' \in A_{j,k}(R)\}} v(R, R^{'}) r(R, R^{'}) P(R, R^{'}) T_{R'}. \tag{3.12}$$

In fact, this claim is similar to the former one (3.8). To see this, we point out that for any $R' \notin B_{j,k}$, $3A^2(2^j Q_1^{'} \times 2^k Q_2^{'}) \cap \Omega^0 = \emptyset$. Thus, for any $R \subset \Omega$, we have that $3A^2(2^j Q_1^{'} \times 2^k Q_2^{'}) \cap 3A^2 R = \emptyset$, which implies that $R' \notin A_{j,k}(R)$. Hence, we can obtain that $\cup_{R \subset \Omega} A_{j,k}(R) \subset B_{j,k}$. This yields that the claim (3.12) holds.

Now we continue to decompose $B_{j,k}$. Let $\mathcal{F}_h^{j,k} = \{R' : \mu(3A^2(2^j Q_1^{'} \times 2^k Q_2^{'}) \cap \Omega^0) \geq \frac{1}{2^h}\mu(3A^2(2^j Q_1^{'} \times 2^k Q_2^{'}))\}$, $\mathcal{D}_h^{j,k} = \mathcal{F}_h^{j,k} \backslash \mathcal{F}_{h-1}^{j,k}$, $h \geq 1$, $\mathcal{F}_0^{j,k} = \emptyset$ and $\Omega_h^{j,k} = \cup_{R' \in \mathcal{D}_h^{j,k}} R'$, $h \geq 1$. From these definitions, we can see that

$$B_{j,k} = \bigcup_{h \geq 1} \mathcal{D}_h^{j,k}.$$

Then (3.12) can be rewritten as

$$a_{j,k} \leq \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h \geq 1} \sum_{R' \in \mathcal{D}_h^{j,k}} \sum_{\{R:\ R \subset \Omega, R' \in A_{j,k}(R)\}} v(R, R^{'}) r(R, R^{'}) P(R, R^{'}) T_{R'}. \quad (3.13)$$

Now we consider

$$\sum_{\{R:\ R \subset \Omega, R' \in A_{j,k}(R)\}} v(R, R^{'}) r(R, R^{'}) P(R, R^{'}). \quad (3.14)$$

Note that when $R' \in A_{j,k}(R)$, we have $3A^2 R \cap 3A^2(2^j Q_1^{'} \times 2^k Q_2^{'}) \neq \emptyset$. Namely, $3AQ_1 \cap 3A2^j Q_1^{'} \neq \emptyset$ and $3AQ_2 \cap 3A2^k Q_2^{'} \neq \emptyset$.

Also we remind that when $R' \in A_{j,k}(R)$, we have $3A^2 R \cap 3A^2(2^{j-1} Q_1^{'} \times 2^k Q_2^{'}) = \emptyset$, $3A^2 R \cap 3A^2(2^j Q_1^{'} \times 2^{k-1} Q_2^{'}) = \emptyset$ and $3A^2 R \cap 3A^2(2^{j-1} Q_1^{'} \times 2^{k-1} Q_2^{'}) = \emptyset$. These imply that $\mathrm{dist}(Q_1^{'}, Q_1) > \mu(2^{j-1} Q_1^{'}) \vee \mu(Q_1)$ and that $\mathrm{dist}(Q_2^{'}, Q_2) > \mu(2^{k-1} Q_2^{'}) \vee \mu(Q_2)$.

Thus we can split (3.14) into four cases:

Case 1: $\mu(2^j Q_1^{'}) \geq \mu(Q_1)$, $\mu(2^k Q_2^{'}) \leq \mu(Q_2)$.

First, it is easy to see that $\mu(Q_1 \times 3A2^k Q_2^{'}) \lesssim \mu(3A^2(2^j Q_1^{'} \times 2^k Q_2^{'}) \cap 3A^2 R)$. So we have

$$\begin{aligned}
\frac{\mu(Q_1)}{\mu(3A2^j Q_1^{'})} \mu(3A^2(2^j Q_1^{'} \times 2^k Q_2^{'})) &\lesssim \mu(3A^2(2^j Q_1^{'} \times 2^k Q_2^{'}) \cap 3A^2 R) \\
&\leq \mu(3A^2(2^j Q_1^{'} \times 2^k Q_2^{'}) \cap \Omega^0) \\
&\leq \frac{1}{2^{h-1}} \mu(3A^2(2^j Q_1^{'} \times 2^k Q_2^{'})),
\end{aligned}$$

which yields that $2^{h-1}\mu(Q_1) \lesssim \mu(3A2^j Q_1^{'}) \lesssim 2^j \mu(Q_1^{'})$.

Now let us consider the measures of Q_1 and $Q_1^{'}$. We can see that there are two subcases.

Subcase 1.1: $\mu(Q_1^{'}) \geq \mu(Q_1)$.

In this subcase, since $2^{h-1-j}\mu(Q_1) \lesssim \mu(Q_1^{'})$, we have that $\mu(Q_1^{'}) \sim 2^{h-1-j+n}\mu(Q_1)$ for some $n \geq 0$. And for each fixed n, the number of such Q_1's must be $\lesssim \frac{C_1}{C_2} \cdot 5A \cdot 2^n$.

Subcase 1.2: $\mu(Q_1^{'}) < \mu(Q_1)$.

In this subcase, we have $\mu(Q_1^{'}) < \mu(Q_1) \le \mu(2^j Q_1^{'})$. So $\mu(2^\ell Q_1^{'}) \sim \mu(Q_1)$ for some positive integer ℓ satisfies $1 \le \ell \le j$. And for each ℓ, the number of such Q_1's must be $\lesssim \frac{C_1}{C_2} \cdot 5A$. Then, from the inequality $2^{h-1}\mu(Q_1) \lesssim 2^j \mu(Q_1^{'})$, we can see that $2^{h-1}2^\ell \mu(Q_1^{'}) \lesssim 2^j \mu(Q_1^{'})$, which yields that $2^{h-1} \lesssim 2^{j-\ell}$. Thus, $h \lesssim j-\ell$. This implies that for each $\ell \le j$, there are at most $C(j-\ell)$ terms in the right-hand side of (3.13).

Now let us consider Q_2 and $Q_2^{'}$. Since $\mu(2^k Q_2^{'}) \le \mu(Q_2)$, we can get that $2^k 2^m \mu(Q_2^{'}) \sim \mu(Q_2)$ for some $m \ge 0$. And for each fixed m, the number of such Q_2's must be $\lesssim \frac{C_1}{C_2} \cdot 5A$ since $3A(2^k Q_2^{'}) \cap 3AQ_2 \ne \emptyset$.

Combining the above estimates, we have that

$$\begin{aligned}
&\sum_{\text{Subcase 1.1}} v(R,R^{'})r(R,R^{'})P(R,R^{'})\\
&= \sum_{\text{Subcase 1.1}} \left(\frac{\mu(Q_1)}{\mu(Q_1^{'})}\right)^{1+\varepsilon^{'}} \left(\frac{\mu(Q_2^{'})}{\mu(Q_2)}\right)^{1+\varepsilon^{'}} \mu(Q_1^{'})\mu(Q_2)\left(1+\frac{\text{dist}(Q_1,Q_1^{'})}{\mu(Q_1^{'})}\right)^{-(1+\theta')}\\
&\quad\times\left(1+\frac{\text{dist}(Q_2,Q_2^{'})}{\mu(Q_2)}\right)^{-(1+\theta')}\\
&\lesssim \sum_{n,m\ge 0} 2^{-(h-1-j+n)(1+\varepsilon^{'})}2^{-(m+k)(1+\epsilon^{'})}2^n\mu(Q_1^{'})2^{m+k}\mu(Q_2^{'})2^{-j(1+\theta')}\\
&\lesssim 2^{-h(1+\varepsilon^{'})}2^{-j(\theta'-\varepsilon')}2^{-k\varepsilon'}\mu(R^{'})
\end{aligned}$$

and that

$$\begin{aligned}
&\sum_{\text{Subcase 1.2}} v(R,R^{'})r(R,R^{'})P(R,R^{'})\\
&= \sum_{\text{Subcase 1.2}} \left(\frac{\mu(Q_1^{'})}{\mu(Q_1)}\right)^{1+\varepsilon^{'}} \left(\frac{\mu(Q_2^{'})}{\mu(Q_2)}\right)^{1+\varepsilon^{'}} \mu(Q_1)\mu(Q_2)\left(1+\frac{\text{dist}(Q_1,Q_1^{'})}{\mu(Q_1)}\right)^{-(1+\theta')}\\
&\quad\times\left(1+\frac{\text{dist}(Q_2,Q_2^{'})}{\mu(Q_2)}\right)^{-(1+\theta')}\\
&\lesssim \sum_{\ell=1}^{j}\sum_{m\ge 0} 2^{-\ell(1+\varepsilon^{'})}2^{-(m+k)(1+\epsilon^{'})}2^\ell\mu(Q_1^{'})2^{m+k}\mu(Q_2^{'})2^{-(j-\ell)(1+\theta')}\\
&\lesssim \sum_{\ell=1}^{j} 2^{-\ell\varepsilon^{'}}2^{-(j-\ell)(1+\theta')}2^{-k\varepsilon'}\mu(R').
\end{aligned}$$

Case 2: $\mu(2^j Q_1^{'}) \le \mu(Q_1)$, $\mu(2^k Q_2^{'}) \ge \mu(Q_2)$.

This can be handled similarly as Case 1. And we have that

$$\sum_{\text{Subcase 2.1}} v(R, R^{'})r(R, R^{'})P(R, R^{'}) \lesssim 2^{-h(1+\varepsilon^{'})}2^{-k(\theta'-\varepsilon')}2^{-j\varepsilon'}\mu(R^{'})$$

and that

$$\sum_{\text{Subcase 2.2}} v(R, R^{'})r(R, R^{'})P(R, R^{'}) \lesssim \sum_{\ell=1}^{k} 2^{-\ell\varepsilon^{'}}2^{-(k-\ell)(1+\theta')}2^{-j\varepsilon'}\mu(R').$$

Also notice that in the subcase 2.2, for each $\ell \le k$, there are at most $C(k-\ell)$ terms in the right-hand side of (3.13).

Case 3: $\mu(2^j Q_1^{'}) \ge \mu(Q_1)$, $\mu(2^k Q_2^{'}) \ge \mu(Q_2)$.

First, it is easy to see that $\mu(R) \lesssim \mu(3A^2 R \cap 3A^2(2^j Q_1^{'} \times 2^k Q_2^{'}))$. So,

$$\mu(R) \lesssim \mu(3A^2(2^j Q_1^{'} \times 2^k Q_2^{'}) \cap \Omega^0) \le \frac{1}{2^{h-1}}\mu(3A^2(2^j Q_1^{'} \times 2^k Q_2^{'})),$$

which yields that $2^{h-1}\mu(R) \lesssim 2^j 2^k \mu(R^{'})$.

Next we consider the measures of Q_1, $Q_1^{'}$ and Q_2, $Q_2^{'}$. We can see that there are four subcases.

Subcase 3.1: $\mu(Q_1^{'}) \ge \mu(Q_1)$, $\mu(Q_2^{'}) \ge \mu(Q_2)$.

In this subcase, we can see that $2^{h-1-j-k+n}\mu(R) \sim \mu(R^{'})$ for some $n \ge 0$. And for each n, the number of such R's must be $\lesssim \frac{C_1^2}{C_2^2}5^2A^2 \cdot 2^n$. Hence

$$\begin{aligned}
&\sum_{\text{Subcase 3.1}} v(R, R^{'})r(R, R^{'})P(R, R^{'}) \\
&= \sum_{\text{Subcase 3.1}} \left(\frac{\mu(R)}{\mu(R^{'})}\right)^{1+\varepsilon^{'}} \mu(R^{'}) \left(1+\frac{\operatorname{dist}(Q_1, Q_1^{'})}{\mu(Q_1^{'})}\right)^{-(1+\theta')} \left(1+\frac{\operatorname{dist}(Q_2, Q_2^{'})}{\mu(Q_2^{'})}\right)^{-(1+\theta')} \\
&\lesssim \sum_{n\ge 0} 2^{-(h-1-j-k+n)(1+\varepsilon^{'})}2^n \mu(R^{'})2^{-j(1+\theta')}2^{-k(1+\theta')} \\
&\lesssim 2^{-h(1+\varepsilon')}2^{-j(\theta'-\varepsilon')}2^{-k(\theta'-\varepsilon')}\mu(R^{'}).
\end{aligned}$$

Subcase 3.2: $\mu(Q_1^{'}) < \mu(Q_1)$, $\mu(Q_2^{'}) \ge \mu(Q_2)$.

In this subcase, we first have that $\mu(Q_1^{'}) < \mu(Q_1) \le \mu(2^j Q_1^{'})$. Similar to the estimate in Subcase 1.2, we have that $\mu(2^{\ell_1} Q_1^{'}) \sim \mu(Q_1)$ for some positive integer ℓ_1 satisfies $1 \le \ell_1 \le j$. And for each ℓ_1, the number of such Q_1's must be $\lesssim \frac{C_1}{C_2} \cdot 5A$.

As to Q_2 and $Q_2^{'}$, we can obtain that $\mu(Q_2^{'}) \sim 2^m \mu(Q_2)$ for some $m \geq 0$. And for each fixed m, the number of such Q_2's must be $\lesssim \frac{C_1}{C_2} \cdot 5A$.

Then, from the inequality $2^{h-1}\mu(R) \lesssim 2^j 2^k \mu(R^{'})$, we can see that $2^h \lesssim 2^{(j-\ell_1)+k+m}$, which yields that $h \lesssim (j-\ell_1)+k+m$. This implies that in this subcase, for each ℓ_1, there are at most $(j-\ell_1)+k+m$ terms in the right-hand side of (3.13).

So, we have

$$\begin{aligned}
&\sum_{\text{Subcase 3.2}} v(R,R^{'})r(R,R^{'})P(R,R^{'}) \\
&= \sum_{\text{Subcase 3.2}} \left(\frac{\mu(Q_1^{'})}{\mu(Q_1)}\right)^{1+\varepsilon^{'}} \left(\frac{\mu(Q_2)}{\mu(Q_2^{'})}\right)^{1+\varepsilon^{'}} \mu(Q_1)\mu(Q_2^{'}) \left(1+\frac{\operatorname{dist}(Q_1,Q_1^{'})}{\mu(Q_1^{'})}\right)^{-(1+\theta')} \\
&\quad \times \left(1+\frac{\operatorname{dist}(Q_2,Q_2^{'})}{\mu(Q_2^{'})}\right)^{-(1+\theta')} \\
&\lesssim \sum_{\ell_1=1}^{j}\sum_{m\geq 0} 2^{-\ell_1(1+\varepsilon^{'})}2^{-m(1+\varepsilon^{'})}2^{\ell_1}\mu(Q_1^{'})\mu(Q_2^{'})2^{-(j-\ell_1)(1+\theta')}2^{-k(1+\theta')}.
\end{aligned}$$

Subcase 3.3: $\mu(Q_1^{'}) \geq \mu(Q_1)$, $\mu(Q_2^{'}) < \mu(Q_2)$.

This subcase can be handled similarly as Subcase 3.2. And we have that

$$\begin{aligned}
&\sum_{\text{Subcase 3.3}} v(R,R^{'})r(R,R^{'})P(R,R^{'}) \\
&\lesssim \sum_{\ell_2=1}^{k}\sum_{n\geq 0} 2^{-\ell_2(1+\varepsilon^{'})}2^{-n(1+\varepsilon^{'})}2^{\ell_2}\mu(Q_1^{'})\mu(Q_2^{'})2^{-(k-\ell_2)(1+\theta')}2^{-j(1+\theta')}.
\end{aligned}$$

Also, we shall point out that in this subcase, for each ℓ_2, there are at most $(k-\ell_2)+j+n$ terms in the right-hand side of (3.13).

Subcase 3.4: $\mu(Q_1^{'}) < \mu(Q_1)$, $\mu(Q_2^{'}) < \mu(Q_2)$.

This subcase can be handled similarly by using the skills as in Subcase 3.1 and Subcase 3.2. And we have that

$$\begin{aligned}
&\sum_{\text{Subcase 3.4}} v(R,R^{'})r(R,R^{'})P(R,R^{'}) \\
&\lesssim \sum_{\ell_1=1}^{j}\sum_{\ell_2=1}^{k} 2^{-\ell_1(1+\varepsilon^{'})}2^{-\ell_2(1+\varepsilon^{'})}2^{\ell_1}2^{\ell_2}\mu(Q_1^{'})\mu(Q_2^{'})2^{-(j-\ell_1)(1+\theta')}2^{-(k-\ell_2)(1+\theta')}.
\end{aligned}$$

Also, we shall point out that in this subcase, for each ℓ_1 and ℓ_2, there are at most $(j-\ell_1)+(k-\ell_2)$ terms in the right-hand side of (3.13).

Case 4: $\mu(2^j Q_1^{'}) \le \mu(Q_1)$, $\mu(2^k Q_2^{'}) \le \mu(Q_2)$.

Similar to Case 3, it is easy to see that $\mu((2^j Q_1^{'} \times 2^k Q_2^{'})) \lesssim \mu(3A^2 R \cap 3A^2(2^j Q_1^{'} \times 2^k Q_2^{'}))$. So, we have

$$\begin{aligned}\mu((2^j Q_1^{'} \times 2^k Q_2^{'})) &\lesssim \mu(3A^2(2^j Q_1^{'} \times 2^k Q_2^{'}) \cap \Omega^0)\\ &\le \frac{1}{2^{h-1}}\mu(3A^2(2^j Q_1^{'} \times 2^k Q_2^{'})),\end{aligned}$$

which implies that $\mu(R^{'}) \le C\frac{1}{2^{h-1}}\mu(R^{'})$, where C is a constant depending only on A, C_1 and C_2. This yields that $h \le h_0 = \left[\log_2(2C)\right] + 1$. Thus we can see that in this case, there are at most h_0 terms in the right-hand side of (3.13).

In this case, we can obtain that $\mu(R) \sim 2^n 2^j 2^k \mu(R')$ for some $n \ge 0$. And for each fixed n, the number of such R's is $\lesssim 5^2 A^2 \frac{C_1^2}{C_2^2}$, since $3A^2 R \cap 3A^2(2^j Q_1^{'} \times 2^k Q_2^{'}) \ne \emptyset$.

Hence

$$\begin{aligned}&\sum_{\text{Case 4}} v(R, R^{'}) r(R, R^{'}) P(R, R^{'})\\ &= \sum_{\text{Case4.1}} \left(\frac{\mu(R^{'})}{\mu(R)}\right)^{1+\varepsilon^{'}} \mu(R) \left(1+\frac{\operatorname{dist}(Q_1, Q_1^{'})}{\mu(Q_1)}\right)^{-(1+\theta')} \left(1+\frac{\operatorname{dist}(Q_2, Q_2^{'})}{\mu(Q_2)}\right)^{-(1+\theta')}\\ &\lesssim \sum_{n\ge 0} 2^{-(n+j+k)(1+\varepsilon^{'})} 2^n 2^j 2^k \mu(R^{'})\\ &\lesssim 2^{-j\varepsilon'} 2^{-k\varepsilon'} \mu(R^{'}).\end{aligned}$$

Now let us come back to (3.13). From the estimates of (3.14) we have

$$\begin{aligned}a_{j,k} \ &\le \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h\ge 1} \sum_{R'\in\mathcal{D}_h^{j,k}} \left(\sum_{R\in\text{Case 1}} + \sum_{R\in\text{Case 2}} + \sum_{R\in\text{Case 3}} + \sum_{R\in\text{Case 4}}\right) v(R, R^{'})\\ &\quad \times r(R, R^{'}) P(R, R^{'}) T_{R'}\\ &=: a_{j,k,1} + a_{j,k,2} + a_{j,k,3} + a_{j,k,4}.\end{aligned}$$

We first consider the term $a_{j,k,1}$. From the subcases in Case 1, we know that $a_{j,k,1}$ can be further divided in to two terms, namely,

$$\begin{aligned}a_{j,k,1} &= \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h\ge 1} \sum_{R'\in\mathcal{D}_h^{j,k}} \left(\sum_{R\in\text{Subcase 1.1}} + \sum_{R\in\text{Subcase 1.2}}\right)\\ &\quad \cdot v(R, R^{'}) r(R, R^{'}) P(R, R^{'}) T_{R'}\\ &=: a_{j,k,1.1} + a_{j,k,1.2}.\end{aligned}$$

Let us estimate $a_{j,k,1.1}$. Noting that $\mu(\Omega_h^{j,k}) \lesssim h2^h\mu(\Omega)$ for $h \geq 1$ and that $1+\varepsilon' > \frac{2}{p}-1$, we have

$$\begin{aligned}
a_{j,k,1.1} &\lesssim \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h\geq 1} 2^{-h(1+\varepsilon')} 2^{-j(\theta'-\varepsilon')} 2^{-k\varepsilon'} \mu(\Omega_h^{j,k})^{\frac{2}{p}-1} \\
&\quad \cdot \frac{1}{\mu(\Omega_h^{j,k})^{\frac{2}{p}-1}} \sum_{R'\subset\Omega_h^{j,k}} \mu(R')T_{R'} \\
&\lesssim \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h\geq 1} 2^{-h(1+\varepsilon')} 2^{-j(\theta'-\varepsilon')} 2^{-k\varepsilon'} (h2^h)^{(\frac{2}{p}-1)} \mu(\Omega)^{\frac{2}{p}-1} \\
&\quad \times \sup_{\bar\Omega} \frac{1}{\mu(\bar\Omega)^{\frac{2}{p}-1}} \sum_{R'\subset\bar\Omega} \mu(R')T_{R'} \\
&\lesssim 2^{-j(\theta'-\varepsilon')} 2^{-k\varepsilon'} \sup_{\bar\Omega} \frac{1}{\mu(\bar\Omega)^{\frac{2}{p}-1}} \sum_{R'\subset\bar\Omega} \mu(R')T_{R'}.
\end{aligned}$$

As a consequence,

$$\sum_{j,k\geq 1} a_{j,k,1.1} \lesssim \sup_{\bar\Omega} \frac{1}{\mu(\bar\Omega)^{\frac{2}{p}-1}} \sum_{R'\subset\bar\Omega} \mu(R')T_{R'}.$$

As to $a_{j,k,1.2}$, we have that

$$\begin{aligned}
a_{j,k,1.2} &\lesssim \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{\ell=1}^{j} \sum_{h\geq 1}^{C(j-\ell)} 2^{-\ell\varepsilon'} 2^{-(j-\ell)(1+\theta')} 2^{-k\varepsilon'} \mu(\Omega_h^{j,k})^{\frac{2}{p}-1} \\
&\quad \cdot \frac{1}{\mu(\Omega_h^{j,k})^{\frac{2}{p}-1}} \sum_{R'\subset\Omega_h^{j,k}} \mu(R')T_{R'} \\
&\lesssim \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{\ell=1}^{j} \sum_{h\geq 1}^{C(j-\ell)} 2^{-\ell\varepsilon'} 2^{-(j-\ell)(1+\theta')} 2^{-k\varepsilon'} (h2^h)^{(\frac{2}{p}-1)} \mu(\Omega)^{\frac{2}{p}-1} \\
&\quad \times \sup_{\bar\Omega} \frac{1}{\mu(\bar\Omega)^{\frac{2}{p}-1}} \sum_{R'\subset\bar\Omega} \mu(R')T_{R'} \\
&\lesssim 2^{-k\varepsilon'} \sum_{\ell=1}^{j} 2^{-\ell\varepsilon'} (j-\ell)^2 2^{-(j-\ell)(1+\theta'-(\frac{2}{p}-1))} \sup_{\bar\Omega} \frac{1}{\mu(\bar\Omega)^{\frac{2}{p}-1}} \sum_{R'\subset\bar\Omega} \mu(R')T_{R'}.
\end{aligned}$$

Hence, noting that $1+\theta' > \frac{2}{p}-1$, we have

$$\sum_{j,k\geq 1} a_{j,k,1.2} \lesssim \sum_{j,k\geq 1} 2^{-k\varepsilon'} \sum_{\ell=1}^{j} 2^{-\ell\varepsilon'}(j-\ell)^2 2^{-(j-\ell)(1+\theta'-(\frac{2}{p}-1))}$$
$$\cdot \sup_{\bar\Omega} \frac{1}{\mu(\bar\Omega)^{\frac{2}{p}-1}} \sum_{R'\subset\bar\Omega} \mu(R')T_{R'}$$
$$\lesssim \sup_{\bar\Omega} \frac{1}{\mu(\bar\Omega)^{\frac{2}{p}-1}} \sum_{R'\subset\bar\Omega} \mu(R')T_{R'}.$$

Similarly, we can obtain the same result for $a_{j,k,2}$, namely,

$$\sum_{j,k\geq 1} a_{j,k,2} \lesssim \sup_{\bar\Omega} \frac{1}{\mu(\bar\Omega)^{\frac{2}{p}-1}} \sum_{R'\subset\bar\Omega} \mu(R')T_{R'}.$$

As to $a_{j,k,3}$, following the step of $a_{j,k,1}$, we can divide it into 4 terms, $a_{j,k,3.1}$, $a_{j,k,3.2}$, $a_{j,k,3.3}$ and $a_{j,k,3.4}$. For the first term, using the same skills as in the estimate of $a_{j,k,1.1}$, we can get that $\sum_{j,k\geq 1} a_{j,k,3.1}$ can be controlled by $\sup_{\bar\Omega} \frac{1}{\mu(\bar\Omega)^{\frac{2}{p}-1}} \cdot \sum_{R'\subset\bar\Omega} \mu(R')T_{R'}$. Now for the second term $a_{j,k,3.2}$, we have:

$$a_{j,k,3.2} \lesssim \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{\ell_1=1}^{j} \sum_{m\geq 0}$$
$$\cdot \sum_{h\geq 1}^{C(j-\ell_1+k+m)} 2^{-\ell_1(1+\varepsilon')} 2^{-m(1+\varepsilon')} 2^{\ell_1} 2^{-(j-\ell_1)(1+\theta')} 2^{-k(1+\theta')} \mu(\Omega_h^{j,k})^{\frac{2}{p}-1}$$
$$\times \frac{1}{\mu(\Omega_h^{j,k})^{\frac{2}{p}-1}} \sum_{R'\subset\Omega_h^{j,k}} \mu(R')T_{R'}$$
$$\lesssim \sum_{\ell=1}^{j} \sum_{h\geq 1}^{C(j-\ell)} 2^{-\ell\varepsilon'} 2^{-(j-\ell)(1+\theta')} 2^{-k\varepsilon'} (h2^h)^{(\frac{2}{p}-1)} \mu(\Omega)^{\frac{2}{p}-1}$$
$$\times \sup_{\bar\Omega} \frac{1}{\mu(\bar\Omega)^{\frac{2}{p}-1}} \sum_{R'\subset\bar\Omega} \mu(R')T_{R'}$$
$$\lesssim 2^{-k\varepsilon'} \sum_{\ell=1}^{j} 2^{-\ell\varepsilon'}(j-\ell)^2 2^{-(j-\ell)(1+\theta'-(\frac{2}{p}-1))} \sup_{\bar\Omega} \frac{1}{\mu(\bar\Omega)^{\frac{2}{p}-1}} \sum_{R'\subset\bar\Omega} \mu(R')T_{R'}.$$

Since $1+\theta' > \frac{2}{p}-1$, it follows that

$$\begin{aligned}\sum_{j,k\geq 1} a_{j,k,3.2} &\lesssim \sum_{j,k\geq 1} 2^{-k\varepsilon'} \sum_{\ell=1}^{j} 2^{-\ell\varepsilon'} (j-\ell)^2 2^{-(j-\ell)(1+\theta'-(\frac{2}{p}-1))} \\ &\quad \cdot \sup_{\bar{\Omega}} \frac{1}{\mu(\bar{\Omega})^{\frac{2}{p}-1}} \sum_{R'\subset\bar{\Omega}} \mu(R')T_{R'} \\ &\lesssim \sup_{\bar{\Omega}} \frac{1}{\mu(\bar{\Omega})^{\frac{2}{p}-1}} \sum_{R'\subset\bar{\Omega}} \mu(R')T_{R'}.\end{aligned}$$

Then, $a_{j,k,3.3}$ and $a_{j,k,3.4}$ can be estimated in the same way and the two terms are both bounded by $\sup_{\bar{\Omega}} \frac{1}{\mu(\bar{\Omega})^{\frac{2}{p}-1}} \sum_{R'\subset\bar{\Omega}} \mu(R')T_{R'}$.

Finally, as to $a_{j,k,4}$, since $0\leq h\leq h_0$ in this case, we have

$$\begin{aligned} a_{j,k,4} &\lesssim \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h=1}^{h_0} 2^{-j\varepsilon'} 2^{-k\varepsilon'} \mu(\Omega_h^{j,k})^{\frac{2}{p}-1} \frac{1}{\mu(\Omega_h^{j,k})^{\frac{2}{p}-1}} \sum_{R'\subset\Omega_h^{j,k}} \mu(R')T_{R'} \\ &\lesssim \frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \sum_{h=0}^{h_0} 2^{-j\varepsilon'} 2^{-k\varepsilon'} (h2^h)^{(\frac{2}{p}-1)} \mu(\Omega)^{\frac{2}{p}-1} \\ &\quad \times \sup_{\bar{\Omega}} \frac{1}{\mu(\bar{\Omega})^{\frac{2}{p}-1}} \sum_{R'\subset\bar{\Omega}} \mu(R')T_{R'} \\ &\lesssim 2^{-j\varepsilon'} 2^{-k\varepsilon'} \sup_{\bar{\Omega}} \frac{1}{\mu(\bar{\Omega})^{\frac{2}{p}-1}} \sum_{R'\subset\bar{\Omega}} \mu(R')T_{R'}.\end{aligned}$$

Then,

$$\sum_{j,k\geq 1} a_{j,k,4} \lesssim \sup_{\bar{\Omega}} \frac{1}{\mu(\bar{\Omega})^{\frac{2}{p}-1}} \sum_{R'\subset\bar{\Omega}} \mu(R')T_{R'}.$$

Combining all the estimates above, we can obtain that

$$IV \lesssim \sup_{\bar{\Omega}} \frac{1}{\mu(\bar{\Omega})^{\frac{2}{p}-1}} \sum_{R'\subset\bar{\Omega}} \mu(R')T_{R'}.$$

Using the same ideas in the estimates of I and IV, we can obtain that

$$II, III \lesssim \sup_{\bar{\Omega}} \frac{1}{\mu(\bar{\Omega})^{\frac{2}{p}-1}} \sum_{R'\subset\bar{\Omega}} \mu(R')T_{R'}.$$

The proof of the Min-Max comparison principle for $\mathrm{CMO}^p(\mathcal{X}\times\mathcal{X})$ is complete. $\square$

Remark 3.3. The operators $D_{k_1'}D_{k_2'}$ in the right hand-side of (3.2) can be replaced by any other operators $E_{k_1'}E_{k_2'}$ which satisfy $E_{k_i}=P_{k_i}-P_{k_i-1}$ for $i=1,2$, where $\{P_{k_i}\}$ are approximation to identity of order $\epsilon\in(0,\theta]$.

Then we can see that the Min-Max comparison principle established above yields that the definition of $\mathrm{CMO}^p(\mathcal{X}\times\mathcal{X})$ is independent of the choice of approximations to identity. More precisely, let $\{S_{k_i}\}$ and $\{P_{k_i}\}$ be approximations to identity of order $\epsilon\in(0,\theta]$ and $D_{k_i}=S_{k_i}-S_{k_i-1}$, $E_{k_i}=P_{k_i}-P_{k_i-1}$ for $i=1,2$. Suppose $\|f\|_{\mathrm{CMO}^p(\mathcal{X}\times\mathcal{X})}$ is defined as (3.1) and $\|f\|_{\mathrm{CMO}^p(\mathcal{X}\times\mathcal{X}),*}$ is defined as (3.1) with D_{k_i} replaced by E_{k_i} for $i=1,2$. Then by using the Min-Max comparison principle, we can see that

$$
\begin{aligned}
&\|f\|_{\mathrm{CMO}^p(\mathcal{X}\times\mathcal{X})}\\
&\quad\le \sup_{\Omega}\Bigg(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\int_{\Omega}\sum_{k_1,k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\\
&\qquad\cdot\sum_{v_1=1}^{N(k_1,\tau_1)}\sum_{v_2=1}^{N(k_2,\tau_2)}\chi_{\{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}\subset\Omega\}}(k_1,k_2,\tau_1,\tau_2,v_1,v_2)\\
&\qquad\times\sup_{u\in Q_{\tau_1}^{k_1,v_1},v\in Q_{\tau_2}^{k_2,v_2}}\big|D_{K_1}D_{K_2}(f)(u,v)\big|^2\chi_{Q_{\tau_1}^{k_1,v_1}}(x_1)\chi_{Q_{\tau_2}^{k_2,v_2}}(x_2)\mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2)\Bigg)^{1/2}\\
&\quad\lesssim \sup_{\Omega}\Bigg(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\int_{\Omega}\sum_{k_1,k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}\\
&\qquad\cdot\sum_{v_2=1}^{N(k_2,\tau_2)}\chi_{\{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}\subset\Omega\}}(k_1,k_2,\tau_1,\tau_2,v_1,v_2)\\
&\qquad\times\inf_{u\in Q_{\tau_1}^{k_1,v_1},v\in Q_{\tau_2}^{k_2,v_2}}\big|E_{K_1}E_{K_2}(f)(u,v)\big|^2\chi_{Q_{\tau_1}^{k_1,v_1}}(x_1)\chi_{Q_{\tau_2}^{k_2,v_2}}(x_2)\mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2)\Bigg)^{1/2}\\
&\quad\le \|f\|_{\mathrm{CMO}^p(\mathcal{X}\times\mathcal{X}),*}.
\end{aligned}
$$

And similarly, we have $\|f\|_{\mathrm{CMO}^p(\mathcal{X}\times\mathcal{X}),*}\lesssim\|f\|_{\mathrm{CMO}^p(\mathcal{X}\times\mathcal{X})}$.

This implies that the Carleson measure space $\mathrm{CMO}^p(\mathcal{X}\times\mathcal{X})$ is well-defined.

4. Product sequence spaces and duality

We introduce the product sequence spaces s^p and c^p as follows.

Definition 4.1. Let $\tilde{\chi}_Q(x) = \mu(Q)^{-1/2}\chi_Q(x)$. The product sequence space s^p, $0 < p \le 1$, is defined as the collection of all complex-value sequences

$$\lambda = \{\lambda_{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}}\}_{k_1,k_2\in\mathbb{Z};\ \tau_1\in I_{k_1},\tau_2\in I_{k_2};\ v_1=1,..N(k_1,\tau_1),v_2=1,..N(k_2,\tau_2)}$$

such that $\|\lambda\|_{s^p}$

$$= \Bigg\| \Bigg\{ \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)} \cdot \sum_{v_2=1}^{N(k_2,\tau_2)} \Big(|\lambda_{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}}| \cdot \tilde{\chi}_{Q_{\tau_1}^{k_1,v_1}}(\cdot)\tilde{\chi}_{Q_{\tau_2}^{k_2,v_2}}(\cdot)\Big)^2 \Bigg\}^{1/2} \Bigg\|_{L^p} < \infty.$$

Similarly, c^p, $0 < p \le 1$, is defined as the collection of all complex-value sequences

$$t = \{t_{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}}\}_{k_1,k_2\in\mathbb{Z};\ \tau_1\in I_{k_1},\tau_2\in I_{k_2};\ v_1=1,..N(k_1,\tau_1),\ v_2=1,..N(k_2,\tau_2)}$$

such that $\|t\|_{c^p}$

$$= \sup_{\Omega}\Bigg(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \int_{\Omega} \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}} \cdot \sum_{v_1=1}^{N(k_1,\tau_1)}\sum_{v_2=1}^{N(k_2,\tau_2)} \chi_{\{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}\subset\Omega\}}(k_1,k_2,\tau_1,\tau_2,v_1,v_2) \times \Big(|t_{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}}| \cdot \tilde{\chi}_{Q_{\tau_1}^{k_1,v_1}}(x_1)\tilde{\chi}_{Q_{\tau_2}^{k_2,v_2}}(x_2)\Big)^2 d\mu(x_1)d\mu(x_2)\Bigg)^{1/2} < \infty.$$

For simplicity, $\forall s \in s^p$, we rewrite $s = \{s_R\}_R$, and

$$\|s\|_{s^p} = \Big\|\Big\{\sum_R |s_R\tilde{\chi}_R(x_1,x_2)|^2\Big\}^{1/2}\Big\|_{L^p}, \tag{4.1}$$

similarly, $\forall t \in c^p$, rewrite $t = \{t_R\}_R$, and

$$\|t\|_{c^p} = \sup_{\Omega}\Bigg(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\sum_{R\subseteq\Omega}|t_R|^2\Bigg)^{1/2}, \tag{4.2}$$

where R runs over all the dyadic rectangles in $\mathcal{X}\times\mathcal{X}$. The main result in this section is the following duality theorem.

Theorem 4.2. $(s^p)' = c^p$ *for* $0 < p \le 1$.

Proof. First, we prove that for all $t \in c^p$, let

$$L(s) = \sum_R s_R \cdot \bar{t}_R, \qquad \forall s \in s^p, \tag{4.3}$$

then $|L(s)| \lesssim \|s\|_{s^p} \|t\|_{c^p}$.

To see this, let

$$\Omega_k = \left\{(x_1, x_2) \in \mathcal{X} \times \mathcal{X} \ : \left\{\sum_R (|s_R|\tilde{\chi}_R(x_1, x_2))^2\right\}^{1/2} > 2^k\right\}.$$

And define

$$B_k = \left\{R : \mu(\Omega_k \cap R) > \frac{1}{2}\mu(R),\ \mu(\Omega_{k+1} \cap R) \le \frac{1}{2}\mu(R)\right\},$$

$$\tilde{\Omega}_k = \left\{(x_1, x_2) \in \mathcal{X} \times \mathcal{X} \ : \mathcal{M}_s(\chi_{\Omega_k}) > \frac{1}{2}\right\},$$

where M_s is the strong maximal function on $\mathcal{X} \times \mathcal{X}$. By (4.3) and the Hölder inequality,

$$\begin{aligned}
|L(s)| &\le \left(\sum_k \left(\sum_{R \in B_k} |s_R|^2\right)^{\frac{p}{2}} \left(\sum_{R \in B_k} |t_R|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \\
&\le \left(\sum_k \mu(\tilde{\Omega}_k)^{1-\frac{p}{2}} \left(\sum_{R \in B_k} |s_R|^2\right)^{\frac{p}{2}} \left(\frac{1}{\mu(\tilde{\Omega}_k)^{\frac{2}{p}-1}} \sum_{R \subset \tilde{\Omega}_k} |t_R|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \qquad (4.4)\\
&\le \left(\sum_k \mu(\tilde{\Omega}_k)^{1-\frac{p}{2}} \left(\sum_{R \in B_k} |s_R|^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \|t\|_{c^p}.
\end{aligned}$$

Combining the fact that $\int_{\tilde{\Omega}_k \backslash \Omega_{k+1}} \sum_{R \in B_k} (|s_R|\tilde{\chi}_R(x))^2 \, \mathrm{d}\mu(x) \le 2^{2(k+1)}\mu(\tilde{\Omega}_k \backslash \Omega_{k+1}) \le C2^{2k}\mu(\Omega_k)$ and that

$$\int_{\tilde{\Omega}_k \backslash \Omega_{k+1}} \sum_{R \in B_k} (|s_R|\tilde{\chi}_R(x))^2 \, d\mu(x) \ge \sum_{R \in B_k} |s_R|^2 \mu(R)^{-1} \mu\left(\tilde{\Omega}_k \backslash \Omega_{k+1} \cap R\right)$$

since $R \in B_k$ then $R$is contained in $\tilde{\Omega}_k$

$$\begin{aligned}
&\ge \sum_{R \in B_k} |s_R|^2 \mu(R)^{-1} \frac{1}{2}\mu(R) \\
&\ge \frac{1}{2} \sum_{R \in B_k} |s_R|^2,
\end{aligned}$$

we obtain $(\sum_{R\in B_k}|s_R|^2)^{\frac{p}{2}} \lesssim 2^{kp}\mu(\Omega_k)^{\frac{p}{2}}$. Substituting this back into the last term of (4.4) yields that $|L(s)| \lesssim \|s\|_{s^p}\|t\|_{c^p}$.

We point out that an idea similar to the one used in the above proof was used earlier to get an atomic decomposition from a wavelet expansion by Meyer in [22].

Conversely, we need to verify that for any $L \in (s^p)'$, there exists $t \in c^p$ with $\|t\|_{c^p} \le \|L\|$ such that for all $s \in s^p$, $L(s) = \sum_R s_R\bar{t}_R$. Here we adapt a similar idea in one-parameter case of Frazier and Jawerth in [9] to our multi-parameter situation.

Now define $s^i_R = 1$ when $R = R_i$ and $s^i_R = 0$ for all other R. Then it is easy to see that $\|S^i_R\|_{s^p} = 1$. Now for all $s \in s^p$, $s = \{s_R\} = \sum_i s_{R_i}s^i_{R_i}$, the limit holds in the norm of s^p, here we index all dyadic rectangles in $\mathcal{X}\times\mathcal{X}$ by $\{R_i\}_{i\in\mathbb{Z}}$. For any $L \in (s^p)'$, let $\bar{t}_{R_i} = L(s^i)$, then $L(s) = L(\sum_i s_{R_i}s^i) = \sum_i s_{R_i}\bar{t}_{R_i} = \sum_R s_R\bar{t}_R$. Let $t = \{t_R\}$. Then we only need to check that $\|t\|_{c^p} \le \|L\|$.

For any open set $\Omega \subset \mathcal{X}\times\mathcal{X}$ with finite measure, let $\bar{\mu}$ be a new measure such that $\bar{\mu}(R) = \frac{\mu(R)}{\mu(\Omega)}$ when $R \subset \Omega$, $\bar{\mu}(R) = 0$ when $R \not\subseteq \Omega$. And let $l^2(\bar{\mu})$ be a sequence space such that when $s \in l^2(\bar{\mu})$, $(\sum_{R\subset\Omega}|s_R|^2\frac{\mu(R)}{\mu(\Omega)^{\frac{p}{2}-1}})^{1/2} < \infty$. It is easy to see that $(l^2(\bar{\mu}))' = l^2(\bar{\mu})$. Then,

$$\begin{aligned}
\left\{\frac{1}{\mu(\Omega)^{\frac{p}{2}-1}}\sum_{R\subset\Omega}|t_R|^2\right\}^{1/2} &= \left\|\mu(R)^{-1/2}|t_R|\right\|_{l^2(\bar{\mu})} \\
&= \sup_{s:\ \|s\|_{l^2(\bar{\mu})}\le 1}\left|\sum_{R\subseteq\Omega}(|t_R|\mu(R)^{-1/2})\cdot s_R\cdot\frac{\mu(R)}{\mu(\Omega)^{\frac{1}{p}-1}}\right| \\
&\le \sup_{s:\ \|s\|_{l^2(\bar{\mu})}\le 1}\left|L\left(\chi_{R\subseteq\Omega}(R)\frac{\mu(R)^{1/2}|s_R|}{\mu(\Omega)^{\frac{p}{2}-1}}\right)\right| \\
&\le \sup_{s:\ \|s\|_{l^2(\bar{\mu})}\le 1}\|L\|\cdot\left\|\chi_{R\subseteq\Omega}(R)\frac{\mu(R)^{1/2}|s_R|}{\mu(\Omega)^{\frac{p}{2}-1}}\right\|_{s^p}.
\end{aligned}$$

By (4.1) and the Hölder inequality, we have

$$\left\|\chi_{R\subseteq\Omega}(R)\frac{\mu(R)^{1/2}|s_R|}{\mu(\Omega)^{\frac{p}{2}-1}}\right\|_{s^p} \le \left(\sum_{R\subseteq\Omega}|s_R|^2\frac{\mu(R)}{\mu(\Omega)^{\frac{p}{2}-1}}\right)^{1/2}.$$

Hence,

$$\|t\|_{c^p} \le \sup_{s:\ \|s\|_{l^2(\bar{\mu})}\le 1}\|L\|\cdot\|s\|_{l^2(\bar{\mu})} \le \|L\|. \qquad \square$$

5. Duality of $H^p(\mathcal{X} \times \mathcal{X})$ with $\mathrm{CMO}^p(\mathcal{X} \times \mathcal{X})$

In this section, we prove Theorem 1.2. First, we define the lifting and projection operators as follows.

Definition 5.1. Let $\{S_{k_i}\}_{k_i \in \mathbb{Z}}$ be an approximation to the identity of order θ, $D_{k_i} = S_{k_i} - S_{k_i - 1}$ for $i = 1, 2$. For any $f \in (\overset{\circ}{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2))'$ with $0 < \beta_i, \gamma_i < \theta$, define the lifting operator S_D by

$$S_D(f) = \left\{ \mu(Q_{\tau_1}^{k_1, v_1})^{1/2} \mu(Q_{\tau_2}^{k_2, v_2})^{1/2} D_{k_1} D_{k_2}(f)(y_1, y_2) \right\}_{Q_{\tau_1}^{k_1, v_1} \; Q_{\tau_2}^{k_2, v_2}}, \quad (5.1)$$

where y_i is the center of $Q_{\tau_i}^{k_i, v_i}$, $k_i \in \mathbb{Z}$, $\tau_i \in I_{k_i}$, $v = 1, \cdots, N(\tau_i, k_i)$ for $i = 1, 2$.

Definition 5.2. Let all the notation be the same as above. For any sequence s, define the projection operator $T_{\tilde{D}}$ by

$$\begin{aligned} T_{\tilde{D}}(s)(x_1, x_2) = \sum_{k_1 = -\infty}^{\infty} \sum_{k_2 = -\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{v_1 = 1}^{N(k_1, \tau_1)} \sum_{v_2 = 1}^{N(k_2, \tau_2)} s_{Q_{\tau_1}^{k_1, v_1} \times Q_{\tau_2}^{k_2, v_2}} \quad (5.2) \\ \times \mu(Q_{\tau_1}^{k_1, v_1})^{1/2} \mu(Q_{\tau_2}^{k_2, v_2})^{1/2} \tilde{D}_{k_1} \tilde{D}_{k_2}(x_1, x_2, y_1, y_2), \end{aligned}$$

where y_i is the center of $Q_{\tau_i}^{k_i, v_i}$ and $\tilde{D}_{k_i}$ is the same operator as in the Calderón reproducing formula (2.8) associated with D_{k_i} for $i = 1, 2$.

To work at the level of product sequences spaces, we still need the following two propositions.

Proposition 5.3. *Let all the notation be the same as above. Then for any* $f \in H^p(\mathcal{X} \times \mathcal{X})$, $\frac{1}{1+\theta} < p \le 1$,

$$\|S_D(f)\|_{s^p} \lesssim \|f\|_{H^p(\mathcal{X} \times \mathcal{X})}. \quad (5.3)$$

Conversely, for any $s \in s^p$,

$$\|T_{\tilde{D}}(s)\|_{H^p(\mathcal{X} \times \mathcal{X})} \lesssim \|s\|_{s^p}. \quad (5.4)$$

Moreover, $T_{\tilde{D}} \circ S_D$ *equals the identity on* $H^p(\mathcal{X} \times \mathcal{X})$.

Proposition 5.4. *Let all the notation be the same as above. Then for any* $f \in \mathrm{CMO}^p(\mathcal{X} \times \mathcal{X})$, $\frac{2}{2+\theta} < p \le 1$,

$$\|S_D(f)\|_{c^p} \lesssim \|f\|_{\mathrm{CMO}^p(\mathcal{X} \times \mathcal{X})}. \quad (5.5)$$

Conversely, for any $t \in c^p$,

$$\|T_{\tilde{D}}(t)\|_{\mathrm{CMO}^p(\mathcal{X} \times \mathcal{X})} \lesssim \|t\|_{c^p}. \quad (5.6)$$

Moreover, $T_{\tilde{D}} \circ S_D$ *is the identity on* $\mathrm{CMO}^p(\mathcal{X} \times \mathcal{X})$.

Assume the above two propositions first, then we give the proof of Theorem 1.2 with $p_0 = \frac{2}{2+\theta}$.

Proof of Theorem 1.2. First, let $\{S_{k_i}\}_{k_i\in\mathbb{Z}}$ be an approximation to the identity of order θ, $D_{k_i} = S_{k_i} - S_{k_i-1}$ for $i = 1, 2$. For any $g \in \overset{\circ}{\mathcal{G}}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ with $\frac{1}{p} - 1 < \beta_i, \gamma_i < \theta$ for $i = 1, 2$ and $f \in \mathrm{CMO}^p(\mathcal{X}\times\mathcal{X})$, from the two propositions above, we have

$$< f, g >=< T_{\tilde{D}} \circ S_D(f), g >=< S_D(f), S_{\tilde{D}}(g) >,$$

where $S_{\tilde{D}}(g) = \left\{\mu(Q_{\tau_1}^{k_1,v_1})^{1/2}\mu(Q_{\tau_2}^{k_2,v_2})^{1/2}\tilde{D}_{k_1,k_2}(g)(y_1, y_2)\right\}_{Q_{\tau_1}^{k_1,v_1}\ Q_{\tau_2}^{k_2,v_2}}$.

By the Definition 4.1 and the Min-Max comparison principle in Lemma 2.7, we obtain $\|S_{\tilde{D}}(g)\|_{s^p} \lesssim \|g\|_{H^p(\mathcal{X}\times\mathcal{X})}$. Hence $| < f, g > | \leq | < S_D(f), S_{\tilde{D}}(g) > | \lesssim \|f\|_{\mathrm{CMO}^p(\mathcal{X}\times\mathcal{X})}\|g\|_{H^p(\mathcal{X}\times\mathcal{X})}$, where the last inequality follows from Proposition 5.3 and 5.4. Since $\mathcal{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)$ is dense in $H^p(\mathcal{X} \times \mathcal{X})$, it follows from a standard density argument that $\mathrm{CMO}^p(\mathcal{X} \times \mathcal{X}) \subseteq (H^p(\mathcal{X} \times \mathcal{X}))'$.

Conversely, suppose $l \in (H^p(\mathcal{X} \times \mathcal{X}))'$. Then $l_1 \equiv l \circ T_{\tilde{D}} \in (s^p)'$ by Proposition 5.3. So by Theorem 4.2, there exists $t \in c^p$ such that $l_1(s) =< t, s >$ for all $s \in s^p$, and $\|t\|_{c^p} \approx \|l_1\| \lesssim \|l\|$, since T_D is bounded. We have $l_1 \circ S_D = l \circ T_D \circ S_D = l$, hence

$$l(g) = l \circ T_D(S_D(g)) =< t, S_D(g) >=< T_D(t), g >,$$

where

$$T_D(t)=\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(\tau_1,k_1)}\sum_{v_2=1}^{N(\tau_2,k_2)} t_{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}}\mu(Q_{\tau_1}^{k_1,v_1})^{\frac{1}{2}}\mu(Q_{\tau_2}^{k_2,v_2})^{\frac{1}{2}} \times D_{k_1,k_1}(x_1, x_2, y_1, y_2).$$

By Definition 4.1 and the Min-Max comparison principle in Theorem 3.2, we obtain that $\|T_D(t)\|_{\mathrm{CMO}^p(\mathcal{X}\times\mathcal{X})} \leq \|t\|_{c^p} \leq \|l\|$. Hence $(H^p(\mathcal{X} \times \mathcal{X}))' \subseteq \mathrm{CMO}^p(\mathcal{X} \times \mathcal{X})$. $\square$

Now we give brief proofs to the above two propositions.

Proof of Proposition 5.3. To show this Proposition, we first point out that the proof is closely related to the Min-Max comparison principle for $H^p(\mathcal{X} \times \mathcal{X})$, namely, Lemma 2.7. (5.3) is a direct consequence of Lemma 2.7 and the proof of (5.4) follows the same routine as the proof of Lemma 2.7.

Now let us go into the details. We first prove (5.3). By Definition 4.1 and 5.1, we can see that for any $f \in H^p(\mathcal{X} \times \mathcal{X})$,

$$
\begin{aligned}
&\|S_D(f)\|_{s^p} \\
&\leq \Bigg\| \Bigg\{ \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{v_1=1}^{N(\tau_1,k_1)} \sum_{v_2=1}^{N(\tau_2,k_2)} \sup_{u \in Q_{\tau_1}^{k_1,v_1}, v \in Q_{\tau_2}^{k_2,v_2}} \big|D_{k_1} D_{k_2}(f)(u,v)\big|^2 \\
&\qquad \times \chi_{Q_{\tau_1}^{k_1,v_1}}(\cdot) \chi_{Q_{\tau_2}^{k_2,v_2}}(\cdot) \Bigg\}^{\frac{1}{2}} \Bigg\|_{L^p} \\
&\lesssim \Bigg\| \Bigg\{ \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{v_1=1}^{N(\tau_1,k_1)} \sum_{v_2=1}^{N(\tau_2,k_2)} \inf_{u \in Q_{\tau_1}^{k_1,v_1}, v \in Q_{\tau_2}^{k_2,v_2}} \big|D_{k_1} D_{k_2}(f)(u,v)\big|^2 \\
&\qquad \times \chi_{Q_{\tau_1}^{k_1,v_1}}(\cdot) \chi_{Q_{\tau_2}^{k_2,v_2}}(\cdot) \Bigg\}^{\frac{1}{2}} \Bigg\|_{L^p} \\
&\leq \Bigg\| \Bigg\{ \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \big|D_{k_1} D_{k_2}(f)(\cdot,\cdot)\big|^2 \Bigg\}^{\frac{1}{2}} \Bigg\|_{L^p} \\
&\lesssim \|f\|_{H^p}.
\end{aligned}
$$

Now let us turn to (5.4). For simplicity, we only need to work with the dyadic cubes of the form $\big\{Q_{\tau_i}^{k_i} : \ k_i \in \mathbb{Z}, \ \tau_i \in I_{k_i+J}\big\}$ for $i = 1, 2$.

To simplify our notation, let $m_{Q_{\tau_1}^{k_1} \times Q_{\tau_2}^{k_2}}(x_1, x_2) = \mu(Q_{\tau_1}^{k_1})^{1/2} \mu(Q_{\tau_2}^{k_2})^{1/2} \tilde{D}_{k_1}(x_1, y_{\tau_1}^{k_1})\, \tilde{D}_{k_2}(x_2, y_{\tau_2}^{k_2})$. Now we first estimate $D_{j_1} D_{j_2}(m_{Q_{\tau_1}^{k_1} \times Q_{\tau_2}^{k_2}})(x_1, x_2)$.

According to the relations between j_i and k_i for $i = 1, 2$, we split $D_{j_1} D_{j_2}(m_{Q_{\tau_1}^{k_1} \times Q_{\tau_2}^{k_2}})(x_1, x_2)$ into four cases as follows.

Case 1: $j_1 \geq k_1,\ j_2 \geq k_2$.

Using the cancellation condition on D_{j_1} and D_{j_2}, we have

$$
\begin{aligned}
&|D_{j_1} D_{j_2}(m_{Q_{\tau_1}^{k_1} \times Q_{\tau_2}^{k_2}})(x_1, x_2)| \\
&= \Bigg| \Bigg(\int_{V_1} + \int_{V_2} + \int_{V_3} + \int_{V_4} \Bigg) D_{j_1}(x_1, y_1) D_{j_2}(x_2, y_2) \Big[m_{Q_{\tau_1}^{k_1} \times Q_{\tau_2}^{k_2}}(x_1, x_2) \\
&\qquad - m_{Q_{\tau_1}^{k_1} \times Q_{\tau_2}^{k_2}}(y_1, x_2) - m_{Q_{\tau_1}^{k_1} \times Q_{\tau_2}^{k_2}}(x_1, y_2) + m_{Q_{\tau_1}^{k_1} \times Q_{\tau_2}^{k_2}}(y_1, y_2) \Big] d\mu(y_1) d\mu(y_2) \Bigg| \\
&= I + II + III + IV,
\end{aligned}
$$

where

$$V_1 = \Big\{(y_1, y_2) : \rho(x_1, y_1) \le \frac{1}{2A}(2^{-k_1} + \rho(x_1, y_{\tau_1}^{k_1})), \rho(x_2, y_2) \le \frac{1}{2A}(2^{-k_2} + \rho(x_2, y_{\tau_2}^{k_2}))\Big\};$$

$$V_2 = \Big\{(y_1, y_2) : \rho(x_1, y_1) \le \frac{1}{2A}(2^{-k_1} + \rho(x_1, y_{\tau_1}^{k_1})), \rho(x_2, y_2) > \frac{1}{2A}(2^{-k_2} + \rho(x_2, y_{\tau_2}^{k_2}))\Big\};$$

$$V_3 = \Big\{(y_1, y_2) : \rho(x_1, y_1) > \frac{1}{2A}(2^{-k_1} + \rho(x_1, y_{\tau_1}^{k_1})), \rho(x_2, y_2) \le \frac{1}{2A}(2^{-k_2} + \rho(x_2, y_{\tau_2}^{k_2}))\Big\};$$

$$V_4 = \Big\{(y_1, y_2) : \rho(x_1, y_1) > \frac{1}{2A}(2^{-k_1} + \rho(x_1, y_{\tau_1}^{k_1})), \rho(x_2, y_2) > \frac{1}{2A}(2^{-k_2} + \rho(x_2, y_{\tau_2}^{k_2}))\Big\}.$$

For term I, we use smoothness condition (2.4) on both $\tilde{D}_{k_1}$ and $\tilde{D}_{k_2}$ with the first variable; for term II, we use smoothness condition (2.4) on $\tilde{D}_{k_1}$ with the first variable, and size condition (2.3) on $\tilde{D}_{k_2}$; similarly, for term III, we use size condition (2.3) on $\tilde{D}_{k_1}$ and smoothness condition (2.4) on $\tilde{D}_{k_2}$ with the first variable; for term IV, we use size condition on both $\tilde{D}_{k_1}$ and $\tilde{D}_{k_2}$. Together with the fact that $\mu(Q_{\tau_i}^{k_i}) \sim 2^{-k_i}$ for $i = 1, 2$, we can get that four terms above can be controlled by

$$\mu(Q_{\tau_1}^{k_1})^{-\frac{1}{2}}\mu(Q_{\tau_2}^{k_2})^{-\frac{1}{2}}\frac{2^{-(j_1-k_1)\varepsilon'}}{(1+2^{k_1}\rho(x_1, y_{\tau_1}^{k_1}))^{1+\varepsilon'}}\frac{2^{-(j_2-k_2)\varepsilon'}}{(1+2^{k_2}\rho(x_2, y_{\tau_2}^{k_2}))^{1+\varepsilon'}}. \tag{5.7}$$

Case 2: $j_1 \ge k_1$, $j_2 < k_2$.

By the cancellation condition on D_{j_1} and $\tilde{D}_{k_2}$, we have

$$\begin{aligned}
&|D_{j_1}D_{j_2}(m_{Q_{\tau_1}^{k_1}\times Q_{\tau_2}^{k_2}})(x_1, x_2)| \\
&= \Big|\Big(\int_{V_1} + \int_{V_2} + \int_{V_3} + \int_{V_4}\Big)\big[D_{j_1}(x_1, y_1)D_{j_2}(x_2, y_2) - D_{j_1}(x_1, y_1)D_{j_2}(x_2, y_{\tau_2}^{k_2})\big] \\
&\quad \times\big[m_{Q_{\tau_1}^{k_1}\times Q_{\tau_2}^{k_2}}(y_1, y_2) - m_{Q_{\tau_1}^{k_1}\times Q_{\tau_2}^{k_2}}(x_1, y_2)\big]d\mu(y_1)d\mu(y_2)\Big| \\
&= I + II + III + IV,
\end{aligned}$$

where

$$V_1 = \Big\{(y_1, y_2) : \rho(x_1, y_1) \le \frac{1}{2A}(2^{-k_1} + \rho(x_1, y^{k_1}_{\tau_1})), \rho(y_2, y^{k_2}_{\tau_2}) \le \frac{1}{2A}(2^{-k_2} + \rho(x_2, y_2))\Big\};$$

$$V_2 = \Big\{(y_1, y_2) : \rho(x_1, y_1) \le \frac{1}{2A}(2^{-k_1} + \rho(x_1, y^{k_1}_{\tau_1})), \rho(y_2, y^{k_2}_{\tau_2}) > \frac{1}{2A}(2^{-k_2} + \rho(x_2, y_2))\Big\};$$

$$V_3 = \Big\{(y_1, y_2) : \rho(x_1, y_1) > \frac{1}{2A}(2^{-k_1} + \rho(x_1, y^{k_1}_{\tau_1})), \rho(y_2, y^{k_2}_{\tau_2}) \le \frac{1}{2A}(2^{-k_2} + \rho(x_2, y_2))\Big\};$$

$$V_4 = \Big\{(y_1, y_2) : \rho(x_1, y_1) > \frac{1}{2A}(2^{-k_1} + \rho(x_1, y^{k_1}_{\tau_1})), \rho(y_2, y^{k_2}_{\tau_2}) > \frac{1}{2A}(2^{-k_2} + \rho(x_2, y_2))\Big\}.$$

For term I, we use the size condition on D_{j_1} and $\tilde{D}_{k_2}$, the smoothness condition on D_{j_2} and $\tilde{D}_{k_1}$; for term II, we use the smoothness condition on $\tilde{D}_{k_1}$ and the size condition on others; for III, we use the smoothness condition on D_{j_2} and the size condition on others; finally, for term IV, we only use the size conditions. Similarly, the four terms above can be controlled by

$$\mu(Q^{k_1}_{\tau_1})^{-\frac{1}{2}}\mu(Q^{k_2}_{\tau_2})^{-\frac{1}{2}}\frac{2^{-(j_1-k_1)\varepsilon'}}{(1+2^{k_1}\rho(x_1, y^{k_1}_{\tau_1}))^{1+\varepsilon'}}\frac{2^{(j_2-k_2)(1+\varepsilon')}}{(1+2^{j_2}\rho(x_2, y^{k_2}_{\tau_2}))^{1+\varepsilon'}}. \tag{5.8}$$

Case 3: $j_1 < k_1$, $j_2 \ge k_2$.

Similarly as Case 2, $D_{j_1}D_{j_2}(m_{Q^{k_1}_{\tau_1}\times Q^{k_2}_{\tau_2}})(x_1, x_2)$ can be controlled by

$$\mu(Q^{k_1}_{\tau_1})^{-\frac{1}{2}}\mu(Q^{k_2}_{\tau_2})^{-\frac{1}{2}}\frac{2^{(j_1-k_1)(1+\varepsilon')}}{(1+2^{j_1}\rho(x_1, y^{k_1}_{\tau_1}))^{1+\varepsilon'}}\frac{2^{-(j_2-k_2)\varepsilon'}}{(1+2^{k_2}\rho(x_2, y^{k_2}_{\tau_2}))^{1+\varepsilon'}}. \tag{5.9}$$

Case 4: $j_1 < k_1$, $j_2 < k_2$.

Similarly as Case 1 with only a change of the positions of $D_{j_1}D_{j_2}$ and $m_{Q^{k_1}_{\tau_1}\times Q^{k_2}_{\tau_2}}$, we can see that $D_{j_1}D_{j_2}(m_{Q^{k_1}_{\tau_1}\times Q^{k_2}_{\tau_2}})(x_1, x_2)$ can be controlled by

$$\mu(Q^{k_1}_{\tau_1})^{-\frac{1}{2}}\mu(Q^{k_2}_{\tau_2})^{-\frac{1}{2}}\frac{2^{(j_1-k_1)(1+\varepsilon')}}{(1+2^{j_1}\rho(x_1, y^{k_1}_{\tau_1}))^{1+\varepsilon'}}\frac{2^{(j_2-k_2)(1+\varepsilon')}}{(1+2^{j_2}\rho(x_2, y^{k_2}_{\tau_2}))^{1+\varepsilon'}}. \tag{5.10}$$

From Definition 2.6 and 5.2, we have

$$\left\|T_{\tilde{D}}(s)(x_1,x_2)\right\|^p_{H^p(\mathcal{X}\times\mathcal{X})} = \|g(T_{\tilde{D}}(s))\|^p_{L^p(\mathcal{X}\times\mathcal{X})}$$

$$\lesssim \Bigg\|\Bigg\{\sum_{j_1,j_2}\Bigg(\Bigg[\sum_{k_1>j_1,k_2>j_2}+\sum_{k_1>j_1,k_2\le j_2}+\sum_{k_1\le j_1,k_2>j_2}+\sum_{k_1\le j_1,k_2\le j_2}\Bigg]$$

$$\times\sum_{\tau_1\in I_{k_1+J_1}}\sum_{\tau_2\in I_{k_2+J_2}}|s_{Q^{k_1}_{\tau_1}\times Q^{k_2}_{\tau_2}}||D_{j_1}D_{j_2}(m_{Q^{k_1}_{\tau_1}\times Q^{k_2}_{\tau_2}})(x_1,x_2)|\Bigg)^2\Bigg\}^{1/2}\Bigg\|^p_{L^p(\mathcal{X}\times\mathcal{X})}$$

$$\lesssim I+II+III+IV.$$

We now first estimate I. From (5.10), we can see that

$$\sum_{k_1>j_1,k_2>j_2}\sum_{\tau_1\in I_{k_1+J_1}}\sum_{\tau_2\in I_{k_2+J_2}}|s_{Q^{k_1}_{\tau_1}\times Q^{k_2}_{\tau_2}}||D_{j_1}D_{j_2}(m_{Q^{k_1}_{\tau_1}\times Q^{k_2}_{\tau_2}})(x_1,x_2)|$$

$$\lesssim \sum_{k_1>j_1,k_2>j_2}\sum_{\tau_1\in I_{k_1+J}}$$

$$\cdot\sum_{\tau_2\in I_{k_2+J}}2^{(j_1-k_1)(1+\varepsilon')}2^{(j_2-k_2)(1+\varepsilon')}|s_{Q^{k_1}_{\tau_1}\times Q^{k_2}_{\tau_2}}|\mu(Q^{k_1}_{\tau_1})^{-1/2}\mu(Q^{k_2}_{\tau_2})^{-1/2}$$

$$\times\frac{1}{(1+2^{j_1}\rho(x_1,y^{k_1}_{\tau_1}))^{1+\varepsilon'}}\frac{1}{(1+2^{j_2}\rho(x_2,y^{k_2}_{\tau_2}))^{1+\varepsilon'}}$$

$$\lesssim \sum_{k_1>j_1,k_2>j_2}2^{(j_1-k_1)(1+\varepsilon'-\frac{1}{r})}2^{(j_2-k_2)(1+\varepsilon'-\frac{1}{r})}\Bigg(\mathcal{M}_1\Bigg[\sum_{\tau_1\in I_{k_1+J}}$$

$$\times\mathcal{M}_2\Bigg(\sum_{\tau_2\in I_{k_2+J}}|s_{Q^{k_1}_{\tau_1}\times Q^{k_2}_{\tau_2}}\mu(Q^{k_1}_{\tau_1})^{-1/2}\mu(Q^{k_2}_{\tau_2})^{-1/2}|^r\chi_{Q^{k_2}_{\tau_2}}(\cdot)\Bigg)(x_2)\chi_{Q^{k_1}_{\tau_1}}(\cdot)\Bigg](x_1)\Bigg)^{\frac{1}{r}},$$

where $\frac{2}{2+\theta}<r<p$ and $\mathcal{M}_i$, $i=1,2$, is the Hardy-Littlewood Maximal function with respect to the first and the second variable, respectively. The last inequality

follows from an iteration of the the result which can be found in [9, pages 147–148], for $\mathbb{R}^n$ and [18, page 93], for spaces of homogeneous type.

Let $k = (k_1, k_2)$, $j = (j_1, j_2)$, $x = (x_1, x_2)$ and

$$\begin{aligned} a(x) &= \{a_k(x)\}_k \\ &= \Bigg(\mathcal{M}_1\Bigg[\sum_{\tau_1 \in I_{k_1+J}} \mathcal{M}_2\Bigg(\sum_{\tau_2 \in I_{k_2+J}} |s_{Q^{k_1}_{\tau_1}\times Q^{k_2}_{\tau_2}}\mu(Q^{k_1}_{\tau_1})^{-1/2}\mu(Q^{k_2}_{\tau_2})^{-1/2}|^r\chi_{Q^{k_2}_{\tau_2}}(\cdot)\Bigg) \\ &\quad \cdot (x_2)\chi_{Q^{k_1}_{\tau_1}}(\cdot)\Bigg](x_1)\Bigg)^{\frac{1}{r}}; \\ b &= \{b_k\}_k = \Big\{2^{k_1(1+\varepsilon'-\frac{1}{r})}2^{k_2(1+\varepsilon'-\frac{1}{r})}\chi_{\{k_1<0\}}(k_1)\chi_{\{k_2<0\}}(k_2)\Big\}_k; \\ (a*b)_j &= \sum_k a_k b_{j-k}. \end{aligned}$$

By the Young inequality and an iterative application of the Fefferman and Stein vector-valued maximal function inequality in [8] on $L^{\frac{p}{r}}(\mathcal{X})$, we have

$$\begin{aligned} IV &\lesssim \Bigg\|\Bigg\{\sum_j |(a*b)_j|^2\Bigg\}^{1/2}\Bigg\|^p_{L^p(\mathcal{X}\times\mathcal{X})} \lesssim \big\|\|a*b\|_{l^2}\big\|^p_{L^p(\mathcal{X}\times\mathcal{X})} \\ &\lesssim \big\|\|a\|_{l^2}\|b\|_{l^1}\big\|^p_{L^p(\mathcal{X}\times\mathcal{X})} \\ &\lesssim \big\|\|a\|_{l^2}\big\|^p_{L^p(\mathcal{X}\times\mathcal{X})} \\ &\lesssim \|s\|^p_{s^p}. \end{aligned}$$

Using the same skills, we can get that $II, III, IV \lesssim \|s\|^p_{s^p}$. Thus

$$\|T_{\tilde{D}}(s)(x_1, x_2)\|_{H^p(\mathcal{X}\times\mathcal{X})} \lesssim \|s\|_{s^p}.$$

Finally, it is easy to check that from the Calderón reproducing formula, $T_{\tilde{D}} \circ S_D$ equals identity on $H^p(\mathcal{X}\times\mathcal{X})$. The proof of proposition is complete. □

Proof of Proposition 5.4. This proposition is similar as the above one since its proof is closely related to the Min-Max comparison principle for $\mathrm{CMO}^p(\mathcal{X}\times\mathcal{X})$, namely, Theorem 3.2. (5.5) is a direct consequence of Theorem 3.2 and the proof of (5.6) follows the same routine as the proof of Theorem 3.2.

Now we give the details of the proof. We first prove (5.5). According to Definition 4.1 and 5.1, for any $f \in \mathrm{CMO}^p(\mathcal{X} \times \mathcal{X})$, we have

$$
\begin{aligned}
&\|S_D(f)\|_{c^p}\\
&\lesssim \sup_{\Omega} \Bigg(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \int_{\Omega} \sum_{k_1,k_2=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{v_1=1}^{N(k_1,\tau_1)} \\
&\quad \cdot \sum_{v_2=1}^{N(k_2,\tau_2)} \chi_{\{Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2} \subset \Omega\}}(k_1, k_2, \tau_1, \tau_2, v_1, v_2)\\
&\quad \times \sup_{u \in Q_{\tau_1}^{k_1,v_1}, v \in Q_{\tau_2}^{k_2,v_2}} \left| D_{k_1} D_{k_2}(f)(u, v) \right|^2 \chi_{Q_{\tau_1}^{k_1,v_1}}(x_1) \chi_{Q_{\tau_2}^{k_2,v_2}}(x_2) \mathrm{d}\mu(x_1) \mathrm{d}\mu(x_2) \Bigg)^{1/2}\\
&\lesssim \sup_{\Omega} \Bigg(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \int_{\Omega} \sum_{k_1,k_2=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \sum_{v_1=1}^{N(k_1,\tau_1)} \\
&\quad \cdot \sum_{v_2=1}^{N(k_2,\tau_2)} \chi_{\{Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2} \subset \Omega\}}(k_1, k_2, \tau_1, \tau_2, v_1, v_2)\\
&\quad \times \inf_{u \in Q_{\tau_1}^{k_1,v_1}, v \in Q_{\tau_2}^{k_2,v_2}} \left| D_{k_1} D_{k_2}(f)(u, v) \right|^2 \chi_{Q_{\tau_1}^{k_1,v_1}}(x_1) \chi_{Q_{\tau_2}^{k_2,v_2}}(x_2) \mathrm{d}\mu(x_1) \mathrm{d}\mu(x_2) \Bigg)^{1/2}\\
&\lesssim \sup_{\Omega} \Bigg(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}} \int_{\Omega} \sum_{k_1,k_2=-\infty}^{\infty} \sum_{\tau_1 \in I_{k_1}} \sum_{\tau_2 \in I_{k_2}} \\
&\quad \cdot \sum_{v_1=1}^{N(k_1,\tau_1)} \sum_{v_2=1}^{N(k_2,\tau_2)} \chi_{\{Q_{\tau_1}^{k_1,v_1} \times Q_{\tau_2}^{k_2,v_2} \subset \Omega\}}(k_1, k_2, \tau_1, \tau_2, v_1, v_2)\\
&\quad \times \left| D_{k_1} D_{k_2}(f)(x_1, x_2) \right|^2 \chi_{Q_{\tau_1}^{k_1,v_1}}(x_1) \chi_{Q_{\tau_2}^{k_2,v_2}}(x_2) \mathrm{d}\mu(x_1) \mathrm{d}\mu(x_2) \Bigg)^{1/2}\\
&\leq \|f\|_{\mathrm{CMO}^p}.
\end{aligned}
$$

Now let us prove (5.6). For any $t \in c^p$, by the definition of norm of CMO^p, we have

$$\|T_{\tilde{D}}(t)\|_{\mathrm{CMO}^p(\mathcal{X}\times\mathcal{X})} \lesssim \sup_{\Omega}\Bigg(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\int_{\Omega}\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}$$

$$\cdot\sum_{v_2=1}^{N(k_2,\tau_2)}\chi_{Q_{\tau_1}^{k_1,v_1}}(x_1)\chi_{Q_{\tau_2}^{k_2,v_2}}(x_2)$$

$$\times\chi_{\{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}\subset\Omega\}}(k_1,k_2,\tau_1,\tau_2,v_1,v_2)$$

$$\cdot\left|D_{k_1}D_{k_2}\left(T_{\tilde{D}}(t)\right)(x_1,x_2)\right|^2 \mathrm{d}\mu(x_1)\mathrm{d}\mu(x_2)\Bigg)^{\frac{1}{2}}.$$

From the definition of $T_{\tilde{D}}(t)$ and the same skill as in the estimate of (3.5), we can obtain that

$$\sup_{x_1\in Q_{\tau_1}^{k_1,v_1},\,x_2\in Q_{\tau_2}^{k_2,v_2}}|D_{k_1}D_{k_2}\left(T_{\tilde{D}}(t)\right)(x_1,x_2)|^2$$

$$\lesssim\sum_{k_1'=-\infty}^{\infty}\sum_{k_2'=-\infty}^{\infty}\sum_{\tau_1'\in I_{k_1'}}\sum_{\tau_2'\in I_{k_2'}}\sum_{v_1'=1}^{N(k_1',\tau_1')}$$

$$\cdot\sum_{v_2'=1}^{N(k_2',\tau_2')}2^{-|k_1-k_1'|\varepsilon'}2^{-|k_2-k_2'|\varepsilon'}\mu(Q_{\tau_1'}^{k_1',v_1'})\mu(Q_{\tau_2'}^{k_2',v_2'}) \tag{5.11}$$

$$\times\frac{2^{-(k_1\wedge k_1')\varepsilon'}}{(2^{-(k_1\wedge k_1')}+\rho(y_1,y_1'))^{1+\varepsilon'}}\frac{2^{-(k_2\wedge k_2')\varepsilon'}}{(2^{-(k_2\wedge k_2')}+\rho(y_2,y_2'))^{1+\varepsilon'}}$$

$$\times\left|t_{Q_{\tau_1'}^{k_1',v_1'}\times Q_{\tau_2'}^{k_2',v_2'}}\mu(Q_{\tau_1'}^{k_1',v_1'})^{-1/2}\mu(Q_{\tau_2'}^{k_2',v_2'})^{-1/2}\right|^2,$$

where y_i is the center of $Q_{\tau_i}^{k_i,v_i}$ and y_i' is the center of $Q_{\tau_i'}^{k_i',v_i'}$ for $i=1,2$.

Comparing (5.11) with (3.5), we can see that the only thing different is that the last term in the right-hand side of(3.5) is $T_{R'}$, while the last term in the right-hand side of (5.11) is $\Big|t_{Q_{\tau_1'}^{k_1',v_1'}\times Q_{\tau_2'}^{k_2',v_2'}}\;\mu(Q_{\tau_1'}^{k_1',v_1'})^{-1/2}\mu(Q_{\tau_2'}^{k_2',v_2'})^{-1/2}\Big|^2$. However, when proving the Theorem 3.2, we can see that the term $T_{R'}$ is fixed throughout the whole proof. This implies that we can prove this proposition just following the proof of Theorem 3.2 without any changes.

Thus, we can obtain that

$$
\begin{aligned}
&\|T_{\tilde{D}}(t)\|_{\mathrm{CMO}^p(\mathcal{X}\times\mathcal{X})}\\
&\lesssim \sup_{\Omega}\Bigg(\frac{1}{\mu(\Omega)^{\frac{2}{p}-1}}\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\sum_{\tau_1\in I_{k_1}}\sum_{\tau_2\in I_{k_2}}\sum_{v_1=1}^{N(k_1,\tau_1)}\\
&\quad\cdot\sum_{v_2=1}^{N(k_2,\tau_2)}\chi_{\{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}\subset\Omega\}}(k_1,k_2,\tau_1,\tau_2,v_1,v_2)\\
&\quad\times\mu(Q_{\tau_1}^{k_1,v_1})\mu(Q_{\tau_2}^{k_2,v_2})\Big|t_{Q_{\tau_1}^{k_1,v_1}\times Q_{\tau_2}^{k_2,v_2}}\;\mu(Q_{\tau_1}^{k_1,v_1})^{-1/2}\mu(Q_{\tau_2}^{k_2,v_2})^{-1/2}\Big|^2\Bigg)^{1/2}\\
&\lesssim \|t\|_{c^p}.
\end{aligned}
$$

Finally, we can easily get that from the Calderón reproducing formula $T_{\tilde{D}}\circ S_D$ is the identity operator on $\mathrm{CMO}^p(\mathcal{X}\times\mathcal{X})$. We finish the proof of the proposition. $\square$

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