Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)
Vol. IX (2010), 459-494

Intrinsic deformation theory of *CR* structures

PAOLO DE BARTOLOMEIS AND FRANCINE MEYLAN

Abstract. Let (V, ξ) be a contact manifold and let J be a strictly pseudoconvex *CR* structure of hypersurface type on (V, ξ) ; starting only from these data, we define and we investigate a Differential Graded Lie Algebra which governs the deformation theory of J.

Mathematics Subject Classification (2010): 32H02 (primary); 32H35 (secondary).

1. Introduction

The present research originates from two facts:

- Deformations Theories represent a fundamental tool to achieve a deeper insight on Geometric Structures;
- Kähler Geometry can be viewed as a Holomorphic Calibrated Geometry J over a symplectic structure κ.

Thus, if we start from a (compact) Kähler manifold (M, κ, J) we can:

1. holomorphically deform J and ask for J-compatible symplectic structures

or

2. symplectically deform κ and look for κ-calibrated holomorphic structures.

The first case provides the celebrated Kodaira-Spencer stability theory of Kähler manifolds: small holomorphic deformations of a Kähler manifold are again Kähler (*cf.* [4]). The second case is entirely covered in [3], where symplectic deformations of a Kähler structure that admit calibrated holomorphic structures are completely characterized.

In this paper, we consider similar questions for contact and *CR* structures on a compact odd dimensional manifold.

The second author was partially supported by Swiss NSF Grant 2100-063464.00/1.
Received July 14, 2008; accepted in revised form July 8, 2009.

First of all, with respect to the even dimensional hierarchy paradigma

complex
holomorphic
Kähler

in the odd dimensional case there is a gap

complex
CR

the notion of purely holomorphic structure being meaningless; moreover, if we want to consider deformations of *CR* structures, because of Gray's stability theorem (*cf.* [5]), we can keep the underlying contact structure fixed. As main result, we define, describe, and investigate a Differential Graded Lie Algebra (DGLA) that governs intrinsically the deformation theory of *CR* structures, *i.e.* starting only from the contact distribution, with no extra choice (*e.g.* of a contact form): consequently we do not invoke any embedding theorem.

In some sense, here, with the hidden help of the symplectic theory, there is a change of point of view with respect to the traditional one: instead of considering $\bar\partial_b$ (which, for vector valued forms, needs a choice of a contact form to project tangentially $\bar\partial$), we consider forms on which $\bar\partial$ act tangentially.

The paper is organized as follow: after some preliminary matter on complex and holomorphic structures, contact structures and related topics, we provide some interesting formulas on curves of complex and *CR* structures (Theorem 4.5), reproving a version of Gray's stability theorem suitable for our further developments (Theorem 4.14). Then we construct our DGLA

$$\mathcal{A}_J(\xi) = \bigoplus_{p\geq 0} \mathcal{A}_J^1(\xi),$$

realizing it as a sub DGLA of the Lie algebra of graded derivations on the algebra of forms: we provide first, starting from a single strictly pseudoconvex *CR* structure J, a complete description of the family $\mathfrak{MC}(\mathcal{A}_J(\xi))$ af all strictly pseudoconvex *CR* structures on ξ (Theorem 5.16); then we consider the action of the gauge equivalence group $\mathcal{G}(\xi)$ on $\mathfrak{MC}(\mathcal{A}_J(\xi))$, describing $\mathfrak{MC}(\mathcal{A}_J(\xi))/\mathcal{G}(\xi)$ and its (virtual) tangent space as the moduli space of *CR* deformation (respectively infinitesimal deformations) of J.

Further discussions on the cohomology of $\mathcal{A}_J(\xi)$, as well as the local unobstructness of the theory are then developed together with some basic examples.

Remark 1.1. Our approach gives also a coordinate free description of the DGLA of $(1, 0)$-vector valued forms on a holomorphic manifold.

ACKNOWLEDGEMENTS. The Authors are pleased to thank the referee for valuable comments and suggestions.

2. Preliminaries

In this section, we recall basic definitions.

2.1. The even dimensional case

Definition 2.1. A linear complex structure (lcs) on a real vector space V is the datum of $J \in Aut(V)$ satisfying $J^2 = -I$.

We have the following properties:

1. J defines on V a structure of complex vector space: $iv := Jv$.
2. J induces a bigraduation on $\wedge^r(V^*)^{\mathbb{C}}$, where

$$\wedge^r(V^*)^{\mathbb{C}} = \bigoplus_{p+q=r} \wedge_J^{p,q} V^*.$$

3. All lcs's are linearly equivalent.
4. Every lcs J (*e.g.* in $\mathbb{R}^{2n}$) satisfying $\det(I - J_n J) \neq 0$, where J_n is the standard lcs on $\mathbb{R}^{2n}$, can be uniquely written as

$$J = (I + S)J_n(I + S)^{-1}, \quad SJ_n + J_n S = 0.$$

Definition 2.2.

$$\begin{aligned}(\wedge_J^{0,p} V^*)^{\mathbb{R}} \otimes V := \{\alpha \in \wedge^p V^* \otimes V | \alpha(X_1, \ldots, JX_j, \ldots, X_p) \\ = -J\alpha(X_1, \ldots, X_p), \ j = 1, \ldots, p\}.\end{aligned}$$

Remark 2.3. $L \in (\wedge_J^{0,1} V^*)^{\mathbb{R}} \otimes V$ if and only if $LJ + JL = 0$.

Definition 2.4. A linear symplectic structure on a $2n$-dimensional real vector space V is the datum of $\kappa \in \wedge^2 V^*$ such that $\kappa^n \neq 0$.

Definition 2.5. Let κ be a linear symplectic structure on a $2n$-dimensional real vector space V. A lcs J is said to be *κ-calibrated* if

$$g_J := \kappa(J\cdot, \cdot)$$

is a positive definite J-Hermitian metric.

In $\mathbb{R}^{2n}$, every J κ_n-calibrated, where κ_n is the standard symplectic structure on $\mathbb{R}^{2n}$, can be uniquely represented as

$$J = (I + L)J_n(I + L)^{-1}$$

with

$$J_n L + LJ_n = 0, ||L|| < 1, L = {}^tL$$

and so, in general, the set $\mathfrak{C}_\kappa(V)$ of κ-calibrated lcs's on V is an (n^2+n)-dimensional cell.

In the sequel, we shall consider the following isomorphism

$$\begin{aligned} m : V &\longrightarrow {V_J}^{1,0} \\ X &\mapsto \frac{1}{2}(X - iJX) \end{aligned} \tag{2.1}$$

and the corresponding isomorphism, again denoted by m, between $(\wedge_J^{0,*}V^*)^{\mathbb{R}} \otimes V$ and $(\wedge_J^{0,*}V^*)^{\mathbb{R}} \otimes {V_J}^{1,0}$, given by

$$m(L) := \frac{1}{2}(L - iJL). \tag{2.2}$$

Note that $m^{-1}(R) = R + \bar{R}$.

Definition 2.6. Let M be a $2n$-dimensional differentiable manifold and let J be a complex structure on TM. Denote by $\mathcal{H}(M)$ the set of smooth vector fields on M. For $X, Y \in \mathcal{H}(M)$, we write

$$\begin{aligned} [X, Y] = &\frac{1}{2}([X, Y] + [JX, JY]) \\ &+ \frac{1}{2}([X, Y] - [JX, JY] + \frac{1}{2}N_J(X, Y)) \\ &- \frac{1}{4}N_J(X, Y) := A(X, Y) + B(X, Y) + C(X, Y), \end{aligned} \tag{2.3}$$

where N_J is the Nijenhuis tensor of J,

$$N_J(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY]. \tag{2.4}$$

We have the following lemma whose easy proof is left to the reader.

Lemma 2.7. $N_J \in (\wedge_J^{0,2}T^*M)^{\mathbb{R}} \otimes TM$.
Moreover, if Z, W are $(1,0)-$vector fields, then

$$[Z, W]^{0,1} = -\frac{1}{4}N_J(Z, W). \tag{2.5}$$

Therefore

$$[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M \Leftrightarrow N_J = 0. \tag{2.6}$$

It is well known that:

$$d : \wedge_J^{p,q} \longrightarrow \wedge_J^{p+2,q-1} \oplus \wedge_J^{p+1,q} \oplus \wedge_J^{p,q+1} \oplus \wedge_J^{p-1,q+2}$$

and so d splits accordingly as

$$d = A_J + \partial_J + \bar{\partial}_J + \bar{A}_J$$

where:

- all the pieces are graded algebra derivations;
- A_J, $\bar{A}_J$ are 0-order differential operators;
- in particular, for $\alpha \in \wedge^1(M)$, we have

$$(A_J(\alpha) + \bar{A}_J(\alpha))(X, Y) = \frac{1}{4}\alpha(N_J(X, Y)).$$

Now we have the following:

Remark 2.8. We have:

1. $A(JX,\ JY) = A(X,\ Y)$;
2. $B(JX,\ Y) = B(X,\ JY) = JB(X,\ Y)$;
3. $C(JX,\ Y) = C(X,\ JY) = -JC(X,\ Y)$.

A is said to be of type $(1,\ 1)$, B of type $(2,\ 0)$, C of type $(0,\ 2)$. We can then view (2.3) as the type decomposition of the bracket [,].

We set the following definition:

Definition 2.9.

$$[[X,\ Y]] := B(X, Y) = \frac{1}{2}([X,\ Y] - [JX,\ JY] + \frac{1}{2}N_J(X,\ Y)). \tag{2.7}$$

In particular, if $\theta \in (\wedge_J^{0,p}T^*M)^{\mathbb{R}} \otimes TM$, then

$$(X_0,\ \ldots,\ X_p) \mapsto \sum_{0\le j\le k\le p} (-1)^{j+k}\theta([[X_j,\ X_k]],\ X_0,\ \ldots,\ \widehat{X}_j,\ \ldots,\ \widehat{X}_k,\ \ldots,\ X_p)$$

defines an element of $(\wedge_J^{0,p+1}T^*M)^{\mathbb{R}} \otimes TM$.

Note also that, for Z, W sections of $T_J^{1,0}M$, we have:

$$\begin{aligned}[[Z, W]] := m[[m^{-1}(Z), m^{-1}(W)]] &= [Z, W] + \frac{1}{4}N_J(Z, W)\\ &= \frac{1}{2}([Z, W] - iJ[Z, W]).\end{aligned}$$

We recall the following definition:

Definition 2.10. Let

$$\bar{\partial}_J : (\wedge_J^{0,p}T^*M)^{\mathbb{R}} \otimes TM \longrightarrow (\wedge_J^{0,p+1}T^*M)^{\mathbb{R}} \otimes TM$$

be the operator defined as follows:

1. For $X \in \mathcal{H}(M)$,

$$(\bar\partial_J X)(Y) := \frac{1}{2}([Y,\ X] + J[JY,\ X]) - \frac{1}{4}N_J(X,\ Y).$$

2. For $\theta \in (\wedge_J^{0,p} T^*M)^{\mathbb{R}} \otimes TM$,

$$\begin{aligned}(\bar\partial_J\theta)(X_0, \dots, X_p) : = &\sum_{j=0}^{p}(-1)^j(\bar\partial_J\theta(X_0, \dots, \hat X_j, \dots, X_p))(X_j)\\ &+\sum_{0\le j\le k\le p}(-1)^{j+k}\theta([[X_j, X_k]], \dots \hat X_j, \dots, \hat X_k, \dots, X_p)).\end{aligned}$$

Remark 2.11. In particular, for $L \in (\wedge_J^{0,1} T^*M)^{\mathbb{R}} \otimes TM$, we obtain

$$(\bar\partial_J L)(X,\ Y) = (\bar\partial_J L(Y))(X) - (\bar\partial_J L(X))(Y) - L([[X,\ Y]]).$$

Note also that, identifying $\bar\partial_J$ with $m \circ \bar\partial_J \circ m^{-1}$, we have for Z, W of type $(1, 0)$:

$$(\bar\partial_J W)(\bar Z) = \frac{1}{2}([\bar Z,\ W] - iJ[\bar Z,\ W]).$$

2.2. The odd dimensional case

Definition 2.12. Let V be a $(2n + 1)$-dimensional differentiable manifold: $\alpha \in \wedge^1(V)$ is called a contact form if

$$\alpha \wedge (d\alpha)^n \neq 0 \text{ everywhere on } V.$$

This is equivalent to say that

1. α never vanishes on V,
2. $d\alpha_{|\ker\alpha}$ is everywhere non degenerate, *i.e.* α restricts to a symplectic form on the $2n$-dim distribution $\xi = \ker\alpha$.

A codimension 1 tangent distribution ξ on V is called a contact structure if it can be locally (and globally in the oriented case) defined by the Pfaffian equation $\alpha = 0$ for some choice of a contact form α ; the pair (V, ξ) is called a contact manifold. $\mathcal{H}(\xi)$ will denote the space of sections of ξ, *i.e.* the space of ξ-valued vector fields on V.

Recall that, given a contact form α on a contact manifold (V, ξ), there exits on V a unique vector field R_α, called the Reeb vector field of α, such that

1. $\iota_{R_\alpha} d\alpha = 0$;
2. $\alpha(R_\alpha) = 1$.

(Recall that ι is the contraction of a form by a vector field.)

Remark 2.13. The Reeb vector field satisfies the following properties

1. $TV = \xi \oplus \mathbb{R}R_\alpha$;
2. $[R_\alpha,\ X] \in \mathcal{H}(\xi)$, for every $X \in \mathcal{H}(\xi)$;
3. if λ is a C^1 function on V, then there exists a vector field $X_{-\lambda} \in \mathcal{H}(\xi)$, such that
$$R_{e^\lambda\alpha} = e^{-\lambda}(R_\alpha + X_{-\lambda}). \tag{2.8}$$

It is easy to see that X_λ is the Hamiltonian vector field [5] of λ with respect to $d\alpha$, *i.e.* on ξ
$$\iota_{X_{-\lambda}} d\alpha - d\lambda = 0.$$

Definition 2.14. Let (V, ξ) be a contact manifold. We define $\mathfrak{C}(\xi)$ to be the set of $d\alpha$-calibrated complex structures on ξ, where α is a contact form for ξ.

Remark 2.15. Notice that $\mathfrak{C}(\xi)$ does not depend on the choice of α.

Remark 2.16. Notice that if $J \in \mathfrak{C}(\xi)$, then
$$d\alpha(JX, JY) = d\alpha(X, Y) \tag{2.9}$$
$$\Leftrightarrow [JX, JY] - [X, Y] \in \mathcal{H}(\xi),$$
$$\Leftrightarrow [JX, Y] + [X, JY] \in \mathcal{H}(\xi), \quad X, Y \in \mathcal{H}(\xi).$$

Therefore
$$N_J \in ({\Lambda_J}^{0,2}\xi^*)^{\mathbb{R}} \otimes \xi. \tag{2.10}$$

Definition 2.17. A strictly pseudoconvex *CR* structure of hypersurface type on (V, ξ) is the datum of $J \in \mathfrak{C}(\xi)$ satisfying
$$N_J(X, Y) = 0, \tag{2.11}$$
for every $X, Y \in \xi$.

We refer to the triple $(V,\ \xi,\ J)$ as a strictly pseudoconvex *CR* manifold.

Remark 2.18. Note that $J \in \mathfrak{C}(\xi)$ is a strictly pseudoconvex structure if and only if
$$[\xi_J^{0,1},\ \xi_J^{0,1}] \subset \xi_J^{0,1},$$
where $\xi_J^{0,1} = \{Z \in \xi \otimes \mathbb{C} | JZ = -iZ\}$.

Using Remark 2.18, we obtain the following lemma:

Lemma 2.19. *Let (V, ξ) be a contact manifold, and let $J \in \mathfrak{C}(\xi)$ be a strictly pseudoconvex structure. Then $(V, \xi_J^{0,1})$ is a strictly pseudoconvex CR manifold.*

Proof. Recall that $(V, \xi_J^{0,1})$ is a strictly pseudoconvex *CR* manifold if its Levi form is (positive or negative) definite, where the Levi form is the map given by
$$L : \xi_J^{0,1} \times \xi_J^{0,1} \longrightarrow TV \otimes \mathbb{C}/(\xi_J^{0,1} \oplus \xi_J^{1,0}) \tag{2.12}$$
$$L(X, Y) = \frac{1}{2i}\pi([X,\ \overline{Y}]),$$
where $X, Y \in \xi_J^{0,1}$, and π is the natural quotient map.

The lemma is proved by combining (2.12) and the following identity

$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]), \tag{2.13}$$

which holds for any C^1 $1-$form ω and vector fields X and Y. □

Finally note that, given $v \in T^{\mathbb{C}}M$, then:

$$v \in \xi^{\mathbb{C}} \iff v^{1,0} \in \xi^{1,0} \iff v^{0,1} \in \xi^{0,1}. \tag{2.14}$$

3. Some generalities on curves of complex structures

In this section, we consider a smooth curve of complex structures J_t on TM, with $J_0 = J$, and study the relationship between N_{J_t} and N_J.

Let J_t be a C^1 curve of complex structures on TM with $J_0 = J$. It is known [2] that J_t can be uniquely written in the following way, for t small,

$$J_t = (I + L_t)J(I + L_t)^{-1} \tag{3.1}$$

with $L_t J + J L_t = 0$ and $L_t = tL + o(t)$.

Remark 3.1. Take $L_t := (I - JJ_t)^{-1}(I + JJ_t)$.

The following lemma is an immediate consequence of (3.1):

Lemma 3.2. *Let J_t be a C^1 curve of complex structures on TM given by* (3.1). *Then*

$$\left(\frac{d}{dt}J_t\right)_{|t=0} = 2LJ. \tag{3.2}$$

Proof. Observe that

$$\left(\frac{d}{dt}(I + L_t)^{-1}\right)_{|t=0} = -L.$$ □

Proposition 3.3. *Let J_t be a C^1 curve of complex structures on TM given by* (3.1), *and let N_{J_t} be the Nijenhuis tensor of J_t. Then the following holds*

$$\frac{d}{dt}N_{J_t}(X,\ Y)_{|t=0} = -4(\bar{\partial}_J L)(X,\ Y) - N_J(LX,\ Y) - N_J(X,\ LY). \tag{3.3}$$

Proof. Using (3.2), we obtain

$$\begin{aligned}\frac{d}{dt}N_{J_t}(X, Y)_{|t=0} &= 2[LJX, JY] + 2[JX, LJY] - 2J[X, LJY] \\ &\quad - 2J[LJX, Y] - 2LJ([JX, Y] + [X, JY]).\end{aligned} \tag{3.4}$$

By definition, we have

$$
\begin{aligned}
4(\bar\partial_J L)(X,Y) = {} & 4(\bar\partial_J L(Y))(X) - 4(\bar\partial_J L(X))(Y) - 2L([X,Y]-[JX,JY]) \\
& - L(N_J(X,Y)) = 2[X,LY] + 2J[JX,LY] + N_J(X,LY) \\
& + 2[LX,Y] + 2J[LX,JY] + N_J(LX,Y) - 2L([X,Y] \\
& - [JX,JY]) - 2LN_J(X,Y).
\end{aligned} \tag{3.5}
$$

By definition, we have

$$
\begin{aligned}
4(\bar\partial_J L)(X,Y) = {} & 2[JX,JLY] - 2J[X,JLY] - N_J(X,LY) + 2[JLX,JY] \\
& - 2J[JLX,Y] - N_J(LX,Y) + 2L(J[X,JY] + J[JX,Y]).
\end{aligned} \tag{3.6}
$$

Therefore, using (3.4), (3.5), and (3.6), we obtain the desired equation (3.3). This achieves the proof of Proposition 3.3. □

4. Curves of *CR* structures

In this section, we first define the *symplectization* of the contact manifold (V,ξ), and then extend any complex structure J defined on ξ to it.

Definition 4.1. Let (V,ξ) be a contact manifold, and let α be a contact form. Then, on W defined as follows

$$
W := V \times \mathbb{R}_\tau, \tag{4.1}
$$

we consider the symplectic form

$$
\kappa_\alpha := d(e^\tau \alpha); \tag{4.2}
$$

$(W,\ \kappa_\alpha)$ is called the *symplectization* of (V,ξ) with respect to α.

We now extend to (W,κ_α) any complex structure J given on ξ in the following way:

Definition 4.2. Let J be a complex structure defined on $\ker\alpha$. We define the extended complex structure on TW, still denoted by J, as follows. Setting $T := \frac{\partial}{\partial\tau}$, we put

$$
JR_\alpha = T, \quad JT = -R_\alpha. \tag{4.3}
$$

Remark 4.3. Notice that, if $J \in \mathfrak{C}(\xi)$, then $\kappa_\alpha(J\cdot,\ \cdot)$ is a positive definite J-Hermitian metric on TW.

We have the following proposition:

Proposition 4.4. *Let $(V,\ \xi)$ be a contact manifold, with α a contact form, and let $J \in \mathfrak{C}(\xi)$. Then the following holds*

$$
N_J(X,\ Y) \in \xi, \tag{4.4}
$$

for every $X, Y \in TW$;

$$\bar{\partial}_J R_\alpha = \frac{1}{4} N_J(R_\alpha, \cdot\,). \tag{4.5}$$

Proof. Since $J \in \mathfrak{C}(\xi)$, we know that (4.4) holds for X and Y in ξ (*cf.* (2.10)).

Also, recall that

$$\iota_{R_\alpha} d\alpha = 0. \tag{4.6}$$

Combining this with the fact that $\alpha(R_\alpha) = 1$ and $\alpha(T) = 0$ (by definition), we obtain that (4.4) holds for any X and Y in TW.

Equation (4.5) follows by direct computation. □

We have the following theorem:

Theorem 4.5. *Let* (V, ξ) *be a contact manifold admitting a strictly pseudoconvex CR structure* J*, and let* ξ_t *be a smooth curve of contact structures on* V*, with* $\xi_0 = \xi$*. Let* J_t *be a smooth curve of complex structures on* ξ_t*, with* $J_0 = J$ *given by* (3.1)*. Let* α_t *be a smooth curve of contact forms for* ξ_t*, with* $\alpha_0 = \alpha$*. Then* $\gamma := \frac{d}{dt}_{t=0}\alpha_t$ *and* L *satisfy the following relation on* ξ

$$\left(\frac{d}{dt} N_{J_t}\right)_{|t=0} = -4\bar{\partial}_J L + 4\gamma^{1,0} \wedge \bar{\partial}_J R_\alpha. \tag{4.7}$$

Remark 4.6. Equation (4.7) tells us also that $\gamma^{1,0} \wedge \bar{\partial}_J R_\alpha$ does not depend on the choice of contact form.

Corollary 4.7. *Let* ξ_t*,* J_t*, and* α_t *as in Theorem* 4.5*. Assume that* J_t *provide strictly pseudoconvex CR structures. Then the following is true on* ξ

$$\gamma^{0,1} \wedge \bar{\partial}_J R_\alpha = -\bar{\partial}_J L. \tag{4.8}$$

The following lemma is immediate:

Lemma 4.8. *Let* α_t *be a smooth curve of contact structures on* V *with*

$$\alpha_t = \alpha + \gamma_t = \alpha + t\gamma + o(t). \tag{4.9}$$

Then

$$\ker \alpha_t = \{X + \beta_t(X) R_\alpha \mid X \in \ker\alpha,\ \beta_t = -(\alpha_t(R_\alpha))^{-1} \alpha_{t|\ker\alpha}\}. \tag{4.10}$$

Recall the following definition:

Definition 4.9. Let $\gamma \in \Lambda^1(V)$ be a 1-form, and let J be a complex structure on ξ. We define the operator $\gamma^{0,1} : \xi \longrightarrow \wedge^{1,0}(W) \otimes TW$ as follows

$$\gamma^{0,1}(X)(Y) = \frac{1}{2}\gamma(X)Y + \frac{1}{2}\gamma(JX)JY, \tag{4.11}$$

where $X \in \xi$ and $Y \in TW$.

Remark 4.10. Notice that $\gamma^{0,1}(JX) = -J\gamma^{0,1}(X)$.

Remark 4.11. Similarily, one defines

$$\gamma^{1,0}(X)(Y) = \frac{1}{2}\gamma(X)Y - \frac{1}{2}\gamma(JX)JY, \tag{4.12}$$

where $X \in \xi$ and $Y \in TW$.

Lemma 4.12. *Let α_t be given by* (4.9)*, and let J_t be a smooth curve of complex structures on* $\ker\alpha_t$ *given by* (3.1)*.*
Then there exists $S \in \mathrm{End}(\xi)$ satisfying $SJ + JS = 0$, such that

$$L(X) = S(X) - \gamma^{0,1}(X)(R_\alpha), \tag{4.13}$$

for every $X \in \xi$.

Proof. Write, for $X \in \xi$,

$$L(X) = S(X) + M(X)R_\alpha + N(X)JR_\alpha, \tag{4.14}$$

where $S \in \mathrm{End}(\xi)$, and $M,\ N$ are linear forms on ξ. Using the fact that $LJ+JL = 0$, we obtain from (4.14) the following equations

$$\begin{aligned} SJ + JS &= 0 \\ M(JX) &= N(X). \end{aligned} \tag{4.15}$$

On the other hand, using the assumption, we have the following equation

$$\alpha_t(J_tX_t) = 0, \tag{4.16}$$

for $X_t \in \ker\alpha_t$. Differentiating (4.16) with respect to t, and putting $t = 0$, we obtain, using (4.10),

$$M(JX) = -\frac{1}{2}\gamma(JX), \tag{4.17}$$

for $X \in \ker\alpha$. Using (4.14), (4.15) and (4.17), we obtain the desired equation (4.13). This achieves the proof of the lemma. □

Proof of Theorem 4.5. Let J_t be complex structures as in the statement of the theorem.
We claim that, for $X, Y \in \xi$,

$$N_J(LX,\ Y) + N_J(X,\ LY) = -4(\gamma^{1,0} \wedge \bar\partial_J R_\alpha)(X,\ Y). \tag{4.18}$$

Indeed, using (4.13) and the assumptions, we obtain

$$-N_J(LX,\ Y) = \frac{1}{2}\gamma(X)N_J(R_\alpha,\ Y) - \frac{1}{2}\gamma(JX)JN_J(R_\alpha,\ Y) \tag{4.19}$$

$$-N_J(X,\ LY) = -\frac{1}{2}\gamma(Y)N_J(R_\alpha,\ X) + \frac{1}{2}\gamma(JY)JN_J(R_\alpha,\ X) \tag{4.20}$$

Using (4.5), (4.12), (4.19) and (4.20), we obtain (4.18), which proves the desired claim. From (4.18) and the hypothesis, we then obtain on ξ

$$\left(\frac{d}{dt}N_{J_t}\right)_{|t=0} = -4\bar{\partial}_J L + 4\gamma^{1,0} \wedge \bar{\partial}_J R_\alpha. \tag{4.21}$$

This achieves the proof of Theorem 4.5. □

Proof of Corollary 4.7. Let J_t as before. We have

$$N_{J_t}(X_t,\ Y_t) = 0, \tag{4.22}$$

for $X_t,\ Y_t \in \ker\alpha_t$. Differentiating (4.22) with respect to t, using (4.10), and putting $t = 0$, we obtain, for X and $Y \in \xi$,

$$\begin{aligned} 0 &= \frac{d}{dt}N_{J_t}(X + \beta_t(X)R_\alpha,\ Y + \beta_t(Y)R_\alpha)_{|t=0} \\ &= \frac{d}{dt}N_{J_t}(X,\ Y)_{|t=0} - \gamma(X)N_J(R_\alpha,\ Y) + \gamma(Y)N_J(R_\alpha, X), \end{aligned}$$

and hence, on $\ker\xi$,

$$\left(\frac{d}{dt}N_{J_t}\right)_{|t=0} = 4\gamma \wedge \bar{\partial}_J R_\alpha. \tag{4.23}$$

Combining (4.7) and (4.23), we obtain the desired conclusion

$$\gamma^{0,1} \wedge \bar{\partial}_J R_\alpha = -\bar{\partial}_J L$$

on ξ. This achieves the proof of Corollary 4.7. □

If we want to consider on compact manifolds deformations of *CR* structures up to diffeomorphisms, we may keep the underlying contact structure fixed. Indeed, we have the following stability result

Theorem 4.13 (Gray's stability Theorem [5]). *Let α_t be a smooth family of contact forms on a compact manifold M. Then there exists a family of diffeomorphisms ψ_t such that*

$$\psi_t{}^*\alpha_t = f_t\alpha_0, \tag{4.24}$$

fore some nonvanishing functions f_t.

For the convenience of the reader, we provide a direct proof of a version of Gray's stability Theorem that is suitable for our purposes (*cf.* Section 8).

Theorem 4.14. *Let (V, ξ) be a compact contact manifold admitting a strictly pseudoconvex CR structure J. Let α be a contact form for ξ. Then, for any $\gamma \in \Lambda^1(V)$, there exists a smooth curve of contact forms α_t on V satisfying*

$$\gamma = \frac{d}{dt}_{t=0}\alpha_t,\ \ \alpha_0 = \alpha,$$

such that $(V, \ker\alpha_t)$ admit strictly pseudoconvex CR structures J_t, with $J_0 = J$.

Proof. Let (V,ξ) be a contact manifold, with contact form α, and let $\gamma\in\Lambda^1(V)$. Set

$$\tilde{\gamma}(X)=\begin{cases}\gamma(X) & \text{for } X\in\ker\alpha\\ 0 & \text{for } X=R_\alpha.\end{cases} \tag{4.25}$$

Since $\tilde{\gamma}$ vanishes on R_α, there exists a vector field $Y\in\mathcal{H}(\xi)$ such that

$$\iota(Y)d\alpha=-\tilde{\gamma}=\mathfrak{L}_Y(\alpha)\,. \tag{4.26}$$

Considering $\{\varphi_t^Y\}_{t\in\mathbb{R}}=\{\varphi_t\}_{t\in\mathbb{R}}$ the induced one-parameter group of diffeomorphisms, we obtain

$$\hat{\gamma}:=\frac{d}{dt}\varphi_t^*(\alpha+t\gamma)_{|t=0}=\mathfrak{L}_Y\alpha+\gamma\,. \tag{4.27}$$

Using (4.25) and (4.27), we obtain $\hat{\gamma}(X)=0$, for $X\in\mathcal{H}(\xi)$, and therefore

$$\hat{\gamma}=\gamma(R_\alpha)\alpha. \tag{4.28}$$

Using (4.28), we get

$$\varphi_t^*(\alpha+t\gamma))=(1+t\gamma(R_\alpha))\alpha+o(t)\,. \tag{4.29}$$

We define α_t by

$$\alpha_t:=\varphi_{-t}^*(\alpha+t\gamma(R_\alpha)\alpha). \tag{4.30}$$

Using (4.29) and (4.30), we obtain

$$\alpha_t=\alpha+t\gamma+o(t). \tag{4.31}$$

Putting

$$J_t=(\varphi_t)_*\circ J\circ(\varphi_{-t})_*, \tag{4.32}$$

and using (4.30), we obtain that $(V,\ker\alpha_t)$ admit a strictly pseudoconvex *CR* structure J_t given by (4.32), with $J_0=J$. This completes the proof of the theorem. □

5. Deformation Theory of *CR* structures

We want to describe and investigate a Differential Graded Lie Algebra that governs Deformation Theory of *CR* Structures.

Let us recall first the definition of DGLA:

Definition 5.1. A Differential Graded Lie Algebra (DGLA)

$$(\mathfrak{g},\,[\,,\,],\,d)$$

is the datum of:

1. a vector space $\mathfrak{g}$ together with decomposition

$$\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p;$$

2. a bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ such that:
   (a) $[\mathfrak{g}_r, \mathfrak{g}_s] \subset \mathfrak{g}_{r+s}$;
   (b) for homogeneous element a, b, c, we have:
      i. $[a, b] = -(-1)^{|a||b|}[b, a]$;
      ii. the graded Jacobi identity:

$$[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]]$$

      or, equivalently:

$$\mathfrak{S}(-1)^{|a||c|}[a, [b, c]] = 0;$$

3. a degree 1 endomorphism $d : \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying:
   (a) $d^2 = 0$;
   (b) $d[a, b] = [da, b] + (-1)^{|a|}[a, db]$.

Example 5.2. $\mathfrak{g} = \mathrm{End}(\wedge^*(M))$, with:

- $[F, G] := F \circ G - (-1)^{|F||G|} G \circ F$
- $dF := [d,\ F]$

where, clearly, $d \in \mathrm{End}(\wedge^*(M))$ is the exterior differential operator. Now, as we mention in the introduction, we perform a change of point of view with respect to the traditional one: instead of considering $\bar\partial_b$ (which, for vector valued forms, needs a choice of a contact form to project tangentially $\bar\partial$), we consider forms on which $\bar\partial_J$ acts "tangentially".

In fact, let (V, ξ) be a (compact) contact manifold and let J be a strictly pseudoconvex *CR* structure of hypersurface type on (V, ξ): we consider the following definition:

Definition 5.3. Let (V, ξ) be a contact manifold equipped with a strictly pseudoconvex *CR* structure of hypersurface type J; fix a contact form α and extend J to the α-symplectization of (V, ξ).
Let $\mathcal{A}^p_J(\xi) \subset (\wedge^{0,p}_J(\xi))^{\mathbb{R}} \otimes \xi$ be defined by

$$\mathcal{A}^p_J(\xi) = \Big\{ \gamma \in (\wedge^{0,p}_J(\xi))^{\mathbb{R}} \otimes \xi \mid \bar\partial_J \gamma \in (\wedge^{0,p+1}_J(\xi))^{\mathbb{R}} \otimes \xi \Big\}. \tag{5.1}$$

We have first the following characterisation:

Lemma 5.4. *Let* $\gamma \in (\wedge_J^{0,p}(\xi))^{\mathbb{R}} \otimes \xi$. *Then* $\gamma \in \mathcal{A}_J^p(\xi)$ *if and only if, for any* $X_0, \dots, X_p \in \xi$,

$$\sum_{j=0}^{p}(-1)^j d\alpha(X_j\,,\ \gamma(X_0, \dots, \widehat{X}_j\,, \dots, X_p)) = 0, \tag{5.2}$$

where α *is a contact form for* ξ. *Consequently, Definition* 5.3 *does not depend on the choice of* α.

Proof. Using Remark 2.16, Definition 2.9, and (4.4), we obtain that $N_J(X, Y)$ and $[[X, Y]] \in \mathcal{H}(\xi)$, for $X,\ Y \in \mathcal{H}(\xi)$. Therefore, using Definition 2.10, we see that it is enough to compute the tangential component of $[X,\ Y]$, denoted by $[X, Y]_\xi$, with respect to the decomposition $TV = \xi \oplus \mathbb{R}R_\alpha$, where R_α is given by Remark 2.13. Using the fact that $\alpha(R_\alpha) = 1$, we obtain that

$$[X, Y]_\xi = [X, Y] + d\alpha(X,\ Y)R_\alpha. \tag{5.3}$$

Using (5.3) and Definition 2.10 again, we obtain the desired conclusion (5.2). The last part of the lemma follows from Remark 2.13. This completes the proof of Lemma 5.4. □

Remark 5.5. Notice that

$$\bar{\partial}_J Y(X)_{\ker e^\lambda \alpha} = \bar{\partial}_J Y(X)_\xi + (\iota_Y d\alpha)^{0,1}(X)X_\lambda,$$

where $X_\lambda \in \mathcal{H}(\xi)$ is the $d\alpha$-Hamiltonian vector field of λ .

Remark 5.6. Notice that

- $\mathcal{A}_J^p(\xi)$ is defined by a tensorial (*i.e.* pointwise) condition;
- $\mathcal{A}_J^0(\xi) = \{0\}$;
- $\mathcal{A}_J^1(\xi) = \{L \in \mathrm{End}(\xi) \mid LJ + JL = 0\,,\ L = {}^tL\}$
  where transposition is taken with respect to $g_J := d\alpha(J\cdot\,,\ \cdot\,)$.

Let $\mathcal{A}_J^p(\xi)$ be given by (5.1). We have the following lemma:

Lemma 5.7.

$$\dim \mathcal{A}_J^p(\xi) = 2n\binom{n}{p} - 2\binom{n}{p+1}. \tag{5.4}$$

Proof. We choose a (local) basis of ξ, and use (5.2) with the basis vectors. The lemma follows, using the fact that $d\alpha$ is everywhere non degenerate on ξ. This achieves the proof of the lemma. □

Let $S_p \subset \wedge_J^{0,p}(\xi)^{\mathbb{R}} \otimes \xi$ be defined (via the identification by (2.1)) by

$$S_p = \{\gamma \in (\wedge_J^{0,p}(\xi))^{\mathbb{R}} \otimes \xi \mid \gamma = \sum_{r=1}^{k} \beta_r \wedge L_r\}, \tag{5.5}$$

where $\beta_1, \dots, \beta_k \in \wedge_J^{0,p-1}(\xi)$, and $L_1, \dots, L_k \in \mathcal{A}_J^1(\xi)$.

Lemma 5.8. *Let S_p be given by* (5.5). *Then $S_p \subset \mathcal{A}_J^p(\xi)$, and*

$$\dim S_p = 2\sum_{k=0}^{n-1}\left(\binom{n}{p} - \binom{k}{p}\right). \tag{5.6}$$

Note that we use the convention $\binom{k}{p} = 0, \ \ k < p$.

Proof. The proof of the lemma is left to the reader. □

Using Lemma 5.7 and Lemma 5.8, the following proposition is immediate, thanks to the formula

$$\binom{m+1}{n+1} = \binom{m}{n+1} + \binom{m}{n}.$$

Proposition 5.9. *Let $\mathcal{A}_J^p(\xi)$ be given by* (5.1)*, and S_p be given by* (5.5). *Then*

$$\mathcal{A}_J^p(\xi) \ = \ S_p. \tag{5.7}$$

Let $\mathcal{A}_J(\xi)$ be defined by

$$\mathcal{A}_J(\xi) = \bigoplus_{p \geq 0} \mathcal{A}_J^p(\xi). \tag{5.8}$$

Theorem 5.10. *Let (V, ξ) be a contact manifold and let J be a strictly pseudoconvex CR structure of hypersurface type; then*

$$\bar{\partial}_J^2 = 0 \ \ on \ \ \mathcal{A}_J(\xi). \tag{5.9}$$

Proof. By (5.7), it is enough to prove (5.9) on $\mathcal{A}_J^1(\xi)$. Recall that $L \in \mathcal{A}_J^1(\xi)$ if and only if $LJ + JL = 0$ and $L = {}^tL$. This implies that

$$[JX, L(Y)] - [JY, L(X)] \in \mathcal{H}(\xi), \tag{5.10}$$

for $X, Y \in \mathcal{H}(\xi)$. A direct computation together with (5.10) yields (5.9). This achieves the proof of Theorem 5.10. □

By means of the identification

$$\xi \longleftrightarrow \xi^{1,0}\ ,\ \ X \mapsto \frac{1}{2}(X - iJX)\,,$$

$\mathcal{A}_J^p(\xi)$ can be viewed as the space of elements

$$\gamma \in \wedge_J^{0,p}(\xi) \otimes \xi^{1,0}$$

such that

$$\bar\partial_J \gamma \in \wedge_J^{0,p+1}(\xi) \otimes \xi^{1,0}$$

or equivalently, such that

$$\sum_{j=0}^{p} (-1)^j d\alpha(\bar Z_j, \gamma(\bar Z_0, \dots, \widehat{\bar Z}_j, \dots, \bar Z_p)) = 0$$

for any $(Z_0, \dots, Z_p) \in \xi^{1,0}$.

Once more, in the complex setting, we have

$$L \in \mathcal{A}_J^1(\xi) \Leftrightarrow d\alpha(L(\bar Z),\ \bar W) + d\alpha(\bar Z,\ L(\bar W)) = 0\,,$$

for every $Z,\ W \in \wedge^{0,0}(\xi) \otimes \xi^{1,0}$.

Thus, for every $Z,\ W \in \wedge^{0,0}(\xi) \otimes \xi^{1,0}$, we have:

$$[L(\bar Z),\ \bar W] + [\bar Z,\ L(\bar W)] \in \xi^{1,0} \oplus \xi^{0,1}\,. \tag{5.11}$$

For any $L \in \mathcal{A}_J^p(\xi)$, we want to define a degree p skew derivation

$$\urcorner_L\ :\ \wedge_J^{0,*}(\xi) \longrightarrow \wedge_J^{0,*}(\xi)\,;$$

first, for $\gamma \in \wedge_J^{l,q}(\xi)$, we set:

$$(\tau(L)\gamma)(U_1, \dots, U_{l+q}) = \sum_{j=1}^{l+q} \gamma(U_1, \dots, L(U_j), \dots, U_{l+q}), U_j \in \xi^{1,0} \oplus \xi^{0,1}$$

- $p = 1$:

  a. For $q = 0$: $\urcorner_L f := \tau(L)(\partial f)$ *i.e.*, for $Z \in T_J^{1,0}\xi$:

$$(\urcorner_L f)(\bar Z) = \partial f(L(\bar Z)) = L(\bar Z) f;$$

b. For $q = 1 : \urcorner_L \gamma := [\tau(L)(\partial\gamma)]^{0,2} = [\tau(L)d\gamma]^{0,2}$ *i.e.*, for $Z,\ W \in T_J^{1,0}\xi$:

$$(\urcorner_L \gamma)(\bar{Z}, \bar{W}) = L(\bar{Z})\gamma(\bar{W}) - L(\bar{W})\gamma(\bar{Z}) - \gamma([L(\bar{Z}), \bar{W}] + [\bar{Z}, L(\bar{W})]);$$

c. Since

$$\urcorner_L (f\gamma) = (\urcorner_L f) \wedge \gamma + f \urcorner_L \gamma,$$

we can extend $\urcorner_L$ as degree 1 skew derivation; it is easy to check that, if $\beta \in \wedge_J^{0,p}(\xi)$, then

$$\urcorner_L \beta = [\tau(L)d\beta]^{0,p+1};$$

- any $p \geq 1$: write L as sum of elements of the form $\alpha \wedge S$, with $S \in \mathcal{A}_J^1(\xi)$ and $\alpha \in \wedge_J^{0,p-1}(\xi)$; then set:

$$\urcorner_{\alpha\wedge S} := \alpha \wedge \urcorner_S.$$

Consider on $\mathcal{A}_J(\xi)$ the following bracket [[,]]:

- for $L \in \mathcal{A}_J^1(\xi)$:

$$[[L, L]](\bar{Z}, \bar{W}) := 2[L(\bar{Z}), L(\bar{W})] - 2L([L(\bar{Z}), \bar{W}] + [\bar{Z}, L(\bar{W})]);$$

then extend to $\mathcal{A}_J^1(\xi)$ by polarization, *i.e.*

$$[[L, S]] = \frac{1}{2}([[L + S, L + S]] - [[L, L]] - [[S, S]]).$$

Lemma 5.11. *Let $L, S \in \mathcal{A}_J^1(\xi)$. Then*

$$[[L, S]] \in \mathcal{A}_J^2(\xi). \tag{5.12}$$

Proof. The lemma follows easily from Lemma 5.4 and Jacobi's Identity: □

- for $\alpha,\ \beta \in \wedge_J^{0,*}(\xi)$, $L,\ S \in \mathcal{A}_J^1(\xi)$:

$$[[\alpha \wedge L, \beta \wedge S]] = (-1)^{|\beta|}\alpha \wedge \beta \wedge [[L, S]] + \alpha \wedge \urcorner_L \beta \wedge S - (-1)^{(|\alpha|+1)(|\beta|+1)}\beta \wedge \urcorner_S \alpha \wedge L.$$

Note also that, more in general, for $\alpha, \beta \in \wedge_J^{0,*}(\xi)$, $L, S \in \mathcal{A}_J(\xi)$, we have:

$$[[\alpha \wedge L, \beta \wedge S]] = (-1)^{|\beta||L|}\alpha \wedge \beta \wedge [[L, S]] + \alpha \wedge \urcorner_L \beta \wedge S - (-1)^{(|\alpha|+|L|)(|\beta|+|S|)}\beta \wedge \urcorner_S \alpha \wedge L.$$

We define $\bar{\partial}_J : \mathcal{A}_J(\xi) \longrightarrow \mathcal{A}_J(\xi)$ as before, *i.e.* as follows:

- for $L \in \mathcal{A}^1_J(\xi)$ and $Z, W \in T^{1,0}_J\xi$:

$$(\bar\partial_J L)(\bar Z, \bar W) := (\bar\partial_J L\bar W)(\bar Z) - (\bar\partial_J L\bar Z)(\bar W) - L([\bar Z, \bar W])$$

where:

$$(\bar\partial_J W)(\bar Z) := \frac{1}{2}([\bar Z, W] - iJ[\bar Z, W])$$

- in the general case:

$$\bar\partial_J(\alpha \wedge L) := \bar\partial_J\alpha \wedge L + (-1)^{|\alpha|}\alpha \wedge \bar\partial_J L.$$

In the space $\mathcal{F}$ of skew symmetric derivations on $\wedge^{0,*}_J(\xi)$, consider the usual bracket defined on homogeneous elements as

$$[F,\ G] := F \circ G - (-1)^{|F||G|} G \circ F$$

and set:

$$\bar\partial_J F := [\bar\partial_J, F];$$

then $(\mathcal{F}, [\ ,\ ], \bar\partial_J)$ is a DGLA; let

$$q : \mathcal{A}_J(\xi) \longrightarrow \mathcal{F}, q : L \mapsto \urcorner_L. \tag{5.13}$$

Theorem 5.12. *q is an injective DGLA homomorphism, i.e. q is an injective map satisfying*

$$[q(L), q(S)] = q([[L, S]]) \tag{5.14}$$

or, equivalently

$$[\urcorner_L, \urcorner_S] = \urcorner_{[[L,S]]} \tag{5.15}$$

and

$$\bar\partial_J q(L) = q(\bar\partial_J L) \tag{5.16}$$

or, equivalently

$$\bar\partial_J \urcorner_L = \urcorner_{\bar\partial_J L}\,. \tag{5.17}$$

Proof. The injectivity of the map q is immediate, using the definition of $\urcorner_L f$, where $f \in \Lambda^0_J(\xi)$.

For the remaining part of the proof, we need the following two lemmata:

Lemma 5.13. *Let $L \in \mathcal{A}^1_J(\xi)$, and $\gamma \in \Lambda^{0,2}_J(\xi)$. Then $\urcorner_L$ satisfies the following*

$$\urcorner_L\gamma(\bar Z, \bar W, \bar U) = \mathfrak{S} L(\bar Z)\gamma(\bar W, \bar U) - \mathfrak{S}\gamma([L(\bar Z), \bar W] + [\bar Z, L(\bar W)], \bar U) \tag{5.18}$$

where $\mathfrak{S}$ denotes the cyclic sum over $Z, W, U \in T^{1,0}_J\xi$.

Proof. Apply the definition of $\urcorner_L$ for $\gamma = \alpha \wedge \beta$, where $\alpha, \beta \in \Lambda^{0,1}_J(\xi)$. □

Lemma 5.14. *Let $S \in \mathcal{A}^2_J(\xi)$, and $\gamma \in \Lambda^{0,1}_J(\xi)$. Then $\daleth_S$ satisfies the following*

$$\daleth_S\gamma(\bar{Z}, \bar{W}, \bar{U}) = \mathfrak{S}S(\bar{Z}, \bar{W})\gamma(\bar{U}) - \mathfrak{S}\gamma([S(\bar{Z}, \bar{W}), \bar{U}]). \tag{5.19}$$

Proof. Apply the definition of $\daleth_S$ for $S = \alpha \wedge L$, where $\alpha \in \Lambda^{0,1}_J(\xi)$ and $L \in \mathcal{A}^1_J(\xi)$, and use the fact that $L(\bar{U}) \in T^{1,0}_J\xi$. □

It is sufficient to prove (5.14) for $L \in \mathcal{A}^1_J(\xi)$, $f \in \Lambda^0_J(\xi)$, and $\gamma \in \Lambda^{0,1}_J(\xi)$.
Let $L \in \mathcal{A}^1_J(\xi)$.

For $f \in \Lambda^0_J(\xi)$, and $Z, W \in T^{1,0}_J\xi$, we have, applying the definition of $\daleth_S$, $S \in \mathcal{A}^2_J(\xi)$,

$$\begin{aligned}\daleth_L\daleth_L f(\bar{Z}, \bar{W}) &= L(\bar{Z})\daleth_L f(\bar{W}) - L(\bar{W})\daleth_L f(\bar{Z})\\ &\quad - \daleth_L f([L(\bar{Z}), \bar{W}] + [\bar{Z}, L(\bar{W})])\\ &= L(\bar{Z})L(\bar{W})f\\ &\quad - L(\bar{W})L(\bar{Z})f - L([L(\bar{Z}), \bar{W}] + [\bar{Z}, L(\bar{W})])f\\ &= [L(\bar{Z}), L(\bar{W})]f - L([L(\bar{Z}), \bar{W}] + [\bar{Z}, L(\bar{W})])f\\ &= \frac{1}{2}[[L, L]](\bar{Z}, \bar{W})f = \frac{1}{2}\daleth_{[[L,L]]}f(\bar{Z}, \bar{W}).\end{aligned} \tag{5.20}$$

For $\gamma \in \Lambda^{0,1}_J(\xi)$, the equality (5.14) is easily shown, using (5.18), (5.19) , and the Identity of Jacobi. This achieves the proof of (5.14).

Again, it is sufficient to prove (5.16) for $L \in \mathcal{A}^1_J(\xi)$, $f \in \Lambda^0_J(\xi)$, and $\gamma \in \Lambda^{0,1}_J(\xi)$.

For $f \in \Lambda^0_J(\xi)$, and $Z, W, U \in T^{1,0}_J\xi$, we first notice, using the definition of $\daleth_S$, $S \in \mathcal{A}^2_J(\xi)$, that

$$\daleth_{\bar{\partial}_J L}(f)(\bar{Z}, \bar{W}) = (\bar{\partial}L)(\bar{Z}, \bar{W})(f). \tag{5.21}$$

Therefore, expanding (5.21), we obtain

$$\begin{aligned}\daleth_{\bar{\partial}_J L}(f)(\bar{Z}, \bar{W}) &= \frac{1}{2}([\bar{Z}, L\bar{W}] + [L\bar{Z}, \bar{W}]\\ &\quad - iJ([\bar{Z}, L\bar{W}] + [L\bar{Z}, \bar{W}]))(f) - L([\bar{Z}, \bar{W}])(f).\end{aligned} \tag{5.22}$$

Also, we have

$$\begin{aligned}(\bar{\partial}_J\daleth_L + \daleth_L\bar{\partial}_J)(f)(\bar{Z}, \bar{W}) &= \bar{Z}(\daleth_L f(\bar{W})) - \bar{W}(\daleth_L f(\bar{Z})) - \daleth_L f([\bar{Z}, \bar{W}])\\ &\quad + L(\bar{Z})\bar{\partial} f(\bar{W}) - L(\bar{W})\bar{\partial} f(\bar{Z})\\ &\quad - \bar{\partial} f([L(\bar{Z}), \bar{W}] + [\bar{Z}, L(\bar{W})])\\ &= \bar{Z}L(\bar{W})f - \bar{W}L(\bar{Z})f - \daleth_L f([\bar{Z}, \bar{W}])\\ &\quad + L(\bar{Z})\bar{W}f - L(\bar{W})\bar{Z}f\\ &\quad - \frac{1}{2}\big([L(\bar{Z}), \bar{W}] + [\bar{Z}, L(\bar{W})] + iJ([L(\bar{Z}), \bar{W}] + [\bar{Z}, L(\bar{W})])\big)(f).\end{aligned} \tag{5.23}$$

Using the definition of $\urcorner_L f$, we see that the expressions (5.22) and (5.23) coincide. This shows that

$$(\bar\partial_J \urcorner_L + \urcorner_L \bar\partial_J)(f)(\bar Z, \bar W) = \urcorner_{\bar\partial_J L}(f)(\bar Z, \bar W). \tag{5.24}$$

For $\gamma \in \Lambda_J^{0,1}(\xi)$, and $Z, W, U \in T_J^{1,0}\xi$, we have, using (5.19),

$$\begin{aligned}\urcorner_{\bar\partial_J L}(\gamma)(\bar Z, \bar W, \bar U) &= \mathfrak{S}\bar\partial_J L(\bar Z, \bar W)\gamma(\bar U) - \mathfrak{S}\gamma([\bar\partial_J L(\bar Z, \bar W), \bar U]) \\ &= \mathfrak{S}(\bar\partial_J L\bar W(\bar Z) - \bar\partial_J L\bar Z(\bar W) - L([\bar Z, \bar W])\gamma(\bar U) \\ &\quad - \mathfrak{S}\gamma([\bar\partial_J L\bar W(\bar Z) - \bar\partial_J L\bar Z(\bar W) - L([\bar Z, \bar W]), \bar U]).\end{aligned} \tag{5.25}$$

Expanding (5.25), we obtain

$$\begin{aligned}\urcorner_{\bar\partial_J L}(\gamma)(\bar Z, \bar W, \bar U) = {}& \mathfrak{S}\frac{1}{2}([\bar Z, L\bar W] - iJ[\bar Z, L\bar W])\gamma(\bar U) \\ & - \mathfrak{S}\frac{1}{2}([\bar W, L\bar Z] - iJ[\bar W, L\bar Z])\gamma(\bar U) \\ & - \mathfrak{S}L([\bar Z, \bar W])\gamma(\bar U) - \mathfrak{S}\gamma([\frac{1}{2}([\bar Z, L\bar W] \\ & - iJ[\bar Z, L\bar W]) - \frac{1}{2}([\bar W, L\bar Z] - iJ[\bar W, L\bar Z]) \\ & - L([\bar Z, \bar W], \bar U]).\end{aligned} \tag{5.26}$$

Also, using (5.18), we have

$$\begin{aligned}(\bar\partial_J \urcorner_L + \urcorner_L \bar\partial_J)(\gamma)(\bar Z, \bar W, \bar U) &= \mathfrak{S}\bar\partial_J \urcorner_L(\gamma)(\bar W, \bar U)(\bar Z) \\ &\quad - \mathfrak{S}\urcorner_L(\gamma)([\bar Z, \bar W], \bar U) + \mathfrak{S}L(\bar Z)\bar\partial_J(\gamma)(\bar W, \bar U) \\ &\quad - \mathfrak{S}\bar\partial_J(\gamma)([L(\bar Z), \bar W] + [\bar Z, L(\bar W)], \bar U) \\ &= \mathfrak{S}\bar Z(L(\bar W)\gamma(\bar U) - L(\bar U)\gamma(\bar W) - \gamma([L\bar W, \bar U] + [\bar W, L\bar U])) \\ &\quad - \mathfrak{S}L([\bar Z, \bar W])\gamma(\bar U) + \mathfrak{S}L(\bar U)\gamma(([\bar Z, \bar W]) + \mathfrak{S}\gamma([L([\bar Z, \bar W]), \bar U]) \\ &\quad + \mathfrak{S}\gamma([[\bar Z, \bar W], L(\bar U)]) + \mathfrak{S}L(\bar Z)(\bar W\gamma(\bar U) - \bar U(\gamma(\bar W)) \\ &\quad - \gamma([\bar W, \bar U])) - \mathfrak{S}\bar\partial_J(\gamma)([L(\bar Z), \bar W] + [\bar Z, L(\bar W)], \bar U).\end{aligned} \tag{5.27}$$

Using the identity of Jacobi, we write

$$\gamma([[\bar Z, \bar W], L(\bar U]) = \gamma([\bar Z, [\bar W, L(\bar U]]) - \gamma([\bar W, [\bar Z, L(\bar U]]). \tag{5.28}$$

Using the fact that $\bar\partial_J\gamma \in \Lambda_J^{0,2}(\xi)$, we write

$$\begin{aligned}&\bar\partial_J(\gamma)([L(\bar Z), \bar W] + [\bar Z, L(\bar W)], \bar U) \\ &= \bar\partial_J(\gamma)\left(\frac{1}{2}\big([L(\bar Z), \bar W] + [\bar Z, L(\bar W)] + iJ([L(\bar Z), \bar W] + [\bar Z, L(\bar W)])\big), \bar U\right).\end{aligned} \tag{5.29}$$

Combining (5.27), (5.28), and (5.29), we obtain

$$
\begin{aligned}
&(\bar\partial_J \urcorner_L + \urcorner_L \bar\partial_J)(\gamma)(\bar Z, \bar W, \bar U) = \mathfrak{S}\bar Z\Big(L(\bar W)\gamma(\bar U) - L(\bar U)\gamma(\bar W) \\
&- \gamma\big([L\bar W, \bar U] + [\bar W, L\bar U]\big)\Big) - \mathfrak{S}L([\bar Z, \bar W])\gamma(\bar U) + \mathfrak{S}L(\bar U)\gamma\Big(([\bar Z, \bar W]) \\
&+ \mathfrak{S}\gamma\big([L([\bar Z, \bar W]), \bar U]\big) + \mathfrak{S}\big(\gamma([\bar Z, [\bar W, L(\bar U)]]) - \gamma\big([\bar W, [\bar Z, L(\bar U)]]\big)\big) \\
&+ \mathfrak{S}L(\bar Z)\Big(\bar W\gamma(\bar U) - \bar U(\gamma(\bar W)) - \gamma([\bar W, \bar U])\Big) \\
&- \mathfrak{S}\frac{1}{2}\Big([L(\bar Z), \bar W] + [\bar Z, L(\bar W)] + iJ\big([L(\bar Z), \bar W] + [\bar Z, L(\bar W)]\big)(\gamma(\bar U))\Big) \\
&+ \mathfrak{S}\bar U\Big(\gamma([L(\bar Z), \bar W] + [\bar Z, L(\bar W)])\Big) \\
&+ \mathfrak{S}\gamma\left(\left[\frac{1}{2}\Big([L(\bar Z), \bar W] + [\bar Z, L(\bar W)] + iJ\big([L(\bar Z), \bar W] + [\bar Z, L(\bar W)]\big)\Big), \bar U\right]\right).
\end{aligned}
\tag{5.30}
$$

Comparing (5.26) and (5.30), we obtain that

$$(\bar\partial_J \urcorner_L + \urcorner_L \bar\partial_J)(\gamma)(\bar Z, \bar W, \bar U) = \urcorner_{\bar\partial_J L}(\gamma)(\bar Z, \bar W, \bar U). \tag{5.31}$$

This achieves the proof of (5.16). □

Corollary 5.15. *Let q be given by* (5.13)*. Then the following holds*:

1. $(Im\, q, [\ ,\ ], \bar\partial_J)$ *is a DGLA.*
2. $(\mathcal{A}_J(M), [[\ ,\ ]], \bar\partial_J)$ *is a DGLA isomorphic to the previous one.*

We are now in the position to prove one of our main results.

Theorem 5.16. *Let $J \in \mathfrak{C}(\xi)$ be a strictly pseudoconvex CR structure of hypersurface type on (V, ξ), and $\tilde J \in \mathfrak{C}(\xi)$ given by*

$$\tilde J = (I + L)J(I + L)^{-1}, \ \ LJ + JL = 0, \ \ {}^tL = L. \tag{5.32}$$

Let $\tilde L \in \mathcal{A}^1_J(\xi)$ be the operator associated to L via the identification

$$\xi \longrightarrow \xi^{1,0} \tag{5.33}$$

$$X \longrightarrow \tilde X = \frac{1}{2}(X - iJX). \tag{5.34}$$

Then

$$N_{\tilde J} = 0 \iff \bar\partial_J \tilde L + \frac{1}{2}[[\tilde L, \ \tilde L]] = 0. \tag{5.35}$$

We need the following lemma:

Lemma 5.17. *Let* $\tilde{J} \in \mathfrak{C}(\xi)$ *given by* (5.32). *Then the following holds*

$$\begin{aligned}(I+L)^{-1}&N_{\tilde{J}}((I+L)(\bar{Z}),(I+L)(\bar{W}))\\&=-4(I-L^2)^{-1}\left(\bar{\partial}_J\tilde{L}+\frac{1}{2}[[\tilde{L},\ \tilde{L}]]\right)(\bar{Z},\bar{W}),\end{aligned} \tag{5.36}$$

where $Z, W \in \xi^{1,0}$.

Proof. Let $Z, W \in \xi^{1,0}$. Using that $N_J(\bar{Z},\bar{W}) = -2[\bar{Z},\bar{W}] + 2iJ[\bar{Z},\bar{W}]$, and using that

$$(I+L)\frac{1}{2}(X+iJX) = \frac{1}{2}((I+L)X + i\tilde{J}(I+L)X), \tag{5.37}$$

we obtain

$$\begin{aligned}(I-L^2)&(I+L)^{-1}N_{\tilde{J}}((I+L)(\bar{Z}),(I+L)(\bar{W}))\\&=(I-L)N_{\tilde{J}}((I+L)(\bar{Z}),(I+L)(\bar{W}))\\&=-2(I-L)([(I+L)(\bar{Z}),(I+L)(\bar{W})]-i\tilde{J}[(I+L)(\bar{Z}),(I+L)(\bar{W})]).\end{aligned} \tag{5.38}$$

On the other hand, using (5.32), we have

$$\begin{aligned}\tilde{J}&=(I+L)J(I+L)^{-1}=(I+L)((I+L)(-J))^{-1}\\&=(I+L)(-J(I-L))^{-1}=(I+L)(I-L)^{-1}J.\end{aligned} \tag{5.39}$$

Combining (5.38) and (5.39), we obtain

$$\begin{aligned}(I-L^2)&(I+L)^{-1}N_{\tilde{J}}((I+L)(\bar{Z}),(I+L)(\bar{W}))\\&=-2(I-L)([(I+L)(\bar{Z}),(I+L)(\bar{W})])\\&\quad+2i(I+L)J[(I+L)(\bar{Z}),(I+L)(\bar{W})]).\end{aligned} \tag{5.40}$$

Expanding (5.40), and using the fact that $\tilde{L}(\bar{Z}) = L(\bar{Z})$, we obtain

$$\begin{aligned}(I-L^2)&(I+L)^{-1}N_{\tilde{J}}((I+L)(\bar{Z}),(I+L)(\bar{W}))\\&=-4\bar{\partial}_J\tilde{L}(\bar{Z},\bar{W})-4[\tilde{L}\bar{Z},\tilde{L}\bar{W}]\\&\quad+2L\Big([\bar{Z},L\bar{W}]+[L\bar{Z},\bar{W}]+iJ([\bar{Z},L\bar{W}]+[L\bar{Z},\bar{W}])\Big).\end{aligned} \tag{5.41}$$

Using the fact that $\tilde{L}X = \frac{1}{2}(LX + iLJX)$, (5.41) becomes

$$\begin{aligned}(I-L^2)&(I+L)^{-1}N_{\tilde{J}}((I+L)(\bar{Z}),(I+L)(\bar{W}))\\&=-4\bar{\partial}_J\tilde{L}(\bar{Z},\bar{W})-4[\tilde{L}\bar{Z},\tilde{L}\bar{W}]+4\tilde{L}([\bar{Z},L\bar{W}]+[L\bar{Z},\bar{W}])\\&=-4\left(\bar{\partial}_J\tilde{L}+\frac{1}{2}[[\tilde{L},\ \tilde{L}]]\right)(\bar{Z},\bar{W}).\end{aligned} \tag{5.42}$$

This achieves the proof of Lemma 5.17 □

Remark 5.18. Note also that, in the same notations, we have on $\wedge_J^{0,*}(\xi)$:

$$\rho^{-1}(I+L)\bar\partial_{\tilde J}\rho(I+L)+\rho^{-1}(I+L)A_{\tilde J}\rho(I+L)\tau(L)=(\bar\partial_J+\urcorner_L)$$

where, given a vector space V and $P\in Aut(V)$ then:

$$\rho(P):=(P^*)^{-1}\otimes P\in Aut(V^*\otimes V).$$

Proof of Theorem 5.16. The proof of Theorem 5.16 follows easily from (5.36) and (5.37). □

Remark 5.19. We recall that the *center* $C(\mathfrak{g})$ of a (super-)Lie algebra $\mathfrak{g}$ is defined as follows:

$$C(\mathfrak{g}):=\{a\in\mathfrak{g}\,|\,[a,\,b]=0\text{ for every }b\in\mathfrak{g}\}$$

it is easy to prove that $C(A_J(\xi))=\{0\}$;
in fact:

- let $R,\ S\in\mathcal{A}_J(\xi)$, $a\in C^\infty(V,\,\mathbb{C})$; then:

$$[[R,\,aS]]=a[[R,\,S]]+\urcorner_R a\wedge S;$$

- let $S\in\mathcal{A}^1_J(\xi)$ defined by $S(\bar Z)=Z$; we have:

$$R\in C(\mathcal{A}_J(\xi))\implies\urcorner_R a\wedge S=0\text{ for every }a\in C^\infty(V,\,\mathbb{C})$$

  and this clearly implies $R=0$.

Consequently, setting

$$\bar\partial_L:=\bar\partial_J+[[L,\,\cdot]]\,,$$

we have:

$$\bar\partial_L^2=0$$
$$\Updownarrow$$
$$\bar\partial_J L+\frac{1}{2}[[L,\,L]]=0$$
$$\Updownarrow$$
$$\bar\partial_J\urcorner_L+\frac{1}{2}[\urcorner_L,\,\urcorner_L]=0$$
$$\Updownarrow$$
$$(\bar\partial_J+\urcorner_L)^2=0.$$

If we set

$$\mathfrak{MC}(\mathcal{A}_J(\xi)):=\left\{L\in\mathcal{A}^1_J(\xi)\,|\,\bar\partial_J L+\frac{1}{2}[[L,\,L]]=0\right\}$$

then

$$L \mapsto (I+L)J(I+L)^{-1}$$

establishes a bijection:

$$\mathfrak{MC}(\mathcal{A}_J(\xi))$$
$$\downarrow$$
$$\{CR \text{ structures on } \xi\}$$

6. Gauge Equivalence

We want to discuss the equivalence of CR structures from the moduli space point of view.

This is a segment of the theory where the CR situation is quite different from the holomorphic one, because the appropriate Lie algebra of vector fields (see below) does not admit a natural intrinsic complexification.

Let $\mathcal{G}(\xi)$ be the group of diffeomorphisms of V keeping ξ fixed; clearly $\mathcal{G}(\xi)$ acts on the right on $\mathfrak{C}(\xi)$ as follows:

$$(\varphi,\ J) \mapsto \varphi^* J := \varphi_*^{-1} J \varphi_*$$

note, in fact, that, given a contact form α, then:

$$\varphi \in \mathcal{G}(\xi) \iff \varphi^*(\alpha) = e^{\lambda}\alpha\,;$$

therefore:

$$\begin{aligned} d\alpha((\varphi^*J)X, (\varphi^*J)Y) &= (\varphi^{-1})^*(d\alpha)(J\varphi_*X, J\varphi_*Y) \\ &= (\varphi^{-1})^*(d\alpha)(\varphi_*X,\ \varphi_*Y) = d\alpha(X,\ Y). \end{aligned}$$

Definition 6.1. We say that two elements of $\mathfrak{C}(\xi)$ are *gauge equivalent* if they are in the same orbit of $\mathcal{G}(\xi)$.

The Lie algebra of the group $\mathcal{G}(\xi)$ is given by:

$$\mathcal{A}_0(\xi) := \{X \in \mathcal{H}(V) \mid [X,\ Y] \in \mathcal{H}(\xi) \text{ for every } Y \in \mathcal{H}(\xi)\};$$

it is immediate to check directly that $\mathcal{A}_0(\xi)$ is a Lie subalgebra of $\mathcal{H}(V)$.

Remark 6.2. One can very easily observe that, if we fix a contact form α, then

$$X \in \mathcal{A}_0(\xi) \iff X = X_\sigma + \sigma R_\alpha$$

with $\sigma \in C^\infty(V,\ \mathbb{R})$ and $X_\sigma \in \mathcal{H}(\xi)$ satisfying

$$\iota_{X_\sigma} d\alpha + d\sigma = 0.$$

Note that $\mathcal{A}^0_J(\xi) = \{0\}$ corresponds to the fact that, clearly:

$$\mathcal{A}_0(\xi) \cap \xi = \{0\}$$

let $X \in \mathcal{A}_0(\xi)$; for every $Y \in \xi$, set:

$$\bar{\partial}_Y X := \frac{1}{2}([Y,\ X] + J[JY, X])$$

clearly $\bar{\partial} X$ is well defined:
moreover:

1. $\bar{\partial} X \in \mathcal{A}^1_J(\xi)$; in fact:
   (a) $\bar{\partial}_{JY} X = -J\bar{\partial}_Y X$;
   (b) for any $U,\ V \in \mathcal{H}(\xi)$:

$$\begin{aligned} 2d\alpha(\bar{\partial}_U X, V) + 2d\alpha(U, \bar{\partial}_V X) &= d\alpha([U, X] - d\alpha([JU, X], JV) \\ &\quad + d\alpha(U, [V, X]) - d\alpha(JU, [JV, X]) \\ &= -\alpha([[U, X], V]) - \alpha([U, [V, X]) \\ &\quad - \alpha([[JU, X], JV]) - \alpha([JU, [JV, X]) \\ &= \alpha([[V, U], X]) + \alpha([X, [JV, JU]]) = 0. \end{aligned}$$

2. $\bar{\partial}^2 X = 0$; in fact, for any $U, V \in \mathcal{H}(\xi)$:

$$\begin{aligned} 4\bar{\partial}(\bar{\partial} X)(U, V) &= 2\bar{\partial}_U(2\bar{\partial}_V X) - 2\bar{\partial}_V(2\bar{\partial}_U X) - 2\bar{\partial}_{[U,\, V]-[JU,\, JV]} X \\ &= 2\bar{\partial}_U([V, X] + J[JV, X]) - 2\bar{\partial}_V([U, X] + J[JU, X]) \\ &\quad - [[U, V]-[JU, JV], X] - J[J[U, V]-J[JU, JV], X] \\ &= [U, [V, X] + J[JV, X]] + J[JU, [V, X] + J[JV, X]] \\ &\quad - [V, [U, X] + J[JU, X]] - J[JV, [U, X] + J[JU, X]] \\ &\quad - [[U, V]-[JU, JV], X] - J[J[U, V]-J[JU, JV], X] \\ &= [U, [V, X]] + [V, [X, U]] + [X, [U, V]] \\ &\quad - [JU, [JV, X]] - [JV, [X, JU]] - [X, [JU, JV]] \\ &\quad + J[U, [JV, X]] + J[JV, [X, U]] + J[X, [U, JV]] \\ &\quad - J[V, [JU, X]] - J[JU, [X, V]] - J[X, [V, JU]] = 0. \end{aligned}$$

$\mathcal{G}(\xi)$ acts on the right on $\mathfrak{MC}(\mathcal{A}_J(\xi))$ as follows:
given $\varphi \in \mathcal{G}(\xi)$ and $L \in \mathfrak{MC}(\mathcal{A}_J(\xi))$ set:

$$\varphi^{\#}(L) := (J + \varphi^* \tilde{J})^{-1}(J - \varphi^* \tilde{J})$$

where:

- $\tilde{J} := (I + L)J(I + L)^{-1}$;

- as before, $\varphi^* \tilde{J} = \varphi_*^{-1} \tilde{J} \varphi_*$.

Consequently:

$$\mathfrak{MC}(\mathcal{A}_J(\xi))/\mathcal{G}(\xi)$$

represents the moduli space of *CR*-deformations of J.
Let now $X \in \mathcal{A}_0$ and let $\{\varphi_t^X\}_{t\in\mathbb{R}}$ be the induced 1-parameter subgroup of $\mathcal{G}(\xi)$; then:

$$\frac{d}{dt}(\varphi_t^X)^\#(L)_{|t=0} = \frac{1}{2}(I+L)J\mathcal{L}_X\tilde{J}(I+L)$$

where, of course, $\mathcal{L}_X$ is the Lie derivative and so

$$(\mathcal{L}_X\tilde{J})(Y) := [X,\ \tilde{J}Y] - \tilde{J}[X,Y]\,;$$

developping, we obtain:

$$\frac{d}{dt}(\varphi_t^X)^\#(L)_{|t=0} = -\bar{\partial}_J X + \frac{1}{2}(\mathcal{L}_X L + J\mathcal{L}_X L J) + \frac{1}{2}JL(\mathcal{L}_X J)L\,.$$

Let $\sigma(t) = tL + o(t)$ be a smooth curve in $\mathfrak{MC}(\mathcal{A}_J(\xi))$; therefore

$$\bar{\partial}_J\sigma(t) + \frac{1}{2}[[\sigma(t),\ \sigma(t)]] = t\bar{\partial}_J L + o(t) = 0$$

and so

$$\bar{\partial}_J L = 0$$

and thus

$$T_0\mathfrak{MC}(\mathcal{A}_J(\xi)) \subset \{L \in \mathcal{A}_J^1(\xi) \mid \bar{\partial}_J L = 0\}\,.$$

Moreover, given $X \in \mathcal{A}_0(\xi)$, let $\hat{\sigma}(t) := (\varphi_t^X)^\#(\sigma(t))$,
then:

$$\hat{\sigma}'(0) = -\bar{\partial}_J X$$

and so:

$$T_{<0>}\mathfrak{MC}(\mathcal{A}_J(\xi))/\mathcal{G}(\xi) \subset \{L \in \mathcal{A}_J^1(\xi) \mid \bar{\partial}_J L = 0\}/\bar{\partial}_J(\mathcal{A}_0(\xi)).$$

7. Further remarks

7.1. Two general remarks

1. $n = 1$ is a very special case: $N_J = 0$ always, $\mathfrak{MC}(\xi)$ coincides with $\mathcal{A}^1(\xi)$ and the deformation theory is totally unobstructed; so, we shall always assume $n > 1$.

2. there is always a huge amount of elements in $\mathcal{A}_J^1(\xi) \cap \mathrm{Ker}\,\bar{\partial}_J$: take any $X \in \mathcal{A}_0(\xi)$, any $a \in C^\infty(V, \mathbb{C})$ satisfying $\bar{\partial}_b a = 0$, then

$$L := a\bar{\partial}_J X \in \mathcal{A}_J^1(\xi) \cap \mathrm{Ker}\,\bar{\partial}_J\,.$$

Note that $a\bar{\partial}_J X$ cannot be written as $\bar{\partial}_J aX$, because aX is meaningless: a great deal of the difference between the holomorphic case and the *CR* case lays here.

7.2. Local Models

Let

$$V(n) := \mathbb{R}^n_x \times \mathbb{R}^n_y \times \mathbb{R}_s \ , \ n \geq 2;$$

let

$$\alpha := ds - \sum_{h=1}^{n} y_n dx_n$$

and, consequently,

$$d\alpha = \sum_{h=1}^{n} dx_h \wedge dy_h;$$

therefore, if $\xi_n = \operatorname{Ker} \alpha$, then we have that

$$\xi_n = \operatorname{span}\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$$

where, if $p = \begin{pmatrix} x \\ y \\ s \end{pmatrix}$

$$X_j(p) = \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial s}, \ Y_j(p) = \frac{\partial}{\partial y_j}, 1 \leq j \leq n;$$

moreover

$$R_\alpha = S := \frac{\partial}{\partial s}$$

and

$$[X_j, \ X_k] = [Y_j, \ Y_k] = 0 \ , \ [X_j, Y_k] = -\delta_{jk} S \ , 1 \leq j, \ k \leq n;$$

note also that the dual basis $\{X_1^*, \ldots, Y_1^*, \ldots, Y_n^*, \ S^*\}$ of the basis $\{X_1, \ldots, X_n,$ $Y_1, \ldots, Y_n, S\}$ is given by $\{X_j^* = dx_j, \ Y_j^* = dy_j \ , \ 1 \leq j \leq n, \ S^* = \alpha\}$ it is well known that any contact structure is locally isomorphic to $(V(n), \ \xi_n)$.

Set now:

$$\begin{cases} JX_j = Y_j \\ & 1 \leq j \leq n \\ JY_j = -X_j \end{cases}$$

it is easy to check that $J \in \mathfrak{C}(\xi)$ and $N_J \equiv 0$; consequently, any strictly pseudoconvex *CR* structure of hypersurface type is locally isomorphic to $(V, \ \xi, \ \tilde{J})$, where $\tilde{J} \in \mathfrak{C}(\xi)$ satisfies $N_{\tilde{J}} \equiv 0$ and thus

$$\tilde{J} = (I + L)J(I + L)^{-1} \text{ with } L = {}^tL, \, JL + LJ = 0 \text{ and } \bar{\partial}_J L + \frac{1}{2}[[L, \ L]] = 0,$$

(note that $L = {}^tL$ amounts to the fact that the matrix representing L with respect to the basis $\{X_1, \ldots, X_n, \ Y_1, \ldots, \ Y_n\}$ is symmetric).

It is therefore highly interesting to give a closer look to the basic structure $(V(n),\ \xi_n,\ J)$.

Let $\sigma \in C^\infty(V,\ \mathbb{R})$: then

$$X_\sigma := \sum_{h=1}^{n}[-(Y_h\sigma)X_h + (X_h\sigma)Y_h]$$

satisfies

$$\iota_{X_\sigma} d\alpha + d\sigma = 0$$

and so

$$T_\sigma := X_\sigma + \sigma S \in \mathcal{A}_0(\xi);$$

for $1 \le j \le n$ we have:

$$[X_j,\ T_\sigma] = \sum_{h=1}^{n}[-(X_j Y_h\sigma)X_h + (X_j X_h\sigma)Y_h]$$

$$[Y_j,\ T_\sigma] = \sum_{h=1}^{n}[-(Y_j Y_h\sigma)X_h + (Y_j X_h\sigma)Y_h]$$

and so:

$$\bar\partial_{X_j} T_\sigma = \frac{1}{2}\sum_{h=1}^{n}[-((X_j Y_h + Y_j X_h)\sigma) + ((X_j X_h - Y_j Y_h)\sigma)Y_h];$$

therefore

1. if $\sigma = -2x_r y_s$, $r < s$, then

$$\bar\partial T_\sigma = X_r^* \otimes X_s + X_s^* \otimes X_r - Y_r^* \otimes Y_s - Y_s^* \otimes Y_r;$$

2. if $\sigma = -x_r y_r$, then

$$\bar\partial T_\sigma = X_r^* \otimes X_r - Y_r^* \otimes Y_r;$$

3. if $\sigma = 2x_r x_s$, $r < s$, then

$$\bar\partial T_\sigma = X_r^* \otimes Y_s + X_s^* \otimes Y_r + Y_r^* \otimes X_s + Y_s^* \otimes X_r;$$

4. if $\sigma = x_r^2$, then

$$\bar\partial T_\sigma = X_r^* \otimes Y_r + Y_r^* \otimes X_r;$$

therefore, setting, for $1 \le j \le n$:

$$Z_j := \frac{1}{2}(X_j - iY_j)\ ,\ \ \bar Z_j^* := X_j^* - iY_j^*,$$

up to the isomorphism $L \leftrightarrow \frac{1}{2}(L - iJL)$, we obtain:

1.
$$\bar{\partial} T_\sigma = \frac{1}{2}(\bar{Z}_r^* \otimes Z_s + \bar{Z}_s^* \otimes Z_r)$$

2.
$$\bar{\partial} T_\sigma = \bar{Z}_r^* \otimes Z_r$$

3.
$$\bar{\partial} T_\sigma = -\frac{i}{2}(\bar{Z}_r^* \otimes Z_s + \bar{Z}_s^* \otimes Z_r)$$

4.
$$\bar{\partial} T_\sigma = -i\bar{Z}_r^* \otimes Z_r.$$

Consequently, if $N = \frac{1}{2}n(n+1)$ and $\sigma_1, \dots \sigma_N$ are the quadratic functions $\{-2x_r y_s, -x_r y_r\}_{1 \le r < s \le n}$, we obtain that, setting $T_h =: T_{\sigma_h}, 1 \le h \le N$, any $L \in \mathcal{A}^1_J(\xi)$ can be uniquely written as

$$L = \sum_{h=1}^{N} a_h \bar{\partial} T_h, a_h \in C^\infty(V, \mathbb{C}), 1 \le h \le N.$$

Note that, for $1 \le h \le N$, $T_h(0) = 0$.
In a neighborhood of $0 \in V$, let $\tilde{J} \in \mathfrak{C}(\xi)$ such that $N_{\tilde{J}} \equiv 0$; then there exists $L \in \mathrm{End}(\xi)$, $L = {}^tL$, $LJ = JL = 0$ such that

$$\tilde{J} = (I+L)J(I+L)^{-1} = (I-L)^{-1}J(I+L);$$

up to a linear change of coordinates, we can assume $L(0) = 0$; for $T \in \mathcal{A}^0(\xi)$, we have:

$$(I-L)(\bar{\partial}_{\tilde{J}} T((I+L)X) = (\bar{\partial}_J T)(X) + \frac{1}{2}([LX, T] + J[JLX, T]) + \frac{1}{2}(JL[JLX, T] - L[X, T] - JL[JX, T] - L[LX, T]);$$

consequently, if $T(0) = 0$
$$\bar{\partial}_{\tilde{J}} T = \bar{\partial}_J + O(|p|)$$

and thus any $L \in \mathcal{A}^1_{\tilde{J}}(\xi)$ can be uniquely written as

$$L = \sum_{h=1}^{N} a_h \bar{\partial}_{\tilde{J}} T_h \quad , \quad a_h \in C^\infty(V, \mathbb{C}) \ , \ 1 \le h \le N.$$

Having all that, it is easy to prove the following

Lemma 7.1. *Let* $(V,\ \xi,\ J)$ *be a compact* $(2n+1)$*-dimensional strictly pseudoconvex CR manifold of hypersurface type and let* $N = \frac{1}{2}n(n+1)$*; fix a contact form* α *; then, there exist two finite open coverings*

$$\mathfrak{U} = (U_j)_{1\le j\le q}\ ,\ \mathfrak{V} = (V_j)_{1\le j\le q}\ ,\ \overline{V}_j \subset U_j\ ,\ 1 \le j \le q$$

such that, for every $j\ ,\ 1 \le j \le q$ *, there exist* $\sigma_1^{(j)},\ \dots,\ \sigma_N^{(j)} \in C^\infty(V,\ \mathbb{R})$ *such that, for* $1 \le j \le q$ *:*

- *for* $1 \le k \le N$ *,* supp $\sigma_k^{(j)} \subset U_j$
- *for every* $x \in V_j$ *setting* $T_k^{(j)} := X_{\sigma_k^{(j)}} + \sigma_k^{(j)} R_\alpha\ ,\ 1 \le k \le N$

$$\{\bar\partial_J(T_k^{(j)})[x]\}_{1\le k\le N}$$

is a basis over $\mathbb{C}$ *of* $\mathcal{A}_J^1(\xi(x))$ *.*

Corollary 7.2. $\{\bar\partial T_k^{(j)}\}_{\substack{1\le j\le q\\ 1\le k\le N}}$ *generate* $\mathcal{A}_J^1(\xi)$ *over* $C^\infty(V,\ \mathbb{C})$.

Proof. let $\mathfrak{W} = \{W_j\}_{1\le j\le q}$ be another open covering of V , with $\overline{W}_j \subset V_j$, $1 \le j \le q$ and let $\{\rho_j\}_{1\le j\le q}$ be a partition of unit subordinated to $\mathfrak{W}$; given $L \in \mathcal{A}_J^1(\xi)$, we have:

$$L = \sum_{j=1}^{q} \rho_j L$$

clearly, globally on V :

$$\rho_j L = \sum_{k=1}^{N} a_k^{(j)} \bar\partial T_k^{(j)}\ ,\ 1 \le j \le q$$

and so

$$L = \sum_{j=1}^{q}\sum_{k=1}^{N} a_k^{(j)} \bar\partial T_k^{(j)}. \qquad \square$$

Back to $(V(n),\ \xi_n,\ J)$ a first question (not really so important): let $\gamma \in \wedge_J^{0,p}(\xi_n)$, $1 \le p \le n-1$ such that $\bar\partial_b\gamma = 0$; is it possible to find a *global* $\beta \in \wedge_J^{0,p-1}(\xi_n)$ such that $\bar\partial_b\beta = \gamma$? (we know that *locally* this is true).

Depending on the answer, the following argument is global or simply local (actually, local is enough).

Let $L \in \mathcal{A}_J^1(\xi_n)$ we can write:

$$L = \sum_{r,s=1}^{n} a_{\bar r s} \overline{Z}_r^* \otimes Z_s, \text{ with } a_{\bar r s} = a_{\bar s r}\ .$$

It is easy to check that

$$\bar{\partial} L = 0 \iff \text{for every } s,\ k,\ r\ ,\ 1 \le s,\ k,\ r \le n\ ,\ \overline{Z}_k a_{\bar{r}s} - \overline{Z}_r a_{\bar{k}s} = 0$$

i.e.

$$\bar{\partial} L = 0 \iff \text{for every } s\ ,\ 1 \le s \le n\ ,\ \bar{\partial}_b < Z_s^*,\ L >= 0$$

($<,\ >$ being the duality pairing); consequently, for every s , $1 \le s \le n$, there exists $c_s \in C^\infty(V(n),\ \mathbb{C})$ such that $\bar{\partial}_b c_s =< Z_s^*,\ L >$; the condition $a_{\bar{r}s} = a_{\bar{s}r}$ implies that, if $\gamma = \sum_{s=1}^n c_s \overline{Z}_s^*$, then $\bar{\partial}_b \gamma = 0$ and so there exists $\sigma \in C^\infty(V(n),\ \mathbb{C})$ such that $\bar{\partial}_b \sigma = \gamma$; in conclusion:

$$\begin{aligned}\bar{\partial} L = 0 \iff &\text{there exists } \sigma \in C^\infty(V(n), \mathbb{C}) \text{ such that,}\\ &\text{for every } r, s, 1 \le r, s \le n, a_{\bar{r}s} = \overline{Z}_r \overline{Z}_s \sigma.\end{aligned}$$

Consequently:

$$H^1(\mathcal{A}_J(\xi_n), \bar{\partial}) = ker \bar{\partial} \cap \mathcal{A}_J^1(\xi_n) = C^\infty(V(n), \mathbb{C})/\{\sigma | \bar{Z}_r \bar{Z}_s \sigma = 0, 1 \le r, s \le n\}.$$

Similar arguments show that

$$H^p(\mathcal{A}_J(\xi_n),\ \bar{\partial}) = 0\ ,\ 2 \le p < n-1\ .$$

Lemma 7.3. *The deformation theory of* $(V(n),\ \xi_n,\ J)$ *is unobstructed.*

Proof (sketch). Given $L \in \mathcal{A}_J^1$ such that $\bar{\partial} L = 0$, we look for a curve $t \mapsto L_t$ in $\mathfrak{MC}_J(\xi_n)$ such that $\frac{d}{dt}_{|t=0} L_t = L$; let:

$$L_t := \sum_{k=1}^{\infty} t^k L_k\ ;$$

then:

$$\bar{\partial} L_t + \frac{1}{2}[[L_t,\ L_t]] = 0$$

$$\Updownarrow$$

$$\text{for every j} \in \mathbb{Z}^+\ \bar{\partial} L_j = -\frac{1}{2} \sum_{r+s=j} [[L_r,\ L_s]]\ ;$$

therefore:

1. set $L_1 = L$;
2. $\bar{\partial}[[L_1,\ L_1]] = 0$ and so we can choose $L_2 \in \mathcal{A}_J^1$ such that

$$\bar{\partial} L_1 = -\frac{1}{2}[[L_1,\ L_1]]\ ;$$

3. recursively, we can choose $L_J \in \mathcal{A}^1_J$ such that

$$\bar{\partial} L_j = -\frac{1}{2} \sum_{r+s=j} [[L_r,\ L_s]]\,;$$

4. then, we can show convergence by means of estimates of solutions of the $\bar{\partial}_b$-equation. □

8. Examples

In this section, we provide examples of contact manifolds which admit strictly pseudoconvex *CR* structures that are not gauge equivalent (*cf.* also [1]); then, in a more specific example, we see gauge equivalence at work.

Definition 8.1. Two strictly pseudoconvex *CR* manifolds (V_1, ξ_1, J_1) and (V_2, ξ_2, J_2) are said to be *CR*-equivalent if there exists a diffeomorphism $\varphi : V_1 \longrightarrow V_2$ such that:

- $\varphi_*(\xi_1) = \xi_2$;
- $\varphi_* \circ J_1 \circ {\varphi_*}^{-1} = J_2$.

Example 8.2. In [6] it is shown that when $n \geq 2$ two ellipsoids, given by

$$\sum_{j=1}^{n} a_j x_j^2 + b_j y_j^2 = 1,\ a_j \geq b_j > 0,\ z_j = x_j + i y_j,$$

are *CR* diffeomorphic if and only if the set of ratios

$$\frac{(a_j - b_j)}{(a_j + b_j)}$$

is the same for the two. Clearly two ellipsoids are diffeomorphic. This in particular implies that the unit sphere S_n is diffeomorphic but not *CR* diffeomorphic to any non-trivial ellipsoid, that is those for which $(a_j - b_j) \neq 0$.

Let $E(a_j, b_j)$ be a non-trivial ellipsoid and let $\psi : (S_n, \xi) \longrightarrow (E(a_j, b_j), \tilde{\xi})$ be the diffeomorphism given by

$$x_j + i y_j \mapsto \frac{x_j}{\sqrt{a}_j} + i \frac{y_j}{\sqrt{b}_j},$$

where ξ (respectively $\tilde{\xi}$) is the complex tangent bundle of S_n (respectively $(E(a_j, b_j))$. Consider on S_n the strictly pseudoconvex *CR* structure given on ${\psi_*}^{-1}(\tilde{\xi})$ by

$$J = {\psi_*}^{-1} \circ J_n \circ \psi_*,$$

where J_n is the standard structure. Using Gray's theorem or Theorem 4.14, we may assume, after composing by a diffeomorphism, that J is a strictly pseudoconcex *CR* structure on ξ.

We claim that J_n and J are not equivalent in the sense of the definition above, that is J_n and J are not gauge equivalent. Indeed, if there is a $\varphi : (S_n, \xi, J_n) \longrightarrow (S_n, \xi, J)$ such that

$$\varphi_* \circ J_n \circ {\varphi_*}^{-1} = J,$$

then, using the definition of J, we obtain that

$$\varphi_* \circ J_n \circ {\varphi_*}^{-1} = {\psi_*}^{-1} \circ J_n \circ \psi_*,$$

and then

$$\psi_* \circ \varphi_* \circ J_n \circ {\varphi_*}^{-1} \circ {\psi_*}^{-1} = J_n.$$

This contradicts the fact that there is no *CR* diffeomorphism between the unit sphere and any non-trivial ellipsoid.

We shall see in a specific example how gauge equivalence works:

Example 8.3. let

$$i : S^3 = \{(x, u, y, v) \in \mathbb{R}^4 | x^2 + u^2 + y^2 + v^2 = 1\} \longrightarrow \mathbb{R}^4$$

and let

$$\alpha := i^*(xdy - ydx + udv - vdu)$$

clearly α is a contact form and $\xi := \ker \alpha$ is globally generated by

$$X := u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u} - v\frac{\partial}{\partial y} + y\frac{\partial}{\partial v}$$

and

$$Y := v\frac{\partial}{\partial x} - y\frac{\partial}{\partial u} - u\frac{\partial}{\partial y} - x\frac{\partial}{\partial v};$$

moreover:

$$R := R_\alpha = -y\frac{\partial}{\partial x} - v\frac{\partial}{\partial u} + x\frac{\partial}{\partial y} + u\frac{\partial}{\partial v}.$$

The standard complex structure J on $\mathbb{R}^4$ clearly satisfies $JX = Y$ and (S^3, ξ, J) is the standard *CR* structure on the unit 3-dimensional sphere.
A straightforward computation gives:

$$[X, Y] = -2R, [X, R] = 2Y, [Y, R] = -2X$$

(after all $S^3 = SU(2)$!).

$C^\infty(S^3, \mathbb{R})$ parametrizes $\mathcal{A}_0(\xi)$ as follows:

$$\sigma \mapsto X_\sigma + \sigma R = -\frac{1}{2}Y\sigma X + \frac{1}{2}X\sigma Y + \sigma R;$$

consequently:

$$4\bar{\partial}_X(X_\sigma + \sigma R) = -(XY\sigma + YX\sigma)X + (XX\sigma - YY\sigma)Y$$

and

$$4\bar{\partial}_Y(X_\sigma + \sigma R) = -(XX\sigma - YY\sigma)X - (XY\sigma + YX\sigma)Y;$$

by dimension reasons, any $\tilde{J} \in \mathfrak{C}(\xi)$ satisfies $N_{\tilde{J}} = 0$ and, accordingly, any $L \in \mathcal{A}^1_J(\xi)$ satisfies $\bar{\partial}_J L = 0 = [[L, L]]$; in terms of the frame $\{X, Y\}$, such an L corresponds to

$$\begin{pmatrix} -a & b \\ b & a \end{pmatrix}$$

with $a,\ b \in C^\infty(S^3,\ \mathbb{R})$ and so:

$$\bar{\partial}_J(X_\sigma + \sigma R) = L$$

corresponds to

$$\begin{cases} (XX\sigma - YY\sigma) = a \\ (XX\sigma - YY\sigma) = b \end{cases} \tag{8.1}$$

or equivalently

$$\bar{Z}\bar{Z}\sigma = \frac{1}{4}(a + ib)$$

where $Z := \frac{1}{2}(X + iY)$.

References

[1] L. BOUTET DE MONVEL, *Integration des equations de Cauchy-Riemann induites formelles*, Seminaire Goulaic-Lions-Schwartz 1974-75, Centre Math. Ecole Polytechnique, Paris, 1975.

[2] P. DE BARTOLOMEIS "Symplectic and Holomorphic Theory in Kähler Geometry", XIII Escola de geometria diferencial, Sao Paulo, 2004.

[3] P. DE BARTOLOMEIS, *Symplectic deformations of Kähler manifolds*, J. Symplectic Geom. **3** (2005), 341–355.

[4] K. KODAIRA and J. MORROW, "Complex Manifolds", Holt, Rinehart and Winston, Inc., 1971.

[5] D. MCDUFF and D. SALAMON, "Introduction to Symplectic Topology", Clarendon Press, Oxford, 1995.

[6] S. M. WEBSTER, *On the mapping problem for algebraic real hypersurfaces*, Invent. Math. **43** (1977), 53–68.

Institut de Mathématiques
Université de Fribourg
1700 Perolles, Fribourg, Switzerland
francine.meylan@unifr.ch

Dipartimento di Matematica Applicata "G. Sansone"
Università degli Studi di Firenze
Via S. Marta, 3
50139 Firenze, Italia
paolo.debartolomeis@unifi.it