Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5)
Vol. IX (2010), 253-295

Shell theories arising as low energy Γ-limit of 3d nonlinear elasticity

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Abstract. We discuss the limiting behavior (using the notion of Γ-limit) of the 3d nonlinear elasticity for thin shells around an arbitrary smooth 2d surface. In particular, under the assumption that the elastic energy of deformations scales like h^4, h being the thickness of a shell, we derive a limiting theory which is a generalization of the von Kármán theory for plates.

Mathematics Subject Classification (2010): 74K20 (primary); 74B20 (secondary).

1. Introduction

The derivation of lower dimensional models for thin structures (such as membranes, shells, or beams) from the three-dimensional theory has been one of the fundamental questions since the beginning of research in elasticity [19]. Recently, a novel variational approach through Γ-*convergence* has lead to the derivation of a hierarchy of limiting theories. Among other features, it provides a rigorous justification of convergence of three-dimensional minimizers to minimizers of suitable lower dimensional limit energies.

In this paper we discuss shell theories arising as Γ-limits of *higher scalings* of the *nonlinear* elastic energy. Given a 2-dimensional surface S, consider a shell S^h of mid-surface S and thickness h, and associate to its deformation u the scaled per unit thickness three dimensional nonlinear elastic energy $E^{\text{elastic}}(u, S^h)$. We are interested in the identification of the Γ-limit I_β of the energies:

$$h^{-\beta} E^{\text{elastic}}(\cdot, S^h),$$

M.L. was partially supported by the NSF grant DMS-0707275. M.G.M. was partially supported by the Italian Ministry of University and Research through the project "Variational problems with multiple scales" 2006 and by GNAMPA through the project "Problemi di riduzione di dimensione per strutture elastiche sottili" 2008. M.R.P. was partially supported by the University of Pittsburgh grant CRDF-9003034, and by the NFS grant DMS-0907844.

Received September 22, 2008; accepted May 7, 2009.

as $h \to 0$, for a given scaling $\beta \geq 0$. As mentioned above, this implies convergence, in a suitable sense, of minimizers u^h of $E^{\text{elastic}}(\cdot, S^h)$ (subject to applied forces) to minimizers of two-dimensional energy I_β, provided $E^{\text{elastic}}(u^h, S^h) \leq Ch^\beta$.

In the case when S is a subset of $\mathbb{R}^2$ (*i.e.*, a plate), such Γ-convergence was first established by LeDret and Raoult [15] for $\beta = 0$, then by Friesecke, James, and Müller [9,10] for all $\beta \geq 2$ (see also [24] for results for $\beta = 2$ under additional conditions). In the case of $0 < \beta < 5/3$ the convergence was recently obtained by Conti and Maggi [5], see also [2]. The regime $5/3 \leq \beta < 2$ remains open and is conjectured to be relevant for crumpling of elastic sheets. Other significant results for plates concern derivation of limit theories for incompressible materials [3, 4, 28], for heterogeneous materials [26], and through establishing convergence of equilibria, rather than strict minimizers [20, 22].

Much less is known in the general case when S is an arbitrary surface. The first result by LeDret and Raoult [16] relates to scaling $\beta = 0$ and models *membrane shells*: the limit I_0 depends only on the stretching and shearing produced by the deformation on the mid-surface S. Another study is due to Friesecke, James, Mora, and Müller [8], who analyzed the case $\beta = 2$. This scaling corresponds to a *flexural shell model*, where the only admissible deformations are those preserving the metric on S. The energy I_2 depends then on the change of curvature produced by the deformation.

All the above mentioned theories (as well as the subsequent results in this paper) should be put in contrast with a large body of literature, devoted to derivations starting from three-dimensional *linear* elasticity (see Ciarlet [1] and references therein). Indeed, since thin structures may undergo large rotations even under the action of very small forces, one cannot assume the small strain condition, on which the linear elasticity is based.

The objective of this work is to discuss the limit energies for scalings $\beta \geq 4$, for arbitrary surfaces S. We now give a heuristic overview of our results, whose precise formulations will be presented in Section 2. If $E^{\text{elastic}}(u, S^h) \approx Ch^\beta$, for any $\beta > 2$, one expects u to be close to a rigid motion R. This argument can be made precise by means of the quantitative rigidity estimate due to Friesecke, James, and Müller [9] (see also Lemma A.1). We further demonstrate that the first term in the expansion of $u - R$, in terms of h, belongs to the space of infinitesimal isometries $\mathcal{V}$. That is, there is no first order change in the Riemannian metric of S under the displacement $V \in \mathcal{V}$. The corresponding bending energy, given in terms of the first order change in the second fundamental form of S, is the Γ-limit I_β if $\beta > 4$ (Theorem 2.3). This limit energy coincides with the so-called *linearly elastic flexural shell model*, derived in [1] from the linear elasticity theory. Our result guarantees therefore that, without any a priori smallness assumption on the strain, the use of the linearized flexural shell model is justified whenever the order of magnitude of the per unit thickness three-dimensional energy is h^β with $\beta > 4$.

When $\beta = 4$, also the second order in h change in the metric on S (stretching) contributes to the limiting energy. This change is induced by V, and additionally, by an "approximate second order displacement" w. This last notion in-

volves studying the *finite strain space* $\mathcal{B}$. For a similar situation where this space emerges see the discussion by Sanchez-Palencia [25] and Geymonat and Sanchez-Palencia [11] under the title of ill-inhibited surfaces, in the context of linear elasticity. In Theorems 2.1 and 2.2 we derive the energy functional I_4, which can be seen as a generalization of the *von Kármán theory* for plates [14], justified in terms of Γ-convergence in [10]. Indeed, if S is a plate, then the normal component of V and w are, respectively, the out-of-plane and the in-plane displacements (modulo a possible in-plane infinitesimal rigid motion). In the general case of shells, the functional I_4 has been, according to our knowledge, so far absent from the literature.

A particular class of surfaces when I_4 simplifies to the bending energy is the hereby introduced class of *approximately robust* surfaces. We say that S is (approximately) robust if any infinitesimal isometry $V \in \mathcal{V}$ can be completed by a second order displacement to an (approximate) second order isometry. In other words, S can always further adjust its deformation, to compensate for the change of metric produced at second order. As a result, the total stretching of second order is insignificant and the Γ-limit consists only of a bending term (Theorem 2.3). We show three general examples of approximately robust surfaces: convex surfaces, surfaces of revolution, and developable surfaces without flat parts. An example of a not approximately robust surface is a plate.

We also address the issue of external forces, depending on the reference configuration, namely the *dead loads* (Theorem 2.5). Under a vanishing average condition and a suitable scaling of the forces f^h applied to S^h, Theorem 2.1 provides information on the deformation of S^h assumed in response to the load. In addition, the appropriate limit force f identifies the set of possible rotations the body will undergo. This phenomenon is easily observed: if f^h is "compressive", then S^h prefers to make a large rotation rather than undergoing a compression, and an alignment of V with the force is energetically preferable.

As noted above, from the mechanical point of view, the class of approximately robust surfaces exhibits a response to loads which is qualitatively different than that of plates, and a better capacity to resist stretching under the same regime of forces. In general, the most important factor in understanding a shell's response to loads is the relationship and properties of spaces $\mathcal{V}$ and $\mathcal{B}$. From the technical point of view these are also the crucial new ingredients of the present paper, improving the analysis of [8–10].

The identification of Γ-limit for any scaling in the range $\beta \in (2, 4)$ and non-flat S is still open. In analogy with the analysis developed in [10] for plates, the construction of a recovery sequence requires finding an exact isometry of S, coinciding with a given second order isometry. Another direction of study concerns shells, whose mid-surface is inhibited (or infinitesimally rigid). Examples of such are closed or partially clamped elliptic surfaces. In this case the limit functionals that our theory yields are identically equal to zero. This suggests looking for higher order terms in the development of the three-dimensional energy in the sense of Γ-convergence. These are subtle issues and we plan to address them in a forthcoming paper.

ACKNOWLEDGEMENTS. We thank Stefan Müller for helpful discussions. A large part of this work was carried out while the second author was visiting the Institute for Mathematics and its Applications in Minneapolis (USA), whose support is gratefully acknowledged.

2. An overview of the main results

Let S be a 2-dimensional surface embedded in $\mathbb{R}^3$. We assume that S is compact, connected, oriented, and of class $\mathcal{C}^{1,1}$, and that its boundary ∂S is the union of finitely many (possibly none) Lipschitz continuous curves. Consider a family $\{S^h\}_{h>0}$ of thin shells of thickness h around S:

$$S^h = \{z = x + t\vec{n}(x);\ x \in S,\ -h/2 < t < h/2\}, \qquad 0 < h < h_0.$$

We will use the following notation: $\vec{n}(x)$ for the unit normal, T_xS for the tangent space, and $\Pi(x) = \nabla\vec{n}(x)$ for the shape operator on S, at a given $x \in S$. The projection onto S along $\vec{n}$ will be denoted by π, so that:

$$\pi(z) = x \qquad \forall z = x + t\vec{n}(x) \in S^h.$$

We will assume that $h < h_0$, with $h_0 > 0$ sufficiently small to have π well defined on each S^h, and so that: $1/2 < |\mathrm{Id} + t\Pi(x)| < 3/2$ for all z as above.

For a $W^{1,2}$ deformation of a thin shell $u^h : S^h \longrightarrow \mathbb{R}^3$, we assume that its elastic energy (scaled per unit thickness) is given by the nonlinear functional:

$$E^{\text{elastic}}(u^h, S^h) = \frac{1}{h}\int_{S^h} W(\nabla u^h).$$

The stored-energy density function $W : \mathbb{R}^{3\times 3} \longrightarrow [0,\infty]$ is $\mathcal{C}^2$ in some open neighborhood of $SO(3)$, in the space $\mathbb{R}^{3\times 3}$ of 3×3 real matrices. Moreover, W is assumed to satisfy the conditions of normalization, frame indifference and nondegeneracy:

$$\forall F \in \mathbb{R}^{3\times 3} \quad \forall R \in SO(3) \qquad W(R) = 0, \quad W(RF) = W(F),$$
$$W(F) \geq C\mathrm{dist}^2(F, SO(3)),$$

with a uniform constant $C > 0$. Here $SO(3)$ denotes the group of proper rotations. Recall that the tangent space to $SO(3)$ at Id is the space of skew-symmetric matrices:

$$so(3) = \left\{F \in \mathbb{R}^{3\times 3};\ \ F = -F^T\right\}.$$

It is convenient to view u^h through their rescalings $y^h \in W^{1,2}(S^{h_0}, \mathbb{R}^3)$, defined on a common domain S^{h_0}:

$$y^h(x + t\vec{n}(x)) = u^h\left(x + th/h_0\vec{n}(x)\right) \qquad \forall x \in S \quad \forall t \in (-h_0/2, h_0/2).$$

Given the rescaled deformation $y^h \in W^{1,2}(S^{h_0}, \mathbb{R}^3)$, its scaled average displacement is given by:

$$(V^h[y^h])(x) = \frac{h}{\sqrt{e^h}} \fint_{-h_0/2}^{h_0/2} y^h(x + t\vec{n}) - x \, \mathrm{d}t.$$

Since we will frequently deal with such vector fields $V \in W^{1,2}(S, \mathbb{R}^3)$ on the surface, we introduce the following notation. By $\operatorname{sym} \nabla V(x)$ we mean a bilinear form on $T_x S$ given by: $(\operatorname{sym} \nabla V(x)\tau)\eta = \frac{1}{2}[(\partial_\tau V(x))\eta + (\partial_\eta V(x))\tau]$, for all $\tau, \eta \in T_x S$. Given a matrix field $A \in L^2(S, \mathbb{R}^{3\times 3})$, by $A_{\tan}(x)$ we denote the tangential minor of A at $x \in S$, that is $[(A(x)\tau)\eta]_{\tau,\eta\in T_x S}$.

We are concerned with the limiting behavior, relative to low energy deformations, of the energies:

$$I^h(y^h) = \frac{1}{h} \int_{S^h} W(\nabla u^h).$$

That is, we want to discuss the limit, as $h \to 0$, of the functionals I^h/e^h, for a given sequence of positive numbers e^h, which we assume to satisfy:

$$\lim_{h\to 0} e^h/h^4 = \kappa^2 < \infty. \tag{2.1}$$

We will prove that the limit under consideration is described (in the sense of Γ-convergence [6]) by the generalized von Kármán functional $I(V, B_{\tan})$ in (2.3), defined for infinitesimal isometries $V \in \mathcal{V}$ and strains $B_{\tan} \in \mathcal{B}$. The crucial space $\mathcal{V}$ consists of infinitesimal isometries [27] $V \in W^{2,2}(S, \mathbb{R}^3)$, that is these vector fields V for whom there exists a matrix field $A \in W^{1,2}(S, \mathbb{R}^{3\times 3})$ so that:

$$\partial_\tau V(x) = A(x)\tau \quad \text{and} \quad A(x) \in so(3) \qquad \text{for a.e. } x \in S, \quad \forall \tau \in T_x S. \tag{2.2}$$

Another crucial space is the finite strain space $\mathcal{B}$, consisting of the following symmetric matrix fields:

$$\mathcal{B} = \left\{ L^2 - \lim_{h\to 0} \operatorname{sym}\nabla w^h; \;\; w^h \in W^{1,2}(S, \mathbb{R}^3) \right\},$$

(clearly, both the weak and the strong convergences yield the same $\mathcal{B}$).

Our first main result is the following:

Theorem 2.1. *Assume* (2.1) *and let* $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ *be a sequence of deformations such that the sequence of scaled energies* $\{\frac{1}{e^h} I^h(y^h)\}$ *is bounded. Then there exist rigid motions of* $\mathbb{R}^3$*, given through proper rotations* $Q^h \in SO(3)$ *and translations* $c^h \in \mathbb{R}^3$*, such that for the normalized deformations:*

$$\tilde{y}^h(x + t\vec{n}) = (Q^h)^T y^h(x + t\vec{n}) - c^h$$

the following holds.

(i) $\tilde{y}^h$ *converge in* $W^{1,2}(S^{h_0})$ *to* π.
(ii) $V^h[\tilde{y}^h]$ *converge (up to a subsequence) in* $W^{1,2}(S)$ *to some* $V \in \mathcal{V}$.
(iii) $\frac{1}{h}\mathrm{sym}\nabla V^h[\tilde{y}^h]$ *converge (up to a subsequence) weakly in* $L^2(S)$ *to some symmetric matrix field* B_{tan} *on* S.
(iv) *There holds:*

$$\liminf_{h\to 0} \frac{1}{e^h} I^h(y^h) \geq I(V, B_{\mathrm{tan}}),$$

where:

$$\begin{aligned} I(V, B_{\mathrm{tan}}) = \frac{1}{2}\int_S \mathcal{Q}_2\left(x, B_{\mathrm{tan}} - \frac{\kappa}{2}(A^2)_{\mathrm{tan}}\right) \\ + \frac{1}{24}\int_S \mathcal{Q}_2\left(x, (\nabla(A\vec{n}) - A\Pi)_{\mathrm{tan}}\right). \end{aligned} \tag{2.3}$$

The following quadratic, nondegenerate forms are of relevance here:

$$\begin{aligned} \mathcal{Q}_3(F) &= D^2W(\mathrm{Id})(F, F), \\ \mathcal{Q}_2(x, F_{\mathrm{tan}}) &= \min\{\mathcal{Q}_3(\tilde{F});\ \ (\tilde{F} - F)_{\mathrm{tan}} = 0\}. \end{aligned} \tag{2.4}$$

The form $\mathcal{Q}_3$ is defined for $F \in \mathbb{R}^{3\times 3}$, while $\mathcal{Q}_2(x, \cdot)$, for a given $x \in S$ is defined on tangential minors F_{tan} of such matrices. Both forms $\mathcal{Q}_3$ and all $\mathcal{Q}_2(x, \cdot)$ are positive definite and depend only on the symmetric parts of their arguments [9].

Theorem 2.1 will be proved in Sections 3 and 4. One of the crucial ingredients is a result on approximating large deformations [9]. For completeness, we sketch its proof, in the setting of shells, in Appendix A. We also note that because of the nontrivial geometry of the shell, the density in the limiting energy I, in general exhibits a dependence on $x \in S$, although the three-dimensional density W is homogeneous.

Our second main result concerns the possibility of recovering the functional $I(V, B_{\mathrm{tan}})$ in (2.3) (or its components) as the limit of scaled energies $\frac{1}{e^h} I^h(y^h)$, for some sequence of deformations.

Theorem 2.2. *Assume* (2.1). *For every* $V \in \mathcal{V}$ *and every* $B_{\mathrm{tan}} \in \mathcal{B}$, *there exists a sequence of deformations* $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ *such that:*

(i) y^h *converge in* $W^{1,2}(S^{h_0})$ *to* π.
(ii) $V^h[y^h]$ *converge in* $W^{1,2}(S)$ *to* V.
(iii) $\frac{1}{h}\mathrm{sym}\,\nabla V^h[y^h]$ *converge in* $L^2(S)$ *to* B_{tan}.
(iv) *Recalling the definition* (2.3) *one has:*

$$\lim_{h\to 0} \frac{1}{e^h} I^h(y^h) = I(V, B_{\mathrm{tan}}).$$

The form of the limiting energy functional I simplifies, when the space $\mathcal{B}$ is large enough to choose B_{tan} so that the first term in (2.3) vanishes. That is, we call S "approximately robust" if for every $V \in \mathcal{V}$ one has $(A^2)_{\mathrm{tan}} \in \mathcal{B}$.

Theorem 2.3. *Assume* (2.1). *Let* $\kappa = 0$ *or let* S *be approximately robust. Then for every* $V \in \mathcal{V}$ *there exists a sequence of deformations* $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ *such that* (i) *and* (ii) *of Theorem* 2.2 *hold. Moreover:*

$$\lim_{h\to 0} \frac{1}{e^h} I^h(y^h) = \tilde{I}(V),$$

where

$$\tilde{I}(V) = \frac{1}{24} \int_S \mathcal{Q}_2\Big(x, \big(\nabla(A\vec{n}) - A\Pi\big)_{\tan}\Big). \tag{2.5}$$

Theorems 2.2 and 2.3 will be proved in Section 6. In Section 5 we discuss the space $\mathcal{B}$. In particular, we shall see that convex surfaces, surfaces of revolution, and non-flat developable surfaces are approximately robust.

Theorems 2.1 and 2.2 (or 2.3) can be summarized (although they provide more information than the below statement), using the language of Γ-convergence. For completeness, the following result will be presented in Appendix B.

Corollary 2.4. *Assume* (2.1).

(i) *Define a sequence of functionals:*

$$\mathcal{F}^h : W^{1,2}(S^{h_0}, \mathbb{R}^3) \times W^{1,2}(S, \mathbb{R}^3) \times L^2(S, \mathbb{R}^{2\times 2}) \longrightarrow \overline{\mathbb{R}}$$

$$\mathcal{F}^h(y^h, V^h, B^h_{\tan}) = \begin{cases} \dfrac{1}{e^h} I^h(y^h) & \text{if } V^h = V^h[y^h] \text{ and } B^h_{\tan} = \frac{1}{h} sym\, \nabla V^h, \\ +\infty & \text{otherwise.} \end{cases}$$

Then $\mathcal{F}^h$ Γ*-converge, as* $h \to 0$*, to the following functional:*

$$\mathcal{F}(y, V, B_{\tan}) = \begin{cases} I(V, B_{\tan}) & \text{if } y = \pi,\ V \in \mathcal{V} \text{ and } B_{\tan} \in \mathcal{B}, \\ +\infty & \text{otherwise.} \end{cases}$$

(ii) *Assume that* $\kappa = 0$ *or let* S *be approximately robust. Define the functionals:*

$$\tilde{\mathcal{F}}^h : W^{1,2}(S^{h_0}, \mathbb{R}^3) \times W^{1,2}(S, \mathbb{R}^3) \longrightarrow \overline{\mathbb{R}}$$

$$\tilde{\mathcal{F}}^h(y^h, V^h) = \begin{cases} \dfrac{1}{e^h} I^h(y^h) & \text{if } V^h = V^h[y^h], \\ +\infty & \text{otherwise.} \end{cases}$$

Then $\tilde{\mathcal{F}}^h$ Γ*-converge, as* $h \to 0$*, to the functional:*

$$\tilde{\mathcal{F}}(y, V) = \begin{cases} \tilde{I}(V) & \text{if } y = \pi \text{ and } V \in \mathcal{V}, \\ +\infty & \text{otherwise.} \end{cases}$$

All statements above remain valid if the product spaces (the domains of functionals $\mathcal{F}^h$, $\tilde{\mathcal{F}}^h$*) are equipped with the weak (instead of strong) topology.*

We further consider a sequence of forces $f^h \in L^2(S^h, \mathbb{R}^3)$, acting on thin shells S^h. For simplicity, we assume that:

$$f^h(x + t\vec{n}(x)) = h\sqrt{e^h} \det(\mathrm{Id} + t\Pi(x))^{-1} f(x),$$

where $f \in L^2(S, \mathbb{R}^3)$ is normalized so that:

$$\int_S f = 0. \tag{2.6}$$

Define m to be the maximized action of force f on S over all rotations of S, and let $\mathcal{M}$ be the corresponding set of maximizers:

$$\mathcal{M} = \left\{ \bar{Q} \in SO(3); \quad \int_S f(x) \cdot \bar{Q}x \, \mathrm{d}x = m = \max_{Q \in SO(3)} \int_S f \cdot Qx \right\}. \tag{2.7}$$

The total energy functional on S^h is given through:

$$J^h(y^h) = I^h(y^h) + m^h - \frac{1}{h} \int_{S^h} f^h u^h,$$

where $m^h = h\sqrt{e^h} m$.

Theorem 2.5. *Assume* (2.1) *and* (2.6). *Then:*

(i) *For every small $h > 0$ one has:*

$$0 \geq \inf \left\{ \frac{1}{e^h} J^h(y^h); \quad u^h \in W^{1,2}(S^h, \mathbb{R}^3) \right\} \geq -C.$$

(ii) *If $u^h \in W^{1,2}(S, \mathbb{R}^3)$ is a minimizing sequence of $\frac{1}{e^h} J^h$, that is:*

$$\lim_{h \to 0} \left(\frac{1}{e^h} J^h(y^h) - \inf \frac{1}{e^h} J^h \right) = 0, \tag{2.8}$$

then there exists $Q^h \in SO(3)$ and $c^h \in \mathbb{R}^3$ such that for the normalized deformations $\tilde{y}^h = (Q^h)^T y^h - c^h$ the convergences of Theorem 2.1 (i) (ii) *and* (iii) *hold. The convergence of (a subsequence of) $\frac{1}{h} sym \nabla V^h[\tilde{y}^h]$ to $B_{\tan}$ in* (iii) *is strong in $L^2(S)$.*
Moreover, the set of accumulation points of $\{Q^h\}$ is contained within $\mathcal{M}$. Any limit $(V, B_{\tan}, \bar{Q})$ minimizes the functional:

$$J(V, B_{\tan}, \bar{Q}) = I(V, B_{\tan}) - \int_S f \cdot \bar{Q}V,$$

over all $V \in \mathcal{V}$, all $B_{\tan} \in \mathcal{B}$ and $\bar{Q} \in \mathcal{M}$.

(iii) *If* $\kappa = 0$ *in* (2.1)*, or if* S *is approximately robust, then for any minimizing sequence as in* (2.8)*, we obtain convergences of* $\tilde{y}^h$*,* $V^h[\tilde{y}^h]$ *and* Q^h *as described in* (ii) *above, and the limit* $(V, \bar{Q})$ *minimizes the functional:*

$$\tilde{J}(V, \bar{Q}) = \tilde{I}(V) - \int_S f \cdot \bar{Q} V$$

over all $V \in \mathcal{V}$ *and all* $\bar{Q} \in \mathcal{M}$.

In Section 7 we prove Theorem 2.5 and explain the significance of the set $\mathcal{M}$ in the setting of dead loads.

The lower bound on the functionals J and $\tilde{J}$, as well as attainment of their infima, can be proved independently, under conditions (2.6) and:

$$\int_S f(x) \cdot \bar{Q} F x \, \mathrm{d}x = 0 \quad \forall \bar{Q} \in \mathcal{M}, \quad \forall F \in so(3). \tag{2.9}$$

Here $\mathcal{M}$ is any closed, nonempty subset of $SO(3)$. When $\mathcal{M}$ has the form (2.7), then (2.9) follows from (2.7) and can be seen as its linearization. This analysis will be carried out in Appendix C.

3. Convergence of low energy deformations

In this section we derive some bounds on families of vector mappings $\{u^h\}_{h>0}$, defined on S^h, under the assumption of smallness on their energy. In what follows, by C we denote an arbitrary positive constant, depending on the geometry of S but not on h or the vector mapping under consideration. In all proofs, the convergences are understood up to a subsequence, unless stated otherwise.

We will work under the following hypothesis:

(H) A sequence of vector mappings $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ and a sequence of positive numbers e^h satisfy, for small $h > 0$:

(i) $\dfrac{1}{h} \displaystyle\int_{S^h} W(\nabla u^h) \leq C e^h$,

(ii) $\displaystyle\lim_{h \to 0} e^h / h^2 = 0$.

As for the flat case in [10], the first crucial step is the following approximation result.

Lemma 3.1. *For each* u^h *as in* **(H)** *there exist a matrix field* $R^h \in W^{1,2}(S, \mathbb{R}^{3\times 3})$ *such that:*

$$R^h(x) \in SO(3) \qquad \forall x \in S,$$

and a matrix $Q^h \in SO(3)$ *such that:*

(i) $\|\nabla u^h - R^h\pi\|_{L^2(S^h)} \leq Ch^{1/2}\sqrt{e^h}$,
(ii) $\|\nabla R^h\|_{L^2(S)} \leq Ch^{-1}\sqrt{e^h}$,
(iii) $\|(Q^h)^T R^h - \mathrm{Id}\|_{L^p(S)} \leq Ch^{-1}\sqrt{e^h}$, *for all* $p \in [1, \infty)$.

The proof follows from Lemma A.1 given in Appendix A, in view of:

$$E(u^h, S^h) = \int_{S^h} \mathrm{dist}^2(\nabla u^h, SO(3)) \leq C \int_{S^h} W(\nabla u^h) \leq Che^h$$

so that $\lim_{h\to 0} h^{-3} E(u^h, S^h) = 0$ by hypothesis **(H)**.

Lemma 3.2. *Assume* **(H)** *and let* R^h, Q^h *be given as in Lemma* 3.1. *There holds:*

(i) $\lim_{h\to 0} (Q^h)^T R^h = \mathrm{Id}$, *in* $W^{1,2}(S)$ *and in* $L^p(S)$.

Moreover, there exists a $W^{1,2}$ *skew-symmetric matrix fields* $A : S \longrightarrow so(3)$ *such that:*

(ii) $\lim_{h\to 0} \dfrac{h}{\sqrt{e^h}} \Big((Q^h)^T R^h - \mathrm{Id}\Big) = A$, *weakly in* $W^{1,2}(S)$ *and (strongly) in* $L^p(S)$.
(iii) $\lim_{h\to 0} \dfrac{h^2}{e^h} \mathrm{sym}\Big((Q^h)^T R^h - \mathrm{Id}\Big) = \dfrac{1}{2}A^2$, *in* $L^p(S)$.

In (ii) *and* (iii) *convergence is up to a subsequence* (*that we do not relabel*). *In* (i), (ii), *and* (iii) *the appropriate convergence holds for all* $p \in [1, \infty)$.

Proof. The convergences in (i) follow from Lemma 3.1 in view of **(H)**. To prove (ii), notice that the sequence:

$$A^h = \frac{h}{\sqrt{e^h}} \Big((Q^h)^T R^h - \mathrm{Id}\Big)$$

is bounded in $W^{1,2}(S)$ and so it has a weakly converging subsequence. In view of the compact embedding of $W^{1,2}(S)$ into $L^p(S)$ the convergence is strong in $L^p(S)$. One has:

$$A^h + (A^h)^T = \frac{h}{\sqrt{e^h}} \Big((Q^h)^T R^h + (R^h)^T Q^h - 2\mathrm{Id}\Big) = -\frac{\sqrt{e^h}}{h} (A^h)^T \cdot A^h.$$

The latter converges to 0 in $L^p(S)$, and therefore the limit matrix field A is skew-symmetric. The above equality proves as well that:

$$\lim_{h\to 0} \frac{h}{\sqrt{e^h}} \mathrm{sym}\, A^h = \frac{1}{2}A^2$$

in $L^p(S)$, which implies (iii). □

Recall the rescaling:

$$y^h(x + t\vec{n}(x)) = u^h\,(x + th/h_0\vec{n}(x)) \qquad \forall x \in S, \quad \forall t \in (-h_0/2, h_0/2),$$

so that $y^h \in W^{1,2}(S^{h_0}, \mathbb{R}^3)$. Also, define:

$$\nabla_h y^h(x + t\vec{n}(x)) = \nabla u^h\,(x + th/h_0\vec{n}(x))\,.$$

By a straightforward calculation we obtain the following proposition.

Proposition 3.3. *For each $x \in S$, $t \in (-h_0/2, h_0/2)$ and $\tau \in T_xS$ there hold:*

$$\partial_\tau y^h(x + t\vec{n}) = \nabla_h y^h\,(x + t\vec{n})\,(\mathrm{Id} + th/h_0\Pi(x))\,(\mathrm{Id} + t\Pi(x))^{-1}\tau$$
$$\partial_{\vec{n}} y^h(x + t\vec{n}) = \frac{h}{h_0}\nabla_h y^h\,(x + t\vec{n})\,\vec{n}(x).$$

Moreover, for $I^h(y^h) = \frac{1}{h}\int_{S^h} W(\nabla u^h)$ one has:

$$I^h(y^h) = \frac{1}{h_0}\int_{S^{h_0}} W(\nabla_h y^h(x + t\vec{n}))\cdot \det\Big[(\mathrm{Id} + th/h_0\Pi)\,(\mathrm{Id} + t\Pi)^{-1}\Big]$$
$$= \int_S \fint_{-h_0/2}^{h_0/2} W(\nabla_h y^h(x + t\vec{n}))\cdot \det\left[\mathrm{Id} + th/h_0\Pi(x)\right]\,\mathrm{d}t\,\mathrm{d}x.$$

Also, directly from Lemma 3.1 (i) and Lemma 3.2 (ii) the next proposition follows:

Proposition 3.4. *Assume* **(H)**. *Then:*

(i) $\|\nabla_h y^h - R^h\pi\|_{L^2(S^{h_0})} \le C\sqrt{e^h}$,

(ii) $\lim\limits_{h\to 0}\dfrac{h}{\sqrt{e^h}}\Big((Q^h)^T\nabla_h y^h - \mathrm{Id}\Big) = A\pi$, *in $L^2(S^{h_0})$ up to a subsequence.*

We will consider the corrected by rigid motions deformations $\tilde{y}^h \in W^{1,2}(S^{h_0}, \mathbb{R}^3)$ and averaged displacements $V^h \in W^{1,2}(S, \mathbb{R}^3)$:

$$\tilde{y}^h = (Q^h)^T y^h - c^h, \qquad V^h = V^h[\tilde{y}^h] = \frac{h}{\sqrt{e^h}}\fint_{-h_0/2}^{h_0/2}\tilde{y}^h(x + t\vec{n}) - x\,\mathrm{d}t,$$

where $c^h = \fint_S \fint_{-h_0/2}^{h_0/2}(Q^h)^T y^h - x\,\mathrm{d}t\,\mathrm{d}x$, so that $\fint_S V^h = 0$.

Lemma 3.5. *Assume* **(H)**. *Then:*

(i) $\lim\limits_{h\to 0}\tilde{y}^h = \pi$, *in $W^{1,2}(S^{h_0})$,*

(ii) $\lim\limits_{h\to 0} V^h = V$, *in $W^{1,2}(S)$ up to a subsequence.*

The vector field V in (ii) *has regularity $W^{2,2}(S,\mathbb{R}^3)$ and it satisfies $\partial_\tau V(x) = A(x)\tau$ for all $\tau \in T_xS$. The $W^{1,2}$ skew-symmetric matrix field $A : S \longrightarrow so(3)$ is as in Lemma* 3.2.

Proof. **1.** In view of Proposition 3.3 and Proposition 3.4 we have:

$$\begin{aligned}&\Big\|\nabla_{\tan}\tilde{y}^h - \Big((Q^h)^T R^h\Big)_{\tan}\cdot(\mathrm{Id}+th/h_0\Pi)(\mathrm{Id}+t\Pi)^{-1}\Big\|_{L^2(S^{h_0})} \leq C\sqrt{e^h}\\ &\Big\|\partial_{\vec{n}}\tilde{y}^h\Big\|_{L^2(S^{h_0})} \leq Ch\|\nabla_h y^h\|_{L^2(S^{h_0})} \leq Ch.\end{aligned} \tag{3.1}$$

To prove convergence of V^h, consider:

$$\begin{aligned}\nabla V^h(x) &= \frac{h}{\sqrt{e^h}}\fint_{-h_0/2}^{h_0/2}\nabla_{\tan}\tilde{y}^h(x+t\vec{n})(\mathrm{Id}+t\Pi)-\mathrm{Id}\,\mathrm{d}t\\ &= \frac{h}{\sqrt{e^h}}\fint_{-h_0/2}^{h_0/2}\Big(\nabla_{\tan}\tilde{y}^h-\Big((Q^h)^T R^h\Big)_{\tan}(\mathrm{Id}+t\Pi)^{-1}\Big)(\mathrm{Id}+t\Pi)\mathrm{d}t\\ &\quad+\frac{h}{\sqrt{e^h}}\Big((Q^h)^T R^h(x)-\mathrm{Id}\Big)_{\tan}.\end{aligned} \tag{3.2}$$

We see that by (3.1) the first term in the right hand side above converges to 0 in $L^2(S^{h_0})$, as $h \to 0$. The second term converges, up to a subsequence, to $A_{\tan}$ by Lemma 3.2 (ii). Therefore ∇V^h converges to $A_{\tan}$ in $L^2(S)$ and since $\fint_S V^h = 0$, we may use Poincaré inequality on S to deduce (ii).

2. To prove (i), notice that by (3.1) and Lemma 3.2 we obtain the following convergences in $L^2(S^{h_0})$:

$$\begin{aligned}&\lim_{h\to 0}\nabla_{\tan}\tilde{y}^h = (\mathrm{Id}+t\Pi)^{-1} = \nabla_{\tan}\pi,\\ &\lim_{h\to 0}\partial_{\vec{n}}\tilde{y}^h = 0.\end{aligned}$$

Therefore $\nabla\tilde{y}^h$ converges to $\nabla\pi$ in $L^2(S^{h_0})$.

Since the sequence $\{V^h\}$ is bounded in $L^2(S)$, it also follows that:

$$\lim_{h\to 0}\Big\|\int_{-h_0/2}^{h_0/2}\tilde{y}^h-\pi\,\mathrm{d}t\Big\|_{L^2(S)} = 0. \tag{3.3}$$

Now, let $g(x+t\vec{n}) = |\det(\mathrm{Id}+t\Pi(x))|^{-1}$. Consider the two terms in the right hand side of:

$$\|\tilde{y}^h-\pi\|_{L^2(S^{h_0})} \leq \Big\|(\tilde{y}^h-\pi)-\int_{S^{h_0}}(\tilde{y}^h-\pi)\cdot\frac{g}{\int_{S^{h_0}}g}\Big\|_{L^2(S^{h_0})}+\Big|\int_{S^{h_0}}(\tilde{y}^h-\pi)\cdot\frac{g}{\int_{S^{h_0}}g}\Big|.$$

The first term can be bounded by means of the weighted Poincaré inequality, by $\|\nabla(\tilde{y}^h - \pi)\|_{L^2(S^{h_0})}$ and therefore it converges to 0 as $h \to 0$. The second term converges to 0 as well, in view of (3.3) and:

$$\left|\int_{S^{h_0}} (\tilde{y}^h - \pi)\cdot g\right| = \left|\int_S \int_{-h_0/2}^{h_0/2} \tilde{y}^h - \pi \,\mathrm{d}t\,\mathrm{d}x\right| \le C \left\|\int_{-h_0/2}^{h_0/2} \tilde{y}^h - \pi \,\mathrm{d}t\right\|_{L^2(S)}.$$

This justifies convergence of $\tilde{y}^h$ to π in $L^2(S^{h_0})$ and ends the proof of (i). □

Towards the proof of Theorem 2.1, we need to consider the following sequence of matrix fields on S^{h_0}:

$$G^h = \frac{1}{\sqrt{e^h}}\Big((R^h)^T \nabla_h y^h - \mathrm{Id}\Big).$$

In view of Proposition 3.4 (i), $2\mathrm{sym}G^h$ is the $\sqrt{e^h}$ order term in the expansion of the nonlinear strain $(\nabla u^h)^T \nabla u^h$, at Id. This expression will also play a major role in the expansion of the energy density at Id: $W(\nabla_h y^h) = W(\mathrm{Id} + \sqrt{e^h} G^h)$.

Lemma 3.6. *Assume* **(H)**. *Then the sequence* $\{G^h\}$ *as above has a subsequence, converging weakly in* $L^2(S^{h_0})$ *to a matrix field* G. *The tangential minor of* G *is, moreover, affine in the* $\vec{n}$ *direction. More precisely:*

$$\forall \tau \in T_x S \qquad G(x + t\vec{n})\tau = G_0(x)\tau + \frac{t}{h_0}\cdot\Big(\nabla(A\vec{n})(x) - A\Pi(x)\Big)\tau,$$

where $G_0(x) = \fint_{-h_0/2}^{h_0/2} G(x + t\vec{n})\,\mathrm{d}t$.

Proof. **1.** The sequence $\{G^h\}$ is bounded in $L^2(S^{h_0})$ by Proposition 3.4 (i). Therefore it has a subsequence (which we do not relabel) converging weakly to some G.

For a fixed $s > 0$, consider now the sequence of vector fields $f^{s,h} \in W^{1,2}(S^{h_0}, \mathbb{R}^3)$:

$$f^{s,h}(x + t\vec{n}) = \frac{1}{s\sqrt{e^h}}\Bigg[\Big(h_0\tilde{y}^h(x + (t+s)\vec{n}) - h(x + (t+s)\vec{n})\Big) - \Big(h_0\tilde{y}^h(x + t\vec{n}) - h(x + t\vec{n})\Big)\Bigg].$$

We claim that $f^{s,h}$ converges in $L^2(S^{h_0})$ (up to a subsequence) to $(A\vec{n})\pi$ as $h \to 0$. Indeed, using Proposition 3.3 one has:

$$\begin{aligned} f^{s,h}(x + t\vec{n}) &= \frac{1}{\sqrt{e^h}} \fint_t^{t+s} \Big(h_0 \partial_{\vec{n}} \tilde{y}^h(x + \sigma\vec{n}) - h\vec{n}\Big)\,\mathrm{d}\sigma \\ &= \frac{h}{\sqrt{e^h}} \fint_t^{t+s} \Big((Q^h)^T \nabla_h y^h(x + \sigma\vec{n}) - \mathrm{Id}\Big)\vec{n}\,\mathrm{d}\sigma, \end{aligned}$$

and the convergence follows by Proposition 3.4 (ii).

2. We claim that this convergence is actually weak in $W^{1,2}(S^{h_0})$. First, notice that the normal derivatives converge to 0 in $L^2(S^{h_0})$ by Proposition 3.4 (ii):

$$\partial_{\vec{n}} f^{s,h}(x+t\vec{n}) = \frac{h}{s\sqrt{e^h}}\,(Q^h)^T\Big(\nabla_h y^h(x+(t+s)\vec{n}) - \nabla_h y^h(x+t\vec{n})\Big)\vec{n}(x).$$

We now find the weak limit of the tangential gradients of $f^{s,h}$. By Proposition 3.3 there holds, for all $\tau \in T_x S$:

$$\begin{aligned}\partial_\tau f^{s,h}(x+t\vec{n}) &= \frac{1}{s\sqrt{e^h}}\Big(h_0\nabla\tilde{y}^h(x+(t+s)\vec{n})(\mathrm{Id}+(t+s)\Pi)(\mathrm{Id}+t\Pi)^{-1}\\ &\qquad - h_0\nabla\tilde{y}^h(x+t\vec{n}) - hs\Pi(\mathrm{Id}+t\Pi)^{-1}\Big)\tau\\ =&\frac{h_0}{s\sqrt{e^h}}(Q^h)^T\Big(\nabla_h y^h(x+(t+s)\vec{n})-\nabla_h y^h(x+t\vec{n})\Big)(\mathrm{Id}+th/h_0\Pi)(\mathrm{Id}+t\Pi)^{-1}\tau\\ &+\frac{h}{s\sqrt{e^h}}\Big((Q^h)^T\nabla_h y^h(x+(t+s)\vec{n})-\mathrm{Id}\Big)s\Pi(\mathrm{Id}+t\Pi)^{-1}\tau.\end{aligned}$$

By Proposition 3.4 (ii), the second term in the right hand side above:

$$\frac{h}{\sqrt{e^h}}\Big((Q^h)^T\nabla_h y^h(x+(t+s)\vec{n})-\mathrm{Id}\Big)\Pi(\mathrm{Id}+t\Pi)^{-1}$$

converges in $L^2(S^{h_0})$ to $A\Pi(\mathrm{Id}+t\Pi)^{-1}$.

On the other hand, the first term equals to:

$$\frac{h_0}{s}(Q^h)^T R^h\Big(G^h(x+(t+s)\vec{n})-G^h(x+t\vec{n})\Big)(\mathrm{Id}+th/h_0\Pi)(\mathrm{Id}+t\Pi)^{-1}$$

and by Lemma 3.2 (i) it converges weakly in $L^2(S^{h_0})$ to

$$\frac{h_0}{s}\Big(G(x+(t+s)\vec{n})-G(x+t\vec{n})\Big)(\mathrm{Id}+t\Pi)^{-1}.$$

This establishes the (weak) convergence of $f^{s,h}$ in $W^{1,2}(S^{h_0})$.

3. Equating the weak limits of tangential derivatives, we obtain, for every $\tau \in T_x S$:

$$\begin{aligned}\partial_\tau(A\vec{n})(x)\cdot(\mathrm{Id}+t\Pi)^{-1} = &\frac{h_0}{s}\Big(G(x+(t+s)\vec{n})-G(x+t\vec{n})\Big)(\mathrm{Id}+t\Pi)^{-1}\tau\\ &+A\Pi(\mathrm{Id}+t\Pi)^{-1}\tau.\end{aligned}$$

This proves the lemma. □

Finally, we have the following bound for convergence of the scaled energies I^h:

Lemma 3.7. *Assume* **(H)**. *Then:*

$$\liminf_{h\to 0}\frac{1}{e^h}I^h(y^h)\geq\frac{1}{2}\int_S\mathcal{Q}_2\left(x,(\operatorname{sym}G_0)_{\tan}\right)+\frac{1}{24}\int_S\mathcal{Q}_2\left(x,(\nabla(A\vec{n})-A\Pi)_{\tan}\right).$$

Proof. By the frame invariance property of W we have:

$$W(\nabla_h y^h)=W((R^h)^T\nabla_h y^h)=W(\mathrm{Id}+\sqrt{e^h}G^h).$$

Consider the sets $\Omega_h=\{x\in S^{h_0};\ \ (e^h)^{1/4}|G^h(x)|\leq 1\}$. Clearly the sequence of characteristic functions χ_{Ω_h} converges to 1 in $L^1(S^{h_0})$, as $\{(e^h)^{1/4}G^h\}$ converges pointwise to 0. Since W is $\mathcal{C}^2$ in a neighborhood of Id, then by the above calculation, in Ω_h (for h sufficiently small) there holds:

$$\begin{aligned}\frac{1}{e^h}W(\nabla_h y^h)&=\frac{1}{e^h}\frac{1}{2}D^2W(\mathrm{Id})(\sqrt{e^h}G^h,\sqrt{e^h}G^h)\\&\quad+\int_0^1(1-s)\left[D^2W(\mathrm{Id}+s\sqrt{e^h}G^h)-D^2W(\mathrm{Id})\right]\mathrm{d}s(G^h,G^h)\qquad(3.4)\\&=\frac{1}{2}\mathcal{Q}_3(G^h)+o(1)|G^h|^2.\end{aligned}$$

Above $o(1)$ is the Landau symbol denoting any quantity uniformly converging to 0, as $h\to 0$. In view of Proposition 3.3 we now obtain:

$$\begin{aligned}\liminf_{h\to 0}\frac{1}{e^h}I^h(y^h)&\geq\liminf_{h\to 0}\frac{1}{e^h}\int_S\fint_{-h_0/2}^{h_0/2}\chi_{\Omega_h}W(\nabla_h y^h)\det\left[\mathrm{Id}+th/h_0\Pi\right]\mathrm{d}t\ \mathrm{d}x\\&=\liminf_{h\to 0}\int_S\fint_{-h_0/2}^{h_0/2}\chi_{\Omega_h}\frac{1}{e^h}W(\nabla_h y^h)\ \mathrm{d}t\ \mathrm{d}x\\&=\liminf_{h\to 0}\frac{1}{2}\int_S\fint_{-h_0/2}^{h_0/2}\mathcal{Q}_3\left(\operatorname{sym}\left(\chi_{\Omega_h}G^h\right)\right)+o(1)\int_{S^{h_0}}|G^h|^2\\&\geq\frac{1}{2}\int_S\fint_{-h_0/2}^{h_0/2}\mathcal{Q}_3(\operatorname{sym}G).\end{aligned}$$

The last inequality follows from positive definiteness of $\mathcal{Q}_3$ on symmetric matrices, and the fact that $\chi_{\Omega_h}G^h$ converges weakly to G, in $L^2(S^{h_0})$.

By the definition of $\mathcal{Q}_2$ and by Lemma 3.6 we get:

$$\begin{aligned}\frac{1}{2}\int_S\fint_{-h_0/2}^{h_0/2}\mathcal{Q}_3(\operatorname{sym}G)&=\frac{1}{2}\int_S\fint_{-h_0/2}^{h_0/2}\mathcal{Q}_2\left(x,(\operatorname{sym}G)_{\tan}\right)\\&=\frac{1}{2}\Bigg[\int_S\fint_{-h_0/2}^{h_0/2}\mathcal{Q}_2\left(x,(\operatorname{sym}G_0)_{\tan}\right)\\&\qquad+\int_S\fint_{-h_0/2}^{h_0/2}\frac{t^2}{h_0^2}\mathcal{Q}_2\left(x,(\nabla(A\vec{n})-A\Pi)_{\tan}\right)\Bigg],\end{aligned}$$

which proves the result. □

4. A proof of Theorem 2.1 and some explanations

To complete the proof of Theorem 2.1, in view of Lemma 3.5 and Lemma 3.7, it remains to understand the structure of the admissible matrices G_0. This is the content of the next lemma.

In addition to the hypothesis **(H)**, we now also assume the existence of the finite limit:

$$\kappa = \lim_{h\to 0} \sqrt{e^h}/h^2 < \infty. \tag{4.1}$$

When $e^h \approx h^\beta$, this corresponds to the case $\beta \geq 4$, with $\kappa > 0$ for $\beta = 4$ and $\kappa = 0$ for $\beta > 4$.

Lemma 4.1. *Assume* **(H)** *and* (4.1)*. Let G_0 be the matrix field on S, as in Lemma* 3.6*. Then we have the following convergence, up to a subsequence, weakly in $L^2(S)$:*

$$\lim_{h\to 0} \frac{1}{h}\operatorname{sym} \nabla V^h = \Big(\operatorname{sym} G_0 + \frac{\kappa}{2}A^2\Big)_{\tan}, \tag{4.2}$$

where the subscript $_{\tan}$ *denotes, as usual, the tangential minor of a given matrix field on S.*

Proof. We use the formula (3.2) to calculate $\frac{1}{h}\operatorname{sym} \nabla V^h$. The last term in the right hand side gives:

$$\frac{1}{\sqrt{e^h}}\operatorname{sym}\Big((Q^h)^T R^h - \mathrm{Id}\Big)_{\tan} = \frac{\sqrt{e^h}}{h^2}\frac{h^2}{e^h}\operatorname{sym}\Big((Q^h)^T R^h - \mathrm{Id}\Big)_{\tan},$$

which converges in $L^2(S)$ to $\kappa/2(A^2)_{\tan}$ by Lemma 3.2 (iii).

To treat the first term in the right hand side of (3.2), notice that for every $\tau \in T_x S$:

$$\begin{aligned}
&\frac{1}{\sqrt{e^h}}\Big[\fint_{-h_0/2}^{h_0/2} \nabla \tilde{y}^h(x+t\vec{n})(\mathrm{Id}+t\Pi) - (Q^h)^T R^h(x)\,\mathrm{d}t\Big]\tau \\
&\quad = \frac{1}{\sqrt{e^h}}(Q^h)^T\Big[\fint_{-h_0/2}^{h_0/2} \nabla_h y^h(x+t\vec{n}) - R^h(x)\,\mathrm{d}t + \fint_{-h_0/2}^{h_0/2} th/h_0 \nabla_h y^h \Pi\,\mathrm{d}t\Big]\tau \\
&\quad = \frac{1}{\sqrt{e^h}}(Q^h)^T R^h(x)\Big[\fint_{-h_0/2}^{h_0/2} (R^h)^T\nabla_h y^h - \mathrm{Id}\,\mathrm{d}t\Big]\tau \\
&\qquad + \frac{h/h_0}{\sqrt{e^h}}(Q^h)^T\Big[\fint_{-h_0/2}^{h_0/2} t\Big(\nabla_h y^h - R^h\pi\Big)\,\mathrm{d}t\Big]\Pi(x)\tau,
\end{aligned}$$

where we used Proposition 3.3. Now, the second term in the right hand side above converges in $L^2(S)$ to 0, by Proposition 3.4 (i). Further, the matrix in the first term equals to:

$$(Q^h)^T R^h(x)\fint_{-h_0/2}^{h_0/2} G^h(x+t\vec{n})\,\mathrm{d}t,$$

and by Lemma 3.2 (i) and Lemma 3.6, it converges weakly in $L^2(S)$ to G_0. This completes the proof. □

We now comment on the regularity and role of various quantities containing V and A, intrinsically related to the geometry of the problem.

Remark 4.2. Notice first that if a vector field $V \in W^{2,2}(S, \mathbb{R}^3)$ has skew-symmetric gradient:

$$\tau \cdot \partial_\tau V(x) = 0 \qquad \forall \tau \in T_x S,$$

then it uniquely determines a $W^{1,2}$ matrix field $A : S \longrightarrow so(3)$ by:

$$\begin{aligned} \forall \tau \in T_x S \qquad & A\tau = \partial_\tau V, \\ & A\vec{n} = \Pi \cdot V_{\tan} - \nabla_{\tan}(V\vec{n}). \end{aligned} \tag{4.3}$$

Regarding the regularity, write V as the sum of its tangential and normal components, to obtain:

$$V = V_{\tan} + (V\vec{n})\vec{n}, \qquad \operatorname{sym} \nabla V = \operatorname{sym} \nabla V_{\tan} + (V\vec{n})\Pi.$$

Hence, assuming sufficient regularity of S (say, S is $\mathcal{C}^{3,1}$ up to its boundary) it follows that $\operatorname{sym} \nabla V_{\tan} = -(V\vec{n})\Pi \in W^{2,2}(S, \mathbb{R}^{2\times 2})$. Using the same calculations as in [1, page 119], we may deduce that the tangential component $V_{\tan}$ enjoys higher regularity than the vector field V. Namely, $V_{\tan} \in W^{3,2}(S, \mathbb{R}^3)$ and:

$$\|V_{\tan}\|_{W^{3,2}(S)} \leq C\Big(\|V_{\tan}\|_{W^{1,2}(S)} + \|V\vec{n}\|_{W^{2,2}(S)}\Big).$$

By Korn's inequality, one can replace the $W^{1,2}$ norm of $V_{\tan}$ by a term of the order $\|V_{\tan}\|_{L^2} + \|\operatorname{sym} \nabla V_{\tan}\|_{L^2}$, so that we finally obtain:

$$\|V_{\tan}\|_{W^{3,2}(S)} \leq C\Big(\|V_{\tan}\|_{L^2(S)} + \|V\vec{n}\|_{W^{2,2}(S)}\Big).$$

For an elementary derivation of Korn's inequality on S from Korn's inequality on open sets, see, *e.g.*, [17].

In the same manner, one can prove the following useful bound, valid under $\mathcal{C}^{2,1}$ regularity of S:

$$\|V_{\tan}\|_{W^{2,2}(S)} \leq C\Big(\|V_{\tan}\|_{L^2(S)} + \|V\vec{n}\|_{W^{1,2}(S)}\Big). \tag{4.4}$$

Remark 4.3. 1. The (scaled) t - derivative of $G\tau$, which is also the argument of the second term in the definition of I (and $\tilde{I}$), may be written as:

$$\big(\nabla(A\vec{n}) - A\Pi\big)\tau = \Big[\big(\nabla(A\vec{n}) - A\Pi\big)\tau\Big]_{\tan} = (\partial_\tau A)\vec{n}.$$

This expression measures the difference of order h between the shape operator Π on S and the shape operator Π^h of the deformed surface $S_h = (\mathrm{id} + hV)(S)$ (see

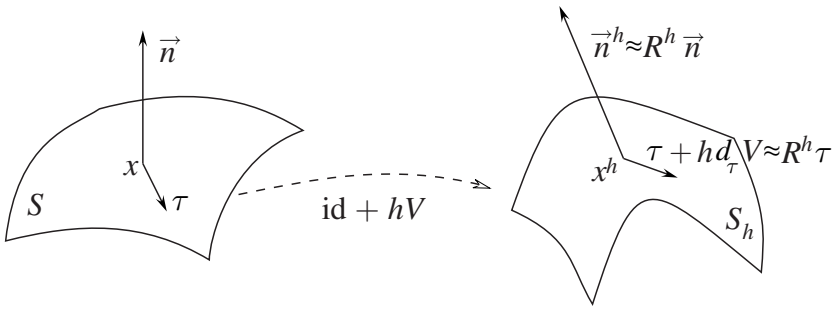

Figure 4.1. The mid-surface S and its deformation.

Figure 4.1). To see this, let $x \in S$ and let $\tau_1, \tau_2 \in T_x S$ be such that $\vec{n}(x) = \tau_1 \times \tau_2$. The tangent map of the deformation $\phi^h(x) = x + hV(x)$ equals $\mathrm{Id} + hA$, and we obtain the following expansion of the (scaled) normal vector $\vec{n}^h$ to S_h at the point $\phi^h(x)$:

$$\begin{aligned}\vec{n}^h &= \Big(\partial_{\tau_1}\phi^h \times \partial_{\tau_2}\phi^h\Big)(x) = \vec{n}(x) + h(\tau_1 \times \partial_{\tau_2} V + \partial_{\tau_1} V \times \tau_2) + \mathcal{O}(h^2)\\ &= \vec{n} + hA\vec{n} + \mathcal{O}(h^2),\end{aligned}$$

where we used the Jacobi identity for vector product and the fact that $A \in so(3)$. Note that $|\vec{n}^h| = 1 + \mathcal{O}(h^2)$ and therefore

$$\Pi^h(\mathrm{Id} + hA)\tau = \partial_\tau \left(\frac{\vec{n}^h}{|\vec{n}^h|}\right) = \partial_\tau \vec{n}^h + \mathcal{O}(h^2).$$

Hence the amount of bending of S, in the direction of $\tau \in T_x S$, can be estimated by:

$$\begin{aligned}(\mathrm{Id} + hA)^{-1}\Pi^h(\mathrm{Id} + hA)\tau - \Pi\tau &= (\mathrm{Id} + hA)^{-1}(\partial_\tau \vec{n}^h + \mathcal{O}(h^2)) - \Pi\tau\\ &= (\mathrm{Id} + hA)^{-1}\Big((\mathrm{Id} + hA)\Pi\tau + h(\partial_\tau A)\vec{n} + \mathcal{O}(h^2)\Big) - \Pi\tau\\ &= (\mathrm{Id} - hA)h(\partial_\tau A)\vec{n} + \mathcal{O}(h^2) = h(\partial_\tau A)\vec{n} + \mathcal{O}(h^2).\end{aligned}$$

A closely related heuristics is the following. By Proposition 3.4 (for simplicity, we assume here that $e^h = h^4$) the tangent map $\nabla u^h(x)$, at $x \in S$, is approximately a rotation $R^h(x) \in SO(3)$. Hence, $\vec{n}^h \approx R^h\vec{n}$. Assuming that $\lim Q^h = \mathrm{Id}$, we may think that $R^h(x) \approx \mathrm{Id} + hA(x)$. The difference of the shape operators on S_h and S satisfies:

$$\begin{aligned}(R^h)^T\nabla\vec{n}^h - \Pi &\approx (\mathrm{Id} + hA^T)\Big(\Pi + h\nabla_{\tan}(A\vec{n})\Big) - \Pi\\ &\approx h\nabla_{\tan}(A\vec{n}) + hA^T\Pi = h\Big(\nabla(A\vec{n}) - A\Pi\Big)_{\tan}.\end{aligned}$$

2. In turn, the role of the first term in the definition of I:

$$(\text{sym } G_0)_{\tan} = \lim_{h\to 0} \frac{1}{h}\text{sym } \nabla V^h - \frac{\kappa}{2}(A^2)_{\tan},$$

is to measure the difference of order h^2 between the metric on S and the metric of the deformed mid-surface. Notice that under the deformation $\text{id} + hV$, as in Figure 4.1, there is no first order change in the length of curves on S because the gradient field ∇V is skew-symmetric. In geometrical terms, vector fields V with this property are known as infinitesimal isometries (see [27, Chapter 12]).

Under the same condition (for simplicity we again assume that $e^h = h^4$), the amount of stretching of S, in the direction $\tau \in T_x S$ and induced by the deformation $\phi^h = \text{id} + hV + h^2 w$ has indeed the following expansion:

$$\begin{aligned}\left|\partial_\tau \phi^h\right|^2 - |\tau|^2 &= h^2\left(2\tau\partial_\tau w + |\partial_\tau V|^2\right) + \mathcal{O}(h^3)\\ &= 2h^2\left(\tau^T(\text{sym}\nabla w)\tau - \frac{1}{2}\tau^T A^2\tau\right) + \mathcal{O}(h^3).\end{aligned}$$

5. The space of finite strains $\mathcal{B}$ and three examples of approximately robust surfaces

The space of limits as in the left hand side of (4.2) plays an important role in defining the exact limiting energy functional on S. With this in mind, we introduce the following definition.

Definition 5.1. The space of finite strains is the following closed subspace of $L^2(S)$:

$$\mathcal{B} = \left\{ \lim_{h\to 0} \text{sym } \nabla w^h;\ \ w^h \in W^{1,2}(S, \mathbb{R}^3)\right\}$$

where limits are taken in $L^2(S)$.

Clearly, by Mazur's theorem, $\mathcal{B}$ contains all weak $L^2(S)$ limits of symmetric gradients of $W^{1,2}$ vector fields on S.

As we shall see in Theorem 2.3, the form of the limiting energy functional simplifies, for surfaces with large space $\mathcal{B}$.

Definition 5.2. We say that S is approximately robust, if for every $V \in \mathcal{V}$ one has: $(A^2)_{\tan} \in \mathcal{B}$.

According to our terminology, S would be called "robust" if every admissible $(A^2)_{\tan}$ as above, equaled sym ∇w for some $w \in W^{1,2}(S, \mathbb{R}^3)$. The notion of robust surfaces will arise at lower scalings, that is when $\kappa = \infty$, which we do not consider in this paper.

Remark 5.3. An equivalent construction of $\mathcal{B}$ is the following. Define the linear space of finite strain displacements:

$$\mathcal{W} = W^{1,2}(S, \mathbb{R}^3)/\{w \in W^{1,2};\ \ \text{sym}\ \nabla w = 0\}.$$

It can be normed by $\|[w]\|_{\mathcal{W}} = \|\text{sym}\ \nabla w\|_{L^2(S)}$. Then $(\mathcal{B}, \|\cdot\|_{L^2(S)})$ is linearly isometric to the completion $\overline{\mathcal{W}}$ of $(\mathcal{W}, \|\cdot\|_{\mathcal{W}})$, so that the elements of $\mathcal{B}$ can be seen as generalized symmetric gradients of elements of $\overline{\mathcal{W}}$.

Such construction is used in [11] or in [1] in the context of derivation of membrane theories from linear elasticity. Note the different regularity of the kernels considered in [11] and the related explanations in [1, page 262].

Remark 5.4. In general, it is complicated to directly determine the exact form of $\mathcal{B}$ or $\overline{\mathcal{W}}$. The crucial step in identifying $\overline{\mathcal{W}}$ is finding the optimal norm $\|\cdot\|_o$ for which a Korn-Poincaré type inequality

$$\inf\Big\{\|u - w\|_o;\ \ w \in W^{1,2},\ \text{sym}\ \nabla w = 0\Big\} \leq C\|\text{sym}\nabla u\|_{L^2(S)} \tag{5.1}$$

holds with a uniform constant C, for all $u \in W^{1,2}(S, \mathbb{R}^3)$. Unlike in the case of tangent vector fields, this optimal norm is usually weaker than L^2. The reason is that the boundedness of the left hand side in:

$$\text{sym}\ \nabla w^h = \text{sym}\ \nabla w^h_{\text{tan}} + (w^h\vec{n})\Pi$$

does not, in general, imply L^2 boundedness of both terms in the right hand side.

This is the case, for example, when S is (a piece of) a cylinder $\mathbb{S}^1 \times [-1/2, 1/2]$. Let τ_1 and τ_2 be the tangent unit vector fields, respectively orthogonal and parallel to the axis x_3 of the cylinder. One can show that there exists a sequence $[w^h] \in \mathcal{W}$ converging in $\overline{\mathcal{W}}$, such that for any choice of representatives $w^h \in W^{1,2}(S, \mathbb{R}^3)$, the norms $\|w^h\tau_1\|_{L^2(S)}$ and $\|w^h\vec{n}\|_{H^{-1}(S)}$ blow up. However, this is the worst case scenario, and one has:

$$\begin{aligned}\overline{\mathcal{W}} = \Big\{ v \in \mathcal{D}'(S, \mathbb{R}^3);\ \ & v\tau_1 \in H^{-1}(S),\ v\tau_2 \in L^2(S),\ v\vec{n} \in H^{-2}(S),\\ & \text{sym}\ \nabla v \in L^2(S),\\ & \int_{-1/2}^{1/2} x_3(v\vec{n}) \equiv \text{const},\ \ \int_{-1/2}^{1/2} v\vec{n} \equiv \text{const},\\ & \int_S v\tau_1 = \int_S v\tau_2 = \int_S x_3(v\tau_1) = 0\Big\}.\end{aligned}$$

In this particular case, however, as we will see below, $\overline{\mathcal{W}}$ is isometric to the space of all symmetric L^2 matrix fields B_{tan} on S.

Remark 5.5. Flat surfaces $S \subset \mathbb{R}^2$ are not approximately robust. Indeed:

$$\begin{aligned}\mathcal{B} &= \left\{\text{sym } \nabla w; \quad w \in W^{1,2}(S, \mathbb{R}^3)\right\} \\ &= \left\{B_{\text{tan}} \in L^2(S, \mathbb{R}^{2\times 2}); \quad B_{\text{tan}}^T = B_{\text{tan}}, \ \text{curl}^T \text{curl } B_{\text{tan}} = 0\right\}.\end{aligned}$$

On the other hand, given $V \in \mathcal{V}$, one has $(A^2)_{\text{tan}} \in \mathcal{B}$ if and only if $V^3 = V\vec{n}$ solves the degenerate Monge-Ampère equation: $\det \nabla^2(V^3) = 0$ (see [10]).

A particular class of approximately robust surfaces are surfaces for which:

$$\mathcal{B} = \left\{B_{\text{tan}} \in L^2(S, \mathbb{R}^{2\times 2}); \quad B_{\text{tan}}^T = B_{\text{tan}}\right\}. \tag{5.2}$$

As we show below, three main examples of such surfaces are: convex surfaces, surfaces of revolution, and developable surfaces without flat regions.

Lemma 5.6. *Assume that S is a simply connected, compact surface of class $\mathcal{C}^{2,1}$ with $\mathcal{C}^1$ boundary, and that its shape operator Π is strictly positive (or strictly negative) definite up to the boundary:*

$$\forall x \in \bar{S} \quad \forall \tau \in T_x S \qquad \frac{1}{C}|\tau|^2 \le \Big(\Pi(x)\tau\Big) \cdot \tau \le C|\tau|^2. \tag{5.3}$$

Then S is approximately robust, and more precisely (5.2) *holds.*

Proof. **1.** We will prove that every compactly supported, smooth symmetric bilinear form B_{tan} on S, must be of the form:

$$B_{\text{tan}} = \text{sym } \nabla w, \tag{5.4}$$

for some $w \in W^{1,2}(S, \mathbb{R}^3)$. This will clearly imply the lemma. In [23] this result is proved under an additional assumption that S is closed. The same method, with a slight modification, can be applied in our case. For convenience of the reader, we present an overview of the argument and for details of calculations we refer to [23] and [12, Section 9.2].

Since S is simply connected, it can be parameterized by a single chart $r \in \mathcal{C}^{2,1}(\bar{\Omega}, \mathbb{R}^3)$, where $\Omega \subset \mathbb{R}^2$ is a simply connected domain with $\mathcal{C}^1$ boundary. The definite form $[g_{ij}]_{i,j:1..2}$ with $g_{ij} = \partial_i r \cdot \partial_j r$ is the pull-back metric on Ω and $\sqrt{|g|} = \sqrt{\det[g_{ij}]}$ is the associated volume form. Also, the shape operator Π expressed in the coordinates $(x_1, x_2) \in \Omega$ is given by $[h_{ij}]_{i,j:1..2}$, where $h_{ij} = \partial_i(\vec{n} \circ r) \cdot \partial_j r$. The inverse of Π is denoted $\Pi^{-1} = [h^{ij}]_{i,j:1..2}$. The mean curvature H of S equals to $\frac{1}{2}\text{tr}\,([g_{ij}]^{-1}\Pi)$.

With the above notation, (5.4) becomes the following system of partial differential equations in Ω:

$$\begin{cases} \partial_1 r \cdot \partial_1 w = B_{11} \\ \partial_1 r \cdot \partial_2 w + \partial_2 r \cdot \partial_1 w = 2B_{12} \\ \partial_2 r \cdot \partial_2 w = B_{22}, \end{cases} \tag{5.5}$$

where we set $B_{ij} = \partial_i r \cdot B_{\tan} \partial_j r$. Since sym ∇w is determined, one concentrates on the skew part of ∇w. Following [23], we let:

$$\omega = \frac{1}{\sqrt{|g|}} \left(\partial_1 w \cdot \partial_2 r - \partial_2 w \cdot \partial_1 r \right),$$

and we observe that ω must satisfy the equation:

$$-\frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} h^{ij} \partial_j \omega \right) - 2H\omega = \mathcal{D}(B_{ij}). \tag{5.6}$$

The operator $\mathcal{D} : W^{2,2}(\Omega, \mathbb{R}^{2\times 2}) \longrightarrow L^2(\Omega, \mathbb{R})$ is a bounded differential operator which depends on the geometry of S. The exact expression of $\mathcal{D}$ is given in the references mentioned before, but for our purposes it is enough to know its stated regularity.

Now, the following crucial relation between problems (5.6) and (5.5) is a direct consequence of calculations in [23].

Proposition 5.7. *Assume that* $[B_{ij}]_{i,j:1..2} \in W^{2,2}(\Omega, \mathbb{R}^{2\times 2})$. *If* (5.6) *has a* (*weak*) *solution* $\omega \in W^{1,2}(\Omega, \mathbb{R})$, *then the system* (5.5) *has a solution* $w \in W^{1,2}(\Omega, \mathbb{R}^3)$.

2. We now show that the hypothesis of Proposition 5.7 is satisfied. Note that we have not imposed any boundary conditions on ω, which makes the argument easier. Extend first the coefficients h^{ij} and $|g|$ to $\tilde{h}^{ij}$ and $|\tilde{g}|$, respectively, defined on $\Omega_\epsilon = \{x \in \mathbb{R}^2;\ \text{dist}\,(x, \Omega) < \epsilon\}$ for a small $\epsilon > 0$. This extension can be made so that $[\tilde{h}^{ij}]_{i,j:1..2}$ satisfies the ellipticity condition (5.3) and that $\tilde{h}^{ij}$, $|\tilde{g}|$ and $1/|\tilde{g}|$ stay bounded in Ω_ϵ.

In order to prove existence of a solution to (5.6), we want to find $f_0 \in C_c^\infty(\Omega_\epsilon \backslash \Omega)$ such that the Dirichlet problem

$$\mathcal{L}\omega = \mathcal{D}(B_{ij}) + f_0 \tag{5.7}$$

has a solution $\omega \in W_0^{1,2}(\Omega_\epsilon, \mathbb{R})$. The restriction of ω to Ω will, clearly, serve our purpose. Here the operator $\mathcal{L}$ is given:

$$\mathcal{L}\omega = -\frac{1}{\sqrt{|\tilde{g}|}} \partial_i \left(\sqrt{|\tilde{g}|} \tilde{h}^{ij} \partial_j \omega \right) - 2\tilde{H}\omega$$

is elliptic and self-adjoint with respect to the scalar product:

$$\langle \omega, \zeta \rangle = \int_{\Omega_\epsilon} \omega\zeta \sqrt{|\tilde{g}|}.$$

Therefore, by the classical theory of elliptic operators (see, *e.g.*, [7, Section 6.2, Theorem 4]), (5.7) has a solution if and only if its right hand side satisfies the orthogonality condition:

$$\langle \mathcal{D}(B_{ij}) + f_0, \zeta \rangle = 0,$$

for all solutions $\zeta \in W_0^{1,2}(\Omega_\epsilon, \mathbb{R})$ of the homogeneous problem: $\mathcal{L}\zeta = 0$ in Ω_ϵ. The solution space of this problem is finite dimensional, say spanned by a basis $\{\zeta_1, \ldots, \zeta_k\}$. For $f_0 \in C_c^\infty(\Omega_\epsilon \setminus \bar{\Omega})$ consider the functional:

$$L(f_0) = \sum_{i=1}^{k} \langle f_0, \zeta_i \rangle e_i \in \mathbb{R}^k.$$

In view of the above, it suffices to prove that L is surjective.

We now argue by contradiction. Assume that there exists a nonzero $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$ orthogonal to the range of L. In other words:

$$\int_{\Omega_\epsilon \setminus \Omega} \left(\sum_{i=1}^{k} \alpha_i \zeta_i \right) f_0 \sqrt{|\tilde{g}|} = 0 \quad \forall f_0 \in C_c^\infty(\Omega_\epsilon \setminus \bar{\Omega}),$$

which clearly implies that $\sum_{i=1}^{k} \alpha_i \zeta_i = 0$ in $\Omega_\epsilon \setminus \bar{\Omega}$. By the Hörmander uniqueness theorem for second order elliptic equations (see [13, Theorem 2.4]), we obtain $\sum_{i=1}^{k} \alpha_i \zeta_i = 0$ in Ω_ϵ, contradicting the linear independence of $\{\zeta_1, \ldots, \zeta_k\}$.

In view of Proposition 5.7, this ends the proof. □

Lemma 5.8. *Assume that S is rotationally invariant, $\mathcal{C}^2$ up to the boundary, and let $\bar{S}$ have no intersection with its axis of rotation. Then* (5.2) *holds.*

Proof. **1.** After a suitable rigid motion, the surface S can be parameterized by:

$$r : (s_0, s_1) \times [0, 2\pi] \to \mathbb{R}^3, \qquad r(s, \theta) := g(s)\gamma(\theta) + se_3,$$

for a positive function $g \in \mathcal{C}^2([s_0, s_1], \mathbb{R})$, $e_3 = (0, 0, 1)$, and $\gamma(\theta) = (\cos\theta, \sin\theta, 0)$.

As in the proof of Lemma 5.6, we will show that (5.4) has a solution for $B_{\tan}$ in an appropriate dense subset of the space in the right hand side of (5.2). Given $w \in W^{1,2}(S, \mathbb{R}^3)$, write:

$$w(s, \theta) := a(s, \theta)\gamma(\theta) + b(s, \theta)\gamma'(\theta) + c(s, \theta)e_3$$

and also let:

$$B_{ij} = \partial_i r \cdot B_{\tan} \partial_j r.$$

The equation (5.4) can now be expressed as the following periodic system of partial differential equations in $(s_0, s_1) \times [0, 2\pi]$ (see [27, Chapter 12]):

$$\begin{cases} g'\partial_s a + \partial_s c = B_{11} \\ \partial_\theta b + a = B_{22} \\ g'(\partial_\theta a - b) + g\partial_s b + \partial_\theta c = 2B_{12}. \end{cases} \tag{5.8}$$

We will prove that (5.8) has a solution $W^{1,2}$, periodic in $\theta \in [0, 2\pi]$, for B_{ij} being finite linear combinations of the Schauder basis for $L^2([s_0, s_1] \times [0, 2\pi])$ consisting of eigenfunctions of Laplacian under the periodic boundary conditions at

$\theta \in \{0, 2\pi\}$ and Neumann boundary conditions in $s \in \{s_0, s_1\}$. By density, this will establish the lemma.

2. Differentiating the third equation in s and using the first two equations in (5.8) we obtain:

$$g\partial_s^2 b - g''(b + \partial_\theta^2 b) = 2\partial_s B_{12} - \partial_\theta B_{11} - g''\partial_\theta B_{22} =: \psi(s, \theta). \tag{5.9}$$

Note that $\psi \in \mathcal{C}^0$ and for all s, $\psi(s, \cdot)$ is a finite linear combination of $\{e^{ik\theta}\}_{k \leq N}$, for some integer N independent of s. Hence:

$$b(s, \theta) = \sum_{-\infty}^{+\infty} b_k(s) e^{ik\theta} \quad \text{and} \quad \psi(s, \theta) = \sum_{-\infty}^{+\infty} \psi_k(s) e^{ik\theta},$$

with $\psi_k = \psi_{-k}$ and $\psi_k = 0$ for $k > N$. Expressing (5.9) in terms of the Fourier coefficients b_k and ψ_k we have:

$$b_k'' - \frac{g''}{g}(1 - k^2) b_k = \frac{\psi_k}{g}.$$

Since the coefficients of the above linear equation are continuous in $[s_0, s_1]$, we deduce that there exist unique solutions $b_k \in \mathcal{C}^2([s_0, s_1], \mathbb{R})$ satisfying $b_k(s_0) = b_k'(s_0) = 0$. Also, $b_k = b_{-k}$ and $b_k = 0$ for $k > N$. Concluding, the finite linear combination $b = \sum b_k(s) e^{ik\theta}$ is a $W^{2,2}$ solution to (5.9), periodic in θ. One can now solve the first two equations in (5.8) for a and then for c, obtaining a $W^{1,2}$ solution to (5.8) and hence also to (5.4). □

Finally, the following result has been proved in [26, Lemma 3.3].

Lemma 5.9. *Let S be a $\mathcal{C}^2$ developable surface without flat regions. That is, assume that for each $x \in S$ the Gauss curvature $\kappa(x) = 0$ while $\Pi(x) \neq 0$. Then S satisfies* (5.2).

6. The recovery sequence: proofs of Theorems 2.2 and 2.3

In this section, we want to prove Theorems 2.2 and 2.3, that is to define a suitable recovery sequence y^h. Recall the definition (2.4). With a slight abuse of notation, one can write:

$$\mathcal{Q}_2(x, F_{\text{tan}}) = \min\{\mathcal{Q}_3(F_{\text{tan}} + c \otimes \vec{n}(x) + \vec{n}(x) \otimes c); \;\; c \in \mathbb{R}^3\}. \tag{6.1}$$

The unique vector c, for which the above minimum is attained will be called $c(x,F_{\text{tan}})$. By uniqueness, the map c is linear in its second argument.

Given $B_{\text{tan}} \in \mathcal{B}$, there exists a sequence of vector fields $w^h \in W^{1,2}(S, \mathbb{R}^3)$ such that sym ∇w^h converge in $L^2(S)$ to B_{tan}. Without loss of generality, we may assume that w^h are smooth, and (by possibly reparameterizing the sequence) that:

$$\lim_{h\to 0} \sqrt{h}\|w^h\|_{W^{2,\infty}(S)} = 0. \tag{6.2}$$

Let $V \in \mathcal{V}$. We approximate V by a sequence $v^h \in W^{2,\infty}(S, \mathbb{R}^3)$ such that, for a sufficiently small, fixed $\epsilon_0 > 0$:

$$\begin{aligned} &\lim_{h\to 0} \|v^h - V\|_{W^{2,2}(S)} = 0, \qquad \frac{\sqrt{e^h}}{h}\|v^h\|_{W^{2,\infty}(S)} \le \epsilon_0, \\ &\lim_{h\to 0} \frac{h^2}{e^h}\,\mu\left\{x \in S;\ \ v^h(x) \neq V(x)\right\} = 0. \end{aligned} \tag{6.3}$$

The existence of such v^h follows by partition of unity and a truncation argument, as a special case of the Lusin-type result for Sobolev functions in [18] (see also [10, Proposition 2]).

Define a sequence of rescaled deformations $y^h \in W^{1,2}(S^{h_0}, \mathbb{R}^3)$:

$$\begin{aligned} y^h(x + t\vec{n}) = x &+ \frac{\sqrt{e^h}}{h} v^h(x) + \sqrt{e^h} w^h(x) \\ &+ th/h_0 \vec{n}(x) + t/h_0 \sqrt{e^h}\Big(\Pi v^h_{\text{tan}} - \nabla(v^h \vec{n})\Big)(x) \\ &- th/h_0 \sqrt{e^h} \vec{n}^T \nabla w^h + th/h_0 \sqrt{e^h} d^{0,h}(x) \\ &+ \frac{t^2}{2h_0^2} h \sqrt{e^h} d^{1,h}(x). \end{aligned} \tag{6.4}$$

Notice that if $V \in W^{2,\infty}(S)$ then one may take $v^h = V$ in which case the term $t/h_0\sqrt{e^h}(\Pi v^h_{\text{tan}} - \nabla(v^h\vec{n}))$ is exactly $t/h_0\sqrt{e^h} A\vec{n}$ (see (4.3) in Remark 4.2).

The vector fields $d^{0,h}, d^{1,h} \in W^{1,\infty}(S, \mathbb{R}^3)$ are defined so that:

$$\lim_{h\to 0} \sqrt{h}\left(\|d^{0,h}\|_{W^{1,\infty}(S)} + \|d^{1,h}\|_{W^{1,\infty}(S)}\right) = 0 \tag{6.5}$$

and:

$$\begin{aligned} &\lim_{h\to 0} d^{0,h} = 2c\left(x,\, B_{\text{tan}} - \frac{\kappa}{2}(A^2)_{\text{tan}}\right) + \kappa A^2\vec{n} - \frac{1}{2}\kappa(\vec{n}^T A^2 \vec{n})\vec{n} \quad \text{in } L^2(S), \\ &\lim_{h\to 0} d^{1,h} = 2c\,(x, \text{sym}\,(\nabla(A\vec{n}) - A\Pi)_{\text{tan}}) + \Big(\vec{n}^T A\Pi - \vec{n}^T\nabla(A\vec{n})\Big) \text{ in } L^2(S). \end{aligned} \tag{6.6}$$

Lemma 6.1. *Assume* (4.1). *For the sequence* $\{y^h\}$ *in* (6.4) *the convergences* (i), (ii) *and* (iii) *of Theorem* 2.2 *hold.*

Proof. (i) follows by the normalization (6.2), (6.3) and (6.5). For (ii) and (iii) notice that:

$$V^h[y^h] = v^h + hw^h + \frac{1}{24}h^2 d^{1,h}$$
$$\frac{1}{h}\text{sym } \nabla V^h[y^h] = \frac{1}{h}\text{sym } \nabla v^h + \text{sym } \nabla w^h + \frac{1}{24}h\text{sym } \nabla d^{1,h}.$$

The proof will be achieved once we establish that:

$$\lim_{h\to 0} \frac{1}{h}\|\text{sym } \nabla v^h\|_{L^2(S)} = 0. \tag{6.7}$$

Since the Lipschitz constant of each ∇v^h is bounded by $\epsilon_0 \frac{h}{\sqrt{e^h}}$, and sym $\nabla v^h = 0$ on the set $\{x \in S;\ v^h(x) = V(x)\}$, we have:

$$|\text{sym } \nabla v^h(x)| \leq C\frac{h}{\sqrt{e^h}}\text{dist}\Big(x, \{v^h = V\}\Big).$$

Now we claim that the right hand side above converges to 0, in $L^\infty(S)$. For otherwise there would be dist $(x^h, \{v^h = V\}) \geq C\frac{\sqrt{e^h}}{h}$, for some sequence $x^h \in S$. Consequently, denoting by $B_{x^h}(r)$ the ball in $\mathbb{R}^3$ centered at x^h and radius r, we would obtain:

$$\mu\{x \in S;\ v^h(x) \neq V(x)\} \geq \left|S \cap B_{x^h}\left(\frac{1}{2}\text{dist}\,(x^h,\ \{v^h = V\})\right)\right| \geq C\frac{e^h}{h^2},$$

contradicting (6.3). In the last inequality above we used that the surface S is of class $\mathcal{C}^1$, with Lipschitz continuous boundary. We thus obtain:

$$\lim_{h\to 0} \|\text{sym } \nabla v^h\|_{L^\infty(S)} = 0.$$

On the other hand:

$$\begin{aligned}\frac{1}{h}\|\text{sym } \nabla v^h\|_{L^2(S)} &\leq \frac{1}{h}\mu\{x \in S;\ v^h(x) \neq V(x)\}^{1/2} \cdot \|\text{sym } \nabla v^h\|_{L^\infty(S)} \\ &\leq C\frac{\sqrt{e^h}}{h^2}\|\text{sym } \nabla v^h\|_{L^\infty(S)}.\end{aligned}$$

The two statements above imply (6.7). □

Proof of Theorem 2.2. We will prove that:

$$\limsup_{h\to 0}\frac{1}{e^h}I^h(y^h)\le I(V,B_{\tan})+\eta, \tag{6.8}$$

where η denotes an error quantity, with the property:

$$\eta\to 0 \quad \text{as } \epsilon_0\to 0. \tag{6.9}$$

In view of Theorem 2.1, this will imply (iv) for a recovery sequence obtained through a diagonal argument, when $\epsilon_0\to 0$. Clearly, the assertions (i) - (iii) will follow as well, by Lemma 6.1.

1. We first look closer at quantities $\nabla_h y^h$. By Proposition 3.3, it follows that:

$$\begin{aligned}
(\nabla_h y^h)(x+t\vec{n})\vec{n}(x) &= \vec{n}+\frac{\sqrt{e^h}}{h}\Big(\Pi v^h_{\tan}-\nabla(v^h\vec{n})\Big)\\
&\quad -\sqrt{e^h}\vec{n}^T\nabla w^h+\sqrt{e^h}d^{0,h}+t/h_0\sqrt{e^h}d^{1,h},\\
(\nabla_h y^h)(x+t\vec{n})\tau &= \nabla y^h(x+t\vec{n})(\mathrm{Id}+t\Pi)(\mathrm{Id}+th/h_0\Pi)^{-1}\tau\\
&= \Big(\mathrm{Id}+\frac{\sqrt{e^h}}{h}\nabla v^h+\sqrt{e^h}\nabla w^h+th/h_0\Pi\\
&\quad +t/h_0\sqrt{e^h}\nabla\Big(\Pi v^h_{\tan}-\nabla(v^h\vec{n})\Big)\\
&\quad -th/h_0\sqrt{e^h}\nabla(\vec{n}^T\nabla w^h)+th/h_0\sqrt{e^h}\nabla d^{0,h}\\
&\quad +\frac{t^2}{2h_0^2}h\sqrt{e^h}\nabla d^{1,h}\Big)(\mathrm{Id}+th/h_0\Pi)^{-1}\tau,
\end{aligned} \tag{6.10}$$

for all $\tau\in T_xS$.

By (6.2), (6.3) and (6.5) one has: $\|\nabla_h y^h-\mathrm{Id}\|_{L^\infty(S^{h_0})}\le C\epsilon_0$. It now follows by polar decomposition theorem (assuming ϵ_0 to be sufficiently small), that $\nabla_h y^h$ is a product of a proper rotation and the well defined square root of $(\nabla_h y^h)^T\nabla_h y^h$. By properties of the energy density function and Taylor expansion, we obtain:

$$W(\nabla_h y^h)=W\left(\sqrt{(\nabla_h y^h)^T\nabla_h y^h}\right)=W\left(\mathrm{Id}+\frac{1}{2}K^h+\mathcal{O}(|K^h|^2)\right),$$

where:

$$K^h=(\nabla_h y^h)^T\nabla_h y^h-\mathrm{Id}.$$

Clearly:

$$\|K^h\|_{L^\infty(S^{h_0})}\le C\epsilon_0, \tag{6.11}$$

and so reasoning as in (3.4), the above expansion in W yields:

$$\frac{1}{e^h}W(\nabla_h y^h)=\frac{1}{2}\mathcal{Q}_3\left(\frac{1}{2\sqrt{e^h}}K^h+\frac{1}{\sqrt{e^h}}\mathcal{O}(|K^h|^2)\right)+\frac{1}{\sqrt{e^h}}\eta\cdot\mathcal{O}(|K^h|^2), \tag{6.12}$$

where η depends only on ϵ_0 and satisfies (6.9).

2. Using (6.10) we now calculate K^h. By Error we will cumulatively denote all the terms with the property:

$$\lim_{h\to 0}\frac{1}{\sqrt{e^h}}\| \text{ Error } \|_{L^2(S^{h_0})}=0. \tag{6.13}$$

We start with the tangential minor of K^h:

$$\begin{aligned}
K^h_{\tan}(x+t\vec{n}) &= (\mathrm{Id}+th/h_0\Pi)^{-1}\Big[\mathrm{Id}+2\frac{\sqrt{e^h}}{h}\mathrm{sym}\,\nabla v^h+2\sqrt{e^h}\mathrm{sym}\,\nabla w^h\\
&\quad +2th/h_0\Pi+2t/h_0\sqrt{e^h}\mathrm{sym}\,\nabla\Big(\Pi v^h_{\tan}-\nabla(v^h\vec{n})\Big)\\
&\quad +\frac{e^h}{h^2}(\nabla v^h)^T\nabla v^h+t^2h^2/h_0^2\Pi^2\\
&\quad +2t\sqrt{e^h}/h_0\mathrm{sym}\,\Big(\Pi\nabla v^h\Big)+\text{Error}\Big](\mathrm{Id}+th/h_0\Pi)^{-1}-\mathrm{Id}\\
&= (\mathrm{Id}+th/h_0\Pi)^{-1}\sqrt{e^h}\Big[2\mathrm{sym}\,\nabla w^h+\frac{\sqrt{e^h}}{h^2}(\nabla v^h)^T\nabla v^h\\
&\quad +2t/h_0\mathrm{sym}\,\nabla\Big(\Pi v^h_{\tan}-\nabla(v^h\vec{n})\Big)\\
&\quad +2t/h_0\mathrm{sym}\,\Big(\Pi\nabla v^h\Big)\Big](\mathrm{Id}+th/h_0\Pi)^{-1}+\text{Error},
\end{aligned}$$

where we used the formulae:

$$\begin{aligned}
(\mathrm{Id}\ +F)^T(\mathrm{Id}\ +F)&=\mathrm{Id}\ +2\mathrm{sym}\,F+F^TF,\\
F_1^{-1}FF_1^{-1}-\mathrm{Id}\ &=F_1^{-1}(F-F_1^2)F_1^{-1}.
\end{aligned}$$

Notice that the quantity Error contains the term $\frac{\sqrt{e^h}}{h}\mathrm{sym}\,\nabla v^h$, resulting from the relaxation of the constraint (2.2) on the small set $\{v^h\neq V\}$, and other product terms, *e.g.*: $\frac{e^h}{h}(\nabla v^h)^T\nabla(\Pi v^h_{\tan}-\nabla(v^h\vec{n}))$. The convergence of $\frac{1}{h}\|\mathrm{sym}\,\nabla v^h\|_{L^2(S^{h_0})}$ to 0 has been proved in (6.7). All other terms in Error can be dealt with by repeated use of (6.3), (6.2), Hölder and Sobolev inequalities, *e.g.*:

$$\begin{aligned}
\frac{\sqrt{e^h}}{h}\|(\nabla v^h)^T\nabla(\Pi v^h_{\tan}-\nabla(v^h\vec{n}))\|_{L^2(S)}&\le C\frac{\sqrt{e^h}}{h}\|\nabla v^h\|_{L^4(S)}\|v^h\|_{W^{2,4}(S)}\\
&\le C\frac{\sqrt{e^h}}{h}\|\nabla v^h\|_{W^{1,2}(S)}\|v^h\|^{1/2}_{W^{2,\infty}(S)}\|v^h\|^{1/2}_{W^{2,2}(S)}\\
&\le C\frac{\sqrt{e^h}}{h}\|v^h\|^{1/2}_{W^{2,\infty}(S)}\longrightarrow 0\quad\text{ as } h\to 0.
\end{aligned}$$

Now, the normal minor of K^h is calculated as:

$$\vec{n}^T K^h(x+t\vec{n})\vec{n} = \sqrt{e^h}\left(\frac{\sqrt{e^h}}{h^2}\left|\Pi v^h_{\tan} - \nabla(v^h\vec{n})\right|^2 + 2d^{0,h}\vec{n} + 2t/h_0 d^{1,h}\vec{n}\right) + \text{Error}\,.$$

The remaining coefficients of the symmetric matrix $K^h(x+t\vec{n})$ are, for $\tau \in T_x S$:

$$\begin{aligned}
\tau^T K^h(x+t\vec{n})\vec{n} = \vec{n}^T \Bigg[& \frac{\sqrt{e^h}}{h}\nabla v^h + t/h_0\sqrt{e^h}\nabla\Big(\Pi v^h_{\tan} - \nabla(v^h\vec{n})\Big)\Bigg](\mathrm{Id} + th/h_0\Pi)^{-1}\tau \\
& + \Bigg[\frac{\sqrt{e^h}}{h}\Big(\Pi v^h_{\tan} - \nabla(v^h\vec{n})\Big)^T + (\sqrt{e^h}d^{0,h} + t/h_0\sqrt{e^h}d^{1,h})^T \\
& + \frac{e^h}{h^2}\Big(\Pi v^h_{\tan} - \nabla(v^h\vec{n})\Big)^T \nabla v^h \\
& + t/h_0\sqrt{e^h}\Big(\Pi v^h_{\tan} - \nabla(v^h\vec{n})\Big)^T \Pi\Bigg](\mathrm{Id} + th/h_0\Pi)^{-1}\tau + \text{Error} \\
= \sqrt{e^h}\Bigg[& t/h_0\vec{n}^T\nabla\Big(\Pi v^h_{\tan} - \nabla(v^h\vec{n})\Big) + \frac{\sqrt{e^h}}{h^2}\Big(\Pi v^h_{\tan} - \nabla(v^h\vec{n})\Big)^T\nabla v^h \\
& + t/h_0\Big(\Pi v^h_{\tan} - \nabla(v^h\vec{n})\Big)^T\Pi \\
& + (d^{0,h} + t/h_0 d^{1,h})^T\Bigg](\mathrm{Id} + th/h_0\Pi)^{-1}\tau + \text{Error}\,.
\end{aligned}$$

We leave the estimation in Error to the reader. The convergence of the most troublesome term:

$$\lim_{h\to 0}\frac{1}{h}\|\vec{n}^T\nabla v^h + (\Pi v^h_{\tan} - \nabla(v^h\vec{n}))^T\|_{L^2(S^{h_0})} = 0$$

can be proved as in (6.7), since the quantity in question vanishes on the set $\{v^h = V\}$. Therefore, $\|\vec{n}^T\nabla v^h + (\Pi v^h_{\tan} - \nabla(v^h\vec{n}))^T\|_{L^\infty(S)}$ converges to 0, as $h \to 0$, and the displayed convergence follows by the last assertion in (6.3).

3. In view of (6.13) we may now write (with a slight abuse of notation)

$$\lim_{h\to 0}\frac{1}{2\sqrt{e^h}}K^h = K_1(x)_{\tan} + \frac{t}{h_0}K_2(x)_{\tan} + (\zeta\otimes\vec{n} + \vec{n}\otimes\zeta) \quad \text{in } L^2(S^{h_0}), \tag{6.14}$$

where the symmetric matrix fields $(K_i)_{\tan} \in L^2(S, \mathbb{R}^{2\times 2})$ and the vector field $\zeta \in L^2(S^{h_0}, \mathbb{R}^3)$ are given by:

$$\begin{aligned}
K_1(x)_{\tan} &= B_{\tan} - \frac{\kappa}{2}(A^2)_{\tan}, \\
K_2(x)_{\tan} &= \text{sym}\ (\nabla(A\vec{n}) - A\Pi)_{\tan}\,, \\
\zeta(x+t\vec{n}) &= c\Big(x, B_{\tan} - \frac{\kappa}{2}(A^2)_{\tan}\Big) + \frac{t}{h_0}c\Big(x, \text{sym}\ (\nabla(A\vec{n}) - A\Pi)_{\tan}\Big).
\end{aligned} \tag{6.15}$$

Further, we observe:

$$\lim_{h\to 0}\frac{1}{e^h}\int_{S^{h_0}}|K^h|^4=0. \tag{6.16}$$

Indeed, (6.14) implies that $\frac{1}{\sqrt{e^h}}K^h$ converges pointwise a.e. in S^{h_0}. Thus $\frac{1}{e^h}|K^h|^4$ converges a.e. to 0. By the boundedness of K^h in (6.11): $\frac{1}{e^h}|K^h|^4\leq C\frac{1}{e^h}|K^h|^2$, and the dominated convergence theorem achieves (6.16).

4. Finally, we prove now (6.8). By (6.16), it follows that the argument of $\mathcal{Q}_3$ in (6.12) converges in $L^2(S^{h_0})$ to the same limit as $\frac{1}{2\sqrt{e^h}}K^h$ in (6.14). Using Proposition 3.3, (6.14) and (6.1), we obtain:

$$\begin{aligned}
\limsup_{h\to 0}\frac{1}{e^h}I^h(y^h)&=\limsup_{h\to 0}\frac{1}{e^h}\int_S\fint_{-h_0/2}^{h_0/2}W(\nabla_h y^h)\cdot\det(\mathrm{Id}+th/h_0\Pi)\,\mathrm{d}t\mathrm{d}x\\
&\leq\frac{1}{2}\limsup_{h\to 0}\int_S\fint_{-h_0/2}^{h_0/2}\mathcal{Q}_3\left(\frac{1}{2\sqrt{e^h}}K^h(x+t\vec{n})\right)\cdot\det(\mathrm{Id}+th/h_0\Pi)\,\mathrm{d}t\mathrm{d}x\\
&\quad+C\eta\limsup_{h\to 0}\frac{1}{e^h}\int_{S^{h_0}}|K^h|^2\\
&=\frac{1}{2}\int_S\fint_{-h_0/2}^{h_0/2}\mathcal{Q}_3\left(\lim_{h\to 0}\frac{1}{2\sqrt{e^h}}K^h\right)+C\eta\left\|\lim_{h\to 0}\frac{1}{2\sqrt{e^h}}K^h\right\|^2_{L^2(S^{h_0})}\\
&\leq\frac{1}{2}\int_S\fint_{-h_0/2}^{h_0/2}\mathcal{Q}_2\left(x,\,K_1(x)_{\tan}+\frac{t}{h_0}K_2(x)_{\tan}\right)\,\mathrm{d}t\mathrm{d}x+C\eta\\
&=\frac{1}{2}\int_S\fint_{-h_0/2}^{h_0/2}\mathcal{Q}_2\left(x,\,K_1(x)_{\tan}\right)+\frac{t^2}{h_0^2}\mathcal{Q}_2\left(x,\,K_2(x)_{\tan}\right)\,\mathrm{d}t\mathrm{d}x+C\eta,
\end{aligned}$$

which implies (6.8) in view of (6.15). □

Remark 6.2. A more careful calculation reveals the exact convergence:

$$\lim_{h\to 0}\frac{1}{e^h}I^h(y^h)=I(V,B_{\tan}),$$

for the recovery sequence (6.4). We have used another argument for the sake of a more transparent presentation.

Proof of Theorem 2.3. When $\kappa=0$, the recovery sequence (for $V\in\mathcal{V}$) is given again by (6.4), where we put $w^h=0$, $B_{\tan}=0$ and $\kappa=0$. That is:

$$\begin{aligned}
y^h(x+t\vec{n})&=x+\frac{\sqrt{e^h}}{h}v^h(x)+th/h_0\vec{n}(x)\\
&\quad+t/h_0\sqrt{e^h}\Big(\Pi v^h_{\tan}-\nabla_{\tan}(v^h\vec{n})\Big)(x)+\frac{t^2}{2h_0^2}h\sqrt{e^h}d^{1,h}(x),
\end{aligned}$$

where $d^{1,h}\in W^{1,\infty}(S,\mathbb{R}^3)$ satisfies (6.5) and the second formula in (6.6).

Clearly, y^h and $V^h[y^h]$ converge in $W^{1,2}(S^{h_0})$ to π and V, respectively, as in Lemma 6.1. The convergence of the scaled energy follows as in Theorem 2.2 (iv). □

7. The convergence of minimizers: proof of Theorem 2.5

Recall that the considered sequence of forces $f^h \in L^2(S^h, \mathbb{R}^3)$ with zero mean: $\int_{S^h} f^h = 0$, has the form:

$$f^h(x + t\vec{n}(x)) = h\sqrt{e^h}\,\det(\mathrm{Id} + t\Pi(x))^{-1} f(x).$$

Lemma 7.1. *Let $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ be a sequence of deformations such that $V^h[y^h]$ converges in $L^2(S)$ to some $V : S \longrightarrow \mathbb{R}^3$ and let $Q^h \in \mathbb{R}^{3\times 3}$ converge to some Q. Then:*

$$\lim_{h\to 0} \frac{1}{e^h}\frac{1}{h}\int_{S^h} f^h \cdot Q^h(u^h - \mathrm{id}) = \int_S f \cdot QV.$$

Proof. We have:

$$\begin{aligned}\frac{1}{e^h}\frac{1}{h}\int_{S^h} f^h \cdot Q^h(u^h - \mathrm{id}) &= \frac{1}{e^h}\int_S f^h(x) \cdot Q^h \fint_{-h/2}^{h/2} u^h(x + t\vec{n}) - x \,\mathrm{d}t\mathrm{d}x \\ &= \frac{1}{e^h}\int_S f^h(x) \cdot Q^h \frac{\sqrt{e^h}}{h} V^h[y^h]\,\mathrm{d}x \\ &= \int_S f \cdot Q^h V^h[y^h],\end{aligned}$$

and the result follows. □

Proof of Theorem 2.5. **1.** We first show that given any $u^h \in W^{1,2}(S^h, \mathbb{R}^3)$ there exists $Q^h \in SO(3)$ and $c^h \in \mathbb{R}^3$ such that $w^h = (Q^h)^T u^h - c^h - \mathrm{id}$ satisfies:

$$\|w^h\|^2_{W^{1,2}(S^h)} \le Ch^{-1} I^h(y^h). \tag{7.1}$$

Indeed, by Lemma A.1 and properties of the energy density W, it follows that:

$$\begin{aligned} I^h(y^h) &\ge Ch^{-1}\int_{S^h} \mathrm{dist}^2(\nabla u^h, SO(3)) \ge Ch^{-1}\int_{S^h} |\nabla u^h - R^h\pi|^2 \\ &\ge Ch^{-1}\int_{S^h} |(Q^h)^T\nabla u^h - \mathrm{Id}|^2 - Ch^{-1}\int_{S^h} |(Q^h)^T R^h\pi - \mathrm{Id}|^2 \qquad (7.2) \\ &\ge Ch^{-1}\int_{S^h} |\nabla w^h|^2 - Ch^{-2} I^h(y^h).\end{aligned}$$

Actually, the assumption of smallness of $h^{-3}E(u^h, S^h)$ cannot be expected to hold here. In this general case one exchanges the $SO(3)$-valued matrix field R^h with $\tilde{R}^h \in W^{1,2}(S, \mathbb{R}^{3\times 3})$ given in the proof of Lemma A.1. Then Q^h for which the above estimates are true may be taken as a rotation in $SO(3)$ with minimal distance from $\fint_S \tilde{R}^h$.

By (7.2) it follows that:

$$\|\nabla w^h\|^2_{L^2(S^h)} \leq ChI^h(y^h) + Ch^{-1}I^h(y^h),$$

which implies (7.1) in view of the Poincaré inequality, for an appropriately chosen constant c^h. A proof of the uniform Poincaré inequality on S^h can be found, for example, in [17].

2. Notice that by the definition of m^h we have:

$$\frac{1}{h}\int_{S^h} f^h(z)\cdot Q^h z \,\mathrm{d}z = h\sqrt{e^h}\int_S \fint_{-h/2}^{h/2} f(x)\cdot Q^h(x+t\vec{n}) \leq m^h.$$

Therefore, in view of (2.6) and (7.1) we obtain:

$$\begin{aligned} J^h(y^h) - I^h(y^h) &= m^h - \frac{1}{h}\int_{S^h} f^h u^h \\ &= -\frac{1}{h}\int_{S^h}(Q^h)^T f^h\cdot w^h + m^h - \frac{1}{h}\int_{S^h} f^h\cdot Q^h z\,\mathrm{d}z \\ &\geq -\frac{1}{h}\int_{S^h}(Q^h)^T f^h\cdot w^h \\ &\geq -Ch^{1/2}\sqrt{e^h}\|f\|_{L^2(S)}\|w^h\|_{L^2(S^h)} \\ &\geq -C\sqrt{e^h}I^h(y^h)^{1/2}. \end{aligned} \tag{7.3}$$

We now prove the first claim of the theorem. Taking $u^h(z) = \bar{Q}z$ for any $\bar{Q}\in\mathcal{M}$, we notice that $J^h(y^h) = 0$. Hence $\inf J^h \leq 0$. The lower bound of $\frac{1}{e^h}J^h$ follows from (7.3):

$$\frac{1}{e^h}J^h(y^h) \geq \frac{1}{e^h}I^h(y^h) - C\left(\frac{1}{e^h}I^h(y^h)\right)^{1/2}, \tag{7.4}$$

which proves (i).

3. To prove (ii), let u^h be a minimizing sequence of $\frac{1}{e^h}J^h$, as defined in (2.8). Then $\{\frac{1}{e^h}J^h(y^h)\}$ is bounded, and therefore, by (7.4) $\{\frac{1}{e^h}I^h(y^h)\}$ is also bounded. The convergences of $\tilde{y}^h$, $V^h[\tilde{y}^h]$ and $\frac{1}{h}\mathrm{sym}\,\nabla V^h[\tilde{y}^h]$ follow from Theorem 2.1. In particular:

$$\liminf_{h\to 0}\frac{1}{e^h}I^h(y^h) \geq I(V, B_{\tan}). \tag{7.5}$$

The strong convergence of $\frac{1}{h}\mathrm{sym}\,\nabla V^h[\tilde{y}^h]$ is deduced from the strong convergence of the sequence $\mathrm{sym}\,G^h_{\tan}$ in Lemma 3.6. This last result is in turn implied by the convergence of $\int_S \mathcal{Q}_3(G^h)$ (valid because the sequence is minimizing), positive definiteness of $\mathcal{Q}_3$ on symmetric matrices, and the weak convergence of G^h. Since the details are exactly the same as in [10, Section 7.2], we omit them.

We now prove that the limit $\bar{Q}$ of any converging subsequence of Q^h belongs to $\mathcal{M}$. By (2.6) we have:

$$\begin{aligned}\frac{1}{e^h}J^h(y^h) - \frac{1}{e^h}I^h(y^h) &= \frac{1}{e^h}\left(m^h - \frac{1}{h}\int_{S^h} f^h u^h\right)\\ &= \frac{h}{\sqrt{e^h}}\left(m - \int_S \fint_{-h/2}^{h/2} f(x)\cdot Q^h\tilde{u}^h \,\mathrm{d}t\mathrm{d}x\right)\\ &= -\frac{1}{he^h}\int_{S^h} f^h\cdot Q^h(\tilde{u}^h - id)\\ &\quad + \frac{h}{\sqrt{e^h}}\left(m - \int_S f\cdot Q^h x\,\mathrm{d}x\right).\end{aligned} \tag{7.6}$$

The first term above is bounded, as it in fact converges to $-\int_S f\cdot\bar{Q}V$, by Lemma 7.1. The quantity in brackets in the second term converges to $m - \int_S f\cdot\bar{Q}x$. Therefore, if $\bar{Q}\notin\mathcal{M}$, this last quantity is uniformly positive, and the second term above converges to $+\infty$ (as $h/\sqrt{e^h}\to\infty$). We observe that, in this situation, $\frac{1}{e^h}J^h(y^h)$ must converge to $+\infty$, contradicting (i) and thus proving that $\bar{Q}\in\mathcal{M}$.

In view of (7.5), (7.6) also implies:

$$\liminf_{h\to 0}\frac{1}{e^h}J^h(y^h) \geq \liminf_{h\to 0}\frac{1}{e^h}I^h(y^h) - \int_S f\cdot\bar{Q}V \geq J(V, B_{\tan}, \bar{Q}).$$

The fact that the limit $(V, B_{\tan}, \bar{Q})$ minimizes the functional J is now a standard consequence of the above inequality. Indeed, if:

$$J(\hat{V}, \hat{B}_{\tan}, \hat{Q}) \leq J(V, B_{\tan}, \bar{Q}) - \epsilon$$

for some $\hat{V}\in\mathcal{V}$, some $\hat{B}_{\tan}\in\mathcal{B}$, $\hat{Q}\in\mathcal{M}$ and $\epsilon>0$, then for a related recovery sequence $\hat{y}^h$ there would be:

$$\lim_{h\to 0}\frac{1}{e^h}J^h(\hat{Q}\hat{y}^h) = J(\hat{V}, \hat{B}_{\tan}, \hat{Q}) \leq J(V, B_{\tan}, \bar{Q}) - \epsilon \leq \liminf_{h\to 0}\frac{1}{e^h}J^h(y^h) - \epsilon,$$

which contradicts (2.8).

Finally, (iii) follows exactly as (i) and (ii). □

Remark 7.2. 1. A dead load (versus a "live load") is any external force which only depends on the reference configuration point, and not on the deformation itself. An

important feature of dead loads, discussed first in [21], is the following. If the load is in a certain average sense compressive, it is advantageous for the body to perform a large rotation rather than undergo a compression. Our analysis identifies $\mathcal{M}$ as the set of candidates for such rotations, which are expected to minimize the total energy J^h among all rigid motions of the body.

This phenomenon may happen even if the average torque of the force is zero:

$$\int_S f(x) \times x \, \mathrm{d}x = 0. \tag{7.7}$$

Note that vanishing of the average torque is necessary for $\mathrm{Id} \in \mathcal{M}$, since (7.7) can be written as: $\int_S f \cdot Fx = 0$ for all $F \in so(3)$. However, it is not sufficient, and if $\mathrm{Id} \notin \mathcal{M}$ then we observe that the minimizers of J^h will not be close to Id. In general, the body chooses an infinitesimal isometric displacement V and a rotation $\bar{Q} \in \mathcal{M}$ which is energetically advantageous in response to the force f. That is, those rotations which allow for a better alignment of infinitesimal isometries with the direction of the dead load, are preferred.

2. The assumption on the sequence of forces f^h can of course be weakened. For example, consider $f^h(x + t\vec{n}) = \det(\mathrm{Id} + t\Pi(x))^{-1} f^h(x)$ and let $\frac{1}{h\sqrt{e^h}} f^h$ converge weakly in $L^2(S)$ to some $f \in L^2(S, \mathbb{R}^3)$. In this situation, one needs to enforce extra assumptions on the asymptotic behavior of the maximizers of the linear functions $SO(3) \ni Q \mapsto \int_S f^h \cdot Qx \, \mathrm{d}x$ with respect to $\mathcal{M}$, to exclude certain degenerate cases. The analysis is as in the proof of Theorem 2.5 and we leave the details to a courageous reader.

3. The lower bound on J and existence of its minimizers can be proved independently, and under the following weaker assumptions:

$$\int_S f = 0 \quad \text{and} \quad \int_S f(x) \cdot \bar{Q} Fx \, \mathrm{d}x = 0 \quad \forall \bar{Q} \in \mathcal{M} \quad \forall F \in so(3),$$

which can be seen as the linearization of (2.7), although it makes perfect sense for any closed nonempty subset $\mathcal{M} \subset SO(3)$. Indeed, the second equality above follows by differentiating the expression $\int_S f(x) \cdot Qx \, \mathrm{d}x$ at $\bar{Q} \in \mathcal{M}$ and using that $so(3)$ is the tangent space to $SO(3)$ at Id. We present the proof of coercivity and the attainment of the minimum by J and $\tilde{J}$ under this condition, for arbitrary $\mathcal{M}$, in Appendix C.

Appendices

A. An approximation theorem on surfaces

For a given vector mapping $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ defined on an open subset $\Omega \subset \mathbb{R}^n$, denote:

$$E(u, \Omega) = \int_\Omega \mathrm{dist}^2(\nabla u(x), SO(3)) \, \mathrm{d}x.$$

Lemma A.1. *Let $u \in W^{1,2}(S^h, \mathbb{R}^n)$ and assume that $h^{-3}E(u, S^h)$ is sufficiently small. Then there exists a matrix field $R \in W^{1,2}(S, \mathbb{R}^{3\times 3})$, such that:*

$$R(x) \in SO(3) \qquad \forall x \in S,$$

and a matrix $Q \in SO(3)$ with the following properties:

(i) $\|\nabla u - R\pi\|_{L^2(S^h)} \leq CE(u, S^h)^{1/2}$,
(ii) $\|\nabla R\|_{L^2(S)} \leq Ch^{-3/2}E(u, S^h)^{1/2}$,
(iii) $\|Q^T R - \mathrm{Id}\|_{L^p(S)} \leq Ch^{-3/2}E(u, S^h)^{1/2}$, *for all* $p \in [1, \infty)$,

where C is independent of u and h (but may depend on p).

The proof of Lemma A.1 uses the following nonlinear quantitative rigidity estimate by Friesecke, James and Müller.

Theorem A.2 ([9]). *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain with Lipschitz boundary. Then, for every $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ one has:*

$$\min_{R\in SO(n)} \int_\Omega |\nabla u(x) - R|^2 \, \mathrm{d}x \leq CE(u, \Omega),$$

where the constant C depends only on Ω. In particular, C is invariant under dilations of Ω, and it is also uniform for the uniform bilipschitz images of a unit ball in $\mathbb{R}^n$.

Proof of Lemma A.1. **1.** For $x \in S$ define 'balls' in S and the corresponding 'cylinders' in S^h:

$$D_{x,h} = B(x, h) \cap S, \qquad B_{x,h} = \pi^{-1}(D_{x,h}) \cap S^h.$$

The main observation is that Theorem A.2 may be applied on each set $B_{x,h}$, yielding matrices $R_{x,h} \in SO(3)$ such that:

$$\int_{B_{x,h}} |\nabla u(z) - R_{x,h}|^2 \, \mathrm{d}z \leq CE(u, B_{x,h}), \tag{A.1}$$

with uniform constant C (independent of x or h).

2. Let $\vartheta \in C_c^\infty([0, 1))$ be a nonnegative cut-off function, equal to a nonzero constant in a neighborhood of 0. For each $x \in S$ define the function $\eta_x : S^h \longrightarrow \mathbb{R}$:

$$\eta_x(z) = \frac{\vartheta(|\pi z - x|/h)}{\int_{S^h} \vartheta(|\pi y - x|/h) \, \mathrm{d}y}.$$

Then $\eta_x(z) = 0$ for $z \notin B_{x,h}$ and:

$$\int_{S^h} \eta_x(z) \, \mathrm{d}z = 1, \qquad \|\eta_x\|_{L^\infty} \leq Ch^{-3}, \qquad \|\nabla_x \eta_x\|_{L^\infty} \leq Ch^{-4}. \tag{A.2}$$

The last inequalities follow from the lipschitzianity of ∂S. In particular, the denominator function in the definition of η_x has Lipschitz constant Ch^2, and hence:

$$\left\| \nabla_x \left(\int_{S^h} \vartheta(|\pi y - x|/h)\, \mathrm{d}y \right)^{-1} \right\|_{L^\infty} \leq Ch^{-4}.$$

Consider the matrix field $\tilde{R} \in W^{1,2}(S, \mathbb{R}^{3\times 3})$:

$$\tilde{R}(x) = \int_{S^h} \eta_x(z) \nabla u(z)\, \mathrm{d}z.$$

By the first two statements in (A.2) we obtain:

$$|\tilde{R}(x) - R_{x,h}|^2 = \left| \int_{S^h} \eta_x(z)(\nabla u(z) - R_{x,h})\, \mathrm{d}z \right|^2 \leq Ch^{-3} E(u, B_{x,h}). \tag{A.3}$$

Similarly:

$$\begin{aligned} |\nabla \tilde{R}(x)|^2 &= \left| \int_{S^h} (\nabla_x \eta_x) \nabla u \right|^2 = \left| \int_{S^h} (\nabla_x \eta_x)(\nabla u - R_{x,h}) \right|^2 \\ &\leq \int_{B_{x,h}} |\nabla_x \eta_x|^2 \cdot \int_{B_{x,h}} \left| \nabla u - R_{x,h} \right|^2 \leq Ch^{-5} E(u, B_{x,h}), \end{aligned} \tag{A.4}$$

and likewise, for any $x' \in D_{x,h}$:

$$|\nabla \tilde{R}(x')|^2 \leq Ch^{-5} E(u, 2B_{x,h}) \tag{A.5}$$

with $2B_{x,h} = \pi^{-1}(D_{x,2h}) \cap S^h$. Therefore, in view of the lipschitzianity of ∂S and by the fundamental theorem of calculus:

$$|\tilde{R}(x'') - \tilde{R}(x)|^2 \leq Ch^{-3} E(u, 2B_{x,h}) \qquad \forall x'' \in D_{x,h}. \tag{A.6}$$

Combining (A.1) with (A.3) and (A.6) yields:

$$\begin{aligned} &\int_{B_{x,h}} |\nabla u(z) - \tilde{R}\pi(z)|^2\, \mathrm{d}z \\ &\quad \leq 2 \left(\int_{B_{x,h}} |\nabla u - R_{x,h}|^2 + \int_{B_{x,h}} |\tilde{R}(x) - R_{x,h}|^2 + \int_{B_{x,h}} |\tilde{R}\pi - \tilde{R}(x)|^2 \right) \\ &\quad \leq CE(u, 2B_{x,h}). \end{aligned} \tag{A.7}$$

Now cover S by $\{D_{x_i,h}\}_{i=1}^{N_h}$ so that the covering number of the family $\{2B_{x_i,h}\}_{i=1}^{N_h}$ is independent of h. Summing the inequalities in (A.7) over $i = 1 \ldots N$ proves:

$$\int_{S^h} |\nabla u - \tilde{R}\pi|^2 \leq CE(u, S^h). \tag{A.8}$$

Also, integrating (A.5) over $D_{x_i,h}$ and summing over $i = 1 \dots N$ gives:

$$\int_S |\nabla \tilde{R}|^2 \leq Ch^{-3} E(u, S^h). \tag{A.9}$$

3. Notice that by (A.3), for every $x \in S$:

$$\text{dist}^2(\tilde{R}(x), SO(3)) \leq Ch^{-3} E(u, S^h).$$

Hence, if $E(u, S^h)/h^3$ is sufficiently small, we may define:

$$R(x) = \mathcal{P}_{SO(3)}(\tilde{R}(x))$$

where $\mathcal{P}_{SO(3)}$ is the orthogonal projection onto the compact manifold $SO(3)$. Clearly $R : S \longrightarrow SO(3)$ is a $W^{1,2}$ matrix field and since:

$$|R(x) - \tilde{R}(x)| = \text{dist}\,(\tilde{R}(x), SO(3)),$$

by (A.8) we conclude that:

$$\int_{S^h} |\nabla u - R\pi|^2 \leq C \left(\int_{S^h} |\nabla u - \tilde{R}\pi|^2 + \int_{S^h} \text{dist}^2(\nabla u, SO(3)) \right) \leq CE(u, S^h),$$

which proves (i) in Lemma A.1. The bound (ii) is deduced directly from (A.9).

4. To deduce (iii), define first the intermediate matrix $\tilde{Q}$ as the average of R on S. Using the Sobolev and Poincaré inequalities, together with (ii) we obtain, for every $p \geq 2$:

$$\left(\int_S |R - \tilde{Q}|^p \right)^{2/p} \leq C \|R - \tilde{Q}\|^2_{W^{1,2}(S)} \leq C \int_S |\nabla R|^2 \leq Ch^{-3} E(u, S^h). \tag{A.10}$$

Now, take $Q \in SO(3)$ such that $|Q - \tilde{Q}| = \text{dist}\,(\tilde{Q}, SO(3))$. As before, (A.10) remains true if $\tilde{Q}$ is replaced with Q. Clearly, the same bound must also hold for $p \in [1, 2)$, and so we conclude that:

$$\forall p \in [1, \infty) \qquad \|R - Q\|^2_{L^p(S)} \leq Ch^{-3} E(u, S^h).$$

The above easily implies (iii). □

B. The Γ-convergence setting

We first recall the notion of Γ-convergence of a sequence of functionals $\mathcal{F}^h : X \longrightarrow \overline{\mathbb{R}}$, defined on a metric space X. Namely, $\mathcal{F}^h$ Γ-converge, as $h \to 0$, to some $\mathcal{F} : X \longrightarrow \overline{\mathbb{R}}$ provided that the following two conditions hold:

(i) For any converging sequence $\{x^h\}$ in X one has:

$$\mathcal{F}\left(\lim_{h\to 0} x^h\right) \leq \liminf_{h\to 0} \mathcal{F}^h(x^h). \tag{B.1}$$

(ii) For every $x \in X$, there exists a sequence $\{x^h\}$ converging to x, such that:

$$\mathcal{F}(x) = \lim_{h\to 0} \mathcal{F}^h(x^h). \tag{B.2}$$

When X is only a topological space, the definition of Γ-convergence involves, naturally, systems of neighborhoods rather than sequences. However, when the functionals $\mathcal{F}^h$ are equi-coercive and X is a reflexive Banach space equipped with weak topology, one can still use (i) and (ii) above (for weakly converging sequences), as an equivalent version of this definition. For details, we refer the reader to [6].

Proof of Corollary 2.4. We only prove (i), in the case when the product space in the domain of $\mathcal{F}$ is equipped with the strong topology. The other statements follow the same.

To obtain (B.1), we take a sequence of $W^{1,2}(S^{h_0})$ vector mappings $\{y^h\}$ such that, writing $B^h_{\text{tan}} = \frac{1}{h}\text{sym}\,\nabla V^h[y^h]$, the sequence $\{\mathcal{F}^h(y^h, V^h[y^h], B^h_{\text{tan}})\}$ is bounded, and such that y^h, $V^h[y^h]$ and B^h_{tan} converge to some y, V and B_{tan} (in $W^{1,2}(S^{h_0})$, $W^{1,2}(S)$ and $L^2(S)$ respectively). By Theorem 2.1 we obtain a sequence of normalized deformations $\tilde{y}^h = (Q^h)^T y^h - c^h$, converging to π. Moreover, $V^h[\tilde{y}^h]$ and $\frac{1}{h}\text{sym}\,\nabla V^h[\tilde{y}^h]$ converge to $\tilde{V}$ and weakly to $\tilde{B}_{\text{tan}}$, respectively. Notice now that:

$$|Q^h - \text{Id}| = Ch^{-1}\sqrt{e^h}\left\|\nabla V^h[\tilde{y}^h] - (Q^h)^T\nabla V^h[y^h]\right\|_{L^2(S)} \leq Ch^{-1}\sqrt{e^h}.$$

In particular, Lemma 3.1 remains true if we put $Q^h = \text{Id}$, for all h. Consequently, all the assertions of Theorem 2.1 still hold for $\tilde{y}^h = y^h - c^h$ (possibly after modifying the constants c^h).

Now, $V^h[y^h] - V^h[\tilde{y}^h] = h/\sqrt{e^h}c^h$ is bounded, so c^h converge to 0, as $h \to 0$. On the other hand $c^h = y^h - \tilde{y}^h$ converge to $y - \pi$. Hence $y = \pi$. Moreover $\nabla V^h[\tilde{y}^h] = \nabla V^h[y^h]$, so $\nabla V = \nabla\tilde{V}$ and $B_{\text{tan}} = \tilde{B}_{\text{tan}}$. By Theorem 2.1 (iv) we conclude that:

$$\mathcal{F}(y, V, B_{\text{tan}}) \leq \liminf_{h\to 0} \mathcal{F}^h(y^h, V^h, B^h_{\text{tan}})$$

which proves (B.1).

The second requirement for Γ-convergence (B.2) follows directly from Theorem 2.2, in view of (B.1). □

We remark that in presence of external forces, the results of Theorem 2.5 can also be formulated in the language of Γ-convergence, similarly as above.

C. On coercivity of the generalized von Kármán functionals J and $\tilde{J}$

In this section, we consider the functionals:

$$J(V, B_{\tan}, \bar{Q}) = \frac{1}{2}\int_S \mathcal{Q}_2\left(x, B_{\tan} - \frac{\kappa}{2}(A^2)_{\tan}\right) + \frac{1}{24}\int_S \mathcal{Q}_2\left(x, (\nabla(A\vec{n}) - A\Pi)_{\tan}\right) - \int_S f\cdot\bar{Q}V,$$

$$\tilde{J}(V, \bar{Q}) = \frac{1}{24}\int_S \mathcal{Q}_2\left(x, (\nabla(A\vec{n}) - A\Pi)_{\tan}\right) - \int_S f\cdot\bar{Q}V,$$

defined for infinitesimal isometries V, matrix fields $B_{\tan} \in \mathcal{B}$ and rotations $\bar{Q} \in \mathcal{M}$, where $\mathcal{M}$ is an arbitrary closed and nonempty subset of $SO(3)$. We prove that J and $\tilde{J}$ attain their finite lower bounds under the following assumptions on $f \in L^2(S, \mathbb{R}^3)$:

$$\int_S f = 0 \quad \text{and} \quad \int_S f(x)\cdot\bar{Q}Fx\,\mathrm{d}x = 0 \quad \forall\bar{Q}\in\mathcal{M} \quad \forall F \in so(3). \tag{C.1}$$

As mentioned in Remark 7.2, the second condition above is a consequence and linearization of (2.7), with $\mathcal{M}$ defined by that formula.

Lemma C.1. *Assume that S is of class $\mathcal{C}^{2,1}$. Then for every $V \in \mathcal{V}$ there exist $D \in so(3)$ and $d \in \mathbb{R}^3$, so that:*

$$\|V - (Dx + d)\|^2_{W^{2,2}(S)} \le C\int_S |(\nabla(A\vec{n}) - A\Pi)_{\tan}|^2.$$

Proof. **1.** We first prove that $\int_S |(\nabla(A\vec{n}) - A\Pi)_{\tan}|^2 = 0$ implies for a $W^{2,2}$ infinitesimal isometry V to have the form $V(x) = Dx + d$, with $D \in so(3)$ and $d \in \mathbb{R}^3$.

To see this, let $c \in W^{1,2}(S, \mathbb{R}^3)$ be such that:

$$A(x)\tau = c(x)\times\tau \qquad \forall x \in S \quad \forall\tau \in T_xS.$$

Since A represents a gradient, it follows that $\partial_\tau c\times\eta = \partial_\eta c\times\tau$ for all $\tau, \eta \in T_xS$. In particular, for any τ and η such that $\tau\times\eta = \vec{n}$, one has:

$$(\partial_\tau c)\cdot\vec{n} = -(\partial_\tau c\times\eta)\cdot\tau = -(\partial_\eta c\times\tau)\cdot\tau = 0.$$

On the other hand:

$$0 = \left(\partial_\tau(A\vec{n}) - A\Pi\tau\right)_{\tan} = \left(\partial_\tau(c\times\vec{n}) - A\Pi\tau\right)_{\tan} = (\partial_\tau c)\times\vec{n}.$$

Hence $\partial_\tau c = 0$ on S, which yields the claim.

2. We prove the result. Arguing by contradiction, we assume that for a sequence of infinitesimal isometries $V^h \in W^{2,2}(S, \mathbb{R}^3)$ there holds:

$$\begin{aligned} \text{dist}_{W^{2,2}(S)}\Big(V^h, \{Dx+d;\ D\in so(3),\ d\in\mathbb{R}^3\}\Big) = 1 \\ \text{and} \quad \lim_{h\to 0}\int_S \Big|(\nabla A^h)\vec{n}\Big|^2 = 0. \end{aligned} \tag{C.2}$$

Since the second condition above involves only higher derivatives of V^h, we may without loss of generality replace the first condition by:

$$\|V^h\|_{W^{2,2}(S)} = 1 \ \text{ and } \ \big\langle V^h, Dx+d\big\rangle_{W^{2,2}(S)} = 0 \quad \forall D\in so(3),\ d\in\mathbb{R}^3. \tag{C.3}$$

In particular, V^h converges weakly in $W^{2,2}(S)$ (up to a subsequence, which we do not relabel) to some vector field $V \in \mathcal{V}$. By (C.2) and the weak lower semicontinuity of the L^2 norm, we deduce that $\int_S |(\nabla A)\vec{n}|^2 = 0$. Hence, in view of the first part of the proof and the second condition in (C.3), it follows that $V = 0$ and so:

$$\lim_{h\to 0}\|V^h\|_{W^{1,2}(S)} = 0. \tag{C.4}$$

By the estimate (4.4) and (C.4) we may deduce:

$$\lim_{h\to 0}\|V^h_{\tan}\|_{W^{2,2}(S)} = 0, \tag{C.5}$$

where $V^h_{\tan} = V^h - (V^h\vec{n})\vec{n}$. Observe that:

$$\begin{aligned} \int_S \Big|(\nabla A^h)\vec{n}\Big|^2 &= \int_S \Big|\nabla\Big(\Pi V^h_{\tan} - \nabla(V^h\vec{n})\Big) - A^h\Pi\Big|^2 \\ &= \int_S \Big|\nabla^2(V^h\vec{n}) + (A^h\Pi - \Pi A^h)_{\tan} - (\nabla\Pi)V^h_{\tan} + (V^h\vec{n})\Pi\Big|^2. \end{aligned}$$

Therefore:

$$\|\nabla^2(V^h\vec{n})\|_{L^2(S)} \le C\Big(\|(\nabla A^h)\vec{n}\|_{L^2(S)} + \|V^h\|_{W^{1,2}(S)}\Big),$$

and in view of (C.4) and the assumption (C.2) we also get:

$$\lim_{h\to 0}\|V^h\vec{n}\|_{W^{2,2}(S)} = 0.$$

The above together with (C.5) contradicts (C.3) and proves the lemma. □

Lemma C.2. *Assume* (C.1) *and let S be of class $\mathcal{C}^{2,1}$. Then the functionals J and $\tilde{J}$, defined for $V \in \mathcal{V}$, $B_{\tan} \in \mathcal{B}$ and $\bar{Q} \in \mathcal{M}$, are bounded from below and attain their infima.*

Proof. **1.** Let $V \in \mathcal{V}$. By Lemma C.1, positive definiteness of $\mathcal{Q}_2$ (on symmetric matrices) and (C.1), we obtain:

$$\begin{aligned} \tilde{J}(V) &\geq C\|\tilde{V}\|^2_{W^{2,2}(S)} - \int_S f \cdot \bar{Q} V = C\|\tilde{V}\|^2_{W^{2,2}(S)} - \int_S f \cdot \bar{Q}\tilde{V} \\ &\geq C\|\tilde{V}\|^2_{W^{2,2}(S)} - \|f\|_{L^2(S)} \cdot \|\tilde{V}\|_{L^2(S)}, \end{aligned} \tag{C.6}$$

for an appropriate modification $\tilde{V} = V - (Dx + d)$. Hence the lower bound on J (and $\tilde{J}$) follows.

2. Let now $(V^h, B^h_{\tan}, \bar{Q}^h)$ be a minimizing sequence of J. Clearly, a subsequence of $\bar{Q}^h$ converges to some $\bar{Q} \in \mathcal{M}$.

Using (C.6) and applying the positive definiteness of $\mathcal{Q}_2$ to the first term in J, there follows the (uniform in h) boundedness of the following expressions:

$$\left(C\|\tilde{V}^h\|^2_{W^{2,2}(S)} - \|f\|_{L^2(S)} \cdot \|\tilde{V}^h\|_{L^2(S)}\right) + C\left\| B^h_{\tan} - \frac{\kappa}{2}((A^h)^2)_{\tan}\right\|^2_{L^2(S)}. \tag{C.7}$$

Again, we put $\tilde{V}^h = V^h - (D^h x + d^h)$ and apply Lemma C.1. In particular, the sequence $\tilde{V}^h$ is bounded in $W^{2,2}(S)$ and so it converges (up to a subsequence), weakly in $W^{2,2}(S)$, to an infinitesimal isometry V. Further, the matrix fields $\tilde{A}^h = A^h - D^h$ converge weakly in $W^{1,2}(S)$ to the field A satisfying (2.2).

Notice that:

$$(A^h)^2 = (\tilde{A}^h)^2 + (D^h)^2 + (D^h A^h + A^h D^h).$$

Hence the boundedness of the second term in (C.7) results in the $L^2(S)$ boundedness of:

$$B^h_{\tan} - \frac{\kappa}{2}\left((D^h)^2 + (D^h A^h + A^h D^h)\right)_{\tan} = B^h_{\tan} - \frac{\kappa}{2}\operatorname{sym}\nabla\left((D^h)^2 x + 2D^h V^h(x)\right).$$

We may now conclude that a subsequence of the above sequence of symmetric matrix fields converges, weakly in $L^2(S)$, to some $B_{\tan} \in \mathcal{B}$. Thus, $B^h_{\tan} - \frac{\kappa}{2}((A^h)^2)_{\tan}$ converges to $B_{\tan} - \frac{\kappa}{2}(A^2)_{\tan}$.

By the weak lower semicontinuity of both quadratic terms in J we conclude that $J(V, B_{\tan}\bar{Q})$ realizes the infimum of J. Likewise, $\tilde{J}(V, \bar{Q})$ realizes the infimum of $\tilde{J}$, had V^h been a minimizing sequence of $\tilde{J}$. □

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