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Asymptotically regular problems II: Partial Lipschitz continuity and a singular set of positive measure

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Abstract. We consider multidimensional variational integrals for vector-valued functions $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$. Assuming that the integrand satisfies the standard smoothness, convexity and growth assumptions **only near** ∞ we investigate the partial regularity of minimizers (and generalized minimizers) u. Introducing the open set

$$R(u) := \{x \in \Omega \,:\, u \text{ is Lipschitz near } x\},$$

we prove that $R(u)$ is dense in Ω, but we demonstrate for $n \geq 3$ by an example that $\Omega \setminus R(u)$ may have positive measure. In contrast, for $n = 2$ one has $R(u) = \Omega$.

Additionally, we establish analogous results for weak solutions of quasilinear elliptic systems.

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1. Introduction

Throughout this paper we fix $n \geq 2$, $N \geq 1$, $p \geq 2$ and a nonempty bounded open subset Ω of $\mathbb{R}^n$. We study the minimization problem for variational integrals

$$F[u] := \int_\Omega f(Du)\,dx \qquad \text{for } u : \Omega \to \mathbb{R}^N, \tag{1.1}$$

where $f : \mathbb{R}^{Nn} \to \mathbb{R}$ is a given integrand of the argument $z \in \mathbb{R}^{Nn}$.

We say that f is regular iff it satisfies a set of standard smoothness, convexity and growth assumptions (see Definition 2.1 below). If f is a regular C^∞-integrand, then there is a well-developed existence and regularity theory for the minimization problem: Precisely, minimizers u of F exist in the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^N)$ and are C^∞ outside a negligible set S; see [1, 2, 5–8, 11, 16, 19, 21, 24, 25, 27, 32]. The smallest such S is called the singular set and need — by famous examples [26, 33, 38, 39] — not be empty, except for $n = 2$ [27, 32] or $N = 1$ [20, 21, 30].

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Here, we restrict our considerations to proving local Lipschitz continuity of minimizers u of F. In this regard it is heuristically plausible that only the behavior of f for large values of the gradient variable z should be relevant. Indeed, it was pointed out by Chipot & Evans [4] that this heuristic idea can be made precise in some particular situations. Moreover, various related results (see for instance [10, 13–15, 20, 29, 34]) have by now been established and we refer the reader to the first part [36] of this work for an extensive discussion of such issues and further references. However, we stress that all these papers are concerned with particular situations where — for some reason — everywhere regularity is available.

In the present paper we focus on the general case, where only partial regularity can be expected. In fact, without imposing any additional structure condition we will merely require that f is locally bounded and asymptotically regular, *i.e.* regular for large values of z (see Definition 2.2 for a precise statement). Introducing the open set

$$R(u) := \{x \in \Omega : u \text{ is Lipschitz near } x\}, \tag{1.2}$$

it would then be natural to expect that $S(u) := \Omega \setminus R(u)$ is negligible. However, we demonstrate that the situation is not that simple: We prove that the regular set $R(u)$ is always dense in Ω, but give for $n \geq 3$ and $N = \frac{1}{2}n(n+1)-1$ an example of a minimizer u such that the singular set $S(u)$ has positive measure. Additionally, we obtain $R(u) = \Omega$ in the cases $n = 2$ and $N = 1$.

2. Statements

In order to state our results, we will now specify our assumptions on the integrand in (1.1), introducing the classes of regular and asymptotically regular integrands. We stress that our notion of an asymptotically regular integrand in the present paper differs from the one in [36], and is in fact considerably stronger; see [12, 36] and Section 4 for further information about notions of asymptotic regularity and their relation.

Definition 2.1 (Regularity). Let $m \in \mathbb{N}$. We say that $f : \mathbb{R}^m \to \mathbb{R}$ is **regular** iff it is C^2 and satisfies the following convexity and growth conditions:

$$D^2 f(z)(\xi, \xi) \geq \gamma(1 + |z|)^{p-2}|\xi|^2,$$
$$|D^2 f(z)| \leq \Gamma(1 + |z|)^{p-2}$$

for all $z, \xi \in \mathbb{R}^m$, and for some positive constants γ and Γ.

Definition 2.2 (Asymptotic regularity). We say that $f : \mathbb{R}^m \to \mathbb{R}$ is **asymptotically regular** iff for some positive constants M, γ and Γ the function f is C^2 outside $\overline{B_M}$ and satisfies, for all $z, \xi \in \mathbb{R}^m$ with $|z| > M$, the conditions

$$D^2 f(z)(\xi, \xi) \geq \gamma|z|^{p-2}|\xi|^2,$$
$$|D^2 f(z)| \leq \Gamma|z|^{p-2}.$$

In the following, we will additionally assume that f is locally bounded, *i.e.* there is some nondecreasing function $\Upsilon : [0, \infty) \to [0, \infty)$ with

$$|f(z)| \leq \Upsilon(|z|) \qquad \text{for all } z \in \mathbb{R}^{Nn}.$$

With this terminology we may now state our first main result:

Theorem 2.3 (Partial Lipschitz regularity for minimizers). *We consider a locally bounded Borel integrand $f : \mathbb{R}^{Nn} \to \mathbb{R}$ and assume that f is asymptotically regular. Then there exists a constant L, depending only on the data n, N, p, γ, Γ, Υ and M, such that for every minimizer $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ of F from* (1.1)*, the domain Ω can be decomposed into three disjoint sets H, B_L and Σ such that*

- *H is an open set with $u \in C^{1,\alpha}_{\text{loc}}(H, \mathbb{R}^N)$ for every $\alpha \in (0, 1)$;*
- *every $x \in B_L$ is a Lebesgue point of Du with $|Du(x)| \leq L$;*
- *Σ is a negligible set.*

In particular, H and the interior of B_L are contained in the regular set $R(u)$, defined in (1.2)*, and thus $R(u)$ is dense in Ω.*

Remark 2.4. The above result does **not** imply that the singular set $S(u)$ is negligible. In fact, it may happen that $S(u) \cap \partial B_L$ has positive measure. We will give an example of a minimizer with this behavior below; see Theorem 2.6.

The reader should note that asymptotic regularity in combination with the requirement that f is locally bounded implies

$$c|z|^p - C \leq f(z) \leq C(1 + |z|^p) \qquad \text{for all } z \in \mathbb{R}^{Nn}, \tag{2.1}$$

which, in turn, ensures that the integral (1.1) is well-defined and finite on $W^{1,p}(\Omega, \mathbb{R}^N)$.

We stress that, in contrast to regularity, asymptotic regularity does not allow to prove the existence of minimizers. Thus, it is desirable to find an appropriate class of generalized minimizers and to extend the above regularity result to this class. Here, we deal with relaxed minimizers in the sense of [36, Section 2.1.5] and establish the following generalization of Theorem 2.3:

Theorem 2.5 (Partial Lipschitz regularity for relaxed minimizers). *The conclusion of Theorem* 2.3 *still holds if u is only a relaxed minimizer of F (instead of being a minimizer).*

The question arises if the above assertion that $R(u)$ is dense — or equivalently that $S(u)$ is nowhere dense — can be improved. As mentioned above, in the case of regular integrands the singular set is negligible, and in fact even its Hausdorff dimension can be bounded away from n [22, 27, 31]. However, our next result shows that such assertions do not carry over to asymptotically regular problems:

Theorem 2.6 (A singular set of positive measure). *For every $n \geq 3$ and $N = \frac{1}{2}n(n+1) - 1$, there exist a smooth integrand $f : \mathbb{R}^{Nn} \to \mathbb{R}$ and a function $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ such that the following holds:*

- *f is asymptotically regular in the sense of Definition 2.2 with $p = 2$;*
- *u is a minimizer of F from* (1.1)*;*
- *$S(u)$ has positive measure.*

Next, we consider quasilinear elliptic systems

$$\operatorname{div} a(Du) = 0 \qquad \text{on } \Omega. \tag{2.2}$$

In the following, we will employ notions of regularity and asymptotic regularity for the structure function $a : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ analogous to the Definitions 2.1 and 2.2. In particular, we say that a is **asymptotically regular** iff there are positive constants M, γ and Γ such that a is C^1 outside $\overline{B_M}$ and satisfies

$$Da(z)\xi \cdot \xi \geq \gamma |z|^{p-2} |\xi|^2,$$
$$|Da(z)| \leq \Gamma |z|^{p-2}$$

for $|z| > M$ and $\xi \in \mathbb{R}^{Nn}$. With these notations we state our main result for systems:

Theorem 2.7 (Partial Lipschitz regularity for weak solutions). *We consider a locally bounded Borel function $a : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ and we assume that a is asymptotically regular. Then the conclusion of Theorem* 2.3 *holds for every weak solution $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ of* (2.2)*.*

In the situation of Theorem 2.7, the singular set $S(u)$ will, in general, not be negligible either. Indeed, this assertion is an immediate consequence of Theorem 2.6 since u solves $\operatorname{div} Df(Du) = 0$.

Additionally, in the cases $n = 2$ and $N = 1$ we can improve the above results obtaining everywhere regularity (compare [15]):

Theorem 2.8 (Two-dimensional problems). *Let $n = 2$. We suppose that u is either a relaxed minimizer of F from* (1.1) *under the assumptions of Theorem* 2.3 *or a weak solution of* (2.2) *under the assumptions of Theorem* 2.7*. Then we have $R(u) = \Omega$.*

We stress that Theorem 2.8, apart from its intrinsic interest, has some applications in the theory of quasiconvex integrals; see the discussion in [28, Section 6.1]

Before stating the result for $N = 1$, let us specify an additional technical assumption, which we will need for $p > 2$, but not for $p = 2$. For the function $\mathcal{A} := D^2 f$ or $\mathcal{A} := Da$, respectively, we require the following uniform continuity condition:

$$|\mathcal{A}(z_2) - \mathcal{A}(z_1)| \leq (|z_1| + |z_2|)^{p-2} \nu\big(|z_2 - z_1|^2\big) \tag{2.3}$$

for all $z_1, z_2 \in \mathbb{R}^n$ with $|z_1| > M$ and $|z_2| > M$ and for some modulus of continuity ν (*i.e.* a continuous function $\nu : [0, \infty) \to [0, \infty)$ with $\nu(0) = 0$).

Theorem 2.9 (Scalar problems). *Let $N = 1$. We suppose that u is either a relaxed minimizer of F from* (1.1) *under the assumptions of Theorem* 2.3 *or a weak solution of* (2.2) *under the assumptions of Theorem* 2.7. *Additionally, in the case $p > 2$ we assume that* (2.3) *holds. Then we have $R(u) = \Omega$.*

Finally, let us briefly outline the plan of the paper and the proofs. Having collected some preliminaries in Section 3, we devote the next sections to the proofs of the regularity theorems Theorem 2.3 and Theorem 2.7. We start taking a closer look at the notion of asymptotic regularity in Section 4. In fact, we prove that f is asymptotically regular if and only if it coincides with a regular integrand for large values of z. This gives us a regular comparison problem. In Section 5 we consider a solution v of this regular problem and establish certain comparison estimates showing that $u-v$ is, in some sense, small. The derivation of these estimates is inspired by ideas in [4,14,20]. In Section 6 we then carry over some partial regularity from v to u, thus completing the proofs of Theorem 2.3 and Theorem 2.7. Additionally, we derive Theorem 2.5, as a consequence of Theorem 2.3 and Section 4. In Section 7, following essentially the same strategy we establish Theorem 2.8 and Theorem 2.9.

Section 8 is devoted to irregularity, specifically to the counterexample in Theorem 2.6. Our starting point here is a recent interesting example of Sverak & Yan [39]. They constructed a minimizer of a regular integral which is not Lipschitz at an isolated singularity. The basic idea of our example is now to construct u by placing rescaled copies of their minimizer on certain balls, with the singularities in the centers. In fact, using a Cantor type construction we may arrange the balls in such a way that the closure of the set of their centers has positive measure. Then we complete the proof of Theorem 2.6 by observing that $S(u)$ coincides with this set and defining an integral which is minimized by u. Here, in order to see that u is actually a minimizer — and not merely a solution of some system — we need to revisit some of the more technical details of [39].

3. Preliminaries

Notation

Constants. We use the notations c and C for positive constants, possibly varying from line to line. The dependences of such constants will only occasionally be highlighted. Anyway, we widely follow the convention that large constants will be denoted by capital letters, and small constants by lowercase letters.

Closure and boundary. We write $\overline{A}$ for the closure and ∂A for the boundary of a set A.

Balls. By $B_r(x)$ we denote the open ball in $\mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius $r > 0$. Here, the center will be omitted if it is 0. Similarly, we write $\overline{B_r(x)}$ and $\overline{B_r}$ for closed balls.

Mean values. We use the common notations f_A and $\fint_A f\,dx$ for the mean value $\frac{1}{|A|}\int_A f\,dx$ of f on A, where $|A|$ is the Lebesgue measure of A. In particular, in the case of balls we abbreviate $f_{x,r} := f_{B_r(x)}$ and $f_r := f_{0,r}$.

Function spaces. As usual we write L^p, $W^{k,p}$ and $C^{k,\alpha}$ for Lebesgue, Sobolev and Hölder spaces, respectively. In addition we write L^p_{loc}, $W^{k,p}_{\mathrm{loc}}$ and $C^{k,\alpha}_{\mathrm{loc}}$ for the localized variants of these spaces.

The (nondegenerate) p-energy. We set

$$e_p(z) := \frac{1}{p}(1+|z|^2)^{\frac{p}{2}}. \tag{3.1}$$

The functions ψ and V. For $z \in \mathbb{R}^{Nn}$ we let

$$\psi(z) := |z|^2 + |z|^p \qquad \text{and} \qquad V(z) := (1+|z|^2)^{\frac{p-2}{4}} z. \tag{3.2}$$

Some inequalities

In the following we state some inequalities for $z_0, z, \xi \in \mathbb{R}^{Nn}$, where all the constants depend only on p. First, computing

$$D^2 e_p(z)(\xi,\xi) = (1+|z|^2)^{\frac{p-2}{2}}|\xi|^2 + (p-2)(1+|z|^2)^{\frac{p-4}{2}}(z\cdot\xi)^2$$

we find the following estimates for the p-energy

$$2^{\frac{2-p}{2}}(1+|z|)^{p-2}|\xi|^2 \le D^2 e_p(z)(\xi,\xi) \le (p-1)(1+|z|)^{p-2}|\xi|^2. \tag{3.3}$$

In addition, we recall the standard inequality (see for instance [19, Lemma 2.1])

$$\int_0^1 (1+|z_0+s(z-z_0)|)^{p-2}\,ds \ge c(1+|z_0|+|z|)^{p-2}. \tag{3.4}$$

By change of variables we deduce the following variant:

$$\int_{\frac{1}{2}}^1 (1+|z_0+s(z-z_0)|)^{p-2}\,ds \ge c(1+|z|)^{p-2}.$$

Furthermore, employing the last two inequalities we derive

$$\begin{aligned}
&\int_0^1\int_0^1 (1+|z_0+t(z-z_0)+st(\xi-z)|)^{p-2}\,ds\,t\,dt\\
&\ge c\int_0^1 (1+|z_0+t(z-z_0)|+|z_0+t(\xi-z_0)|)^{p-2}t\,dt\\
&\ge c\Big[\int_{\frac{1}{2}}^1 (1+|z_0+t(z-z_0)|)^{p-2}\,dt + \int_{\frac{1}{2}}^1 (1+|z_0+t(\xi-z_0)|)^{p-2}\,dt\Big]\\
&\ge c\big[(1+|z|)^{p-2}+(1+|\xi|)^{p-2}\big] \ge c(1+|z|+|\xi|)^{p-2}.
\end{aligned} \tag{3.5}$$

Finally, we record the following inequalities for the function V from (3.2) (see [19, Lemma 2.2]):

$$c(1+|z_0|+|z|)^{p-2}|z-z_0|^2 \leq |V(z)-V(z_0)|^2 \leq C(1+|z_0|+|z|)^{p-2}|z-z_0|^2. \quad (3.6)$$

In particular, we have

$$|z-z_0|^2 + |z-z_0|^p = \psi(z-z_0) \leq C|V(z)-V(z_0)|^2. \quad (3.7)$$

4. Asymptotic regularity

In this section we prove that asymptotic regularity of an integrand f is in fact equivalent with the existence of a regular g such that $f(z)$ equals $g(z)$ for large values of z; see Corollary 4.3. This characterization of asymptotic regularity will be crucial in the following sections since it enables us to work with a suitable regular comparison problem. We also establish analogous results for structure functions a in the case of systems; see Corollary 4.6.

Starting with a technical lemma we initially treat the case of integrands.

Lemma 4.1. *We consider a function $f : \mathbb{R}^m \to \mathbb{R}$ that is bounded from below. Moreover, we assume that there are positive constants γ and M such that f is C^2 outside $\overline{B_M}$ with*

$$D^2 f(z)(\xi,\xi) \geq \gamma |z|^{p-2}|\xi|^2 \qquad \text{for } |z| > M \text{ and } \xi \in \mathbb{R}^m.$$

Then, there is a positive constant $\widetilde{M}$ such that

$$f(z) = Cf(z) \qquad \text{holds for } |z| > \widetilde{M},$$

where Cf denotes the convex hull of f.

Proof. We assume $f \geq 0$ on $\mathbb{R}^m$ and recall the definition of the convex hull

$$Cf(z) := \sup\{h(z) : h \text{ is convex with } h \leq f \text{ on } \mathbb{R}^m\}. \quad (4.1)$$

Clearly, we have $Cf(z) \leq f(z)$. To prove the opposite inequality, we fix a large constant $\widetilde{M}$ with $\frac{1}{2}\gamma(M+1)^{p-2}(\widetilde{M}-M-1)^2 \geq \sup_{|\zeta|=M+1} f(\zeta)$. We claim that for $|z| > \widetilde{M}$ the affine function $h(\xi) = f(z) + Df(z)(\xi - z)$ is admissible in the definition of $Cf(z)$. To see this we fix a point $\xi \in \mathbb{R}^m$. If the line from ξ to z does not intersect $\overline{B_M}$ we clearly have $f(z) + Df(z)(\xi - z) \leq f(\xi)$ since f is convex along this line. If the line intersects $\overline{B_M}$, we denote by ξ_* the closest point to z in the intersection of the line with $\overline{B_{M+1}}$. Recalling $|z| > \widetilde{M} \geq M+1$ and $|\xi_*| = M+1$

we have

$$
\begin{aligned}
&f(z) + Df(z)(\xi_* - z)\\
&\qquad = f(\xi_*) - \int_0^1 [Df(z + t(\xi_* - z)) - Df(z)]\, dt\, (\xi_* - z)\\
&\qquad = f(\xi_*) - \int_0^1 \int_0^1 D^2 f(z + st(\xi_* - z))\, ds\, t\, dt\, (\xi_* - z, \xi_* - z)\\
&\qquad \le f(\xi_*) - \gamma \int_0^1 \int_0^1 |st\xi_* + (1 - st)z|^{p-2}\, ds\, t\, dt\, |\xi_* - z|^2\\
&\qquad \le f(\xi_*) - \frac{1}{2}\gamma (M + 1)^{p-2} |\xi_* - z|^2.
\end{aligned}
$$

Keeping in mind $|z| > \widetilde{M}$, $|\xi_*| = M + 1$ and the choice of $\widetilde{M}$, we deduce

$$
f(z) + Df(z)(\xi_* - z) \le 0. \tag{4.2}
$$

In particular, (4.2) implies $Df(z)(\xi_* - z) \le 0$ and noting $\xi - \xi_* = r(\xi_* - z)$ for some $r > 0$ we get $Df(z)(\xi - \xi_*) \le 0$. Combining this with (4.2) we finally arrive at $h(\xi) = f(z) + Df(z)(\xi - z) \le f(\xi)$ in any case. Thus, h is admissible as claimed and $Cf(z) \ge f(z)$ for $|z| > \widetilde{M}$. □

Theorem 4.2. *For a function $f : \mathbb{R}^m \to \mathbb{R}$ the following statements are equivalent:*

(i) *There are positive constants γ and M and a map $g \in C^2(\mathbb{R}^m)$ such that*

$$
f(z) = g(z) \qquad \textit{for } |z| > M
$$

and

$$
D^2 g(z)(\xi, \xi) \ge \gamma (1 + |z|)^{p-2} |\xi|^2 \qquad \textit{for all } z, \xi \in \mathbb{R}^m.
$$

(ii) *f is C^2 outside a large ball and there are a positive constant γ and a map $g \in C^2(\mathbb{R}^m)$ such that*

$$
\lim_{|z| \to \infty} \frac{|D^2 f(z) - D^2 g(z)|}{|z|^{p-2}} = 0
$$

and

$$
D^2 g(z)(\xi, \xi) \ge \gamma (1 + |z|)^{p-2} |\xi|^2 \qquad \textit{for all } z, \xi \in \mathbb{R}^m.
$$

(iii) *There are positive constants γ and M such that f is C^2 outside $\overline{B_M}$ and*

$$
D^2 f(z)(\xi, \xi) \ge \gamma |z|^{p-2} |\xi|^2 \qquad \textit{holds for } |z| > M \textit{ and all } \xi \in \mathbb{R}^m.
$$

Proof. Clearly, (i) implies (ii) and (ii) implies (iii).

Now we assume that f satisfies (iii) with constants γ and M. Clearly, we may take $M \geq 1$. It is not difficult to show that f is bounded from below on $\mathbb{R}^m \setminus B_{M+1}$. Since the above properties depend only on the values of f outside large balls, we may assume that f is bounded from below on $\mathbb{R}^m$. Letting

$$\tilde{f}(z) := f(z) - \frac{2^{1-p}}{p-1}\gamma e_p(z)$$

and recalling (3.1) and (3.3) we have for $|z| > M \geq 1$

$$D^2\tilde{f}(z)(\xi,\xi) \geq \gamma|z|^{p-2}|\xi|^2 - 2^{1-p}\gamma(1+|z|)^{p-2}|\xi|^2 \geq \frac{1}{2}\gamma|z|^{p-2}|\xi|^2.$$

Thus, for

$$f^* := C\tilde{f},$$

Lemma 4.1 implies the existence of a constant $\widetilde{M} > M$ with $f^*(z) = \tilde{f}(z)$ for $|z| > \widetilde{M}$. In particular, f^* is C^2 outside $\overline{B_{\widetilde{M}}}$ and we have

$$D^2 f^*(z)(\xi,\xi) = D^2\tilde{f}(z)(\xi,\xi) \geq \frac{1}{2}\gamma|z|^{p-2}|\xi|^2 \qquad \text{for } |z| > \widetilde{M}. \tag{4.3}$$

Mollifying f^* with smoothing radius $0 < \varepsilon < 1$ we obtain a C^2-function f^*_ε on $\mathbb{R}^m$. One checks that f^*_ε is again convex and satisfies

$$D^2 f^*_\varepsilon(z)(\xi,\xi) \geq \frac{1}{2}\gamma|z|^{p-2}|\xi|^2 \qquad \text{for } |z| \geq \widetilde{M}+1 > 2. \tag{4.4}$$

We choose a cut-off function $\varphi \in C^\infty(\mathbb{R}, [0,1])$ with $\varphi \equiv 1$ on $[0, \widetilde{M}+1]$, $\varphi \equiv 0$ on $[\widetilde{M}+2, \infty)$ and $\|\varphi\|_{C^2} \leq 8$. Furthermore, we define a C^2-function

$$\tilde{g}(z) := \varphi(|z|)\, f^*_\varepsilon(z) + (1-\varphi(|z|))\, f^*(z).$$

Then, we clearly have $D^2\tilde{g}(z)(\xi,\xi) \geq 0$ for $|z| < \widetilde{M}+1$ and for $|z| > \widetilde{M}+2$. Moreover, we compute for $\widetilde{M}+1 \leq |z| \leq \widetilde{M}+2$

$$\begin{aligned} D^2\tilde{g}(z)(\xi,\xi) &\geq \varphi(|z|)D^2 f^*_\varepsilon(z)(\xi,\xi) + (1-\varphi(|z|))D^2 f^*(z)(\xi,\xi) \\ &\quad - C\|f^*_\varepsilon - f^*\|_{C^1(B_{\widetilde{M}+2}\setminus B_{\widetilde{M}+1})}|\xi|^2 \\ &\geq \Big(c - C\|f^*_\varepsilon - f^*\|_{C^1(B_{\widetilde{M}+2}\setminus B_{\widetilde{M}+1})}\Big)|\xi|^2 \end{aligned}$$

by (4.3) and (4.4), where c and C are positive constants depending only on n, N, p, γ and $\widetilde{M}$. Choosing $\varepsilon > 0$ small enough, the last expression is positive and we conclude that $\tilde{g}$ is convex on $\mathbb{R}^m$ with $\tilde{g}(z) = \tilde{f}(z)$ for $|z| > \widetilde{M} + 2$. Finally, we define

$$g(z) := \tilde{g}(z) + \frac{2^{1-p}}{p-1}\gamma e_p(z),$$

and deduce from (3.3) that g has the properties from (i) with constants $\frac{2^{\frac{4-3p}{2}}}{p-1}\gamma$ and $\widetilde{M}+2$. □

As a special case of the preceding theorem we obtain the following characterization of asymptotic regularity:

Corollary 4.3. *A function $f : \mathbb{R}^m \to \mathbb{R}$ is asymptotically regular if and only if there exist a constant $M > 0$ and a regular $g : \mathbb{R}^m \to \mathbb{R}$ such that $f(z) = g(z)$ holds for $|z| > M$.*

We also record a slight refinement of Corollary 4.3, which will be convenient later:

Lemma 4.4. *Let $f : \mathbb{R}^m \to \mathbb{R}$ be locally bounded and asymptotically regular. Then there exist a constant $M > 0$ and a regular $g : \mathbb{R}^m \to \mathbb{R}$ such that we have $f(z) = g(z)$ for $|z| > M$ and additionally $f(z) \le g(z)$ for all $z \in \mathbb{R}^m$.*

Proof. By Corollary 4.3 there are a constant M^* and a regular g^* (say with constants γ and Γ) such that $f(z) = g^*(z)$ for $|z| > M^*$. Next, let $L := \sup_{\overline{B_{M^*}}}(f - g^*)$. If $L \le 0$ holds we are done. If L is positive we consider a smooth and compactly supported cut-off function $h : \mathbb{R}^m \to \mathbb{R}$ with $h \equiv L$ on $\overline{B_{M^*}}$ and $|D^2h| \le \frac{\gamma}{2}$ on $\mathbb{R}^m$. Then $g := g^* + h$ has the claimed properties. □

Next, using a quite different construction, we deal with the case of systems.

Theorem 4.5. *For $a : \mathbb{R}^m \to \mathbb{R}^m$ with*

$$\limsup_{|z|\to\infty} \frac{|a(z)|}{|z|^{p-1}} < \infty \tag{4.5}$$

the following statements are equivalent:

(i) *There are positive constants γ and M and a map $b \in C^1(\mathbb{R}^m, \mathbb{R}^m)$ such that*

$$a(z) = b(z) \qquad \text{for } |z| > M$$

and

$$Db(z)\xi \cdot \xi \ge \gamma(1 + |z|)^{p-2}|\xi|^2 \qquad \text{for all } z, \xi \in \mathbb{R}^m.$$

(ii) *The function a is C^1 outside a large ball and there are a positive constant γ and a map $b \in C^1(\mathbb{R}^m, \mathbb{R}^m)$ such that*

$$\lim_{|z|\to\infty} \frac{|Da(z) - Db(z)|}{|z|^{p-2}} = 0$$

and

$$Db(z)\xi \cdot \xi \geq \gamma(1+|z|)^{p-2}|\xi|^2 \qquad \text{for all } z, \xi \in \mathbb{R}^m.$$

(iii) *There are positive constants γ and M such that a is C^1 outside $\overline{B_M}$ and*

$$Da(z)\xi \cdot \xi \geq \gamma |z|^{p-2}|\xi|^2 \qquad \text{holds for } |z| > M \text{ and all } \xi \in \mathbb{R}^m.$$

Proof. Clearly, (i) implies (ii) and (ii) implies (iii). Now we assume that a satisfies (iii) with constants γ and M. In view of (4.5) — enlarging M if necessary — we may assume

$$\sup_{|\zeta|\geq M} \frac{|a(\zeta)|}{|\zeta|^{p-1}} < \infty.$$

For $\varepsilon > 0$ to be chosen later we consider a cut-off function $\tilde{\varphi} \in C^\infty(\mathbb{R}, [0,1])$ with $\tilde{\varphi} \equiv 1$ on $(-\infty, \log(M)]$, $\tilde{\varphi} \equiv 0$ on $[\log(M) + 1/\varepsilon, \infty)$ and $|\tilde{\varphi}'| \leq 2\varepsilon$ on $\mathbb{R}$. Setting $\varphi(t) := \tilde{\varphi}(\log(t))$ for $t > 0$ and $\varphi(0) := 1$ we have constructed a smooth function $\varphi : [0,\infty) \to [0,1]$ with the following properties:

$$\begin{gathered}
\varphi \equiv 1 \text{ on } [0, M], \\
\varphi \equiv 0 \text{ on } [M\exp(1/\varepsilon), \infty), \\
\varphi'(t) \leq \frac{2\varepsilon}{t} \text{ for all } t > 0.
\end{gathered}$$

We define

$$b(z) := \varphi(|z|) De_p(z) + (1 - \varphi(|z|))a(z),$$

where e_p is defined in (3.1). The function b is in $C^1(\mathbb{R}^m, \mathbb{R}^m)$ and (3.3) gives

$$Db(z)\xi \cdot \xi \geq 2^{\frac{2-p}{2}}(1+|z|)^{p-2}|\xi|^2$$

for $|z| < M$. In addition, for $|z| \geq M$ we have

$$\begin{aligned}
Db(z)\xi \cdot \xi &= \varphi(|z|) D^2 e_p(z)(\xi,\xi) + (1-\varphi(|z|))Da(z)\xi \cdot \xi \\
&\qquad + \varphi'(|z|)(1+|z|^2)^{\frac{p-2}{2}} \frac{(z\cdot\xi)^2}{|z|} - \varphi'(|z|)\frac{(z\cdot\xi)(a(z)\cdot\xi)}{|z|} \\
&\geq c(1+|z|)^{p-2}|\xi|^2 - C\varepsilon(1+|z|)^{p-2}\left[1 + \sup_{|\zeta|\geq M}\frac{|a(\zeta)|}{|\zeta|^{p-1}}\right]|\xi|^2
\end{aligned}$$

with positive constants c, C depending only on p, γ and M. Choosing ε small enough we end up with

$$Db(z)\xi \cdot \xi \geq c(1+|z|)^{p-2}|\xi|^2$$

for all $z, \xi \in \mathbb{R}^m$. Thus, (i) holds. □

Noting that asymptotic regularity of a implies (4.5) we deduce the following characterization:

Corollary 4.6. *A function $a : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is asymptotically regular if and only if there exist a constant $M > 0$ and a regular $b : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ such that $a(z) = b(z)$ holds for $|z| > M$.*

5. Comparison estimates

In this section we prove that solutions of the asymptotically regular problems (1.1) and (2.2) can be approximated, close to infinity, by solutions of the regular comparison problems (5.4) and (5.8). For the proofs we modify techniques of [14, 20].

We start with an auxiliary result.

Lemma 5.1. *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ and let $M > 0$. Then there is a constant $K_M > M$, depending only on M and p, such that for $K \geq K_M$, the estimate*

$$\fint_\Omega |Du|^p \, dx > K^p$$

implies

$$|\{y \in \Omega : |Du(y)| \leq M\}| \leq \frac{2^p}{K} \int_\Omega |Du - (Du)_\Omega|^p \, dx.$$

Proof. At first, observe that the assumption and Minkowski's inequality yield

$$K < \left(\fint_\Omega |Du|^p \, dx\right)^{1/p} \leq \left(\fint_\Omega |Du - (Du)_\Omega|^p \, dx\right)^{1/p} + |(Du)_\Omega|. \quad (5.1)$$

Now we choose $K_M > M$ large enough so that $K \geq K_M$ implies $K^{-1}(K-M)^p \geq 1$. From this and (5.1) we conclude for every $y \in \Omega$ with $|Du(y)| \leq M$

$$\begin{aligned} 1 &\leq \frac{1}{K}\big[K - |Du(y)|\big]^p \\ &\leq \frac{1}{K}\left[\left(\fint_\Omega |Du - (Du)_\Omega|^p \, dx\right)^{1/p} + |(Du)_\Omega| - |Du(y)|\right]^p \\ &\leq \frac{2^{p-1}}{K}\left[\fint_\Omega |Du - (Du)_\Omega|^p \, dx + |Du(y) - (Du)_\Omega|^p\right]. \end{aligned}$$

Integrating this inequality over the set $\{y \in \Omega : |Du(y)| \le M\}$, we infer the desired result

$$|\{y \in \Omega : |Du(y)| \le M\}| \le \frac{2^p}{K} \int_\Omega |Du - (Du)_\Omega|^p \, dx. \qquad \square$$

In view of Lemma 4.4 we impose the following hypotheses:

Assumption 5.2. $f : \mathbb{R}^{Nn} \to \mathbb{R}$ is a locally bounded Borel integrand and $g : \mathbb{R}^{Nn} \to \mathbb{R}$ is regular. Moreover, we have $f(z) = g(z)$ whenever $|z|$ is larger than some constant M, and there holds $f \le g$ on $\mathbb{R}^{Nn}$.

To fix notations let us record that these assumptions imply, in particular,

$$D^2 g(z)(\xi, \xi) \ge \gamma(1 + |z|)^{p-2}|\xi|^2, \tag{5.2}$$

$$|D^2 g(z)| \le \Gamma(1 + |z|)^{p-2},$$

$$0 \le g(z) - f(z) \le \Gamma_1 \tag{5.3}$$

for all $z, \xi \in \mathbb{R}^{Nn}$ with positive constants γ, Γ and Γ_1.

In this setting we may now introduce, for some ball $B_R(x_0) \subset \Omega$, the regular comparison problem

$$G[v] := \int_{B_R(x_0)} g(Dv) \, dx. \tag{5.4}$$

After these preparations we will now derive the comparison estimates for the case of integrals.

Lemma 5.3. *Let f and g be as in Assumption* 5.2*. Then for any $\varepsilon > 0$ there is a $K(\varepsilon) > M$ with the following property: For every minimizer $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ of F from* (1.1)*, every ball $B_R(x_0) \subset \Omega$ and every minimizer $v \in u + W_0^{1,p}(B_R(x_0), \mathbb{R}^N)$ of G from* (5.4)*, there holds either*

$$\fint_{B_R(x_0)} |Du|^p \, dx \le K^p(\varepsilon)$$

or

$$\fint_{B_R(x_0)} (1 + |Du| + |Dv|)^{p-2}|Du - Dv|^2 \, dx \le \varepsilon \fint_{B_R(x_0)} |Du - (Du)_{x_0,R}|^p \, dx.$$

Here, the constant $K(\varepsilon)$ depends only on the data p, γ, Γ_1, M and on ε.

Proof. In this proof, we use the notation $\{|Du| \le M\}$ for the set $\{y \in B_R(x_0) : |Du(y)| \le M\}$. Setting $w := \frac{1}{2}(u + v)$ we use the minimizing properties of u and v together with the hypotheses from Assumption 5.2 to derive

$$\begin{aligned} \int_{B_R(x_0)} g(Dv) + g(Du) - 2g(Dw) \, dx &\le \int_{B_R(x_0)} g(Du) - g(Dw) \, dx \\ &\le \int_{B_R(x_0)} g(Du) - f(Dw) \, dx \\ &\le \int_{\{|Du| \le M\}} g(Du) - f(Du) \, dx. \end{aligned}$$

The left-hand side can be estimated by the inequalities (5.2) and (3.5):

$$
\begin{aligned}
&g(Dv) + g(Du) - 2g(Dw) \\
&= \frac{1}{2}\int_0^1 \big[Dg(Dw + t(Dv-Dw)) - Dg(Dw + t(Du-Dw))\big]\,dt\,(Dv-Du) \\
&= \frac{1}{2}\int_0^1\int_0^1 D^2 g(Dw+t(Du-Dw)+st(Dv-Du))\,ds\,t\,dt\,(Dv-Du,\, Dv-Du) \\
&\geq \frac{\gamma}{2}\int_0^1\int_0^1 (1+|Dw+t(Du-Dw)+st(Dv-Du)|)^{p-2}\,ds\,t\,dt\,|Du-Dv|^2 \\
&\geq c\gamma(1+|Du|+|Dv|)^{p-2}|Du-Dv|^2
\end{aligned}
$$

for some positive constant c, depending only on p. Combining the last two estimates and recalling (5.3), we arrive at

$$
c\gamma \int_{B_R(x_0)} (1+|Du|+|Dv|)^{p-2}|Du-Dv|^2\,dx \leq \Gamma_1|\{|Du| \leq M\}|.
$$

Now by Lemma 5.1, for any $K \geq K_M$ the condition

$$
⨍_{B_R(x_0)} |Du|^p\,dx > K^p
$$

implies

$$
\int_{B_R(x_0)} (1+|Du|+|Dv|)^{p-2}|Du-Dv|^2\,dx \leq \frac{2^p\Gamma_1}{c\gamma K}\int_{B_R(x_0)} |Du-(Du)_{x_0,R}|^p\,dx.
$$

We have thus proven the lemma if we let $K(\varepsilon) := \max\left\{\frac{2^p\Gamma_1}{c\gamma\varepsilon},\, K_M\right\}$. ☐

Next, keeping Corollary 4.6 in mind, we proceed similarly in the case of systems.

Assumption 5.4. $a : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is a locally bounded Borel function and $b : \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is regular. Moreover, we have $a(z) = b(z)$ whenever $|z|$ is larger than some constant M. In particular, these assumptions imply

$$
Db(z)\xi\cdot\xi \geq \gamma(1+|z|)^{p-2}|\xi|^2, \tag{5.5}
$$

$$
|Db(z)| \leq \Gamma(1+|z|)^{p-2}, \tag{5.6}
$$

$$
|b(z)-a(z)| \leq \Gamma_1 \tag{5.7}
$$

for all $z, \xi \in \mathbb{R}^{Nn}$ with positive constants γ, Γ and Γ_1.

Here, introducing the regular system

$$
\operatorname{div} b(Dv) = 0 \qquad \text{on } B_R(x_0), \tag{5.8}
$$

the comparison estimate reads:

Lemma 5.5. *Let a and b be as in Assumption* 5.4. *Then for any* $\varepsilon > 0$ *there exists a constant* $K(\varepsilon) > M$ *with the following property: For every weak solution* $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ *of* (2.2)*, every ball* $B_R(x_0) \subset \Omega$ *and every weak solution* $v \in u + W_0^{1,p}(B_R(x_0), \mathbb{R}^N)$ *of* (5.8)*, there holds either*

$$\fint_{B_R(x_0)} |Du|^p \, dx \leq K^p(\varepsilon)$$

or

$$\fint_{B_R(x_0)} (1 + |Du| + |Dv|)^{p-2} |Du - Dv|^2 \, dx \leq \varepsilon \fint_{B_R(x_0)} |Du - (Du)_{x_0,R}|^p \, dx.$$

Here, the constant $K(\varepsilon)$ *depends only on the data* p, γ, Γ_1, M *and on* ε.

Proof. At first, using (5.5) and (3.4) we derive the pointwise estimate

$$\begin{aligned}
&b(Du) \cdot (Du - Dv) - b(Dv) \cdot (Du - Dv) \\
&\quad = \int_0^1 Db(Dv + t(Du - Dv)) \, dt \, (Du - Dv, Du - Dv) \\
&\quad \geq \gamma \int_0^1 (1 + |Dv + t(Du - Dv)|)^{p-2} \, dt \, |Du - Dv|^2 \\
&\quad \geq c\gamma (1 + |Du| + |Dv|)^{p-2} |Du - Dv|^2.
\end{aligned}$$

Since we have $u - v \in W_0^{1,p}(B_R(x_0), \mathbb{R}^N)$, this function is an admissible test function in the weak formulation of (5.8) and (2.2). Therefore, integrating the above inequality yields

$$\begin{aligned}
c\gamma &\int_{B_R(x_0)} (1 + |Du| + |Dv|)^{p-2} |Du - Dv|^2 \, dx \\
&\leq \int_{B_R(x_0)} b(Du) \cdot (Du - Dv) \, dx \\
&= \int_{B_R(x_0)} (b(Du) - a(Du)) \cdot (Du - Dv) \, dx \\
&\leq \Gamma_1 \int_{\{|Du| \leq M\}} |Du - Dv| \, dx.
\end{aligned}$$

Here, we used the bound (5.7) in the last step. Applying the Hölder inequality we arrive at

$$\begin{aligned}
&\int_{B_R(x_0)} (1 + |Du| + |Dv|)^{p-2} |Du - Dv|^2 \, dx \\
&\leq \frac{\Gamma_1}{c\gamma} |\{|Du| \leq M\}|^{1-\frac{1}{p}} \left(\int_{B_R(x_0)} |Du - Dv|^p \, dx \right)^{\frac{1}{p}} \\
&\leq \frac{\Gamma_1}{c\gamma} |\{|Du| \leq M\}|^{1-\frac{1}{p}} \left(\int_{B_R(x_0)} (1 + |Du| + |Dv|)^{p-2} |Du - Dv|^2 \, dx \right)^{\frac{1}{p}}.
\end{aligned}$$

The last estimate implies

$$\int_{B_R(x_0)} (1+|Du|+|Dv|)^{p-2}|Du-Dv|^2\,dx \le \left(\frac{\Gamma_1}{c\gamma}\right)^{\frac{p}{p-1}} |\{|Du| \le M\}|. \tag{5.9}$$

Now if we assume that for some $K \ge K_M$ there holds

$$\fint_{B_R(x_0)} |Du|^p\,dx > K^p$$

then the estimate (5.9) implies by Lemma 5.1

$$\begin{aligned}\int_{B_R(x_0)} &(1+|Du|+|Dv|)^{p-2}|Du-Dv|^2\,dx \\ &\le \left(\frac{\Gamma_1}{c\gamma}\right)^{\frac{p}{p-1}} \frac{2^p}{K} \int_{B_R(x_0)} |Du-(Du)_{x_0,R}|^p\,dx.\end{aligned}$$

Thus, choosing $K(\varepsilon) := \max\left\{\left(\frac{\Gamma_1}{c\gamma}\right)^{\frac{p}{p-1}}\frac{2^p}{\varepsilon}, K_M\right\}$ we have established the claim of the lemma. □

6. Partial regularity

6.1. Regular problems

We start with introducing the excess

$$\Psi_v(y,r) := \fint_{B_r(y)} \psi\big(Dv-(Dv)_{y,r}\big)\,dx \tag{6.1}$$

where ψ is defined in (3.2), and recall the following ε-regularity result for regular problems.

Theorem 6.1. *Let a regular structure function $b \in C^1(\mathbb{R}^{Nn}, \mathbb{R}^{Nn})$ be given, that is the properties* (5.5) *and* (5.6) *are satisfied. Then there is a constant C_0, depending only on n, N and $\frac{\Gamma}{\gamma}$, such that the following holds. For all $T > 0$ and $\tau \in (0,1)$ there exists a positive constant κ_0, depending on n, N, p, T, τ and b, such that for any ball $B_R(x_0) \subset \mathbb{R}^n$ and every weak solution $v \in W^{1,p}(B_R(x_0), \mathbb{R}^N)$ of the regular system* (5.8)*, the conditions*

$$\Psi_v(x_0,R) \le \kappa_0^2 \qquad \textit{and} \qquad |(Dv)_{x_0,R}| \le T$$

imply

$$\Psi_v(x_0,\tau R) \le C_0\tau^2\Psi_v(x_0,R).$$

The theorem follows from techniques in [25], where a more general case was considered; compare with [5, 6, 8] and [23, Chapter 9] for the case of integrals. Since the dependences of the constants as stated in the theorem are crucial for our proofs, we repeat the relevant arguments from [25] for the convenience of the reader.

Sketch of proof. We claim that the theorem holds with the constant $C_0 := C_1 + 1$, where $C_1 = C_1(n, N, \frac{\Gamma}{\gamma}) > 0$ is the corresponding constant for linear systems with constant coefficients, determined by (6.9). Assume that the theorem does not hold with C_0, some parameters $\tau \in (0, 1)$, $T > 0$ and a structure function b. Then there exist sequences of radii $r_m > 0$, of centers $x_m \in \mathbb{R}^n$ and of weak solutions $v_m \in W^{1,p}(B_{r_m}(x_m), \mathbb{R}^N)$ of

$$\operatorname{div} b(Dv_m) = 0$$

with

$$\lambda_m^2 := \Psi_{v_m}(x_m, r_m) \underset{m\to\infty}{\longrightarrow} 0 \quad \text{and} \quad |(Dv_m)_{x_m,r_m}| \le T \text{ for all } m \in \mathbb{N}, \tag{6.2}$$

but

$$\Psi_{v_m}(x_m, \tau r_m) > C_0 \tau^2 \lambda_m^2 \qquad \text{for all } m \in \mathbb{N}. \tag{6.3}$$

We rescale the solutions v_m as follows. With

$$\bar{v}_m := (v_m)_{x_m,r_m} \qquad \text{and} \qquad P_m := (Dv_m)_{x_m,r_m},$$

we let

$$w_m(x) := \frac{v_m(x_m + r_m x) - \bar{v}_m - r_m P_m x}{r_m \lambda_m}$$

for all $m \in \mathbb{N}$ and $x \in B_1$. The definition implies in particular

$$Dw_m(x) = \frac{Dv_m(x_m + r_m x) - P_m}{\lambda_m},$$
$$(w_m)_{0,1} = 0 \qquad \text{and} \qquad (Dw_m)_{0,1} = 0.$$

Thus, the conditions (6.2) and (6.3) become

$$\fint_{B_1} \big(|Dw_m|^2 + \lambda_m^{p-2}|Dw_m|^p\big)\, dx = 1, \quad \text{but} \tag{6.4}$$

$$\fint_{B_\tau} \big(|Dw_m - (Dw_m)_{0,\tau}|^2 + \lambda_m^{p-2}|Dw_m - (Dw_m)_{0,\tau}|^p\big)\, dx > C_0\tau^2. \tag{6.5}$$

From (6.4) we infer, possibly after extracting a subsequence, that there is a limit map $w \in W^{1,2}(B_1, R^N)$ with $w_m \rightharpoonup w$ weakly in $W^{1,2}(B_1, \mathbb{R}^N)$ and $w_m \to w$ strongly in $L^2(B_1, \mathbb{R}^N)$ as $m \to \infty$. Furthermore, since $\sup_m |P_m| \le T$, we may assume $P_m \to P_0$ as $m \to \infty$ for some $P_0 \in \mathbb{R}^{Nn}$. As in [25, Section 4], one can show furthermore

$$Dw_m \to Dw \quad \text{strongly in } L^2_{\mathrm{loc}}(B_1, \mathbb{R}^N) \text{ and} \tag{6.6}$$

$$\lambda_m^{(p-2)/p} Dw_m \to 0 \quad \text{strongly in } L^p_{\mathrm{loc}}(B_1, \mathbb{R}^N) \text{ in the case } p > 2. \tag{6.7}$$

Moreover, the reasoning of [25, Section 3] implies

$$\operatorname{div}(Db(P_0)Dw) \equiv 0 \qquad \text{on } B_1. \tag{6.8}$$

Here, the hypotheses on b imply the following properties of the constant structure function $Db(P_0)$.

$$\begin{aligned} |Db(P_0)| &\le \Gamma(1+|P_0|)^{p-2} && \text{and} \\ Db(P_0)\xi \cdot \xi &\ge \gamma(1+|P_0|)^{p-2} && \text{for all } \xi \in \mathbb{R}^{Nn}. \end{aligned}$$

This implies that the dispersion ratio of the linear system (6.8) is bounded by $\frac{\Gamma}{\gamma}$. We can thus apply linear theory (compare [17, Theorem III.2.1 and Remarks III.2.2, III.2.3]) with the result

$$⨍_{B_\tau} |Dw - (Dw)_{0,\tau}|^2\,dx \le C_1\tau^2 ⨍_{B_1} |Dw - (Dw)_{0,1}|^2\,dx < C_0\tau^2 \tag{6.9}$$

for a constant C_1 depending only on n, N and the dispersion ratio of $Db(P_0)$, that is on $\frac{\Gamma}{\gamma}$. The fact that C_1 does not depend on $|P_0|$ can alternatively be checked by a scaling argument. By the strong convergence (6.6) and (6.7), the decay estimate (6.9) clearly contradicts the assumption (6.5). This completes the proof. □

Remark 6.2. In particular, taking into account the Euler equation, the conclusion of Theorem 6.1 holds for minimizers v of regular integrals.

6.2. Asymptotically regular problems

This subsection is devoted to the proofs of Theorem 2.3 and Theorem 2.7. Throughout this section, we suppose that the Assumptions 5.2 and 5.4 are satisfied.

Theorem 6.3. *There is a constant $L = L(n, N, p, \gamma, \Gamma, \Gamma_1, M)$ such that the following holds. Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be either a minimizer of the functional F from* (1.1) *or a weak solution of* (2.2). *Then, in every Lebesgue point $x_0 \in \Omega$ of Du with*

$$\liminf_{r \searrow 0} ⨍_{B_r(x_0)} |Du - (Du)_{x_0,r}|^p\,dx = 0, \tag{6.10}$$

there holds either $|Du(x_0)| \le L$, or there is a neighborhood U of x_0 with $u \in C^{1,\alpha}(U, \mathbb{R}^N)$ for every $\alpha \in (0, 1)$.

Remark 6.4. We point out that the size of the neighborhood U and the $C^{1,\alpha}$-norm of u may not be controlled by the data.

The proof is based on the following decay estimate near infinity for the excess Ψ_u defined in (6.1).

Lemma 6.5. *For every $\alpha \in (0,1)$ there are constants $\tau \in (0,1)$, and K_0, and for every $T > 0$ there is a constant $\kappa_T \in (0,1)$, such that the following holds. Every minimizer $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ of F and every solution $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ of* (2.2) *with*

$$\Psi_u(x_0, R) \leq \kappa_T^2 \qquad \text{and} \qquad K_0 < \fint_{B_R(x_0)} |Du|\, dx \leq T$$

for some ball $B_R(x_0) \subset \Omega$ satisfies

$$\Psi_u(x_0, \tau R) \leq \tau^{2\alpha} \Psi_u(x_0, R).$$

Here, the dependences are given by $\tau(n, N, p, \frac{\Gamma}{\gamma}, \alpha)$, $K_0(n, N, p, \gamma, \Gamma, \Gamma_1, M, \alpha)$ and $\kappa_T(T, n, N, p, \frac{\Gamma}{\gamma}, \alpha, b$ respectively $g)$.

Proof. For a given $\alpha \in (0,1)$, we fix constants $\tau \in (0,1)$ and $\varepsilon \in (0, \frac{1}{2})$ small enough to ensure

$$16^p C_0 \tau^2 \leq \tau^{2\alpha} \qquad \text{and} \qquad 4^p \varepsilon \leq \frac{1}{2} \tau^{n+2\alpha}, \tag{6.11}$$

with the constant $C_0 = C_0\big(n, N, \frac{\Gamma}{\gamma}\big)$ from Theorem 6.1. Accordingly, the above choices depend only on $n, N, p, \frac{\Gamma}{\gamma}$ and α. With these choices of ε and τ and a given constant $T > 0$, we claim that Lemma 6.5 holds with $K_0 := K(\varepsilon)$ and $\kappa_T := 4^{-p/2}\kappa_0$, where the constants $K(\varepsilon)$ and κ_0 are given by Lemma 5.3, Lemma 5.5 and Theorem 6.1, respectively.

In the case of minimizers, we employ the direct method to choose the comparison map $v \in W^{1,p}(B_R(x_0), \mathbb{R}^N)$ as the minimizer of the regular functional G from (5.4) in the Dirichlet class $u + W_0^{1,p}(B_R(x_0), \mathbb{R}^N)$. Similarly, in the case of systems, we choose a solution $v \in u + W_0^{1,p}(B_R(x_0), \mathbb{R}^N)$ of the regular system (5.8). Since b is regular, such a solution can be constructed by Galerkin's method; see *e.g.* [40, Chapter 26]. Note that by assumption,

$$\left(\fint_{B_R(x_0)} \big|Du\big|^p\, dx \right)^{1/p} \geq \fint_{B_R(x_0)} |Du|\, dx > K_0.$$

By the choice of K_0, we may thus apply Lemma 5.3 and Lemma 5.5, respectively, deducing

$$\fint_{B_R(x_0)} \psi(Du - Dv)\, dx \leq 2\varepsilon \fint_{B_R(x_0)} \psi\big(Du - (Du)_{x_0,R}\big)\, dx. \tag{6.12}$$

Combining this with the assumptions on u, we infer

$$\begin{aligned}
\Psi_v(x_0, R) &= \fint_{B_R(x_0)} \psi\big(Dv - (Dv)_{x_0,R}\big)\,dx \\
&\le 2^p \fint_{B_R(x_0)} \psi(Dv - (Du)_{x_0,R})\,dx \\
&\le 2^{2p-1} \fint_{B_R(x_0)} \psi\big(Du - (Du)_{x_0,R}\big)dx + 2^{2p-1} \fint_{B_R(x_0)} \psi(Du - Dv)\,dx \\
&\le 2^{2p-1}(1+2\varepsilon) \fint_{B_R(x_0)} \psi\big(Du - (Du)_{x_0,R}\big)\,dx \\
&\le 4^p\, \Psi_u(x_0, R) \le 4^p \kappa_T^2 \le \kappa_0^2
\end{aligned} \tag{6.13}$$

by the choice of κ_T. Keeping in mind that

$$\fint_{B_R(x_0)} |Du|\,dx \le T,$$

Theorem 6.1 and Remark 6.2 yield the following excess estimate for the comparison map v.

$$\fint_{B_{\tau R}(x_0)} \psi\big(Dv - (Dv)_{x_0,\tau R}\big)\,dx \le C_0 \tau^2 \fint_{B_R(x_0)} \psi\big(Dv - (Dv)_{x_0,R}\big)\,dx$$

for the value of τ chosen above and a constant $C_0 = C_0\big(n, N, \frac{\Gamma}{\gamma}\big)$. This implies

$$\begin{aligned}
&\Psi_u(x_0, \tau R) \\
&\le 2^p \fint_{B_{\tau R}(x_0)} \psi\big(Du - (Dv)_{x_0,\tau R}\big)\,dx \\
&\le 2^{2p-1} \fint_{B_{\tau R}(x_0)} \psi\big(Dv - (Dv)_{x_0,\tau R}\big)\,dx + 2^{2p-1} \fint_{B_{\tau R}(x_0)} \psi(Du - Dv)\,dx \\
&\le 2^{2p-1} C_0 \tau^2 \fint_{B_R(x_0)} \psi\big(Dv - (Dv)_{x_0,R}\big)\,dx + 2^{2p-1} \tau^{-n} \fint_{B_R(x_0)} \psi(Du - Dv)\,dx \\
&\le \big(2^{4p-1} C_0 \tau^2 + 2^{2p} \varepsilon \tau^{-n}\big) \Psi_u(x_0, R) \\
&\le \tau^{2\alpha} \Psi_u(x_0, R),
\end{aligned}$$

where we first employed (6.13) and (6.12), and then the choice of τ and ε according to (6.11). □

Proof of Theorem 6.3. We may fix an arbitrary $\alpha \in (0, 1)$, for simplicity we let $\alpha := \frac{1}{2}$. This choice determines the constants K_0 and τ from the preceding lemma. Next we let $L := K_0 + 2$, which depends only on $n, N, p, \gamma, \Gamma, \Gamma_1$ and M, and fix a Lebesgue point $x_0 \in \Omega$ with (6.10). For the value $T := |Du(x_0)| + 3$, we choose the constant κ_T according to the preceding lemma and let $\varepsilon_0 := \frac{1}{2}\tau^n(1-\sqrt{\tau})\kappa_T < \frac{1}{2}\kappa_T$. We start by choosing a good radius. In the first step, we let

$$R_1 := \inf\left\{ r \in (0, \operatorname{dist}(x_0, \partial\Omega)) \, : \, \left| \fint_{B_r(x_0)} Du\, dx \right| \le K_0 + 2 \right\},$$

which is to be interpreted as $R_1 := \operatorname{dist}(x_0, \partial\Omega)$ if the above set is empty. If $R_1 = 0$, we deduce $|Du(x_0)| \le K_0 + 2 = L$, since x_0 is a Lebesgue point. Thus, in this case the first alternative of the theorem holds. Now we consider the remaining case $R_1 > 0$. We choose $R_0 \in (0, R_1]$ small enough that

$$\fint_{B_{R_0}(x_0)} \psi\big(Du - (Du)_{x_0,R_0}\big)\, dx < \varepsilon_0^2 \quad \text{and} \quad \fint_{B_{R_0}(x_0)} |Du|\, dx \le T - 2.$$

Note that by the choice of R_1 and $0 < R_0 \le R_1$, we have furthermore

$$\fint_{B_{R_0}(x_0)} |Du|\, dx \ge |(Du)_{x_0,R_0}| \ge K_0 + 2.$$

By the absolute continuity of the integral, we can choose a neighborhood U of x_0 in such a way that for all $y \in U$,

$$\Psi_u(y, R_0) = \fint_{B_{R_0}(y)} \psi\big(Du - (Du)_{y,R_0}\big)\, dx \le 4\varepsilon_0^2 < \kappa_T^2 \qquad \text{and}$$

$$K_0 + 1 \le |(Du)_{y,R_0}| \le \fint_{B_{R_0}(y)} |Du|\, dx \le T - 1.$$

Thus, we are in the situation of Lemma 6.5, which yields with the value of $\tau \in (0, 1)$ fixed above and $\alpha = \frac{1}{2}$

$$\Psi_u(y, \tau R_0) \le \tau\, \Psi_u(y, R_0) \le 4\varepsilon_0^2 < \kappa_T^2.$$

Furthermore we estimate

$$\begin{aligned}
\left| \fint_{B_{\tau R_0}(y)} Du\, dx \right| &\ge |(Du)_{y,R_0}| - \fint_{B_{\tau R_0}(y)} |Du - (Du)_{y,R_0}|\, dx \\
&\ge K_0 + 1 - \tau^{-n} \left(\fint_{B_{R_0}(y)} \psi\big(Du - (Du)_{y,R_0}\big)\, dx \right)^{1/2} \\
&\ge K_0 + 1 - \frac{2\varepsilon_0}{\tau^n} \ge K_0 + 1 - \kappa_T > K_0,
\end{aligned}$$

where we used the Cauchy-Schwarz inequality in the second estimate. Similarly, we estimate the mean value from above.

$$\begin{aligned}
\fint_{B_{\tau R_0}(y)} |Du|\, dx &\le |(Du)_{y,R_0}| + \fint_{B_{\tau R_0}(y)} |Du - (Du)_{y,R_0}|\, dx \\
&\le T - 1 + \frac{2\varepsilon_0}{\tau^n} \le T - 1 + \kappa_T \le T.
\end{aligned}$$

Consequently, we can apply Lemma 6.5 again on the ball $B_{\tau R_0}(y)$. In this manner, we establish successively the following estimates on the balls with radii $r_k := \tau^k R_0$.

$$\fint_{B_{r_k}(y)} \psi\big(Du - (Du)_{y,r_k}\big)\,dx \le 4\tau^k \varepsilon_0^2 \qquad \text{for } k \in \mathbb{N} \tag{6.14}$$

and

$$\begin{aligned}\left|\fint_{B_{r_k}(y)} Du\,dx\right| &\ge K_0 + 1 - \frac{2\varepsilon_0}{\tau^n}\sum_{\ell=0}^{k-1}\tau^{\ell/2}\\ &> K_0 + 1 - \frac{2\varepsilon_0}{\tau^n(1-\sqrt{\tau})} = K_0 + 1 - \kappa_T > K_0,\end{aligned} \tag{6.15}$$

as well as

$$\begin{aligned}\fint_{B_{r_k}(y)} |Du|\,dx &\le T - 1 + \frac{2\varepsilon_0}{\tau^n}\sum_{\ell=0}^{k-1}\tau^{\ell/2}\\ &\le T - 1 + \frac{2\varepsilon_0}{\tau^n(1-\sqrt{\tau})} = T - 1 + \kappa_T \le T\end{aligned}$$

for all $k \in \mathbb{N}$. From the estimate (6.14) we infer that for every $y \in U$ and $\rho \le R_0$,

$$\fint_{B_\rho(y)} |Du - (Du)_{y,\rho}|^2\,dx \le \fint_{B_\rho(y)} \psi\big(Du - (Du)_{y,\rho}\big)\,dx \le C\varepsilon_0^2 \frac{\rho}{R_0}$$

for some constant $C > 0$. This implies $u \in C^{1,1/2}(U, \mathbb{R}^N)$ by Morrey's lemma. From (6.15) we infer furthermore $|Du| > K_0 > M$ on U, so that u is actually a solution of the regular system

$$\operatorname{div} Dg(Du) \equiv 0 \quad \text{respectively} \quad \operatorname{div} b(Du) \equiv 0 \qquad \text{on } U.$$

Classical regularity theory[1] thus implies $u \in C^{1,\alpha}(U, \mathbb{R}^N)$ for every $\alpha \in (0, 1)$. The proof is complete. □

Proof of Theorems 2.3 *and* 2.7. By Lemma 4.4 and Corollary 4.6 we can assume that the Assumptions 5.2 and 5.4 are satisfied, so that Theorem 6.3 is applicable. For a map $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ as in the theorems to be proved, we define $\Sigma := \Sigma_1 \cup \Sigma_p$, where Σ_1 denotes the set of non-Lebesgue points of Du and

$$\Sigma_p := \left\{x_0 \in \Omega \,:\, \liminf_{r \searrow 0} \fint_{B_r(x_0)} |Du - (Du)_{x_0,r}|^p\,dx > 0\right\}.$$

[1] For instance one might use Theorem 6.1 combined with a standard iteration argument.

This set satisfies $|\Sigma| = 0$, see *e.g.* [18, Theorem 6.13]. With the above choice of Σ, Theorem 6.3 yields the claimed decomposition

$$\Omega = \Sigma \cup B_L \cup H,$$

where $|Du| \le L$ on B_L for the constant L determined in Theorem 6.3, and H is open with $u \in C^{1,\alpha}_{\mathrm{loc}}(H, \mathbb{R}^N)$ for every $\alpha \in (0, 1)$. □

Proof of Theorem 2.5. Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a relaxed minimizer of F, that is a minimizer of the functional

$$QF[u] := \int_\Omega Qf(Du)\, dx,$$

where f is a locally bounded Borel integrand that is asymptotically regular in the sense of Definition 2.2 and Qf denotes its quasiconvex hull. From Lemma 4.1 we infer that $Qf(z) = f(z)$ for $|z| \gg 1$, so that Qf is itself asymptotically regular. Theorem 2.5 is thus a special case of Theorem 2.3. □

7. Everywhere regularity

In this section we turn our attention to the special cases $n = 2$ and $N = 1$, for which we will prove everywhere regularity as stated in Theorem 2.8 and Theorem 2.9. For this purpose, it is more convenient to choose a slightly different excess than in the preceding section, namely

$$\Phi_v(y, r) := ⨍_{B_r(y)} |V(Dv) - V(Dv)_{y,r}|^2\, dx, \tag{7.1}$$

where V is defined in (3.2). Here, we abbreviated $V(Dv)_{y,r} := [V(Dv)]_{y,r}$.

We start with excess estimates for the regular case.

Theorem 7.1 (Excess estimate for $n = 2$). *Let $n = 2$. There is a number $\beta > 0$ such that every weak solution $v \in W^{1,p}(B_R(x_0), \mathbb{R}^N)$ of the regular system* (5.8) *on a ball $B_R(x_0) \subset \mathbb{R}^2$ satisfies*

$$\Phi_v(x_0, \rho) \le C \left(\frac{\rho}{R}\right)^{2\beta} \Phi_v(x_0, R) \tag{7.2}$$

for all $\rho \in (0, R]$. Here, the constants β and C depend only on p, γ and Γ.

This estimate is a consequence of $V(Dv) \in W^{1,2+\kappa}_{\mathrm{loc}}(B_R(x_0), \mathbb{R}^N)$ for some $\kappa > 0$, which follows from Gehring's higher integrability lemma; see [3, Theorem 1.V]. For an elementary proof of Theorem 7.1 that avoids Gehring's lemma, we refer to [36, Lemma 8.2].

Theorem 7.2 (Excess estimate for N= 1). *Let $N = 1$. Theorem 7.1 holds analogously — with constants depending additionally on n and for $p > 2$ also on ν — for solutions $v \in W^{1,p}(B_R(x_0))$ of regular equations of the form (5.8), if one assumes in the case $p > 2$ additionally the existence of a continuous function $\nu : [0,\infty) \to [0,\infty)$ with $\nu(0) = 0$ and*

$$|Db(z_2) - Db(z_1)| \le (1+|z_1|+|z_2|)^{p-2}\nu\big(|z_2 - z_1|^2\big) \quad \text{for all } z_1, z_2 \in \mathbb{R}^n. \tag{7.3}$$

Proof. We begin with the case $p = 2$. The estimate (7.2) holds trivially for $\rho \in [\frac{R}{2}, R]$, so we need to consider only the case $\rho < \frac{R}{2}$. By the difference quotient method, one checks that the partial derivatives satisfy $\partial_k v \in W^{1,2}(B_{R/2}(x_0))$ and solve the linear equation

$$\int_{B_{R/2}(x_0)} Db(Dv)D\partial_k v \cdot D\varphi \, dx = 0 \qquad \text{for all } \varphi \in W_0^{1,2}(B_{R/2}(x_0)),$$

for any $1 \le k \le n$. Since $p = 2$, the assumptions (5.5) and (5.6) imply that the coefficients $Db(Dv)$ are in $L^\infty(B_{R/2}(x_0))$ and are uniformly elliptic. Therefore, the claim (7.2) is a consequence of the De Giorgi-Nash-Moser theorem, see *e.g.* [18, Section 8.3] or [23, Chapter 7.3].

In the case $p > 2$, under the additional assumption (7.3), the claim can be established following the proof of [20, Theorem 6.2]. ☐

Remark 7.3. As in Remark 6.2 we note that Theorem 7.1 and Theorem 7.2 hold analogously for minimizers v of regular integrals.

For asymptotically regular problems, we have the following excess estimate close to infinity.

Lemma 7.4. *Let $n = 2$ or $N = 1$. There is an exponent $\alpha > 0$, depending only on n, p, γ and Γ, and there are numbers $\tau \in (0,1)$ and $K_0 > 0$, depending additionally on Γ_1 and M, with the following property. Let $u \in W^{1,p}(B_R(x_0), \mathbb{R}^N)$ be a minimizer of F from (1.1) or a solution of (2.2) under the Assumptions 5.2 and 5.4, respectively. Additionally, in the case $N = 1$, $p > 2$ suppose that (2.3) holds. Then we have either*

$$\Phi_u(x_0, \tau R) \le \tau^{2\alpha}\Phi_u(x_0, R) \qquad \text{or} \qquad \fint_{B_R(x_0)} |Du|^p \, dx \le K_0^p.$$

Once more, for $N = 1$, $p > 2$ all the constants depend additionally on ν.

Proof. As in the proof of Lemma 6.5, we choose the comparison map $v \in u + W_0^{1,p}(B_R(x_0), \mathbb{R}^N)$ as a minimizer of the regular functional G from (5.4) or as a solution of the regular system (5.8), respectively. Since we have assumed $n = 2$

or $N = 1$, Theorem 7.1, Theorem 7.2[2] and Remark 7.3 yield the following decay estimate for the comparison map.

$$\fint_{B_{\tau R}(x_0)} |V(Dv)-V(Dv)_{x_0,\tau R}|^2 dx \le C\tau^{2\beta} \fint_{B_R(x_0)} |V(Dv)-V(Dv)_{x_0,R}|^2 dx \quad (7.4)$$

for all $\tau \in (0,1)$ and some constants $\beta \in (0,1)$ and $C > 0$, both depending only on n, p, γ and Γ (and for $N = 1$, $p > 2$ on ν). For an $\varepsilon \in (0,1)$ to be fixed later, we let $K_0 := K(\varepsilon)$ with the constant $K(\varepsilon)$ from Lemma 5.3 or 5.5, respectively. If the second alternative of the lemma does not hold with the constant K_0, that is if

$$\fint_{B_R(x_0)} |Du|^p \, dx > K_0^p,$$

then we infer from Lemma 5.3 or 5.5, respectively, combined with the inequalities (3.6) and (3.7), that

$$\fint_{B_R(x_0)} |V(Du) - V(Dv)|^2 \, dx \le C\varepsilon \fint_{B_R(x_0)} |V(Du) - V(Du)_{x_0,R}|^2 \, dx \quad (7.5)$$

with a constant C depending only on p. Combining the excess estimate (7.4) and (7.5), we arrive at

$$\begin{aligned}
&\Phi_u(x_0, \tau R) \\
&\le 2\left[\fint_{B_{\tau R}(x_0)} |V(Dv) - V(Dv)_{x_0,\tau R}|^2 \, dx + \fint_{B_{\tau R}(x_0)} |V(Du) - V(Dv)|^2 \, dx\right] \\
&\le C\left[\tau^{2\beta} \fint_{B_R(x_0)} |V(Dv) - V(Dv)_{x_0,R}|^2 \, dx + \fint_{B_{\tau R}(x_0)} |V(Du) - V(Dv)|^2 \, dx\right] \\
&\le C\left[\tau^{2\beta} \fint_{B_R(x_0)} |V(Du) - V(Du)_{x_0,R}|^2 \, dx + \tau^{-n} \fint_{B_R(x_0)} |V(Du) - V(Dv)|^2 \, dx\right] \\
&\le C\left(\tau^{2\beta} + \varepsilon\tau^{-n}\right) \Phi_u(x_0, R).
\end{aligned}$$

This implies the first alternative of the lemma if we fix any $\alpha \in (0,\beta)$ and choose first $\tau \in (0,1)$ and then $\varepsilon \in (0,1)$ sufficiently small. □

Proof of Theorem 2.8 *and Theorem* 2.9. As in the proof of Theorem 2.5, we observe that the case of relaxed minimizers is a special case of minimizers of an

[2] In the case $N = 1$, $p > 2$ we additionally need to check that (7.3) holds. However, enlarging ν if necessary this follows from (2.3) and the fact that Db and $D^2 g$, respectively, are uniformly continuous on bounded sets.

asymptotically regular functional. Therefore, we consider from now on a map $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ that is either a minimizer of F or a solution of (2.2), under the assumptions of Theorem 2.8 or Theorem 2.9. As above, in view of Section 4 we may suppose that Assumptions 5.2 and 5.4 hold.

We fix a Lebesgue point $x_0 \in B$ of Du and define, with the constant K_0 from the preceding lemma,

$$R := \inf\left\{r \in (0, \operatorname{dist}(x_0, \partial\Omega)) : \fint_{B_r(x_0)} |Du|^p\,dx \le K_0^p + 1\right\}.$$

If the above set is empty, we simply let $R := \operatorname{dist}(x_0, \partial\Omega)$. In the case $R = 0$, we readily deduce the desired bound $|Du(x_0)|^p \le K_0^p + 1$. Thus, we consider from now on the case $R > 0$. First of all we deduce the estimate

$$\fint_{B_R(x_0)} |Du|^p\,dx \le \max\left(K_0^p + 1,\ \frac{\|Du\|^p_{L^p(\Omega)}}{\operatorname{dist}(x_0, \partial\Omega)^n}\right) \tag{7.6}$$

by distinguishing the cases $R < \operatorname{dist}(x_0, \partial\Omega)$ and $R = \operatorname{dist}(x_0, \partial\Omega)$ and using the absolute continuity of the integral in the first case. We choose the constants $\alpha, \tau \in (0,1)$ according to the preceding lemma and let $r_k := \tau^k R$ for $k \in \mathbb{N}_0$. Keeping in mind that x_0 is a Lebesgue point, we estimate

$$\begin{aligned}
&|Du(x_0)| \\
&\le |(Du)_{x_0,R}| + \sum_{k=0}^{\infty} |(Du)_{x_0,r_{k+1}} - (Du)_{x_0,r_k}| \\
&\le \fint_{B_R(x_0)} |Du|\,dx + \sum_{k=0}^{\infty} \fint_{B_{r_{k+1}}(x_0)} |Du - (Du)_{x_0,r_k}|\,dx \\
&\le \left(\fint_{B_R(x_0)} |Du|^2\,dx\right)^{1/2} + \frac{C}{\tau^n} \sum_{k=0}^{\infty} \left(\fint_{B_{r_k}(x_0)} |V(Du) - V(Du)_{x_0,r_k}|^2\,dx\right)^{1/2} \\
&= \left(\fint_{B_R(x_0)} |Du|^2\,dx\right)^{1/2} + \frac{C}{\tau^n} \sum_{k=0}^{\infty} \Phi_u^{1/2}(x_0, r_k).
\end{aligned}$$

Here, we used the Cauchy-Schwarz inequality and (3.7) in the third step. By the choice of R and since $r_k \le R$, the first alternative of Lemma 7.4 holds on all balls $B_{r_k}(x_0)$, which implies

$$\Phi_u(x_0, r_k) \le \tau^{2\alpha k} \Phi_u(x_0, R) \le \tau^{2\alpha k} \fint_{B_R(x_0)} \big(1 + |Du|\big)^p\,dx.$$

Plugging this into the above estimate, we arrive at

$$|Du(x_0)| \le \left[1 + \frac{C}{\tau^n} \sum_{k=0}^{\infty} \tau^{\alpha k}\right] \left(\fint_{B_R(x_0)} \big(1 + |Du|^p\big)\,dx\right)^{1/2},$$

which implies by (7.6)

$$|Du(x_0)|^2 \le C(n, p, \gamma, \Gamma) \max\left(K_0^p + 2, \frac{\|Du\|^p_{L^p(\Omega)}}{\operatorname{dist}(x_0, \partial\Omega)^n} + 1 \right).$$

The proof is complete. □

8. Irregularity

Our construction of the counterexample in Theorem 2.6 is based on a recent interesting result of Sverak & Yan [39], which we restate next.

8.1. A counterexample of Sverak & Yan, revisited

For $\varepsilon > 0$ consider the map w_ε on the unit ball B_1 in $\mathbb{R}^n$ with values in the space of $n \times n$ matrices, defined by

$$w_\varepsilon(x) := \frac{x \otimes x}{|x|^{1+\varepsilon}} - \frac{1}{n}|x|^{1-\varepsilon} E_n, \tag{8.1}$$

where E_n denotes the $n \times n$ unit matrix. Clearly, w_ε is homogeneous of degree $1-\varepsilon$ and has its values in the symmetric and trace-free matrices. Thus, w_ε can be regarded as an element of $W^{1,q}(B_1, \mathbb{R}^N)$ for $N = \frac{1}{2}n(n+1) - 1$ and every $q < \frac{n}{\varepsilon}$.

Theorem 8.1 (Sverak & Yan [39]). *Let* $N = \frac{1}{2}n(n+1) - 1$*. There is a quadratic null Lagrangian* $\ell : \mathbb{R}^{Nn} \to \mathbb{R}$ *and for every*

$$0 < \varepsilon < \frac{1}{2}\left[\sqrt{3(n+1)(n-1)} - (n+1)\right] \tag{8.2}$$

a smooth integrand $g_\varepsilon : \mathbb{R}^{Nn} \to \mathbb{R}$ *with*

$$\gamma|\xi|^2 \le D^2 g_\varepsilon(z)(\xi, \xi) \le \Gamma|\xi|^2$$

for all $z, \xi \in \mathbb{R}^{Nn}$ *such that*

$$Dg_\varepsilon(Dw_\varepsilon(x)) = D\ell(Dw_\varepsilon(x)) \tag{8.3}$$

holds for every $0 \ne x \in B_1$.

In the situation of the theorem we deduce the Euler equation

$$\int_{B_1} Dg_\varepsilon(Dw_\varepsilon) D\varphi \, dx = \int_{B_1} D\ell(Dw_\varepsilon) D\varphi \, dx = 0$$

for every $\varphi \in W_0^{1,2}(B_1, \mathbb{R}^N)$. From the strict convexity of g_ε we then infer that w_ε is a minimizer leading to the following remark which in fact highlights the main feature of Theorem 8.1.

Remark 8.2. Since the right-hand side of (8.2) is positive for every $n \geq 3$, Theorem 8.1 implies the existence of a non-Lipschitz minimizer of the regular variational integral

$$\int_{B_1} g_\varepsilon(Du)\,dx.$$

Furthermore, for every $n \geq 5$ it is possible to choose $\varepsilon > 1$, which yields unbounded minimizers.

Remark 8.3. By an elementary computation, the condition (8.2) turns out to be equivalent with the condition

$$0 < \varepsilon < \frac{n+1-\sqrt{\frac{3(n+1)}{n-1}}}{\sqrt{\frac{3(n+1)}{n-1}}+1}$$

appearing originally in [39]. Moreover, the reader should note that (8.2) implies $\varepsilon < \frac{n}{2}$ and thus $w_\varepsilon \in W^{1,2}(B_1, \mathbb{R}^N)$.

For our purposes, we will need the following refinement of Theorem 8.1.

Theorem 8.4. *The smooth integrand g_ε and the null Lagrangian ℓ from Theorem 8.1 can be chosen in such a way that there holds*

$$\ell(z) \leq g_\varepsilon(z) \qquad \text{for all } z \in \mathbb{R}^{Nn} \tag{8.4}$$

and

$$\ell(Dw_\varepsilon(x)) = g_\varepsilon(Dw_\varepsilon(x)) \qquad \text{for all } 0 \neq x \in B_1, \tag{8.5}$$

where w_ε is defined by (8.1).

The proof of Theorem 8.4 is based on the same constructions used by Sverak & Yan [39, Section 3] for the proof of Theorem 8.1; compare also [38]. Before implementing the details let us fix some notation:

As already said above, the map w_ε defined in (8.1) takes its values in the space of symmetric and trace-free matrices, which we interpreted as elements of $\mathbb{R}^N$. Similarly, we can identify $\mathbb{R}^{Nn}$ with the space

$$T := \{A \in (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \otimes (\mathbb{R}^n)^* \,:\, A_{ijk} = A_{jik},\ \textstyle\sum_{i=1}^n A_{iik} = 0\},$$

where we used the abbreviation $A_{ijk} := A(e_i, e_j, e_k)$. Furthermore, we employ the notation

$$K_1^\varepsilon := \{Dw_\varepsilon(x) \,:\, x \in S^{n-1}\}.$$

As in [39] we decompose $T = T' \oplus T_3$, where T' denotes the subspace of trace-free tensors and T_3 its orthogonal complement. Here, the trace is to be taken with respect to the last two components. The space T' is decomposed further into the space T_1 of the symmetric tensors and its orthogonal complement T_2, so that $T = T_1 \oplus T_2 \oplus T_3$. It is shown in [39] that the quadratic function

$$\ell(z_1 + z_2 + z_3) := -2|z_1|^2 + |z_2|^2 + n|z_3|^2 \qquad \text{for } z_i \in T_i,\ i = 1, 2, 3$$

is a null Lagrangian. One checks that the derivative of w_ε in a point $x \in S^{n-1}$ can be written as $Dw_\varepsilon(x) = W^{(1)}(x) + W^{(2)}(x) + W^{(3)}(x)$ with $W^{(i)}(x) \in T_i$ for $i = 1, 2, 3$, where $W^{(2)} \equiv 0$ and

$$\begin{aligned} W^{(1)}_{ijk}(x) &= (1+\varepsilon)\left(-x_i x_j x_k + \frac{1}{n+2}(x_i\delta_{jk} + x_j\delta_{ik} + x_k\delta_{ij})\right), \\ W^{(3)}_{ijk}(x) &= \frac{n+1-\varepsilon}{n+2}\left(x_i\delta_{jk} + x_j\delta_{ik} - \frac{2}{n}\delta_{ij}x_k\right) \end{aligned}$$

for $1 \le i, j, k \le n$. Consequently,

$$\left|W^{(1)}\right|^2 \equiv \frac{(1+\varepsilon)^2(n-1)}{n+2} \qquad \text{and} \qquad \left|W^{(3)}\right|^2 \equiv \frac{2(n+1-\varepsilon)^2(n-1)}{n(n+2)}$$

on S^{n-1}. It follows easily that there are positive constants ℓ_ε, m_ε and N_ε with

$$\ell(z) = \ell_\varepsilon, \qquad |D\ell(z)| = m_\varepsilon \qquad \text{and} \qquad |z| = N_\varepsilon \tag{8.6}$$

for all $z \in K_1^\varepsilon$. Here, the constant ℓ_ε is positive because (8.2) implies $\varepsilon < \frac{n}{2}$. Furthermore, there holds $\sum_{i=1}^n \partial_i (w_\varepsilon)_{ij}(x) = \frac{1}{n}(n+1-\varepsilon)(n-1)x_j$ for all $1 \le j \le n$ and $x \in S^{n-1}$, from which we infer that K_1^ε is diffeomorphic to S^{n-1}. In particular, K_1^ε is a smooth submanifold of $T_1 \oplus T_3$.

From now on, we restrict ourselves to the subspace $T_1 \oplus T_3 \subset T \cong \mathbb{R}^{Nn}$ and we introduce an enlarged version S^ε of K_1^ε in the following way: For a given $\mu > 0$ and every $z \in K_1^\varepsilon$, let $z' := z - \mu\, D\ell(z)$ and $r_\mu = \mu|D\ell(z)| = \mu m_\varepsilon$. Then we write

$$S^\varepsilon := \bigcup_{z \in K_1^\varepsilon} \overline{B_{r_\mu}(z')} \subset T_1 \oplus T_3,$$

where $B_{r_\mu}(z')$ denotes the ball in $T_1 \oplus T_3$ with center z' and radius r_μ. We point out that the set S^ε is defined in such a way that $K_1^\varepsilon \subset \partial S^\varepsilon$. Moreover, by the symmetry of w_ε, the set S^ε is point-symmetric with respect to the origin. Since K_1^ε is a smooth submanifold, we know furthermore that ∂S^ε is smooth if we choose $\mu > 0$ small enough.

With the notations introduced above we restate [39, Lemma 3.2]:

Lemma 8.5. *For every $\varepsilon > 0$ with* (8.2) *there is a constant $\delta(\varepsilon) > 0$ such that for sufficiently small values of $\mu > 0$ there holds*

$$D\ell(z) \cdot (\tilde\xi - z) \le -2\delta(\varepsilon)|\tilde\xi - z|^2 \qquad \textit{for all } z \in K_1^\varepsilon,\ \tilde\xi \in S^\varepsilon. \tag{8.7}$$

Now we are in the position to prove Theorem 8.4.

Proof of Theorem 8.4. In the following we will write $L : T \times T \to \mathbb{R}$ for the symmetric bilinear form defined by

$$L(z,z) = \ell(z) \qquad \text{for } z \in T.$$

Since ℓ is a quadratic function, the inequality (8.7) can be written in the form

$$L(z, \tilde{\xi} - z) \le -\delta(\varepsilon)|\tilde{\xi} - z|^2 \qquad \text{for all } z \in K_1^\varepsilon, \tilde{\xi} \in S^\varepsilon, \tag{8.8}$$

which implies in particular $L(z, \tilde{\xi}) \le \ell_\varepsilon$ for all $z \in K_1^\varepsilon$ and $\tilde{\xi} \in S^\varepsilon$. We claim that, furthermore, there holds

$$L(\tilde{z}, \tilde{\xi}) \le \ell_\varepsilon \qquad \text{for all } \tilde{z}, \tilde{\xi} \in S^\varepsilon, \tag{8.9}$$

provided $\mu > 0$ is small enough. For the proof of this claim, we fix $\tilde{z}, \tilde{\xi} \in S^\varepsilon$ and choose a point $z \in K_1^\varepsilon$ with $\tilde{z} \in B_{r_\mu}(z')$, where $z' = z - \mu D\ell(z)$ and $r_\mu = \mu|D\ell(z)|$. We begin with the observation

$$\begin{aligned} \mu^2|D\ell(z)|^2 &\ge |\tilde{z} - z'|^2 = |\tilde{z} - z + \mu\, D\ell(z)|^2 \\ &= |\tilde{z} - z|^2 + 2\mu\, D\ell(z) \cdot (\tilde{z} - z) + \mu^2|D\ell(z)|^2, \end{aligned}$$

which implies

$$L(z, \tilde{z} - z) = \frac{1}{2} D\ell(z) \cdot (\tilde{z} - z) \le -\frac{1}{4\mu}|\tilde{z} - z|^2. \tag{8.10}$$

Applying (8.8) and (8.10), we can estimate

$$\begin{aligned} L(\tilde{z}, \tilde{\xi}) &= L(z, \tilde{\xi}) + L(\tilde{z} - z, \tilde{\xi} - z) + L(\tilde{z} - z, z) \\ &\le \ell_\varepsilon - \delta(\varepsilon)|\tilde{\xi} - z|^2 + (2+n)|\tilde{z} - z|\,|\tilde{\xi} - z| - \frac{1}{4\mu}|\tilde{z} - z|^2 \\ &\le \ell_\varepsilon + \left(C(n,\varepsilon) - \frac{1}{4\mu}\right)|\tilde{z} - z|^2, \end{aligned}$$

where we applied Young's inequality in the last step. This implies the claim (8.9) if we choose $\mu > 0$ small enough. Next we consider the convex hull H^ε of $S^\varepsilon \subset T_1 \oplus T_3$ and we will show $\ell(z) \le \ell_\varepsilon$ for all $z \in H^\varepsilon$. Every $z \in H^\varepsilon$ can be written in the form $z = \sum_{i=0}^n \lambda_i \tilde{z}_i$ with $\tilde{z}_i \in S^\varepsilon$ and $\lambda_i \ge 0$ with $\sum_{i=0}^n \lambda_i = 1$. Since L is bilinear, we can estimate, using (8.9),

$$\ell(z) = L(z,z) = \sum_{i,j=0}^n \lambda_i \lambda_j L(\tilde{z}_i, \tilde{z}_j) \le \ell_\varepsilon \sum_{i,j=0}^n \lambda_i \lambda_j = \ell_\varepsilon \tag{8.11}$$

for all $z \in H^\varepsilon$.

Next we will construct the convex integrand g_ε. Following [39] once more we employ the Minkowski function, also called the gauge, of the convex set H^ε. For more details about this function we refer to [35]. Precisely, we define

$$G_\varepsilon(z) := \ell_\varepsilon \chi^2(z), \qquad \text{where} \quad \chi(z) = \inf\{t \geq 0 \,:\, z \in tH^\varepsilon\}$$

for $z \in T_1 \oplus T_3$. Since H^ε is a convex neighborhood of the origin in $T_1 \oplus T_3$, it follows from the definition that $G^\varepsilon \equiv \ell_\varepsilon$ on the boundary ∂H^ε. Keeping in mind (8.11), we infer

$$\ell(z) \leq G_\varepsilon(z) \tag{8.12}$$

first for all $z \in \partial H^\varepsilon$ and finally for all $z \in T_1 \oplus T_3$, since ℓ and G_ε are both homogeneous of degree two and H^ε is a neighborhood of the origin.

Letting $\nu(z) := \frac{D\ell(z)}{|D\ell(z)|}$ for $z \in K_1^\varepsilon$, we point out that by (8.7), the vector field ν, defined on K_1^ε, is a field of unit normal vectors of ∂S^ε. For a suitable neighborhood U^ε of K_1^ε, we extend ν to a smooth vector field of unit normal vectors on $U^\varepsilon \cap \partial S^\varepsilon$. Diminishing the neighborhood U^ε if necessary, we deduce from (8.7) that for all $z, \xi \in U^\varepsilon \cap \partial S^\varepsilon$ there holds

$$\frac{\nu(z) - \nu(\xi)}{|\xi - z|} \cdot \frac{\xi - z}{|\xi - z|} \leq -3\frac{\delta(\varepsilon)}{m_\varepsilon}.$$

Letting $\xi \to z$, we infer

$$-D_V\nu(z) \cdot V \leq -3\frac{\delta(\varepsilon)}{m_\varepsilon} < 0$$

for all $z \in U^\varepsilon \cap \partial S^\varepsilon$ and $V \in \mathrm{Tan}_z(\partial S^\varepsilon)$. We recall that the principal curvatures of $\partial S^\varepsilon \cap U^\varepsilon$ are defined as the eigenvalues of the shape operator $A_\nu(V) = -(D_V\nu)^\top$. Thus, we conclude from the above estimate that all the principal curvatures are negative and bounded away from zero, which implies that $\partial S^\varepsilon \cap U^\varepsilon$ is strictly elliptic. Combining this with (8.7), we infer

$$\partial H^\varepsilon \cap U^\varepsilon = \partial S^\varepsilon \cap U^\varepsilon$$

if U_ε is chosen sufficiently small. In particular, $K_1^\varepsilon \subset \partial H^\varepsilon$, so that

$$G_\varepsilon(z) = \ell_\varepsilon = \ell(z) \qquad \text{for all } z \in K_1^\varepsilon. \tag{8.13}$$

Moreover, the strict ellipticity of ∂H^ε implies that G_ε is strictly convex close to K_1^ε in the sense

$$D^2G_\varepsilon(z)(\xi, \xi) \geq c_\varepsilon|\xi|^2 \qquad \text{for all } z \in U_\varepsilon \text{ and } \xi \in T_1 \oplus T_3, \tag{8.14}$$

with a suitable constant $c_\varepsilon > 0$. Following [39] we obtain the convex integrand g_ε from G_ε by a smoothing technique which preserves the identity (8.13) and does not diminish the function G_ε away from the origin. We provide a rereading of this smoothing process for the convenience of the reader. First, we choose a kernel $\varphi \in C^\infty_{\mathrm{cpt}}([\frac{1}{2}, 1])$ with $\int_{T_1\oplus T_3} \varphi(|\xi|)\, d\xi = 1$ and let $\varphi_\delta(\xi) := \delta^{-\dim(T_1\oplus T_3)}\varphi(\delta^{-1}\xi)$. Since χ is convex and homogeneous of degree one, the function

$$\tilde{\chi}(z) := \int_{T_1\oplus T_3} \chi(z + |z|\xi)\varphi_\delta(|\xi|)\, d\xi \qquad \text{for } z \in T_1 \oplus T_3$$

is also convex and homogeneous of degree one, and additionally smooth; see [37, Thm. 3.3.1]. Consequently, the function

$$\widetilde{G}_\varepsilon(z) := \ell_\varepsilon \tilde{\chi}^2(z) + \tau|z|^2 \qquad \text{for } z \in T_1 \oplus T_3$$

is strictly convex and homogeneous of degree two for any $\tau > 0$. Note that $\widetilde{G}_\varepsilon$ depends on δ and τ, but we suppress these dependences to facilitate the reading. Since $\widetilde{G}_\varepsilon \to G_\varepsilon$ in the C^2-norm as $\delta, \tau \searrow 0$, we conclude

$$D^2\widetilde{G}_\varepsilon(z)(\xi, \xi) \ge \frac{c_\varepsilon}{2}|\xi|^2 \qquad \text{for all } z \in U^\varepsilon \text{ and } \xi \in T_1 \oplus T_3, \tag{8.15}$$

if δ and τ are chosen sufficiently small. We stress that the constant c_ε can be chosen independent of τ if we restrict ourselves to $z \in U^\varepsilon$. We record that as a consequence of Jensen's inequality and (8.12), there holds

$$\widetilde{G}_\varepsilon(z) \ge \ell_\varepsilon\chi^2\left(\int_{T_1\oplus T_3} (z + |z|\xi)\varphi_\delta(|\xi|)\, d\xi\right) = \ell_\varepsilon\chi^2(z) = G_\varepsilon(z) \ge \ell(z) \tag{8.16}$$

for all $z \in T_1 \oplus T_3$. Since the above smoothing process might have changed G_ε in K_1^ε, we choose a cut-off function $\tilde{\eta} \in C_0^\infty(U^\varepsilon)$ with $\tilde{\eta} \equiv 1$ in a smaller neighborhood $V^\varepsilon \subset U^\varepsilon$ of K_1^ε, and let $\eta(z) := \tilde{\eta}(N_\varepsilon \frac{z}{|z|})$, where N_ε is defined by (8.6). We write

$$G_\varepsilon^*(z) := (1 - \eta(z))\, \widetilde{G}_\varepsilon(z) + \eta(z)G^\varepsilon(z) \qquad \text{for } 0 \ne z \in T_1 \oplus T_3$$

and $G_\varepsilon^*(0) := 0$. Obviously, G_ε^* is homogeneous of degree two. In order to show that G_ε^* is strictly convex on U^ε, we calculate for $z \in U^\varepsilon$ and an arbitrary $\xi \in T_1 \oplus T_3$

$$\begin{aligned} D^2G_\varepsilon^*(z)(\xi, \xi) &\ge (1 - \eta(z))D^2\widetilde{G}_\varepsilon(z)(\xi, \xi) + \eta(z)D^2G_\varepsilon(z)(\xi, \xi) \\ &\qquad - C\|\widetilde{G}_\varepsilon - G_\varepsilon\|_{C^1(U^\varepsilon)}\, |\xi|^2 \\ &\ge \Big(\frac{c_\varepsilon}{2} - C\|\widetilde{G}_\varepsilon - G_\varepsilon\|_{C^1(U^\varepsilon)}\Big)\, |\xi|^2 \end{aligned} \tag{8.17}$$

by (8.14) and (8.15). Since $\|\widetilde{G}_\varepsilon - G_\varepsilon\|_{C^1(U^\varepsilon)}$ can be made arbitrarily small by choosing the parameters $\delta, \tau > 0$ small enough, we infer for suitable choices of δ

and τ that G^*_ε is strictly convex on U^ε. Keeping in mind the strict convexity of $\widetilde{G}_\varepsilon$ and the homogeneity of G^*_ε, we deduce that G^*_ε is strictly convex and smooth away from the origin. Recalling (8.16), we know furthermore

$$\ell(z) \le G^*_\varepsilon(z) \qquad \text{for all } z \in T_1 \oplus T_3, \tag{8.18}$$

and from (8.13), combined with the homogeneity of G^*_ε, ℓ and w_ε, we infer

$$\ell(Dw_\varepsilon(x)) = G^*_\varepsilon(Dw_\varepsilon(x)) \qquad \text{for all } 0 \neq x \in B_1. \tag{8.19}$$

In order to make G^*_ε smooth and strictly convex on all of $T_1 \oplus T_3$, we choose a cut-off function $\zeta \in C^\infty_0(B_{N_\varepsilon/2})$ with $\zeta \equiv 1$ on $B_{N_\varepsilon/4}$ and a radial mollifying kernel ψ_ρ with smoothing radius $\rho > 0$. For a parameter $\beta > 0$, let

$$\tilde{g}_\varepsilon(z) := \zeta(z)\big[\psi_\rho * G^*_\varepsilon(z) + \beta|z|^2\big] + (1 - \zeta(z))G^*_\varepsilon(z).$$

This function is smooth on $T_1 \oplus T_3$ and a similar computation as in (8.17) shows that for sufficiently small values of the parameters $\rho, \beta > 0$, $\tilde{g}_\varepsilon$ is strictly convex on $T_1 \oplus T_3$. Finally, letting

$$g_\varepsilon(z) := \tilde{g}_\varepsilon(z_1 + z_3) + |z_2|^2 \qquad \text{for } z = z_1 + z_2 + z_3 \in T \cong \mathbb{R}^{Nn},$$

where $z_i \in T_i$ for $i = 1, 2, 3$, we arrive at a function $g_\varepsilon : \mathbb{R}^{Nn} \to \mathbb{R}$ which is smooth and strictly convex on $\mathbb{R}^{Nn}$. The inequality (8.18) implies, by another application of Jensen's inequality as in (8.16), that $\ell \le g_\varepsilon$ on $\mathbb{R}^{Nn}$, so that the first claim (8.4) is satisfied. Since $|Dw_\varepsilon(x)| = |x|^{-\varepsilon} N_\varepsilon \ge N_\varepsilon$ for $x \in B_1$, we infer from (8.19) that the second claim (8.5) of the theorem is valid. This completes the proof. □

8.2. A singular set of positive measure

The remainder of this section will be devoted to the proof of Theorem 2.6. In fact, introducing first some additional notation we will even establish a somewhat stronger result.

In the following, cubes will be considered as open subsets of $\mathbb{R}^n$. We write Q_0 for the cube $]-\frac{1}{2}, \frac{1}{2}[^n$.

Definition 8.6. For a function u on Q_0 and a regularity class $\mathscr{C}$ we define the regular set

$$R_{\mathscr{C}}(u) := \{x \in Q_0 : u \text{ is of class } \mathscr{C} \text{ in a neighborhood of } x\}$$

and the singular set

$$S_{\mathscr{C}}(u) := Q_0 \setminus R_{\mathscr{C}}(u).$$

Moreover, as in (1.2) we abbreviate $R(u) := R_{C^{0,1}}(u)$ and $S(u) := S_{C^{0,1}}(u)$.

With these notations we formulate the main result of this section providing wild solutions of asymptotically regular problems:

Theorem 8.7. *Let $N = \frac{1}{2}n(n+1) - 1$. For every*

$$0 < \varepsilon < \frac{1}{2}\left[\sqrt{3(n+1)(n-1)} - (n+1)\right]$$

there are positive constants M, γ and Γ, a function $u_\varepsilon \in W_0^{1,2}(Q_0, \mathbb{R}^N)$ and a smooth function $f_\varepsilon : \mathbb{R}^{Nn} \to \mathbb{R}$ depending only on n and ε with the following three properties:

$$\gamma|\xi|^2 \le D^2 f_\varepsilon(z)(\xi,\xi) \le \Gamma|\xi|^2 \qquad \text{for } |z| > M \text{ and } \xi \in \mathbb{R}^{Nn};$$

$$\int_{Q_0} f_\varepsilon(Du_\varepsilon)\,dx \le \int_{Q_0} f_\varepsilon(D\varphi)\,dx \qquad \text{for every } \varphi \in W_0^{1,2}(Q_0, \mathbb{R}^N);$$

$$|S_{W^{1,\frac{n}{\varepsilon}}}(u_\varepsilon)| > 0 \text{ and } |S_{C^{0,\alpha}}(u_\varepsilon)| > 0 \qquad \text{for every } \max\{1-\varepsilon, 0\} < \alpha \le 1.$$

Moreover, if $\varepsilon > 1$ holds we even have

$$|S_{L^{\frac{n}{\varepsilon-1}}}(u_\varepsilon)| > 0.$$

Remark 8.8. In particular, $|S(u_\varepsilon)| > 0$ may occur for every $n \ge 3$, and $|S_{L^\infty}(u_\varepsilon)| > 0$ for every $n \ge 5$. These restrictions on the dimension are optimal in the case $p = 2$ considered here. To be more precise, if $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ is a solution of an asymptotically regular problem, Theorem 2.8 implies $S(u) = \emptyset$ for $n = 2$, while for $n \in \{3, 4\}$ we have $S_{C^{0,\alpha}}(u) = \emptyset$ for some $\alpha > 0$ by [36, Theorem 8.1].

The proof of Theorem 8.7 is based on the results of Section 8.1 and a Cantor type construction[3]. It will be carried out in a series of lemmas.

We define $v_\varepsilon : B_1 \to \mathbb{R}^N$ by

$$v_\varepsilon(x) := \begin{cases} w_\varepsilon(x) & \text{for } x \in B_{1/2} \\ 2(1-|x|)w_\varepsilon\left(\dfrac{x}{2|x|}\right) & \text{for } x \in B_1 \setminus B_{1/2}, \end{cases}$$

where w_ε is defined in (8.1). Clearly, we have

$$v_\varepsilon \in W_0^{1,q}(B_1, \mathbb{R}^N) \tag{8.20}$$

for every $q < \frac{n}{\varepsilon}$ and involving the homogeneity of w_ε it is not difficult to see

$$|Dv_\varepsilon| \le L \qquad \text{on } B_1 \setminus B_{1/8} \tag{8.21}$$

for some constant L depending only on n and ε.

[3] We refer the reader to [9] for another Cantor type construction of a minimizer with a large singular set.

Next, we choose a smooth cut-off function $\eta : \mathbb{R}^{Nn} \to [0, 1]$ with $\eta(z) = 0$ for $|z| \leq L$ and $\eta(z) = 1$ for $|z| \geq L + 1$ and define $f_\varepsilon : \mathbb{R}^{Nn} \to \mathbb{R}$ by

$$f_\varepsilon(z) := \eta(z) g_\varepsilon(z) + (1 - \eta(z))\ell(z),$$

where g_ε and ℓ are the integrands from Section 8.1. Clearly, we have

$$\ell(z) = f_\varepsilon(z) \qquad \text{for } |z| \leq L \tag{8.22}$$

and

$$\gamma|\xi|^2 \leq D^2 f_\varepsilon(z)(\xi, \xi) \leq \Gamma|\xi|^2 \qquad \text{for } |z| > L + 1.$$

In particular, f_ε is asymptotically regular. Moreover, (8.4) and (8.5) give

$$\ell(z) \leq f_\varepsilon(z), \tag{8.23}$$

$$\ell(Dw_\varepsilon(x)) = f_\varepsilon(Dw_\varepsilon(x)) \tag{8.24}$$

for all $z \in \mathbb{R}^{Nn}$ and $0 \neq x \in B_1$.

Next we will arrange the minimizers of Section 8.1 on a wild set: For every $k \in \mathbb{N} \cup \{0\}$ we subdivide Q_0 into $\prod_{i=0}^{k} 3^{in}$ disjoint cubes with edges of length $\prod_{i=0}^{k} 3^{-i}$ and define W_k as the collection of these cubes and Y_k as the set of their centers. We point out that for every cube $Q \in W_k$, its predecessors are unique; that is for every $i \leq k$ there is a unique cube in W_i containing Q. Moreover, we note $\{0\} = Y_0 \subset Y_1 \subset Y_2 \subset Y_3 \subset \ldots$.

We define another collection of cubes inductively. We set $V_0 := \varnothing$ and for $k \in \mathbb{N}$ we denote by V_k the collection of all cubes Q in W_k with centers in Y_{k-1} such that Q is not contained in some cube of $\bigcup_{i=0}^{k-1} V_i$. Finally we write X_k for the set of centers of cubes in V_k.

Our first goal is now to show that the closure of $\bigcup_{i=0}^{\infty} X_i$ has positive measure. Actually, this will follow from the next two lemmas.

Lemma 8.9. *Denote by A the union of the closures of all cubes in $\bigcup_{i=0}^{\infty} V_i$. Then, the set $Q_0 \setminus A$ is contained in the closure of $\bigcup_{i=0}^{\infty} X_i$.*

Proof. Consider an arbitrary point $x \in Q_0 \setminus A$ and fix $k \in \mathbb{N}$. Then, x is contained in the closure of a cube $Q_k \in W_{k-1}$ with center $x_k \in Y_{k-1}$. By the definition of A, the cube Q_k is not contained in some cube of $\bigcup_{i=0}^{k-1} V_i$. Now consider the cube $\tilde{Q}_k \in W_k$ with center at x_k. Since the predecessors are unique, $\tilde{Q}_k$ is not contained in a cube of $\bigcup_{i=0}^{k-1} V_i$. Thus we have $\tilde{Q}_k \in V_k$ and $x_k \in X_k$. Finally, for the sequence $x_k \in \bigcup_{i=0}^{\infty} X_i$ just defined we have

$$|x_k - x| \leq \frac{1}{2}\text{diam } Q_k \leq \frac{\sqrt{n}}{2} 3^{1-k}.$$

Thus, x is in the closure of $\bigcup_{i=0}^{\infty} X_i$ and the lemma is proved. □

Lemma 8.10. *Denote by A the union of the closures of all cubes in* $\bigcup_{k=0}^{\infty} V_k$. *Then, we have* $|Q_0 \setminus A| > 0$.

Proof. Clearly, Y_{k-1} has $\prod_{i=0}^{k-1} 3^{in}$ elements and thus V_k contains at most $\prod_{i=0}^{k-1} 3^{in}$ cubes. Furthermore, the closure of every cube in W_k — and in particular of every cube in V_k — has measure $\prod_{i=0}^{k} 3^{-in}$. Thus, we get

$$|A| \le \sum_{k=1}^{\infty} 3^{-kn} = \frac{3^{-n}}{1-3^{-n}} < 1$$

and the claim is established. □

We will now construct a minimizer u_ε on Q_0 that is singular near all points of $\bigcup_{k=0}^{\infty} X_k$. Setting

$$r_k := \prod_{i=0}^{k} 6^{-i}$$

we note that all the balls $B_{r_k}(x)$ with $x \in X_k$ and $k \in \mathbb{N}$ are mutually disjoint. Finally, we introduce u_ε setting

$$u_\varepsilon(y) := \begin{cases} r_k v_\varepsilon\left(\dfrac{y-x}{r_k}\right) & \text{if } y \in B_{r_k}(x) \text{ for some } x \in X_k \text{ and some } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$

Lemma 8.11. *For every* $q < \frac{n}{\varepsilon}$ *we have* $u_\varepsilon \in W_0^{1,q}(Q_0, \mathbb{R}^N)$.

Proof. The claim follows easily from (8.20) and the calculation

$$\begin{aligned} \int_{Q_0} \left[|u_\varepsilon|^q + |Du_\varepsilon|^q\right] dx &= \sum_{k=1}^{\infty} \sum_{y \in X_k} r_k^n \int_{B_1} \left[r_k^q |v_\varepsilon|^q + |Dv_\varepsilon|^q\right] dx \\ &\le \sum_{k=1}^{\infty} \left(\frac{3}{6}\right)^{kn} \int_{B_1} \left[|v_\varepsilon|^q + |Dv_\varepsilon|^q\right] dx < \infty. \end{aligned}$$

Here, in the last estimate we used the definition of r_k and the fact that $|X_k|$ contains at most $\prod_{i=0}^{k} 3^{in}$ elements. □

Proposition 8.12. *Let*

$$0 < \varepsilon < \frac{1}{2}\left[\sqrt{3(n+1)(n-1)} - (n+1)\right].$$

Then, we have $u_\varepsilon \in W_0^{1,2}(Q_0, \mathbb{R}^N)$ *and*

$$\int_{Q_0} f_\varepsilon(Du_\varepsilon)\,dx \le \int_{Q_0} f_\varepsilon(D\varphi)\,dx \qquad \textit{for every } \varphi \in W_0^{1,2}(Q_0, \mathbb{R}^N).$$

Proof. Recalling $\varepsilon < \frac{n}{2}$, Lemma 8.11 gives $u_\varepsilon \in W_0^{1,2}(Q_0, \mathbb{R}^N)$. For the proof of the minimizing property we first claim

$$\ell(Du_\varepsilon) = f_\varepsilon(Du_\varepsilon) \qquad \text{on } Q_0. \tag{8.25}$$

To verify (8.25) we fix a Lebesgue point $y \in Q_0$ of Du_ε and distinguish two cases: If y is in $B_{r_k/2}(x)$ for some $x \in X_k$ and some $k \in \mathbb{N}$, then we have $Du_\varepsilon(y) = Dw_\varepsilon(\frac{y-x}{r_k})$ and (8.24) gives $f_\varepsilon(Du_\varepsilon(y)) = \ell(Du_\varepsilon(y))$. Otherwise, we see from the construction of u_ε that $|Du_\varepsilon(y)| \leq L$, so that $f_\varepsilon(Du_\varepsilon(y)) = \ell(Du_\varepsilon(y))$ follows from the definition of f_ε. Thus, (8.25) is proved in any case.

Consequently, exploiting in turn (8.25), the fact that ℓ is a null Lagrangian and (8.23) we find

$$\int_{Q_0} f_\varepsilon(Du_\varepsilon)\,dx = \int_{Q_0} \ell(Du_\varepsilon)\,dx = \int_{Q_0} \ell(D\varphi)\,dx \leq \int_{Q_0} f_\varepsilon(D\varphi)\,dx \tag{8.26}$$

for any $\varphi \in W_0^{1,2}(Q_0, \mathbb{R}^N)$, thus completing the proof. □

Proof of Theorem 8.7. In the situation of the theorem let $\mathscr{C}$ denote one of the regularity classes $W^{1,\frac{n}{\varepsilon}}$, $C^{0,\alpha}$, $L^{\frac{n}{\varepsilon-1}}$, where $\max\{1-\varepsilon, 0\} < \alpha \leq 1$. In view of the preceding considerations, in particular of Proposition 8.12, it just remains to prove $|S_{\mathscr{C}}(u_\varepsilon)| > 0$. To this aim we first note that, by the definition of w_ε in Section 8.1, the function w_ε is not of class $\mathscr{C}$ near 0. From the construction of u_ε we deduce $\bigcup_{i=0}^{\infty} X_i \subset S_{\mathscr{C}}(u_\varepsilon)$. Since $S_{\mathscr{C}}(u_\varepsilon)$ is — by definition — closed in Q_0 we infer that the closure of $\bigcup_{i=0}^{\infty} X_i$ is still contained in $S_{\mathscr{C}}(u_\varepsilon)$. Thus, Lemma 8.9 and Lemma 8.10 give $|S_{\mathscr{C}}(u_\varepsilon)| > 0$. □

Remark 8.13. In Theorem 8.7 one may additionally choose f_ε such that f_ε is quasiconvex. To see this it suffices to replace f_ε with its quasiconvex hull observing that (8.22), (8.23) and (8.24) are preserved.

Proof of Theorem 2.6. Theorem 2.6 is essentially a particular case of Theorem 8.7 and Remark 8.8 with $f := f_\varepsilon$. It just remains to reason that the cube Q_0 may be replaced by any non-empty bounded open set Ω. To see this we fix such a set Ω and assume — by scaling and translation — that $Q_0 \subset \Omega$ holds. Then we define $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ as the extension by 0 of the map $u_\varepsilon \in W_0^{1,2}(Q_0, \mathbb{R}^N)$ from Theorem 8.7. Recalling the above construction, in particular $f(0) = f_\varepsilon(0) = \ell(0)$, one easily sees that (8.25) and (8.26) still hold if we replace u_ε by u and Q_0 by Ω. Hence, u is a minimizer of F on Ω. The remaining claims of Theorem 2.6 are now obvious. □

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