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A priori estimates for weak solutions of complex Monge-Ampère equations

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Abstract. Let X be a compact Kähler manifold and ω be a smooth closed form of bidegree $(1,1)$ which is nonnegative and big. We study the classes $\mathcal{E}_\chi(X,\omega)$ of ω-plurisubharmonic functions of finite weighted Monge-Ampère energy. When the weight χ has fast growth at infinity, the corresponding functions are close to be bounded.

We show that if a positive Radon measure is suitably dominated by the Monge-Ampère capacity, then it belongs to the range of the Monge-Ampère operator on some class $\mathcal{E}_\chi(X,\omega)$. This is done by establishing a priori estimates on the capacity of sublevel sets of the solutions.

Our result extends those of U. Cegrell's and S. Kolodziej's and puts them into a unifying frame. It also gives a simple proof of S. T. Yau's celebrated a priori $\mathcal{C}^0$-estimate.

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1. Introduction

Let X be a compact connected Kähler manifold of dimension $n \in \mathbb{N}^*$. Throughout the article ω denotes a smooth closed form of bidegree $(1,1)$ which is nonnegative and *big*, *i.e.* such that $\int_X \omega^n > 0$. We continue the study started in [10], [8] of the complex Monge-Ampère equation

$$(\omega + dd^c\varphi)^n = \mu, \qquad \text{(MA)}_\mu$$

where φ, the unknown function, is ω-plurisubharmonic: this means that $\varphi \in L^1(X)$ is upper semi-continuous and $\omega + dd^c\varphi \geq 0$ is a positive current. We let $PSH(X,\omega)$ denote the set of all such functions (see [9] for their basic properties). Here μ is a fixed positive Radon measure of total mass $\mu(X) = \int_X \omega^n$, and $d = \partial + \overline{\partial}$, $d^c = \frac{1}{2i\pi}(\partial - \overline{\partial})$.

Following [10] we say that a ω-plurisubharmonic function φ has finite weighted Monge-Ampère energy, $\varphi \in \mathcal{E}(X,\omega)$, when its Monge-Ampère measure

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$(\omega + dd^c\varphi)^n$ is well defined, and there exists an increasing function $\chi : \mathbb{R}^- \to \mathbb{R}^-$ such that $\chi(-\infty) = -\infty$ and $\chi \circ \varphi \in L^1((\omega + dd^c\varphi)^n)$. In general χ has very slow growth at infinity, so that φ is far from being bounded.

The purpose of this article is twofold. First we extend one of the main results of [10] by showing:

Theorem A. *There exists $\varphi \in \mathcal{E}(X, \omega)$ such that $\mu = (\omega + dd^c\varphi)^n$ if and only if μ does not charge pluripolar sets.*

This results has been established in [10] when ω is a Kähler form. It is important for applications to complex dynamics and Kähler geometry to consider as well forms ω that are less positive (see [8]).

We then look for conditions on the measure μ which insure that the solution φ is almost bounded. Following the seminal work of S. Kolodziej [12, 13], we say that μ is dominated by the Monge-Ampère capacity Cap_ω if there exists a function $F : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to 0^+} F(t) = 0$ and

$$\mu(K) \leq F(\mathrm{Cap}_\omega(K)), \quad \text{for all Borel subsets } K \subset X. \tag{$*$}$$

Here Cap_ω denotes the global version of the Monge-Ampère capacity introduced by E. Bedford and A. Taylor [3] (see Section 2).

Observe that μ does not charge pluripolar sets since $F(0) = 0$. When $F(x) \lesssim x^\alpha$ vanishes at order $\alpha > 1$ and ω is Kähler, S. Kolodziej has proved [12] that the solution $\varphi \in PSH(X, \omega)$ of $(\mathrm{MA})_\mu$ is *continuous*. The boundedness part of this result was extended in [8] to the case when ω is merely big and nonnegative. If $F(x) \lesssim x^\alpha$ with $0 < \alpha < 1$, two of us have proved in [10] that the solution φ has finite χ−energy, where $\chi(t) = -(-t)^p$, $p = p(\alpha) > 0$. This result was first established by U. Cegrell in a local context [7].

Another objective of this article is to fill in the gap inbetween Cegrell's and Kolodziej's results, by considering all intermediate dominating functions F. Write $F_\varepsilon(x) = x[\varepsilon(-\ln(x)/n)]^n$ where $\varepsilon : \mathbb{R} \to [0, \infty[$ is nonincreasing. Our second main result is:

Theorem B. *If $\mu(K) \leq F_\varepsilon(\mathrm{Cap}_\omega(K))$ for all Borel subsets $K \subset X$, then $\mu = (\omega + dd^c\varphi)^n$ where $\varphi \in PSH(X, \omega)$ satisfies $\sup_X \varphi = 0$ and*

$$\mathrm{Cap}_\omega(\varphi < -s) \leq \exp(-nH^{-1}(s)).$$

Here H^{-1} is the reciprocal function of $H(x) = e\int_0^x \varepsilon(t)dt + s_0$, where $s_0 = s_0(\varepsilon, \omega) \geq 0$ only depends on ε and ω.

This general statement has several useful consequences:

- if $\int_0^{+\infty} \varepsilon(t)dt < +\infty$, then $H^{-1}(s) = +\infty$ for $s \geq s_\infty := e\int_0^{+\infty} \varepsilon(t)dt + s_0$, hence $\mathrm{Cap}_\omega(\varphi < -s) = 0$. This means that φ is bounded from below by $-s_\infty$. This result is due to S. Kolodziej [12, 13] when ω is Kähler, and [8] when $\omega \geq 0$ is merely big;

- condition $(*)$ is easy to check for measures with density in L^p, $p > 1$. Our result thus gives a simple proof (Corollary 3.2), following the seminal approach of S. Kolodziej ([12]), of the $\mathcal{C}^0$-a priori estimate of S. T. Yau [19], which is crucial for proving the Calabi conjecture (see [18] for an overview);
- when $\int_0^{+\infty} \varepsilon(t)dt = +\infty$, the solution φ is generally unbounded. The faster $\varepsilon(t)$ decreases towards zero, the faster the growth of H^{-1} at infinity, hence the closer is φ from being bounded;
- the special case $\varepsilon \equiv 1$ is of particular interest. Here $\mu(\cdot) \leq \mathrm{Cap}_\omega(\cdot)$, and our result shows that $\mathrm{Cap}_\omega(\varphi < -s)$ decreases exponentially fast, hence φ has " loglog-singularities". These are the type of singularities of the metrics used in Arakelov geometry in relation with measures $\mu = f dV$ whose density has Poincaré-type singularities (see [5, 15]).

We prove Theorem B in Section 3 , after establishing Theorem A in Subection 2.1 and recalling some useful facts from [8, 10] in Subection 2.2. We then test the sharpness of our estimates in Section 4, where we give examples of measures fulfilling our assumptions: these are absolutely continuous with respect to ω^n, and their density do not belong to L^p, for any $p > 1$.

2. Weakly singular quasiplurisubharmonic functions

The class $\mathcal{E}(X, \omega)$ of ω-psh functions with finite weighted Monge-Ampère energy has been introduced and studied in [10]. It is the largest subclass of $PSH(X, \omega)$ on which the complex Monge-Ampère operator $(\omega + dd^c \cdot)^n$ is well-defined and the comparison principle is valid. Recall that $\varphi \in \mathcal{E}(X, \omega)$ if and only if $(\omega + dd^c\varphi_j)^n(\varphi \leq -j) \to 0$, where $\varphi_j := \max(\varphi, -j)$.

2.1. The range of the Monge-Ampère operator

The range of the operator $(\omega + dd^c \cdot)^n$ acting on $\mathcal{E}(X, \omega)$ has been characterized in [10] when ω is a *Kähler* form. We extend here this result to the case when ω is merely nonnegative and big.

Theorem 2.1. *Assume ω is a smooth closed nonnegative* $(1, 1)$ *form on X, and μ is a positive Radon measure such that $\mu(X) = \int_X \omega^n > 0$.*

Then there exists $\varphi \in \mathcal{E}(X, \omega)$ such that $\mu = (\omega + dd^c\varphi)^n$ if and only if μ does not charge pluripolar sets.

Proof. We can assume without loss of generality that μ and ω are normalized so that $\mu(X) = \int_X \omega^n = 1$. Consider, for $A > 0$,

$$\mathcal{C}_A(\omega) := \{\nu \text{ probability measure } / \ \nu(K) \leq A \cdot \mathrm{Cap}_\omega(K), \text{for all } K \subset X\},$$

where Cap_ω denotes the Monge-Ampère capacity introduced by E. Bedford and A. Taylor in [3] (see [9] for this compact setting). Recall that

$$\mathrm{Cap}_\omega(K) := \sup\left\{\int_K (\omega + dd^c u)^n \;/\; u \in PSH(X,\omega),\ 0 \le u \le 1\right\}.$$

We first show that a measure $\nu \in \mathcal{C}_A(\omega)$ is the Monge-Ampère of a function $\psi \in \mathcal{E}^p(X,\omega)$, for any $0 < p < 1$, where

$$\mathcal{E}^p(X,\omega) := \{\psi \in \mathcal{E}(X,\omega) \;/\; \psi \in L^p\big((\omega + dd^c\psi)^n\big)\}.$$

Indeed, fix $\nu \in \mathcal{C}_A(\omega)$, $0 < p < 1$, and $\omega_j := \omega + \varepsilon_j \Omega$, where Ω is a Kähler form on X, and $\varepsilon_j > 0$ decreases towards zero. Observe that $PSH(X,\omega) \subset PSH(X,\omega_j)$, hence $\mathrm{Cap}_\omega(.) \le \mathrm{Cap}_{\omega_j}(.)$, so that $\nu \in \mathcal{C}_A(\omega_j)$. It follows from [9, Proposition 3.6 and 2.7] that there exists $C_0 > 0$ such that for any $v \in PSH(X,\omega_j)$ normalized by $\sup_X v = -1$, we have

$$\mathrm{Cap}_{\omega_j}(v < -t) \le \frac{C_0}{t}, \text{ for all } t \ge 1.$$

This yields $\mathcal{E}^p(X,\omega_j) \subset L^p(\nu)$: if $v \in \mathcal{E}^p(X,\omega_j)$ with $\sup_X v = -1$, then

$$\begin{aligned}\int_X (-v)^p d\nu &= p \cdot \int_0^{+\infty} t^{p-1} \nu(v < -t) dt \\ &\le pA \cdot \int_1^{+\infty} t^{p-1} \mathrm{Cap}_\omega(v < -t) dt + C_p \\ &\le \frac{pAC_0}{1-p} + C_p < +\infty.\end{aligned}$$

It follows therefore from [10, Theorem 4.2] that there exists $\varphi_j \in \mathcal{E}^p(X,\omega_j)$ with $\sup_X \varphi_j = -1$ and $(\omega_j + dd^c\varphi_j)^n = c_j \cdot \nu$, where $c_j = \int_X \omega_j^n \ge 1$ decreases towards 1 as ε_j decreases towards zero. We can assume without loss of generality that $1 \le c_j \le 2$. Observe that the φ_j's have uniformly bounded energies, namely

$$\int_X (-\varphi_j)^p (\omega_j + dd^c\varphi_j)^n \le 2\int_X (-\varphi_j)^p d\nu \le 2\left[\frac{pAC_0}{1-p} + C_p\right].$$

Since $\sup_X \varphi_j = -1$, we can assume (after extracting a convergent subsequence) that $\varphi_j \to \varphi$ in $L^1(X)$, where $\varphi \in PSH(X,\omega)$, $\sup_X \varphi = -1$.

Set $\phi_j := (\sup_{l \ge j} \varphi_l)^*$. Thus $\phi_j \in PSH(X,\omega_j)$, and ϕ_j decreases towards φ. Since $\phi_j \ge \varphi_j$, it follows from the "fundamental inequality" ([10, Lemma 2.3]) that

$$\int_X (-\phi_j)^p (\omega_j + dd^c\phi_j)^n \le 2^n \int_X (-\varphi_j)^p (\omega_j + dd^c\varphi_j)^n \le C' < +\infty.$$

Hence it follows from stability properties of the class $\mathcal{E}^p(X,\omega)$ that $\varphi \in \mathcal{E}^p(X,\omega)$ (see [10, Proposition 5.6]). Moreover

$$(\omega_j + dd^c\phi_j)^n \geq \inf_{l\geq j}(\omega_l + dd^c\varphi_l)^n \geq \nu,$$

hence $(\omega+dd^c\varphi)^n = \lim(\omega_j+dd^c\phi_j)^n \geq \nu$. Since $\int_X \omega^n = \nu(X) = 1$, this yields $\nu = (\omega + dd^c\varphi)^n$ as claimed above.

We can now prove the statement of the theorem. One implication is obvious: if $\mu = (\omega+dd^c\varphi)^n$, $\varphi \in \mathcal{E}(X,\omega)$, then μ does not charge pluripolar sets, as follows from [10, Theorem 1.3].

So we assume now μ that does not charge pluripolar sets. Since $\mathcal{C}_1(\omega)$ is a compact convex set of probability measures which contains all measures $(\omega + dd^c u)^n$, $u \in PSH(X,\omega)$, $0 \leq u \leq 1$, we can project μ onto $\mathcal{C}_1(\omega)$ and get, by a generalization of Radon-Nikodym theorem (see [7, 16]),

$$\mu = f\cdot\nu,\ \nu \in \mathcal{C}_1(\omega),\ 0 \leq f \in L^1(\nu).$$

Now $\nu = (\omega + dd^c\psi)^n$ for some $\psi \in \mathcal{E}^{1/2}(X,\omega)$, $\psi \leq 0$, as follows from the discussion above. Replacing ψ by e^{ψ} shows that we can actually assume ψ to be bounded (see [10, Lemma 4.5]). We can now apply line by line the same proof as that of [10, Theorem 4.6] to conclude that $\mu = (\omega + dd^c\varphi)^n$ for some $\varphi \in \mathcal{E}(X,\omega)$. □

2.2. High energy and capacity estimates

Given $\chi : \mathbb{R}^- \to \mathbb{R}^-$ an increasing function, we consider, following [10],

$$\mathcal{E}_\chi(X,\omega) := \left\{\varphi \in \mathcal{E}(X,\omega) \,/\, \int_X (-\chi)(-|\varphi|)\,(\omega + dd^c\varphi)^n < +\infty\right\}.$$

Alternatively a function $\varphi \leq 0$ belongs to $\mathcal{E}_\chi(X,\omega)$ if and only if

$$\sup_j \int_X (-\chi)\circ\varphi_j\,(\omega + dd^c\varphi_j)^n < +\infty,\ \text{where } \varphi_j := \max(\varphi, -j)$$

is the *canonical approximation* of φ by bounded ω-psh functions. When $\chi(t) = -(-t)^p$, $\mathcal{E}_\chi(X,\omega)$ is the class $\mathcal{E}^p(X,\omega)$ used in previous section.

The properties of classes $\mathcal{E}_\chi(X,\omega)$ are quite different whether the weight χ is convex (slow growth at infinity) or concave. In previous works [10], two of us were mainly interested in weights χ of moderate growth at infinity (at most polynomial). Our main objective in the sequel is to construct solutions φ of $(MA)_\mu$ which are "almost bounded", *i.e.* in classes $\mathcal{E}_\chi(X,\omega)$ for concave weights χ of arbitrarily high growth.

For this purpose it is useful to relate the property $\varphi \in \mathcal{E}_\chi(X,\omega)$ to the speed of decreasing of $\mathrm{Cap}_\omega(\varphi < -t)$, as $t \to +\infty$. We set

$$\hat{\mathcal{E}}_\chi(X,\omega) := \left\{ \varphi \in PSH(X,\omega) \,/\, \int_0^{+\infty} t^n \chi'(-t)\mathrm{Cap}_\omega(\varphi < -t)dt < +\infty \right\}.$$

An important tool in the study of classes $\mathcal{E}_\chi(X,\omega)$ are the "fundamental inequalities" ([10, Lemmas 2.3 and 3.5]), which allow to compare the weighted energy of two ω-psh functions $\varphi \le \psi$. These inequalities are only valid for weights of slow growth (at most polynomial), while they become immediate for classes $\hat{\mathcal{E}}_\chi(X,\omega)$. So are the convexity properties of $\hat{\mathcal{E}}_\chi(X,\omega)$. We summarize this and compare these classes in the following:

Proposition 2.2. *The classes $\hat{\mathcal{E}}_\chi(X,\omega)$ are convex and stable under maximum: if $\hat{\mathcal{E}}_\chi(X,\omega) \ni \varphi \le \psi \in PSH(X,\omega)$, then $\psi \in \hat{\mathcal{E}}_\chi(X,\omega)$.*

One always has $\hat{\mathcal{E}}_\chi(X,\omega) \subset \mathcal{E}_\chi(X,\omega)$, while

$$\mathcal{E}_{\hat{\chi}}(X,\omega) \subset \hat{\mathcal{E}}_\chi(X,\omega), \ \textit{where}\ \chi'(t-1) = t^n \hat{\chi}'(t).$$

Since we are mainly interested in the sequel in weights with (super) fast growth at infinity, the previous proposition shows that $\hat{\mathcal{E}}_\chi(X,\omega)$ and $\mathcal{E}_\chi(X,\omega)$ are roughly the same: a function $\varphi \in PSH(X,\omega)$ belongs to one of these classes if and only if $\mathrm{Cap}_\omega(\varphi < -t)$ decreases fast enough, as $t \to +\infty$.

Proof. The convexity of $\hat{\mathcal{E}}_\chi(X,\omega)$ follows from the following simple observation: if $\varphi, \psi \in \hat{\mathcal{E}}_\chi(X,\omega)$ and $0 \le a \le 1$, then

$$\{a\varphi + (1-a)\psi < -t\} \subset \{\varphi < -t\} \cup \{\psi < -t\}.$$

The stability under maximum is obvious.

Assume $\varphi \in \hat{\mathcal{E}}_\chi(X,\omega)$. We can assume without loss of generality $\varphi \le 0$ and $\chi(0) = 0$. Set $\varphi_j := \max(\varphi, -j)$. It follows from Lemma 2.3 below that

$$\begin{aligned}\int_X (-\chi)\circ\varphi_j\,(\omega + dd^c\varphi_j)^n &= \int_0^{+\infty} \chi'(-t)(\omega + dd^c\varphi_j)^n(\varphi_j < -t)dt \\ &\le \int_0^{+\infty} \chi'(-t)t^n \mathrm{Cap}_\omega(\varphi < -t)dt < +\infty.\end{aligned}$$

This shows that $\varphi \in \mathcal{E}_\chi(X,\omega)$. The other inclusion goes similarly, using the second inequality in Lemma 2.3 below. □

If $\varphi \in \mathcal{E}_\chi(X,\omega)$ (or $\hat{\mathcal{E}}_\chi(X,\omega)$), then the bigger the growth of χ at $-\infty$, the smaller $\mathrm{Cap}_\omega(\varphi < -t)$ when $t \to +\infty$, hence the closer φ is from being bounded.

Indeed $\varphi \in PSH(X, \omega)$ is bounded iff it belongs to $\mathcal{E}_\chi(X, \omega)$ for all weights χ, as was observed in [10, Proposition 3.1]. Similarly

$$PSH(X, \omega) \cap L^\infty(X) = \bigcap_\chi \hat{\mathcal{E}}_\chi(X, \omega),$$

where the intersection runs over all concave increasing functions χ.

We will make constant use of the following result:

Lemma 2.3. *Fix $\varphi \in \mathcal{E}(X, \omega)$. Then for all $s > 0$ and $0 \le t \le 1$,*

$$t^n \mathrm{Cap}_\omega(\varphi < -s - t) \le \int_{(\varphi < -s)} (\omega + dd^c \varphi)^n \le s^n \mathrm{Cap}_\omega(\varphi < -s),$$

where the second inequality is true only for $s \ge 1$.

The proof is a direct consequence of the comparison principle (see [8, Lemma 2.2] and [10]).

3. Measures dominated by capacity

From now on μ denotes a positive Radon measure on X whose total mass is $\mathrm{Vol}_\omega(X)$: this is an obvious necessary condition in order to solve $(MA)_\mu$. To simplify numerical computations, we assume in the sequel that μ and ω have been normalized so that

$$\mu(X) = \mathrm{Vol}_\omega(X) = \int_X \omega^n = 1.$$

When $\mu = e^h \omega^n$ is a smooth volume form and ω is a Kähler form, S. T. Yau has proved [19] that $(MA)_\mu$ admits a unique *smooth* solution $\varphi \in PSH(X, \omega)$ with $\sup_X \varphi = 0$. Smooth measures are easily seen to be nicely dominated by the Monge-Ampère capacity (see the proof of Corollary 3.2 below).

Measures dominated by the Monge-Ampère capacity have been extensively studied by S.Kolodziej in [12–14]. Following S. Kolodziej ([13, 14]) with slightly different notations, fix $\varepsilon : \mathbb{R} \to [0, \infty[$ a continuous decreasing function and set

$$F_\varepsilon(x) := x[\varepsilon(-\ln x/n)]^n, x > 0.$$

We will consider probability measures μ satisfying the following condition : for all Borel subsets $K \subset X$,

$$\mu(K) \le F_\varepsilon(\mathrm{Cap}_\omega(K)).$$

The main result achieved in [12], can be formulated as follows: If ω is a Kähler form and $\int_0^{+\infty} \varepsilon(t)dt < +\infty$ then $\mu = (\omega + dd^c\varphi)^n$ for some *continuous* function $\varphi \in PSH(X, \omega)$.

The condition $\int_0^{+\infty} \varepsilon(t)dt < +\infty$ means that ε decreases fast enough towards zero at infinity. This gives a quantitative estimate on how fast $\varepsilon(-\ln \mathrm{Cap}_\omega(K)/n)$, hence $\mu(K)$, decreases towards zero as $\mathrm{Cap}_\omega(K) \to 0$.

When $\int_0^{+\infty} \varepsilon(t)dt = +\infty$, it follows from Theorem 2.1 that $\mu = (\omega + dd^c\varphi)^n$ for some function $\varphi \in \mathcal{E}(X, \omega)$, but φ will generally be unbounded. Our second main result measures how far φ is from being bounded:

Theorem 3.1. *Assume for all compact subsets $K \subset X$,*

$$\mu(K) \leq F_\varepsilon(\mathrm{Cap}_\omega(K)). \tag{3.1}$$

Then $\mu = (\omega + dd^c\varphi)^n$ where $\varphi \in \mathcal{E}(X, \omega)$ is such that $\sup_X \varphi = 0$ *and*

$$\mathrm{Cap}_\omega(\varphi < -s) \leq \exp(-nH^{-1}(s)), \text{ for all } s > 0.$$

Here H^{-1} is the reciprocal function of $H(x) = e\int_0^x \varepsilon(t)dt + s_0$, where $s_0 = s_0(\varepsilon, \omega) \geq 0$ is a constant which only depends on ε and ω.

In particular $\varphi \in \mathcal{E}_\chi(X, \omega)$ where $-\chi(-t) = \exp(nH^{-1}(t)/2)$.

Recall that here, and troughout the article, $\omega \geq 0$ is merely big.

Before proving this result we make a few observations.

- It is interesting to consider as well the case when $\varepsilon(t)$ increases towards $+\infty$. One can then obtain solutions φ such that $\mathrm{Cap}_\omega(\varphi < -t)$ decreases at a polynomial rate. When *e.g.* ω is Kähler and $\mu(K) \leq \mathrm{Cap}_\omega(K)^\alpha$, $0 < \alpha < 1$, it follows from [10, Proposition 5.3] that $\mu = (\omega + dd^c\varphi)^n$ where $\varphi \in \mathcal{E}^p(X, \omega)$ for some $p = p_\alpha > 0$. Here $\mathcal{E}^p(X, \omega)$ denotes the Cegrell type class $\mathcal{E}_\chi(X, \omega)$, with $\chi(t) = -(-t)^p$.
- When $\varepsilon(t) \equiv 1$, $F_\varepsilon(x) = x$ and $H(x) \asymp e.x$. Thus Theorem 3.1 reads $\mu \leq \mathrm{Cap}_\omega \Rightarrow \mu = (\omega + dd^c\varphi)^n$, where

$$\mathrm{Cap}_\omega(\varphi < -s) \lesssim \exp\left(-ns/e\right).$$

  This is precisely the rate of decreasing corresponding to functions which look locally like $-\log(-\log ||z||)$, in some local chart $z \in U \subset \mathbb{C}^n$. This class of ω-psh functions with "loglog-singularities" is important for applications (see [5, 15]).
- If $\varepsilon(t)$ decreases towards zero, then $\mathrm{Cap}_\omega(\varphi < -t)$ decreases at a superexponential rate. The faster $\varepsilon(t)$ decreases towards zero, the slower the growth of H, hence the faster the growth of H^{-1} at infinity. When $\int^{+\infty} \varepsilon(t)dt < +\infty$, the function ε decreases so fast that $\mathrm{Cap}_\omega(\varphi < -t) = 0$ for $t >> 1$, thus φ is bounded. This is the case when $\mu(K) \leq \mathrm{Cap}_\omega(K)^\alpha$ for some $\alpha > 1$ [8, 12].
- When $\int^{+\infty} \varepsilon(t)dt = +\infty$, the solution φ may well be unbounded (see examples in Section 4). At the critical case where $\mu \leq F_\varepsilon(\mathrm{Cap}_\omega)$ for all functions ε such that $\int^{+\infty} \varepsilon(t)dt = +\infty$, we obtain

$$\mu = (\omega + dd^c\varphi)^n \text{ with } \varphi \in PSH(X, \omega) \cap L^\infty(X),$$

as follows from [10, Proposition 3.1]. This partially explains the difficulty in describing the range of Monge-Ampère operators on the set of *bounded* (quasi-)psh functions.

Proof. The assumption on μ implies in particular that it vanishes on pluripolar sets. It follows from Theorem 2.1 that there exists a function $\varphi \in \mathcal{E}(X, \omega)$ such that $\mu = (\omega + dd^c\varphi)^n$ and $\sup_X \varphi = 0$. Set

$$g(s) := -\frac{1}{n} \log \mathrm{Cap}_\omega(\varphi < -s), \quad \forall s > 0.$$

The function g is increasing on $[0, +\infty]$ and $g(+\infty) = +\infty$, since Cap_ω vanishes on pluripolar sets. Observe also that $g(s) \geq 0$ for all $s \geq 0$, since

$$g(0) = -\frac{1}{n} \log \mathrm{Cap}_\omega(X) = -\frac{1}{n} \log \mathrm{Vol}_\omega(X) = 0.$$

It follows from Lemma 2.3 and (3.1) that for all $s > 0$ and $0 \leq t \leq 1$,

$$t^n \mathrm{Cap}_\omega(\varphi < -s - t) \leq \mu(\varphi < -s) \leq F_\varepsilon\left(\mathrm{Cap}_\omega(\varphi < -s)\right).$$

Therefore for all $s > 0$ and $0 \leq t \leq 1$,

$$\log t - \log \varepsilon \circ g(s) + g(s) \leq g(s + t). \tag{3.2}$$

We define an increasing sequence $(s_j)_{j \in \mathbb{N}}$ by induction setting

$$s_{j+1} = s_j + e\varepsilon \circ g(s_j), \text{ for all } j \in \mathbb{N}.$$

The choice of s_0

Recall that (3.2) is only valid for $0 \leq t \leq 1$. We choose $s_0 \geq 0$ large enough so that

$$e\varepsilon \circ g(s_0) \leq 1. \tag{3.3}$$

This will allow us to use (3.2) with $t = t_j = s_{j+1} - s_j \in [0, 1]$, since $\varepsilon \circ g$ is decreasing, while $s_j \geq s_0$ is increasing, hence

$$0 \leq t_j = e\varepsilon \circ g(s_j) \leq e\varepsilon \circ g(s_0) \leq 1.$$

We must insure that $s_0 = s_0(\varepsilon, \omega)$ can be chosen to be independent of φ. This is a consequence of [9, Proposition 2.7]: since $\sup_X \varphi = 0$, there exists $c_1(\omega) > 0$ so that $0 \leq \int_X (-\varphi)\omega^n \leq c_1(\omega)$, hence

$$g(s) := -\frac{1}{n} \log \mathrm{Cap}_\omega(\varphi < -s) \geq \frac{1}{n} \log s - \frac{1}{n} \log(n + c_1(\omega)).$$

Therefore $g(s_0) \geq \varepsilon^{-1}(1/e)$ for $s_0 = s_0(\varepsilon, \omega) := (n + c_1(\omega)) \exp(n\varepsilon^{-1}(1/e))$, which is independent of φ. This yields $e\varepsilon \circ g(s_0) \leq 1$, as desired.

The growth of s_j

We can now apply (3.2) and get $g(s_j) \geq j + g(s_0) \geq j$. Thus $\lim g(s_j) = +\infty$. There are two cases to be considered.

If $s_\infty = \lim s_j \in \mathbb{R}^+$, then $g(s) \equiv +\infty$ for $s > s_\infty$, *i.e.* $\mathrm{Cap}_\omega(\varphi < -s) = 0$, $\forall s > s_\infty$. Therefore φ is bounded from below by $-s_\infty$, in particular $\varphi \in \mathcal{E}_\chi(X, \omega)$ for all χ.

Assume now (second case) that $s_j \to +\infty$. For each $s > 0$, there exists $N = N_s \in \mathbb{N}$ such that $s_N \leq s < s_{N+1}$. We can estimate $s \mapsto N_s$:

$$s \leq s_{N+1} = \sum_0^N (s_{j+1} - s_j) + s_0 = \sum_{j=0}^N e\,\varepsilon \circ g(s_j) + s_0$$
$$\leq e \sum_0^N \varepsilon(j) + s_0 \leq e.\varepsilon(0) + e \int_0^N \varepsilon(t)dt + s_0 =: H(N).$$

Therefore $H^{-1}(s) \leq N \leq g(s_N) \leq g(s)$, hence

$$\mathrm{Cap}_\omega(\varphi < -s) \leq \exp(-nH^{-1}(s)).$$

Set now $-\chi(-t) = \exp(nH^{-1}(t)/2)$. Then

$$\int_0^{+\infty} t^n \chi'(-t)\mathrm{Cap}_\omega(\varphi < -t)dt$$
$$\leq \frac{n}{2} \int_0^{+\infty} t^n \frac{1}{\varepsilon(H^{-1}(t)) + \tilde{s}_0} \exp(-nH^{-1}(t)/2)dt$$
$$\leq C \int_0^{+\infty} t^n \exp(-nt/2)dt < +\infty.$$

This shows that $\varphi \in \mathcal{E}_\chi(X, \omega)$ where $\chi(t) = -\exp(nH^{-1}(-t)/2)$.

It follows from the proof above that when $\int_0^{+\infty} \varepsilon(t)dt < +\infty$, the solution φ is bounded since in this case we have

$$s_\infty := \lim_{j\to+\infty} s_j \leq s_0(\varepsilon, \omega) + e\,\varepsilon(0) + e \int_0^{+\infty} \varepsilon(t)dt < +\infty$$

where $s_0(\varepsilon, \omega)$ is an absolute constant satisfying (3.3) (see above). □

Let us emphasize that Theorem 3.1 also yields a slightly simplified proof of the following result [8, 12]: if $\mu(K) \leq F_\varepsilon(\mathrm{Cap}_\omega(K))$ for some decreasing function $\varepsilon : \mathbb{R} \to \mathbb{R}^+$ such that $\int^{+\infty} \varepsilon(t)dt < +\infty$, then the sequence (s_j) above is convergent, hence $\mu = (\omega + dd^c\varphi)^n$, where $\varphi \in PSH(X, \omega)$ is *bounded*. For the reader's convenience we indicate a proof of the following important particular case:

Corollary 3.2. *Let $\mu = f\omega^n$ be a measure with density $0 \le f \in L^p(\omega^n)$, where $p > 1$ and $\int_X f\omega^n = \int_X \omega^n$. Then there exists a unique bounded function $\varphi \in PSH(X,\omega)$ such that $(\omega + dd^c\varphi)^n = \mu$, $\sup_X \varphi = 0$ and*

$$0 \le ||\varphi||_{L^\infty(X)} \le C(p,\omega).||f||_{L^p(\omega^n)}^{1/n},$$

where $C(p,\omega) > 0$ only depends on p and ω.

This a priori bound is a crucial step in the proof by S. T. Yau of the Calabi conjecture (see [2,4,6,18,19]). The proof presented here follows Kolodziej's new and decisive pluripotential approach (see [12]). Let us stress that the dependence $\omega \longmapsto C(p,\omega)$ is quite explicit, as we shall see in the proof. This is important when considering degenerate situations [8].

Proof. We claim that there exists $C_1(\omega)$ such that

$$\mu(K) \le \left[C_1(\omega)||f||_{L^p(\omega^n)}^{1/n}\right]^n \left[\mathrm{Cap}_\omega(K)\right]^2, \quad \text{for all Borel sets } K \subset X. \tag{3.4}$$

Assuming this for the moment, we can apply Theorem 3.1 with

$$\varepsilon(x) = C_1(\omega)||f||_{L^p(\omega^n)}^{1/n} \exp(-x),$$

which yields, as observed at the end of the proof of Theorem 3.1

$$||\varphi||_{L^\infty(X)} \le M(f,\omega),$$

where

$$M(f,\omega) := s_0(\varepsilon,\omega) + e\,\varepsilon(0) + e\int_0^{+\infty} \varepsilon(t)dt = s_0(\varepsilon,\omega) + 2eC_1(\omega)||f||_{L^p(\omega^n)}^{1/n}$$

and $s_0 = s_0(\varepsilon,\omega)$ is a large number $s_0 > 1$ satisfying the inequality (3.3).

In order to give the precise dependence of the uniform bound $M(f,\omega)$ on the L^p−norm of the density f, we need to choose s_0 more carefully. Observe that condition (3.3) can be written

$$\mathrm{Cap}_\omega(\{\varphi \le -s_0\}) \le \exp(-n\varepsilon^{-1}(1/e)).$$

Since $n\varepsilon^{-1}(1/e) = \log\Big(e^n C_1(\omega)^n \|f\|_{L^p(\omega^n)}\Big)$, we must choose $s_0 > 0$ so that

$$\mathrm{Cap}_\omega(\{\varphi \le -s_0\}) \le \frac{1}{e^n C_1(\omega)^n \|f\|_{L^p(\omega^n)}}. \tag{3.5}$$

We claim that for any $N \ge 1$ there exists a uniform constant $C_2(N,p,\omega) > 0$ such that for any $s > 0$,

$$\mathrm{Cap}_\omega(\{\varphi \le -s\}) \le C_2(N,p)\, s^{-N}\, \|f\|_{L^p(\omega^n)}. \tag{3.6}$$

Indeed observe first that by Hölder inequality,

$$\int_X (-\varphi)^N \omega_\varphi^n = \int_X (-\varphi)^N f \omega^n \leq \|f\|_{L^p(\omega^n)} \|\varphi\|^N_{L^{Nq}(\omega^n)}.$$

Since φ belongs to the compact family $\{\psi \in PSH(X,\omega);\, \sup_X \psi = 0\}$ ([10]), there exists a uniform constant $C_2'(N,p,\omega) > 0$ such that $\|\varphi\|^N_{L^{Nq}(\omega^n)} \leq C_2'(N,p,\omega)$, hence

$$\int_X (-\varphi)^N \omega_\varphi^n \leq C_2'(N,p,\omega) \|f\|_{L^p(\omega^n)}.$$

Fix $u \in PSH(X,\omega)$ with $-1 \leq u \leq 0$ and $N \geq 1$ to be specified later. If follows from Tchebysheff and energy inequalities ([10]) that

$$\begin{aligned}\int_{\{\varphi \leq -s\}} (\omega + dd^c u)^n &\leq s^{-N} \int_X (-\varphi)^N (\omega + dd^c u)^n \\ &\leq c_N\, s^{-N} \max\left\{ \int_X (-\varphi)^N \omega_\varphi^n, \int_X (-u)^N \omega_u^n \right\} \\ &\leq c_N\, s^{-N} \max\left\{ C_2'(N,p,\omega), 1 \right\} \|f\|_{L^p(\omega^n)}.\end{aligned}$$

We have used here the fact that $\|f\|_{L^p(\omega^n)} \geq 1$, which follows from the normalization : $1 = \int_X \omega^n = \int_X f \omega^n \leq \|f\|_{L^p(\omega^n)}$. This proves the claim.

Set $N = 2n$, it follows from (3.6) that $s_0 := C_1(\omega)^n e^n C_2(2n,p,\omega) \|f\|^{1/n}_{L^p(\omega^n)}$ satisfies the required condition (3.5), which implies the estimate of the theorem.

We now establish the estimate (3.4). Observe first that Hölder's inequality yields

$$\mu(K) \leq ||f||_{L^p(\omega^n)} \left[\mathrm{Vol}_\omega(K)\right]^{1/q}, \text{ where } 1/p + 1/q = 1. \tag{3.7}$$

Thus it suffices to estimate the volume $\mathrm{Vol}_\omega(K)$. Recall the definition of the Alexander-Taylor capacity, $T_\omega(K) := \exp(-\sup_X V_{K,\omega})$, where

$$V_{K,\omega}(x) := \sup\{\psi(x) \,/\, \psi \in PSH(X,\omega),\, \psi \leq 0 \text{ on } K\}.$$

This capacity is comparable to the Monge-Ampère capacity, as was observed by H. Alexander and A. Taylor [1] (see [9, Proposition 7.1] for this compact setting):

$$T_\omega(K) \leq e \exp\left[-\frac{1}{\mathrm{Cap}_\omega(K)^{1/n}} \right]. \tag{3.8}$$

It thus remains to show that $\mathrm{Vol}_\omega(K)$ is suitably bounded from above by $T_\omega(K)$. This follows from Skoda's uniform integrability result: set

$$\nu(\omega) := \sup\{\nu(\psi,x) \,/\, \psi \in PSH(X,\omega),\ x \in X\},$$

where $\nu(\psi, x)$ denotes the Lelong number of ψ at point x. This actually only depends on the cohomology class $\{\omega\} \in H^{1,1}(X, \mathbb{R})$. It is a standard fact that goes back to H. Skoda (see [20]) that there exists $C_2(\omega) > 0$ so that

$$\int_X \exp\left(-\frac{1}{\nu(\omega)}\psi\right) \omega^n \leq C_2(\omega),$$

for all functions $\psi \in PSH(X, \omega)$ normalized by $\sup_X \psi = 0$. We infer

$$\mathrm{Vol}_\omega(K) \leq \int_K \exp\left(-\frac{1}{\nu(\omega)} V^*_{K,\omega}\right) \omega^n \leq C_2(\omega)[T_\omega(K)]^{1/\nu(\omega)}. \tag{3.9}$$

It now follows from (3.7), (3.8), (3.9), that

$$\mu(K) \leq ||f||_{L^p}[C_2(\omega)]^{1/q} e^{1/q\nu(\omega)} \exp\left[-\frac{1}{q\nu(\omega)\mathrm{Cap}_\omega(K)^{1/n}}\right].$$

The conclusion follows by observing that $\exp(-1/x^{1/n}) \leq C_n x^2$ for some explicit constant $C_n > 0$. □

4. Examples

4.1. Measures invariant by rotations

In this section we produce examples of radially invariant functions/measures which show that our previous results are essentially sharp. The first example is due to S. Kolodziej [11].

Example 4.1. We work here on the Riemann sphere $X = \mathbb{P}^1(\mathbb{C})$, with $\omega = \omega_{FS}$, the Fubini-Study volume form. Consider $\mu = f\omega$ a measure with density f which is smooth and positive on $X \setminus \{p\}$, and such that

$$f(z) \simeq \frac{c}{|z|^2(\log|z|)^2}, \ c > 0,$$

in a local chart near $p = 0$. A simple computation yields $\mu = \omega + dd^c\varphi$, where $\varphi \in PSH(\mathbb{P}^1, \omega)$ is smooth in $\mathbb{P}^1 \setminus \{p\}$ and $\varphi(z) \simeq -c' \log(-\log|z|)$ near $p = 0$, $c' > 0$, hence

$$\log \mathrm{Cap}_\omega(\varphi < -t) \simeq -t,$$

Here $a \simeq b$ means that a/b is bounded away from zero and infinity.

This is to be compared to our estimate $\log \mathrm{Cap}_\omega(\varphi < -t) \lesssim -t/e$ (Theorem 3.1) which can be applied, as it was shown by S.Kolodziej in [11] that $\mu \lesssim \mathrm{Cap}_\omega$. Thus Theorem 3.1 is essentially sharp when $\varepsilon \equiv 1$.

We now generalize this example and show that the estimate provided by Theorem 3.1 is essentially sharp in all cases.

Example 4.2. Fix ε as in Theorem 3.1. Consider $\mu = f\omega$ on $X = \mathbb{P}^1(\mathbb{C})$, where $\omega = \omega_{FS}$ is the Fubini-Study volume form, $f \geq 0$ is continuous on $\mathbb{P}^1 \setminus \{p\}$, and

$$f(z) \simeq \frac{\varepsilon(\log(-\log|z|))}{|z|^2(\log|z|)^2}$$

in local coordinates near $p = 0$. Here $\varepsilon : \mathbb{R} \to \mathbb{R}^+$ decreases towards 0 at $+\infty$. We claim that there exists $A > 0$ such that

$$\mu(K) \leq A\mathrm{Cap}_\omega(K)\varepsilon(-\log \mathrm{Cap}_\omega(K)), \text{ for all } K \subset X. \tag{4.1}$$

This is clear outside a small neighborhood of $p = 0$ since the measure μ is there dominated by a smooth volume form. So it suffices to establish this estimate when K is included in a local chart near $p = 0$. Consider

$$\tilde{K} := \{r \in [0, R]\,;\; K \cap \{|z| = r\} \neq \emptyset\}.$$

It is a classical fact (see *e.g.* [17]) that the logarithmic capacity $c(K)$ of K can be estimated from below by the length of $\tilde{K}$, namely

$$\frac{l(\tilde{K})}{4} \leq c(\tilde{K}) \leq c(K).$$

Using that ε is decreasing, hence $0 \leq -\varepsilon'$, we infer

$$\begin{aligned}
\mu(K) &\leq 2\pi \int_0^{l(\tilde{K})} f(r) r dr \\
&\leq 2\pi \int_0^{l(\tilde{K})} \frac{\varepsilon(\log(-\log r)) - \varepsilon'(\log - \log r)}{r(\log r)^2} dr \\
&= 2\pi \frac{\varepsilon(\log(-\log l(\tilde{K})))}{-\log l(\tilde{K})} \leq 2\pi \frac{\varepsilon(\log(-\log 4c(K)))}{-\log 4c(K)}.
\end{aligned}$$

Recall now that the logarithmic capacity c(K) is equivalent to Alexander-Taylor's capacity $T_\Delta(K)$, which in turn is equivalent to the global Alexander-Taylor capacity $T_\omega(K)$ (see [9]): $c(K) \simeq T_\Delta(K) \simeq T_\omega(K)$. The Alexander-Taylor's comparison theorem [1] reads

$$-\log 4c(K) \simeq -\log T_\omega(K) \simeq 1/\mathrm{Cap}_\omega(K),$$

thus $\mu(K) \leq A\mathrm{Cap}_\omega(K)\varepsilon(-\log \mathrm{Cap}_\omega(K))$.

We can therefore apply Theorem 3.1. It guarantees that $\mu = (\omega + dd^c\varphi)$, where $\varphi \in PSH(\mathbb{P}^1, \omega)$ satisfies $\log \mathrm{Cap}_\omega(\varphi < -s) \simeq -nH^{-1}(s)$, with $H(s) =$

$eA\int_0^s \varepsilon(t)dt + s_0$. On the other hand a simple computation shows that φ is continuous in $\mathbb{P}^1 \setminus \{p\}$ and

$$\varphi \simeq -H(\log(-\log|z|)) \ , \quad \text{near } p = 0.$$

The sublevel set $(\varphi < -t)$ therefore coincides with the ball of radius

$$\exp(-\exp(H^{-1}(t))),$$

hence $\log \mathrm{Cap}_\omega(\varphi < -s) \simeq -H^{-1}(s)$.

4.2. Measures with density

Here we consider the case when $\mu = f dV$ is absolutely continuous with respect to a volume form.

Proposition 4.3. *Assume $\mu = f\omega^n$ is a probability measure whose density satisfies $f[\log(1+f)]^n \in L^1(\omega^n)$. Then $\mu \lesssim \mathrm{Cap}_\omega$.*

More generally if $f[\log(1+f)/\varepsilon(\log(1+|\log f|))]^n \in L^1(\omega^n)$ for some continuous decreasing function $\varepsilon : \mathbb{R} \to \mathbb{R}^+_*$, then for all $K \subset X$,

$$\mu(K) \leq F_\varepsilon(\mathrm{Cap}_\omega(K)), \text{ where } F_\varepsilon(x) = Ax\left[\varepsilon\left(-\frac{\ln x}{n}\right)\right]^n, \ A > 0.$$

Proof. With slightly different notations, the proof is identical to that of Lemma 4.2 in [14] to which we refer the reader. □

We now give examples showing that Proposition 4.3 is almost optimal.

Example 4.4. For simplicity we give local examples. The computations to follow can also be performed in a global compact setting.

Consider $\varphi(z) = -\log(-\log||z||)$, where $||z|| = \sqrt{|z_1|^2 + \ldots + |z_n|^2}$ denotes the Euclidean norm in $\mathbb{C}^n$. One can check that φ is plurisubharmonic in a neighborhood of the origin in $\mathbb{C}^n$, and that there exists $c_n > 0$ so that

$$\mu := (dd^c\varphi)^n = f\, dV_{\mathrm{eucl}}, \text{ where } f(z) = \frac{c_n}{||z||^{2n}(-\log||z||)^{n+1}}.$$

Observe that $f[\log(1+f)]^{n-\alpha} \in L^1, \ \forall\alpha > 0$ but $f[\log(1+f)]^n \notin L^1$.

When $n = 1$ it was observed by S. Kolodziej [11] that $\mu(K) \lesssim \mathrm{Cap}_\omega(K)$. Proposition 4.3 yields here

$$\mu(K) \lesssim \mathrm{Cap}_\omega(K)(|\log \mathrm{Cap}_\omega(K)| + 1).$$

For $n \geq 1$, it follows from Proposition 4.3 and Theorem 3.1 that

$$\log \mathrm{Cap}_\omega(\varphi < -s) \lesssim -nH^{-1}(s).$$

On the other hand, one can directly check that $\log \mathrm{Cap}_\omega(\varphi < -s) \simeq -nH^{-1}(s)$.

One can get further examples by considering $\varphi(z) = \chi \circ \log||z||$, so that

$$(dd^c\varphi)^n = \frac{c_n(\chi' \circ \log||z||)^{n-1}\chi''(\log||z||)}{||z||^{2n}} dV_{\mathrm{eucl}}.$$

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